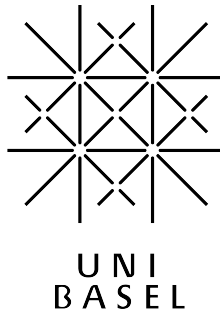UNIVERSITÉ DE RENNES 1

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Subgroups of Cremona groups

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Chapter 1

Résumé en français

1.1 Introduction et contexte historique

Soit X une variété algébrique sur un corps $\mathbf{k}$ et $\mathrm{Bir}(X)$ son groupe des transformations birationnelles. Les buts principaux pour étudier $\mathrm{Bir}(X)$ sont:

(1) Comprendre la géométrie de X en étudiant la structure de groupe de $\mathrm{Bir}(X)$.

(2) Comprendre la structure de groupe de $\mathrm{Bir}(X)$ en étudiant la géométrie de X.

Souvent on se concentrera sur le cas où X est une variété rationnelle. Dans ce cas, la structure de $\mathrm{Bir}(X)$ est particulièrement riche et intéressante. En l'honneur du mathématicien italien Luigi Cremona (1830-1903), le groupe $\mathrm{Bir}(X)$ est dans ce cas appelé *groupe de Cremona en n-variables*, où n est la dimension de X. On utilise la notation

$$\mathrm{Cr}_n(\mathbf{k}) := \mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}}).$$

Si $n = 2$ on parle souvent du *groupe de Cremona du plan*.

Si on fixe des coordonnées homogènes $[x_0 : \cdots : x_n]$ de $\mathbb{P}^n_{\mathbf{k}}$, chaque élément $f \in \mathrm{Cr}_n(\mathbf{k})$ est représenté par des polynômes homogènes du même degré $f_0, \ldots, f_n \in \mathbf{k}[x_0, \ldots, x_n]$ sans facteur commun non constant. Alors f est donné par

$$f \colon [x_0 : \cdots : x_n] \dashrightarrow [f_0 : \cdots : f_n].$$

En utilisant des coordonnées affines $[1 : X_1 : \cdots : X_n] = (X_1, \ldots, X_n)$, la transformation birationnelle f est donnée par

$$f : (X_1, \ldots, X_n) \dashrightarrow (F_1, \ldots, F_n),$$

où les $F_i(X_1, \ldots, X_n) \in \mathbf{k}(X_1, \ldots, X_n)$ sont les quotients

$$F_i(X_1, \ldots, X_n) = f_i(1, X_1, \ldots, X_n)/f_0(1, X_1, \ldots, X_n).$$

Ceci définit un (anti-)isomorphisme entre $\mathrm{Cr}_n(\mathbf{k})$ et le groupe des $\mathbf{k}$-automorphismes du corps des fonctions $\mathbf{k}(X_1, \ldots, X_n)$. Plus généralement, on peut identifier $\mathrm{Bir}(X)$ avec le groupe des $\mathbf{k}$-automorphismes du corps des fonctions $\mathbf{k}(X)$.

Les éléments de $\mathrm{Cr}_n(\mathbf{k})$ sont appellés des *transformations de Cremona*. Les premiers exemples étaient déjà utilisé par les mathématiciens dans la Grèce antique. Mais ce ne fut qu'au 19ème siècle, dans le cadre des recherches en géométrie projective, en vogue dans ce temps-là, que les transformations de Cremona furent étudiées dans le contexte de la géométrie algébrique, notamment dans les deux articles [Cre63] et [Cre65] publiés par Cremona en 1863 et 1865. Cette époque est aujourd'hui appelée l'école italienne de la géométrie algébrique. Des travaux importants étaient conduits par Bertini, Castelnuovo, Enriques, Geiser, Hudson, de Jonquières, Kantor, Noether, Segre, Wiman et beaucoup d'autres. On renvoie à [Dol12], [Hud27], [Sny28] et [Dés06a] pour des références et des remarques historiques plus complètes.

De nombreuses techniques et des résultats importants développés par ces géomètres algébriques classiques continuent d'inspirer les mathématiciens aujourd'hui (voir [Dol12] pour un résumé et des références). Un des résultats les plus connus et importants est le théorème de Noether et Castelnuovo:

Théorème 1.1.1 ([Noe70], [Cas01], [Ale16]). Le groupe de Cremona du plan sur un corps algébriquement clos $\mathbf{k}$ est engendré par le groupe des automorphismes $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}}) \simeq \mathrm{PGL}_3(\mathbf{k})$ et l'involution quadratique standard

$$\sigma := [x_0 : x_1 : x_2] \dashrightarrow [x_1 x_2 : x_0 x_2 : x_0 x_1].$$

Remarque 1.1.1. L'énoncé du théorème de Noether et Castelnuovo n'est plus vrai, si $\mathbf{k}$ n'est pas algébriquement clos (voir par example [Isk83], [BM14] pour des généralisations aux cas où $\mathbf{k}$ est parfait, respectivement le cas $\mathbf{k} = \mathbb{R}$).

La classification des sous-groupes finis de $\mathrm{Cr}_2(\mathbf{k})$ est un autre problème classique important. Une classification pour le cas $\mathbf{k} = \mathbb{C}$ a été obtenue en 2007 par Dolgachev et Iskovskikh avec des outils modernes ([DI09], voir aussi [Bla07] et [Tsy13]). La classification pour le cas où la caractéristique de $\mathbf{k}$ est positive est toujours ouverte et fait l'objet de recherches. Le problème de classifier les sous-groupes finis de $\mathrm{Cr}_3(\mathbf{k})$ est plus difficile. Prokhorov a décrit tous les groupes finis simples qui se plongent dans $\mathrm{Cr}_3(\mathbb{C})$ ([Pro12]). D'autres travaux concernant ce sujet sont [CS16] et [Pro11]. En dimension plus grande que trois, les sous-groupes finis des groupes de Cremona sont peu connus. Cependant Birkar a récemment démontré la conjecture BAB (dénommée après A. Borisov, V. Alexeev and L. Borisov) qui implique, par un résultat de Prokhorov et Shramov, que $\mathrm{Cr}_2(\mathbb{C})$ satisfait la propriété de Jordan, c'est-à-dire que pour chaque entier n il existe une constante $C(n)$ telle que chaque sous-groupe de $\mathrm{Cr}_n(\mathbb{C})$ est abélien d'indice fini plus petit que $C(n)$ ([PS16], [Bir16]).

Les groupes de Cremona peuvent être munis de structures algébriques. On peut définir notamment la *topologie de Zariski* (voir [BF13] pour une vue d'ensemble sur le sujet et les propriétés de cette topologie). Cette topologie permet de définir des sous-groupes *algébriques*. Enriques a classé tous les sous-groupes algébriques connexes maximaux de $\mathrm{Cr}_2(\mathbb{C})$ ([Enr93]). Cette classification a été réécrite dans un langue plus moderne par Umemura ([Ume82b]) et étendue par Blanc à une classification des sous-groupes algébriques maximaux, pas nécessairement connexes ([Bla09]). Umemura a aussi classé les sous-groupes algébriques connexes maximaux de $\mathrm{Cr}_3(\mathbb{C})$ ([Ume85, Ume85]). Parmi les travaux sur les sous-groupes algébriques

des groupes de Cremona il faut aussi mentionner ceux de Demazure [Dem70] et
Bialynicki-Birula [BB67].

Vers le milieu du 20ème siècle la recherche en géométrie algébrique s'est plutôt
concentrée sur les fondations du sujet et les questions classiques n'étaient plus au
centre de l'intérêt. La recherche en géométrie birationnelle et notamment sur les
groupes des transformations birationnelles a prospéré de nouveau quelques décades
plus tard en Russie avec Danilov, Gizatullin, Iskovskikh, Manin et d'autres. Suite
aux travaux de Mori et d'autres dans les années 1980, la géométrie birationnelle a
gagné encore plus d'attention et est devenue un des sujets principaux de la géométrie
algébrique. En 2008 deux séminaires Bourbaki avaient comme sujet le groupe de
Cremona du plan ([Fav10], [Ser10]) et ont remis le sujet de nouveau en vogue.

Des nouvelles techniques, qui venaient principalement de la dynamique, ont
conduit à des résultats qui ont contribué considérablement à la compréhension
du groupe de Cremona du plan. Une des percées principales fut sans doute la
construction d'une action de $\mathrm{Bir}(S)$ par isométries sur un espace hyperbolique $\mathbb{H}^\infty$
de dimension infinie par Manin ([Man86]) et l'application des outils de la théorie
géométrique des groupes sur cette action pour obtenir des résultats sur la structure
de groupe de $\mathrm{Bir}(S)$ par Cantat ([Can11a]). Ici, S est une surface projective sur un
corps $\mathbf{k}$. Un élément $f \in \mathrm{Cr}_2(\mathbf{k})$ est appelé respectivement *elliptique, parabolique*
ou *loxodromique*, si l'isométrie de $\mathbb{H}^\infty$ correspondante est *elliptique, parabolique* ou
loxodromique. Cette distinction se traduit au niveau du comportement dynamique
de f. Si X est une variété projective lisse et H une polarisation de X, c'est-à-dire
un diviseur ample, alors le *degré* $\deg_H(f) \in \mathbb{Z}_+$ d'une transformation rationnelle
dominante f de X par rapport à H est défini par

$$\deg_H(f) = f^*H \cdot H^{d-1},$$

oú d est la dimension de X et f^*H est la transformation totale de H par f.

Théorème 1.1.2 (Gizatullin; Cantat; Diller and Favre)**.** Soit $\mathbf{k}$ un corps algé-
briquement clos, S une surface projective sur $\mathbf{k}$ avec une polarisation H et $f \in$
$\mathrm{Bir}(S)$. Alors on se trouve dans un des cas suivants:

(1) f est elliptique, la suite $\{\deg_H(f^n)\}$ est bornée, il existe un entier $k \in \mathbb{Z}_+$ et
une application birationnelle $\varphi \colon S \dashrightarrow S'$ vers une surface projective lisse S'
telle que $\varphi f^k \varphi^{-1}$ est contenu dans $\mathrm{Aut}^0(S')$, la composante neutre du groupe
des automorphismes $\mathrm{Aut}(S')$.

(2a) f est parabolique et $\deg_H(f^n) \sim cn$ pour une constante positive c et f
préserve une fibration rationnelle, c'est-à-dire qu'il existe une surface pro-
jective lisse S', une application birationnelle $\varphi \colon S \dashrightarrow S'$, une courbe B et
une fibration rationnelle $\pi \colon S' \to B$, telles que $\varphi f \varphi^{-1}$ permute les fibres de
π.

(2b) f est parabolique et $\deg_H(f^n) \sim cn^2$ pour une constante positive c et f
préserve une fibration elliptique ou quasi-elliptique (si $\mathrm{car}(\mathbf{k}) = 2$ or 3), c'est-
à-dire qu'il existe une surface projective lisse S', une application birationnelle
$\varphi \colon S \dashrightarrow S'$, une courbe B et une fibration elliptique ou quasi-elliptique
$\pi \colon S' \to B$, telles que $\varphi f \varphi^{-1}$ permute les fibres de π.

(3) f est loxodromique et $\deg_H(f^n) = c\lambda(f)^n + O(1)$ pour une constante positive c. Ici, $\lambda(f)$ est le degré dynamique de f. Dans ce cas f ne préserve aucune fibration.

Le théorème 1.1.2 a mené à des résultats remarquables sur la structure de groupe de $\mathrm{Bir}(S)$, où S est une surface. Du point de vue de la théorie géométrique des groupes, le groupe de Cremona agissant sur $\mathbb{H}^\infty$ partage des propriétés avec d'autres groupes agissant sur des espaces hyperboliques comme par example le groupe modulaire d'une surface agissant sur le complexe des courbes ou le groupe des automorphismes extérieurs d'un groupe libre à n générateurs agissant sur l'outre espace.

Dans [Can11a], Cantat a démontré, entre autres résultats, que le groupe de Cremona du plan satisfait *l'alternative de Tits pour les sous-groupes de type fini*, c'est-à-dire que chaque sous-groupe de type fini de $\mathrm{Cr}_2(\mathbb{C})$ est soit résoluble à indice fini près, soit il contient un groupe libre à deux générateurs. Blanc a démontré que le groupe de Cremona du plan ne contient pas de sous-groupe propre distingué fermé par rapport à la topologie de Zariski (voir [Bla10], ou [BZ15] pour une généralisation de ce résultat à la dimension quelconque). Dans [CLdC13], Cantat et Lamy ont démontré que le groupe de Cremona du plan n'est pas simple et ainsi ont répondu à une des questions les plus importantes du sujet. L'outil principal de leur article consiste à coupler l'action de $\mathrm{Cr}_2(\mathbb{C})$ sur $\mathbb{H}^\infty$ avec les résultats du Théorème 1.1.2 et puis d'appliquer la théorie de petite simplification (voir par exemple [Cou16]).

Cette thèse contribue à la recherche sur les groupes de Cremona en considérant des classes de "grands" sous-groupes.

Dans le **chapitre 3** on considère des plongements algébriques de $\mathrm{Cr}_2(\mathbb{C})$ vers le groupe des transformations birationnelles $\mathrm{Bir}(M)$ d'une variété algébrique M. D'abord on étudie les propriétés géométriques d'un exemple d'un plongement de $\mathrm{Cr}_2(\mathbb{C})$ dans $\mathrm{Cr}_5(\mathbb{C})$ décrit par Gizatullin. Dans une deuxième étape on donne une classification des homomorphismes algébriques de $\mathrm{Cr}_2(\mathbb{C})$ vers $\mathrm{Bir}(M)$, où M est une variété de dimension 3 et on généralise partiellement ce résultat à des homomorphismes algébriques de $\mathrm{Cr}_2(\mathbb{C})$ vers $\mathrm{Bir}(M)$, où M est une variété algébrique de dimension $n + 1$. Cela donne notamment une classification de tous les actions régulières de $\mathrm{PGL}_{n+1}(\mathbb{C})$ sur des variétés projectives lisses de dimension $n + 1$ qui s'étendent à une action rationnelle de $\mathrm{Cr}_n(\mathbb{C})$. Le chapitre 3 est le texte d'un article qui est accepté pour publication dans les "Annales de l'Institut Fourier" avec un appendice additionnel.

Les suites de degrés d'itérés des transformation birationnelles jouent un rôle principal pour comprendre la structure des groupes de transformations birationnelles des surfaces. Très peu est connu sur ces suites dans les dimensions plus grandes que deux. Au **chapitre 4** on considère quelques questions concernant les suites de degrés des transformations dominantes rationnelles des variétés de dimension quelconque. On démontre notamment de nouvelles contraintes sur leur croissance et on étudie quelques exemples. En plus on démontre que *l'ensemble de tous les suites de degrés est dénombrable*. Ceci généralise un résultat de Bonifant et Fornaess ([BF00]). Le chapitre 4 est le texte d'un article qui est accepté pour publication dans "Mathematical Research Letters".

Dans le **chapitre 5** on explique quelques résultats connus sur le groupe de Cremona du plan, dont on aura besoin aux chapitres 6 et 7.

Le chapitre 6 est dédié aux sous-groupes de $\mathrm{Cr}_2(\mathbb{C})$ qui ne contiennent que des éléments elliptiques. Jusqu' à présent les résultats sur ce type de groupes n'existent que pour le cas de groupes de type fini ([Can11a]). L'idée principale de ce chapitre est d'utiliser le théorème de compacité de la théorie des modèles pour généraliser des résultats sur les groupes de type fini à des résultats sur des groupes quelconque. Ceci nous permet de déduire des résultats sur la structure des groupes qui ne contiennent que des éléments elliptiques. On verra notamment que $\mathrm{Cr}_2(\mathbb{C})$ *satisfait l'alternative de Tits pour tous les sous-groupes*, c'est-à-dire que chaque sous-groupe de $\mathrm{Cr}_2(\mathbb{C})$ est soit résoluble à indice fini près, soit il contient un groupe libre à deux générateurs. On démontre également que *tous les sous-groupes de torsion de* $\mathrm{Cr}_2(\mathbb{C})$ *sont abéliens à indice fini près* et que l*a longueur dérivée des sous-groupes résolubles de* $\mathrm{Cr}_2(\mathbb{C})$ *est bornée.*

Dans le **chapitre 7** on considère des sous-groupes simples de $\mathrm{Cr}_2(\mathbb{C})$. On démontre que *tous les sous-groupes simples de type fini de* $\mathrm{Cr}_2(\mathbb{C})$ *sont finis*. Si on suppose qu'une conjecture ouverte est vraie, on peut démontrer qu'un sous-groupe de $\mathrm{Cr}_2(\mathbb{C})$ qui contient un élément loxodromique n'est pas simple. Les ingrédients principaux pour cette preuve sont des résultats de la théorie de la petite simplification de [CLdC13] et des travaux plus récents de Shepherd-Barron [SB13]. Le résultat principal de ce chapitre, est que *si une conjecture ouverte est vraie (conjecture 7.1.1), alors un sous-groupe simple se plonge dans* $\mathrm{Cr}_2(\mathbb{C})$ *si et seulement s'il se plonge dans* $\mathrm{PGL}_3(\mathbb{C})$.

Dans le **chapitre 8** on résume les questions ouvertes les plus importantes qui se sont posées pendant la thèse et à laquelles je ne sais pas encore répondre.

Dans ce qui suit on donne un résumé de chaque chapitre qui décrit les principaux résultats originaux.

1.2   Homomorphismes entre groupes de Cremona

Dans ce chapitre on travaille toujours sur le corps des nombres complexes. Le point de départ de nos considérations est la théorie des représentations de dimension finie des groupes de Lie. Il existe une théorie assez complète sur les homomorphismes de groupes de Lie entre les groupes classiques. Ces homomorphismes peuvent être décrits de manière satisfaisante en fonction des vecteurs de poids maximal (voir par exemple [FH91]). La stratégie principale consiste à étudier les groupes suivants qu'on peut associer à un groupe de Lie complexe G:

- un tore fixe T de rang maximal, c'est-à-dire un sous-groupe de la forme $(\mathbb{C}^*)^n$, où n est maximal entre tous les sous-groupe de cette forme:

- le groupe de Weyl associé à T, c'est-à-dire le groupe $\mathrm{Norm}_G(T)/T$;

- les sous-groupes de racines, c'est-à-dire les sous-groupes additifs à un paramètre de G qui sont normalisés par T.

Parfois, les groupes de Cremona sont vus comme des analogues en dimension infinie des groupes de Lie complexes. De cette perspective on pourrait rêver d'une théorie des "représentations de Cremona", d'une classification complète de tous les

homomorphismes de $\mathrm{Cr}_n(\mathbb{C})$ vers $\mathrm{Bir}(M)$ qui préservent des structures algébriques, pour des variétés projectives quelconques M.

Il existe des analogues des tores maximaux et des groupes de Weyl dans $\mathrm{Cr}_n(\mathbb{C})$. Soit $D_n \subset \mathrm{Cr}_n(\mathbb{C})$ le sous-groupe de dimension n formé des automorphismes diagonaux. C'est le *tore de rang maximal* selon le sens suivant: soit D un sous-groupe algébrique de $\mathrm{Cr}_n(\mathbb{C})$ qui est isomorphe à $(\mathbb{C}^*)^k$. Alors $k \leq n$ et si $k = n$, alors D est conjugué à D_n ([BB67]).

Soit $A = (a_{ij}) \in M_n(\mathbb{Z})$ une matrice d'entiers. La matrice A détermine une application rationnelle de l'espace affine vers lui-même:

$$f_A = (x_1^{a_{11}} x_2^{a_{12}} \cdots x_n^{a_{1n}}, x_1^{a_{21}} x_2^{a_{22}} \cdots x_n^{a_{2n}}, \ldots, x_1^{a_{n1}} x_2^{a_{n2}} \cdots x_n^{a_{nn}}).$$

On a $f_A \circ f_B = f_{AB}$ pour $A, B \in M_n(\mathbb{Z})$ et on peut observer que f_A est une transformation birationnelle si et seulement si $A \in \mathrm{GL}_n(\mathbb{Z})$. Ceci défini un homomorphisme injectif $M \colon \mathrm{GL}_n(\mathbb{Z}) \to \mathrm{Cr}_n(\mathbb{C})$; l'image de M dans $\mathrm{Cr}_n(\mathbb{C})$ on appelle le *groupe de Weyl* et on écrit $\mathrm{W}_n \coloneqq M(\mathrm{GL}_n(\mathbb{Z}))$. Cette terminologie est justifiée par le fait que le normalisateur de D_n dans $\mathrm{Cr}_n(\mathbb{C})$ est le produit semi-direct $\mathrm{Norm}_{\mathrm{Cr}_n(\mathbb{C})}(D_n) = D_n \rtimes \mathrm{W}_n$. Parfois W_n est appelé *groupe des transformations monomiales*.

Observons que le théorème de Noether et Castelnuovo implique notamment que $\mathrm{Cr}_2(\mathbb{C}) = \langle \mathrm{W}_2, \mathrm{PGL}_3(\mathbb{C}) \rangle$. Par contre, Hudson et Pan ([Hud27], [Pan99]) ont démontré que pour $n \geq 3$ le groupe de Cremona $\mathrm{Cr}_n(\mathbb{C})$ n'est plus engendré par $\mathrm{PGL}_{n+1}(\mathbb{C})$ et W_n. On définit:

$$\mathrm{H}_n \coloneqq \langle \mathrm{PGL}_{n+1}(\mathbb{C}), \mathrm{W}_n \rangle.$$

Malgré l'analogie avec les groupes de Lie, il n'est pas clair que l'on puisse créer une théorie des représentations de Cremona. Les exemples suivantes illustrent que la situation est plus compliquée que pour les groupes de Lie classiques et qu'en général il existe beaucoup d'homomorphismes différents entre des groupes de Cremona:

Exemple 1.2.1. Supposons qu'une variété projective M est birationnellement équivalente à $\mathbb{P}^n \times N$ pour une variété N. L'action standard sur le premier facteur induit un homomorphisme injectif de $\mathrm{Cr}_n(\mathbb{C})$ vers $\mathrm{Bir}(\mathbb{P}^n \times N)$ et donc aussi vers $\mathrm{Bir}(M)$. On appelle les plongements de cette forme *plongements standards*. On obtient en particulier un plongement standard $\mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}^n \times \mathbb{P}^m)$ pour tout entier positif m.

Exemple 1.2.2. Une variété M est appelé *stablement rationnelle* s'il existe un entier n tel que $M \times \mathbb{P}^n$ est rationnelle. Il existe des variétés de dimension ≥ 3 qui sont stablement rationnelles mais pas rationnelles (voir [BCTSSD85]). On va démontrer que deux plongements standard $f_1 \colon \mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}^n \times N)$ et $f_2 \colon \mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}^n \times M)$ sont conjugués si et seulement si N et M sont birationnellement équivalentes. Alors, toutes les classes d'équivalence birationnelle des variétés stablement rationnelles de dimension k définissent une classe différente de plongement $\mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}^m)$ pour $m = n + k$.

1.2.1 Le cas $\dim(M) \leq n$

Soit M une variété complexe projective de dimension n et $\rho\colon \mathrm{PGL}_{r+1}(\mathbb{C}) \to \mathrm{Bir}(M)$ un plongement. Alors $n \geq r$ et si $n = r$ on a que M est rationnelle et que ρ et le plongement standard à conjugaison et à homomorphismes du corps $\mathbb{C}$ près (voir [Can14] et [Dés06b]). Ceci implique notamment qu'il n'existe pas de plongement de $\mathrm{Cr}_n(\mathbb{C})$ vers $\mathrm{Bir}(M)$ si $\dim(M) < n$. On rappellera ces résultats et on verra que la restriction d'un automorphisme de $\mathrm{Cr}_n(\mathbb{C})$ au sous-groupe H_n est le plongement standard à conjugaison et à automorphisme du corps $\mathbb{C}$ près (section 3.5.5).

1.2.2 Homomorphismes algébriques

On dit qu'un homomorphisme de groupes $\Psi\colon \mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(M)$ est *algébrique* si sa restriction à $\mathrm{PGL}_{n+1}(\mathbb{C})$ est un morphisme algébrique. Pour la définition des morphismes algébriques on utilisera la *topologie de Zariski* sur $\mathrm{Bir}(M)$ (voir section 3.2). Un élément $f \in \mathrm{Cr}_n(\mathbb{C})$ est appellé *algébrique*, si la suite $\{\deg(f^n)\}_{n \in \mathbb{Z}_+}$ est bornée.

Définition 1.2.1. Soit M une variété et $\varphi_M\colon \mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(M)$ un homomorphisme algébrique non trivial. On dit que φ_M est *réductible* s'il existe une variété N telle que $0 < \dim(N) < \dim(M)$ et un morphisme algébrique $\varphi_N\colon \mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(N)$ avec une application rationnelle dominante $\pi\colon M \dashrightarrow N$ qui est $\mathrm{Cr}_n(\mathbb{C})$-équivariante par rapport aux actions rationnelles induites par φ_M et φ_N, c'est-à-dire $\pi \circ \varphi_M(g) = \varphi_N(g) \circ \pi$ pour tous les $g \in \mathrm{Cr}_n(\mathbb{C})$.

1.2.3 Un exemple dû à Gizatullin

Dans [Giz99] Gizatullin considère la question suivante: est-ce qu'on peut étendre une représentation linéaire $\psi\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_{n+1}(\mathbb{C})$ vers un homomorphisme des groupes $\Psi\colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Cr}_n(\mathbb{C})$? Il démontre que les actions de $\mathrm{PGL}_3(\mathbb{C})$ sur l'espace des coniques, cubiques et quartiques s'étendent à des homomorphismes de $\mathrm{Cr}_2(\mathbb{C})$ vers $\mathrm{Cr}_5(\mathbb{C})$, $\mathrm{Cr}_9(\mathbb{C})$ et $\mathrm{Cr}_{14}(\mathbb{C})$.

Dans la section 3.3 on étudiera en détail quelques propriétés géométriques de l'homomorphisme de Gizatullin

$$\Phi\colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Cr}_5(\mathbb{C}).$$

Par construction, la restriction de Φ à $\mathrm{PGL}_3(\mathbb{C})$ donne la représentation linéaire $\varphi\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_6(\mathbb{C})$ qui est donné par l'action de $\mathrm{PGL}_3(\mathbb{C})$ sur les coniques. On démontrera notamment le théorème suivant:

Théorème 1.2.3. L'homomorphisme de Gizatullin $\Phi\colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Cr}_5(\mathbb{C})$ a les propriétés suivantes:

(1) Φ est injectif et irréductible.

(2) L'action rationnelle de $\mathrm{Cr}_2(\mathbb{C})$ sur $\mathbb{P}^5$ induite par Φ préserve la surface de Veronese V et sa variété des sécantes $S \subset \mathbb{P}^5$. La restriction de cette action induit une action rationnelle de $\mathrm{Cr}_2(\mathbb{C})$ sur V et sur S.

(3) Le plongement de Veronese $v\colon \mathbb{P}^2 \to \mathbb{P}^5$ est $\mathrm{Cr}_2(\mathbb{C})$-équivariant par rapport à l'action standard sur $\mathbb{P}^2$.

(4) L'application rationnelle dominante des sécantes $s\colon \mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow S \subset \mathbb{P}^5$ (voir section 3.3.4) est $\mathrm{Cr}_2(\mathbb{C})$-équivariant par rapport à l'action diagonale de $\mathrm{Cr}_2(\mathbb{C})$ sur $\mathbb{P}^2 \times \mathbb{P}^2$.

(5) L'action rationnelle de $\mathrm{Cr}_2(\mathbb{C})$ sur $\mathbb{P}^5$ préserve une forme volume sur $\mathbb{P}^5$ avec un pôle d'ordre 3 le long de la variété des sécantes S.

(6) Φ envoie le groupe d'automorphismes polynomiaux $\mathrm{Aut}(\mathbb{A}^2) \subset \mathrm{Cr}_2(\mathbb{C})$ vers $\mathrm{Aut}(\mathbb{A}^5) \subset \mathrm{Cr}_5(\mathbb{C})$.

(7) Pour tout élément $f \in \mathrm{Cr}_2(\mathbb{C})$ on a $\deg(f) \leq \deg(\Phi(f))$,

(8) Pour tout élément $g \in \mathrm{Aut}(\mathbb{A}^2) \subset \mathrm{Cr}_2(\mathbb{C})$ on a $\deg(g) = \deg(\Phi(g))$.

La représentation $\varphi^\vee$ de $\mathrm{PGL}_3(\mathbb{C})$ vers $\mathrm{PGL}_6(\mathbb{C})$ donnée par $\psi \circ \alpha$, où α est l'homomorphisme algébrique $g \mapsto {}^t g^{-1}$, est conjuguée dans $\mathrm{Cr}_5(\mathbb{C})$ à la représentation φ. Cette conjugaison nous donne un homomorphisme

$$\Phi^\vee\colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Cr}_5(\mathbb{C}),$$

dont l'image préserve S et induit une action rationnelle de $\mathrm{Cr}_2(\mathbb{C})$. Comme S est rationnelle, Φ et $\Phi^\vee$ induisent deux homomorphismes de $\mathrm{Cr}_2(\mathbb{C})$ vers $\mathrm{Cr}_4(\mathbb{C})$, qu'on appelle Ψ_1 et Ψ_2. On démontrera l'énoncé suivant:

Proposition 1.2.4. *Les deux homomorphismes $\Psi_1, \Psi_2\colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Cr}_4(\mathbb{C})$ ne sont pas conjugués dans $\mathrm{Cr}_4(\mathbb{C})$; ils sont de plus irréductibles et donc non conjugués au plongement standard.*

1.2.4   Plongements algébriques en codimension 1

On donnera une classification partielle des plongements algébriques de $\mathrm{Cr}_n(\mathbb{C})$ vers $\mathrm{Bir}(M)$, où M est une variété de dimension $n+1$ pour $n \geq 2$.

Exemple 1.2.5. Pour une courbe C de genre ≥ 1, la variété $\mathbb{P}^n \times C$ n'est pas rationnelle et on a le plongement standard $\Psi_C\colon \mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}^n \times C)$.

Exemple 1.2.6. $\mathrm{Cr}_n(\mathbb{C})$ agit rationnellement sur l'espace total du fibré canonique de $\mathbb{P}^n$:

$$K_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-(n+1)) \simeq \bigwedge^n (\mathrm{T}\,\mathbb{P}^n)^\vee$$

par $f(p, \omega) = (f(p), \omega \circ (df_p)^{-1})$, où $p \in \mathbb{P}^n$ et $\omega \in \bigwedge^n (\mathrm{T}_p\,\mathbb{P}^n)^\vee$. Plus généralement, on obtient une action rationnelle de $\mathrm{Cr}_n(\mathbb{C})$ sur l'espace total du fibré canonique $K_{\mathbb{P}^n}^{\otimes l} \simeq \mathcal{O}_{\mathbb{P}^n}(-(n+1)l)$ et sa complétion projective

$$F_l := \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-l(n+1))$$

pour tout entier $l \in \mathbb{Z}_{\geq 0}$. Ceci induit une famille dénombrable d'homomorphismes

$$\Psi_l\colon \mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(F_l).$$

Exemple 1.2.7. Soit $\mathbb{P}(\mathrm{T}\,\mathbb{P}^2)$ l'espace total de la projectivisation du fibré tangent de $\mathbb{P}^2$. Alors $\mathbb{P}(\mathrm{T}\,\mathbb{P}^2)$ est rationnel et il existe un homomorphisme injectif

$$\Psi_B \colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}(\mathrm{T}\,\mathbb{P}^2))$$

défini par $\Psi_B(f)(p,v) \coloneqq (f(p), \mathbb{P}(df_p)(v))$. Ici, $\mathbb{P}(df_p) \colon \mathbb{P}\,\mathrm{T}_p \to \mathbb{P}\,\mathrm{T}_{f(p)}$ est défini comme la projectivisation de la différentielle df_p de f au point $p \in \mathbb{P}^2$.

Exemple 1.2.8. Soit $\mathbb{G}(1,3)$ la Grassmannienne des droites dans $\mathbb{P}^3$. La variété $\mathbb{G}(1,3)$ est rationnelle de dimension 4 munie d'une action régulière transitive de $\mathrm{PGL}_4(\mathbb{C})$. Ceci nous donne un plongement algébrique de $\mathrm{PGL}_4(\mathbb{C})$ vers $\mathrm{Cr}_4(\mathbb{C})$. On démontrera *qu'une action régulière de* $\mathrm{PGL}_4(\mathbb{C})$ *sur* $\mathbb{G}(1,3)$ *ne peut pas être étendue en une action rationnelle de* $\mathrm{Cr}_3(\mathbb{C})$.

La classification des actions régulières de $\mathrm{PGL}_{n+1}(\mathbb{C})$ sur des variétés projectives lisses de dimension $n+1$ est connue; dans la section 3.4 on étudiera leurs classes de conjugaison dans $\mathrm{Cr}_{n+1}(\mathbb{C})$. On démontrera que les exemples 1.2.5 à 1.2.8 décrivent, à conjugaison birationnelle et à automorphisme algébrique de $\mathrm{PGL}_{n+1}(\mathbb{C})$ près, tous les actions régulières de $\mathrm{PGL}_{n+1}(\mathbb{C})$ sur des variétés projectives lisses de dimension $n+1$ et que ces actions ne sont pas birationellement conjuguées. Cela nous donne une classification des homomorphismes algébriques de $\mathrm{PGL}_{n+1}(\mathbb{C})$ vers $\mathrm{Bir}(M)$, où M est projective lisse de dimension $n+1$. À la section 3.5 on étudiera comment ces actions régulières s'étendent en des actions rationnelles de $\mathrm{Cr}_n(\mathbb{C})$ sur M. On note $\alpha \colon \mathrm{PGL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$ l'automorphisme algébrique donné par $g \mapsto {}^t(g^{-1})$.

Théorème 1.2.9. Soit $n \geq 2$ et M une variété projective complexe de dimension $n+1$ et soit $\varphi \colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{Bir}(M)$ un homomorphisme algébrique non trivial. Alors

(1) φ est conjugué, à l'automorphisme α près, à l'un des plongements décrits dans les exemples 1.2.5 à 1.2.8.

(2) Si $n = 3$ et φ est conjugué à l'action décrite dans l'exemple 1.2.8, alors ni φ ni $\varphi \circ \alpha$ ne s'étendent à un homomorphisme de H_3 vers $\mathrm{Bir}(M)$.

(3) Si φ est conjugué à l'un des plongements décrits dans les exemples 1.2.5 à 1.2.7 alors un et seulement un des plongements φ ou $\varphi \circ \alpha$ s'étend en un homomorphisme de $\mathrm{Cr}_n(\mathbb{C})$ vers $\mathrm{Bir}(M)$.

(4) φ s'étend à H_n si et seulement si φ s'étend à $\mathrm{Cr}_n(\mathbb{C})$; dans ce cas l'extension à H_n est unique.

Ici, H_n est défini par $\mathrm{H}_n \coloneqq \langle \mathrm{PGL}_{n+1}(\mathbb{C}), \mathrm{W}_n \rangle$, et on rappelle que W_n est le sous-groupe isomorphe à $\mathrm{GL}_n(\mathbb{Z})$ des transformations monomiales de $\mathrm{Cr}_n(\mathbb{C})$.

Le théorème 1.2.9 classifie tous les homomorphismes de groupes $\Psi \colon \mathrm{H}_n \to \mathrm{Bir}(M)$ pour des variétés projectives lisses M de dimension $n+1$ tels que la restriction à $\mathrm{PGL}_{n+1}(\mathbb{C})$ est un morphisme algébrique. Par le théorème de Noether et Castelnuovo, on obtient notamment une classification complète de tous les homomorphismes de $\mathrm{Cr}_2(\mathbb{C})$ vers $\mathrm{Bir}(M)$ pour les variétés projectives M de dimension 3:

Corollaire 1.2.10. *Soit M une variété projective de dimension 3 et $\Psi\colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Bir}(M)$ un homomorphisme algébrique non trivial. Alors Ψ est conjugué à exactement un des homomorphismes décrits aux exemples 1.2.5 à 1.2.8.*

Les observations suivantes sont immédiates:

Corollaire 1.2.11. *Soit M une variété projective de dimension 3 et $\Psi\colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Bir}(M)$ un homomorphisme algébrique non trivial. Alors,*

(1) *Ψ est injectif.*

(2) *Il existe une application rationnelle $f\colon M \dashrightarrow \mathbb{P}^2$ qui est $\mathrm{Cr}_2(\mathbb{C})$-équivariante par rapport à l'action rationnelle induite par Ψ sur M et l'action standard sur $\mathbb{P}^2$. En particulier tous les homomorphismes algébriques de $\mathrm{Cr}_2(\mathbb{C})$ vers $\mathrm{Bir}(M)$ sont réductibles.*

(3) *Il existe un entier $C_\Psi \in \mathbb{Z}$ tel que*

$$1/C_\Psi \deg(f) \leq \deg \Psi(f) \leq C_\Psi \deg(f).$$

On observe que la partie (3) du corollaire 1.2.11 rassemble à la partie (8) du théorème 1.2.3 et nous mène vers la question suivante:

Question 1.2.1. Soit $\Phi\colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Cr}_n(\mathbb{C})$ un plongement algébrique. Existe-il une constante C qui ne dépend que de Φ telle que $1/C \deg(f) \leq \deg(\Phi(f)) \leq C \deg(f)$ pour tous les $f \in \mathrm{Cr}_2(\mathbb{C})$?

On peut considérer un analogue de la question 1.2.1 pour les représentations de groupes de Lie de $\mathrm{SL}_2(\mathbb{C})$. Soit $\|A\|$ la norme d'opérateur d'une matrice $A \in \mathrm{SL}_2(\mathbb{C})$. L'application $d\colon A \mapsto |\log(\|A\|)|$ est symétrique et sous-additive et peut donc être vue comme analogue de la fonction des degrés sur $\mathrm{Cr}_2(\mathbb{C})$. Les représentations irréductibles de dimension finie de $\mathrm{SL}_2(\mathbb{C})$ sont exactement les représentations données par l'action de $\mathrm{SL}_2(\mathbb{C})$ sur l'espace vectoriel $\mathbb{C}[x,y]_d$ des polynômes de degré d. Soit $\rho\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(\mathbb{C}[x,y]_d)$ une représentation irréductible. Alors il existe une constante C telle que

$$1/C\|A\| \leq \|\rho(A)\| \leq C\|A\|.$$

1.3 Suites de degrés

Soit $X_{\mathbf{k}}$ une variété projective définie sur un corps $\mathbf{k}$ et soit $\mathrm{Bir}(X)$ le groupe des transformations birationnelles de $X_{\mathbf{k}}$. Un groupe Γ est appelé *groupe des transformations birationnelles* s'il existe un corps $\mathbf{k}$ et une variété projective $X_{\mathbf{k}}$ sur $\mathbf{k}$ tel que $\Gamma \subset \mathrm{Bir}(X_{\mathbf{k}})$. Plus généralement on considère $\mathrm{Rat}(X_{\mathbf{k}})$, le monoïde des applications rationnelles dominantes de $X_{\mathbf{k}}$ sur lui-même. Par conséquent, on appelle un monoïde Δ *monoïde des applications rationnelles dominantes*, s'il existe un corps $\mathbf{k}$ et une variété projective $X_{\mathbf{k}}$ sur $\mathbf{k}$ telle que $\Delta \subset \mathrm{Rat}(X_{\mathbf{k}})$.

Si $X_{\mathbf{k}}$ est une variété projective lisse, un outil intéressant pour étudier la structure des monoïdes des applications rationnelles dominantes sont les *fonctions de*

degré. On fixe une polarisation de $X_{\mathbf{k}}$, c'est-à-dire la classe H d'un diviseur ample sur $X_{\mathbf{k}}$. Alors on associe à chaque élément $f \in \mathrm{Rat}(X_{\mathbf{k}})$ son degré $\deg_H(f) \in \mathbb{Z}_+$ par rapport à H, qui est défini par

$$\deg_H(f) = f^*H \cdot H^{d-1},$$

où d est la dimension de $X_{\mathbf{k}}$ et f^*H est la transformé totale de H par f. Si $X_{\mathbf{k}}$ est une variété projective lisse sur un corps $\mathbf{k}$ de caractéristique zéro, on a

$$\deg_H(f \circ g) \le C(X_{\mathbf{k}}, H) \deg_H(f) \deg_H(g),$$

pour tout $f, g \in \mathrm{Bir}(X_{\mathbf{k}})$, où $C(X_{\mathbf{k}}, H)$ est une constante qui ne dépend que de $X_{\mathbf{k}}$ et de la polarisation H (voir [DS05]). Pour une généralisation de ce résultat aux corps de caractéristique positive, voir [Tru15], [Tru16] ou, pour une approche alternative plus récente, [Dan17].

Soit $\Delta \subset \mathrm{Rat}(X_{\mathbf{k}})$ un monoïde de transformations rationnelles dominantes de type fini d'une variété projective lisse $X_{\mathbf{k}}$ avec un ensemble fini S de générateurs. On définit

$$D_{S,H} \colon \mathbb{Z}_+ \to \mathbb{Z}_+$$

par

$$D_{S,H}(n) := \max_{\delta \in B_S(n)} \{\deg_H(\gamma)\},$$

où $B_S(n)$ est l'ensemble de tous les éléments dans Δ dont la longueur de mot par rapport aux générateurs S est $\le n$. On appelle une application $\mathbb{Z}_+ \to \mathbb{Z}_+$ qui peut être réalisée pour un corps $\mathbf{k}$ et un quadruplet $(X_{\mathbf{k}}, H, \Delta, S)$ comme une telle application une *suite de degrés*. Notre définition de suite de degrés inclut notamment les suites de degrés qui sont données par des groupes de type fini $\Gamma \subset \mathrm{Bir}(X_{\mathbf{k}})$ de transformations birationnelles.

Dans le chapitre 4 on démontrera que l'ensemble de toutes les suites de degrés est dénombrable, on donnera des nouvelles contraintes sur la croissance d'une suite de degrés et on donnera de nouveaux exemples.

1.3.1 Dénombrabilité des suites de degrés

Bonifant et Fornaess on démontré que l'ensemble des suites $\{d_n\}$ telles qu'il existe une application rationnelle dominante f de $\mathbb{P}^2_{\mathbb{C}}$ vers lui-même satisfaisant $\deg(f^n) = d_n$ est dénombrable ([BF00]), ce qui a répondu à une question de Ghys. On généralisera ce résultat de Bonifant et Fornaess à toutes les suites de degrés sur toutes les variétés projectives lisses, tous les corps, toutes les polarisations et tous les ensembles finis générateurs des monoïdes de type fini des applications rationnelles dominantes:

Théorème 1.3.1. L'ensemble de toutes les suites de degrés est dénombrable.

1.3.2 Questions

En dimension 2 la croissance des degrés des transformations birationnelles est bien comprise (Theorem 1.1.2). Dans le cas des automorphismes polynomiaux de $\mathbb{A}^2$ on a même un résultat plus précis:

Théorème 1.3.2 ([Fur99])**.** Soit $f \in \mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$. Alors la suite de degrés $\{\deg(f^n)\}$ est soit périodique (dans ce cas la suite est bornée), soit une progression géométrique.

Par contre, en dimension supérieure on ne dispose pas de beaucoup de résultats. Par exemple, Lo Bianco a traité le cas des automorphismes de variétés kähleriennes compactes de dimension 3 ([LB14]). Mais les questions suivantes semblent rester ouvertes:

Question 1.3.1. Est-ce qu'il existe une transformation birationnelle f d'une variété projective $X_{\mathbf{k}}$ tel que $\deg_H(f^n)$ est de croissance intermédiaire, par exemple $\deg_H(f^n) \sim e^{\sqrt{n}}$?

Question 1.3.2. Est-ce qu'il existe une transformation f tel que $\deg_H(f^n)$ croît "lentement"? Par exemple, est-ce qu'on peut avoir $\deg_H(f^n) \sim \sqrt{n}$? Ou est-ce que la croissance d'une suite de degrés qui n'est pas bornée, est au moins linéaire?

Question 1.3.3. Si une transformation birationnelle f satisfait $\deg_H(f^n) \sim \lambda^n$, alors est-ce que λ est un nombre algébrique?

Question 1.3.4. Est-ce que tous les transformations birationnelles de croissance polynomiale préservent des fibrations?

1.3.3 Suites de degrés des automorphismes polynomiaux

Un bon point de départ pour étudier les suites de degrés est le groupe d'automorphismes polynomiaux $\mathrm{Aut}(\mathbb{A}^d)$ de l'espace affine $\mathbb{A}^d$ de dimension d. À la section 4.3.2 on va démontrer l'observation suivante (une démonstration de ce résultat se trouve aussi dans [Dés16b]):

Proposition 1.3.3. *Soit* $\mathbf{k}$ *un corps et* $f \in \mathrm{Aut}(\mathbb{A}^d_{\mathbf{k}})$ *un automorphisme polynomial tel que* $\deg(f^d) = \deg(f)^d$. *Alors* $\deg(f^n) = \deg(f)^n$ *pour tout* $n \in \mathbb{Z}_+$.

Le monoïde $\mathrm{End}(\mathbb{A}^d_{\mathbf{k}})$ peut être vu comme espace vectoriel sur $\mathbf{k}$, sur lequel la fonction de degré induit une filtration d'espaces vectoriels de dimension finie. Cette structure supplémentaire donne la possibilité d'employer une nouvelle technique, qu'on utilisera pour démontrer que les suites de degrés non bornés des groupes d'automorphismes polynomiaux divergent et ne peuvent pas être de croissance trop petite:

Théorème 1.3.4. Soit $f \in \mathrm{End}(\mathbb{A}^d_{\mathbf{k}})$ un endomorphisme et supposons que $\{\deg(f^n)\}$ n'est pas borné. Alors pour tout entier K, on a

$$\# \{m \mid \deg(f^m) \leq K\} < C_d \cdot K^d,$$

où $C_d = \frac{(1+d)^d}{(d-1)!}$. En particulier $\deg(f^n)$ converge vers ∞ lorsque n tend vers ∞.

Le théorème 1.3.4 implique par un résultat d'Olshanskii ([Ols99]) qu'une suite de degrés non bornée d'un automorphisme polynomial se comporte comme une fonction de longueur des mots. Le corollaire suivant se déduit directement du théorème 1.3.4:

Corollaire 1.3.5. *Soit* $\Gamma \subset \mathrm{End}(\mathbb{A}_{\mathbf{k}}^d)$ *un monoïde engendré par un ensemble fini* S. *Si* $D_S(n) < C_d \cdot n^{1/d}$ *pour un ensemble infini des entiers* n, *alors* Γ *est de degré borné.*

Malheureusement, nos techniques pour démontrer le théorème 1.3.4 ne s'appliquent pas aux transformations birationnelles arbitraires de $\mathbb{P}_{\mathbf{k}}^d$. Par contre, si on suppose que le corps de base $\mathbf{k}$ est fini, on obtient des résultats similaires:

Théorème 1.3.6. Soit $\mathbb{F}_q$ un corps fini avec q éléments et soit $f \in \mathrm{Rat}(\mathbb{P}_{\mathbb{F}_q}^d)$ tel que la suite $\{\deg(f^n)\}$ n'est pas bornée. Alors, pour tout entier K,

$$\# \{m \mid \deg(f^m) \leq K\} \leq q^{C(K,d)},$$

où $C(K,d) = (d+1) \cdot \binom{d+K}{K}$. En particulier $\deg(f^n)$ diverge vers ∞ lorsque n tend vers ∞.

Corollaire 1.3.7. *Soit* $\Gamma \subset \mathrm{Rat}(\mathbb{P}_{\mathbb{F}_q}^d)$ *un monoïde de type fini et* S *un ensemble fini de générateurs de* Γ. *Il existe une constante positive* $C_{d,q}$ *tel que si*

$$D_S(n) < C_{d,q} \cdot \log(n)^{1/d}$$

pour tous les n, *alors* Γ *est de degré borné.*

1.3.4   Types de croissance de degrés

Définition 1.3.1. Soit $X_{\mathbf{k}}$ une variété projective lisse avec une polarisation H sur un corps $\mathbf{k}$ et soit $f \in \mathrm{Bir}(X_{\mathbf{k}})$. On définit *l'ordre de croissance* de $\deg_H(f^n)$ par

$$\mathrm{dpol}(f) := \limsup_{n \to \infty} \frac{\log(\deg_H(f^n))}{\log(n)}.$$

L'ordre de croissance peut être infini.

Des résultats de Truong, Dinh et Sibony impliquent que l'ordre de croissance ne dépend pas du choix de la polarisation (section 4.2.5):

Proposition 1.3.8. *Soit* $X_{\mathbf{k}}$ *une variété projective sur un corps* $\mathbf{k}$ *et soit* $f \in \mathrm{Bir}(X_{\mathbf{k}})$. *Alors* $\mathrm{dpol}(f)$ *ne dépend pas du choix de la polarisation.*

Soit f une transformation birationnelle d'une surface. Le théorème 1.1.2 implique que $\mathrm{dpol}(f) = 0, 1, 2$ ou ∞. On peut se poser la question suivante:

Question 1.3.5. Est-ce qu'il existe une constante $C(d)$ qui ne dépend que de d, tel que pour tous les variétés $X_{\mathbf{k}}$ de dimension d on a pour tous les $f \in \mathrm{Bir}(X_{\mathbf{k}})$ si $\mathrm{dpol}(f)$ est fini, alors $\mathrm{dpol}(f) < C(d)$?

On donnera quelques exemples de suites de degrés qui indiquent que la croissance des degrés en dimension supérieure est plus riche qu'en dimension 2. On constate tout d'abord qu'il existe des automorphismes polynomiaux dont les suites de degrés ont une croissance polynomiale, ce qui n'est pas le cas en dimension 2:

Exemple 1.3.9. Soit $\mathbf{k}$ un corps. On définit $f, g, h \in \mathrm{Aut}(\mathbb{A}_{\mathbf{k}}^d)$ par $g = (x+yz, y, z)$, $h = (x, y + xz, z)$ et

$$f = g \circ h = (x + z(y + xz), y + xz, z).$$

On peut observer par induction que $\deg(f^n) = 2n + 1$; notamment, $\mathrm{dpol}(f) = 1$.

Plus général, pour tout $l \leq d/2$ il existe des éléments $f_l \in \mathrm{Aut}(\mathbb{A}_{\mathbf{k}}^d)$ tel que $\mathrm{dpol}(f) = l$ (Section 4.3.4).

Exemple 1.3.10. La transformation birationnelle $f = (x_1, x_1 x_2, \ldots, x_1 x_2 \cdots x_n)$ de $\mathbb{P}_{\mathbf{k}}^d$ défini par rapport à des coordonnées affines $(x_1, \ldots, x_d)$, satisfait $\deg(f^n) = n^{d-1}$, c'est-à-dire $\mathrm{dpol}(f) = d - 1$.

L'observation suivante est de Serge Cantat:

Exemple 1.3.11. On définit la racine de l'unité $\omega := \frac{-1+\sqrt{-3}}{2}$ et la courbe elliptique $E_\omega := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\omega)$. Soit

$$X := E_\omega \times E_\omega \times E_\omega$$

et $s\colon X \to X$ l'automorphisme d'ordre fini donné par multiplication diagonale par $-\omega$. Dans [OT15] Oguiso et Truong on démontré que $Y := X/s$ est une variété rationnelle de dimension 3. Soit $f\colon X \to X$ l'automorphisme défini par $(x_1, x_2, x_3) \mapsto (x_1, x_1 + x_2, x_2 + x_3)$. Comme f commute avec s, il induit un automorphisme de Y, qu'on appelle $\hat{f}$. Soit $\phi_1\colon \tilde{Y} \to Y$ une résolution des singularités de Y. On définit $\tilde{f} \in \mathrm{Bir}(\tilde{Y})$ par

$$\tilde{f} := \phi_1^{-1} \circ \hat{f} \circ \phi_1.$$

On va voir à la section 4.3.5 que $\mathrm{dpol}(\tilde{f}) = 4$.

1.4 Sous-groupes des éléments elliptiques du groupe de Cremona du plan

Dans ce chapitre on s'intéresse aux groupes $G \subset \mathrm{Cr}_2(\mathbb{C})$ tels que tous les éléments de G sont elliptiques. Par définition, cela signifie que l'isométrie sur l'espace $\mathbb{H}^\infty$ qui correspond à $g \in G$ a un point fixe dans $\mathbb{H}^\infty$. Dans [Can11a], Proposition 3.10, Cantat démontre qu'un élément $g \in \mathrm{Cr}_2(\mathbb{C})$ est elliptique si et seulement si g est algébrique, c'est-à-dire que $\{\deg(f^n)\}$ est borné ou, ce qui est équivalent, il existe un $k \in \mathbb{Z}_+$ tel que g^k est conjugué à un automorphisme dans $\mathrm{Aut}^0(S)$, la composante connexe de l'identité du groupe d'automorphismes d'une surface projective lisse S. Le fait que tous les éléments de G soient elliptiques n'implique pas que G est contenu dans un groupe algébrique, c'est-à-dire que le degré des éléments est uniformément borné. Ceci est illustré par les exemples suivants:

Exemple 1.4.1. Soit $G \subset \mathrm{Cr}_2(\mathbb{C})$ le groupe des éléments de la forme $(x, y + p(x))$, où $p(x) \in \mathbb{C}(x)$ est une fonction rationnelle. Alors chaque élément dans G est algébrique, mais G contient des éléments de degrés arbitrairement grands.

Exemple 1.4.2. Dans [Wri79], Wright construit des exemples de sous-groupes de torsion de $\mathrm{Aut}(\mathbb{A}^2)$ (donc en particulier de $\mathrm{Cr}_2(\mathbb{C})$) qui contiennent des éléments de degré arbitrairement grands. En fait, il démontre qu'il existe un sous-groupe G de $\mathrm{Cr}_2(\mathbb{C})$ qui est isomorphe au sous-groupe des racines de l'unité dans $\mathbb{C}^*$ mais qui n'est pas borné. Dans [Lam01a], Lamy démontre que certaines exemples de Wright ne préservent aucune fibration.

Jusqu'à présent il n'existe des résultats sur des sous-groupes des éléments elliptiques que dans le cas où les sous-groupes sont de type fini ou bornés. Notre stratégie consiste à utiliser ces résultats en les combinant avec le théorème de compacité de la théorie des modèles afin de démontrer le théorème suivant qui donne des informations sur la structure des sous-groupes de $\mathrm{Cr}_2(\mathbb{C})$ qui consistent en des éléments elliptiques:

Théorème 1.4.3. Soit $G \subset \mathrm{Cr}_2(\mathbf{k})$ un sous-groupe formé d'éléments elliptiques. Alors, on est dans un des cas suivants:

(1) G n'est pas un groupe de torsion et G contient un sous-groupe de type fini qui ne préserve aucune fibration. Dans ce cas, G est conjugué à un sous-groupe de $\mathrm{Aut}(\mathbb{P}^2)$, de $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ ou de $\mathrm{Aut}(S_6)$, où S_6 est la surface de del Pezzo de degré 6.

(2) G est un groupe de torsion.

(3) G préserve une fibration rationnelle et G est conjugué à un sous-groupe du groupe de Jonquières $\mathcal{J} \simeq \mathrm{PGL}_2(\mathbb{C}(t)) \rtimes \mathrm{PGL}_2(\mathbb{C})$, qui est le sous-groupe de $\mathrm{Cr}_2(\mathbb{C})$ des éléments qui préservent une fibration rationnelle donnée.

(4) Tous les sous-groupes de type fini G préservent une fibration rationnelle et il existe un sous-groupe de torsion $G_0 \subset G$ et une suite exacte

$$1 \to G_0 \to G \to \mathrm{PGL}_2(\mathbb{C}).$$

Dans le cas où G est un sous-groupe de torsion de $\mathrm{Cr}_2(\mathbb{C})$, on peut préciser le résultat:

Théorème 1.4.4. Soit $G \subset \mathrm{Cr}_2(\mathbb{C})$ un sous-groupe de torsion. Alors G se plonge dans $\mathrm{GL}_N(\mathbb{C})$ pour un $N \leq 48$ et $N \leq 36$ si G est infini.

Remarque 1.4.1. Il existe un $N \in \mathbb{Z}_+$ tel que chaque sous-groupe fini de $\mathrm{Cr}_2(\mathbb{C})$ se plonge dans $\mathrm{GL}_N(\mathbb{C})$. Dans ce texte on donnera le borne $N \leq 48$, mais on suppose que cette borne peut être baissée sensiblement. On verra notamment qu'un sous-groupe fini de $\mathrm{Cr}_2(\mathbb{C})$ se plonge dans $\mathrm{GL}_{36}(\mathbb{C})$ ou il est isomorphe à une extension de $\mathcal{S}_4$ par le groupe de Klein.

Théorème 1.4.5 (Schur, voir [CR62, p.258]). Soit $G \subset \mathrm{GL}_n(\mathbb{C})$ un sous-groupe de torsion. Alors G contiens un sous-groupe abélien d'indice plus petit ou égal à

$$(\sqrt{8n} + 1)^{2n^2} - (\sqrt{8n} - 1)^{2n^2}.$$

À l'aide du théorème 1.4.4 et du théorème 1.4.5, on déduit directement que la même propriété est vraie pour les sous-groupe de torsion de $\mathrm{Cr}_2(\mathbb{C})$:

Corollaire 1.4.6. *Un sous-groupe de torsion G de $\mathrm{Cr}_2(\mathbb{C})$ est soit fini, soit isomorphe à un sous-groupe de $\mathrm{GL}_N(\mathbb{C})$ pour $N \leq 36$. En particulier, G contient un sous-groupe abélien d'indice plus petit ou égal à*

$$(96\sqrt{2}+1)^{2592} - (96\sqrt{2}-1)^{2592} < 10^{5537}.$$

La suite dérivée d'un groupe G est la suite des groupes définie par

- $G^{(0)} \coloneqq G$

- $G^{(i+1)} \coloneqq [G^{(i)}, G^{(i)}],$

où le *groupe dérivé* $[H, H]$ d'un groupe H est le sous-groupe engendré par tous les éléments de la forme $aba^{-1}b^{-1}$, $a, b \in H$.

Un groupe G est résoluble si et seulement si sa suite dérivée se termine par l'identité après un nombre fini d'itérations. La longueur dérivée de G est le nombre minimal k tel que $G^{(k)} = \{\mathrm{id}\}$.

Un autre résultat qui se déduit du théorème 1.4.3 concerne l'alternative de Tits. En [Tit72], Tits a démontré l'énoncé suivant:

Théorème 1.4.7 ([Tit72]). Soit $\mathbf{k}$ un corps de caractéristique 0 et $n \in \mathbb{Z}_+$. Alors tout sous-groupe G de $\mathrm{GL}_n(\mathbf{k})$ est soit résoluble à indice fini près, soit il contient un sous-groupe libre non-abélien.

Ce résultat a mené aux définitions suivantes:

Définition 1.4.1.

(1) Un groupe G satisfait *l'alternative de Tits* si tout sous-groupe G est soit résoluble à indice fini près, soit contient un sous-groupe libre non-abélien.

(2) Un groupe G satisfait *l'alternative de Tits pour les sous-groupes de type fini* si tout sous-groupe de type fini de G est soit résoluble à indice fini près, soit contient un sous-groupe libre non-abélien.

Le théorème 1.4.7 énonce que tous les groupes linéaires sur un corps de caractéristique 0 satisfont l'alternative de Tits. Les groupes linéaires en caractéristique positive satisfont l'alternative de Tits pour les sous-groupes de type fini ([Tit72]).

Dans sa thèse, Lamy a démontré l'alternative de Tits pour $\mathrm{Aut}(\mathbb{A}_{\mathbb{C}}^2)$:

Théorème 1.4.8 ([Lam01b]). $\mathrm{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ satisfait l'alternative de Tits.

La preuve du théorème 1.4.8 emploie la structure de produit amalgamé de $\mathrm{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ qui est donnée par le théorème de Jung et van der Kulk (voir par exemple [Lam02]) et la théorie de Bass-Serre (voir [Ser77]).

Cantat a démontré l'alternative de Tits pour les sous-groupes de type fini de $\mathrm{Cr}_2(\mathbb{C})$:

Théorème 1.4.9 ([Can11a]). $\mathrm{Cr}_2(\mathbb{C})$ satisfait l'alternative de Tits pour les sous-groupes de type fini.

Le théorème 1.4.9 fait partie d'une série de résultats profonds sur la structure du groupe de Cremona du plan, que Cantat déduit par l'action isométrique de $\mathrm{Cr}_2(\mathbb{C})$ sur l'espace de Picard-Manin $\mathbb{H}^\infty$ ([Can11a]). L'obstacle principale pour généraliser le théorème 1.4.9 à des sous-groupes quelconques était imposé par les sous-groupes des éléments elliptiques qui ne sont pas bornés. On peut appliquer le théorème 1.4.3 à ce cas afin de démontrer le résultat suivant:

Théorème 1.4.10. $\mathrm{Cr}_2(\mathbb{C})$ satisfait l'alternative de Tits.

En [Dés15], Déserti donne une description des classes de sous-groupes résolubles de $\mathrm{Cr}_2(\mathbb{C})$. On complémentera son résultat par l'observation suivante:

Théorème 1.4.11. Il existe une constante $K \leq 35$ telle que la longueur dérivée de tous les sous-groupes résolubles de $\mathrm{Cr}_2(\mathbb{C})$ est $\leq K$.

1.5 Sous-groupes simples du groupe de Cremona du plan

Cantat et Lamy ont démontré en 2012 que le groupe de Cremona du plan n'est pas simple, une question ouverte depuis longtemps. L'idée principale pour démontrer ce résultat consistait d'utiliser l'action isométrique de $\mathrm{Cr}_2(\mathbb{C})$ sur l'espace de Picard-Manin $\mathbb{H}^\infty$ et d'appliquer des techniques de la théorie de petite simplification. Dans ce chapitre on raffinera ces techniques avec le but de classifier tous les sous-groupes simples de $\mathrm{Cr}_2(\mathbb{C})$. En travaillant sur ce sujet je suis tombé sur un problème que je ne pouvais pas résoudre et qu'on énonce maintenant comme conjecture. On dénote par $\mathrm{Ind}(f) \subset \mathbb{P}^2$ l'ensemble des points d'indétermination d'une transformation birationnelle $f \in \mathrm{Cr}_2(\mathbb{C})$.

Conjecture 1.5.1. *Soit $f \in \mathrm{Cr}_2(\mathbb{C})$ un élément loxodromique, $p \in \mathbb{P}^2$ un point qui n'est pas contenu dans un axe de coordonnée de $\mathbb{P}^2$, et k un entier positif. Alors l'ensemble constructible*

$$\{d(fd)^k(p) \mid d \in D_2 \text{ tel que } p \notin \mathrm{Ind}((fd)^l) \text{ pour tous les } 1 \leq l \leq k\}$$

est ouvert et dense dans $\mathbb{P}^2$.

Le résultat principal serait:

Théorème 1.5.2. On suppose que la conjecture 1.5.1 soit vrai.
Soit $G \subset \mathrm{Cr}_2(\mathbb{C})$ un sous-groupe simple. Alors:

(1) G ne contient pas d'élément loxodromique.

(2) Si G contient un élément parabolique, alors G fixe une fibration rationnelle, c'est-à-dire il existe une application rationnelle G-invariante $\pi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ telle que les fibres générales sont rationnelles. Dans ce cas, G est isomorphe à un sous-groupe de $\mathrm{PGL}_2(\mathbb{C}(t))$.

(3) Si tous les éléments de G sont elliptiques, alors G est un sous-groupe simple d'un sous-groupe algébrique de $\mathrm{Cr}_2(\mathbb{C})$ ou G est conjugué à un sous-groupe du groupe de Jonquières $\mathcal{J} \simeq \mathrm{PGL}_2(\mathbb{C}) \ltimes \mathrm{PGL}_2(\mathbb{C}(t))$.

Remarque 1.5.1. Seulement la partie (1) du théorème 1.5.2 dépend de la conjecture 1.5.1. Si on suppose que le sous-groupe simple G ne contient pas d'élément loxodromique, les parties (2) et (3) peuvent être démontrées sans utilisation de la conjecture 1.5.1.

En considérant les sous-groupes algébriques maximaux de $\mathrm{Cr}_2(\mathbb{C})$, on peut déduire du théorème 1.5.2 l'observation suivante:

Corollaire 1.5.3. *Supposons que la conjecture* 1.5.1 *soit vraie.*

Un sous-groupe simple G se plonge dans $\mathrm{Cr}_2(\mathbb{C})$ si et seulement si G est isomorphe à un sous-groupe de $\mathrm{PGL}_3(\mathbb{C})$.

Le corollaire 1.5.3 mène vers la question suivante qui semble être ouverte:

Question 1.5.1. Quels sont les sous-groupes simples de $\mathrm{PGL}_3(\mathbb{C})$ et $\mathrm{PGL}_2(\mathbb{C})$?

Les sous-groupes de la forme $\mathrm{PSL}_2(\mathbf{k})$, où $\mathbf{k} \subset \mathbb{C}$ est un sous-corps, et les sous-groupes simples finis sont des exemples des sous-groupes simples de $\mathrm{PGL}_2(\mathbb{C})$. Il n'est pas évident qu'il existe d'autres exemples:

Question 1.5.2. Quels sont les sous-groupes simples de $\mathrm{PSL}_2(\mathbb{Q})$? Est-ce que $\mathrm{PSL}_2(\mathbb{Q})$ contient des sous-groupes infinis propres?

1.5.1   Sous-groupes simples de type fini

Un groupe G satisfait la *propriété de Malcev*, si chaque sous-groupe de type fini $\Gamma \subset G$ est *résiduellement fini*, c'est-à-dire, pour chaque $g \in \Gamma$ il existe un groupe fini H et un homomorphisme $\varphi \colon \Gamma \to H$ tel que g n'est pas contenu dans le noyau de φ. Malcev a démontré que les groupes linéaires satisfont cette propriété ([Mal40a]). Parmi les groupes qui satisfont la propriété de Malcev on trouve les groupes d'automorphismes des variétés algébriques sur des corps de caractéristique 0. Une question de Cantat, qui est toujours ouverte, demande si le groupe de Cremona du plan satisfait la propriété de Malcev.

Un sous-groupe simple d'un groupe qui satisfait la propriété de Malcev est toujours simple. On démontrera que cette dernière propriété est vraie pour le groupe de Cremona du plan:

Théorème 1.5.4. Tous les sous-groupes simples de type fini de $\mathrm{Cr}_2(\mathbb{C})$ sont finis.

Pour démontrer le théorème 1.5.4 on n'a pas besoin de la conjecture 1.5.1.

Avec la classification des sous-groupes finis de $\mathrm{Cr}_2(\mathbb{C})$ ([DI09]) on obtient:

Corollaire 1.5.5. *Les sous-groupes simples non-abéliens de type fini de $\mathrm{Cr}_2(\mathbb{C})$ sont les groupes*

$$\mathcal{A}_5, \quad \mathcal{A}_6, \quad \mathrm{PSL}_2(7).$$

Chapter 2

Introduction and summary of results

2.1 Introduction and historical context

To an algebraic variety X over a field $\mathbf{k}$ one can associate $\mathrm{Bir}(X)$, its group of birational transformations. The study of $\mathrm{Bir}(X)$ includes the following two aims:

(1) Understanding the geometry of X through the group structure of $\mathrm{Bir}(X)$.

(2) Understanding the group structure of $\mathrm{Bir}(X)$ through the geometry of X.

   We will mostly focus on the case where X is a rational variety. In this instance the group structure of $\mathrm{Bir}(X)$ is particularly rich and interesting. In honor of the Italian mathematician Luigi Cremona (1830-1903), it is called the *Cremona group in n-variables*, where n is the dimension of n. We denote it by

$$\mathrm{Cr}_n(\mathbf{k}) := \mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}}).$$

If $n = 2$ we will frequently speak of the *plane Cremona group*.

   If we fix homogeneous coordinates $[x_0 : \cdots : x_n]$ of $\mathbb{P}^n_{\mathbf{k}}$, every element $f \in \mathrm{Cr}_n(\mathbf{k})$ can be described by homogeneous polynomials of the same degree $f_0, \ldots, f_n \in \mathbf{k}[x_0, \ldots, x_n]$ without non-constant common factor, such that f is given by

$$f \colon [x_0 : \cdots : x_n] \dashrightarrow [f_0 : \cdots : f_n].$$

With respect to affine coordinates $[1 : X_1 : \cdots : X_n] = (X_1, \ldots, X_n)$, the birational transformation f is given by

$$f : (X_1, \ldots, X_n) \dashrightarrow (F_1, \ldots, F_n),$$

where the $F_i(X_1, \ldots, X_n) \in \mathbf{k}(X_1, \ldots, X_n)$ are the quotients

$$F_i(X_1, \ldots, X_n) = f_i(1, X_1, \ldots, X_n)/f_0(1, X_1, \ldots, X_n).$$

This yields a natural (anti-)isomorphism between $\mathrm{Cr}_n(\mathbf{k})$ and the group of $\mathbf{k}$-field automorphisms of the function field $\mathbf{k}(X_1, \ldots, X_n)$. More generally, one can identify $\mathrm{Bir}(X)$ with the group of $\mathbf{k}$-field automorphisms of the function field $\mathbf{k}(X)$.

Elements of $\mathrm{Cr}_n(\mathbf{k})$ are called *Cremona transformations*. First examples of Cremona transformations, such as the inversion at a circle, had already been used in antiquity. But it was mainly in the 19th century, after the rise of projective and algebraic geometry, that the systematic study of Cremona transformations in the context of algebraic geometry was initiated. In particular, the two groundbreaking papers [Cre63] and [Cre65] by Cremona, published in 1863 and 1865, laid the foundations for considerable research activities conducted by numerous algebraic geometers from what is called the Italian school of algebraic geometry. Some important work of this time was done by Bertini, Castelnuovo, Enriques, Geiser, Hudson, de Jonquières, Kantor, Noether, Segre, and Wiman amongst others. We refer to [Dol12], [Hud27], [Sny28] or [Dés06a] for more exhaustive historical remarks and references.

A considerable number of important techniques and results have been produced by these classical algebraic geometers that keep inspiring researchers up to today (see [Dol12] for an overview and references). One of the most celebrated results is the Theorem of Noether and Castelnuovo:

Theorem 2.1.1 ([Noe70], [Cas01], [Ale16]). *The plane Cremona group over an algebraically closed field $\mathbf{k}$ is generated by the group of automorphisms $\mathrm{Aut}(\mathbb{P}_{\mathbf{k}}^2) \simeq \mathrm{PGL}_3(\mathbf{k})$ and the standard quadratic involution*

$$\sigma := [x_0 : x_1 : x_2] \dashrightarrow [x_1 x_2 : x_0 x_2 : x_0 x_1].$$

Remark 2.1.1. The statement of the Theorem of Noether and Castelnuovo does not hold anymore, if $\mathbf{k}$ is not algebraically closed (see for example [Isk83], [BM14] for some generalizations to perfect fields and the fields of real numbers respectively).

Another important classical problem was the classification of finite subgroups of $\mathrm{Cr}_2(\mathbf{k})$. A full classification for $\mathbf{k} = \mathbb{C}$ has been obtained by Dolgachev and Iskovskikh with the help of modern techniques in 2007 ([DI09], see also [Bla07] and [Tsy13]). The classification for the case that the characteristic of $\mathbf{k}$ is positive is still open and subject to research. The problem of classifying finite subgroups of $\mathrm{Cr}_3(\mathbf{k})$ is more difficult. Prokhorov described all finite simple groups that can be embedded into $\mathrm{Cr}_3(\mathbb{C})$ ([Pro12]). Other works on the subject include [CS16], [Pro11]. In dimensions strictly larger than 3 very little is known about finite subgroups of Cremona groups. However, recently Birkar proved the BAB conjecture (named after A. Borisov, V. Alexeev and L. Borisov), which implies, by a result of Prokhorov and Shramov, that $\mathrm{Cr}_n(\mathbb{C})$ satisfies the Jordan property, i.e. for each n there exists a constant $C(n)$ such that every finite subgroup of $\mathrm{Cr}_n(\mathbb{C})$ is abelian up to index at most $C(n)$ ([PS16], [Bir16]).

Cremona groups can be equipped with additional algebraic structures. In particular, we can define the so called Zariski topology on it (see [BF13] for an overview and the main properties of the topology). Enriques classified the maximal connected algebraic subgroups of $\mathrm{Cr}_2(\mathbb{C})$ ([Enr93], see Section 3.2.2 for the definition of algebraic subgroups). This classification has been rewritten in a more modern

language by Umemura ([Ume82b]) and been extended to the classification of all maximal algebraic subgroups of $\mathrm{Cr}_2(\mathbb{C})$ by Blanc ([Bla09]). Umemura also classified all maximal connected algebraic subgroups of $\mathrm{Cr}_3(\mathbb{C})$ ([Ume85, Ume85]). Other works on algebraic subgroups include [Dem70] and [BB67].

Towards the middle of the 20th century the research focus in algebraic geometry shifted away from the classical subjects and rather towards the work on rigorous foundations. Research in birational geometry and in particular groups of birational transformations bloomed again in Russia some decades later with researchers including Iskovskikh, Manin, Gizatullin and Danilov. With the works of Mori and others in the 1980s birational geometry moved again into the center of research in algebraic geometry. In 2008 two Bourbaki talks were held on the plane Cremona group ([Fav10], [Ser10]) which definitely brought the research on Cremona groups back into fashion.

Major new techniques, mostly from dynamics, yielded novel results that contributed substantially to the understanding of the plane Cremona group. One of the most remarkable breakthroughs was probably the construction of an action of $\mathrm{Bir}(S)$ by isometries on an infinite dimensional hyperbolic space $\mathbb{H}^\infty$ by Manin ([Man86]) and the application of methods from geometric group theory on this action in order to deduce results on the structure of $\mathrm{Bir}(S)$ by Cantat ([Can11a]). Here, S is a projective surface over a field $\mathbf{k}$. An element $f \in \mathrm{Cr}_2(\mathbf{k})$ is called *elliptic, parabolic* or *loxodromic* if the isometry on $\mathbb{H}^\infty$ corresponding to f is *elliptic, parabolic* or *loxodromic* respectively. The crucial point is, that this distinction corresponds to the dynamical behavior of f. If X is a smooth projective variety and H an ample divisor class of X, then the *degree* $\deg_H(f) \in \mathbb{Z}_+$ of a rational dominant transformation f of X with respect to H is defined by

$$\deg_H(f) = f^*H \cdot H^{d-1},$$

where d is the dimension of X and f^*H is the total transform of H under f.

Theorem 2.1.2 (Gizatullin; Cantat; Diller and Favre)**.** *Let $\mathbf{k}$ be an algebraically closed field, S a projective surface over $\mathbf{k}$ with a fixed polarization H and $f \in \mathrm{Bir}(S)$. Then one of the following is true:*

(1) *f is elliptic, the sequence $\{\deg_H(f^n)\}$ is bounded and there exists a $k \in \mathbb{Z}_+$ and a birational map $\varphi\colon S \dashrightarrow S'$ to a smooth projective surface S' such that $\varphi f^k \varphi^{-1}$ is contained in $\mathrm{Aut}^0(S')$, the neutral component of the automorphism group $\mathrm{Aut}(S')$.*

(2a) *f is parabolic and $\deg_H(f^n) \sim cn$ for some positive constant c and f preserves a rational fibration, i.e. there exists a smooth projective surface S', a birational map $\varphi\colon S \dashrightarrow S'$, a curve B and a fibration $\pi\colon S' \to B$, such that a general fiber of π is rational and such that $\varphi f \varphi^{-1}$ permutes the fibers of π.*

(2b) *f is parabolic and $\deg_H(f^n) \sim cn^2$ for some positive constant c and f preserves a rational fibration, i.e there exists a smooth projective surface S', a birational map $\varphi\colon S \dashrightarrow S'$, a curve B and a fibration $\pi\colon S' \to B$, such that $\varphi f \varphi^{-1}$ permutes the fibers of π and such that π is an elliptic fibration, or a quasi-elliptic fibration (the latter only occurs if $\mathrm{char}(\mathbf{k}) = 2$ or 3).*

(3) f is loxodromic and $\deg_H(f^n) = c\lambda(f)^n + O(1)$ for some positive constant c, where $\lambda(f)$ is the dynamical degree of f. In this case f does not preserve any fibration.

Theorem 2.1.2 has lead to various remarkable results on the group structure of $\mathrm{Bir}(S)$ for surfaces S. From the point of view of geometric group theory, the plane Cremona group acting on $\mathbb{H}^\infty$ has some analogies with other groups acting on hyperbolic spaces such as the mapping class group of a surface acting on the complex of curves or groups of outer automorphisms of a free group with n generators acting on the outer space.

In [Can11a], Cantat showed amongst other results that the plane Cremona group satisfies *Tits' alternative for finitely generated subgroups*, i.e. every finitely generated subgroup either contains a free group or is virtually solvable. Blanc showed that the plane Cremona group does not contain any non-trivial closed normal subgroup ([Bla10], see [BZ15] for a generalization of this result for arbitrary dimensions). In [CLdC13], Cantat and Lamy proved that the plane Cremona group is not simple and thus answered one of the most significant questions in the field. The main ingredients for their paper come from combining the action of $\mathrm{Cr}_2(\mathbb{C})$ on $\mathbb{H}^\infty$ with the results from Theorem 2.1.2 and then applying small cancellation theory (see for example [Cou16]).

This thesis contributes to the research on Cremona groups by studying certain classes of "large" subgroups. We briefly outline the structure and the main results. Chapter 3 and Chapter 4 are self-contained texts, whereas Chapter 5 to 7 depend on each other.

In **Chapter 3** we look at algebraic embeddings of $\mathrm{Cr}_n(\mathbb{C})$ to the group of birational transformations $\mathrm{Bir}(M)$ of an algebraic variety M. First we study geometrical properties of an example of an embedding of $\mathrm{Cr}_2(\mathbb{C})$ into $\mathrm{Cr}_5(\mathbb{C})$ that has been discovered by Gizatullin. In a second part, we give a full classification of all algebraic embeddings of $\mathrm{Cr}_2(\mathbb{C})$ into $\mathrm{Bir}(M)$, where M is a variety of dimension 3 and generalize this result partially to algebraic embeddings of $\mathrm{Cr}_n(\mathbb{C})$ into $\mathrm{Bir}(M)$, where the dimension of M is $n + 1$, for arbitrary n. In particular, this yields a classification of all algebraic $\mathrm{PGL}_{n+1}(\mathbb{C})$-actions on smooth projective varieties of dimension $n + 1$ that can be extended to rational actions of $\mathrm{Cr}_n(\mathbb{C})$. Chapter 3 consists of the text of an article that has been accepted for publication in "Annales de l'Institut Fourier" with an additional appendix.

Degree sequences of birational transformations play an important role in understanding the structure of groups of birational transformations of surfaces. Only little is known about these sequences in higher dimensions. In **Chapter 4** we will look at some questions regarding degree sequences of dominant rational transformations of varieties in dimension ≥ 2, in particular we will discover some new constraints on their growth and investigate properties of certain examples. We will also see that *the set of all possible degree sequences is countable*; this generalizes a result of Bonifant and Fornaess ([BF00]). Chapter 4 consists of the text of an article that has been accepted for publication in "Mathematical Research Letters".

In **Chapter 5** we recall some results about the plane Cremona group that we use in Chapter 6 and 7.

Chapter 6 is concerned with subgroups of $\mathrm{Cr}_2(\mathbb{C})$ consisting only of elliptic elements. Up to this point, results on such groups existed only if they were finitely generated ([Can11a]) or contained in an algebraic group. The main idea of this chapter is to use the compactness theorem from model theory to generalize results about finitely generated subgroups to results about arbitrary subgroups. This allows us to deduce results about the group structure of groups consisting of elliptic elements. In particular, we show that $\mathrm{Cr}_2(\mathbb{C})$ *satisfies Tits' alternative for arbitrary subgroups*, i.e. every subgroup of $\mathrm{Cr}_2(\mathbb{C})$ is either virtually solvable or contains a free subgroup. Moreover, we prove that *torsion subgroups are virtually abelian* and that *the derived length of solvable subgroups of* $\mathrm{Cr}_2(\mathbb{C})$ *is bounded*.

In **Chapter 7** we look at simple subgroups of $\mathrm{Cr}_2(\mathbb{C})$. We show that *all finitely generated simple groups of* $\mathrm{Cr}_2(\mathbb{C})$ *are finite*. Under the additional assumption of an unproven conjecture on a dynamical property of loxodromic maps (Conjecture 7.1.1) we moreover show that a subgroup of $\mathrm{Cr}_2(\mathbb{C})$ that contains a loxodromic element is not simple. The main ingredients for this proof are the techniques from [CLdC13] and the more recent work by Shepherd-Barron [SB13]. The main result of this section is that, *under the assumption of an unproven conjecture, a simple subgroup can be embedded into* $\mathrm{Cr}_2(\mathbb{C})$ *if and only if it can be embedded into* $\mathrm{PGL}_3(\mathbb{C})$.

Chapter 8 briefly summarizes the main questions I have encountered during my thesis, which I was not able to solve.

In what follows we will give a detailed description of the results from each part.

2.2 On homomorphisms between Cremona groups

The starting point of our considerations is the finite dimensional representation theory of Lie groups. There exists a rather complete theory on Lie group homomorphisms between classical groups that has a satisfying description in terms of vectors of highest weights (see for example [FH91]). The main strategy consists in considering the following groups associated to a complex Lie group G:

- a fixed torus T of maximal rank, i.e. a subgroup of the form $(\mathbb{C}^*)^n$, where n is maximal among all such subgroups;

- the Weyl group of T. i.e. the group $\mathrm{Norm}_G(T)/T$;

- root subgroups, i.e. additive one-parameter subgroups of G that are normalized by T.

Sometimes Cremona groups are seen as infinite dimensional analogues of complex Lie groups. From that perspective one could dream of a theory of "Cremona representations", of a complete description of group homomorphisms from $\mathrm{Cr}_n(\mathbb{C})$ to $\mathrm{Bir}(M)$, for a given variety M, that preserve some algebraic structure.

There exist analogues in $\mathrm{Cr}_n(\mathbb{C})$ of tori of maximal rank and the Weyl group. Let $D_n \subset \mathrm{PGL}_{n+1}(\mathbb{C}) \subset \mathrm{Cr}_n(\mathbb{C})$ be the n-dimensional subgroup consisting of diagonal automorphisms. It is the *torus of highest rank* of $\mathrm{Cr}_n(\mathbb{C})$ in the following sense: Let D be an algebraic subgroup of $\mathrm{Cr}_n(\mathbb{C})$ isomorphic to $(\mathbb{C}^*)^k$, then $k \leq n$ and if $k = n$ then D is conjugate to D_n ([BB66]).

Let $A = (a_{ij}) \in M_n(\mathbb{Z})$ be a matrix of integers. The matrix A determines a rational self map of the affine space

$$f_A = (x_1^{a_{11}} x_2^{a_{12}} \cdots x_n^{a_{1n}}, x_1^{a_{21}} x_2^{a_{22}} \cdots x_n^{a_{2n}}, \ldots, x_1^{a_{n1}} x_2^{a_{n2}} \cdots x_n^{a_{nn}}).$$

We have $f_A \circ f_B = f_{AB}$ for $A, B \in M_n(\mathbb{Z})$. One observes that f_A is a birational transformation if and only if $A \in \mathrm{GL}_n(\mathbb{Z})$. This yields an injective homomorphism $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{Cr}_n(\mathbb{C})$ whose image we call the *Weyl group* and denote by W_n. This terminology is justified by the fact that the normalizer of D_n in $\mathrm{Cr}_n(\mathbb{C})$ is the semidirect product $\mathrm{Norm}_{\mathrm{Cr}_n(\mathbb{C})}(D_n) = D_n \rtimes \mathrm{W}_n$. Sometimes W_n is also called the *group of monomial transformations*.

Note that Theorem 2.1.1 implies in particular that $\mathrm{Cr}_2(\mathbb{C}) = \langle \mathrm{W}_2, \mathrm{PGL}_3(\mathbb{C}) \rangle$. Results of Hudson and Pan ([Hud27], [Pan99]) show that for $n \geq 3$ the Cremona group $\mathrm{Cr}_n(\mathbb{C})$ is not generated by $\mathrm{PGL}_{n+1}(\mathbb{C})$ and W_n anymore. We define:

$$\mathrm{H}_n := \langle \mathrm{PGL}_{n+1}(\mathbb{C}), \mathrm{W}_n \rangle.$$

Despite this analogy to Lie groups, it is unclear, whether a theory of Cremona representations can be developed. With the following examples we would like to illustrate that the situation is more complicated and that in general there exist many different homomorphisms between various Cremona groups:

Example 2.2.1. Assume that a variety M is birationally equivalent to $\mathbb{P}^n \times N$ for some variety N. The standard action on the first factor yields an injective homomorphism of $\mathrm{Cr}_n(\mathbb{C})$ into $\mathrm{Bir}(\mathbb{P}^n \times N)$ and therefore also into $\mathrm{Bir}(M)$. We call embeddings of this type *standard embeddings*. In particular, we obtain in that way for all nonnegative integers m a standard embedding $\mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}^n \times \mathbb{P}^m)$.

Example 2.2.2. A variety M is called *stably rational* if there exists a n such that $M \times \mathbb{P}^n$ is rational. There exist varieties of dimension larger than or equal to 3 that are stably rational but not rational (see [BCTSSD85]). We will see that two standard embeddings $f_1 \colon \mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}^n \times N)$ and $f_2 \colon \mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}^n \times M)$ are conjugate if and only if N and M are birationally equivalent. So every class of birationally equivalent stably rational varieties of dimension k defines a different conjugacy class of embeddings $\mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}^m)$ for $m = n + k$.

As a first step in exploring Cremona representations, we consider in this chapter group homomorphisms from $\mathrm{Cr}_n(\mathbb{C})$ to $\mathrm{Bir}(M)$. In particular, we will study an embedding of $\mathrm{Cr}_2(\mathbb{C})$ into $\mathrm{Cr}_5(\mathbb{C})$ that was described by Gizatullin [Giz99] and consider the case, where $\dim(M) = n + 1$. Throughout the whole chapter we work over the field of complex numbers $\mathbb{C}$.

2.2.1 The case $\dim(M) \leq n$

Let M be a complex projective variety of dimension n and $\rho \colon \mathrm{PGL}_{r+1}(\mathbb{C}) \to \mathrm{Bir}(M)$ an embedding. Then $n \geq r$ and if $n = r$ it follows that M is rational and that up to a field automorphism, ρ is the standard embedding (see [Can14] and [Dés06b]). This implies in particular that there are no embeddings of $\mathrm{Cr}_n(\mathbb{C})$ into $\mathrm{Bir}(M)$ if $\dim(M) < n$. In the Appendix we recall these results and show that the restriction of an automorphism of $\mathrm{Cr}_n(\mathbb{C})$ to the subgroup H_n is inner up to a field automorphism.

2.2.2   Algebraic homomorphisms

We call a group homomorphism $\Psi\colon \mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(M)$ *algebraic* if its restriction to $\mathrm{PGL}_{n+1}(\mathbb{C})$ is an algebraic morphism, i.e. the map $\mathrm{PGL}_{n+1}(\mathbb{C}) \times M \dashrightarrow M$ given by $(g, x) \mapsto g(x)$ is a rational map. The algebraic structure of $\mathrm{Bir}(M)$ and some properties of algebraic homomorphisms will be discussed in Section 3.2. Recall that an element $f \in \mathrm{Cr}_n(\mathbb{C})$ is called *algebraic*, if the sequence $\{\deg(f^n)\}_{n\in\mathbb{Z}_+}$ is bounded.

Definition 2.2.1. Let M be a variety and $\varphi_M\colon \mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(M)$ a non-trivial algebraic group homomorphism. We say that φ_M is *reducible* if there exists a variety N such that $0 < \dim(N) < \dim(M)$ and an algebraic homomorphism $\varphi_N\colon \mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(N)$ together with a dominant rational map $\pi\colon M \dashrightarrow N$ that is $\mathrm{Cr}_n(\mathbb{C})$-equivariant with respect to the rational actions induced by φ_M and φ_N respectively, i.e. $\pi \circ \varphi_M(g) = \varphi_N(g) \circ \pi$ for all $g \in \mathrm{Cr}_n(\mathbb{C})$.

Remark 2.2.1. In [Zha10], Zhang uses the terminology *primitive action* for irreducible actions in the sense of Definition 2.2.1; in [Can03], Cantat says that an action *admits a non-trivial factor* if it is reducible.

Note that if we look at the induced action of $\mathrm{Cr}_n(\mathbb{C})$ on the function field $\mathbb{C}(M)$ of M, reducibility is equivalent to the existence of a nontrivial invariant function field $\mathbb{C}(N) \subset \mathbb{C}(M)$.

2.2.3   An example by Gizatullin

In [Giz99], Gizatullin looks at the following question: Consider a linear representation $\psi\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_{n+1}(\mathbb{C})$. Does the homomorphism ψ extend to a homomorphism $\Psi\colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Cr}_n(\mathbb{C})$? He shows that the linear representations given by the action of $\mathrm{PGL}_3(\mathbb{C})$ on conics, cubics and quartics can be extended to homomorphisms from $\mathrm{Cr}_2(\mathbb{C})$ to $\mathrm{Cr}_5(\mathbb{C})$, $\mathrm{Cr}_9(\mathbb{C})$ and $\mathrm{Cr}_{14}(\mathbb{C})$, respectively. These homomorphisms are related to the rational action of $\mathrm{Cr}_2(\mathbb{C})$ on moduli spaces of certain vector bundles on $\mathbb{P}^2$ that were discovered by Artamkin ([Art90]).

In Section 3.3 we study in detail some geometrical properties of the homomorphism

$$\Phi\colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Cr}_5(\mathbb{C})$$

that was described by Gizatullin; by construction, the restriction of Φ to $\mathrm{PGL}_3(\mathbb{C})$ yields the linear representation $\varphi\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_6(\mathbb{C})$ given by the action of $\mathrm{PGL}_3(\mathbb{C})$ on plane conics. Among other things, we prove the following:

Theorem 2.2.3. *Let $\Phi\colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Cr}_5(\mathbb{C})$ be the Gizatullin homomorphism. Then the following is true:*

(1) *The group homomorphism Φ is injective and irreducible.*

(2) *The rational action of $\mathrm{Cr}_2(\mathbb{C})$ on $\mathbb{P}^5$ that is induced by Φ preserves the Veronese surface V and its secant variety $S \subset \mathbb{P}^5$ and induces rational actions of $\mathrm{Cr}_2(\mathbb{C})$ on V and S.*

(3) *The Veronese embedding $v \colon \mathbb{P}^2 \to \mathbb{P}^5$ is $\mathrm{Cr}_2(\mathbb{C})$-equivariant with respect to the standard rational action on $\mathbb{P}^2$.*

(4) *The dominant secant map $s \colon \mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow S \subset \mathbb{P}^5$ (see Section 3.3.4) is $\mathrm{Cr}_2(\mathbb{C})$-equivariant with respect to the diagonal action of $\mathrm{Cr}_2(\mathbb{C})$ on $\mathbb{P}^2 \times \mathbb{P}^2$.*

(5) *The rational action of $\mathrm{Cr}_2(\mathbb{C})$ on $\mathbb{P}^5$ preserves a volume form on $\mathbb{P}^5$ with poles of order three along the secant variety S.*

(6) *The group homomorphism Φ sends the group of polynomial automomorphisms $\mathrm{Aut}(\mathbb{A}^2) \subset \mathrm{Cr}_2(\mathbb{C})$ to $\mathrm{Aut}(\mathbb{A}^5)$.*

Note that the injectivity of Φ follows from (3); in Section 3.3.8 irreducibility is proved. Part (2) - (4) of Theorem 3.1.3 will be proved in Section 3.3.4, part (5) in Section 3.3.6 and part (6) in Section 3.3.7

The representation $\varphi^\vee$ of $\mathrm{PGL}_3(\mathbb{C})$ into $\mathrm{PGL}_6(\mathbb{C})$ given by $\psi \circ \alpha$, where α is the algebraic homomorphism $g \mapsto {}^t g^{-1}$, is conjugate in $\mathrm{Cr}_5(\mathbb{C})$ to the representation φ. This conjugation yields the embedding $\Phi^\vee \colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Cr}_5(\mathbb{C})$, whose image preserves the secant variety S as well and induces a rational action on it. As the secant variety S is rational, Φ and $\Phi^\vee$ induce two non-standard homomorphisms of $\mathrm{Cr}_2(\mathbb{C})$ into Cr_4, which we denote by Ψ_1 and Ψ_2 respectively. In Section 3.3.5 we prove the following:

Proposition 2.2.4. *The two homomorphisms $\Psi_1, \Psi_2 \colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Cr}_4(\mathbb{C})$ are not conjugate in Cr_4; moreover they are irreducible and therefore not conjugate to the standard embedding.*

Remark 2.2.2. The homomorphism Ψ_1 is injective, since it restricts to the standard action on the Veronese surface. However, it seems to be unclear, whether Ψ_2 is injective as well. Since the restriction of Ψ_2 to $\mathrm{PGL}_3(\mathbb{C})$ is injective, it seems unlikely that Ψ_2 is not injective. In fact, it seems that one could use results form [BZ15] to show that non-trivial algebraic homomorphisms are always injective. However, I haven't proved it yet.

Since Φ is algebraic, the images of algebraic elements under Φ are algebraic again (see Proposition 3.2.6). Calculation of the degrees of some examples suggests that Φ might even preserve the degrees of all elements in $\mathrm{Cr}_2(\mathbb{C})$. However, we were only able to prove the following (Section 3.3.7):

Theorem 2.2.5. *Let $\Phi \colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Cr}_5(\mathbb{C})$ be the Gizatullin-embedding. Then*

(1) *for all elements $f \in \mathrm{Cr}_2(\mathbb{C})$ we have $\deg(f) \leq \deg(\Phi(f))$,*

(2) *for all $g \in \mathrm{Aut}(\mathbb{A}^2) \subset \mathrm{Cr}_2(\mathbb{C})$ we have $\deg(g) = \deg(\Phi(g))$.*

The image of the Weyl group W_2 under Φ is not contained in the Weyl group W_5. More generally, it can be shown that there exists no algebraic homomorphism from $\mathrm{Cr}_2(\mathbb{C})$ to $\mathrm{Cr}_5(\mathbb{C})$ that preserves automorphisms, diagonal automorphisms and the Weyl group:

Theorem 2.2.6. *There is no non-trivial algebraic homomorphism $\Psi \colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Cr}_5(\mathbb{C})$ such that:*

(1) $\Psi(\mathrm{PGL}_3(\mathbb{C})) \subset \mathrm{PGL}_6(\mathbb{C})$

(2) $\Psi(D_2) \subset D_5$ and

(3) $\Psi(\mathrm{W}_2) \subset \mathrm{W}_5$.

2.2.4 Algebraic embeddings in codimension 1

In Section 3.4 and Section 3.5 we look at algebraic homomorphisms $\mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(M)$ in the case where M is a smooth projective variety of dimension $n+1$ for $n \geq 2$.

Example 2.2.7. For all curves C of genus ≥ 1, the variety $\mathbb{P}^n \times C$ is not rational and there exists the standard embedding $\Psi_C \colon \mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}^n \times C)$.

Example 2.2.8. $\mathrm{Cr}_n(\mathbb{C})$ acts rationally on the total space of the canonical bundle of $\mathbb{P}^n$

$$K_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-(n+1)) \simeq \bigwedge^n (\mathrm{T}\,\mathbb{P}^n)^\vee$$

by $f(p, \omega) = (f(p), \omega \circ (df_p)^{-1})$, where $p \in \mathbb{P}^n$ and $\omega \in \bigwedge^n (\mathrm{T}_p\,\mathbb{P}^n)^\vee$. More generally, we obtain a rational action of $\mathrm{Cr}_n(\mathbb{C})$ on the total space of the bundle $K_{\mathbb{P}^n}^{\otimes l} \simeq \mathcal{O}_{\mathbb{P}^n}(-(n+1)l)$ and on its projective completion

$$F_l := \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-l(n+1))$$

for all $l \in \mathbb{Z}_{\geq 0}$. This yields a countable family of injective homomorphisms

$$\Psi_l \colon \mathrm{Cr}_n(\mathbb{C}) \to \mathrm{Bir}(F_l).$$

Note that the restriction of this rational action to $\mathrm{PGL}_n(\mathbb{C})$ is regular, hence these embeddings are algebraic.

We can choose affine coordinates $(x_1, \ldots, x_n, x_{n+1})$ of F_l such that Ψ_l is given by

$$\Psi_l(f)(x_1, \ldots, x_n, x_{n+1}) = (f(x_1, \ldots, x_n), J(f(x_1, \ldots, x_n))^{-l} x_{n+1}),$$

where, $J(f(x_1, \ldots, x_n))$ denotes the determinant of the Jacobian of f at the point $(x_1, \ldots, x_n)$. Observe that Ψ_0 is conjugate to the standard embedding.

Example 2.2.9. Let $\mathbb{P}(\mathrm{T}\,\mathbb{P}^2)$ be the total space of the fiberwise projectivisation of the tangent bundle over $\mathbb{P}^2$. Then $\mathbb{P}(\mathrm{T}\,\mathbb{P}^2)$ is rational and there is an injective group homomorphism

$$\Psi_B \colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}(\mathrm{T}\,\mathbb{P}^2))$$

defined by $\Psi_B(f)(p, v) := (f(p), \mathbb{P}(df_p)(v))$. Here, $\mathbb{P}(df_p) \colon \mathbb{P}\,\mathrm{T}_p \to \mathbb{P}\,\mathrm{T}_{f(p)}$ defines the projectivisation of the differential df_p of f at the point $p \in \mathbb{P}^2$.

Example 2.2.10. The Grassmannian of lines in the projective 3-space $\mathbb{G}(1, 3)$ is a rational variety of dimension 4 with a transitive algebraic $\mathrm{PGL}_4(\mathbb{C})$-action. This action induces an algebraic embedding of $\mathrm{PGL}_4(\mathbb{C})$ into Cr_4. In Proposition 3.5.2 we will show that the image of this embedding does not lie in any subgroup isomorphic to Cr_3. *So no group action of* $\mathrm{PGL}_4(\mathbb{C})$ *on* $\mathbb{G}(1, 3)$ *by automorphisms can be extended to a rational action of* Cr_3.

The classification of $\mathrm{PGL}_{n+1}(\mathbb{C})$-actions on smooth projective varieties of dimension $n + 1$ is well known to the experts; in Section 3.4 we study their conjugacy classes. We will see that Examples 2.2.7 to 2.2.10 describe up to birational conjugation and up to algebraic homomorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$ all possible $\mathrm{PGL}_{n+1}(\mathbb{C})$-actions on smooth projective varieties of dimension $n + 1$ and that these actions are not birationally conjugate to each other. This yields a classification of algebraic homomorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$ to $\mathrm{Bir}(M)$ up to birational conjugacy, for smooth projective M of dimension $n + 1$. We will study in Section 3.5 how these actions extend to rational actions of $\mathrm{Cr}_n(\mathbb{C})$ on M. Denote by $\alpha \colon \mathrm{PGL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$ the algebraic automorphism given by $g \mapsto {}^t(g^{-1})$.

Theorem 2.2.11. *Let $n \geq 2$, let M be a complex projective variety of dimension $n + 1$ and let $\varphi \colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{Bir}(M)$ be a non-trivial algebraic homomorphism. Then*

(1) *φ is conjugate, up to the automorphism α, to exactly one of the embeddings described in Example 2.2.7 to 2.2.10.*

(2) *If $n = 3$ and φ is conjugate to the action described in Example 2.2.10, then neither φ nor $\varphi \circ \alpha$ can be extended to a homomorphism of H_3 to $\mathrm{Bir}(M)$.*

(3) *If φ is conjugate to one of the embeddings described in Example 2.2.7 to 2.2.9 then exactly one of the embeddings φ or $\varphi \circ \alpha$ extends to a homomorphism of $\mathrm{Cr}_n(\mathbb{C})$ to $\mathrm{Bir}(M)$.*

(4) *φ extends to H_n if and only if it extends to $\mathrm{Cr}_n(\mathbb{C})$; moreover, in this case the extension to H_n is unique.*

Theorem 2.2.11 classifies all group homomorphisms $\Psi \colon \mathrm{H}_n \to \mathrm{Bir}(M)$ for projective varieties M of dimension $n + 1$ such that the restriction to $\mathrm{PGL}_{n+1}(\mathbb{C})$ is an algebraic morphism. By the theorem of Noether and Castelnuovo, we obtain in particular a full classification of all algebraic homomorphisms from $\mathrm{Cr}_2(\mathbb{C})$ to $\mathrm{Bir}(M)$ for projective varieties M of dimension 3:

Corollary 2.2.12. *Let M be a projective variety of dimension 3 and $\Psi \colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Bir}(M)$ a non-trivial algebraic group homomorphism. Then Ψ is conjugate to exactly one of the homomorphisms described in Example 2.2.7 to 2.2.9.*

The following observations are now immediate:

Corollary 2.2.13. *Let M be a projective variety of dimension 3 and $\Psi \colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Bir}(M)$ a non-trivial algebraic homomorphism. Then*

(1) *Ψ is injective.*

(2) *There exists a $\mathrm{Cr}_2(\mathbb{C})$-equivariant rational map $f \colon M \dashrightarrow \mathbb{P}^2$ with respect to the rational action induced by Ψ and the standard action respectively. In particular, all algebraic homomorphisms from $\mathrm{Cr}_2(\mathbb{C})$ to $\mathrm{Bir}(M)$ are reducible.*

(3) *There exists an integer $C_\Psi \in \mathbb{Z}$ such that*

$$1/C_\Psi \deg(f) \leq \deg(\Psi(f)) \leq C_\Psi \deg(f).$$

Note that Part (3) of Corollary 2.2.13 resembles in some way Theorem 2.2.5 and leads to the following question:

Question 2.2.1. Let $\Phi\colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Cr}_n(\mathbb{C})$ be an algebraic embedding. Does there always exist a constant C depending only on Φ such that $1/C \deg(f) \le \deg(\Phi(f)) \le C \deg(f)$?

One could consider an analogue of Question 2.2.1 for Lie group representations of $\mathrm{SL}_2(\mathbb{C})$. Let $||A||$ be the operator norm of a matrix $A \in \mathrm{SL}_2(\mathbb{C})$. The function $d\colon A \mapsto |\log(||A||)|$ can be seen as an analogue of the degree function on $\mathrm{Cr}_2(\mathbb{C})$ in the sense that it is symmetric and subadditive. The finite-dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{C})$ are exactly the representations given by the action on the vector space $\mathbb{C}[x,y]_d$ of polynomials of degree d. Let $\rho\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(\mathbb{C}[x,y]_d)$ be an irreducible representation. One can check that there exists a constant C such that

$$1/C ||A|| \le ||\rho(A)|| \le C ||A||.$$

2.3 Degree sequences

Let $X_{\mathbf{k}}$ be a projective variety defined over a field $\mathbf{k}$; denote by $\mathrm{Bir}(X_{\mathbf{k}})$ the group of birational transformations of $X_{\mathbf{k}}$. A group Γ is called a *group of birational transformations* if there exists a field $\mathbf{k}$ and a projective variety $X_{\mathbf{k}}$ over $\mathbf{k}$ such that $\Gamma \subset \mathrm{Bir}(X_{\mathbf{k}})$. More generally, one can consider $\mathrm{Rat}(X_{\mathbf{k}})$, the monoid of dominant rational self-maps of $X_{\mathbf{k}}$. Accordingly, we call a monoid Δ a *monoid of rational dominant transformations*, if there exists a field $\mathbf{k}$ and a projective variety $X_{\mathbf{k}}$ over $\mathbf{k}$ such that $\Delta \subset \mathrm{Rat}(X_{\mathbf{k}})$.

If $X_{\mathbf{k}}$ is a smooth projective variety, an interesting tool to study the structure of monoids of rational dominant transformations are *degree functions*. Fix a polarization of $X_{\mathbf{k}}$, i.e. an ample divisor class H of $X_{\mathbf{k}}$. Then one can associate to every element $f \in \mathrm{Rat}(X_{\mathbf{k}})$ its degree $\deg_H(f) \in \mathbb{Z}_+$ with respect to H, which is defined by

$$\deg_H(f) = f^* H \cdot H^{d-1},$$

where d is the dimension of $X_{\mathbf{k}}$ and $f^* H$ is the total transform of H under f. For a smooth projective variety $X_{\mathbf{k}}$ over a field $\mathbf{k}$ of characteristic zero, one has for $f, g \in \mathrm{Bir}(X_{\mathbf{k}})$

$$\deg_H(f \circ g) \le C(X_{\mathbf{k}}, H) \deg_H(f) \deg_H(g),$$

where $C(X_{\mathbf{k}}, H)$ is a constant only depending on $X_{\mathbf{k}}$ and the choice of polarization H (see [DS05]). For a generalization of this result to fields of positive characteristic, see [Tru15] and [Tru16] and the more recent [Dan17] for an alternative approach.

Let f be a rational self map of $\mathbb{P}_{\mathbf{k}}^d$. With respect to homogeneous coordinates $[x_0 : \cdots : x_d]$ of $\mathbb{P}_{\mathbf{k}}^d$, f is given by $[x_0 : x_1 : \cdots : x_d] \mapsto [f_0 : f_1 : \cdots : f_d]$, where $f_0, \ldots, f_d \in \mathbf{k}[x_0, \ldots, x_d]$ are homogeneous polynomials of the same degree and without common factor. We define $\deg(f) = \deg(f_i)$. If f is dominant, then $\deg(f) = \deg_H(f)$ for $H = \mathcal{O}(1)$. Note that if f is an endomorphism of $\mathbb{A}_{\mathbf{k}}^d$ defined by $(x_1, \ldots, x_d) \mapsto (f_1, \ldots, f_d)$ with respect to affine coordinates $(x_1, \ldots, x_d)$ of $\mathbb{A}_{\mathbf{k}}^d$, where $f_1, \ldots, f_d \in \mathbf{k}[x_1, \ldots, x_d]$, then $\deg(f)$ is the maximal degree of the f_i.

Let $\Delta \subset \mathrm{Rat}(X_{\mathbf{k}})$ be a finitely generated monoid of rational dominant transformations of a smooth projective variety $X_{\mathbf{k}}$ with a finite set of generators S. We define

$$D_{S,H} \colon \mathbb{Z}_+ \to \mathbb{Z}_+$$

by

$$D_{S,H}(n) := \max_{\delta \in B_S(n)} \{\deg_H(\gamma)\},$$

where $B_S(n)$ denotes all elements in Δ of word length $\leq n$ with respect to the generating set S. We call a map $\mathbb{Z}_+ \to \mathbb{Z}_+$ that can be realized for some field $\mathbf{k}$ and some $(X_{\mathbf{k}}, H, \Delta, S)$ as such a function a *degree sequence*. Note that our definition of degree sequences includes in particular degree sequences that are given by finitely generated groups of birational transformations $\Gamma \subset \mathrm{Bir}(X_{\mathbf{k}})$.

In this chapter we show that there exist only countably many degree sequences, display certain constraints on their growth and give some new examples.

2.3.1 Countability of degree sequences

In [BF00], Bonifant and Fornaess proved that the set of sequences $\{d_n\}$ such that there exists a rational self map f of $\mathbb{P}^d_{\mathbb{C}}$ satisfying $\deg(f^n) = d_n$ for all n, is countable, which answered a question of Ghys. We generalize the result of Bonifant and Fornaess to all degree sequences over all smooth projective varieties, all fields, all polarizations and all finite generating sets S of finitely generated monoids of rational dominant maps:

Theorem 2.3.1. *The set of all degree sequences is countable.*

2.3.2 Questions

In dimension 2 the degree growth of birational transformations is well understood (Theorem 2.1.2). In the case of polynomial automorphisms, we have an even more precise statement:

Theorem 2.3.2 ([Fur99]). *Let $f \in \mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$, then the degree sequence $\{\deg(f^n)\}$ is either periodic (in particular bounded) or a geometric progression.*

However, in higher dimensions there are only few results on the degree growth of birational transformations. For instance, in [LB14] Lo Bianco treats the case of automorphisms of compact Kähler threefolds. The following questions seem to be open:

Question 2.3.1. Does there exist a birational transformation f of a projective variety $X_{\mathbf{k}}$ such that $\deg_H(f^n)$ is of intermediate growth, for instance $\deg_H(f^n) \sim e^{\sqrt{n}}$?

Question 2.3.2. Does there exist a birational transformation f such that $\deg_H(f^n)$ is unbounded, but grows "slowly"? For instance, can we have $\deg_H(f^n) \sim \sqrt{n}$? Or do unbounded degree sequences grow at least linearly?

Question 2.3.3. If there is a birational transformation f such that $\deg_H(f^n) \sim \lambda^n$, is λ always an algebraic number?

Question 2.3.4. Do birational transformations of polynomial growth always preserve some non-trivial rational fibration?

2.3.3   Degree sequences of polynomial automorphisms

A good place to start the examination of degree sequences seems to be the group of polynomial automorphisms of the affine d-space $\mathrm{Aut}(\mathbb{A}_{\mathbf{k}}^d)$. In Section 4.3.2 we will show the following observation (the proof of which can be found as well in [Dés16a]):

Proposition 2.3.3. *Let $\mathbf{k}$ be a field and $f \in \mathrm{Aut}(\mathbb{A}_{\mathbf{k}}^d)$ a polynomial automorphism such that $\deg(f^d) = \deg(f)^d$, then $\deg(f^n) = \deg(f)^n$ for all $n \in \mathbb{Z}_+$.*

The monoid $\mathrm{End}(\mathbb{A}_{\mathbf{k}}^d)$ has the additional structure of a $\mathbf{k}$-vector space, on which the degree function induces a filtration of finite dimensional vector spaces. This gives rise to a new technique, which we will use to prove that unbounded degree sequences of groups of polynomial automorphisms diverge and can not grow arbitrarily slowly:

Theorem 2.3.4. *Let $f \in \mathrm{End}(\mathbb{A}_{\mathbf{k}}^d)$ be an endomorphism and assume that the sequence $\{\deg(f^n)\}$ is unbounded. Then for all integers K*

$$\# \{m \mid \deg(f^m) \le K\} < C_d \cdot K^d,$$

where $C_d = \frac{(1+d)^d}{(d-1)!}$. In particular, $\deg(f^n)$ converges to ∞ as n goes to ∞.

By a result of Ol'shanskii ([Ols99]), Theorem 2.3.4 shows that an unbounded degree sequence of a polynomial automorphism behaves in some ways like a word length function. The following corollary is immediate:

Corollary 2.3.5. *Let $\Gamma \subset \mathrm{End}(\mathbb{A}_{\mathbf{k}}^d)$ be a finitely generated monoid with generating system S. If $D_S(n) < C_d \cdot n^{1/d}$ for infinitely many n then Γ is of bounded degree.*

Unfortunately our methods to prove Theorem 2.3.4 do not work for arbitrary birational transformations of $\mathbb{P}_{\mathbf{k}}^d$. However, if we assume the ground field to be finite, we obtain similar results:

Theorem 2.3.6. *Let $\mathbb{F}_q$ be a finite field with q elements and let $f \in \mathrm{Rat}(\mathbb{P}_{\mathbb{F}_q}^d)$ such that the sequence $\{\deg(f^n)\}$ is unbounded. Then, for all integers K,*

$$\# \{m \mid \deg(f^m) \le K\} \le q^{C(K,d)},$$

where $C(K,d) = (d+1) \cdot \binom{d+K}{K}$. In particular, $\deg(f^n)$ converges to ∞ as n goes to ∞.

Corollary 2.3.7. *Let $\Gamma \subset \mathrm{Rat}(\mathbb{P}_{\mathbb{F}_q}^d)$ be a finitely generated monoid with generating system S. There exists a positive constant $C_{d,q}$ such that if $D_S(n) < C_{d,q} \cdot \log(n)^{1/d}$ for all n, then Γ is of bounded degree.*

2.3.4 Types of degree growth

Definition 2.3.1. Let $X_{\mathbf{k}}$ be a smooth projective variety with polarization H over a field $\mathbf{k}$ and let $f \in \operatorname{Bir}(X_{\mathbf{k}})$. We denote the *order of growth* of $\deg_H(f^n)$ by

$$\operatorname{dpol}(f) := \limsup_{n \to \infty} \frac{\log(\deg_H(f^n))}{\log(n)}.$$

The order of growth can be infinite.

By results of Truong, Dinh and Sibony, the order of growth does not depend on the choice of polarization (see Section 4.2.4):

Proposition 2.3.8. *Let $X_{\mathbf{k}}$ be a smooth projective variety over a field $\mathbf{k}$ and let $f \in \operatorname{Bir}(X_{\mathbf{k}})$. Then $\operatorname{dpol}(f)$ does not depend on the choice of polarization.*

Let f be a birational transformation of a surface. As recalled above, in that case $\operatorname{dpol}(f) = 0, 1, 2$ or ∞. This gives rise to the following question:

Question 2.3.5. Does there exist a constant $C(d)$ depending only on d such that for all varieties $X_{\mathbf{k}}$ of dimension d we have $\operatorname{dpol}(f) < C(d)$ for all $f \in \operatorname{Bir}(X_{\mathbf{k}})$ with $\operatorname{dpol}(f)$ finite?

We give some examples of degree sequences that indicate that the degree growth in higher dimensions is richer than in dimension 2. First of all, note that in general elements in $\operatorname{Aut}(\mathbb{A}_{\mathbf{k}}^d)$ can have polynomial growth, which is not the case for $d = 2$ by Theorem 2.3.2.

Example 2.3.9. Let $\mathbf{k}$ be any field and define $f, g, h \in \operatorname{Aut}(\mathbb{A}_{\mathbf{k}}^3)$ by $g = (x + yz, y, z)$, $h = (x, y + xz, z)$ and

$$f = g \circ h = (x + z(y + xz), y + xz, z).$$

One sees by induction that $\deg(f^n) = 2n + 1$; in particular, $\operatorname{dpol}(f) = 1$.

More generally, for all $l \leq d/2$ there exist elements $f_l \in \operatorname{Aut}(\mathbb{A}_{\mathbf{k}}^d)$ such that $\operatorname{dpol}(f) = l$ (Section 4.3.4).

Other interesting examples of degree sequences of polynomial automorphisms and the dynamical behavior of the corresponding maps are described in [Dés16a]. For birational transformations of $\mathbb{P}_{\mathbf{k}}^d$ we can obtain even faster growth (see [Lin12] for more details):

Example 2.3.10. The birational transformation $f = (x_1, x_1 x_2, \ldots, x_1 x_2 \cdots x_n)$ of $\mathbb{P}_{\mathbf{k}}^d$ defined with respect to affine coordinates $(x_1, \ldots, x_d)$ satisfies $\deg(f^n) = n^{d-1}$, i.e. $\operatorname{dpol}(f) = d - 1$.

The following interesting observation is due to Serge Cantat:

Example 2.3.11. Define $\omega := \frac{-1+\sqrt{-3}}{2}$ and the elliptic curve $E_\omega := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\omega)$. Let

$$X := E_\omega \times E_\omega \times E_\omega$$

and $s\colon X \to X$ the automorphism of finite order given by diagonal multiplication with $-\omega$. In [OT15] Oguiso and Truong prove that the quotient $Y := X/s$ is a rational threefold. Let $f\colon X \to X$ be the automorphism defined by $(x_1, x_2, x_3) \mapsto (x_1, x_1 + x_2, x_2 + x_3)$. Since f commutes with s, it induces an automorphism on Y, which we denote by $\hat{f}$. Let $\phi_1\colon \tilde{Y} \to Y$ be a resolution of the singularities of Y and define $\tilde{f} \in \mathrm{Bir}(\tilde{Y})$ by

$$\tilde{f} := \phi_1^{-1} \circ \hat{f} \circ \phi_1.$$

We will show in Section 4.3.5 that $\mathrm{dpol}(\tilde{f}) = 4$.

Remark 2.3.1. In [Dés16a] Déserti shows that for all $l \leq d$ there exists a birational transformation $f \in \mathrm{Bir}(\mathbb{P}^d_{\mathbb{C}})$ such that $\mathrm{dpol}(f) = l$.

2.4   Groups of elliptic elements of the plane Cremona group

In this chapter we will be interested in groups $G \subset \mathrm{Cr}_2(\mathbb{C})$ such that every element in G is elliptic. An element g is elliptic if and only if $\{\deg(f^n)\}$ is bounded or, equivalently, there exists a $k \in \mathbb{Z}_+$ such that g^k is conjugate to an automorphism in $\mathrm{Aut}^0(S)$, the neutral component of the automorphism group of a projective surface S ([Can11a, Proposition 3.10]).

Definition 2.4.1. A group $G \subset \mathrm{Cr}_2(\mathbb{C})$ is a *group of elliptic elements* if every element in G is elliptic.

Definition 2.4.2. A group $G \subset \mathrm{Cr}_2(\mathbb{C})$ is *bounded* if there exists a constant K such that $\deg(g) \leq K$ for all elements $g \in G$.

Algebraic groups are always bounded, but bounded groups do not need to be algebraic. However, a subgroup of $\mathrm{Cr}_2(\mathbb{C})$ is bounded if and only if it is contained in an algebraic group. All bounded groups are groups of elliptic elements. But the converse is not true as the following examples illustrate:

Example 2.4.1. Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be the group of elements of the form $(x, y+p(x))$, where $p(x) \in \mathbb{C}(x)$ is a rational function. Then every element in G is algebraic, but G contains elements of arbitrarily high degrees.

Example 2.4.2. In [Wri79], Wright constructs examples of torsion subgroups of $\mathrm{Aut}(\mathbb{A}^2)$ and hence in particular of $\mathrm{Cr}_2(\mathbb{C})$ that contain elements of arbitrary high degree. In fact, he shows that there is a subgroup G of $\mathrm{Cr}_2(\mathbb{C})$ that is isomorphic to the subgroup of roots of unity in $\mathbb{C}^*$ but that is not bounded. In [Lam01a], Lamy shows that some of the examples of Wright do not preserve any fibration.

So far there only exist results about subgroups consisting of elliptic elements if they are bounded or finitely generated. Our strategy will be to use these results together with the compactness theorem from model theory in order to prove the following theorem that gives some information about the structure of subgroups of $\mathrm{Cr}_2(\mathbb{C})$ consisting only of elliptic elements:

Theorem 2.4.3. *Let $G \subset \mathrm{Cr}_2(\mathbf{k})$ be a subgroup of elliptic elements. Then we are in one of the following cases:*

(1) *G is not a torsion group and G contains a finitely generated subgroup that does not preserve any fibration. In this case, G is conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$, of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ or of $\mathrm{Aut}(S_6)$, where S_6 is the del Pezzo surface of degree 6.*

(2) *G is a torsion group.*

(3) *G preserves a rational fibration and is therefore conjugate to a subgroup of the de Jonquières group $\mathcal{J} \simeq \mathrm{PGL}_2(\mathbb{C}(t)) \rtimes \mathrm{PGL}_2(\mathbb{C})$, which is the subgroup of $\mathrm{Cr}_2(\mathbb{C})$ that preserves a given rational fibration.*

(4) *Every finitely generated subgroup of G preserves a rational fibration and there exists a torsion subgroup $G_0 \subset G$ and an exact sequence*

$$1 \to G_0 \to G \to \mathrm{PGL}_2(\mathbb{C}).$$

In the case, where G is a torsion subgroup of $\mathrm{Cr}_2(\mathbb{C})$, we can say more:

Theorem 2.4.4. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a torsion subgroup. Then G can be embedded into $\mathrm{GL}_N(\mathbb{C})$ for some $N \leq 48$ and $N \leq 24$ if G is infinite.*

Remark 2.4.1. There exists an $N \in \mathbb{Z}_+$ such that every finite subgroup of $\mathrm{Cr}_2(\mathbb{C})$ can be embedded into $\mathrm{GL}_N(\mathbb{C})$. In this text we will give the rough upper bound $N \leq 48$, but we expect that this bound can be lowered ignificantly. We will see in particular, that every finite subgroup of $\mathrm{Cr}_2(\mathbb{C})$ can be eitehr embedded into $\mathrm{GL}_{36}(\mathbb{C})$ or it is isomorphic to an extension of $\mathcal{S}_4$ by the Klein four-group.

Let us give some consequences of Theorem 2.4.3. It has been proven by Schur that every torsion subgroup of $\mathrm{GL}_n(\mathbb{C})$ is abelian up to finite index. More precisely:

Theorem 2.4.5 (Schur, see [CR62, p.258])**.** *Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a torsion subgroup. Then G contains an abelian subgroup of index at most*

$$(\sqrt{8n} + 1)^{2n^2} - (\sqrt{8n} - 1)^{2n^2}.$$

So from Theorem 2.4.4 (2) and Theorem 2.4.5 we can deduce directly that the same property holds for torsion subgroups of $\mathrm{Cr}_2(\mathbb{C})$:

Corollary 2.4.6. *Every torsion subgroup G of $\mathrm{Cr}_2(\mathbb{C})$ is finite or isomorphic to a subgroup of $\mathrm{GL}_N(\mathbb{C})$ for some $N \leq 24$. In particular, G contains an abelian subgroup of index at most*

$$(96\sqrt{2} + 1)^{2592} - (96\sqrt{2} - 1)^{2592} < 10^{5537}.$$

Recall that the derived series of a group G is the series of groups defined by

- $G^{(0)} := G$

- $G^{(i+1)} := [G^{(i)}, G^{(i)}],$

where the *commutator subgroup* $[H, H]$ of a group H is the subgroup generated by all elements of the form $aba^{-1}b^{-1}$, $a, b \in H$.

A group G is solvable if and only if its derived series terminates at the identity element after finitely many steps. The derived length of G is the smallest k such that $G^{(k)} = \{\mathrm{id}\}$.

Another result that can be deduced from Theorem 2.4.3 concerns the Tits alternative. In [Tit72], Tits showed the following:

Theorem 2.4.7 ([Tit72])**.** *Let* $\mathbf{k}$ *be a field of characteristic zero and* $n \in \mathbb{Z}_+$*. Then every subgroup* G *of* $\mathrm{GL}_n(\mathbf{k})$ *is either virtually solvable or contains a non-abelian free subgroup.*

This result has lead to the following definitions:

Definition 2.4.3.

(1) A group G satisfies *Tits' alternative* if every subgroup of G is either virtually solvable or contains a non-abelian free subgroup.

(2) A group G satisfies *Tits' alternative for finitely generated subgroups* if every finitely generated subgroup of G either is virtually solvable or contains a non-abelian free subgroup.

Theorem 2.4.7 states that linear groups over fields of characteristic zero satisfy Tits' alternative. Linear groups over fields of positive characteristic only satisfy Tits' alternative for finitely generated subgroups ([Tit72]). Other well-known examples of groups that satisfy Tits' alternative include mapping class groups of surfaces ([Iva84]), $\mathrm{Out}(\mathbb{F}_n)$, the outer automorphism group of the free group of finite rank n ([BFH00]) or hyperbolic groups in the sense of Gromov ([Gro87]).

In his PhD thesis Lamy showed Tits' alternative for subgroups of $\mathrm{Aut}(\mathbb{A}_\mathbb{C}^2)$:

Theorem 2.4.8 ([Lam01b])**.** $\mathrm{Aut}(\mathbb{A}_\mathbb{C}^2)$ *satisfies Tits' alternative.*

The proof of Theorem 2.4.8 relies on the amalgamated product structure of $\mathrm{Aut}(\mathbb{A}_\mathbb{C}^2)$ that is given by the Theorem of Jung and van der Kulk (see for example [Lam02]) and Bass-Serre theory (see [Ser77]).

Cantat established Tits' alternative for finitely generated subgroups of $\mathrm{Cr}_2(\mathbb{C})$:

Theorem 2.4.9 ([Can11a])**.** $\mathrm{Cr}_2(\mathbb{C})$ *satisfies Tits' alternative for finitely generated subgroups.*

Theorem 2.4.9 is part of a series of profound results about the group structure of the plane Cremona group that Cantat deduces from the action by isometries on the Picard-Manin space $\mathbb{H}^\infty$ by $\mathrm{Cr}_2(\mathbb{C})$ ([Can11a]). The main obstacle to generalize Theorem [Can11a] to arbitrary subgroups was caused by unbounded groups of elliptic elements that do not preserve any fibration. At this point, Theorem 2.4.3 steps in. It turns out that it yields the techniques to complete the result:

Theorem 2.4.10. $\mathrm{Cr}_2(\mathbb{C})$ *satisfies Tits' alternative.*

In [Dés15], Déserti gives a description of solvable subgroups of $\mathrm{Cr}_2(\mathbb{C})$. We complement her result with the following observation:

Theorem 2.4.11. *There exists a constant $K \leq 35$ such that the derived length of solvable subgroups of $\mathrm{Cr}_2(\mathbb{C})$ is $\leq K$.*

Remark 2.4.2. Also here, the upper bound 35 is only a rough estimate and we expect that it can be lowered by a detailed examination of the derived length of finite subgroups of $\mathrm{Cr}_2(\mathbb{C})$.

For each $n \in \mathbb{Z}_+$ there exists an $N \in \mathbb{Z}_+$ such that every solvable subgroup of $\mathrm{GL}_n(\mathbb{C})$ has derived length $\leq N$. This result seems to go back to Zassenhaus, see for example [New72], where explicit bounds are computed. The following bound is attributed to Huppert:

Theorem 2.4.12 ([New72, p.1])**.** *Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a solvable subgroup. The derived length of G is $\leq \min\{2n, 1 + 7\log_2(n)\}$.*

In [FP16], Furter and Poloni show that the maximal derived length of a solvable subgroup of $\mathrm{Aut}(\mathbb{A}_\mathbb{C}^2)$ is 5 (and that this bound is optimal).

2.5 Simple subgroups of the plane Cremona group

It had been a long-standing open question, whether the plane Cremona group is simple as a group until Cantat and Lamy showed in 2012 that it is not. The main idea to prove this result was to use the action of $\mathrm{Cr}_2(\mathbb{C})$ on the Picard-Manin space $\mathbb{H}^\infty$ by isometries and use techniques from small cancellation theory. In this chapter, we refine these techniques with the aim of classifying all simple subgroups of $\mathrm{Cr}_2(\mathbb{C})$. However, during the work I have encountered a problem that I was not able to solve. We formulate it in the following conjecture, which will be discussed in more detail in Section 7.2.3. Denote by $\mathrm{Ind}(f) \subset X$ the indeterminacy locus of a birational transformation of a variety X.

Conjecture 2.5.1. *Let $f \in \mathrm{Cr}_2(\mathbb{C})$ be a loxodromic element, $p \in \mathbb{P}^2$ a point that is not contained in any of the coordinate lines of $\mathbb{P}^2$, k a positive integer. Then the constructible set*

$$\{d(fd)^k(p) \mid d \in D_2 \text{ such that } p \notin \mathrm{Ind}((fd)^l) \text{ for all } 1 \leq l \leq k\}$$

is open and dense in $\mathbb{P}^2$.

The main result of this chapter will be:

Theorem 2.5.2. *Assume that Conjecture 2.5.1 holds.*
Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a simple group. Then:

(1) *G does not contain any loxodromic element.*

(2) *If G contains a parabolic element, then G fixes a rational fibration, i.e. there exists a G-invariant rational map $\pi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ with rational fibers. In that case, G is isomorphic to a subgroup of $\mathrm{PGL}_2(\mathbb{C}(t))$.*

(3) *If all elements in G are elliptic then either G is a simple subgroup of an algebraic subgroup of $\mathrm{Cr}_2(\mathbb{C})$ or it is conjugate to a subgroup of the de Jonquières group $\mathcal{J} \simeq \mathrm{PGL}_2(\mathbb{C}) \ltimes \mathrm{PGL}_2(\mathbb{C}(t))$.*

Remark 2.5.1. Only part (1) of Theorem 7.1.2 depends on Conjecture 2.5.1. If we assume that the simple subgroup G does not contain any loxodromic element, Part (2) and (3) can be proven without assuming Conjecture 2.5.1 (Lemma 7.3.1 and 7.3.2).

As for the case (3) of Theorem 2.5.2, one observes that a simple subgroup of $\mathrm{PGL}_2(\mathbb{C}) \ltimes \mathrm{PGL}_2(\mathbb{C}(t))$ is abstractly isomorphic to a simple subgroup of $\mathrm{PGL}_2(\mathbb{C})$. However, we are not able to give a description of the conjugacy classes of simple subgroups of $\mathrm{PGL}_2(\mathbb{C}) \ltimes \mathrm{PGL}_2(\mathbb{C}(t))$.

From Theorem 2.5.2 one can deduce the following observation by looking at the classification of maximal algebraic subgroups of $\mathrm{Cr}_2(\mathbb{C})$ (see Theorem 6.2.1):

Corollary 2.5.3. *Assume that Conjecture 2.5.1 holds.*

A simple group G can be embedded into $\mathrm{Cr}_2(\mathbb{C})$ if and only if G is isomorphic to a subgroup of $\mathrm{PGL}_3(\mathbb{C})$.

Corollary 2.5.3 naturally leads to the following question, which is, to our knowledge, an open problem:

Question 2.5.1. What are the simple subgroups of $\mathrm{PGL}_3(\mathbb{C})$ and $\mathrm{PGL}_2(\mathbb{C})$?

Obvious classes of simple subgroups of $\mathrm{PGL}_2(\mathbb{C})$ are subgroups of the form $\mathrm{PSL}_2(\mathbf{k})$, where $\mathbf{k} \subset \mathbb{C}$ is a subfield, or finite simple subgroups. It is unclear, whether there exist other examples:

Question 2.5.2. What are the simple subgroups of $\mathrm{PSL}_2(\mathbb{Q})$? Does $\mathrm{PSL}_2(\mathbb{Q})$ contain a proper infinite simple subgroup?

2.5.1 Finitely generated simple subgroups

A group G satisfies the *property of Malcev*, if every finitely generated subgroup $\Gamma \subset G$ is *residually finite*, i.e. for every element $g \in \Gamma$ there exists a finite group H and a homomorphism $\varphi \colon \Gamma \to H$ such that g is not contained in the kernel of φ. Malcev showed that linear groups satisfy this property ([Mal40a]). Other groups that fulfill the property of Malcev include automorphism groups of algebraic varieties over fields of characteristic zero. In [Can11a], Cantat asked whether the plane Cremona group has the property of Malcev, a question that is still open.

Note that finitely generated simple subgroups of groups with the property of Malcev are always finite. We will prove the following theorem for the plane Cremona group:

Theorem 2.5.4. *Every finitely generated simple subgroup of $\mathrm{Cr}_2(\mathbb{C})$ is finite.*

Note that in order to prove Theorem 2.5.4 we do not need Conjecture 2.5.1.

From the classification of finite subgroups of $\mathrm{Cr}_2(\mathbb{C})$ (see [DI09]) we obtain:

Corollary 2.5.5. *A finitely generated simple subgroup of $\mathrm{Cr}_2(\mathbb{C})$ is isomorphic to*

$$\mathbb{Z}/p\mathbb{Z}, \text{ for some prime } p, \quad \mathcal{A}_5, \quad \mathcal{A}_6, \quad \mathrm{PSL}_2(7).$$

The conjugacy classes of these finite groups are classified in [DI09].

Chapter 3

On homomorphisms between Cremona groups

3.1 Introduction and statement of the results

3.1.1 Cremona groups

Let M be a complex algebraic variety and $\mathrm{Bir}(M)$ the group of birational transformations of M. Denote by $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$ the complex projective space of dimension n. The group

$$\mathrm{Cr}_n := \mathrm{Bir}(\mathbb{P}^n)$$

is called the *Cremona group*. In this paper we are interested in group homomorphisms from Cr_n to $\mathrm{Bir}(M)$. In particular, we will study an embedding of Cr_2 into Cr_5 that was described by Gizatullin [Giz99] and consider the case, where $\dim(M) = n + 1$.

A birational transformation $A\colon M \dashrightarrow N$ between varieties M and N induces an isomorphism $\mathrm{Bir}(M) \to \mathrm{Bir}(N)$ by conjugating elements of $\mathrm{Bir}(M)$ with A. Two homomorphisms $\Phi\colon \mathrm{Bir}(M) \to \mathrm{Bir}(N_1)$ and $\Psi\colon \mathrm{Bir}(M) \to \mathrm{Bir}(N_2)$ are called *conjugate* if there exists a birational transformation $A\colon N_1 \dashrightarrow N_2$ such that $\Psi(g) = A \circ \Phi(g) \circ A^{-1}$ for all $g \in \mathrm{Bir}(M)$.

Example 3.1.1. Assume that a variety M is birationally equivalent to $\mathbb{P}^n \times N$ for some variety N. The standard action on the first factor yields an injective homomorphism of Cr_n into $\mathrm{Bir}(\mathbb{P}^n \times N)$ and therefore also into $\mathrm{Bir}(M)$. We call embeddings of this type *standard embeddings*. In particular, we obtain in that way for all nonnegative integers m a standard embedding $\mathrm{Cr}_n \to \mathrm{Bir}(\mathbb{P}^n \times \mathbb{P}^m)$.

Example 3.1.2. A variety M is called *stably rational* if there exists a n such that $M \times \mathbb{P}^n$ is rational. There exist varieties of dimension larger than or equal to 3 that are stably rational but not rational (see [BCTSSD85]). We will see that two standard embeddings $f_1\colon \mathrm{Cr}_n \to \mathrm{Bir}(\mathbb{P}^n \times N)$ and $f_2\colon \mathrm{Cr}_n \to \mathrm{Bir}(\mathbb{P}^n \times M)$ are conjugate if and only if N and M are birationally equivalent (Lemma 3.3.3). So

every class of birationally equivalent stably rational varieties of dimension k defines a different conjugacy class of embeddings $\mathrm{Cr}_n \to \mathrm{Bir}(\mathbb{P}^m)$ for $m = n + k$.

If we fix homogeneous coordinates $[x_0 : \cdots : x_n]$ of $\mathbb{P}^n$, every element $f \in \mathrm{Cr}_n$ can be described by homogeneous polynomials of the same degree $f_0, \ldots, f_n \in \mathbb{C}[x_0, \ldots, x_n]$ without non-constant common factor, such that

$$f([x_0 : \cdots : x_n]) = [f_0 : \cdots : f_n].$$

The *degree* of f is the degree of the f_i.

With respect to affine coordinates $[1 : X_1 : \cdots : X_n] = (X_1, \ldots, X_n)$, we have

$$f(X_1, \ldots, X_n) = (F_1, \ldots, F_n),$$

where $F_i(X_1, \ldots, X_n) = f_i(1, X_1, \ldots, X_n)/f_0(1, X_1, \ldots, X_n)$ is an element of the field $\mathbb{C}(X_1, \ldots, X_n)$. The subgroup of Cr_n consisting of elements F such that all the F_i are polynomials as well as all the entries of F^{-1}, is exactly $\mathrm{Aut}(\mathbb{A}^n)$, the group of polynomial automorphisms of the affine space $\mathbb{A}^n$. When we look at $\mathrm{Aut}(\mathbb{A}^n)$ as a subgroup of Cr_n, we will always consider the embedding given by the affine coordinates $x_0 \neq 0$.

An important subgroup of Cr_n is the automorphism group

$$\mathrm{Aut}(\mathbb{P}^n) \simeq \mathrm{PGL}_{n+1}(\mathbb{C}).$$

The n-dimensional subgroup of $\mathrm{Aut}(\mathbb{P}^n)$ consisting of diagonal automorphisms will be denoted by D_n. It is the *torus of highest rank* of Cr_n in the following sense: Let D be an algebraic subgroup of Cr_n isomorphic to $(\mathbb{C}^*)^k$, then $k \leq n$ and if $k = n$ then D is conjugate to D_n ([BB66]).

Let $A = (a_{ij}) \in M_n(\mathbb{Z})$ be a matrix of integers. The matrix A determines a rational self map of the affine space

$$f_A = (x_1^{a_{11}} x_2^{a_{12}} \cdots x_n^{a_{1n}}, x_1^{a_{21}} x_2^{a_{22}} \cdots x_n^{a_{2n}}, \ldots, x_1^{a_{n1}} x_2^{a_{n2}} \cdots x_n^{a_{nn}}).$$

We have $f_A \circ f_B = f_{AB}$ for $A, B \in M_n(\mathbb{Z})$. One observes that f_A is a birational transformation if and only if $A \in \mathrm{GL}_n(\mathbb{Z})$. This yields an injective homomorphism $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{Cr}_n$ whose image we call the *Weyl group* and denote it by W_n. This terminology is justified by the fact that the normalizer of D_n in Cr_n is the semidirect product $\mathrm{Norm}_{\mathrm{Cr}_n}(D_n) = D_n \rtimes \mathrm{W}_n$. Note that $D_n \rtimes \mathrm{W}_n$ is the automorphism group of $(\mathbb{C}^*)^n$. Sometimes, W_n is also called the *group of monomial transformations*.

The well known theorem of Noether and Castelnuovo (see for example [AC02]) states that over an algebraically closed field k the Cremona group in two variables is generated by $\mathrm{PGL}_3(k)$ and the standard quadratic involution

$$\sigma := [x_1 x_2 : x_0 x_2 : x_0 x_1] \in \mathrm{W}_2.$$

Results of Hudson and Pan ([Hud27], [Pan99]) show that for $n \geq 3$ the Cremona group Cr_n is not generated by $\mathrm{PGL}_{n+1}(\mathbb{C})$ and W_n. Let

$$\mathrm{H}_n := \langle \mathrm{PGL}_{n+1}(\mathbb{C}), \mathrm{W}_n \rangle.$$

Blanc and Hedén studied the subgroup G_n of Cr_n generated by $\mathrm{PGL}_{n+1}(\mathbb{C})$ and the element $\sigma_n := [x_0^{-1} : \cdots : x_n^{-1}]$ ([BH14]). In particular, they show that G_n is a finite index subgroup of H_n and that it is strictly contained in H_n if and only if n is odd. Further results about the group structure of G_n can be found in [Dés14].

3.1.2 The case $\dim(M) \leq n$

Let M be a complex projective variety of dimension n and $\rho\colon \mathrm{PGL}_{r+1}(\mathbb{C}) \to \mathrm{Bir}(M)$ an embedding. Then $n \geq r$ and if $n = r$ it follows that M is rational and that up to a field automorphism, ρ is the standard embedding (see [Can14] and [Dés06b]). This implies in particular that there are no embeddings of Cr_n into $\mathrm{Bir}(M)$ if $\dim(M) < n$. In the Appendix we recall these results and show that the restriction of an automorphism of Cr_n to the subgroup H_n is inner up to a field automorphism.

3.1.3 Algebraic homomorphisms

We call a group homomorphism $\Psi\colon \mathrm{Cr}_n \to \mathrm{Bir}(M)$ *algebraic* if its restriction to $\mathrm{PGL}_{n+1}(\mathbb{C})$ is an algebraic morphism. The algebraic structure of $\mathrm{Bir}(M)$ and some properties of algebraic homomorphisms will be discussed in Section 3.2. Recall that an element $f \in \mathrm{Cr}_n$ is called *algebraic*, if the sequence $\{\deg(f^n)\}_{n \in \mathbb{N}}$ is bounded.

Definition 3.1.1. Let M be a variety and $\varphi_M\colon \mathrm{Cr}_n \to \mathrm{Bir}(M)$ a non-trivial algebraic group homomorphism. We say that φ_M is *reducible* if there exists a variety N such that $0 < \dim(N) < \dim(M)$ and an algebraic homomorphism $\varphi_N\colon \mathrm{Cr}_n \to \mathrm{Bir}(N)$ together with a dominant rational map $\pi\colon M \dashrightarrow N$ that is Cr_n-equivariant with respect to the rational actions induced by φ_M and φ_N respectively, i.e. $\pi \circ \varphi_M(g) = \varphi_N(g) \circ \pi$ for all $g \in \mathrm{Cr}_n$.

Remark 3.1.1. In [Zha10], Zhang uses the terminology *primitive action* for irreducible actions in the sense of Definition 3.1.1; in [Can03], Cantat says that an action *admits a non-trivial factor* if it is reducible.

Note that if we look at the induced action of Cr_n on the function field $\mathbb{C}(M)$ of M, reducibility is equivalent to the existence of a non-trivial Cr_n-invariant function field $\mathbb{C}(N) \subset \mathbb{C}(M)$.

3.1.4 An example by Gizatullin

In [Giz99], Gizatullin looks at the following question: Consider a linear representation $\psi\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_{n+1}(\mathbb{C})$. Does ψ extend to a homomorphism $\Psi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_n$? He shows that the linear representations given by the action of $\mathrm{PGL}_3(\mathbb{C})$ on conics, cubics and quartics can be extended to homomorphisms from Cr_2 to Cr_5, Cr_9 and Cr_{14}, respectively. These homomorphisms are related to the rational action of Cr_2 on moduli spaces of certain vector bundles on $\mathbb{P}^2$ that were discovered by Artamkin ([Art90]).

In Section 3.3 we study in detail some geometrical properties of the homomorphism

$$\Phi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$$

that was described by Gizatullin; by construction, the restriction of Φ to $\mathrm{PGL}_3(\mathbb{C})$ yields the linear representation $\varphi\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_6(\mathbb{C})$ given by the action of $\mathrm{PGL}_3(\mathbb{C})$ on plane conics. Among other things, we prove the following:

Theorem 3.1.3. *Let $\Phi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$ be the Gizatullin homomorphism. Then the following is true:*

(1) *The group homomorphism Φ is injective and irreducible.*

(2) *The rational action of Cr_2 on $\mathbb{P}^5$ that is induced by Φ preserves the Veronese surface V and its secant variety $S \subset \mathbb{P}^5$ and induces rational actions of Cr_2 on V and S.*

(3) *The Veronese embedding $v\colon \mathbb{P}^2 \to \mathbb{P}^5$ is Cr_2-equivariant with respect to the standard rational action on $\mathbb{P}^2$.*

(4) *The dominant secant map $s\colon \mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow S \subset \mathbb{P}^5$ (see Section 3.3.4) is Cr_2-equivariant with respect to the diagonal action of Cr_2 on $\mathbb{P}^2 \times \mathbb{P}^2$.*

(5) *The rational action of Cr_2 on $\mathbb{P}^5$ preserves a volume form on $\mathbb{P}^5$ with poles of order three along the secant variety S.*

(6) *The group homomorphism Φ sends the group of polynomial automomorphisms $\mathrm{Aut}(\mathbb{A}^2) \subset \mathrm{Cr}_2$ to $\mathrm{Aut}(\mathbb{A}^5)$.*

Note that the injectivity of Φ follows from (3); in Section 3.3.8 irreducibility is proved. Part (2) - (4) of Theorem 3.1.3 will be proved in Section 3.3.4, part (5) in Section 3.3.6 and part (6) in Section 3.3.7

The representation $\varphi^\vee$ of $\mathrm{PGL}_3(\mathbb{C})$ into $\mathrm{PGL}_6(\mathbb{C})$ given by $\psi \circ \alpha$, where α is the algebraic homomorphism $g \mapsto {}^t g^{-1}$, is conjugate in $\mathrm{Cr}_5(\mathbb{C})$ to the representation φ. This conjugation yields the embedding $\Phi^\vee\colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Cr}_5(\mathbb{C})$, whose image preserves the secant variety S as well and induces a rational action on it. As the secant variety S is rational, Φ and $\Phi^\vee$ induce two non-standard homomorphisms of $\mathrm{Cr}_2(\mathbb{C})$ into Cr_4, which we denote by Ψ_1 and Ψ_2 respectively. In Section 3.3.5 we prove the following:

Proposition 3.1.4. *The two homomorphisms $\Psi_1, \Psi_2\colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Cr}_4(\mathbb{C})$ are not conjugate in Cr_4; moreover they are irreducible and therefore not conjugate to the standard embedding.*

Remark 3.1.2. The homomorphism Ψ_1 is injective, since it restricts to the standard action on the Veronese surface. However, it seems to be unclear, whether Ψ_2 is injective as well. Since the restriction of Ψ_2 to $\mathrm{PGL}_3(\mathbb{C})$ is injective, it seems unlikely that Ψ_2 is not injective. In fact, it seems that one could use results from [BZ15] to show that non-trivial algebraic homomorphisms are always injective. However, I haven't proved it yet.

Since Φ is algebraic, the images of algebraic elements under Φ are algebraic again (see Proposition 3.2.6). Calculation of the degrees of some examples suggests that Φ might even preserve the degrees of all elements in Cr_2. However, we were only able to prove the following (Section 3.3.7):

Theorem 3.1.5. *Let $\Phi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$ be the Gizatullin-embedding. Then*

(1) *for all elements $f \in \mathrm{Cr}_2$ we have $\deg(f) \leq \deg(\Phi(f))$,*

(2) *for all $g \in \mathrm{Aut}(\mathbb{A}^2) \subset \mathrm{Cr}_2$ we have $\deg(g) = \deg(\Phi(g))$.*

The image of the Weyl group W_2 under Φ is not contained in the Weyl group W_5. More generally, we will show in the appendix that there exists no algebraic homomorphism from $\mathrm{Cr}_2(\mathbb{C})$ to $\mathrm{Cr}_5(\mathbb{C})$ that preserves automorphisms, diagonal automorphisms and the Weyl group:

Theorem 3.1.6. *There is no non-trivial algebraic homomorphism $\Psi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$ such that:*

(1) $\Psi(\mathrm{PGL}_3(\mathbb{C})) \subset \mathrm{PGL}_6(\mathbb{C})$

(2) $\Psi(D_2) \subset D_5$ *and*

(3) $\Psi(\mathrm{W}_2) \subset \mathrm{W}_5$.

3.1.5 Algebraic embeddings in codimension 1

In Section 3.4 and Section 3.5 we look at algebraic homomorphisms $\mathrm{Cr}_n \to \mathrm{Bir}(M)$ in the case where M is a smooth projective variety of dimension $n+1$ for $n \geq 2$.

Example 3.1.7. For all curves C of genus ≥ 1, the variety $\mathbb{P}^n \times C$ is not rational and there exists the standard embedding $\Psi_C\colon \mathrm{Cr}_n \to \mathrm{Bir}(\mathbb{P}^n \times C)$.

Example 3.1.8. Cr_n acts rationally on the total space of the canonical bundle of $\mathbb{P}^n$

$$K_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-(n+1)) \simeq \bigwedge^n (\mathrm{T}\,\mathbb{P}^n)^\vee$$

by $f(p, \omega) = (f(p), \omega \circ (df_p)^{-1})$, where $p \in \mathbb{P}^n$ and $\omega \in \bigwedge^n (\mathrm{T}_p\,\mathbb{P}^n)^\vee$. More generally, we obtain a rational action of Cr_n on the total space of the bundle $K_{\mathbb{P}^n}^{\otimes l} \simeq \mathcal{O}_{\mathbb{P}^n}(-(n+1)l)$ and on its projective completion

$$F_l := \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-l(n+1))$$

for all $l \in \mathbb{Z}_{\geq 0}$. This yields a countable family of injective homomorphisms

$$\Psi_l\colon \mathrm{Cr}_n \to \mathrm{Bir}(F_l).$$

Note that the restriction of this rational action to $\mathrm{PGL}_n(\mathbb{C})$ is regular, hence these embeddings are algebraic.

We can choose affine coordinates $(x_1, \ldots, x_n, x_{n+1})$ of F_l such that Ψ_l is given by

$$\Psi_l(f)(x_1, \ldots, x_n, x_{n+1}) = (f(x_1, \ldots, x_n), J(f(x_1, \ldots, x_n))^{-l} x_{n+1}),$$

where, $J(f(x_1, \ldots, x_n))$ denotes the determinant of the Jacobian of f at the point $(x_1, \ldots, x_n)$. Observe that Ψ_0 is conjugate to the standard embedding.

Example 3.1.9. Let $\mathbb{P}(\mathrm{T}\,\mathbb{P}^2)$ be the total space of the fiberwise projectivisation of the tangent bundle over $\mathbb{P}^2$. Then $\mathbb{P}(\mathrm{T}\,\mathbb{P}^2)$ is rational and there is an injective group homomorphism

$$\Psi_B\colon \mathrm{Cr}_2 \to \mathrm{Bir}(\mathbb{P}(\mathrm{T}\,\mathbb{P}^2))$$

defined by $\Psi_B(f)(p, v) := (f(p), \mathbb{P}(df_p)(v))$. Here, $\mathbb{P}(df_p)\colon \mathbb{P}\,\mathrm{T}_p \to \mathbb{P}\,\mathrm{T}_{f(p)}$ defines the projectivisation of the differential df_p of f at the point $p \in \mathbb{P}^2$.

Example 3.1.10. The Grassmannian of lines in the projective 3-space $\mathbb{G}(1,3)$ is a rational variety of dimension 4 with a transitive algebraic $\mathrm{PGL}_4(\mathbb{C})$-action. This action induces an algebraic embedding of $\mathrm{PGL}_4(\mathbb{C})$ into Cr_4. In Proposition 3.5.2 we will show that the image of this embedding does not lie in any subgroup isomorphic to Cr_3. *So no group action of $\mathrm{PGL}_4(\mathbb{C})$ on $\mathbb{G}(1,3)$ by automorphisms can be extended to a rational action of Cr_3.*

The classification of $\mathrm{PGL}_{n+1}(\mathbb{C})$-actions on smooth projective varieties of dimension $n+1$ is well known to the experts; in Section 3.4 we study their conjugacy classes. We will see that Examples 3.1.7 to 3.1.10 describe up to birational conjugation and up to algebraic automorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$ all possible $\mathrm{PGL}_{n+1}(\mathbb{C})$-actions on smooth projective varieties of dimension $n+1$ and that these actions are not birationally conjugate to each other. This yields a classification of algebraic homomorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$ to $\mathrm{Bir}(M)$ up to birational conjugacy, for smooth projective M of dimension $n+1$. We will study in Section 3.5 how these actions extend to rational actions of Cr_n on M. Denote by $\alpha\colon \mathrm{PGL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$ the algebraic automorphism given by $g \mapsto {}^t(g^{-1})$.

Theorem 3.1.11. *Let $n \geq 2$, let M be a complex projective variety of dimension $n+1$ and let $\varphi\colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{Bir}(M)$ be a non-trivial algebraic homomorphism. Then*

(1) *φ is conjugate, up to the automorphism α, to exactly one of the embeddings described in Example 3.1.7 to 3.1.10.*

(2) *If $n = 3$ and φ is conjugate to the action described in Example 3.1.10, then neither φ nor $\varphi \circ \alpha$ can be extended to a homomorphism of H_3 to $\mathrm{Bir}(M)$.*

(3) *If φ is conjugate to one of the embeddings described in Example 3.1.7 to 3.1.9 then exactly one of the embeddings φ or $\varphi \circ \alpha$ extends to a homomorphism of Cr_n to $\mathrm{Bir}(M)$.*

(4) *φ extends to H_n if and only if it extends to Cr_n; moreover, in this case the extension to H_n is unique.*

Theorem 3.1.11 classifies all group homomorphisms $\Psi\colon \mathrm{H}_n \to \mathrm{Bir}(M)$ for projective varieties M of dimension $n+1$ such that the restriction to $\mathrm{PGL}_{n+1}(\mathbb{C})$ is an algebraic morphism. By the theorem of Noether and Castelnuovo, we obtain in particular a full classification of all algebraic homomorphisms from Cr_2 to $\mathrm{Bir}(M)$ for projective varieties M of dimension 3:

Corollary 3.1.12. *Let M be a projective variety of dimension 3 and $\Psi\colon \mathrm{Cr}_2 \to \mathrm{Bir}(M)$ a non-trivial algebraic group homomorphism. Then Ψ is conjugate to exactly one of the homomorphisms described in Example 3.1.7 to 3.1.9.*

The following observations are now immediate:

Corollary 3.1.13. *Let M be a projective variety of dimension 3 and $\Psi\colon \mathrm{Cr}_2 \to \mathrm{Bir}(M)$ a non-trivial algebraic homomorphism. Then*

(1) *Ψ is injective.*

(2) *There exists a Cr_2-equivariant rational map $f\colon M \dashrightarrow \mathbb{P}^2$ with respect to the rational action induced by Ψ and the standard action respectively. In particular, all algebraic homomorphisms from Cr_2 to $\mathrm{Bir}(M)$ are reducible.*

(3) *There exists an integer $C_\Psi \in \mathbb{Z}$ such that*

$$1/C_\Psi \deg(f) \leq \deg(\Psi(f)) \leq C_\Psi \deg(f).$$

Note that Part (3) of Corollary 3.1.13 resembles in some way Theorem 3.1.5 and leads to the following question:

Question 3.1.1. Let $\Phi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_n$ be an algebraic embedding. Does there always exist a constant C depending only on Φ such that $1/C \deg(f) \leq \deg(\Phi(f)) \leq C \deg(f)$?

3.1.6 Acknowledgements

3.2 Algebraic homomorphisms

In this section we recall some results on the algebraic structure of $\mathrm{Bir}(M)$ and of some of its subgroups and we discuss our notion of algebraic homomorphisms.

3.2.1 The Zariski topology

We can equip $\mathrm{Bir}(M)$ with the so-called Zariski topology. Let A be an algebraic variety and

$$f\colon A \times M \dashrightarrow A \times M$$

an A-birational map (i.e. a map of the form $(a,x) \dashrightarrow (a, f(a,x))$ that induces an isomorphism between open subsets U and V of $A \times M$ such that the projections from U and from V to A are both surjective). For each $a \in A$ we obtain therefore an element of $\mathrm{Bir}(M)$ defined by $x \mapsto p_2(f(a,x))$, where p_2 is the second projection. Such a map $A \to \mathrm{Bir}(M)$ is called a *morphism* or *family of birational transformations parametrized by A*.

Definition 3.2.1. The Zariski topology on $\mathrm{Bir}(M)$ is the finest topology such that all morphisms $f\colon A \to \mathrm{Bir}(M)$ for all algebraic varieties A are continuous (with respect to the Zariski topology on A).

The map $\iota\colon \mathrm{Bir}(M) \to \mathrm{Bir}(M)$, $x \mapsto x^{-1}$ is continuous as well as the maps $x \mapsto g \circ x$ and $x \mapsto x \circ g$ for any $g \in \mathrm{Bir}(M)$. This follows from the fact that the inverse of an A-birational map as above is again an A-birational map as is the right/left-composition with an element of $\mathrm{Bir}(M)$. The Zariski topology was introduced in [Dem70] and [Ser08] and studied in [BF13].

3.2.2 Algebraic subgroups

An algebraic subgroup of $\mathrm{Bir}(M)$ is the image of an algebraic group G by a morphism $G \to \mathrm{Bir}(M)$ that is also an injective group homomorphism. It can be shown that algebraic groups are closed in the Zariski topology and of bounded degree in the case of $\mathrm{Bir}(M) = \mathrm{Cr}_n$. Conversely, closed subgroups of bounded degree in Cr_n are always algebraic subgroups with a unique algebraic group structure that is compatible with the Zariski topology (see [BF13]). We will also use the fact that all algebraic subgroups of Cr_n are linear ([BF13]).

Let N be a smooth projective variety that is birationally equivalent to M. Let G be an algebraic group acting regularly and faithfully on N. This yields a morphism $G \to \mathrm{Bir}(M)$, so G is an algebraic subgroup of $\mathrm{Bir}(M)$. On the other hand, a theorem by Weil, Rosenlicht and Sumihiro states that in characteristic zero all algebraic subgroups of $\mathrm{Bir}(M)$ have this form.

Theorem 3.2.1 ([Wei55], [Ros56, Theorem 1]). *Let M be an algebraic variety over a field k and let $G \subset \mathrm{Bir}(M)$ be an algebraic subgroup. Then there exists an algebraic variety X and a birational map $f\colon M \dashrightarrow X$ that conjugates G to a subgroup of $\mathrm{Aut}(X)$ such that the induced action on X is algebraic.*

However, Theorem 3.2.1, while giving no restrictions on the base-field, does not say anything about the structure of the variety X. The following two theorems show that under additional assumptions Theorem 3.2.1 can be improved:

Theorem 3.2.2 ([Sum74, Theorem 3]). *Let X be a normal variety over an algebraically closed field on which a linear group G acts algebraically. Then there exists a G-equivariant completion $\overline{X}$ of X.*

Theorem 3.2.3 ([Kol07]). *Let X be a projective variety over a field k of characteirstic 0 on which an algebraic group G acts algebraically. There exists a G-equivariant resolution of singularities $\pi\colon \tilde{X} \to X$, i.e. $\tilde{X}$ is a smooth projective G-variety and π is a G-equivariant birational morphism.*

We can now put everything together. Let M be a complex variety and $G \subset \mathrm{Bir}(M)$ a linear algebraic subgroup. By Theorem 3.2.1, there exists a variety X and a birational map $f\colon M \dashrightarrow X$ that conjugates G to a subgroup of $\mathrm{Aut}(X)$ such that the induced action on X is algebraic. After normalizing G-equivariantly, we may assume that X is normal. By Theorem 3.2.2, there exists a G-equivariant completion $\overline{X}$ of X. By the G-equivariant Chow Lemma ([Sum74, Theorem 2]), there exists a projective G-variety X' and a G-equivariant birational morphism $\pi'\colon X' \to X$. Finally, by Theorem 3.2.3, there exists a G-equivariant resolution of singularities of $\pi\colon N \to X'$. Hence we obtain the following version of Weil's theorem:

Theorem 3.2.4 (Weil, Rosenlicht, Sumihiro)**.** *Let M be a complex variety and $G \subset \mathrm{Bir}(M)$ a linear algebraic subgroup. Then there exists a smooth projective variety N and a birational map $f \colon M \to N$ that conjugates G to a subgroup of $\mathrm{Aut}(N)$ such that the induced action on N is algebraic.*

It can be shown (see for example, [BF13]) that the sets $(\mathrm{Cr}_n)_{\leq d} \subset \mathrm{Cr}_n$ consisting of all birational transformations of degree $\leq d$ are closed with respect to the Zariski topology. So the closure of a subgroup of bounded degree in Cr_n is an algebraic subgroup. Since all algebraic subgroups of Cr_n are linear ([BF13]), it can therefore be regularized in the sense of the above theorem. We obtain:

Corollary 3.2.5. *Let $G \subset \mathrm{Cr}_n$ be a subgroup that is contained in some $(\mathrm{Cr}_n)_{\leq d}$, then there exists a smooth projective variety N and a birational transformation $f \colon \mathbb{P}^n \dashrightarrow N$ such that $fGf^{-1} \subset \mathrm{Aut}(N)$.*

The maximal algebraic subgroups of Cr_2 have been classified together with the rational surfaces on which they act as automorphisms ([Enr93], [Bla09]). In dimension 3, a classification for maximal connected algebraic subgroups exists: [Ume82b], [Ume85], [Ume82a].

3.2.3 Algebraic homomorphisms and continuous homomorphisms

We defined a group homomorphism from Cr_n to $\mathrm{Bir}(M)$ to be algebraic if its restriction to $\mathrm{PGL}_{n+1}(\mathbb{C})$ is a morphism. Note that this is a priori a weaker notion than being continuous with respect to the Zariski topology. It is not clear, whether algebraic homomorphisms are always continuous. However, for dimension 2 we have the following partial result, which will proved in Section 3.2.5:

Proposition 3.2.6. *Let $\Phi \colon \mathrm{Cr}_2 \to \mathrm{Bir}(M)$ be a homomorphism of groups. The following are equivalent:*

(1) *Φ is algebraic.*

(2) *The restriction of Φ to any algebraic subgroup of Cr_2 is algebraic.*

(3) *The restriction of Φ to one positive dimensional algebraic subgroup of Cr_2 is algebraic.*

3.2.4 One-parameter subgroups

A one-parameter subgroup is a connected algebraic group of dimension 1. It is well known (see for example [Hum75]) that all linear one-parameter subgroups are isomorphic to either $\mathbb{C}$ or $\mathbb{C}^*$. The group $\mathbb{C}$ is unipotent, the group $\mathbb{C}^*$ semi-simple.

Proposition 3.2.7 shows that, up to conjugation by birational maps, there exists only one birational action of $\mathbb{C}$ and only one of $\mathbb{C}^*$ on $\mathbb{P}^2$:

Proposition 3.2.7. *In Cr_2 all one-parameter subgroups isomorphic to $\mathbb{C}$ are conjugate and all one-parameter subgroups isomorphic to $\mathbb{C}^*$ are conjugate.*

The first part of Proposition 3.2.7 follows from results in [BD15] and [Bla06] (see also [Bru97]). The second part is a special case of Theorem 3.2.8.

Theorem 3.2.8 ([BB66], [Pop13])**.** *In* Cr_n *all tori of dimension* $\geq n - 2$ *are conjugate to a subtorus of* D_n. *Moreover, two subtori of* D_n *are conjugate in* Cr_n *to each other if and only if they are isomorphic.*

Let G be a connected linear algebraic group and $\{U_i\}_{i \in I}$ the set of one-parameter subgroups of G. Then the subgroup $H \subset G$ generated by all the U_i is closed and connected and there exist one-parameter subgroups $U_1, U_2 \ldots, U_n$ such that $U_1 \cdot U_2 \cdots U_n = H$ ([Hum75, Proposition 7.5]). On the other hand, if $\mathfrak{g}$ is the Lie algebra of G, then the exponential map $\exp\colon \mathfrak{g} \to G$ induces a diffeomorphism from an analytically open set of $\mathfrak{g}$ to an analytically open neigborhood V of the identity in G. For all elements $A \in \mathfrak{g}$, the closure of the abelian subgroup $\{\exp(tA) \mid t \in \mathbb{C}\}$ in G is abelian and connected and therefore of the form $(\mathbb{C}^*)^m \times \mathbb{C}^n$ for some integers m and n (see for example [Kra14, Chapter 3.8]), hence it is contained in H. We obtain that V is contained in G and hence that H is open in the analytic topology. This yields $H = G$ and thus

$$U_1 \cdot U_2 \cdots U_n = G.$$

The following Lemma is a classical result (see for example [Sta13]):

Lemma 3.2.9. *Let* G *be a linear algebraic group and* $U_1, \ldots, U_n$ *be algebraic subgroups such that* $U_1 U_2 \cdots U_n = G$. *Let* H *be a linear algebraic group and* $\varphi\colon G \to H$ *a homomorphism of abstract groups such that* $\varphi|_{U_i}$ *is a homomorphism of algebraic groups for all* i. *Then* φ *is a homomorphism of algebraic groups.*

3.2.5　Algebraic and abstract group homomorphisms

Let G and H be algebraic groups that are isomorphic as abstract groups. The question whether G and H are also isomorphic as algebraic groups has been treated in detail in [BT73] (see also [Die71] and [Dés06a]). We will use the following result:

Proposition 3.2.10. *Let* G *be an algebraic group that is isomorphic to* $\mathrm{PGL}_n(\mathbb{C})$ *as an abstract group. Then* G *is isomorphic to* $\mathrm{PGL}_n(\mathbb{C})$ *as an algebraic group. Moreover, for every abstract isomorphism*

$$\rho\colon \mathrm{PGL}_n(\mathbb{C}) \to G$$

there exists an automorphism of fields $\tau\colon \mathbb{C} \to \mathbb{C}$ *such that* $\rho \circ \tau$ *is an algebraic isomorphism.*

Remark 3.2.1. It is well known that the automorphisms of $\mathrm{PGL}_n(\mathbb{C})$ as an algebraic group are compositions of inner automorphisms and the automorphism

$$\alpha\colon \mathrm{PGL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C}), \ \ g \mapsto {}^t g^{-1}.$$

Proof of Proposition 3.2.6. We first show how (1) implies (2). Let G be an algebraic subgroup of Cr_2. We can assume that G is connected. By the above remark,

there exist one parameter subgroups $U_1, \ldots, U_k \subset G$ such that $U_1 \cdots U_k = G$. Since, by Proposition 3.2.7, the group U_i is conjugate to a one parameter subgroup of $\mathrm{PGL}_3(\mathbb{C})$ for all i, we obtain that the restriction of φ to any of the U_i is an algebraic homomorphism of groups and that $\varphi(G) \subset \mathrm{Cr}_n$ is of bounded degree. Then $\overline{\varphi(G)} \subset \mathrm{Cr}_n$ is an algebraic group. We can now apply Lemma 3.2.9 and conclude that the restriction of φ to G is a homomorphism of algebraic groups.

Statement (3) follows immediately from statement (2), so it only remains to prove that (3) implies (1). Let $\varphi \colon \mathrm{Cr}_2 \to \mathrm{Bir}(M)$ be a homomorphism of abstract groups and let $G \subset \mathrm{Cr}_2$ be a positive dimensional algebraic subgroup such that the restriction of φ to G is a morphism. Since G is infinite, it contains a one parameter subgroup $U \subset G$.

Let $U_1, \ldots, U_n \subset \mathrm{PGL}_3(\mathbb{C})$ be unipotent one parameter subgroups such that $U_1 \cdots U_n = \mathrm{PGL}_3(\mathbb{C})$. If U is unipotent, all the subgroups U_i are conjugate to U. Hence the restriction of φ to U_i is a morphism for all i. The image $\varphi(\mathrm{PGL}_3(\mathbb{C})) \subset \mathrm{Cr}_n$ is of bounded degree, so $\overline{\varphi(\mathrm{PGL}_3(\mathbb{C}))} \subset \mathrm{Cr}_n$ is an algebraic group and with Lemma 3.2.9 it follows that the restriction of φ to $\mathrm{PGL}_3(\mathbb{C})$ is a morphism.

Denote by $D_1 \subset \mathrm{PGL}_3(\mathbb{C})$ the subgroup given by elements of the form $[cx_0 : x_1 : x_2]$, $c \in \mathbb{C}^*$ and by $T \subset \mathrm{PGL}_3(\mathbb{C})$ the subgroup of all elements of the form $[x_0 : x_1 + cx_0 : x_2]$, $c \in \mathbb{C}$; we have $D_1 \simeq \mathbb{C}^*$ and $T \simeq \mathbb{C}$. If U is semi-simple, it is, again by Proposition 3.2.7, conjugate to D_1, hence the restriction of φ to D_1 is a morphism well. Note that

$$T = \{[x_0 : x_1 + cx_0 : x_2] \mid c \in \mathbb{C}\} = \{dgd^{-1} \mid d \in D_1\} \cup \{\mathrm{id}\}$$

where $g = [x_0 : x_1 + x_0 : x_2]$. We obtain that $\varphi(T)$ is of bounded degree and contained in the algebraic group $\overline{\varphi(T)} \subset \mathrm{Cr}_n$. As $\varphi(T)$ consists of two $\varphi(D_1)$-orbits, it is constructible and therefore closed. We obtain that the images of all unipotent subgroups of Cr_2 under φ are algebraic subgroups. The map $\varphi(U_1) \times \cdots \times \varphi(U_n) \to \mathrm{Cr}_n$ is a morphism, so its image is a constructible set and therefore closed since it is a group. Hence $\varphi(\mathrm{PGL}_3(\mathbb{C})) = \varphi(U_1) \cdots \varphi(U_n)$ is an algebraic subgroup. By Proposition 3.2.10 it is isomorphic as an algebraic group to $\mathrm{PGL}_3(\mathbb{C})$ and there exists an automorphism of fields $\tau \colon \mathbb{C} \to \mathbb{C}$ such that $\varphi \circ \tau \colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_3(\mathbb{C})$ is an isomorphism of algebraic groups. But since the restriction of φ to T is already an algebraic homomorphism, it follows that τ is the identity. $\square$

Remark 3.2.2. Proposition 3.2.6 shows in particular that algebraic homomorphisms $\Phi \colon \mathrm{Cr}_2 \to \mathrm{Bir}(M)$ send algebraic elements to algebraic elements. This result follows also directly from the fact that a birational transformation $f \in \mathrm{Cr}_2$ of degree d can be written as the product of at most $4d$ linear maps and $4d$ times the standard quadratic involution σ (see for example [AC02]); we therefore obtain that the sequence $\{\deg(\Phi(f)^n)\}$ is bounded if $\{\deg(f^n)\}$ is bounded.

3.3 An example by Gizatullin

3.3.1 Projective representations of $\mathrm{PGL}_n(\mathbb{C})$

The results from representation theory of linear algebraic groups that we use in this section can be found, for example, in [FH91], [Pro07].

Proposition 3.3.1. *There is a bijection between homomorphisms of algebraic groups from $\mathrm{SL}_n(\mathbb{C})$ to $\mathrm{SL}_m(\mathbb{C})$ such that the image of the center is contained in the center and homomorphisms of algebraic groups from $\mathrm{PGL}_n(\mathbb{C})$ to $\mathrm{PGL}_m(\mathbb{C})$.*

From Proposition 3.3.1 and some elementary representation theory of $\mathrm{SL}_3(\mathbb{C})$ it follows that $n = 6$ is the smallest number such that there exist non-trivial and non-standard homomorphisms of algebraic groups from $\mathrm{PGL}_3(\mathbb{C})$ to $\mathrm{PGL}_n(\mathbb{C})$. In fact, up to automorphisms of $\mathrm{PGL}_3(\mathbb{C})$ there are exactly two non-trivial representations from $\mathrm{PGL}_3(\mathbb{C})$ to $\mathrm{PGL}_6(\mathbb{C})$.

The first one is reducible. Let $\psi' \colon \mathrm{GL}_3 \to \mathrm{GL}_6$ be the linear representation given by the diagonal action on $\mathbb{C}^3 \times \mathbb{C}^3$; we denote by $\psi \colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_6(\mathbb{C})$ its projectivisation.

The second one is given by the action of $\mathrm{PGL}_3(\mathbb{C})$ on the space of conics. The latter one can be parametrized by the space $\mathbb{P}\, M_3$ of symmetric 3×3-matrices up to scalar multiple and is isomorphic to $\mathbb{P}^5$. Let $g \in \mathrm{PGL}_3(\mathbb{C})$, we define $\varphi(g) \in \mathrm{PGL}_6(\mathbb{C})$ by $(a_{ij}) \mapsto g(a_{ij})({}^t g)$.

In this section we identify the space of conics with $\mathbb{P}^5$ in the following way:

$$(a_{ij}) \mapsto [a_{00} : a_{11} : a_{22} : a_{12} : a_{02} : a_{01}]$$

In other words, the conic C given by the zeroes of the equation

$$F = a_{00}X^2 + a_{11}Y^2 + a_{22}Z^2 + 2a_{12}YZ + 2a_{02}XZ + 2a_{01}XY$$

is identified with the point $[a_{00} : a_{11} : a_{22} : a_{12} : a_{02} : a_{01}] \in \mathbb{P}^5$.

Observe that with our definition, $\varphi(g)$ sends the conic C to the conic given by the zero set of the polynomial $F \circ ({}^t g)$.

Let

$$\alpha \colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_3(\mathbb{C})$$

be the algebraic automorphism $g \mapsto ({}^t g)^{-1}$. Then $\varphi(\alpha(g))$ maps the conic C to $g(C)$, which is the conic given by the zero set of the polynomial $F \circ g^{-1}$. Accordingly, $\varphi(\alpha(g)) \in \mathrm{PGL}_6(\mathbb{C})$ maps the matrix $(a_{ij}) \in M_3$ to $({}^t g)^{-1}(a_{ij})g^{-1}$.

The action of $\mathrm{PGL}_3(\mathbb{C})$ on $\mathbb{P}^5$ induced by φ has exactly three orbits that are characterized by the rank of the corresponding symmetric matrix in M_3. Geometrically they correspond to the sets of smooth conics, pairs of distinct lines and double lines. The set of double lines is a surface isomorphic to $\mathbb{P}^2$ and called the *Veronese surface*; we denote it by V. The set of singular conics S is the secant variety of V and has dimension 4.

To describe the $\mathrm{PGL}_3(\mathbb{C})$-orbits with respect to the action induced by ψ, consider a point $p = [x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \in \mathbb{P}^5$. Then p can either be mapped by an element of $\psi(\mathrm{PGL}_3(\mathbb{C}))$ to a point of the form $[a : 0 : 0 : b : 0 : 0]$, where $[a : b] \in \mathbb{P}^1$, or to the point $[1 : 0 : 0 : 0 : 0 : 1]$ and these points are all in different $\psi(\mathrm{PGL}_3(\mathbb{C}))$-orbits. The stabilizer of $[1 : 0 : 0 : 0 : 0 : 1]$ in $\psi(\mathrm{PGL}_3(\mathbb{C}))$ is the subgroup of matrices of the form

$$\begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix}, \text{ where } g \in \mathrm{PGL}_3(\mathbb{C}) \text{ has the form } \begin{bmatrix} 1 & a & 0 \\ 0 & b & 0 \\ 0 & c & 1 \end{bmatrix}.$$

Therefore, the orbit of $[1:0:0:0:0:1]$ under $\psi(\mathrm{PGL}_3(\mathbb{C}))$ has dimension 5. The orbit of a point of the form $[a:0:0:b:0:0]$, on the other hand, has dimension 2. So we have a family parametrized by $\mathbb{P}^1$ of orbits of dimension 2 and one orbit of dimension 5. In particular, there is no $\psi(\mathrm{PGL}_3(\mathbb{C}))$-invariant subset of dimension 4.

The following observation is easy but useful. We leave its proof to the reader.

Lemma 3.3.2. *Let X and Y be two projective varieties with biregular actions of a group G and let $f\colon X \dashrightarrow Y$ be a G-equivariant rational map. Then the indeterminacy locus $I_f \subset X$ and the exceptional locus $\mathrm{Exc}(f) \subset X$ are G-invariant sets.*

Note that Lemma 3.3.2 implies in particular that all equivariant rational maps between smooth projective varieties with respect to actions without orbits of codimension ≥ 2 are morphisms.

Lemma 3.3.3. *Let M and M' be irreducible complex projective varieties such that $M \times \mathbb{P}^n$ et $M' \times \mathbb{P}^n$ are birationally equivalent. Then the standard embeddings*

$$\Psi\colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}^n \times M) \ \text{and}\ \Psi'\colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}^n \times M')$$

are conjugate if and only if M and M' are birationally equivalent.

Proof. If M and M' are birationally equivalent it follows directly that Ψ and Ψ' are conjugate. On the other hand, assume that there exists a birational map $A\colon \mathbb{P}^n \times M \dashrightarrow \mathbb{P}^n \times M'$ that conjugates Ψ to Ψ', i.e. $A \circ \Psi(g) = \Psi'(g) \circ A$ for all $g \in \mathrm{PGL}_{n+1}(\mathbb{C})$. The images $\Psi(\mathrm{PGL}_{n+1}(\mathbb{C}))$ and $\Psi'(\mathrm{PGL}_{n+1}(\mathbb{C}))$ permute the fibers $\{p\} \times M$, $p \in \mathbb{P}^n$ and $\{p\} \times M'$, $p \in \mathbb{P}^n$ respectively. By Lemma 3.3.2, no fiber is fully contained in the exceptional locus of A.

The fiber

$$F := [1:1:\cdots:1] \times M \subset \mathbb{P}^n \times M$$

consists of all fixed points of the image of the subgroup of coordinate permutations $\Psi(\mathcal{S}_{n+1})$ and it is isomorphic to M. Correspondingly, the fiber

$$F' := [1:1:\cdots:1] \times M' \subset \mathbb{P}^n \times M'$$

consists of all fixed points of $\Psi'(\mathcal{S}_{n+1})$ and is isomorphic to M'. Hence the strict transform of F under A is F' and we obtain that M and M' are birationally equivalent. $\qquad \square$

Proposition 3.3.4. *Let $\varphi, \psi\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_6(\mathbb{C})$ be the homomorphisms defined in Section 3.3.1. The subgroups $\varphi(\mathrm{PGL}_3(\mathbb{C}))$ and $\psi(\mathrm{PGL}_3(\mathbb{C}))$ are not conjugate in Cr_5.*

Proof. Assume that there is an element $f \in \mathrm{Cr}_5$ conjugating $\varphi(\mathrm{PGL}_3(\mathbb{C}))$ to $\psi(\mathrm{PGL}_3(\mathbb{C}))$. Note that $\mathbb{P}^5$ has no $\psi(\mathrm{PGL}_3(\mathbb{C}))$-invariant subset of dimension 4. Hence, by Lemma 3.3.2, f must be a birational morphism and therefore an automorphism. But this isn't possible since the action of $\varphi(\mathrm{PGL}_3(\mathbb{C}))$ has an orbit of dimension 4 and the action of $\psi(\mathrm{PGL}_3(\mathbb{C}))$ does not. $\qquad \square$

3.3.2   A rational action on the space of plane conics

Our goal is to extend the group homomorphism $\varphi\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_6(\mathbb{C})$ to a group homomorphism

$$\Phi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5\,.$$

A first naive idea is to check whether the map $\Psi\colon \{\mathrm{PGL}_3(\mathbb{C}), \sigma\} \to \mathrm{Cr}_5$ defined by $\Psi(g) = \varphi(g)$ for $g \in \mathrm{PGL}_3(\mathbb{C})$ and $\Psi(\sigma) = [x_0^{-1} : x_2^{-1} : \cdots : x_5^{-1}]$ extends to a group homomorphism $\mathrm{Cr}_2 \to \mathrm{Cr}_5$. However, $\Psi(\sigma)$ and $\Psi(h)$ don't satisfy relation (3) of Lemma 3.5.12. Let $h = [Z - X : Z - Y : Z] \in \mathrm{Cr}_2$, then

$$([x_0^{-1} : x_1^{-1} : \cdots : x_5^{-1}] \circ \varphi(h))^3 \neq \mathrm{id}\,.$$

In [Giz99], Gizatullin constructs an extension $\Phi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$ of φ, defined by $\Phi|_{\mathrm{PGL}_3(\mathbb{C})} = \varphi$ and

$$\Phi(\sigma) = [x_1x_2 : x_0x_2 : x_0x_1 : x_3x_0 : x_4x_1 : x_5x_2].$$

He shows the following:

Proposition 3.3.5 ([Giz99])**.** *The map $\Phi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$ is a group homomorphism.*

3.3.3   The dual action

We can also look at the representation $\varphi^\vee\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_6(\mathbb{C})$ that is defined by

$$\varphi^\vee(g) := {}^t\varphi(g)^{-1}.$$

In other words, $\varphi^\vee = \varphi \circ \alpha$, where $\alpha\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_3(\mathbb{C})$ is the algebraic automorphism $g \mapsto ({}^tg)^{-1}$.

Let $A = (a_{ij})$ be a 3×3 matrix. The cofactor matrix $C(A)$ of A is given by

$$C_{ij}(A) = (-1)^{i+j}A_{ij},$$

where A_{ij} is the i, j-minor of A, i.e. the determinant of the 2×2-matrix obtained by removing the i-th row and j-th column of A. We denote by

$$Ad(A) := {}^tC(A)$$

the *adjugate* matrix of A. This is a classical construction and it is well known that $Ad(AB) = Ad(B)Ad(A)$ and that if A is invertible, then $Ad(A) = \det(A)A^{-1}$. In particular, $Ad\colon \mathbb{P}M_3 \dashrightarrow \mathbb{P}M_3$ is a birational map. The conic corresponding to the symmetric matrix A is the dual of the conic corresponding to the symmetric matrix A. This is one of the birational maps that A. R. Williams described in 1938 in his paper "Birational transformations in 4-space and 5-space" ([Wil38]).

Lemma 3.3.6. *We identify $\mathbb{P}^5$ with the projectivized space of symmetric 3×3 matrices $\mathbb{P}M_3$. The birational transformation $Ad \in \mathrm{Cr}_5$ is given by*

$$Ad := [x_1x_2 - x_3^2 : x_0x_2 - x_4^2 : x_0x_1 - x_5^2 : x_4x_5 - x_0x_3 : x_3x_5 - x_1x_4 : x_3x_4 - x_2x_5].$$

Moreover, Ad conjugates φ to $\varphi^\vee$.

Proof. It is a straightforward calculation that the rational map Ad from $\mathbb{P}^5$ to itself that corresponds to Ad is given by

$$Ad := [x_1x_2 - x_3^2 : x_0x_2 - x_4^2 : x_0x_1 - x_5^2 : x_4x_5 - x_0x_3 : x_3x_5 - x_1x_4 : x_3x_4 - x_2x_5].$$

The actions of $\mathrm{PGL}_3(\mathbb{C})$ on $\mathbb{P} M_3$ induced by φ and $\varphi^\vee$ are given by $\varphi(g)(X) = gX({}^tg)$ and $\varphi^\vee(g)X = {}^t(g^{-1})Xg^{-1}$ respectively, for all $X \in \mathbb{P} M_3$. We obtain

$$Ad(\varphi(g)(X)) = Ad({}^tg)Ad(X)Ad(g) = ({}^tg)^{-1}Ad(X)g^{-1} = \varphi^\vee(g)Ad(X).$$

$\square$

Remark 3.3.1. The blow-up Q of $\mathbb{P}^5$ along the Veronese surface is the so called *space of complete conics*. Let $U \subset \mathbb{P}^5$ be the open orbit of the PGL_3-action on $\mathbb{P}^5$ given by φ, i.e. $U = \mathbb{P}^5 \setminus S$. Then U can be embedded into $\mathbb{P}(\mathbb{C}^6) \times \mathbb{P}((\mathbb{C}^6)^\vee)$ by sending a conic $C \in U$ to the pair $(C, C^\vee)$, where $\mathbb{C}^\vee$ denotes the dual conic of C. It turns out that Q is isomorphic to the closure of U in $\mathbb{P}(\mathbb{C}^6) \times \mathbb{P}((\mathbb{C}^6)^\vee)$. Moreover, the PGL_3-action on $\mathbb{P}^5$ given by φ lifts to an algebraic action on Q and the birational map ad to an automorphism of Q. More details on this subject can be found for example in [Bri89].

Lemma 3.3.6 shows that the representations φ and $\varphi^\vee$ are conjugate to each other in Cr_5 by the birational transformation Ad. By conjugating $\Phi(\sigma)$ with Ad we can extend $\varphi^\vee$ to the *dual embedding* $\Phi^\vee \colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$ and obtain

$$\Phi^\vee(\sigma) = [(x_1x_2 - x_3^2)^2 x_0 : (x_0x_2 - x_4^2)^2 x_1 : (x_0x_1 - x_5^2)^2 x_2 :$$

$$(x_0x_2 - x_4^2)(x_0x_1 - x_5^2)x_3 : (x_1x_2 - x_3^2)(x_0x_1 - x_5^2)x_4 : (x_1x_2 - x_3^2)(x_0x_2 - x_4^2)x_5].$$

3.3.4 Geometry of Φ

The embedding Φ induces a rational action of Cr_2 on the space of conics on $\mathbb{P}^2$. The action of $\Phi(\sigma)$ can be seen geometrically as follows (compare with [Giz99, Introduction]): Let $Q_0 := [1:0:0]$, $Q_1 := [0:1:0]$ and $Q_2 := [0:0:1]$. Let $C \subset \mathbb{P}^2$ be a conic that doesn't pass through any of the points Q_i. Write

$$C = \{a_{00}X^2 + a_{11}Y^2 + a_{22}Z^2 + 2a_{12}YZ + 2a_{02}XZ + 2a_{01}XY = 0\} \subset \mathbb{P}^2$$

Denote by P_{ij}, $i \in \{1,2,3\}$, $j \in \{1,2\}$ the points of intersection of C with the lines l_i, where $l_0 := \{X = 0\}$, $l_1 := \{Y = 0\}$ and $l_2 := \{Z = 0\}$. Denote by f_{ij} the line passing through Q_i and P_{ij}. The images $\sigma(f_{ij})$ are again lines passing through the point Q_i. Let P'_{ij} be the intersection of $\sigma(f_{ij})$ with l_i. One checks that the conic D defined by the equation

$$a_{11}a_{22}x_0^2 + a_{00}a_{22}x_1^2 + a_{00}a_{11}x_2^2 + 2a_{00}a_{12}x_1x_2 + 2a_{11}a_{02}x_0x_2 + 2a_{22}a_{01}x_0x_2 = 0$$

passes through the points P'_{ij}. Since no 4 of the 6 points P'_{ij} lie on the same line, D is the unique conic through the points P'_{ij}. We have thus proven the following:

Proposition 3.3.7. *For a general conic $C \subset \mathbb{P}^2$ there exists a unique conic D through the six points P'_{ij} and D is the image of C under $\Phi(\sigma)$.*

Notice as well that the indeterminacy points of $\Phi(\sigma)$ in $\mathbb{P}^5$ correspond to the subspace of dimension 2 of conics passing through the points Q_1, Q_2, Q_3 and the subspaces of dimension 2 of conics consisting of one l_i and any other line. The three subspaces of dimension 4 of conics passing through one of the points Q_i are contracted by the action of $\Phi(\sigma)$ and form the exceptional divisor.

In homogeneous coordinates of $\mathbb{P}^5$, the four planes of indeterminacy locus of $\Phi(\sigma)$ can be described as follows

$$E_0 = \{x_1 = x_2 = x_3 = 0\}, E_1 = \{x_0 = x_2 = x_4 = 0\}, E_2 = \{x_0 = x_1 = x_5 = 0\}$$

$$\text{and } F = \{x_1 = x_2 = x_3 = 0\}.$$

The exceptional divisor of $\Phi(\sigma)$ consists of the three hyperplanes

$$H_0 = \{x_0 = 0\}, H_1 = \{x_1 = 0\}, H_2 = \{x_2 = 0\},$$

The hyperplanes H_0, H_1 and H_2 are contracted by $\Phi(\sigma)$ onto the planes E_0, E_1 and E_2 respectively. Note as well that E_0, E_1 and E_2 are contained in the secant variety $S \subset \mathbb{P}^5$ of the Veronese surface V and they are tangent to V.

The geometrical description of the rational action of $\Phi^\vee(\sigma)$ on the space of conics is the dual of the construction described above. If C is a conic not passing through any of the points Q_0, Q_1, Q_2, we get $\Phi^\vee(\sigma)(C)$ in the following way: let $l_{i,1}, l_{i,2}$ be the tangents of C passing through the point Q_i. Then the images of the $l_{i,1}$ and $l_{i,2}$ under σ are lines again. There exists a unique conic having all the lines $\sigma(l_{i,1})$ and $\sigma(l_{i,2})$ for all i as tangents.

These geometrical constructions show that $\Phi(\mathrm{Cr}_2)$ preserves the space of conics consisting of double lines and therefore the Veronese surface V in $\mathbb{P}^5$. The injective morphism

$$v \colon \mathbb{P}^2 \to \mathbb{P}^5, \ [X : Y : Z] \mapsto [X^2 : Y^2 : Z^2 : YZ : XZ : XY]$$

is called the *Veronese morphism*. It is an isomorphism onto its image, which is V. It is well known that v is $\mathrm{PGL}_3(\mathbb{C})$-equivariant with respect to the standard action and the action induced by Φ respectively. The restriction of $\Phi(\sigma)$ to V is a birational transformation. We therefore obtain a rational action of Cr_2 on $V \simeq \mathbb{P}^2$. Since the restriction of this rational action to $\mathrm{PGL}_3(\mathbb{C})$ is the standard action, we obtain by Corollary 3.5.11 from the appendix that v is Cr_2-equivariant.

We observe as well that $\Phi(\mathrm{Cr}_2)$ preserves the secant variety $S \subset \mathbb{P}^5$ of V. Note that S is the closure of the image of the rational map:

$$\mathcal{S} \colon \mathbb{P}^2 \times \mathbb{P}^2 \to S \subset \mathbb{P}^5,$$

that maps the point $[X : Y : Z], [U : V : W] \in \mathbb{P}^2 \times \mathbb{P}^2$ to the point

$$[XU : YV : ZW : 1/2(YW + UZ) : 1/2(XW + ZU) : 1/2(XV + YU)].$$

Note that $\mathcal{S}$ is generically $2 : 1$. Again, the geometrical construction above shows that $\mathcal{S}$ is Cr_2-equivariant with respect to the diagonal action on $\mathbb{P}^2 \times \mathbb{P}^2$ and the action given by Φ on $\mathbb{P}^5$ respectively.

We obtain the following sequence of Cr_2-equivariant maps:

$$\mathbb{P}^2 \xrightarrow{\Delta} \mathbb{P}^2 \times \mathbb{P}^2 \xdashrightarrow{\mathcal{S}} \mathbb{P}^5,$$

where Δ is the diagonal embedding. This proves part (2) to (4) of Theorem 3.1.3.

Let $f \in \mathrm{Cr}_n$ be a birational transformation and let $l \subset \mathbb{P}^n$ be a general line and $H \subset \mathbb{P}^5$ a general hyperplane. Then, $f^{-1}(H)$ intersects l in $\deg(f)$ points, which is equivalent to $f(l)$ intersecting H in $\deg(f)$ points. More generally, if $C \subset \mathbb{P}^n$ is a general curve of degree d, then $f(C)$ intersects H in $d \cdot \deg(f)$ points. If C and H are not in general position, but C is not contained in the exceptional locus of f and $f(C)$ is not contained in H, we only have that $f(C)$ and H intersect in $\leq d \cdot \deg(f)$ points. With this and the observation that $\Phi(\mathrm{Cr}_2)$ preserves the Veronese surface and extends the canonical rational action of Cr_2 we are able to prove part (1) of Theorem 3.1.5:

Proposition 3.3.8. *Let $f \in \mathrm{Cr}_2$. Then $\deg(f) \leq \deg(\Phi(f))$.*

Proof. Denote by $v \colon \mathbb{P}^2 \to \mathbb{P}^5$ the Veronese embedding. Let $C \subset \mathbb{P}^2$ be a general conic. The image $v(C) \subset \mathbb{P}^5$ is a curve of degree 4 given by the intersection of a hyperplane $H \subset \mathbb{P}^5$ and the Veronese surface. Let $f \in \mathrm{Cr}_2$ be a birational transformation of degree d. The strict transform $f(C')$ of a general conic $C' \subset \mathbb{P}^2$ intersects C in $4d$ different points. By the above results (namely, (3) in Theorem 3.1.3) we know that $v(f(C')) = \Phi(f)(v(C'))$. The curve $v(C')$ is a curve of degree 4 and since $v(f(C')) = \Phi(f)(v(C'))$ intersects the hyperplane H in $4d$ points we obtain by the above remark that $\deg(\Phi(f)) \geq d$. $\qquad\square$

3.3.5 Two induced embeddings from Cr_2 into Cr_4

The birational map $Ad \in \mathrm{Cr}_5$ contracts the secant variety $S \subset \mathbb{P}^5$ onto the Veronese surface $V \subset \mathbb{P}^5$. However, the exceptional locus of $\Psi^\vee(\sigma) = Ad\Phi(\sigma)Ad$ consists of the three hyperplanes

$$G_0 = \{z_1 z_2 - z_3^2 = 0\}, G_1 = \{z_0 z_2 - z_4^2 = 0\}, G_2 = \{z_0 z_1 - z_5^2 = 0\},$$

with respect to homogeneous coordinates $[z_0 : z_1 : z_2 : z_3 : z_4 : z_5]$ of $\mathbb{P}^5$.

This implies in particular that the restriction of $\Phi^\vee(\sigma)$ to S induces a birational map of S and therefore that any element in $\Phi^\vee(\mathrm{Cr}_2)$ restricts to a birational map of S.

Since S is a cubic hypersurface and contains the two disjoint planes

$$E_1 = \{z_1 = z_2 = z_3 = 0\}, E_2 = \{z_0 = z_4 = z_5 = 0\},$$

it is rational. Explicitly, projection onto E_1 and E_2 yields the birational map $A \colon S \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$[z_0 : z_1 : z_2 : z_3 : z_4 : z_5] \mapsto [z_1 : z_2 : z_3], [z_0 : z_4 : z_5].$$

The inverse transformation A^{-1} is given by

$$[x_0 : x_1 : x_2], [y_0 : y_1 : y_2] \mapsto [p_2 y_0, p_1 x_0, p_1 x_1, p_1 x_2, p_2 y_1, p_2 y_2],$$

where $p_1 = (x_0 y_1^2 + x_1 y_2^2 - 2x_2 y_1 y_2)$ and $p_2 = y_0(x_0 x_1 - x_2^2)$.

Let $f \in \mathrm{Cr}_2$. As seen above, both images $\Phi(f)$ and $\Phi^\vee(f)$ restrict to a birational map of S. So conjugation of Φ and $\Phi^\vee$ by A yields two embeddings from Cr_2 into $\mathrm{Bir}(\mathbb{P}^2 \times \mathbb{P}^2) \simeq \mathrm{Cr}_4$, which we denote by Ψ_1 and Ψ_2 respectively.

Proof of Proposition 3.1.4. Irreducibility is proved in Section 3.3.8.

By Theorem 3.2.8, all tori $D_2 \subset \mathrm{Cr}_4$ are conjugate to the standard torus $D_2 \subset \mathrm{Cr}_4$. We calculate the map that conjugates $\Psi_1(D_2) = \Psi_2(D_2)$ to the image of the standard embedding of D_2 explicitely. Let $\rho \colon \mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ be the birational transformation defined by

$$([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([x_2 y_0 : x_0 y_1 : x_2 y_1], [x_0 y_1^2 : x_1 y_2^2 : x_2 y_1 y_2]).$$

The inverse map ρ^{-1} is given by

$$([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([x_1^2 y_2^2 : x_2^2 y_0 y_1 : x_1 x_2 y_2^2], [x_0 y_0 : x_2 y_0 : x_1 y_2]).$$

One calculates that $\rho A \Psi_1([aX : bY : cZ]) A^{-1} \rho^{-1}$ maps $([x_0 : x_1 : x_2], [y_0 : y_1 : y_2])$ to $([ax_0 : bx_1 : cx_2], [y_0 : y_1 : y_2])$. Correspondingly, $\rho A \Psi_2([aX : bY : cZ]) A^{-1} \rho^{-1}$ maps $([x_0 : x_1 : x_2], [y_0 : y_1 : y_2])$ to $([a^{-1} x_0 : b^{-1} x_1 : c^{-1} x_2], [y_0 : y_1 : y_2])$. So the second coordinates parametrize the closures of the D_2-orbits. Since W_2 normalizes D_2, its image preserves the D_2-orbits. We thus obtain two homomorphisms

$$\chi_1 \colon W_2 \to \mathrm{Cr}_2, \chi_2 \colon W_2 \to \mathrm{Cr}_2$$

by just considering the rational action of W_2 on the second coordinate.

Assume that there exists an element $A \in \mathrm{Bir}(\mathbb{P}^2 \times \mathbb{P}^2)$ that conjugates Ψ to $\Psi^\vee$. As A normalizes $\Psi_1(D_2) = \Psi_2(D_2)$, it preserves the $\Psi_1(D_2)$-orbits as well. Hence by restriction on the second coordinate, it conjugates χ_1 to χ_2. It therefore suffices to show that χ_1 and χ_2 are not conjugate.

In Cr_2 we have

$$f := [XY : YZ : Z^2] = \tau_1 g_0 \sigma g_0 \sigma g_0 \tau_2,$$

where $\tau_1 = [Z : Y : X], \tau_2 = [Y : Z : X]$ and $g_0 = [Y - X : Y : Z]$. By calculating the corresponding images under Φ we obtain

$$\Phi(f) = \Phi(\tau_1 g_0 \sigma g_0 \sigma g_0 \tau_2) = [x_0 x_1 : x_1 x_2 : x_2^2 : x_2 x_3 : -x_2 x_5 + 2x_3 x_4 : x_1 x_4]$$

and $\Phi^\vee(f) = [g_0 : g_1 : g_2 : g_3 : g_4 : g_5]$, where

$$g_0 = (x_0 x_1 - x_5^2)^2 x_0,$$

$$g_1 = x_0^2 x_1^2 x_2 - 2x_0 x_1 x_2 x_5^2 - 4x_0 x_1 x_3 x_4 x_5 + 4x_0 x_3^2 x_5^2 + 4x_1 x_4^2 x_5^2 + x_2 x_5^4 - 4x_3 x_4 x_5^3,$$

$$g_2 = (x_0 x_2 - x_4^2)^2 x_1,$$

$$g_3 = (x_0 x_2 - x_4^2)(x_0 x_1 x_3 - 2x_1 x_4 x_5 + x_3 x_5^2),$$

$$g_4 = -(x_0 x_2 - x_4^2)(x_0 x_1 - x_5^2) x_5,$$

$$g_5 = (x_0 x_1 - x_5^2)(x_0 x_1 x_4 - 2x_0 x_3 x_5 + x_4 x_5^2).$$

This yields

$$\chi_1(f) = [(y_1 - 2y_2)^2 : y_0 y_1 : -y_2(y_1 - 2y_2)]$$

and

$$\chi_2(f) = [y_0^2 y_1 + 4y_0 y_1^2 - 6y_0 y_1 y_2 - 3y_1 y_2^2 + 4y_2^3 : y_0(y_0 + 2y_1 - 3y_2)^2 :$$
$$(2y_0 y_1 - y_0 y_2 - y_2^2)(y_0 + 2y_1 - 3y_2)].$$

We show that these two transformations are not conjugate in Cr_2. With respect to affine coordinates $[y_0 : y_1 : 1]$ one calculates

$$\chi_1(f)^2 = \left(\frac{y_0 y_1 - 2y_1 + 4}{y_1 - 2}, y_1 \right).$$

From this we see that the integer sequence $\deg(\chi_1(f)^n)$ grows linearly in n and is, in particular, not bounded.

Let $A = [y_0 - y_2 : y_1 - y_2 : y_2]$. Then

$$A\chi_2(f)^2 A^{-1} = [-y_0^2 y_1^2 (2y_1 + y_0) : y_0^2 y_1^2 (3y_1 + 2y_0) : p(y_0, y_1, y_2)(3y_1 + 2y_0)(2y_1 + y_0)],$$

where $p(y_0, y_1, y_2) = (6y_1^2 y_2 + 7y_2 y_0 y_1 + 6y_0 y_1^2 + 2y_0^2 y_2 + 2y_0^2 y_1)$. We claim that

$$f_A^n = A\chi_2(f)^{2n} A^{-1} = [-y_0^2 y_1^2 (2ny_1 + (2n-1)y_0) : y_0^2 y_1^2 ((2n+1)y_1 + 2ny_0) : f_n],$$

where $f_n = (2ny_1 + (2n-1)y_0)((2n+1)y_1 + 2ny_0)p_n(y_0, y_1, y_2)$ for some homogeneous $p_n \in \mathbb{C}[y_0, y_1, y_2]$ of degree 3. Note that this claim implies in particular that $\deg(\chi_2(f)^n)$ is bounded for all n and hence that $\chi_1(f)$ and $\chi_2(f)$ are not conjugate.

To prove the claim we proceed by induction. Assume that f_A^n has the desired form. One calculates that the first coordinate of $f_A^{n+1} = A\chi_2(f)^2 A^{-1} \circ f_A^n$ is

$$-ry_0^2 y_1^2 ((2n+2)y_1 + (2n+1)y_0),$$

the second coordinate is

$$ry_0^2 y_1^2 ((2n+3)y_1 + (2n+1)y_0)$$

and the third coordinate

$$r((2n+2)y_1 + (2n+1)y_0)((2n+3)y_1 + (2n+1)y_0)p_{n+1}(x_0, x_1, x_2),$$

where $r = y_0^4 y_1^4 (2ny_0 + (2n-1)y_1)^2 ((2n+1)y_0 + 2ny_1)^2$ and $p_n \in \mathbb{C}[x_0, x_1, x_2]$ is homogeneous of degree 3. This proves the claim. $\square$

3.3.6 A volume form

Let M be a complex projective manifold. It is sometimes interesting to study subgroups of $\mathrm{Bir}(M)$ that preserve a given form. In [Bla13] and [DL16] the authors study for example birational maps of surfaces that preserve a meromorphic symplectic form (see [CK15] for the 3-dimensional case). In [Giz08] and [CD16] Cremona transformations in dimension 3 preserving a contact form are studied.

Define

$$F := \det \begin{pmatrix} x_0 & x_5 & x_4 \\ x_5 & x_1 & x_3 \\ x_4 & x_3 & x_2 \end{pmatrix}$$

and let

$$\Omega := \frac{x_5^6}{F^2} \cdot dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

Then Ω is a 5-form on $\mathbb{P}^5$ with a double pole along the secant variety of the Veronese surface. Note that the total volume of $\mathbb{P}^5$ is infinite.

Proposition 3.3.9. *All elements in $\Phi(\mathrm{Cr}_2)$ preserve Ω.*

Proof. We show that $\Phi(\mathrm{PGL}_3(\mathbb{C}))$ and $\Phi(\sigma)$ preserve Ω.

Let $g = [-X : -Y : Z] \in \Phi(\mathrm{PGL}_3(\mathbb{C}))$. One checks that $\Phi(g)$ preserves Ω. Since $\Phi(\mathrm{PGL}_3(\mathbb{C}))$ preserves F, we have that $\Phi(fgf^{-1})$ preserves Ω as well. As $\Phi(\mathrm{PGL}_3(\mathbb{C}))$ is simple, the whole group preserves Ω.

With respect to affine coordinates given by $x_5 = 1$, we have

$$\Phi(\sigma) = (x_1, x_0, x_0 x_1 x_2^{-1}, x_0 x_3 x_2^{-1}, x_1 x_4 x_2^{-1}).$$

A direct calculation yields $\Omega \circ \Phi(\sigma) = \Omega$. □

3.3.7   Polynomial automorphisms

In this section we will prove Claim (6) of Theorem 3.1.3 as well as Theorem 3.1.5. Let $\mathrm{Aut}(\mathbb{A}^2) \subset \mathrm{Cr}_2$ be the subgroup of automorphisms of the affine plane with respect to the affine coordinates $[1 : X : Y]$. By the theorem of Jung and van der Kulk (see for example [Lam02]), $\mathrm{Aut}(\mathbb{A}^2)$ has the following amalgamated product structure

$$\mathrm{Aut}(\mathbb{A}^2) = \mathrm{Aff}_2 *_\cap \mathcal{J}_2,$$

where $\mathcal{J}_2$ denotes the subgroup of elementary automorphisms, which is the subgroup of all elements of the form

$$\{(c_1 X + b, c_2 Y + p(X)) \mid c_1, c_2, b \in \mathbb{C}, p(X) \in \mathbb{C}[X]\}.$$

Let $f \in \mathrm{Aut}(\mathbb{A}^2)$ and assume that $f = a_1 j_1 a_2 j_2 \cdots j_{n-1} a_n$, where $a_1, a_n \in \mathrm{Aff}_2$, $a_i \in \mathrm{Aff}_2 \backslash \mathcal{J}_2$ for $2 \leq i \leq n-1$ and $j_i \in \mathcal{J}_2 \backslash \mathrm{Aff}_2$. It is well known that $\deg(f) = \deg(j_1) \deg(j_2) \cdots \deg(j_{n-1})$.

Let $\mathrm{Aut}(\mathbb{A}^5) \subset \mathrm{Cr}_5$ be given by the affine coordinates $[1 : x_1 : \cdots : x_5]$. Lemma 3.3.10 follows from a direct calculation.

Lemma 3.3.10. *The image $\Phi(\mathrm{Aff}_2)$ is contained in Aff_5.*

We consider the following elements in $\mathcal{J}_2$:

$$f_n^\lambda := (X, Y + \lambda X^n),$$

where $n \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \mathbb{C}$.

Lemma 3.3.11. *For all $n \in \mathbb{Z}_{\geq 0}$ we have*

$$\Phi(f_n^\lambda) = (x_1, x_2 + \lambda^2 x_1^n + \lambda x_3 A_n - \lambda x_4 x_1 A_{n-1}, x_3 + \lambda x_1 B_{n-1}, x_4 + \lambda B_n, x_5),$$

where

$$A_n = 2 \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k+1} x_5^{n-2k-1} (x_5^2 - x_1)^k$$

and

$$B_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x_5^{n-2k} (x_5^2 - x_1)^k.$$

Moreover, the following recursive identities hold:

$$A_n = 2x_5 A_{n-1} - x_1 A_{n-2},$$

$$B_n = 2x_5 B_{n-1} - x_1 B_{n-2}.$$

Proof. For $n = 0$ and $n = 1$ the claim follows from a direct calculation.

Let $s := (X, XY) \in \mathrm{Cr}_2$. Then we have $f_{n+1}^\lambda = s f_n^\lambda s^{-1}$. In Cr_2 the identity $s = \tau_1 g_0 \sigma g_0 \sigma g_0 \tau_2$ holds, where $\tau_1 = (XY^{-1}, Y^{-1}), \tau_2 = (Y^{-1}, XY^{-1})$ and $g_0 = (X, X_Y)$. Note that τ_1 and τ_2 are elements of PGL_3. If we calculate the corresponding images under Φ we obtain

$$\Phi(s) = \Phi(\tau_1 g_0 \sigma g_0 \sigma g_0 \tau_2) = (x_1, x_1 x_2, x_1 x_4, 2x_4 x_5 - x_3, x_5)$$

and

$$\Phi(s^{-1}) = (x_1, x_2 x_1^{-1}, 2x_3 x_5 x_1^{-1} - x_4, x_3 x_1^{-1}, x_5).$$

One calculates

$$s f_n^\lambda s^{-1} = (x_1, x_2 + \lambda^2 x_1^{n+1} + \lambda x_3 (2x_5 - x_1) A_{n-1} - \lambda x_4 x_1 A_n, x_3 + \lambda x_1 B_n,$$

$$x_4 - \lambda(2x_5 B_n - x_1 B_{n-1}).$$

This shows by induction that

$$\Phi(f_n^\lambda) = (x_1, x_2 + \lambda^2 x_1^n + \lambda x_3 A_n - \lambda x_4 x_1 A_{n-1}, x_3 + \lambda x_1 B_{n-1}, x_4 + \lambda B_n, x_5),$$

where

$$A_n = 2x_5 A_{n-1} - x_1 A_{n-2}, A_0 = 0, A_1 = 2;$$

$$B_n = 2x_5 B_{n-1} - x_1 B_{n-2}, B_0 = 1, B_1 = x_5.$$

These recursive formulas have the following closed form:

$$A_n = \frac{\left(x_5 + \sqrt{x_5^2 - x_1}\right)^n - \left(x_5 - \sqrt{x_5^2 - x_1}\right)^n}{\sqrt{x_5^2 - x_1}},$$

$$B_n = \frac{1}{2}\left(x_5 - \sqrt{x_5^2 - x_1}\right)^n + 1/2\left(x_5 + \sqrt{x_5^2 - x_1}\right)^n.$$

The claim follows. $\square$

Since Aff_n together with all the elements f_n^λ, $n \in \mathbb{N}$, $\lambda \neq 0$ generates $\mathrm{Aut}(\mathbb{A}^2)$, Lemma 3.3.11 shows that $\Phi(\mathrm{Aut}(\mathbb{A}^2))$ is contained in $\mathrm{Aut}(\mathbb{A}^5)$ and thus claim (6) of Theorem 3.1.3.

Lemma 3.3.12. *Let n and m be positive integers and A_n, B_m as in Lemma 3.3.11. Then*

$$A_n B_{m-1} - A_{n-1} B_m = P(x_1, x_5),$$

where $P \in \mathbb{C}[x_1, x_5]$ is a polynomial of degree $< \max\{m, n\}$.

Proof. If $n = 1$ or $m = 1$ the claim is true, since $A_0 = 0, A_1 = 2, B_0 = 1, B_1 = x_5$ and $\deg(A_k) = k - 1, \deg(B_k) = k$. By the identities from Lemma 3.3.11, one obtains

$$
\begin{aligned}
A_n B_{m-1} - A_{n-1} B_m &= (2x_5 A_{n-1} - x_1 A_{n-2}) B_{m-1} - A_{n-1}(2x_5 B_{m-1} - x_1 B_{m-2}) \\
&= x_1(A_{n-1} B_{m-2} - A_{n-2} B_{m-1}).
\end{aligned}
$$

The claim follows by induction on m and n. $\square$

Lemma 3.3.13. *Let*

$$f = f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_n^{\lambda_n}, \ \text{where } \lambda_n \neq 0.$$

Then

$$\Phi(f) = (x_1, x_2 + F, x_3 + p_3(x_1, x_5) + \lambda_n x_1 B_{n-1}, x_4 + p_4(x_1, x_5) + \lambda_n B_n, x_5),$$

where $F = p_2(x_1, x_5) + x_3(\lambda_1 A_1 + \cdots + \lambda_n A_n) - x_4 x_1(\lambda_1 A_{n-1} + \cdots + \lambda_n A_n)$ and $p_2, p_3, p_4 \in \mathbb{C}[x_1, x_5]$ are polynomials of degree $\leq n$. In particular, $\deg(\Phi(f)) = \deg(f)$.

Proof. It is easy to see that the third and fourth coordinate of $\Phi(f)$ have the claimed form. The more difficult part is the second coordinate.

For $n = 1$ the claim follows directly from Lemma 3.3.11. We proceed now by induction. Let $\lambda_{n+1} \neq 0$ and m be the largest number, such that $m \leq n$ and $n \neq 0$. By the induction hypothesis we may assume that the second coordinate of $\Phi(f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_m^{\lambda_m})$ has the form

$$x_2 + p_2(x_1, x_5) + x_3(\lambda_1 A_1 + \cdots + \lambda_m A_m) - x_4 x_1(\lambda_1 A_0 + \cdots + \lambda_m A_{m-1}).$$

The second coordinate of $\Phi(f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_m^{\lambda_m}) \circ \Phi(f_n^{\lambda_n})$ is therefore

$$x_2 + p_2(x_1, x_5) + x_3(\lambda_1 A_1 + \cdots + \lambda_m A_m + \lambda_n A_n) - x_4 x_1(\lambda_1 A_0 + \cdots + \lambda_m A_{m-1} + \lambda_n A_n)$$

$$+ x_1 \sum_{k=1}^{m} \lambda_k (A_k B_{n-1} - A_{k-1} B_n).$$

By Lemma 3.3.12, $x_1 \sum_{k=1}^{m} \lambda_k (A_k B_{n-1} - A_{k-1} B_n)$ is a polynomial in x_1 and x_5 of degree $\leq n$. $\square$

Proof of Theorem 3.1.5. The first claim was proved in Proposition 3.3.8.

To prove the second part, we show in a first step that $\deg(\Phi(f)) = \deg(f)$ for all elements $f \in \mathcal{J}_2$. Composition with an element in Aff_2 doesn't change the degree, by Lemma 3.3.10. So it is enough to consider elements in $\mathcal{J}_2$ of the form $f = (X, Y + P(X)), P \in \mathbb{C}[X]$. For suitable $\lambda_i \in \mathbb{C}$ we have $f = f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_n^{\lambda_n}$, where $\lambda_n \neq 0$. In Lemma 3.3.13 we've seen that Φ preserves the degree of these elements.

Let now $f \in \mathrm{Aut}(\mathbb{A}^2)$ be arbitrary and assume that $f = a_1 j_1 a_2 j_2 \cdots j_{n-1} a_n$, where $a_1, a_n \in \mathrm{Aff}_2$, $a_i \in \mathrm{Aff}_2 \setminus \mathcal{J}_2$ for $2 \leq i \leq n-1$ and $j_i \in \mathcal{J}_2 \setminus \mathrm{Aff}_2$. So $\deg(f) = \deg(j_1) \deg(j_2) \cdots \deg(j_{n-1})$ and

$$\Phi(f) = \Phi(a_1)\Phi(j_1)\Phi(a_2)\Phi(j_2) \cdots \Phi(j_{n-1})\Phi(a_n).$$

This implies in particular that

$$\deg(\Phi(f)) \leq \deg(\Phi(a_1)) \deg(\Phi(j_1)) \cdots \deg(\Phi(j_{n-1})) \deg(\Phi(a_n)) = \deg(f).$$

With this and part (1) of Theorem 3.1.5, part (2) follows. $\qquad\square$

3.3.8 Irreducibility of Φ, Ψ_1 and Ψ_2

First we show that Φ is irreducible. Assume that there is a rational dominant map $\pi \colon \mathbb{P}^5 \dashrightarrow M$ to a variety M with an algebraic embedding $\varphi_M \colon \mathrm{Cr}_2 \to \mathrm{Bir}(M)$ such that A is Cr_2-equivariant. Since φ_M is algebraic, we may assume that $\mathrm{PGL}_3(\mathbb{C})$ acts regularly on M. We obtain that the restriction of A to the open $\mathrm{PGL}_3(\mathbb{C})$-invariant subset $U \subset \mathbb{P}^5$ consisting of all smooth conics is a $\mathrm{PGL}_3(\mathbb{C})$-equivariant morphism, whose image is an open dense subset of M on which $\mathrm{PGL}_3(\mathbb{C})$ acts transitively. Note that this implies $\dim(M) > 1$.

If $\dim M = 2$, we obtain by Theorem 3.5.9 that $M \simeq \mathbb{P}^2$ with the standard action of $\mathrm{PGL}_3(\mathbb{C})$. The stabilizer in $\mathrm{PGL}_3(\mathbb{C})$ of a point in $U \subset \mathbb{P}^5$ is isomorphic to $\mathrm{SO}_3(\mathbb{C})$. On the other hand the stabilizer in $\mathrm{PGL}_3(\mathbb{C})$ of a point in $\mathbb{P}^2$ is isomorphic to the group of affine transformations $\mathrm{Aff}_2 = \mathrm{GL}_2(\mathbb{C}) \ltimes \mathbb{C}^2$. Since $\mathrm{SO}_3(\mathbb{C})$ can not be embedded into Aff_2, the case $\dim(M) = 2$ is not possible.

If $\dim(M) = 3$, we find, by Theorem 3.4.1, a $\mathrm{PGL}_3(\mathbb{C})$-equivariant projection $M \dashrightarrow \mathbb{P}^2$ and are again in the case $\dim(M) = 2$.

If $\dim(M) = 4$, let $p \in M$ be a general point and $F_p := A^{-1}(p) \subset \mathbb{P}^5$ the fiber of A. Let $q \in F_p$ be a point that is only contained in one connected component C of F_p. Again, the stabilizer of q is isomorphic to $\mathrm{SO}_3(\mathbb{C})$. This implies that $\mathrm{SO}_3(\mathbb{C})$ acts regularly on the curve C with a fixpoint. The neutral component of the group of birational transformations of C is isomorphic to $\mathrm{PGL}_2(\mathbb{C})$, is abelian or is trivial. In all cases we obtain that the connected group $\mathrm{SO}_3(\mathbb{C})$ fixes C pointwise. In other words, the group $\mathrm{SO}_3(\mathbb{C})$ preserves each conic of the family of conics in $\mathbb{P}^2$ parametrized by C. This is not possible.

The proof that Ψ_1 and Ψ_2 are irreducible is done analogously.

3.4 $\mathrm{PGL}_{n+1}(\mathbb{C})$-actions in codimension 1

In this section we look at algebraic embeddings of $\mathrm{PGL}_{n+1}(\mathbb{C})$ into $\mathrm{Bir}(M)$ for complex projective varieties M of dimension $n+1$. Our aim is to prove Theorem 3.4.1.

Theorem 3.4.1. *Let $n \geq 2$ and let M be a smooth projective variety of dimension $n + 1$ with an algebraic rational non-trivial $\mathrm{PGL}_{n+1}(\mathbb{C})$-action. Then, up to birational conjugation and automorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$, we have one of the following:*

(1) *$M \simeq F_l = \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-l(n+1)))$ for a unique element $l \in \mathbb{Z}_{\geq 0}$ and $\mathrm{PGL}_{n+1}(\mathbb{C})$ acts as in Example 3.1.8.*

(2) *$M \simeq \mathbb{P}^n \times C$ for a unique smooth curve C and $\mathrm{PGL}_{n+1}(\mathbb{C})$ acts on the first factor as in Example 3.1.1.*

(3) *$M \simeq \mathbb{P}(T\,\mathbb{P}^2)$ and $\mathrm{PGL}_3(\mathbb{C})$ acts as in Example 3.1.9.*

(4) *$M \simeq \mathbb{G}(1, 3)$ and $\mathrm{PGL}_4(\mathbb{C})$ acts as in Example 3.1.10.*

Moreover, these actions are not birationally conjugate to each other.

Remark 3.4.1. If M is rational and of dimension 2 or 3, this result can be deduced directly from the classification of maximal algebraic subgroups of Cr_2 and Cr_3 by Enriques, Umemura and Blanc ([Enr93], [Ume82b], [Ume85], [Ume82a], [Bla09]).

3.4.1   Classification of varieties and groups of automorphisms

With some geometric invariant theory and using results of Freudenthal about topological ends, the following classification can be made (see [CZ12, Theorem 4.8] and the references in there):

Theorem 3.4.2. *Let M be a smooth projective variety of dimension $n + 1$ with a non-trivial regular action of $\mathrm{PGL}_{n+1}(\mathbb{C})$, where $n \geq 2$. Then we are in one of the following cases:*

(1) *$M \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k))$ for some $k \in \mathbb{Z}_{\geq 0}$.*

(2) *$M \simeq \mathbb{P}^n \times C$ for a curve C of genus ≥ 1.*

(3) *$M \simeq \mathbb{P}(T\,\mathbb{P}^2) \simeq \mathrm{PGL}_3(\mathbb{C})/B$, where $B \subset \mathrm{PGL}_3(\mathbb{C})$ is a Borel subgroup.*

(4) *$M \simeq \mathbb{G}(1, 3) \simeq \mathrm{PGL}_4(\mathbb{C})/P$, where $P \subset \mathrm{PGL}_4(\mathbb{C})$ is the parabolic subgroup consisting of matrices of the form*

$$
\begin{pmatrix}
* & * & * & * \\
* & * & * & * \\
0 & 0 & * & * \\
0 & 0 & * & *
\end{pmatrix}.
$$

The neutral components $\mathrm{Aut}^0(M)$ of the automorphism groups of the varieties M that appear in Theorem 3.4.2 are well known. Proofs of the following Proposition can be found in [Akh95, Proposition 2.4.1, 2.4.2, Example 2.4.2, and Theorem 3.3.2].

Proposition 3.4.3.

- $\mathrm{Aut}^0(\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k))) \simeq (\mathrm{GL}_{n+1}(\mathbb{C})/\mu_k) \rtimes \mathbb{C}[x_0, \dots, x_n]_k$, where $\mathbb{C}[x_0, \dots, x_n]_k$ denotes the additive group of homogeneous polynomials of degree k and $\mu_k \subset \mathbb{C}^*$ the group of all elements $c \in \mathbb{C}^*$ satisfying $c^k = 1$,

- $\mathrm{Aut}^0(\mathbb{P}^n \times C) \simeq \mathrm{PGL}_{n+1}(\mathbb{C}) \times \mathrm{Aut}^0(C)$,

- $\mathrm{Aut}^0(\mathbb{P}(T\,\mathbb{P}^2)) \simeq \mathrm{PGL}_3(\mathbb{C})$,

- $\mathrm{Aut}^0(\mathbb{G}(1,3)) \simeq \mathrm{PGL}_4(\mathbb{C})$.

To describe the $\mathrm{PGL}_{n+1}(\mathbb{C})$-actions on these varieties we recall some results about group cohomology.

3.4.2 Group cohomology

Let H be a group that acts by automorphisms on a group N. A *cocycle* is a map $\tau \colon H \to N$ such that $\tau(gh) = \tau(g)(g \cdot \tau(h))$ for all $g, h \in H$. Two cocycles τ and ν are *cohomologous* if there exists an $a \in N$ such that

$$\tau(g) = a^{-1}\nu(g)(g \cdot a) \ \text{ for all } g \in H.$$

This defines an equivalence relation on the cocycles. The set of cocycles up to cohomology will be denoted by $H^1(H, N)$. If H acts trivially on N, the set $H^1(H, N)$ corresponds to the set of group homomorphisms $H \to N$ up to conjugation. The following lemma is well known.

Lemma 3.4.4. *Let $G := N \rtimes H$ be a semi direct product of groups and $\pi \colon G \to H$ the canonical projection on H. Then there exists a bijection between $H^1(H, N)$ and the sections of π up to conjugation in N.*

There always exists the trivial cocycle $\tau_0 \colon H \to N$, $g \mapsto e_N$. The set $H^1(G, N)$ is therefore a pointed set with basepoint τ_0. Assume that G acts on two groups A and B by automorphisms. A G-homomorphism $\phi \colon A \to B$ induces a homomorphism of pointed sets

$$\phi_* \colon H^1(G, A) \to H^1(G, B)$$

given by $\phi_*(\tau) = \phi \circ \tau$.

Proposition 3.4.5 ([Ser79], p. 125, Proposition 1)**.** *Let G be a group that acts by automorphisms on groups A, B and C. Every exact sequence of G-homomorphisms*

$$1 \to A \to B \to C \to 1$$

induces an exact sequence of pointed sets

$$H^1(G, A) \to H^1(G, B) \to H^1(G, C).$$

3.4.3   Proof of Theorem 3.4.1

We use the classification of smooth projective varieties of dimension $n + 1$ with a regular $\mathrm{PGL}_{n+1}(\mathbb{C})$ action of Theorem 3.4.2. We have to show that case (1) appears only if $k = l(n+1)$ for some integer l. Examples 3.1.7 to 3.1.10 show that for all the other varieties there exist $\mathrm{PGL}_{n+1}(\mathbb{C})$-actions. So we have to show that these actions are unique and not birationally conjugate to each other.

Uniqueness of the actions

By Proposition 3.4.3, $\mathrm{Aut}^0(\mathbb{P}(T\,\mathbb{P}^2)) \simeq \mathrm{PGL}_3(\mathbb{C})$ and $\mathrm{Aut}^0(\mathbb{G}(1,3)) \simeq \mathrm{PGL}_4(\mathbb{C})$. The uniqueness of the embedding is clear in these cases since $\mathrm{PGL}_{n+1}(\mathbb{C})$ is a simple group. If $M \simeq \mathbb{P}^n \times C$ uniqueness follows directly from the fact that $\mathrm{PGL}_{n+1}(\mathbb{C})$ does not embed into $\mathrm{Aut}(C)$.

   Now we show that $\mathrm{PGL}_{n+1}(\mathbb{C})$ can be embedded into $\mathrm{Aut}^0(\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k))$ if and only if $n \mid k$. Then we show that in this case, up to conjugation and algebraic automorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$, the embedding is unique.

Lemma 3.4.6. *A non-trivial algebraic group homomorphism from $\mathrm{PGL}_n(\mathbb{C})$ to $\mathrm{GL}_n(\mathbb{C})/\mu_k$ exists if and only if $n \mid k$, where*

$$\mu_k = \{\lambda\,\mathrm{id} \mid \lambda \in \mathbb{C}, \lambda^k = 1\}.$$

Proof. If $n \mid k$, we have $\mathrm{PGL}_n(\mathbb{C}) \simeq \mathrm{SL}_n(\mathbb{C})/(\mu_k \cap \mathrm{SL}_n(\mathbb{C})) = \mathrm{SL}_n(\mathbb{C})/(\mu_k \cap \mu_n(\mathbb{C})) \subset \mathrm{GL}_n(\mathbb{C})/\mu_k$.

   On the other hand, assume that there exists a non-trivial algebraic homomorphism $\phi \colon \mathrm{PGL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})/\mu_k$. Since $\mathrm{PGL}_n(\mathbb{C})$ is simple, the image $\phi(\mathrm{PGL}_n(\mathbb{C}))$ is contained in the derived subgroup of $\mathrm{GL}_n(\mathbb{C})/\mu_k$, which is the group $\mathrm{SL}_n(\mathbb{C})/(\mu_k \cap \mathrm{SL}_n(\mathbb{C})) = \mathrm{SL}_n(\mathbb{C})/(\mu_k \cap \mu_n)$. Both, $\phi(\mathrm{PGL}_n(\mathbb{C}))$ and $\mathrm{SL}_n(\mathbb{C})/(\mu_k \cap \mu_n(\mathbb{C}))$, are irreducible and have the same dimension. Since the center of $\phi(\mathrm{PGL}_n(\mathbb{C}))$ is trivial, it follows that $\mu_n \subset \mu_k$ and hence that n divides k. $\qquad\square$

   Let n and k be positive integers such that $(n+1) \mid k$. Denote by $\mathbb{C}[x_0, \ldots, x_n]_k$ the vector space of homogeneous polynomials of degree k . We define

$$G := \mathbb{C}[x_0, \ldots, x_n]_k \rtimes \mathrm{PGL}_{n+1}(\mathbb{C}),$$

where the semi direct product is taken with respect to the action $g \cdot p = p \circ g^{-1}$. Here we look at $\mathrm{PGL}_{n+1}(\mathbb{C}) \subset \mathrm{GL}_{n+1}(\mathbb{C})/\mu_k$ as described in Lemma 3.4.6. Let $\pi \colon G \to \mathrm{PGL}_{n+1}(\mathbb{C})$ be the standard projection.

Lemma 3.4.7. *Up to conjugation, there exists a unique section $\iota \colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to G$ of π.*

Proof. Let

$$\iota \colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to G$$

be the standard section of π and let

$$\varphi \colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to G$$

be an arbitrary section of π. We show that φ is conjugate to ι. Note that it is enough to show that $\varphi(\text{PGL}_{n+1}(\mathbb{C})) = \iota(\text{PGL}_{n+1}(\mathbb{C}))$.

The group G acts by affine transformation on the affine space $\mathbb{C}[x_0, \ldots, x_n]_k$ and $\iota(\text{PGL}_{n+1}(\mathbb{C}))$ is the subgroup of G consisting of all elements that fix 0. Let $H \subset D_{n-1}$ be the finite subgroup consisting of the identity and all elements of order d for some d large enough and not divisible by k. Then the action of $\varphi(H)$ on $\mathbb{C}[x_0, \ldots, x_n]_k$ has a fixed point p. Indeed, let $p \in \mathbb{C}[x_0, \ldots, x_n]_k$ and consider its finite orbit $p_1, \ldots, p_r$ of p under H. The barycenter of $p_1, \cdots, p_r$, i.e. the point $q = 1/r(p_1 + p_2 + \cdots + p_r)$ is then invariant under the transformations in H. After conjugation, we can assume that $q = 0$ and hence

$$\varphi(H) \subset \iota(\text{PGL}_{n+1}(\mathbb{C}))$$

and therefore $\varphi(H) = \iota(H)$.

The centralizer of H in $\text{PGL}_{n+1}(\mathbb{C})$ is the diagonal subgroup D_n. The centralizer of $\iota(H)$ in G is $\iota(D_n)$, since, by assumption, d is no multiple of k. This implies $\varphi(D_n) = \iota(D_n)$. The normalizer of D_n in $\text{PGL}_{n+1}(\mathbb{C})$ is $D_n \rtimes \mathcal{S}_{n+1}$ and the normalizer of $\iota(D_n)$ in G is $\iota(D_n \rtimes \mathcal{S}_{n+1})$. Hence we obtain $\varphi(D_n \rtimes \mathcal{S}_{n+1}) = \iota(D_n \rtimes \mathcal{S}_{n+1})$ and since both ι and φ are sections,

$$\varphi|_{D_n \rtimes \mathcal{S}_{n+1}} = \iota|_{D_n \rtimes \mathcal{S}_{n+1}}.$$

Let $g := (x_0 + x_1, x_1, \ldots, x_n) \in \text{PGL}_{n+1}(\mathbb{C})$. Let $E \subset D_n$ be the centralizer of g in D_n. So E is the subgroup of elements of the form

$$(c_0 x_0, c_0 x_1, c_2 x_2, \ldots, c_n x_n)$$

with $c_i \in \mathbb{C}^*$ such that $c_0^2 c_2 c_3 \cdots c_n = 1$. Denote $\varphi(g) = (v, g)$, for some $v \in \mathbb{C}[x_0, \ldots, x_n]_k$. Take a $d \in E$. Then $\iota(d) = (0, d)$ and $(0, d)(v, g)(0, d^{-1}) = (v \circ d^{-1}, g)$ yields $v \circ d^{-1} = v$. Therefore, all summands of v are of the form $x_0^{r_0} x_1^{r_1}(x_0 \cdots x_n)^s$, where $r_0 + r_1 = s$. Assume that

$$v = \sum_{s, r_0 + r_1 = s} a_{r_0 r_1 s} x_0^{r_0} x_1^{r_1}(x_0 \cdots x_n)^s.$$

We calculate

$$(v, g)^2 = (v + v \circ g^{-1}, g^2),$$

and

$$v + v \circ g^{-1} = \sum_{s, r_0 + r_1 = s} a_{r_0 r_1 s} x_0^{r_0} x_1^{r_1}(x_0 \cdots x_n)^s$$

$$+ \sum_{s, r_0 + r_1 = s} a_{r_0 r_1 s}(x_0 - x_1)^{r_0} x_1^{r_1}((x_0 - x_1) x_1 \cdots x_n)^s.$$

On the other hand, we have $g^2 = f \circ g \circ f^{-1}$ for $f = (x_0, 1/2 x_1, 2x_2, x_3, \ldots, x_n)$. Hence

$$(0, f)(v, g)(0, f^{-1}) = (v, g)^2 = (v + v \circ g^{-1}, g^2),$$

and therefore

$$v + v \circ g^{-1} = \sum_{s, r_0 + r_1 = s} 2^{r_1} a_{r_0 r_1 s} x_0^{r_0} x_1^{r_1}(x_0 \cdots x_n)^s$$

This yields $a_{r_0 r_1 s} = 0$ for all r_1, r_2 and s and thus $\varphi(g) \in \iota(\mathrm{PGL}_{n+1}(\mathbb{C}))$.

Since $\mathrm{PGL}_{n+1}(\mathbb{C})$ is generated as an abstract group by $D_n \rtimes \mathcal{S}_{n+1}$ and g, this yields $\varphi(\mathrm{PGL}_{n+1}(\mathbb{C})) = \iota(\mathrm{PGL}_{n+1}(\mathbb{C}))$, which finishes the prove. $\qquad\square$

Lemma 3.4.8. $\mathrm{PGL}_{n+1}(\mathbb{C})$ *acts non-trivially on the fibration* $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k))$ *with basis* $\mathbb{P}^n$ *if and only if* $k = l(n+1)$ *for some nonnegative* l. *Moreover, in this case the action is unique up to conjugation and up to algebraic automorphisms of* $\mathrm{PGL}_{n+1}(\mathbb{C})$.

Proof. Let $\phi\colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{Aut}^0(\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k)))$ be an algebraic embedding. By Proposition 3.4.3, there exists an exact sequence of algebraic homomorphisms

$$1 \to \mathbb{C}[x_0, \dots, x_n]_k \to \mathrm{Aut}^0(\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k))) \to \mathrm{GL}_{n+1}(\mathbb{C})/\mu_k \to 1.$$

If ϕ is non-trivial, this induces a non-trivial algebraic homomorphism from $\mathrm{PGL}_{n+1}(\mathbb{C})$ into $\mathrm{GL}_{n+1}(\mathbb{C})/\mu_k$ and by Lemma 3.4.6 this is possible if and only if $(n+1) \mid k$. So assume that $k = l(n+1)$ for an integer l. It remains to show that in this case ϕ is unique up to conjugation and up to algebraic automorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$. Let

$$F_l := \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k)).$$

We look at F_l as a $\mathbb{P}^1$-fibration over the basis $\mathbb{P}^n$. So there is an exact sequence

$$1 \to \mathrm{Aut}^0_{\mathbb{P}^n}(F_l) \to \mathrm{Aut}^0(F_l) \xrightarrow{\pi} \mathrm{PGL}_n(\mathbb{C}) \to 1.$$

Here, $\mathrm{Aut}^0_{\mathbb{P}^n}(F_l) \simeq \mathbb{C}^* \ltimes \mathbb{C}[x_0, \dots, x_n]_k$ denotes the subgroup of automorphisms of F_l that fix the basis $\mathbb{P}^n$ pointwise.

Let $H := \mathrm{PGL}_{n+1}(\mathbb{C})$. By Lemma 3.4.4, the sections of π up to conjugation are in bijection with

$$H^1(H, \mathrm{Aut}^0_{\mathbb{P}^n}(F_l)) = H^1(H, \mathbb{C}[x_0, \dots, x_n]_k \rtimes \mathbb{C}^*/\mu_k).$$

By Proposition 3.4.5, there is an exact sequence of pointed sets

$$H^1(H, \mathbb{C}[x_0, \dots, x_n]_k) \to H^1(H, \mathrm{Aut}_{\mathbb{P}^n}(F_l)) \to H^1(H, \mathbb{C}^*/\mu_k).$$

The action of H on $\mathbb{C}^*/\mu_k$ is trivial, so $H^1(H, \mathbb{C}^*/\mu_k)$ is the set of homomorphisms $H \to \mathbb{C}^*/\mu_k$. Hence $H^1(H, \mathbb{C}^*/\mu_m) = \{1\}$. By Lemma 3.4.7, we obtain $H^1(H, \mathbb{C}[x_0, \dots, x_n]_k) = \{1\}$ and thus $H^1(H, \mathrm{Aut}_{\mathbb{P}^n}(F_l)) = \{1\}$. So, all sections of π are conjugate.

Now, since H is simple and not contained in $\mathrm{Aut}^0_{\mathbb{P}^n}(F_l)$, we obtain $\pi \circ \phi(H) \subset H$. Both ϕ and π are are algebraic morphisms, so $\pi \circ \phi(H) = H$. Therefore, up to the algebraic automorphism $\pi \circ \phi$, the homomorphism ϕ is a section of π. $\qquad\square$

Non conjugacy

It remains to show that the actions from Theorem 3.4.1 are not birationally conjugate.

Let M be a variety of dimension $n+1$ on which $\mathrm{PGL}_{n+1}(\mathbb{C})$ acts faithfully.

If M is not rational, then M is isomorphic to $\mathbb{P}^n \times C$ for some smooth curve C. Recall that $\mathbb{P}^n \times C$ is birationally equivalent to $\mathbb{P}^n \times C'$ for smooth curves C and

C' if and only if C and C' are birationally equivalent which again implies that C and C' are isomorphic. So, if $\mathrm{PGL}_{n+1}(\mathbb{C})$ acts rationally and non trivially on a non rational variety M of dimension $n+1$, then this one is uniquely determined up to algebraic automorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$ and up to birational conjugation in $\mathrm{Bir}(M)$.

In the case that M is rational, we have to show that the $\mathrm{PGL}_{n+1}(\mathbb{C})$-actions listed in Theorem 3.4.1 are not conjugate to each other. For this, note that none of them has an orbit of codimension ≥ 1. Lemma 3.3.2 induces therefore that any birational transformation conjugating one action to another one must be an isomorphism. As the varieties listed in Theorem 3.4.1 are not isomorphic we conclude that the actions are not conjugate.

3.5 Extension to Cr_n and H_n

In this section we study how the $\mathrm{PGL}_{n+1}(\mathbb{C})$-actions described in the above section extend to rational Cr_n-actions. Our goal is to prove Theorem 3.1.11. We proceed case by case.

3.5.1 The case $\mathbb{G}(1,3)$

Let

$$s_1 := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{ and } s_2 := \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{Z}).$$

Lemma 3.5.1. *Let G be a group. There exists no homomorphism $\rho\colon \mathrm{GL}_3(\mathbb{Z}) \to G$ such that $\rho(s_1)$ has order 3 and $s_2 \in \ker(\rho)$.*

Proof. Assume that such a ρ exists. Let

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, B := \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, T := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{Z}).$$

One calculates $(A(s_2(Bs_2B^{-1}))A^{-1}) = s_1T$. So s_1T is contained in the kernel of ρ and we get $\rho(T) = \rho(s_1^{-1})$. But this is a contradiction since the order of T is 2. □

The following construction comes up in the context of tetrahedral line complexes (see [Dol12]). Consider the 4 hyperplanes in $\mathbb{P}^3$

$$E_0 := \{x_0 = 0\},\ E_1 := \{x_1 = 0\},\ E_2 := \{x_2 = 0\},\ E_3 := \{x_3 = 0\}.$$

A line $l \in \mathbb{G}(1,3)$ that is not contained in any of the E_i, intersects each plane E_i in one point p_i. We thus obtain a rational surjective map

$$cr\colon \mathbb{G}(1,3) \dashrightarrow \mathbb{P}^1$$

that is defined by associating to the line l the cross ratio between the points p_i.

The closure $\overline{cr^{-1}([a:b])}$ in $\mathbb{G}(1,3)$ is irreducible if and only if $[a:b] \in \mathbb{P}^1 \setminus \{[0:1], [1:0], [1:1]\}$, whereas $cr^{-1}([a:b])$ consists of two irreducible components in all the other cases ([Dol12, Chapter 10.3.6]).

Recall that α is the automorphism of PGL_4 given by $g \mapsto {}^t g^{-1}$.

Proposition 3.5.2. *There exists no non-trivial group homomorphism*

$$\Phi \colon \langle \mathrm{PGL}_4(\mathbb{C}), \mathrm{W}_3 \rangle \to \mathrm{Bir}(\mathbb{G}(1,3))$$

such that $\Phi(\mathrm{PGL}_4(\mathbb{C})) \subset \mathrm{Aut}(\mathbb{G}(1,3))$.

In particular, neither the action of $\mathrm{PGL}_4(\mathbb{C})$ *on* $\mathbb{G}(1,3)$ *given by the embedding* φ_G *(see Example* 3.1.10*) nor the action given by* $\varphi_G \circ \alpha$ *can be extended to a rational action of* Cr_4.

Proof. Lemma 3.5.14 implies that if $\mathrm{PGL}_4(\mathbb{C})$ is contained in the kernel of a homomorphism $\Phi \colon \langle \mathrm{PGL}_4(\mathbb{C}), \mathrm{W}_3 \rangle \to \mathrm{Bir}(\mathbb{G}(1,3))$, then Φ is trivial. So we may assume that Φ embeds $\mathrm{PGL}_4(\mathbb{C})$ into $\mathrm{Aut}^0(\mathbb{G}(1,3))$. By Theorem 3.4.1, it is therefore enough to show that φ_G and $\varphi_G \circ \alpha$ do not extend to a homomorphism of $\langle \mathrm{PGL}_4(\mathbb{C}), \mathrm{W}_3 \rangle$.

The $\varphi_G(D_3)$-orbit of a line that is not contained in one of the planes E_i and that does not pass through any of the points $[1:0:0:0], [0:1:0:0], [0:0:1:0], [0:0:0:1]$, has dimension 3 and these are all $\varphi_G(D_3)$-orbits of dimension 3.

Since $\varphi_G(D_3)$ stabilizes the hyperplanes E_i and since the cross ratio is invariant under linear transformations, we obtain that cr is $\varphi_G(D_3)$-invariant. By the above remark, the rational map cr therefore parametrizes all but finitely many $\varphi_G(D_3)$-orbits of dimension 3 by $\mathbb{P}^1 \setminus \{[0:1], [1:0], [1:1]\}$.

The image $\varphi_G(\mathcal{S}_4)$, where $\mathcal{S}_4 \subset \mathrm{PGL}_4$ is the subgroup of coordinate permutations, normalizes $\varphi_G(D_3)$ and therefore it permutes its 3-dimensional orbits. Since $\mathcal{S}_4$ permutes the hyperplanes E_i, we can describe its action on the 3-dimensional $\varphi_G(D_3)$-orbits by its action on the cross ratio of the intersection of general lines with the planes E_i.

Let r be the cross ratio between the points p_0, p_1, p_2, p_3 on a line. One calculates that the cross ratio between p_3, p_1, p_2, p_0 is again r and that the cross ratio between the points p_2, p_0, p_1, p_3 is $\frac{1}{1-r}$. Hence the image of $\tau_1 := [x_3 : x_1 : x_2 : x_0]$ leaves cr invariant, whereas for the permutation $\tau_2 := [x_2 : x_0 : x_1 : x_3]$ we have $cr \circ \varphi(\tau_2) \neq cr$ and $cr \circ \varphi(\tau_2)^2 \neq cr$.

Let $f \colon \mathbb{G}(1,3) \dashrightarrow \mathbb{P}^4$ be a birational transformation and let $\varphi'_G := f \circ \varphi_G \circ f^{-1}$. The image $\varphi'_G(D_3) \subset \mathrm{Cr}_4$ is an algebraic torus of rank 3 and therefore, by Proposition 3.2.8, conjugate to the standard subtorus $D_3 \subset D_4$ of rank 3. In other words, there exists a rational map $\mathbb{P}^4 \dashrightarrow \mathbb{P}^1$ whose fibers consist of the closure of the $\varphi'_G(D_3)$-orbits. The image $\varphi'_G(\mathcal{S}_4)$ permutes the torus orbits, hence we obtain a homomorphism $\rho \colon \mathcal{S}_4 \to \mathrm{PGL}_2(\mathbb{C})$. By what we observed above, the permutation τ_1 is contained in the kernel of ρ, whereas the image $\rho(\tau_2)$ has order 3. The matrix representation in $\mathrm{GL}_3(\mathbb{Z})$ of τ_1 corresponds to s_1 and the matrix representation of τ_2 corresponds to s_2.

It follows now from Lemma 3.5.1 that ρ can not be extended to a homomorphism from $\mathrm{GL}_3(\mathbb{Z}) \simeq \mathrm{W}_3$ to $\mathrm{PGL}_2(\mathbb{C})$, which implies that there exists no homomorphism

$\Phi\colon \langle\mathrm{PGL}_4(\mathbb{C}), \mathrm{W}_3\rangle \to \mathrm{Cr}_4$ such that $\Phi(\mathrm{PGL}_4(\mathbb{C})) = \varphi'_G(\mathrm{PGL}_4(\mathbb{C}))$, since W_3 normalizes the torus and its image would therefore permute the torus orbits as well. The statement follows. $\square$

3.5.2 The case $\mathbb{P}(T\mathbb{P}^2)$

Recall that matrices of order two in $\mathrm{PGL}_2(\mathbb{C})$ have the form

$$\begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 1 & b \\ c & -1 \end{bmatrix}, \text{ where } a \in \mathbb{C}^*,\, b, c \in \mathbb{C},\, bc \neq -1. \qquad (3.1)$$

Proposition 3.5.3. *The embedding $\varphi_B\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}(T\mathbb{P}^2))$ extends in a unique way to an embedding*

$$\Phi_B\colon \mathrm{Cr}_2 \to \mathrm{Bir}(\mathbb{P}(T\mathbb{P}^2)).$$

Proof. We show that every extension coincides with the one given in Example 3.1.9. For this it is enough to show that the image of σ is uniquely determined. Assume that there is an extension $\psi\colon \mathrm{Cr}_2 \to \mathrm{Bir}(\mathbb{P}(T\mathbb{P}^2))$ of φ_B. Our goal is to show $\psi(\sigma) = \Psi_B(\sigma)$.

Let $d \in D_2$, $d = (ax_1, bx_2)$ with respect to affine coordinates given by $x_0 = 1$. Then $\varphi_B(d) = (ax_1, bx_2, (b/a)x_3)$, with respect to suitable local affine coordinates of $\mathbb{P}(T\mathbb{P}^2)$. Let $\phi\colon \mathbb{P}(T\mathbb{P}^2) \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^1$ be the birational map given by

$$\phi\colon (x_1, x_2, x_3) \mapsto (x_1, x_2, \frac{x_1}{x_2}x_3),$$

with respect to affine coordinates.

Let $\psi_1\colon \mathrm{Cr}_2 \to \mathrm{Bir}(\mathbb{P}^2 \times \mathbb{P}^1)$ be the algebraic embedding $\psi_1 = \phi \circ \psi \circ \phi^{-1}$. This gives us a $\mathbb{P}^2$-fibration, which we call the horizontal fibration, and a $\mathbb{P}^1$-fibration, which we call the vertical fibration. The image $\psi_1(D_2)$ acts canonically on the first factor and leaves the second one invariant. The horizontal fibers thus consist of the closures of D_2-orbits. Since W_2 normalizes D_2, the image $\psi_1(\mathrm{W}_2)$ permutes the orbits of $\psi_1(D_2)$. Hence it preserves the horizontal fibration and we obtain a homomorphism

$$\rho\colon \mathrm{W}_2 \simeq \mathrm{GL}_2(\mathbb{Z}) \to \mathrm{Bir}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbb{C}).$$

In what follows we identify W_2 with $\mathrm{GL}_2(\mathbb{Z})$.

The images of the three transpositions in $\mathcal{S}_3 = \mathrm{W}_2 \cap \mathrm{PGL}_3(\mathbb{C})$ under ρ are:

$$\rho\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \rho\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

$$\text{and } \rho\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}.$$

The image $\rho(\sigma)$ is either the identity or it has order 2. The elements of the form (3.1) do not commute with the images of $\mathcal{S}_3$ described above. Since σ is contained in the center of W_2, we obtain $\rho(\sigma) = \mathrm{id}$.

It remains to show that the action of $\psi_1(\sigma)$ on the first factor of $\mathbb{P}^2 \times \mathbb{P}^1$ is the standard action. Let $M = \mathbb{P}^2$ be a horizontal fiber. It is stabilized by $\psi(D_2)$ and $\psi(\sigma)$, so we obtain a homomorphism

$$\gamma\colon \langle D_2, \sigma \rangle \to \mathrm{Bir}(M) = \mathrm{Cr}_2 \,.$$

Since $\sigma d \sigma^{-1} = d^{-1}$ for all $d \in D_2$, there exists a $d \in D_2$ such that $\gamma(\sigma) = d\sigma$. This is true for all horizontal fibers, so $\psi_1(\sigma)$ induces an automorphism of $U \times \mathbb{P}^1$, where

$$U = \{[x_0 : x_1 : x_2] \mid x_0, x_1, x_2 \neq 0\} \subset \mathbb{P}^2 \,.$$

Let $S \simeq \mathbb{P}^1 \subset U \times \mathbb{P}^1$ be a vertical fiber and $\pi\colon U \times \mathbb{P}^1 \to U$ the projection onto the first factor. Then $\pi \circ \psi_1(\sigma)(S)$ is a regular map from $\mathbb{P}^1$ to the affine set U and is therefore constant. We obtain that $\psi_1(\sigma)$ preserves the vertical fibration.

The image $\psi_1(\mathrm{PGL}_3(\mathbb{C}))$ preserves the vertical fibration as well and projection onto $\mathbb{P}^2$ yields a homomorphism from $\mathrm{PGL}_3(\mathbb{C})$ to Cr_2 that is the standard embedding. Hence $\psi_1(\mathrm{Cr}_2)$ preserves the vertical fibration and we obtain an algebraic homomorphism from Cr_2 to Cr_2, which is uniquely determined by its restriction to $\mathrm{PGL}_3(\mathbb{C})$ by Corollary 3.5.11 in the appendix. So the image $\psi_1(\sigma)$ is uniquely determined. $\qquad\square$

Proposition 3.5.4. *There exists no homomorphism $\Phi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_3$ such that*

$$\Phi|_{\mathrm{PGL}_3(\mathbb{C})} = \varphi_B \circ \alpha,$$

where φ_B denotes the embedding of $\mathrm{PGL}_3(\mathbb{C})$ into Cr_3 from Example 3.1.9 and α the algebraic automorphism of $\mathrm{PGL}_3(\mathbb{C})$ given by $g \mapsto {}^t g^{-1}$.

Proof. Assume that such an extension $\Phi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_3$ of $\varphi_B \circ \alpha$ exists.

Observe that $\alpha(D_2) = D_2$ and that $\alpha|_{\mathcal{S}_3} = \mathrm{id}_{\mathcal{S}_3}$. Therefore, we can repeat the same argument as in the proof of Proposition 3.5.3 to obtain $\Psi(\sigma) = \Phi_B(\sigma)$. But we have

$$\Psi(\sigma)\Psi(g)\Psi(\sigma)\Psi(g)\Psi(\sigma)\Psi(g) \neq \mathrm{id}$$

for $g = [z - x : z - y : z]$ - this contradicts the relations in Cr_2 (Proposition 3.5.12). $\qquad\square$

3.5.3 The case $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k(n+1)))$

Proposition 3.5.5. *The algebraic homomorphism $\varphi_l\colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{Bir}(F_l)$ extends uniquely to the embedding*

$$\Psi_l\colon \mathrm{H}_n \to \mathrm{Bir}(F_l) \text{ (see Example 3.1.8)}.$$

Proof. Suppose that there is an extension $\psi\colon \mathrm{H}_n \to \mathrm{Bir}(F_l)$ of φ_l. We will show that ψ is unique and therefore that $\psi = \Psi_l$.

Let $(x_1, \ldots, x_{n-1}, w)$ be affine coordinates of F_l such that for every $g \in \mathrm{PGL}_{n+1}(\mathbb{C})$ the image $\varphi_l(g)$ acts by

$$(x_1, \ldots, x_n, w) \mapsto (g(x_1, \ldots, x_n), J(g(x_1, \ldots, x_n))^{-l} w).$$

In particular, the image under ψ of $(d_1 x_1, \ldots, d_n x_n) \in D_n$ acts by

$$(x_1, \ldots, x_n, w) \mapsto (d_1 x_1, \ldots, d_n x_n, (d_1 \cdots d_n)^{-l} w).$$

Define $\phi \colon F_l \to \mathbb{P}^n \times \mathbb{P}^1$ by

$$\phi \colon (x_1, \ldots, x_n, w) \mapsto (x_1, \ldots, x_n, (x_1 \cdots x_n)^l w)$$

with respect to affine coordinates. Let $\psi_1 \colon \mathrm{Cr}_n \to \mathrm{Bir}(\mathbb{P}^n \times \mathbb{P}^1)$ be the algebraic embedding $\psi_1 := \phi \circ \psi \circ \phi^{-1}$. Then the image $\psi_1(D_n)$ acts canonically on the first factor and leaves the second one invariant. Since W_n normalises D_n, the image $\psi_1(\mathrm{W}_n)$ permutes the orbits of $\psi_1(D_n)$. Hence $\psi_1(\mathrm{W}_n)$ preserves the horizontal fibration. This induces a homomorphism

$$\rho \colon \mathrm{W}_n \simeq \mathrm{GL}_n(\mathbb{Z}) \to \mathrm{PGL}_2(\mathbb{C}).$$

In what follows, we identify W_n with $\mathrm{GL}_n(\mathbb{Z})$. Let $\mathcal{A}_{n+1} \subset \mathcal{S}_{n+1} \subset \mathrm{PGL}_{n+1}(\mathbb{C})$ be the subgroup of coordinate permutations $s \in \mathcal{S}_{n+1}$ such that $J(s) = 1$. Hence $\mathcal{A}_{n+1} \in \ker(\rho)$. Note that the fixed point set of $\psi_1(\mathcal{A}_{n+1})$ is the vertical fiber

$$L := [1 : \cdots : 1] \times \mathbb{P}^1 \subset \mathbb{P}^n \times \mathbb{P}^1.$$

Since σ_n commutes with $\mathcal{A}_{n+1}$, the image $\psi_1(\sigma_n)$ stabilises L. The group $\psi_1(D_n)$ acts transitively on an open dense subset of vertical fibers that contains L. Since $\psi_1(\sigma_n)$ normalizes $\psi_1(D_n)$, we obtain that $\psi_1(\sigma_n)$ preserves the vertical fibration. Therefore $\langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle$ preserves the vertical fibration. We obtain a homomorphism $\langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle \to \mathrm{Cr}_n$, which is, by Corollary 3.5.11 and Remark 3.5.1 in the appendix, the standard embedding.

Let

$$f_A = (\frac{1}{x_1}, x_2, \ldots, x_n).$$

In [BH14] it is shown that f_A is contained in $\langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle$, which implies that $\psi_1(f_A)$ preserves the vertical fibration and that its action on $\mathbb{P}^n$ is the standard action.

Recall that $(h f_A)^3 = \mathrm{id}$ for $h = (1 - x_1, x_2, \ldots, x_{n-1}) \in \mathrm{Cr}_n$. The image $\psi_1(h)$ is

$$\psi_1(h) \colon (x_1, \ldots, x_n, z) \mapsto (1 - x_1, x_2, \ldots, x_n, (-1)^l z).$$

Denote by $A \in \mathrm{GL}_n(\mathbb{Z})$ the integer matrix corresponding to f_A. We have $\rho(A) = \mathrm{id}$ or $\rho(A)$ is of order two, i.e. has the form (3.1).

Suppose that $\rho(A) = \mathrm{id}$. Then

$$\psi_1(f_A) \colon (x_1, \ldots x_n, z) \mapsto (\frac{1}{x_1}, x_2 \ldots x_n, z).$$

The relation $(h f_A)^3 = \mathrm{id}$ then implies that l is even.

Suppose that

$$\rho(f_A) = \begin{bmatrix} 1 & b \\ c & -1 \end{bmatrix}, \quad \text{where } b, c \in \mathbb{C}, bc \neq -1,$$

hence

$$\psi_1(f_A)\colon (x_1,\ldots x_n, z) \mapsto \left(\frac{1}{x_1}, x_2 \ldots x_n, \frac{z+b}{cz-1}\right)$$

and therefore

$$\psi_1(hf_A) = (x_1,\ldots x_n, z) \mapsto \left(1 - \frac{1}{x_1}, \ldots x_n, \frac{(-1)^l z + (-1)^l b}{cz-1}\right),$$

One calculates that if l is even, then the relation $(hf_A)^3 = \mathrm{id}$ is not satisfied. So assume that l is odd. This gives

$$\psi_1(hf_A)^3 = (x_1,\ldots x_n, z) \mapsto \left(x_1,\ldots x_n, \frac{a_1 z + a_2}{a_3 z - a_4}\right),$$

where $a_1 = 3bc - 1, a_2 = (bc-1)b - 2b, a_3 = (1-bc)c + 2c$ and $a_4 = 3bc - 1$. So $(hf_A)^3 = \mathrm{id}$ yields either l odd and $b = c = 0$ or l odd and $bc = 3$. However, the latter is not possible. Consider the transformation

$$\tau = (x_1,\ldots, x_{n-2}, x_n, x_{n-1}) \in \mathcal{S}_n.$$

We have $f_A \tau = \tau f_A$. Note that

$$\psi_1(\tau)\colon (x_1,\ldots, x_n, z) \mapsto (x_1,\ldots, x_{n-2}, x_n, x_{n-1},\ldots, x_n, (-1)^l z)$$

and this transformation does not commute with $\left(x_1,\ldots x_n, \frac{a_1 z + a_2}{a_3 z - a_4}\right)$ in the second case. Hence $c = b = 0$ and l is odd.

Finally, assume that

$$\rho(f_A) = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}, \text{ where } a \in \mathbb{C}^*.$$

This implies

$$\psi_1(f_A)\colon (x_1,\ldots x_n, z) \mapsto \left(\frac{1}{x_1}, x_2 \ldots x_n, \frac{1}{az}\right)$$

and hence $\psi(hf_A)^3 \neq \mathrm{id}$.

We conclude that

$$\rho(f_A) = \begin{bmatrix} 1 & 0 \\ 0 & (-1)^l \end{bmatrix}$$

and therefore that the action of $\psi(f_A)$ is uniquely determined by l. Hence

$$\psi|_{\langle \mathrm{PGL}_n(\mathbb{C}), \sigma_{n-1}\rangle} = \Psi_l|_{\langle \mathrm{PGL}_n(\mathbb{C}), \sigma_{n-1}\rangle}.$$

Let f_B, f_C, f_D and $f_E \in \mathrm{Cr}_n$ be as in the proof of Corollary 3.5.10. By Lemma 3.5.13 it remains to show that the image $\psi(f_B)$ is uniquely determined. We use once more the relation

$$f_B = f_D f_C f_E f_D^{-1}.$$

Since $\rho(CE) = \mathrm{id}$ and since f_D has order two, we obtain $\rho(B) = \mathrm{id}$.

Let $c \in \mathbb{P}^1$ such that the restriction of $\psi_1(f_B)$ to the hyperplane

$$\{c\} \times \mathbb{P}^n \subset \mathbb{P}^1 \times \mathbb{P}^n$$

is a birational map. Then the restriction of $\psi_1(f_B)$ to $\{c\} \times \mathbb{P}^n$ has to fulfill the relations with the group $\langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle$. By Corollary 3.5.11 we obtain that this restriction has to be f_B. Hence the image $\psi_1(f_B)$ is unique. $\square$

Proposition 3.5.6. *There exists no group homomorphism $\psi \colon \mathrm{H}_n \to \mathrm{Bir}(F_l)$ such that $\psi|_{\mathrm{PGL}_{n+1}(\mathbb{C})} = \varphi_l \circ \alpha$.*

Proof. Assume that such an extension $\psi \colon \mathrm{H}_n \to \mathrm{Cr}_n$ exists. Let ϕ be as in Proposition 3.5.5 and $\psi_2 \colon \mathrm{H}_n \to \mathrm{Bir}(\mathbb{P}^1 \times \mathbb{P}^n)$,

$$\psi_2 := \phi \circ \varphi_l \circ \alpha \circ \phi^{-1}.$$

Similarly as in the proof of Proposition 3.5.5 one can show that $\psi_2(\sigma_n)$ preserves the vertical fibration. In that way we obtain an algebraic homomorphism

$$A \colon \langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma \rangle \to \mathrm{Cr}_n$$

such that $A|_{\mathrm{PGL}_{n+1}(\mathbb{C})} = \alpha$. Such a homomorphism does not exist by Corollary 3.5.11. $\square$

3.5.4 The case $C \times \mathbb{P}^n$

Throughout this section, C denotes a projective curve. For the proof of Theorem 3.4.1 it is enough to consider non rational curves, however, the following propositions hold in the more general case.

Proposition 3.5.7. *The embedding $\varphi_C \colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{Bir}(C \times \mathbb{P}^n)$ extends uniquely to the standard embedding*

$$\Phi_C \colon \mathrm{H}_n \to \mathrm{Bir}(C \times \mathbb{P}^n) \text{ (see Example 3.1.7).}$$

Proof. Let $\pi \colon C \times \mathbb{P}^n \to C$ be the first projection. Suppose that there is an extension $\Psi \colon \mathrm{H}_n \to \mathrm{Bir}(C \times \mathbb{P}^n)$ of φ_C. By definition, $\Psi(\mathrm{PGL}_{n+1}(\mathbb{C}))$ fixes the fibers of π. Moreover, each fiber of π is a closure of a $\Psi(D_n)$-orbit. Since the elements of W_n commute with D_n, we conclude that $\Psi(\mathrm{W}_n)$ preserves the fibration given by π. Hence H_n preserves the fibration given by π and we obtain a homomorphism

$$\rho \colon \mathrm{H}_n \to \mathrm{Bir}(C)$$

such that $\mathrm{PGL}_{n+1}(\mathbb{C}) \subset \ker(\rho)$. In the Appendix it is shown that the normal subgroup generated by $\mathrm{PGL}_{n+1}(\mathbb{C})$ in H_n is all of H_n. Hence ρ is trivial and $\Psi(\mathrm{H}_n)$ preserves every fiber of π. The restriction $\Psi(\mathrm{H}_n)|_{c \times \mathbb{P}^n}$ for any $c \in C$ defines a homomorphism from H_n to Cr_n such that the restriction to $\mathrm{PGL}_{n+1}(\mathbb{C})$ is the standard embedding. By Corollary 3.5.11, this is the standard embedding. Hence Ψ is unique. $\square$

Proposition 3.5.8. *There exists no group homomorphism* $\Psi\colon H_n \to \mathrm{Bir}(\mathbb{C} \times \mathbb{P}^n)$ *such that* $\Psi|_{\mathrm{PGL}_{n+1}(\mathbb{C})} = \varphi_C \circ \alpha$.

Proof. Assume there exists such a Ψ. As in the proof of Proposition 3.5.7 one can show that $\Psi(H_n)$ fixes the horizontal fibration. The restriction $\Psi(H_n)|_{c \times \mathbb{P}^{n-1}}$ defines for each $c \in C$ a homomorphism from H_n to Cr_n such that the restriction to $\mathrm{PGL}_{n+1}(\mathbb{C})$ is given by $g \mapsto \alpha(g)$. By Corollary 3.5.11, there exists no such homomorphism. $\qquad\square$

3.5.5 Proof of Theorem 3.1.11

Statement (1) is the content of Theorem 3.4.1. Statement (2) follows from Proposition 3.5.2. Statement (3) and (4) follow from the Propositions 3.5.3, 3.5.4, 3.5.5, 3.5.6, 3.5.7 and 3.5.8. Note that the Propositions 3.5.2, 3.5.4, 3.5.6 and 3.5.8 show that the regular actions of $\mathrm{PGL}_{n+1}(\mathbb{C})$ that do not extend to a rational action of Cr_n also do not extend to a rational action of H_n.

Appendix

3.5.6 Embeddings of H_n into Cr_n

Let $\gamma\colon \mathbb{C} \to \mathbb{C}$ be a field homomorphism. By acting on the coordinates, γ induces a bijective map $\Gamma\colon \mathbb{P}^n \to \mathbb{P}^n$. Conjugation with Γ yields a group homomorphism of Cr_n that preserves degrees. Observe that we obtain the image of $g \in \mathrm{Cr}_n$ by letting γ operate on the coefficients of g. By abuse of notation we denote this group homomorphism by γ as well. In [Dés06b] Déserti showed that all automorphisms of Cr_2 are inner up to field automorphisms of that type. A generalization of this result is the following theorem by Cantat:

Theorem 3.5.9 ([Can14])**.** *Let M be a smooth projective variety of dimension n and $r \in \mathbb{N}$. Let $\rho\colon \mathrm{PGL}_{r+1}(\mathbb{C}) \to \mathrm{Bir}(M)$ be a non-trivial group homomorphism. Then $n \geq r$ and if $n = r$ then M is rational and there exists a homomorphism of fields $\gamma\colon \mathbb{C} \to \mathbb{C}$ such that ρ is up to conjugation the standard embedding of $\mathrm{PGL}_{n+1}(\mathbb{C})$ into Cr_n followed by the group homomorphism $\gamma\colon \mathrm{Cr}_n \to \mathrm{Cr}_n$.*

The goal of this appendix is to prove the following two corollaries of Theorem 3.5.9:

Corollary 3.5.10. *Let $n > m$ and let $\Phi\colon \mathrm{Cr}_n \to \mathrm{Cr}_m$ be a group homomorphism. Then the normal subgroup of Cr_n containing H_n is contained in the kernel of Φ.*

No such non-trivial homomorphism is known so far. In fact, the existence of a non-trivial homomorphism $\Phi\colon \mathrm{Cr}_n \to \mathrm{Cr}_m$ for $n > m \geq 2$ would imply that Cr_n is not simple, a question that is still open.

Let $\alpha\colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{PGL}_{n+1}(\mathbb{C})$ be the algebraic automorphism defined by $g \mapsto {}^t g^{-1}$.

Corollary 3.5.11. *Let $\Psi\colon H_n \to \mathrm{Cr}_n$ be a non-trivial group homomorphism. Then there exists a homomorphism of fields γ of $\mathbb{C}$ and an element $g \in \mathrm{Cr}_n$ such that $g\Psi g^{-1}$ is the standard embedding followed by the group homomorphism γ.*

Moreover, the extension of the standard embedding $\varphi\colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{Cr}_n$ as well as the extension of the embedding $\gamma \circ \varphi$, to the group H_n is unique, where γ is any field homomorphism of $\mathbb{C}$. The embedding $\varphi \circ \alpha$, does not extend to a homomorphism from H_n to Cr_n.

Remark 3.5.1. The proof of Corollary 3.5.11 shows that the statement of Corollary 3.5.11 holds as well if we replace the group H_n by the group $G_n \subset \mathrm{Cr}_n$, i.e. the group generated by $\mathrm{PGL}_{n+1}(\mathbb{C})$ and the element $\sigma_n := (x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1})$.

By the theorem of Noether and Castelnuovo, Corollary 3.5.11 implies in particular the theorem of Déserti about automorphisms of Cr_2.

We often use the following relations between elements of the Cremona group:

Lemma 3.5.12. *In Cr_2 the following relations hold:*

(1) $\sigma\tau(\tau\sigma)^{-1} = \mathrm{id}$ *for all $\tau \in \mathcal{S}_3$,*

(2) $\sigma d = d^{-1}\sigma$ *for all diagonal maps $d \in D_2$ and*

(3) $(\sigma h)^3 = \mathrm{id}$ *for $h = [x_2 - x_0 : x_2 - x_1 : x_2]$.*

Proof. One checks the identities by a direct calculation. $\square$

Denote by $\mathrm{Cr}_n^0 \subset \mathrm{Cr}_n$ the subgroup consisting of elements that contract only rational hypersurfaces. We have $\mathrm{H}_n \subset \mathrm{Cr}_n^0$. On the other hand, it seems to be an interesting open question, whether there exist elements in Cr_n^0 that are not contained in H_n for any $n \geq 3$ (cf. [Lam14]).

Lemma 3.5.13. *The group H_n is generated by $\mathrm{PGL}_{n+1}(\mathbb{C})$ and the two birational transformations $\sigma_n := (x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1})$ and $f_B := (x_1 x_2, x_2, x_3, \ldots, x_n)$.*

Proof. It is known that $\mathrm{GL}_n(\mathbb{Z})$ is generated by the subgroup of permutation matrices in $\mathrm{GL}_n(\mathbb{Z})$ and the two elements

$$
A := \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}
$$

(see for example [dlH00, III.A.2]). Notice that f_B is the birational transformation in W_n corresponding to B. Let f_A be the birational transformation corresponding to A. In [BH14] it is shown that f_A is contained in $\langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle$. $\square$

Lemma 3.5.14. *The two birational transformations $\sigma_n := (x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1})$ and $f_B := (x_1 x_2, x_2, x_3, \ldots, x_n)$ are contained in the normal subgroup of H_n generated by $\mathrm{PGL}_{n+1}(\mathbb{C})$.*

Proof. Let

$$g_n := [x_n - x_0 : x_n - x_1 : \cdots : x_n - x_{n-1} : x_n] \in \mathrm{PGL}_{n+1}(\mathbb{C}).$$

Then $\sigma_n g_n \sigma_n g_n \sigma_n g_n = \mathrm{id}$. In particular, $\sigma_n g_n$ conjugates σ_n to g_n.

Let

$$C := \begin{pmatrix} -1 & 2 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \ D := \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

$$E := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathrm{GL}_n(\mathbb{Z})$$

and let f_C, f_D and f_E be the corresponding elements in W_n. It is shown in [BH14] that f_C is contained in $\langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle$. Moreover, one calculates that

$$f_B = f_D f_C f_E f_D^{-1},$$

which implies that f_B is conjugate to an element in $\langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle$. □

Proof of Corollary 3.5.10. The corollary follows directly from Lemma 3.5.13 and from Lemma 3.5.14 □

Proof of Corollary 3.5.11. By Theorem 3.5.9 we may assume that there exists a field homomorphism $\gamma \colon \mathbb{C} \to \mathbb{C}$, such that, up to conjugation, the restriction of Ψ to $\mathrm{PGL}_{n+1}(\mathbb{C})$ is the standard embedding composed with γ or the standard embedding composed with γ and the automorphism α of $\mathrm{PGL}_{n+1}(\mathbb{C})$ given by $\alpha(g) = {}^t g^{-1}$.

Therefore, after conjugation, the restriction of Ψ to D_2 is the standard embedding composed with γ. In particular, $\Psi(D_2) = \gamma(D_2)$ is dense in D_2 and therefore $\Psi(\mathrm{W}_n)$ is contained in $D_n \rtimes \mathrm{W}_n$. Assume that $\Psi(\sigma_n) = d\tau$ for some $d \in D_n$ and $\tau \in \mathrm{W}_n$. The relation $\Psi(\sigma_n)\Psi(e)\Psi(\sigma_n) = \Psi(e)^{-1}$ for all $e \in D_n$ implies $\Psi(\sigma_n)e\Psi(\sigma_n) = e^{-1}$ for all $e \in D_2$ and hence, $\tau = \sigma_n$. Note that the restriction of Ψ to $\mathcal{S}_{n+1}$ is the standard embedding. So for all $\tau \in \mathcal{S}_{n+1}$ we obtain

$$\tau d \sigma_n = d \sigma_n \tau = d \tau \sigma_n.$$

The only element in D_n that commutes with $\mathcal{S}_{n+1}$ is the identity. Hence we obtain $\Psi(\sigma_n) = \sigma_n$ (and this shows the statement from Remark 3.5.1).

Let g_n be as in the proof of Corollary 3.5.10. The relation $\sigma_n g_n \sigma_n g_n \sigma_n g_n = \mathrm{id}$ implies that $\Psi|_{\mathrm{PGL}_{n+1}(\mathbb{C})}$ is the standard embedding composed with γ, since

$$\sigma_n \alpha(g_n) \sigma_n \alpha(g_n) \sigma_n \alpha(g_n) \neq \mathrm{id}.$$

It remains to show that $\Psi(f_B) = f_B$. Let $d \in D_n$, and $\rho \in W_n$ such that $\Psi(f_B) = d\rho$. The image $\Psi(f_B)$ acts on $\Psi(D_n)$ by conjugation. The action of $\Psi(f_B)$ on $\Psi(D_n)$ is determined by ρ. Since $\Psi|_{D_n}$ is the standard embedding composed with γ, we obtain $\rho = f_B$. Let $d = (d_1 x_1, \ldots, d_n x_n)$. The image $\Psi(f_B)$ commutes with σ_n. We obtain

$$d^{-1} \sigma_n f_B = \sigma_n df_B = df_B \sigma_n = d\sigma_n f_B$$

and hence $d_i = \pm 1$ for all i.

The image $\Psi(f_B)$ commutes with all elements of $\mathcal{S}_{n+1}$ that fix the coordinates x_1 and x_2. Similarly as above, this yields that d commutes with all elements of $\mathcal{S}_{n+1}$ that fix the coordinates x_1 and x_2 and we get $d_i = 1$ for $i \neq 1$ and $i \neq 2$.

In [BH14] it is shown that f_B^2 is contained in $\langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle$. By what we proved above, this gives

$$\Psi(f_B^2) = f_B^2 = df_B df_B = dd' f_B^2,$$

where $d' = (d_1 d_2 x_1, d_2 x_2, \ldots, d_n x_n)$. So $dd' = \mathrm{id}$, which yields $d_1^2 d_2 = 1$ and therefore $d_2 = 1$. This means that we have either $\Psi(f_B) = f_B$ or $\Psi(f_B) = df_B$ with $d = (-x_1, x_2, \ldots, x_n)$.

Let

$$r_1 := [x_0 : x_1 : \cdots : x_{n-1} : x_n + x_1], r_2 := [x_n : x_1 : \cdots : x_{n-1} : x_0],$$

$$r_3 := [x_n : x_0 : x_2 : \cdots : x_{n-1} : x_1], t := [x_n : x_0 : \cdots : x_{n-1}].$$

We have the relation

$$(r_2 t f_B t^{-1} r_3) r_1 (r_2 t f_B t^{-1} r_3) = r_1$$

and therefore

$$(r_2 t \Psi(f_B) t^{-1} r_3) r_1 (r_2 t \Psi(f_B) t^{-1} r_3) = r_1.$$

One calculates that if $\Psi(f_B) = (-x_1, x_2, \ldots, x_n) f_B$ then this relation is not satisfied. Hence $\Psi(f_B) = f_B$. $\square$

3.5.7 The Weyl group is not preserved

There are many embeddings δ of the group of monomial maps in dimension 2 which is given by $W_2 \ltimes D_2$ into the group $W_n \ltimes D_n$ such that $\delta(D_2) \subset D_n$ and $\delta(W_2) \subset W_n$. Indeed, by a theorem of Popov ([Pop13]), every embedding $D_2 \to D_n$ is conjugate by elements of W_n to the standard embedding

$$\alpha \colon \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & 0 & \ldots & 0 \\ 0 & b & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & \ldots & & & 0 \\ 0 & \ldots & & 0 & 1 \end{pmatrix}.$$

Therefore, any group homomorphism $f \colon W_2 \to W_{n-2}$ induces a homomorphism $F \colon W_2 \to W_n$ given by

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & f(g) \end{pmatrix}.$$

We see now that F extends to an embedding $H\colon \mathrm{W}_2 \ltimes D_2 \to \mathrm{W}_n \ltimes D_n$ such that the restriction of H to D_2 is the standard embedding. Moreover, every embedding $\mathrm{W}_2 \ltimes D_2 \to \mathrm{W}_n \ltimes D_n$ that sends W_2 into W_n and D_2 into D_n is conjugate to an embedding of this form. The goal of this section is to prove Theorem 3.1.6.

Lemma 3.5.15. *Let $\mathcal{S}_n \subset \mathrm{GL}_n$ be the symmetric subgroup of permutation matrices. A matrix $(a_{ij}) \in M_n$ commutes with all elements in $\mathcal{S}_n$ if and only if there are $a, b \in \mathbb{C}$ such that $a_{ii} = a$ for all i and $a_{rs} = b$ for all $r \neq s$.*

Proof. All matrices of this form commute with all transpositions of $\mathcal{S}_n$ and therefore with $\mathcal{S}_n$. On the other hand, let a_{ij} be a matrix that commutes with all elements of $\mathcal{S}_n$. Let $\tau = (rs) \in \mathcal{S}_n$ be a transposition. Then $\tau(a_{ij}) = (a_{ij})\tau$ implies $a_{\tau(k)l} = a_{k\tau(l)}$ for all $1 \leq k, l \leq n$. So we get $a_{rr} = a_{ss}$ and $a_{rk} = a_{sk}$ as well as $a_{kr} = a_{ks}$ for all k. The result follows with $a := a_{11}$ and $b := a_{12}$. □

Lemma 3.5.16. *The homomorphism $\psi\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_6(\mathbb{C})$ cannot be extended to a homomorphism $\Psi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$ such that $\Psi(\sigma) \in D_5 \ltimes \mathrm{W}_5$.*

Proof. Assume that there is an element $tf_A \in \mathrm{W}_5$ of order two, where $t \in D_5$ and $f_A \in \mathrm{W}_5$, which satisfies the relations of Proposition 3.5.12. Let $A \in \mathrm{GL}_5(\mathbb{Z})$ be the integer matrix corresponding to f_A with respect to the affine coordinates $x_2 = 1$.

We look at the conditions that the relations (2)-(5) of Proposition 3.5.12 impose on tf_A. Note that $(\delta f_A)^2 = \mathrm{id}$ implies that $f_A^2 = \mathrm{id}$, that

$$tf_A \tau = \tau t f_A$$

for all $\tau \in \psi(\mathcal{S}_3)$ implies $f_A \tau = \tau f_A$ and that $tf_A d = d^{-1}tf_A$ for all $d \in \psi(D_2)$ implies $f_A d = d^{-1} f_A$.

First we observe that we can associate to each element $w \in \mathrm{W}_5$ a 6×6 integer matrix, i.e. a matrix $N = (n_{ij}) \in M_6(\mathbb{Z})$ with integer entries such that

$$w = [x_0^{n_{00}} x_1^{n_{01}} \cdots x_5^{n_{05}} : \cdots : x_0^{n_{50}} \cdots x_5^{n_{55}}].$$

On the other hand, each 6×6 integer matrix $N \in M_6(\mathbb{Z})$ yields a rational map $g_N\colon \mathbb{P}^5 \dashrightarrow \mathbb{P}^5$. For two matrices $N_1, N_2 \in M_6(\mathbb{Z})$ we have

$$g_{N_1} g_{N_2} = g_{N_1 N_2}.$$

Moreover $g_{N_1} = g_{N_2}$ if and only if we obtain N_1 by adding some integer row-vector of length 6 to every row of N_2. So for every element $w \in \mathrm{W}_5$ there is a unique matrix $N \in M_6(\mathbb{Z})$ with nonnegative entries and at least one zero in each column satisfying $w = g_N$. In this proof we will call an integer matrix of this form *reduced*.

Let now M be a 6×6 integer matrix such that $f_A = g_M$. Note that the matrix A can be obtained from M by subtracting the third row of M from all the other rows and erasing the third row and the third column. We assume M to be reduced. Let $s \in \mathcal{S}_6$ be a permutation of coordinates. The corresponding reduced matrix $S \in M_6(\mathbb{Z})$ is the permutation matrix corresponding to s. Notice that the matrices SM and MS are reduced again.

We look at M in block form

$$M := \left[\begin{array}{c|c} \alpha & \beta \\ \hline \gamma & \delta \end{array} \right] \in M_6(\mathbb{N}),$$

where $\alpha = (a_{ij})$, $\beta = (b_{ij})$, $\gamma = (c_{ij})$ and $\delta = (d_{ij})$ are 3×3 matrices. Let $\tau \in \mathcal{S}_3$. Then $\psi(\tau)$ is a permutation again. By relation (2), we get

$$\psi(\tau) f_A = f_A \psi(\tau).$$

Since the 6×6 integer matrices corresponding to $\psi(\tau) f_A$ and $f_A \psi(\tau)$ are both reduced, they are the same. Looking at them in block form, we obtain

$$\left[\begin{array}{c|c} \tau\alpha & \tau\beta \\ \hline \tau\gamma & \tau\delta \end{array} \right] = \left[\begin{array}{c|c} \alpha\tau & \beta\tau \\ \hline \gamma\tau & \delta\tau \end{array} \right].$$

By Lemma 3.5.15 it follows that there exist $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{Z}$ such that $a_{ii} = a_1$, $b_{ii} = b_1$, $c_{ii} = c_1$, $d_{ii} = d_1$ for all i and $a_{ij} = a_2$, $b_{ij} = b_2$, $c_{ij} = c_2$ and $d_{ij} = d_2$ for $i \neq j$. If we pass on to the matrix $A \in \mathrm{GL}_5(\mathbb{Z})$ with respect to affine coordinates given by $x_2 = 1$ we obtain:

$$A = \left[\begin{array}{cc|ccc} a & 0 & b_1 - b_2 & 0 & b_2 - b_1 \\ 0 & a & 0 & b_1 - b_2 & b_2 - b_1 \\ \hline c_1 & c_2 & d_1 - b_2 & d_2 - b_2 & d_2 - b_1 \\ c_2 & c_1 & d_2 - b_2 & d_1 - b_2 & d_2 - b_1 \\ c_2 & c_2 & d_2 - b_2 & d_2 - b_2 & d_1 - b_1 \end{array} \right]$$

for $a := a_1 - a_2$.

The image under ψ with respect to these affine coordinates of a diagonal element $[\alpha x_0 : \beta x_1 : \gamma x_2]$ is

$$\left(\frac{\alpha}{\gamma} x_0, \frac{\beta}{\gamma} x_1, \frac{\alpha}{\gamma} x_3, \frac{\beta}{\gamma} x_4, x_5 \right).$$

Relation (3) yields

$$\left(\frac{\alpha}{\gamma} \right)^a \left(\frac{\beta}{\gamma} \right)^{b_1 - b_2} = \frac{\gamma}{\alpha}.$$

We therefore get

$$a = -1 \text{ and } b_1 - b_2 = 0.$$

So the matrix A has the form

$$A = \left[\begin{array}{cc|ccc} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline c_1 & c_2 & d & e & e \\ c_2 & c_1 & e & d & e \\ c_2 & c_2 & e & e & d \end{array} \right],$$

where $d := d_1 - b_1, e := d_2 - b_2$.

Since A is of order two, we obtain

$$d^2 + 2e^2 = 1,$$

which implies $d = \pm 1$ and $e = 0$.

Using again relation (4), we obtain

$$\left(\frac{\alpha}{\gamma} \right)^{c_1 + d} \left(\frac{\beta}{\gamma} \right)^{c_2} = \frac{\gamma}{\alpha}.$$

Hence $c_2 = 0$ and $c_1 + d = -1$. If $d = -1$, this implies $c_1 = 0$ and thus

$$tf_A = \left(t_1 {x_0}^{-1}, t_2 {x_1}^{-1}, t_3 {x_3}^{-1}, t_4 {x_4}^{-1}, t_5 {x_5}^{-1}\right),$$

for some $t_i \in \mathbb{C}^*$. Whereas if $d = 1$, we obtain $c_1 = -2$ and therefore

$$f_A = \left(t_1 {x_0}^{-1}, t_2 {x_1}^{-1}, t_3 x_3 x_0^{-2}, t_4 x_4 x_1^{-2}, t_5 x_5\right).$$

Neither of these transformations satisfies relation (5) with

$$\psi(h) = (x_0 - 1, x_1 - 1, x_3 - x_5, x_2 - x_5, x_5)$$

nor with

$$\psi^\vee(h) = \left(\frac{x_0}{1 - x_0 - x_1}, \frac{x_1}{1 - x_0 - x_1}, \frac{x_3}{1 - x_0 - x_1}, \frac{x_4}{1 - x_0 - x_1}, \frac{x_5 - x_3 - x_4}{1 - x_0 - x_1} \right).$$

$\square$

Proposition 3.5.17. *The map $\Phi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$ is the unique homomorphism such that $\Phi|_{\mathrm{PGL}_3(\mathbb{C})} = \varphi$ and such that $\Phi(\sigma)$ is contained in $D_5 \ltimes W_5$.*
Moreover there exists no homomorphism $\Psi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$ such that

$$\Psi|_{\mathrm{PGL}_3(\mathbb{C})} = \varphi^\vee$$

and such that $\Psi(\sigma)$ is contained in $D_5 \ltimes W_5$.

Proof. Assume that there is an element $tf_A \in D_5 \ltimes W_5$ of order two, where $t \in D_5$ and $f_A \in W_5$, which satisfies the relations of Proposition 3.5.12. Let $A \in \mathrm{GL}_5(\mathbb{Z})$ be the integer matrix corresponding to f_A with respect to the affine coordinates $x_2 = 1$.

We look at the conditions that the relations (2)-(5) of Proposition 3.5.12 impose on tf_A. Note that $(tf_A)^2 = \mathrm{id}$ implies that $f_A^2 = \mathrm{id}$ and that $tf_A d = d^{-1} tf_A$ for all $d \in \psi(D_2)$ implies $f_A d = d^{-1} f_A$. The relation $tf_A \tau = \tau tf_A$ for all $\tau \in \psi(S_3)$ implies $f_A \tau = \tau f_A$ as well as $\tau t = t\tau$. Hence t is of the form

$$t = [x_0 : x_1 : x_2 : t_1 x_3 : t_1 x_4 : t_1 x_5]$$

for a $t_1 \in \mathbb{C}^*$.

Moreover, the relation $tf_A tf_A = \mathrm{id}$ implies that $tf_A = f_A t^{-1}$.

Note that

$$\varphi|_{S_3} = \psi|_{S_3}.$$

We can therefore apply the same arguments as in the proof of Lemma 3.5.16 to obtain that A has the form

$$A = \left[\begin{array}{cc|ccc} a & 0 & b_1 - b_2 & 0 & b_2 - b_1 \\ 0 & a & 0 & b_1 - b_2 & b_2 - b_1 \\ \hline c_1 & c_2 & d_1 - b_2 & d_2 - b_2 & d_2 - b_1 \\ c_2 & c_1 & d_2 - b_2 & d_1 - b_2 & d_2 - b_1 \\ c_2 & c_2 & d_2 - b_2 & d_2 - b_2 & d_1 - b_1 \end{array}\right],$$

where $a, b_1, b_2, c_1, c_2, d_1, d_2$ are integers.

The image under Φ with respect to these affine coordinates of a diagonal element $[\alpha x_0 : \beta x_1 : \gamma x_2]$ is

$$\left(\frac{\alpha^2}{\gamma^2} x_0, \frac{\beta^2}{\gamma^2} x_1, \frac{\beta}{\gamma} x_3, \frac{\alpha}{\gamma} x_4, \frac{\alpha\beta}{\gamma^2} x_5 \right).$$

Relation (3) yields

$$\left(\frac{\alpha^2}{\gamma^2} \right)^a \left(\frac{\beta}{\gamma} \right)^{b_1 - b_2} \left(\frac{\alpha\beta}{\gamma^2} \right)^{b_2 - b_1} = \frac{\gamma^2}{\alpha^2}.$$

We therefore get

$$b_1 - b_2 = 2a + 2.$$

The matrix A has order 2. Multiplication of the first row with the last column gives therefore 0 and we obtain

$$(b_1 - b_2)(-a + d_2 - d_1) = 0.$$

Case 1: $b_1 - b_2 = 0$. Then we have $a = -1$.

So our matrix has the form

$$A = \left[\begin{array}{cc|ccc} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline c_1 & c_2 & \delta_1 & \delta_2 & \delta_2 \\ c_2 & c_1 & \delta_2 & \delta_1 & \delta_2 \\ c_2 & c_2 & \delta_2 & \delta_2 & \delta_1 \end{array} \right],$$

for some $c_1, c_2, \delta_1, \delta_2 \in \mathbb{Z}$. Since A is of order 2, we get $\delta_2 = 0$ and $\delta_1 = \pm 1$.

Assume that $\delta_1 = -1$. Then, by considering the third row and using relation (3),

$$\left(\frac{\alpha^2}{\gamma^2} \right)^{c_1} \left(\frac{\beta^2}{\gamma^2} \right)^{c_2} \left(\frac{\beta}{\gamma} \right)^{-1} = \frac{\gamma}{\beta}.$$

Hence $c_1 = c_2 = 0$. and we find that $A = -\,\mathrm{id}$. However, this transformation does not satisfy relation (3) from Lemma 3.5.12, nor does $t f_A$ for any $t \in D_5$ of the form described above. So we have $\delta_1 = 1$ and

$$\left(\frac{\alpha^2}{\gamma^2} \right)^{c_1} \left(\frac{\beta^2}{\gamma^2} \right)^{c_2} \left(\frac{\beta}{\gamma} \right)^1 = \frac{\gamma}{\beta}$$

yields $c_1 = 0, c_2 = -1$. So

$$A = \left[\begin{array}{cc|ccc} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \end{array} \right].$$

This is exactly the matrix corresponding to $\Phi(\sigma)$ with respect to these affine coordinates. Direct calculations show that $t f_A$ satisfies relation (5) if and only if $t = \mathrm{id}$.

Case 2: $-a + d_2 - d_1 = 0$. If we consider the fifth row and use again relation (3), we obtain:

$$\left(\frac{\alpha^2}{\gamma^2}\right)^{c_2} \left(\frac{\beta^2}{\gamma^2}\right)^{c_2} \left(\frac{\beta}{\gamma}\right)^{d_2-b_2} \left(\frac{\alpha}{\gamma}\right)^{d_2-b_2} \left(\frac{\alpha\beta}{\gamma^2}\right)^{d_1-b_1} = \frac{\gamma^2}{\alpha\beta}.$$

Looking at the exponents of γ this yields

$$-4c_2 - 2(d_2 - b_2) - 2(d_1 - b_1) = 2.$$

We use $b_1 = b_2 + 2a + 2$ and $d_1 = d_2 + a$ and get

$$b_2 = c_2 - \frac{a+1}{2} + d_1.$$

Considering the third row and relation (3) we obtain

$$\left(\frac{\alpha^2}{\gamma^2}\right)^{c_1} \left(\frac{\beta^2}{\gamma^2}\right)^{c_2} \left(\frac{\beta}{\gamma}\right)^{d_1-b_2} \left(\frac{\alpha}{\gamma}\right)^{d_2-b_2} \left(\frac{\alpha\beta}{\gamma^2}\right)^{d_2-b_1} = \frac{\gamma}{\beta}$$

and therefore

$$-2c_1 - 2c_2 - (d_1 - b_2) - (d_2 - b_2) - 2(d_2 - b_1) = 1.$$

From this we conclude

$$c_1 = b_2 - d_1 + 1.$$

This yields

$$A = \begin{bmatrix} a & 0 & 2a+2 & 0 & -2a-2 \\ 0 & a & 0 & 2a+2 & -2a-2 \\ \frac{-a+1}{2}+c_2 & c_2 & \frac{a+1}{2}-c_2 & \frac{3a+1}{2}-c_2 & \frac{-a-3}{2}-c_2 \\ c_2 & \frac{-a+1}{2}+c_2 & \frac{3a+1}{2}-c_2 & \frac{a+1}{2}-c_2 & \frac{-a-3}{2}-c_2 \\ c_2 & c_2 & \frac{3a+1}{2}-c_2 & \frac{3a+1}{2}-c_2 & \frac{-3a-3}{2}-c_2 \end{bmatrix}.$$

By multiplying the third row with the first column we obtain

$$3c_2 a - (3/4)a^2 + (1/2)a + 1/4 - 3c_2^2 - c_2 = 0.$$

This equation has two solutions for a:

$$a = 2c_2 + 1 \text{ and } a + 2c_2 - 1/3.$$

Since c_2 and a have to be integers this yields $a = 2c_2 + 1$. Thus

$$A = \begin{bmatrix} 2c_2+1 & 0 & 4c_2+4 & 0 & -4c_2-4 \\ 0 & 2c_2+1 & 0 & 4c_2+4 & -4c_2-4 \\ 0 & c_2 & 1 & 2c_2+2 & -2c_2-2 \\ c_2 & 0 & 2c_2+2 & 1 & -2c_2-2 \\ c_2 & c_2 & 2c_2+2 & 2c_2+2 & -3-4c_2 \end{bmatrix}.$$

So we get

$$f_A = (x_0^{2c_2+1} x_3^{4+4c_2} x_5^{-4-4c_2}, x_1^{2c_2+1} x_4^{4+4c_2} x_5^{-4-4c_2},$$

$$x_1^{c_2} x_3 x_4^{2c_2+2} x_5^{-2c_2-2}, x_0^{c_2} x_3^{2c_2+2} x_4 x_5^{-2c_2-2}, x_0^{c_2} x_1^{c_2} x_3^{2c_2+2} x_4^{2c_2+2} x_5^{-3-4c_2}).$$

Since $f_A t = t^{-1} f_A$, we obtain in this case that $t_1 = 1$ and therefore that $t = \mathrm{id}$.

By relation (5), we have

$$f_A \varphi(h) f_A = \varphi(h) f_A \varphi(h).$$

Remember that

$$\varphi(h) = (x_0 + 1 - x_4, x_1 + 1 - x_3, 2 - x_3, 2 - x_4, 2 - x_3 - x_4 + x_5).$$

Let $U := \{(u, 1, 1, 1, 1) \mid u \in \mathbb{C}\}$. We observe that $\varphi(h)(U) = U$, so in particular, $\varphi(U)$ is not contained in the indeterminacy locus of f_A, which is contained in the hyperplanes given by $x_i = 0$ for some i. The restriction of $\varphi(h) f_A \varphi(h)$, respectively of $f_A \varphi(h) f_A$ to U is therefore a rational map. One calculates that

$$\varphi(h) f_A \varphi(h)(u, 1, 1, 1, 1) = (u^{2c_2+1} + 1 - u^{c_2}, 1, 1, 2 - u^{c_2}, 1),$$

and that the first coordinate of $f_A \varphi(h) f_A$ is

$$(u^{2c_2+1} + 1 - u^{c_2})^{2c_2+1} (1 - 2u^{c_2})^{-4-4c_2}$$

So this case is not possible.

For the second claim we assume that there is an element $f_A \in W_5$ fulfilling the relations (2)-(4) with $\varphi^\vee(\mathrm{PGL}_3(\mathbb{C}))$. Since $\varphi^\vee|_{\mathcal{S}_3} = \varphi|_{\mathcal{S}_3}$ and $\varphi(d)^{-1} = \varphi^\vee(d)$ for all elements $d \in D_2$, we can repeat all the steps of the proof that only required relations (2)-(4). So in case 1 we just check by calculations that neither of the transformations $f_A, f_{-\mathrm{id}} \in W_5$ corresponding to $-\mathrm{id}$ or A satisfies relation (5) with respect to the image of h under $\varphi^\vee$, nor does f_A or f_{-id}.

For case 2, let

$$g := [X - Z : Y : Z].$$

Then $\sigma g \sigma g \sigma g = \mathrm{id}$. We have

$$\varphi^\vee(g) = \left(\frac{x_0}{p}, \frac{x_2}{p}, \frac{x_3 + x_5}{p}, \frac{-x_0 - x_4}{p}, \frac{-x_5}{p} \right)$$

where $p = x_0 + 2x_4 + 1$. Here again, $\varphi^\vee(h)(U)$ is not contained in the indeterminacy locus of f_A and therefore the restriction of $f_A \varphi^\vee(g) f_A$ to U is rational. One calculates that the first entry of $\varphi^\vee(g) f_A \varphi^\vee(g)(u, 1, 1, 1, 1)$ is

$$\frac{2^{4c_2+4} u^{2c_2+1}}{2^{4c_2+4} u^{2c_2+1} - 2^{2c_2+3}(u+3)^{c_2}(u+1) + u(u+3)^{c_2} + 3(u+3)^{2c_2}},$$

whereas the first entry of $f_A \varphi^\vee(g) f_A(u, 1, 1, 1, 1)$ is

$$\frac{u(2 + u^{c_2})^{4c_2+4}}{(u^{2c_2+1} + 4 + 4u^{c_2})^{2c_2+1}}.$$

One calculates that the two expressions are not the same for all $c_2 \in \mathbb{Z}$, so for no choice of c_2 the map f_A fulfills relation (5). $\qquad\square$

Lemma 3.5.18. *The normalizer of $\varphi(D_2)$ in $\mathrm{PGL}_6(\mathbb{C})$ is the group generated by $\mathcal{S}_6$ and D_5.*

Proof. The fixed points of the action of $\varphi(D_2)$ on $\mathbb{P}^5$ are exactly the points

$$P_0 = [1:0:0:0:0:0], P_1 = [0:1:0:0:0:0], \ldots, P_5 = [0:0:0:0:0:1].$$

If an element $f \in \mathrm{PGL}_6(\mathbb{C})$ normalizes $\varphi(D_2)$, then it has to preserve the set of fixed points of $\varphi(D_2)$. The elements of $\mathrm{PGL}_6(\mathbb{C})$ that permute the points P_i are exactly the linear monomial maps. $\qquad\square$

We also recall the following fusion theorem about tori:

Lemma 3.5.19 (Lemma 2 in [Pop13]). *Let $T_1 \subset D_n$ and $T_2 \subset D_n$ be two subtori. If T_1 and T_2 are conjugate in GL_n then they are conjugate in the normalizer $N_{\mathrm{GL}_n}(D_n)$.*

Proposition 3.5.20. *There is no homomorphism of groups $\Psi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$ such that $\Psi(\mathrm{PGL}_3(\mathbb{C})) = \psi(\mathrm{PGL}_3(\mathbb{C}))$ or $\Psi(\mathrm{PGL}_3(\mathbb{C})) = \varphi(\mathrm{PGL}_3(\mathbb{C}))$ and $\Psi(\mathrm{W}_2) \subset D_5 \rtimes \mathrm{W}_5$.*

Proof. First we show that φ can not be extended to a homomorphism from Cr_2 to Cr_5 that sends W_2 into $D_5 \rtimes \mathrm{W}_5$. As seen in the proof of Proposition 3.1.4, we have

$$\Phi([xy : yz : z^2]) = [x_0x_1 : x_1x_2 : x_2^2 : x_2x_3 : -x_2x_5 + 2x_3x_4 : x_1x_4].$$

So $\Phi(\mathrm{W}_2)$ is not contained in $D_5 \rtimes \mathrm{W}_5$ and by Proposition 3.5.17, there is no other extension of φ to Cr_2 such that W_2 is sent to $D_5 \rtimes \mathrm{W}_5$.

Lemma 3.5.16 shows that ψ can not be extended to a homomorphism from Cr_2 to Cr_5 such that W_2 is sent to W_5.

Let $\Psi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$ be an automorphism such that $\Psi(\mathrm{PGL}_3(\mathbb{C})) = \varphi(\mathrm{PGL}_3(\mathbb{C}))$ or $\Psi(\mathrm{PGL}_3(\mathbb{C})) = \psi(\mathrm{PGL}_3(\mathbb{C}))$. By Proposition 3.2.10 there exists an automorphism of fields $\gamma\colon \mathbb{C} \to \mathbb{C}$ such that $\Psi \circ \gamma$ is an algebraic homomorphism. The restriction of $\Psi \circ \gamma$ to $\mathrm{PGL}_3(\mathbb{C})$ is either $\psi, \psi \circ \alpha, \varphi$ or $\varphi \circ \alpha$, up to conjugation by an element $g \in \mathrm{PGL}_6$. By assumption, $\Psi \circ \gamma(D_2)$ is contained in D_5, so $g \in D_5 \rtimes \mathrm{W}_5$, by Lemma 3.5.18.

Since neither of $\psi, \psi \circ \alpha, \varphi$ or $\varphi \circ \alpha$ can be extended to an embedding of Cr_2 such that W_2 is sent to $D_5 \rtimes \mathrm{W}_5$ by Proposition 3.5.16 and Proposition 3.5.17 it follows that $\Psi \circ \gamma$ and thus Ψ do not map W_2 into W_5. $\qquad\square$

Proof of Theorem 3.1.6. Assume that there is a homomorphism of groups

$$\Psi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$$

such that $\Psi(\mathrm{PGL}_3(\mathbb{C})) \subset \mathrm{PGL}_6(\mathbb{C})$, $\Psi(D_2) \subset D_5$ and $\Psi(\mathrm{W}_2) \subset \mathrm{W}_5$. By definition, $\Psi|_{\mathrm{PGL}_3(\mathbb{C})}$ is a homomorphism of algebraic groups. So there exists an automorphism $g \in \mathrm{PGL}_6(\mathbb{C})$ such that the restriction of $g\Psi g^{-1}$ to $\mathrm{PGL}_3(\mathbb{C})$ is either φ, ψ or $\varphi \circ \alpha$, respectively $\psi \circ \alpha$. By Lemma 3.5.18 and Lemma 3.5.19 the automorphism g is contained in $\mathcal{S}_6 \ltimes D_5$.

But then $g \circ \Psi \circ g^{-1}$ preserves $D_2 \rtimes \mathrm{W}_2$, which is not possible, by Proposition 3.5.20. $\qquad\square$

Chapter 4

Remarks on the degree growth of birational transformations

4.1 Introduction and results

4.1.1 Groups of birational transformations and degree sequences

Let X_k be a projective variety defined over a field k. Here, a projective variety is a separated, geometrically integral, projective scheme of finite type over k. Denote by $\mathrm{Bir}(X_k)$ the group of birational transformations of X_k. A group Γ is called a *group of birational transformations* if there exists a field k and a projective variety X_k over k such that $\Gamma \subset \mathrm{Bir}(X_k)$. More generally, one can consider $\mathrm{Rat}(X_k)$, the monoid of dominant rational self-maps of X_k. Accordingly, we call a monoid Δ a *monoid of rational dominant transformations*, if there exists a field k and a projective variety X_k over k such that $\Delta \subset \mathrm{Rat}(X_k)$.

If X_k is a smooth projective variety, an interesting tool to study the structure of monoids of rational dominant transformations are *degree functions*. Fix a polarization of X_k, i.e. an ample divisor class H of X_k. Then one can associate to every element $f \in \mathrm{Rat}(X_k)$ its degree $\deg_H(f) \in \mathbb{Z}^+$ with respect to H, which is defined by

$$\deg_H(f) = f^*H \cdot H^{d-1},$$

where d is the dimension of X_k and f^*H is the total transform of H under f. For a smooth projective variety X_k over a field of characteristic zero k, one has for $f, g \in \mathrm{Bir}(X_k)$

$$\deg_H(f \circ g) \leq C(X_k, H) \deg_H(f) \deg_H(g),$$

where $C(X_k, H)$ is a constant only depending on X_k and the choice of polarization H (see [DS05]). For a generalization of this result to fields of positive character-

istic, see [Tru15] and [Tru16] (see also the more recent [Dan17] for an alternative approach).

Let f be a rational self map of $\mathbb{P}_k^d$. With respect to homogeneous coordinates $[x_0 : \cdots : x_d]$ of $\mathbb{P}_k^d$, f is given by $[x_0 : x_1 : \cdots : x_d] \mapsto [f_0 : f_1 : \cdots : f_d]$, where $f_0, \ldots, f_d \in k[x_0, \ldots, x_d]$ are homogeneous polynomials of the same degree and without a common factor. We define $\deg(f) = \deg(f_i)$. If f is dominant, then $\deg(f) = \deg_H(f)$ for $H = \mathcal{O}(1)$. So in case $X_k = \mathbb{P}_k^d$ we can extend the notion of degree to all rational self maps. Note that if f is an endomorphism of $\mathbb{A}_k^d$ defined by $(x_1, \ldots, x_d) \mapsto (f_1, \ldots, f_d)$ with respect to affine coordinates $(x_1, \ldots, x_d)$ of $\mathbb{A}_k^d$, where $f_1, \ldots, f_d \in k[x_1, \ldots, x_d]$, then $\deg(f)$ is the maximal degree of the f_i.

Let $\Delta \subset \mathrm{Rat}(X_k)$ be a finitely generated monoid of rational dominant transformations of a smooth projective variety X_k with a finite set of generators S. We define

$$D_{S,H} \colon \mathbb{Z}^+ \to \mathbb{Z}^+$$

by

$$D_{S,H}(n) := \max_{\delta \in B_S(n)} \{\deg_H(\gamma)\},$$

where $B_S(n)$ denotes all elements in Δ of word length $\leq n$ with respect to the generating set S. We call a map $\mathbb{Z}^+ \to \mathbb{Z}^+$ that can be realized for some field k and some (X_k, H, Δ, S) as such a function a *degree sequence*. Note that our definition of degree sequences includes in particular degree sequences that are given by finitely generated groups of birational transformations $\Gamma \subset \mathrm{Bir}(X_k)$.

In this paper we show that there exist only countably many degree sequences, display certain constraints on their growth and give some new examples.

4.1.2   Countability of degree sequences

In [BF00], Bonifant and Fornaess proved that the set of sequences $\{d_n\}$ such that there exists a rational self map f of $\mathbb{P}_{\mathbb{C}}^d$ satisfying $\deg(f^n) = d_n$ for all n, is countable, which answered a question of Ghys. We generalize the result of Bonifant and Fornaess to all degree sequences over all smooth projective varieties, all fields, all polarizations and all finite generating sets S of finitely generated monoids of rational dominant maps:

Theorem 4.1.1. *The set of all degree sequences is countable.*

4.1.3   Previous results on degree growth

In dimension 2 the degree growth of birational transformations is well understood and is a helpful tool to understand the group structure of $\mathrm{Bir}(S_k)$ for projective surfaces S_k over a field k.

Theorem 4.1.2 (Gizatullin; Cantat; Diller and Favre)**.** *Let k be an algebraically closed field, S_k a projective surface over k with a fixed polarization H and $f \in \mathrm{Bir}(S_k)$. Then one of the following is true:*

- *the set $\{\deg_H(f^n)\}$ is bounded;*

- $\deg_H(f^n) \sim cn$ for some positive constant c and f preserves a rational fibration;

- $\deg_H(f^n) \sim cn^2$ for some positive constant c and f preserves an elliptic fibration;

- $\deg_H(f^n) \sim \lambda^n$, where λ is a Pisot or Salem number.

For more details on this rich subject and references to the proof of Theorem 4.1.2 see [Can12a]. In the case of polynomial automorphisms of the affine plane, the situation is even less complicated. Let $f \in \mathrm{Aut}(\mathbb{A}_k^2)$. Then the sequence $\{\deg(f^n)\}$ is either bounded or it grows exponentially in n. See [Fur99] for this and more results on the degree growth in $\mathrm{Aut}(\mathbb{A}_k^2)$

In higher dimensions there are only few results on the degree growth of birational transformations. In particular, the following questions are open:

Question 4.1.1. Does there exist a birational transformation f of a projective variety X_k such that $\deg_H(f^n)$ is of intermediate growth, for instance $\deg_H(f^n) \sim e^{\sqrt{n}}$?

Question 4.1.2. Does there exist a birational transformation f such that $\deg_H(f^n)$ is unbounded, but grows "slowly"? For instance, can we have $\deg_H(f^n) \sim \sqrt{n}$? Or do unbounded degree sequences grow at least linearly?

Question 4.1.3. If there is a birational transformation f such that $\deg_H(f^n) \sim \lambda^n$, is λ always an algebraic number?

Question 4.1.4. Do birational transformations of polynomial growth always preserve some non-trivial rational fibration?

In [LB14] Lo Bianco treats the case of automorphisms of compact Kähler threefolds.

4.1.4 Degree sequences of polynomial automorphisms

A good place to start the examination of degree sequences seems to be the group of polynomial automorphisms of the affine d-space $\mathrm{Aut}(\mathbb{A}_k^d)$. In Section 4.3.2 we will show the following observation (the proof of which can be found as well in [Dés16a]):

Proposition 4.1.3. *Let k be a field and $f \in \mathrm{Aut}(\mathbb{A}_k^d)$ a polynomial automorphism such that $\deg(f^d) = \deg(f)^d$, then $\deg(f^n) = \deg(f)^n$ for all $n \in \mathbb{Z}^+$.*

The monoid $\mathrm{End}(\mathbb{A}_k^d)$ has the additional structure of a k-vector space, on which the degree function induces a filtration of finite dimensional vector spaces. This gives rise to a new technique, which we will use to prove that unbounded degree sequences of groups of polynomial automorphisms diverge and can not grow arbitrarily slowly:

Theorem 4.1.4. *Let* $f \in \mathrm{End}(\mathbb{A}_k^d)$ *be an endomorphism and assume that the sequence* $\{\deg(f^n)\}$ *is unbounded. Then for all integers* K

$$\#\{m \mid \deg(f^m) \leq K\} < C_d \cdot K^d,$$

where $C_d = \frac{(1+d)^d}{(d-1)!}$. *In particular,* $\deg(f^n)$ *converges to* ∞ *as* n *goes to* ∞.

By a result of Ol'shanskii ([Ols99]), Theorem 4.1.4 shows that an unbounded degree sequence of a polynomial automorphism behaves in some ways like a word length function. The following corollary is immediate:

Corollary 4.1.5. *Let* $\Gamma \subset \mathrm{End}(\mathbb{A}_k^d)$ *be a finitely generated monoid with generating system* S. *If* $D_S(n) < C_d \cdot n^{1/d}$ *for infinitely many* n *then* Γ *is of bounded degree.*

Unfortunately our methods to prove Theorem 4.1.4 do not work for arbitrary birational transformations of $\mathbb{P}_k^d$. However, if we assume the ground field to be finite, we obtain similar results:

Theorem 4.1.6. *Let* $\mathbb{F}_q$ *be a finite field with* q *elements and let* $f \in \mathrm{Rat}(\mathbb{P}_{\mathbb{F}_q}^d)$ *such that the sequence* $\{\deg(f^n)\}$ *is unbounded. Then, for all integers* K,

$$\#\{m \mid \deg(f^m) \leq K\} \leq q^{C(K,d)},$$

where $C(K,d) = (d+1) \cdot \binom{d+K}{K}$. *In particular,* $\deg(f^n)$ *converges to* ∞ *as* n *goes to* ∞.

Corollary 4.1.7. *Let* $\Gamma \subset \mathrm{Rat}(\mathbb{P}_{\mathbb{F}_q}^d)$ *be a finitely generated monoid with generating system* S. *There exists a positive constant* $C_{d,q}$ *such that if* $D_S(n) < C_{d,q} \cdot \log(n)^{1/d}$ *for all* n, *then* Γ *is of bounded degree.*

4.1.5   Types of degree growth

Definition 4.1.1. Let X_k be a smooth projective variety with polarization H over a field k and let $f \in \mathrm{Bir}(X_k)$. We denote the *order of growth* of $\deg_H(f^n)$ by

$$\mathrm{dpol}(f) := \limsup_{n \to \infty} \frac{\log(\deg_H(f^n))}{\log(n)}.$$

The order of growth can be infinite.

By results of Truong, Dinh and Sibony, the order of growth does not depend on the choice of polarization (see Section 4.2.4):

Proposition 4.1.8. *Let* X_k *be a smooth projective variety over a field* k *and let* $f \in \mathrm{Bir}(X_k)$. *Then* $\mathrm{dpol}(f)$ *does not depend on the choice of polarization.*

Let f be a birational transformation of a surface. As recalled above, in that case $\mathrm{dpol}(f) = 0, 1, 2$ or ∞. This gives rise to the following question:

Question 4.1.5. Does there exist a constant $C(d)$ depending only on d such that for all varieties X_k of dimension d we have $\mathrm{dpol}(f) < C(d)$ for all $f \in \mathrm{Bir}(X_k)$ with $\mathrm{dpol}(f)$ finite?

We give some examples of degree sequences that indicate that the degree growth in higher dimensions is richer than in dimension 2. First of all, note that elements in $\mathrm{Aut}(\mathbb{A}_k^d)$ can have polynomial growth:

Example 4.1.9. Let k be any field and define $f, g, h \in \mathrm{Aut}(\mathbb{A}_k^d)$ by $g = (x + yz, y, z)$, $h = (x, y + xz, z)$ and

$$f = g \circ h = (x + z(y + xz), y + xz, z).$$

One sees by induction that $\deg(f^n) = 2n + 1$; in particular, $\mathrm{dpol}(f) = 1$.

More generally, for all $l \leq d/2$ there exist elements $f_l \in \mathrm{Aut}(\mathbb{A}_k^d)$ such that $\mathrm{dpol}(f) = l$ (Section 4.3.4).

Other interesting examples of degree sequences of polynomial automorphisms and the dynamical behavior of the corresponding maps are described in [Dés16a]. For birational transformations of $\mathbb{P}_k^d$ we can obtain even faster growth (see [Lin12] for more details):

Example 4.1.10. The birational transformation $f = (x_1, x_1 x_2, \ldots, x_1 x_2 \cdots x_n)$ of $\mathbb{P}_k^d$ defined with respect to local affine coordinates $(x_1, \ldots, x_d)$ satisfies $\deg(f^n) = n^{d-1}$, i.e. $\mathrm{dpol}(f) = d - 1$.

The following interesting observation is due to Serge Cantat:

Example 4.1.11. Define $\omega := \frac{-1 + \sqrt{3}}{2}$ and the elliptic curve $E_\omega := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\omega)$. Let

$$X := E_\omega \times E_\omega \times E_\omega$$

and $s \colon X \to X$ the automorphism of finite order given by diagonal multiplication with $-\omega$. In [OT15] Oguiso and Truong prove that the quotient $Y := X/s$ is a rational threefold. Let $f \colon X \to X$ be the automorphism defined by $(x_1, x_2, x_3) \mapsto (x_1, x_1 + x_2, x_2 + x_3)$. Since f commutes with s, it induces an automorphism on Y, which we denote by $\hat{f}$. Let $\phi_1 \colon \tilde{Y} \to Y$ be a resolution of the singularities of Y and define $\tilde{f} \in \mathrm{Bir}(\tilde{Y})$ by

$$\tilde{f} := \phi_1^{-1} \circ \hat{f} \circ \phi_1.$$

We will show in Section 4.3.5 that $\mathrm{dpol}(\tilde{f}) = 4$.

Remark 4.1.1. In [Dés16a] Déserti shows that for all $l \leq d$ there exists a birational transformation $f \in \mathrm{Bir}(\mathbb{P}_{\mathbb{C}}^d)$ such that $\mathrm{dpol}(f) = l$.

4.2 Preliminaries

4.2.1 Monoids of rational dominant transformations

Let X_k be a projective variety over a field k (i.e. a separated, geometrically integral, projective scheme of finite type over k). There is a one to one correspondence between rational dominant self maps of X_k and k-endomorphisms of the function field $k(X_k)$. The field $k(X_k)$ is the field of fractions of a k-algebra of finite type $k[x_1, \ldots, x_n]/I$, where $I \subset k[x_1, \ldots, x_n]$ is a prime ideal generated

by elements $f_1, \ldots, f_l \in k[x_1, \ldots, x_n]$. A field extension $k \to k'$ induces a base change $X_{k'} \to X_k$. The function field of $X_{k'}$ is the field of fractions of the k'-algebra $k'[x_1, \ldots, x_n]/I'$, where I' is the ideal generated by $f_1, \ldots, f_l$. Note that $k(X_k) \subset k'(X_{k'})$. We say that a k'-endomorphism of $k'(X_{k'})$ is defined over k, if it restricts to a k-endomorphism of $k(X_k)$. Consider a k'-endomorphism G of $k'(X_{k'})$ sending generators $(x_1, \ldots, x_n)$ of $k'(X_{k'})$ to $(g_1, \ldots, g_n)$, where $g_i \in k'(X_{k'})$. Then G is defined over k if and only if $g_i \in k(X_k)$ for all i. On the other hand, let $g_i, \ldots, g_n \in k(X_k)$ and let $(x_1, \ldots, x_d) \mapsto (g_1, \ldots, g_d)$ be a k-endomorphism of $k(X_k)$. Then $(x_1, \ldots, x_d) \mapsto (g_1, \ldots, g_d)$ defines as well a k'-endomorphism of $k'(X_{k'})$. So a k-endomorphism $(x_1, \ldots, x_n) \mapsto (g_1, \ldots, g_n)$ of $k(X_k)$ extends uniquely to a k'-endomorphism of $k'(X_{k'})$. This yields the following observation:

Lemma 4.2.1. *Let X_k be a projective variety over a field k and $\varphi\colon k \to k'$ a homomorphism of fields. Then φ induces a natural injection of monoids $\Psi_\varphi\colon \mathrm{Rat}(X_k) \to \mathrm{Rat}(X_{k'})$.*

Recall that there are uncountably many isomorphism classes of finitely generated groups and thus in particular uncountably many isomorphism classes of finitely generated monoids. The following observation by de Cornulier shows that being a monoid of rational dominant transformations is in some sense a special property (cf. [Cor13] and [Can12b]).

Proposition 4.2.2. *There exist only countably many isomorphism classes of finitely generated monoids of rational dominant transformations.*

Proof. Let $\Delta \subset \mathrm{Rat}(X_k)$ be a monoid of rational dominant transformations with a finite generating set $f_1, \ldots, f_n \in \Delta$, where X_k is a projective variety defined over a field k. Denote by $S \subset k$ the finite set of coefficients necessary to define X_k and the rational dominant transformations f_i. Let k' be the field $\mathbb{F}_p(S)$, where $p = \mathrm{char}(k)$, or $\mathbb{Q}(S)$ if $\mathrm{char}(k) = 0$.

We consider the function field $k'(X_{k'})$ as a subfield of the function field $k(X_k)$. Note that the action of the elements of Δ on $k'(X_k)$ preserves $k'(X_{k'})$ and that $f_{i_1} f_{i_2} \cdots f_{i_m} = f_{j_1} f_{j_2} \cdots f_{j_m}$ in $\mathrm{Rat}(X_k)$ if and only if $f_{i_1} f_{i_2} \cdots f_{i_m} = f_{j_1} f_{j_2} \cdots f_{j_m}$ in $\mathrm{Rat}(X_{k'})$. So without loss of generality we may assume $\Delta \subset \mathrm{Rat}(X_{k'})$, where k' is a finitely generated field extension of some $\mathbb{F}_p$ or of $\mathbb{Q}$.

A rational dominant transformation of a given variety X_k is defined by finitely many coefficients in k. So the cardinality of the set of all finitely generated monoids of rational dominant transformations of a variety X_k is either countable, if k is finite, or it has at most the cardinality of k.

Recall that the cardinality of the set of all finitely generated field extensions of $\mathbb{F}_p$ and $\mathbb{Q}$ is countable. Since a projective variety is defined by a finite set of coefficients, we obtain that there are only countably many isomorphism classes of projective varieties defined over a field k' that is a finitely generated field extension of $\mathbb{F}_p$ or $\mathbb{Q}$. The claim follows.   □

4.2.2   Intersection form

Let X_k be a smooth projective variety of dimension d over an algebraically closed field k and let D be a Cartier divisor on X_k. The *Euler characteristic of D* is the

integer

$$\chi(X_k, D) = \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X_k, D).$$

Let $D_1, \ldots, D_d$ be Cartier divisors on X_k. The function

$$(m_1, \ldots, m_d) \mapsto \chi(X_k, m_1 D_1 + \cdots + m_d D_d)$$

is a polynomial in $(m_1, \ldots, m_d)$. The *intersection number* of $D_1, \ldots, D_d$ is then defined to be the coefficient of the term $m_1 m_2 \cdots m_d$ in this polynomial and we denote it by $D_1 \cdot D_2 \cdots D_d$. One can show that intersection numbers are always integers and that the intersection form is symmetric and linear in all d arguments. Moreover, if $D_1, \ldots, D_d$ are effective and meet properly in a finite number of points, $D_1 \cdots D_d$ is the number of points in $D_1 \cap \cdots \cap D_d$ counted with multiplicities and the intersection form is the unique bilinear form with this property. For the details on this construction we refer to [Deb01] and the references in there. The intersection number is preserved under linear equivalence, therefore it is well defined on classes of Cartier divisors. Note as well that an isomorphism between algebraically closed fields does not change the cohomology dimensions and hence that the intersection numbers are invariant under such an isomorphism.

Let X_k be a smooth projective variety of dimension d over an arbitrary field k and let $D_1, \ldots, D_d$ be Cartier divisors on X_k. Denote by $\overline{k}$ an algebraic closure of k. We define the intersection number $D_1 \cdot D_2 \cdots D_d$ as the intersection number of $D_1, D_2, \ldots, D_d$ on $X_{\overline{k}}$ after base extension $k \to \overline{k}$. By the remark above, this does not depend on the choice of the algebraic closure $\overline{k}$. Every field isomorphism $k \to k'$ extends to an isomorphism between the algebraic closures of k and k', hence the intersection number is invariant under field isomorphisms. Since the intersection form is unique, it also does not change under a base extension $k \to k'$ between algebraically closed fields.

We summarize these properties in the following proposition:

Proposition 4.2.3. *Let X_k be a smooth projective variety of dimension d over a field k. Then there exists a symmetric d-linear form on the group of divisors of X_k:*

$$\mathrm{Div}(X_k) \times \cdots \times \mathrm{Div}(X_k) \to \mathbb{Z}, \ (D_1, \ldots, D_d) \mapsto D_1 \cdot D_2 \cdots D_d,$$

such that if $D_1, \ldots, D_d$ are effective and meet properly in a finite number of points over the algebraic closure of k, $D_1 \cdots D_d$ is the number of points in $D_1 \cap \cdots \cap D_d$ over the algebraic closure of k counted with multiplicity. Moreover, this intersection form is invariant under base change $k \to k'$ of fields.

Recall that two divisors D_1 and D_2 are called numerically equivalent if $D_1 \cdot \gamma = D_2 \cdot \gamma$ for all curves γ on X. Denote by $N^1(X)$ the Neron-Severi group, which is the group of divisors modulo numerical equivalence. The intersection number of divisors $D_1, \ldots, D_d$ is invariant under numerical equivalence, so we obtain an intersection form on $N^1(X)$.

4.2.3   Polarizations and degree functions

A polarization on a smooth projective variety X_k of dimension d is an ample divisor class H. This implies in particular that $Y \cdot H^{d-1} > 0$ for all effective divisors Y.

Let $f \in \mathrm{Rat}(X_k)$ and denote by $\mathrm{dom}(f)$ the maximal open subset of X_k on which f is defined. The *graph* Γ_f of f is the closure of $\{(x, f(x)) \mid x \in \mathrm{dom}(f))\} \subset X_k \times X_k$. Let p_1 and $p_2 \colon \Gamma_f \to X$ be the natural projections on the first respectively second factor, then $f = p_2 \circ p_1^{-1}$. Note that p_1 is birational. The *total transform* of a divisor D under f is the divisor $p_{1*}(p_2^* D)$, where $p_2^* D$ is the pullback of D as a Cartier divisor and $p_{1*}(p_2^* D)$ is the pushforward of $p_2^* D$ as a Weil divisor. Note that if k is algebraically closed and D is effective, then the support of $f^* D$ is the closure of all the points in $\mathrm{dom}(f)$ that are mapped to D by f. The *degree* of a dominant rational transformation f of X with respect to an ample divisor class H of X is defined as the integer $\deg_H(f) = f^* H \cdot H^{d-1}$, where d is the dimension of X and $f^* H$ is the total transform of H under f.

4.2.4   Degrees in the case of complex varieties

Let $X_{\mathbb{C}}$ be a smooth complex projective variety of dimension d. The following constructions and results can be found in [Laz04]. By taking the first Chern class one can embed $N^1(X_{\mathbb{C}})$ into $H^2(X; \mathbb{Z})_{t.f.}$, which is $H^2(X_{\mathbb{C}}; \mathbb{Z})$ modulo its torsion part. It turns out that $N^1(X_{\mathbb{C}}) = H^2(X_{\mathbb{C}}; \mathbb{Z})_{t.f.} \cap H^{1,1}(X_{\mathbb{C}})$, i.e. we can associate to each divisor class D in $N^1(X_{\mathbb{C}})$ its Chern class $c_1(D)$ which is a unique class of closed $(1,1)$-forms in $H^{1,1}(X_{\mathbb{C}})$. Let $D_1, \ldots, D_d$ be divisors and choose for each i a representative ω_i of the class $c_1(D_i)$. Then

$$D_1 \cdot D_2 \cdots D_d = \int_{X_{\mathbb{C}}} \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_d.$$

A divisor is ample if and only if its Chern class is a Kähler class, i.e. it is represented by a Kähler form. This notion makes it possible to study degrees of meromorphic maps of compact Kähler manifolds.

Let $f \colon X \dashrightarrow Y$ be a meromorphic transformation between compact Kähler varieties. Denote by $\Gamma_f \subset X \times Y$ its graph and by π_X and π_Y the canonical projections from $X \times Y$ to X and Y. Let ω be a $(1,1)$-form on Y and let $\pi_Y^* \omega|_{\Gamma_f}$ be the restriction of the pullback of ω to Γ_f. One can define $f^* \omega$ as the closed positive current

$$f^* \omega = (\pi_X)_* (\pi_Y^* \omega|_{\Gamma_f}),$$

where $(\pi_X)_*$ is the push forward of $\pi_Y^* \omega|_{\Gamma_f}$ as a current. With respect to local coordinates $(x_1, x_2, \ldots, x_d)$ one can write $f^* \omega = \sum_{i,j} a_{ij} dx_i \wedge d\overline{x_j}$, where the a_{ij} are local L^1-functions. We refer to [DN11] for details on this construction. If H is a divisor on Y and ω represents its chern class, then $f^* \omega$ represents the Chern class of $f^* H$.

If ω is a Kähler form of a compact Kähler variety $X_{\mathbb{C}}$, one can define the degree of a birational map $f \in \mathrm{Bir}(X_{\mathbb{C}})$ by

$$\deg_{\omega_{X_{\mathbb{C}}}}(f) = \int_{X_{\mathbb{C}}} f^* \omega_{X_{\mathbb{C}}} \wedge \omega_{X_{\mathbb{C}}} \wedge \cdots \wedge \omega_{X_{\mathbb{C}}}.$$

This definition coincides with our previous definition of degree (Section 4.2.3) and extends this notion to compact Kähler manifolds.

4.2.5 Invariance of the degree by semi-conjugacy

In [Tru16] the following is shown (see also [DS05] and [DN11] for the characteristic 0 case):

Theorem 4.2.4. *Let X_k be a smooth projective variety over a field k with polarization H and $f \in \mathrm{Bir}(X)$. There exists a constant $A > 0$ such that*

$$A^{-1} \deg_H(f^n) \le \|(f^n)^*\| \le A \deg_H(f^n),$$

where $\| \cdot \|$ denotes any norm on $\mathrm{End}(N^1(X_k) \otimes_{\mathbb{Z}} \mathbb{R})$.

The following corollary follows from Theorem 4.2.4 and proves Proposition 4.1.8:

Corollary 4.2.5. *Let X_k be a smooth projective variety over a field k, then*

$$\mathrm{dpol}(f) = \limsup_{n \to \infty} \frac{\log(\|(f^n)^*\|)}{\log(n)}.$$

In particular, $\mathrm{dpol}(f)$ does not depend on the choice of polarization.

Proposition 4.2.6. *Let X_k and Y_k be smooth projective varieties over a field k of dimension d, let $f \in \mathrm{Bir}(X_k)$, $g \in \mathrm{Bir}(Y_k)$ and $\pi \colon X \to Y$ a dominant morphism such that $\pi \circ f = g \circ \pi$. Then $\mathrm{dpol}(f) = \mathrm{dpol}(g)$.*

Proposition 4.2.6 follows from [Tru16, Lemma 4.5] by looking at the maps $\pi \colon X \to Y$ and $\pi^{-1} \colon Y \to X$ as correspondences. Let us describe another strategy to prove Proposition 4.2.6, which has been communicated to us by the referee (see also [Dan17]). Let H_X and H_Y be ample divisors on X and Y respectively. The divisor $H := \pi^* H_Y$ is big and nef. Siu's inequality (see [Laz04, 2.2.13], [Dan17]) implies

$$H_X \le d \cdot \frac{H_X \cdot H^{d-1}}{H^d} H.$$

This means that $H_X \cdot \gamma \le d \cdot \frac{H_X \cdot H^{d-1}}{H^d} H \cdot \gamma$ for all effective curves γ on X. Let C be equal to $d \cdot \frac{(H_X \cdot H^{d-1})}{H^d}$. Then

$$\deg_{H_X}(f^n) \le C^d (f^n)^* H \cdot H^{d-1} = C^d \pi^*(g^n)^* H_Y \cdot (\pi^* H_Y)^{d-1}$$
$$\le C \deg(\pi)(g^n)^* H_Y \cdot H_Y^{d-1} = C \deg(\pi) \deg_{H_Y}(g^n).$$

On the other hand, $\deg_{H_Y}(g^n) \le \pi^*(g^n)^* H_Y \cdot (\pi^* H_Y)^{d-1} = (f^n)^* H \cdot H^{d-1}$. By Theorem 4.2.4, there exists a constant A such that $(f^n)^* H \cdot H^{d-1} \le A \deg_{H_X}(f^n)$.

4.3 Proofs

4.3.1 Proof of Theorem 4.1.1

Let k be a field, X_k a smooth projective variety defined over k, H a polarization of X_k and $\Delta \subset \mathrm{Rat}(X_k)$ a finitely generated monoid of rational dominant transformations with generating set T. Then X_k, H and T are defined by a finite set S of coefficients from k. Let $k' \subset k$ be the field $\mathbb{F}_p(S)$, where $p = \mathrm{char}(k)$, or $k' = \mathbb{Q}(S)$ if $\mathrm{char}(k) = 0$. By Proposition 4.2.3, the degree of elements in Γ considered as rational dominant transformations of X_k with respect to the polarization H is the same as the degree of elements in Δ considered as rational dominant transformations of $X_{k'}$. So without loss of generality, we may assume that Δ is a submonoid of $\mathrm{Rat}(X_{k'})$, where k' is a finitely generated field extension of $\mathbb{F}_p$ or of $\mathbb{Q}$.

As in the proof of Proposition 4.2.2, we use the fact that there are only countably many isomorphism classes of such varieties. Polarizations and rational self maps are defined by a finite set of coefficients, so the cardinality of the set of all $(k', X_{k'}, H, T)$ up to isomorphism, where k' is a finitely generated field extension of $\mathbb{F}_p$ or $\mathbb{Q}$, $X_{k'}$ a smooth projective variety over k', H a polarization of $X_{k'}$ and T a finite set of elements in $\mathrm{Rat}(X_{k'})$ is countable. It follows in particular that the set of all degree sequences is countable.

$\square$

4.3.2 Proof of Proposition 4.1.3

Let k be a field and $f \in \mathrm{Bir}(\mathbb{P}_k^d)$ a birational map that is given by $[f_0 : \cdots : f_d]$, with respect to homogeneous coordinates $[x_0 : \cdots : x_d]$, where the f_i are homogenous polynomials of the same degree without common factors. There are two important closed subsets of $\mathbb{P}_k^d$ associated to f, the indeterminacy locus $\mathrm{Ind}(f)$, consisting of all the points where f is not defined and the exceptional divisor $\mathrm{Exc}(f)$, the set of all the points where f is not a local isomorphism. If f is not an automorphism, the indeterminacy locus is a closed set of codimension ≥ 2 and the exceptional divisor a closed set of codimension 1. Note that $\mathrm{Ind}(f)$ is exactly the set of points, where all the f_i vanish. Let $X \subset \mathbb{P}_k^d$ be an irreducible closed set that is not contained in $\mathrm{Ind}(f)$. We denote by $f(X)$ the closure of $f(X \setminus \mathrm{Ind}(f))$ and we say that f *contracts* X if $\dim(f(X)) < \dim(X)$.

The following lemma is well known (see for example [FS94]):

Lemma 4.3.1. *Let k be a field and $g, f \in \mathrm{Bir}(\mathbb{P}_k^d)$. Then $\deg(f \circ g) \leq \deg(f)\deg(g)$ and $\deg(f \circ g) < \deg(f)\deg(g)$ if and only if g contracts a hypersurface to a subset of $\mathrm{Ind}(f)$.*

Proof. Let $f = [f_0 : \cdots : f_d]$ and $g = [g_0 : \cdots : g_d]$. Then $\deg(f \circ g) < \deg(f)\deg(g)$ if and only if the polynomials $f_0(g_0, \ldots, g_d), \ldots, f_d(g_0, \ldots, g_d)$ have a non constant common factor $h \in k[x_0, \ldots, x_d]$. Let $M \subset \mathbb{P}_k^d$ be the hypersurface defined by $h = 0$. Then f is not defined along $g(M)$. This implies that the codimension of $g(M)$ is ≥ 2 and therefore that g contracts M to a subset of $\mathrm{Ind}(f)$.

On the other hand, let M be an irreducible component of a hypersurface that is contracted by g to a subset $\mathrm{Ind}(f)$. Assume that M is the zero set of an

irreducible polynomial h. Since $g(M) \subset \mathrm{Ind}(f)$, we obtain that the polynomials $f_0(g_0, \ldots, g_d), \ldots, f_d(g_0, \ldots, g_d)$ vanish all on M and therefore that h divides $f_0(g_0, \ldots, g_d), \ldots, f_d(g_0, \ldots, g_d)$. This implies $\deg(f \circ g) < \deg(f) \deg(g)$. $\square$

In order to prove Proposition 4.1.3, we consider an element $f \in \mathrm{Aut}(\mathbb{A}_k^d)$ as a birational transformation of $\mathbb{P}_k^d$ whose exceptional divisor is the hyperplane at infinity $H := \mathbb{P}_k^d \setminus \mathbb{A}_k^d$ and whose indeterminacy points are contained in H.

Note that $\deg(f^d) = \deg(f)^d$ implies $\deg(f^l) = \deg(f)^l$ for $l = 1, \ldots, d$. We look at f as an element of $\mathrm{Bir}(\mathbb{P}_k^d)$. If f is an automorphism of $\mathbb{P}_k^d$, its degree is 1 and the claim follows directly. Otherwise, f contracts the hyperplane H. By Lemma 4.3.1, $\deg(f^{l+1}) = \deg(f^l) \deg(f)$ is equivalent to $f^l(H)$ not being contained in $\mathrm{Ind}(f)$. In particular, if $\deg(f^l) = \deg(f)^l$ and $f^l(H) = f^{l+1}(H)$ for some l then $\deg(f^i) = \deg(f)^i$ for all $i \geq l$.

Let $1 \leq l \leq d$. By Lemma 4.3.1, $f^l(H)$ is not contained in $\mathrm{Ind}(f)$. Observe that $f^l(H)$ is irreducible and $f^{l+1}(H) \subset f^l(H)$. This implies $\dim(f^{l+1}(H)) \leq \dim(f^l(H))$ and $\dim(f^{l+1}(H)) = \dim(f^l(H))$ if and only if $f^{l+1}(H) = f^l(H)$. It follows that the chain $H \supset f(H) \supset f^2(H) \supset \cdots$ becomes stationary within the first d iterations. In particular, $f^d(H) = f^{d+i}(H)$ for all i and therefore $\deg(f^i) = \deg(f)^i$ for all i.

$\square$

4.3.3   Proof of Theorem 4.1.4 and Theorem 4.1.6

We start by proving Theorem 4.1.4. Let $f \in \mathrm{End}(\mathbb{A}_k^d)$ be an endomorphism such that the sequence $\{\deg(f^n)\}$ is unbounded. Our first remark is that the elements of $\{f^n\}$ are linearly independent in the vector space of polynomial endomorphisms $\mathrm{End}(\mathbb{A}_k^d)$. Indeed, assume that for any n we have

$$f^n = \sum_{l < n} c_l f^l,$$

for some $c_1, \ldots, c_{n-1} \in k$. It follows by induction that $\deg(f^{n+i})$ is smaller or equal to $\max_{l < n} \deg(f^l)$ for all i .

Denote by $\mathrm{End}(\mathbb{A}^d)_{\leq K}$ the k-vector space of polynomial endomorphisms of degree $\leq K$, which has dimension $d \cdot \binom{d+K}{K}$. One calculates

$$d \cdot \binom{d + K}{K} < \frac{1}{(d-1)!}(K+d)^d \leq \frac{(1 + d/K)^d}{(d-1)!} K^d \leq C_d K^d,$$

where $C_d = \frac{(1+d)^d}{(d-1)!}$. Since the elements in $\{f^k\} \cap \mathrm{End}(\mathbb{A}^d)_{\leq K}$ are linearly independent, the cardinality of $\{f^k\} \cap \mathrm{End}(\mathbb{A}^d)_{\leq K}$ is at most $C_d K^{d-1}$.

For the proof of Theorem 4.1.6 note that in the case of finite fields, there are only finitely many birational transformations of a given degree. If $f^l = f$ for some $l > 1$, then $\{\deg(f^n)\}$ is bounded. There are $\binom{d+K}{K}$ monomials of degree $\leq K$. A birational transformation of degree $\leq K$ is given by $d+1$ polynomials of degree $\leq K$, so by $C(K, d) = (d+1)\binom{d+K}{K}$ coefficients from $\mathbb{F}_q$. Hence there are less than $q^{C(K,d)}$ birational transformations of degree $\leq K$. This proves the claim.

$\square$

4.3.4   Proof of Example 4.1.9

Let d be an integer and $l = \lfloor d/2 \rfloor$. For $d = 3$ the automorphism $f_3 := (x + z(y + xz), y + xz, z)$ from Example 4.1.9 satisfies $\deg(f_3^n) \sim n$. Moreover, the first coordinate of f_3^n is the coordinate with highest degree. Assume now that $d \geq 5$ and that we are given an automorphism $f_{d-2} \in \mathrm{Aut}(\mathbb{A}^{d-2})$ such that $\deg(f_{d-2}^n)$ grows like n^{l-1} and that the first coordinate of f_{d-2}^n is the entry with highest degree. Let

$$f_d := (x_1 + x_3(x_2 + x_1 x_3), x_2 + x_1 x_3, f_{d-2}(x_3, \ldots, x_d)).$$

One sees by induction that the degree of f_d^n grows like n^l and that the first coordinate of f_d^n is the coordinate of f_d^n with highest degree.

4.3.5   Proof of Example 4.1.11

In this section we use the notation introduced in Example 4.1.11.

Let dz_1, dz_2, dz_3 be a basis of $H^{1,0}(X)$. Then $d\bar{z}_1, d\bar{z}_2, d\bar{z}_3$ is a basis of $H^{0,1}(X)$ and the $(1,1)$-forms $dz_i \wedge d\bar{z}_j$ form a basis of $H^{1,1}(X)$. This allows us to identify $H^{1,1}(X)$ with $H^{1,0}(X) \otimes H^{0,1}(X)$. The automorphism f^n induces automorphisms on both $H^{1,0}(X)$ and $H^{0,1}(X)$ whose norm grows like n^2. This implies that the norm of the induced action of f^n on $H^{1,1}(X)$ grows like n^4. By Corollary 4.2.5, we obtain $\mathrm{dpol}(f) = 4$.

Denote by $\pi\colon X \to Y = X/s$ the quotient map. Let $\tilde{X}$ be a smooth projective variety, $\phi_2\colon \tilde{X} \to X$ a birational morphism and $\psi\colon \tilde{X} \to \tilde{Y}$ a dominant morphism such that the following diagram commutes:

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\ \psi\ } & \tilde{Y} \\
\downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} \\
X & \xrightarrow{\ \pi\ } & Y.
\end{array}
$$

Note that ψ is generically finite. By Proposition 4.2.6, we have $\mathrm{dpol}(\phi_2^{-1} \circ f \circ \phi_2) = \mathrm{dpol}(f) = 4$ and hence, again by Proposition 4.2.6, $\mathrm{dpol}(\tilde{f}) = \mathrm{dpol}(\psi \circ \phi_2^{-1} \circ f \circ \phi_2 \circ \psi^{-1}) = 4$, which proves the claim of Example 4.1.11.

$\square$

4.4   Remarks

4.4.1   Other degree functions

One can define more general degree functions. Let X_k be a smooth projective variety over a field k with polarization H and $1 \leq l \leq d-1$, then

$$\deg_H^l(f) := (f^* H)^l \cdot H^{d-l}.$$

These degree functions play an important role in dynamics. In characteristic 0, we still have $\deg_H^l(fg) \leq C \deg_H^l(f) \deg_H^l(g)$ for a constant C not depending on f and

g (see [DS05] for characteristic 0, and [Tru15], [Tru16] for generalizations to fields of positive characteristic).

Our proof of Theorem 4.1.1 works as well if we replace the function $\deg_H$ by $\deg_H^l$ for any l. Let $\Gamma \subset \mathrm{Bir}(X_k)$ be a finitely generated group of birational transformations with a finite symmetric set of generators S. We define $D_{S,H}^l \colon \mathbb{Z}^+ \to \mathbb{Z}^+$ by $D_{S,H}^l(n) := \max_{\gamma \in B_S(n)}\{\deg_H(\gamma)\}$ and we call a map $\mathbb{Z}^+ \to \mathbb{Z}^+$ that can be realized for some (X_k, H, Γ, S, l) as such a function a *general degree sequence*.

Theorem 4.4.1. *The set of all general degree sequences is countable.*

Proof. Analogous to the proof of Theorem 4.1.1. $\square$

Acknowledgements

Chapter 5

Preliminaries

5.1 Introduction

To a smooth projective surface S over a field $\mathbf{k}$ one can associate its group of birational transformations $\mathrm{Bir}(S)$. If S is rational, this group is particularly rich and interesting. In this case it is isomorphic to the *plane Cremona group*

$$\mathrm{Cr}_2(\mathbf{k}) \coloneqq \mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}).$$

In this section we always assume that $\mathbf{k}$ is a fixed algebraically closed field. If we choose homogeneous coordinates $[x : y : z]$ of $\mathbb{P}^2_{\mathbf{k}}$, every element $f \in \mathrm{Cr}_2(\mathbf{k})$ is given by

$$[x : y : z] \dashrightarrow [f_0(x,y,z) : f_1(x,y,z) : f_2(x,y,z)],$$

where $f_0, f_1, f_2 \in \mathbf{k}[x,y,z]$ are homogeneous polynomials of the same degree and without a non-constant common factor. We will always assume that the homogeneous coordinates are fixed and identify f with $[f_0 : f_1 : f_2]$.

In the last decade many results about the group structure of the plane Cremona group have been proven. One of the main techniques to better understand infinite subgroups of $\mathrm{Cr}_2(\mathbf{k})$ was the construction of an action by isometries of the plane Cremona group on an infinite dimensional hyperbolic space and the use of results from hyperbolic geometry and group theory. The aim of this chapter is to gather some results we need for our purposes. Most of the times we will refer to other sources for the proofs.

5.2 An infinite dimensional hyperbolic space

5.2.1 The bubble space

Let X be a smooth projective surface. The *bubble space* $\mathcal{B}(X)$ is, roughly speaking, the set of all points that belong to X or are infinitely near to X. More precisely, $\mathcal{B}(X)$ can be defined as the set of all triples (y, Y, π), where Y is a smooth projective surface, $y \in Y$ and $\pi \colon Y \to X$ a birational morphism modulo the following equivalence relation: A triple (y, Y, π) is equivalent to (y', Y', π') if there exists a

birational transformation $\varphi\colon Y \dashrightarrow Y'$ that restricts to an isomorphism in a neighborhood of y, maps y to y' and satifies $\pi' \circ \varphi = \pi$. A point $p \in \mathcal{B}(X)$ that is equivalent to (x, X, id) is called a *proper point* of X. All points in $\mathcal{B}(X)$ that are not proper are called *infinitely near*. If there is no ambiguity, we will sometimes denote a point in the bubble space by y instead of (y, Y, π).

Let $f\colon X \dashrightarrow Y$ be a birational transformation. By Zariski's theorem (see [Bea96, Corollary II.12]) we can write $f = \pi_2 \circ \pi_1^{-1}$, where $\pi_1\colon Z \to X, \pi_2\colon Z \to Y$ are finite sequences of blow ups. We may assume that there is no (-1)-curve in Z that is contracted by both, π_1 and π_2. The *base-points* $\mathcal{B}(f)$ of f are the points in $\mathcal{B}(X)$ that are blown up by π_1. The proper base-points are sometimes called *indeterminacy points* of f in the above sense.

A birational morphism $\pi\colon X \to Y$ induces a bijection $(\pi_1)_\bullet\colon \mathcal{B}(X) \to \mathcal{B}(Y) \setminus \mathcal{B}(\pi^{-1})$ by sending a point represented by (x, X, φ) to the point represented by $(x, X, \pi \circ \varphi)$. A birational transformation of smooth projective surfaces $f\colon X \dashrightarrow Y$ induces a bijection $f_\bullet\colon \mathcal{B}(X) \setminus \mathcal{B}(f) \to \mathcal{B}(Y) \setminus \mathcal{B}(f^{-1})$ by $f_\bullet := (\pi_2)_\bullet \circ (\pi_1)_\bullet^{-1}$, where $\pi_1\colon Z \to X$, $\pi_2\colon Z \to Y$ is a minimal resolution of f.

5.2.2   The Picard-Manin space

Let X be a smooth projective surface and $\mathrm{Pic}(X)$ its Picard group. The intersection between curves extends to a quadratic form $\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}$, $(C, D) \mapsto C \cdot D$. The Néron-Severi group $\mathrm{NS}(X)$ is the quotient of $\mathrm{Pic}(X)$ modulo numerical equivalence, i.e. C is equivalent to D if $C \cdot E = D \cdot E$ for all divisors E. Recall that in the case of rational surfaces, we have $\mathrm{NS}(X) = \mathrm{Pic}(X)$.

The Néron-Severi group is a finitely generated free abelian group, whose rank $\rho(X)$ is called the *Picard number*. By the Hodge index theorem, the signature of the intersection form on $\mathrm{NS}(X)$ has signature $(1, \rho(X) - 1)$.

The pull-back of a birational morphism $\pi\colon Y \to X$ yields an injection of $\mathrm{Pic}(X)$ into $\mathrm{Pic}(Y)$ that preserves the intersection form, so we obtain an injection from $\mathrm{NS}(X)$ into $\mathrm{NS}(Y)$. The morphism $\pi\colon Y \to X$ can be written as a finite sequence of blow-ups. Let $e_1, \ldots, e_k \subset Y$ be the classes of the irreducible components of the exceptional divisor of π, i.e. the classes contracted by π. Then we have a decomposition

$$\mathrm{NS}(Y) = \mathrm{NS}(X) \oplus \mathbb{Z}\, e_1 \oplus \cdots \oplus \mathbb{Z}\, e_k, \tag{5.1}$$

which is orthogonal with respect to the intersection form.

Let $\pi_1\colon Y \to X$ and $\pi_2\colon Y' \to X$ be birational morphisms of smooth projective surfaces. We say that π_1 is *above* π_2 if $\pi_2^{-1} \circ \pi_1$ is a morphism. In other words, π_1 lies above π_2 if all the points that are blown up by π_2 are also blown up by π_1. For any two birational morphisms $\pi_1\colon Y \to X$ and $\pi_2\colon Y' \to X$ there exists a birational morphism $\pi_3\colon Y'' \to X$ that lies above π_1 and π_2.

Consider the set of all birational morphisms of smooth projective surfaces $\pi\colon Y \to X$. Our remark above shows that the corresponding embeddings of the Néron-Severi groups $\pi\colon \mathrm{NS}(X) \to \mathrm{NS}(Y)$ form a directed family, so the direct limit

$$\mathcal{Z}(X) := \varinjlim_{\pi\colon Y \to X} \mathrm{NS}(X)$$

exists. It is called the *Picard-Manin space* of X. The intersection forms on the groups $\mathrm{NS}(Y)$ induce a quadratic form on $\mathcal{Z}(X)$ of signature $(1, \infty)$.

Let $p \in \mathcal{B}(X)$ be a point in the bubble space of X and denote by e_p the divisor class of the exceptional divisor of the blow-up of p in the corresponding Néron-Severi group, i.e. e_p can be seen as a point in $\mathcal{Z}(X)$. From Equation 5.1 one deduces that the Picard-Manin space has the following decomposition

$$\mathcal{Z}(X) = \mathrm{NS}(X) \oplus \bigoplus_{p \in \mathcal{B}(X)} \mathbb{Z} \, e_p.$$

Moreover, $e_p \cdot e_p = -1$ and $e_p \cdot e_q = 0$ for all $p \neq q$, as well as $e_p \cdot l = 0$ for all $l \in \mathrm{NS}(X)$.

5.2.3 Some hyperbolic geometry

Let $\mathcal{H}$ be a real Hilbert space of dimension n, where n can also be infinite (in which case $n-1$ is infinite as well), and $e_0 \in \mathcal{H}$ a unit vector. Let $e_0^\perp$ be the orthogonal complement of the space $\mathbb{R} \, e_0$ and $(e_i)_{i \in I}$ an orthonormal basis of $e_0^\perp$. We define a scalar product $\langle \cdot \mid \cdot \rangle_h$ with signature $(1, n-1)$ on $\mathcal{H}$ by setting

$$\langle u \mid v \rangle_h := a_0 b_0 - \sum_{i \in I} a_i b_i,$$

for any two elements $u = a_0 e_0 + \sum_{i \in I} a_i e_i$ and $v = b_0 e_0 + \sum_{i \in I} b_i e_i$ in $\mathcal{H}$. The set of all elements in $v \in \mathcal{H}$ such that $\langle v \mid v \rangle_h = 1$ defines a hyperboloid with two connected components. We define $\mathbb{H}^{n-1}$ to be the connected component of this hyperboloid that contains e_0. We define a metric on $\mathbb{H}^{n-1}$ by

$$d(u, v) := \mathrm{arccosh}(\langle u \mid v \rangle_h).$$

It turns out that $\mathbb{H}^{n-1}$ with the metric d is a complete metric space that is δ-hyperbolic in the sense of Gromov with $\delta = \log(1 + \sqrt{2})$. Indeed, if we intersect $\mathbb{H}^{n-1}$ with a subspace of dimension $m \leq n-1$ of $\mathcal{H}$ such that the intersection is not empty, we obtain a hyperbolic space $\mathbb{H}^{m-1}$. In particular, every triangle in $\mathbb{H}^{n-1}$ lies in a subspace isometric to $\mathbb{H}^2$. The space $\mathbb{H}^\infty$ is hyperbolic (see for example [CLdC13]).

The *boundary* $\partial \mathbb{H}^{n-1}$ of $\mathbb{H}^{n-1}$ consists of the one-dimensional vector subspaces in the *light cone*, i.e. the cone of isotropic vectors with respect to $\langle \cdot \mid \cdot \rangle_h$.

Let $\mathrm{O}_{1,n}(\mathbb{R})$ be the group of linear transformations of $\mathcal{H}$ that preserve the form $\langle \cdot \mid \cdot \rangle_h$ and $\mathrm{O}_{1,n}^+(\mathbb{R}) \subset \mathrm{O}_{1,n}(\mathbb{R})$ the index two subgroup that preserves $\mathbb{H}^{n-1}$. Then $\mathrm{O}_{1,n}^+(\mathbb{R})$ acts by isometries on $\mathbb{H}^{n-1}$. On the other hand, one can also show that every isometry of $\mathbb{H}^{n-1}$ is induced by an element of $\mathrm{O}_{1,n}^+(\mathbb{R})$.

To an isometry h of $\mathbb{H}^{n-1}$ we associate $L(h) := \inf\{d(h(p), p) \mid p \in \mathbb{H}^{n-1}\}$. If $L(h) = 0$ and the infimum is achieved, i.e. h has a fixed point in $\mathbb{H}^{n-1}$, then h is called *elliptic*. If $L(h) = 0$ but the infimum is not achieved, we call h *parabolic*. It can be shown that a parabolic isometry fixes exactly one point p on the border $\partial \mathbb{H}^{n-1}$. If $L(h) > 0$ we call h *loxodromic*. In this case the set

$$\{p \in \mathbb{H}^{n-1} \mid d(h(p), p) = L(h)\}$$

is a geodesic line in $\mathbb{H}^{n-1}$. It is called the *axis* $\mathrm{Ax}(h)$ of h and $L(h)$ is called the *translation length*. A loxodromic isometry has exactly two fixed points in $\partial \mathbb{H}^\infty$, one of them attractive and the other repulsive (see [Can11a]).

5.2.4 An isometric action of $\mathrm{Bir}(X)$

Now we consider again a smooth projective surface X and its Picard-Manin space $\mathcal{Z}(X)$. Let $\mathrm{Z}(X)$ be the completion of the real vector space $\mathcal{Z}(X) \otimes \mathbb{R}$ that we obtain in the following way:

$$\mathrm{Z}(X) \coloneqq \{v + \sum_{p \in \mathcal{B}(X)} a_p e_p \mid v \in \mathrm{NS}(X) \otimes \mathbb{R}, a_p \in \mathbb{R}, \sum_{p \in \mathcal{B}(X)} a_p^2 < \infty\}.$$

The intersection form extends continuously to a quadratic form on $\mathrm{Z}(X)$ with signature $(1, \infty)$. Let $e_0 \in \mathrm{Z}(X)$ be a vector that corresponds to an ample class. We define $\mathbb{H}^\infty(X)$ to be the set of all vectors v in $\mathrm{Z}(X)$ such that $v \cdot v = 1$ and $e_0 \cdot v > 0$. This yields a distance d on $\mathbb{H}^\infty(X)$ by

$$d(u, v) \coloneqq \mathrm{arccosh}(u \cdot v).$$

This yields the same space as described above, for $n = \infty$. As before, the boundary $\partial \mathbb{H}^\infty(X)$ consists of the lines in the isotropic cone. Often, we will just write $\mathbb{H}^\infty$ and $\partial \mathbb{H}^\infty$ instead of $\mathbb{H}^\infty(X)$ and $\partial \mathbb{H}^\infty(X)$, if it is clear from the context, over which surface we are working.

Let us explain, how $\mathrm{Bir}(X)$ acts on $\mathbb{H}^\infty$. A birational morphism $f \colon Y \to X$ of smooth projective surfaces induces an isomorphism $f_* \colon \mathcal{Z}(Y) \to \mathcal{Z}(X)$. Let $\mathrm{NS}(Y) = \mathrm{NS}(X) \oplus \mathbb{Z} e_{p_1} \oplus \cdots \oplus \mathbb{Z} e_{p_n}$, where $p_1, \ldots, p_n \in \mathcal{B}(X)$ are the points blown up by f and e_{p_i} is the irreducible component in the exceptional divisor that is contracted to p_i. The map f_* is then defined by $f_*(e_p) = e_{f_\bullet(p)}$ for all $p \in \mathcal{B}(Y)$, $f_*(e_{p_i}) = e_{p_i}$ and $f_*(D) = D$ for all $D \in \mathrm{NS}(X) \subset \mathrm{NS}(Y)$ (where the inclusion is given by the pull back of f).

A birational map $f \colon Y \dashrightarrow X$ induces an isomorphism $f_* \colon \mathcal{Z}(Y) \to \mathcal{Z}(X)$, which is defined by $f_* = (\pi_2)_* \circ (\pi_1)_*^{-1}$, where $\pi_1 \colon Z \to Y, \pi_2 \colon Z \to X$ are birational morphisms such that $f = \pi_2 \circ \pi_1^{-1}$.

Now assume that $f \in \mathrm{Bir}(X)$. Then f_* yields an automorphism of $\mathcal{Z}(X) \otimes \mathbb{R}$, which extends to an automorphism of the completion $\mathrm{Z}(X)$ and preserves the intersection form. This automorphism thus preserves the hyperboloid $\mathbb{H}^\infty$ and it induces an isometry on $\mathbb{H}^\infty$. This gives an action by isometries of $\mathrm{Bir}(X)$ on $\mathbb{H}^\infty$.

We refer to [Man86], where this construction was developed for the first time and [Can11a] for details and proofs.

An element $f \in \mathrm{Cr}_2(\mathbb{C})$ is called *elliptic*, if the corresponding isometry on $\mathbb{H}^\infty$ is elliptic, *parabolic* if the corresponding isometry is parabolic and *loxodromic* if the corresponding isometry is loxodromic. The *axis* $\mathrm{Ax}(f)$ of a loxodromic element $f \in \mathrm{Cr}_2(\mathbb{C})$ is the axis in $\mathbb{H}^\infty$ of the isometry of $\mathbb{H}^\infty$ corresponding to f.

5.2.5 Dynamical degrees

Let X be a projective surface with a polarization H and $f \in \mathrm{Bir}(X)$. The *dynamical degree* of f is defined by

$$\lambda(f) \coloneqq \lim_{n \to \infty} \deg_H(f^n)^{\frac{1}{n}}.$$

The following result is well known. A proof can be found for example in [Can15, Lemma 4.5]:

Proposition 5.2.1. *The dynamical degree $\lambda(f)$ of a birational transformation $f \in \mathrm{Bir}(X)$ does not depend on the choice of the polarization H. Moreover, f is loxodromic if and only if $\lambda(f) > 1$. In this case the translation length of the isometry of $\mathbb{H}^\infty$ induced by f is $\log(\lambda(f))$.*

In [BC16], Blanc and Cantat studied the spectrum of possible values that can be obtained as dynamical degrees of birational transformations of a given projective surface.

Theorem 5.2.2 ([BC16, Corollary 1.7], [DF01]). *Let X be a projective surface over an algebraically closed field $\mathbf{k}$ and let $f \in \mathrm{Bir}(X)$ with $\lambda(f) > 1$. Then $\lambda(f)$ is either a Pisot or Salem number. Moreover, $\lambda(f) \geq \lambda_L$, where $\lambda_L > 1$ is the Lehmer number.*

Recall that the Lehmer number $\lambda_L \simeq 1.1762$ is the unique root > 1 of the irreducible polynomial $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$. The fact that there is no birational transformation f of a surface such that $1 < \lambda(f) < \lambda_L$ is usually refered to as the *gap property*.

One of the consequences of Theorem 5.2.2 is the following:

Corollary 5.2.3 ([BC16, Corollary 4.7]). *Let f be a birational transformation of a projective surface S. If f is loxodromic, the infinite cyclic group generated by f is a finite index subgroup of the centralizer of f in $\mathrm{Bir}(X)$.*

5.2.6 Degrees and types

The importance of the action of $\mathrm{Bir}(X)$ by isometries on $\mathbb{H}^\infty$ is a result of the following correspondence between the dynamical behavior of a birational transformation f of X, in particular its degree, and the type of the induced isometry on $\mathbb{H}^\infty$:

Theorem 5.2.4 (Gizatullin; Cantat; Diller and Favre). *Let X be a smooth projective surface over an algebraically closed field $\mathbf{k}$ with a fixed polarization H and $f \in \mathrm{Bir}(X)$. Then one of the following is true:*

(1) *f is elliptic, the sequence $\{\deg_H(f^n)\}$ is bounded and there exists a $k \in \mathbb{Z}_+$ and a birational map $\varphi \colon X \dashrightarrow Y$ to a smooth projective surface Y such that $\varphi f^k \varphi^{-1}$ is contained in $\mathrm{Aut}^0(Y)$, the neutral component of the automorphism group $\mathrm{Aut}(Y)$.*

(2a) *f is parabolic and $\deg_H(f^n) \sim cn$ for some positive constant c and f preserves a rational fibration, i.e. there exists a smooth projective surface Y, a birational map $\varphi \colon X \dashrightarrow Y$, a curve B and a fibration $\pi \colon Y \to B$, such that a general fiber of π is rational and such that $\varphi f \varphi^{-1}$ permutes the fibers of π.*

(2b) *f is parabolic and $\deg_H(f^n) \sim cn^2$ for some positive constant c and f preserves a fibration of genus 1 curves, i.e there exists a smooth projective surface Y, a birational map $\varphi \colon X \dashrightarrow Y$, a curve B and a fibration $\pi \colon Y \to B$, such that $\varphi f \varphi^{-1}$ permutes the fibers of π and such that π is an elliptic fibration, or a quasi-elliptic fibration (the latter only occurs if $\mathrm{char}(\mathbf{k}) = 2$ or 3).*

(3) f is loxodromic and $\deg_H(f^n) = c\lambda(f)^n + O(1)$ for some positive constant c, where $\lambda(f)$ is the dynamical degree of f. In this case, f does not preserve any fibration.

A first main step towards Theorem 5.2.4 has been done by Gizatullin in [Giz80] (see as well [Gri16]), where he classified parabolic automorphisms of surfaces. In [DF01], Diller and Favre proved the main result about the possible degree growths (see also [Fav10]).

Theorem 5.2.4 has lead to various remarkable results on the group structure of $\mathrm{Bir}(X)$; we will state some of them in the following sections. From the point of view of geometric group theory, the plane Cremona group acting on $\mathbb{H}^\infty$ has some analogies with other groups acting on hyperbolic spaces such as for example the mapping class group of a surface acting on the complex of curves or groups of outer automorphisms of a free group with n generators acting on the outer space.

Let us also recall the following result, which we will use later:

Theorem 5.2.5 ([Can11a, Theorem 6.6]). *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a subgroup. If G does not contain any loxodromic element, then G fixes a point in $\mathbb{H}^\infty \cup \partial \mathbb{H}^\infty$.*

5.3   The parabolic case

5.3.1   The de Jonquières subgroup

A fibration of a surface S is a rational map $\pi\colon S \dashrightarrow C$, where C is a curve such that the general fibers are one-dimensional. We will identify two fibrations $\pi_1\colon S \dashrightarrow C$ and $\pi_2\colon S \dashrightarrow C'$ with each other if there exists an open dense subset $U \subset S$ that is contained in the domain of π_1 and π_2 such that the restrictions of π_1 and π_2 to U define the same set of fibers. We say that a group $G \subset \mathrm{Bir}(S)$ *preserves* a fibration π if G permutes the fibers, i.e. there exists a rational G-action on C such that π is a G-equivariant map. A *rational fibration* of a rational surface X is a rational map $\pi\colon S \dashrightarrow \mathbb{P}^1$ such that the general fiber is rational.

The following theorem is due to Noether and Enriques. We refer to [Bea96, III.4] for a proof.

Theorem 5.3.1. *Let $\pi\colon X \dashrightarrow C$ be a rational fibration. Then there exists a birational map $\varphi\colon C \times \mathbb{P}^1 \dashrightarrow X$ such that $\pi \circ \varphi$ is the projection onto the first factor.*

In other words, Theorem 5.3.1 states that, up to birational transformations, there exists just one rational fibration of $\mathbb{P}^2$.

Definition 5.3.1. The *de Jonquières subgroup* $\mathcal{J}$ of $\mathrm{Cr}_2(\mathbb{C})$ is the subgroup of elements that preserve the pencil of lines through the point $[0:0:1] \in \mathbb{P}^2$.

With respect to affine coordinates $[x:y:1]$ an element in $\mathcal{J}$ is of the form

$$(x,y) \dashmapsto \left(\frac{ax+b}{cx+d}, \frac{\alpha(x)y + \beta(x)}{\gamma(x)y + \delta(x)} \right),$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C})$ and $\begin{pmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & \delta(x) \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C}(x))$. This induces an isomorphism

$$\mathcal{J} \simeq \mathrm{PGL}\,2(\mathbb{C}) \ltimes \mathrm{PGL}_2(\mathbb{C}(x)).$$

By Theorem 5.3.1, every subgroup of $\mathrm{Cr}_2(\mathbb{C})$ that preserves a rational fibration is conjugate to a subgroup of $\mathcal{J}$.

5.3.2 Halphen surfaces

Consider two smooth cubic curves C and D in $\mathbb{P}^2$. Then C and D intersect in 9 points $p_1, \ldots, p_9$ and there is a pencil of cubic curves passing through these 9 points. By blowing up $p_1, \ldots, p_9$, we obtain a rational surface X with a fibration $\pi \colon X \to \mathbb{P}^1$ whose fibers are genus 1 curves. More generally, we can consider a pencil of curves of degree $3m$ for any $m \in \mathbb{Z}_+$ and blow up its base-points to obtain a surface X. Such a pencil of genus 1 curves is called a *Halphen pencil* and the surface X a *Halphen surface of index m*. A surface X is Halphen if and only if the linear system $|-mK_X|$ is one-dimensional, has no fixed component and is base-point free. Up to conjugacy by birational maps, every pencil of genus 1 curves of $\mathbb{P}^2$ is a Halphen pencil and Halphen surfaces are the only examples of rational elliptic surfaces. We refer to [CD12] and [IS96, Chapter 10] for proofs and more details. A subgroup G of $\mathrm{Cr}_2(\mathbb{C})$ that preserves a pencil of genus 1 curves is therefore conjugate to a subgroup of the automorphism group of some Halphen surface.

Lemma 5.3.2. *Let X be a Halphen surface and $f \in \mathrm{Bir}(X)$ a birational transformation that preserves the Halphen pencil, then $f \in \mathrm{Aut}(X)$.*

Proof. Since the Halphen pencil is defined by a multiple of $-K_X$, the class of the anticanonical divisor, every birational transformation of a Halphen surface that preserves the Halphen fibration, preserves K_X, the class of the canonical divisor. Assume that f is not an automorphism and let Z be a minimal resolution of indeterminacies of f and $\pi, \eta \colon Z \to X$ such that $f = \pi \circ \eta^{-1}$. We have

$$\eta^*(K_X) + \sum E_i = K_Z = \pi^*(K_X) + \sum F_i,$$

where the E_i and F_i are the total pull-backs of the exceptional curves; in particular, $E_i^2 = -1$, $F_i^2 = -1$ and $E_i E_j = 0$, $F_i F_j = 0$ for $i \neq j$. Since f preserves K_X, we have that $\eta^*(K_X) = \pi(K_X)$ and hence $\sum E_i = \sum F_i$. Note that $\sum E_i$ contains at least one (-1)-curve E_k. Hence $E_k \cdot (\sum E_i) = -1 = E_k \cdot (\sum F_i)$. But this implies that E_k is contained in the support of $\sum F_i$, which contradicts the minimality of the resolution. $\qquad \square$

The automorphism groups of Halphen surfaces are studied in [Giz80] and in [CD12], see also [Gri16]. We need the following result, which can be found in [CD12, Remark 2.11]:

Theorem 5.3.3. *Let X be a Halphen surface. Then, there exists a homomorphism $\rho \colon \mathrm{Aut}(X) \to \mathrm{PGL}_2(\mathbb{C})$ with finite image such that $\ker(\rho)$ is an extension of an abelian group of rank ≤ 8 by a cyclic group of order dividing 24.*

We also recall the following result from [Can11a]:

Lemma 5.3.4. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a group that does not contain any loxodromic element but contains a parabolic element. Then G is conjugate to a subgroup of the de Jonquières group $\mathcal{J}$ or to a subgroup of $\mathrm{Aut}(Y)$, where Y is a Halphen surface.*

Proof. By Theorem 5.2.5, G fixes a point $q \in \mathbb{H}^\infty \cup \partial \mathbb{H}^\infty$. Let $f \in G$ be a parabolic element. By definition, f has no fixed point in $\mathbb{H}^\infty$ and a unique fixed point $p \in \partial \mathbb{H}^\infty$. It follows that $p = q$. By Theorem 5.2.4, there exists a birational map $\varphi \colon \mathbb{P}^2 \dashrightarrow Y$, a curve C and a fibration $\pi \colon Y \to C$ such that $\varphi f \varphi^{-1}$ permutes the fibers of π. In particular, $\varphi f \varphi^{-1}$ preserves the divisor class of a fiber F of π. Being the class of a fiber, F has self-intersection 0. The point $A \in \mathrm{Z}(\mathbb{P}^2)$ corresponding to F satisfies therefore $\langle A \mid A \rangle = 0$ and we obtain that $p \in \partial \mathbb{H}^\infty$ corresponds to the line passing through the origin and A. It follows that every element in G fixes A and hence that every element in G preserves the divisor class of F, i.e. every element in $\varphi G \varphi^{-1}$ permutes the fibers of the fibration $\pi \colon Y \to C$. If the fibration is rational, G is conjugate to a subgroup of $\mathcal{J}$. If it is a fibration of genus 1 curves, there exists a Halphen surface Y such that G is conjugate to a subgroup of $\mathrm{Bir}(Y)$ and such that G preserves the Halphen fibration. By Lemma 5.3.2, G is therefore contained in $\mathrm{Aut}(Y)$. $\qquad\square$

5.4   The Zariski topology and algebraic subgroups

Let X be a complex projective variety. The group $\mathrm{Bir}(X)$ can be equipped with some additional geometric and algebraic structure. We recall some definitions and results.

Let A be an algebraic variety and

$$f \colon A \times X \dashrightarrow A \times X$$

an A-birational map (i.e. a map of the form $(a, x) \mapsto (a, f(a, x))$ that induces an isomorphism between open subsets U and V of $A \times X$ such that the projections from U and from V to A are both surjective. For each $a \in A$ we obtain therefore an element of $\mathrm{Bir}(X)$ defined by $x \mapsto p_2(f(a, x))$, where p_2 is the second projection. Such a map $A \to \mathrm{Bir}(X)$ is called a *morphism* or *family of birational transformations parametrized by A*.

Definition 5.4.1. The *Zariski topology* on $\mathrm{Bir}(X)$ is the finest topology such that all morphisms $f \colon A \to \mathrm{Bir}(X)$ for all algebraic varieties A are continuous (with respect to the Zariski topology on A).

The map $\iota \colon \mathrm{Bir}(X) \to \mathrm{Bir}(X)$, $x \mapsto x^{-1}$ is continuous as well as the maps $x \mapsto g \circ x$ and $x \mapsto x \circ g$ for any $g \in \mathrm{Bir}(X)$. This follows from the fact that the inverse of an A-birational map as above is again an A-birational map as is the right/left-composition with an element of $\mathrm{Bir}(X)$. The Zariski topology was introduced in [Dem70] and [Ser08] and studied in [BF13].

An algebraic subgroup of $\mathrm{Bir}(X)$ is the image of an algebraic group G by a morphism $G \to \mathrm{Bir}(X)$ that is also an injective group homomorphism. It can be

shown that algebraic groups are closed in the Zariski topology and of bounded degree in the case of $\mathrm{Bir}(X) = \mathrm{Cr}_n(\mathbb{C})$. Conversely, closed subgroups of bounded degree in $\mathrm{Cr}_n(\mathbb{C})$ are always algebraic subgroups with a unique algebraic group structure that is compatible with the Zariski topology (see [BF13]). In [BF13], it is shown moreover, that all algebraic subgroups of $\mathrm{Cr}_n(\mathbb{C})$ are linear.

Let Y be a smooth projective variety that is birationally equivalent to X. Let G be an algebraic group acting regularly and faithfully on Y. This yields a morphism $G \to \mathrm{Bir}(X)$, so G is an algebraic subgroup of $\mathrm{Bir}(X)$. On the other hand, recall the theorem of Weil, Rosenlicht and Sumihiro states, that in the complex case all algebraic subgroups have this form:(cf. [Theorem 3.2.4] in Section 3 for further explanations and references):

Theorem 5.4.1 (Weil, Rosenlicht, Sumihiro)**.** *Let M be a complex variety and $G \subset \mathrm{Bir}(M)$ a linear algebraic subgroup. Then there exists a smooth projective variety N and a birational map $f\colon M \to N$ that conjugates G to a subgroup of $\mathrm{Aut}(N)$ such that the induced action on N is algebraic.*

It can be shown (see for example, [BF13]) that the sets $\mathrm{Cr}_n(\mathbb{C})_{\leq d} \subset \mathrm{Cr}_n(\mathbb{C})$ consisting of all birational transformations of degree $\leq d$ are closed with respect to the Zariski topology. Hence the closure of a subgroup of bounded degree in $\mathrm{Cr}_n(\mathbb{C})$ is an algebraic subgroup and since all algebraic subgroups of $\mathrm{Cr}_n(\mathbb{C})$ are linear ([BF13]), it can therefore be regularized by Theorem 3.2.4. We obtain:

Corollary 5.4.2. *Let $G \subset \mathrm{Cr}_n(\mathbb{C})$ be a subgroup that is contained in some set $\mathrm{Cr}_n(\mathbb{C})_{\leq d}$. Then there exists a smooth projective variety Y and a birational transformation $f\colon \mathbb{P}^n \dashrightarrow Y$ such that $fGf^{-1} \subset \mathrm{Aut}(Y)$.*

The maximal algebraic subgroups of $\mathrm{Cr}_2(\mathbb{C})$ have been classified together with the rational surfaces on which they act as automorphisms ([Enr93], [Bla09]). In dimension 3, a classification for maximal connected algebraic subgroups exists: [Ume82b], [Ume85], [Ume82a]. We will discuss and use the classification of maximal algebraic subgroups in Chapter 6.

Lemma 5.4.3. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a group that fixes a point in $\mathbb{H}^\infty$. Then the degree of all elements in G is uniformly bounded and there exists a smooth projective variety X and a birational transformation $\varphi\colon \mathbb{P}^2 \dashrightarrow X$ such that $\varphi G \varphi^{-1} \subset \mathrm{Aut}(X)$.*

Proof. Let $p \in \mathbb{H}^\infty$ be the fixed point of G and denote by $e_0 \in \mathbb{H}^\infty$ the class of a line in $\mathbb{P}^2$. Let $g \in G$ be arbitrary. Since the action of G on $\mathbb{H}^\infty$ is isometric, $d(g(e_0), p) = d(e_0, p)$, in particular, $d(g(e_0), e_0) \leq 2d(e_0, p)$. This implies $\langle g(e_0), e_0 \rangle \leq \cosh(2d(e_0, p))$ for all $g \in G$. As $\langle g(e_0), e_0 \rangle = \deg(g)$, the degree of all elements in G is uniformly bounded. The statement follows now from Corollary 5.4.2. $\qquad\square$

Lemma 5.4.4. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a positive dimensional connected algebraic subgroup. If an element $f \in \mathrm{Cr}_2(\mathbb{C})$ normalizes G then G has either a dense orbit, or the group generated by f and G is conjugate to a subgroup of the de Jonquières subgroup $\mathcal{J}$.*

Proof. By Theorem 3.2.4, there exists a smooth projective variety Y and a birational map $f \colon \mathbb{P}^2 \dashrightarrow Y$ that conjugates the rational action of G on $\mathbb{P}^2$ to a regular action on Y. By geometric invariant theory, there exists a G-invariant dominant rational map $\pi \colon Y \dashrightarrow X$, for some variety X, such that every general fiber of π contains an orbit of G that is dense in the fiber. As G is positive dimensional, the dimension of X is either 0 or 1. If the dimension of X is 0, then G has an open and dense orbit on Y. If X has dimension 1, then the general fibers have dimension 1. Since G is an algebraic subgroup of $\mathrm{Cr}_2(\mathbb{C})$, it is linear. Therefore, the general fibers of π are rational curves. As f normalizes G, it permutes the orbits of G and therefore preserves the rational fibration given by π. By Theorem 5.3.1, f is conjugate to a subgroup of the de Jonquières group. $\qquad\square$

5.5   Tori and monomial maps

An algebraic torus of rank n is an algebraic subgroup isomorphic to $(\mathbb{C}^*)^n$. The subgroup of diagonal automorphisms $D_2 \subset \mathrm{PGL}_3(\mathbb{C})$ is a torus of rank 2. All algebraic tori in $\mathrm{Cr}_2(\mathbb{C})$ are of rank ≤ 2 and are conjugate in $\mathrm{Cr}_2(\mathbb{C})$ to a subtorus of D_2 ([BB67], [Dem70]).

An integer matrix $A = (a_{ij}) \in M_2(\mathbb{Z})$ determines a rational map f_A of $\mathbb{P}^2$, which we define, with respect to local coordinates (x, y), by

$$f_A = (x^{a_{11}} y^{a_{12}}, x^{a_{21}} y^{a_{22}}).$$

We have $f_A \circ f_B = f_{AB}$ for $A, B \in M_2(\mathbb{Z})$. One observes that f_A is a birational transformation if and only if $A \in \mathrm{GL}_2(\mathbb{Z})$. This yields an injective homomorphism $\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{Cr}_2(\mathbb{C})$. By abuse of notation, we will identify its image with $\mathrm{GL}_2(\mathbb{Z})$. The normalizer of D_2 in $\mathrm{Cr}_2(\mathbb{C})$ is the semidirect product

$$\mathrm{Norm}_{\mathrm{Cr}_2(\mathbb{C})}(D_2) = \mathrm{GL}_2(\mathbb{Z}) \ltimes D_2.$$

We call $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$ the *group of monomial transformations* and its elements *monomial maps*. We say that $f \in \mathrm{Cr}_2(\mathbb{C})$ is of *monomial type*, if f is conjugate to an element in $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$. We call a matrix $A \in \mathrm{GL}_2(\mathbb{Z})$ *loxodromic*, if the corresponding birational monomial map in $\mathrm{Cr}_2(\mathbb{C})$ is loxodromic.

We will prove a couple of easy lemmas for later use.

Lemma 5.5.1. *Let $m \in \mathrm{GL}_2(\mathbb{Z}) \subset \mathrm{Cr}_2(\mathbb{C})$ be a loxodromic monomial map and $\Delta_2 \subset D_2$ an infinite subgroup that is normalized by m. Then Δ_2 is dense in D_2 with respect to the Zariski topology.*

Proof. Let $\overline{\Delta}_2^0$ be the neutral component of the Zariski-closure of Δ_2. If $\overline{\Delta}_2^0$ has dimension 2, we are done. Otherwise, $\overline{\Delta}_2^0$ is of dimension 1, since Δ_2 is not finite. It follows, that $\overline{\Delta}_2^0$ has no dense orbit and therefore, by Lemma 5.4.4, that m is conjugate to an element of the de Jonquières group. This is not possible since m is loxodromic, by assumption. $\qquad\square$

Lemma 5.5.2. *Let $f \in \mathrm{Cr}_2(\mathbb{C})$ be a birational transformation such that $f D_2 f^{-1} \subset \mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$. Then $f \in \mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$, i.e. there exists, up to automorphisms of D_2, a unique embedding of D_2 into $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$.*

Proof. Since fD_2f^{-1} is an algebraic subgroup, it is of bounded degree. Since $\mathrm{GL}_2(\mathbb{Z})$ contains only finitely many elements of a given degree, fD_2f^{-1} is therefore contained in a group of the form $H \ltimes D_2$, where H is finite. Since fD_2f^{-1} is connected, it is contained in D_2, which implies that f normalizes D_2. □

Lemma 5.5.3. *Let $g \in \mathrm{GL}_2(\mathbb{Z})$ be a loxodromic element and $f \in \mathrm{Cr}_2(\mathbb{C})$ an element such that fgf^{-1} is contained in $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$. Then $f \in \mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$.*

Proof. Assume that $fgf^{-1} \in \mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$. Then the axis of the loxodromic element fgf^{-1} is fixed pointwise by both, fD_2f^{-1} and D_2 (see Example 5.6.1). Hence, the group A generated by fgf^{-1} and D_2 is bounded. By Theorem 5.5.4, A is conjugate to a subgroup of D_2. This implies that $fD_2f^{-1} \subset D_2$ and therefore, that $f \in \mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$. □

Let $M \in \mathrm{GL}_2(\mathbb{Z})$ and f_M be the corresponding birational transformation. The dynamical degree $\lambda(f_M)$ of f_M is exactly the spectral radius of the matrix M. This shows in particular that the dynamical degree of a monomial matrix is always a quadratic algebraic integer.

If $M \in \mathrm{GL}_2(\mathbb{Z})$ has spectral radius strictly larger than 1, the birational map f_M is loxodromic. This yields examples of loxodromic elements that normalize an infinite elliptic subgroup. The following theorem by Cantat shows that, up to conjugacy, these are the only examples with this property:

Theorem 5.5.4 ([DP12, Appendix]). *Let N be a subgroup of $\mathrm{Cr}_2(\mathbb{C})$ containing at least one loxodromic element. Assume that there exists a short exact sequence*

$$1 \to A \to N \to B \to 1,$$

where A is infinite and of bounded degree. Then N is conjugate to a subgroup of $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$.

Later, we will generalize Theorem 5.5.4 (see Theorem 6.6.3) to the case where A is a group that only contains elliptic elements.

5.6 Small cancellation

Let $\epsilon, B > 0$. Two geodesic lines L and L' in $\mathbb{H}^\infty$ are called (ϵ, B)-close, if the diameter of the set $S = \{x \in L \mid d(x, L') \leq \epsilon\}$ is at least B, i.e. there exist two points in S with distance at least B.

Definition 5.6.1. Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a subgroup. A loxodromic element $g \in G$ is called *rigid in G* if there exist $\epsilon, B > 0$ such that for every element $h \in G$ we have: $h(\mathrm{Ax}(g))$ is (ϵ, B)-close to $\mathrm{Ax}(g)$ if and only if $h(\mathrm{Ax}(g)) = \mathrm{Ax}(g)$.

Definition 5.6.2. Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a subgroup. A loxodromic element $g \in G$ is called *tight in G* if it is rigid and if, for all $h \in G$, $h(\mathrm{Ax}(g)) = \mathrm{Ax}(g)$ implies $hgh^{-1} = g$ or $hgh^{-1} = g^{-1}$.

Example 5.6.1. Let $m \in \mathrm{GL}_2(\mathbb{Z}) \subset \mathrm{Cr}_2(\mathbb{C})$ be a loxodromic element. Then the group D_2 fixes the axis of m pointwise and no power of m is tight ([Can15, Example 7.1]).

The following theorem has been developed in various contexts. In [CLdC13] the authors use a version of it to show that $\mathrm{Cr}_2(\mathbb{C})$ is not simple, by showing that a generic element in $\mathrm{Cr}_2(\mathbb{C})$ is tight in $\mathrm{Cr}_2(\mathbb{C})$. This result was later generalized by Lonjou to show that $\mathrm{Cr}_2(\mathbf{k})$ is not simple, where $\mathbf{k}$ is an arbitrary field ([Lon16]). Dahmani, Guirardel and Osin used a similar statement in the context of mapping class groups ([DGO17]). We refer to [Cou16] for an overview of the subject.

Theorem 5.6.2 ([CLdC13, Theorem 2.10]). *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a subgroup and $g \in G$ an element that is tight in G. Then every element h in $\langle\langle g \rangle\rangle$, the smallest normal subgroup of G containing g, satisfies the following alternative: Either h is a conjugate of g or h is a loxodromic element with larger translation length than g. In particular, for $n \geq 2$, g is not contained in $\langle\langle g^n \rangle\rangle$.*

In [SB13], Shepherd-Barron classifies tight elements in $G = \mathrm{Cr}_2(\mathbb{C})$ using Theorem 5.5.4:

Theorem 5.6.3 ([SB13]). *In $\mathrm{Cr}_2(\mathbb{C})$ every loxodromic element is rigid. If g is conjugate to a monomial map, then no power of g is tight. In all the other cases, there exists an integer n such that g^n is tight.*

Note that if $G \subset \mathrm{Cr}_2(\mathbb{C})$ is a subgroup and $g \in \mathrm{Cr}_2(\mathbb{C})$ is a rigid element, then g is rigid in G as well. The same is true for tight elements. However, there might be loxodromic elements $g \in G$ such that g is tight in G but not in $\mathrm{Cr}_2(\mathbb{C})$. From the proof of Theorem 5.6.3 (see [SB13, p.18]) and Lemma 5.5.1 the following Theorem follows:

Theorem 5.6.4. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a subgroup and $g \in G$ a loxodromic element. The following two conditions are equivalent:*

(1) *no power of g is tight in G;*

(2) *there is a subgroup $\Delta_2 \subset G$ that is normalized by g and a birational transformation $f \in \mathrm{Cr}_2(\mathbb{C})$ such that $f \Delta_2 f^{-1} \subset D_2$ is a dense subgroup and $f g f^{-1} \in \mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$.*

$\square$

From Theorem 5.6.2 and Theorem 5.6.4 one deduces directly the following lemma, which we will need later:

Lemma 5.6.5. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a simple subgroup. Then for every loxodromic element $g \in G$ there exists an infinite subgroup $\Delta_2^g \subset G$ and an element $f \in \mathrm{Cr}_2(\mathbb{C})$ that conjugates Δ_2^g to a subgroup of D_2 and g to an element of $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$.* $\square$

Chapter 6

Subgroups of elliptic elements

6.1 Introduction

In this chapter we are interested in groups $G \subset \mathrm{Cr}_2(\mathbb{C})$ such that every element in G is elliptic. In Lemma 5.4.3, we have seen that g is elliptic if and only if $\{\deg(f^n)\}$ is bounded or, equivalently, there exists a $k \in \mathbb{Z}_+$ such that g^k is conjugate to an automorphism in $\mathrm{Aut}^0(S)$, the neutral component of the automorphism group of a projective surface S ([Can11a, Proposition 3.10]).

Definition 6.1.1. A group $G \subset \mathrm{Cr}_2(\mathbb{C})$ is a *group of elliptic elements* if every element in G is elliptic.

Definition 6.1.2. A group $G \subset \mathrm{Cr}_2(\mathbb{C})$ is *bounded* if there exists a constant K such that $\deg(g) \leq K$ for all elements $g \in G$.

Algebraic groups are always bounded, but bounded groups do not need to be algebraic. However, a subgroup of $\mathrm{Cr}_2(\mathbb{C})$ is bounded if and only if it is contained in an algebraic group (see Section 5.4). Note that all bounded groups are groups of elliptic elements. But the converse is not true as the following examples illustrate:

Example 6.1.1. Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be the group of elements of the form $(x, y+p(x))$, where $p(x) \in \mathbb{C}(x)$ is a rational function. Then every element in G is algebraic, but G contains elements of arbitrarily high degrees.

Example 6.1.2. In [Wri79], Wright constructs examples of torsion subgroups of $\mathrm{Aut}(\mathbb{A}^2)$ and hence in particular of $\mathrm{Cr}_2(\mathbb{C})$ that contain elements of arbitrarily high degree. In fact, he shows that there is a subgroup G of $\mathrm{Cr}_2(\mathbb{C})$ that is isomorphic to the subgroup of roots of unity in $\mathbb{C}^*$ but that is not bounded. In [Lam01a], Lamy shows that some of the examples of Wright do not preserve any fibration.

So far there only exist results about subgroups consisting of elliptic elements if they are bounded or finitely generated. Our strategy will be to use these results together with the compactness theorem from model theory in order to prove the

following theorem that gives some information about the structure of subgroups of $\mathrm{Cr}_2(\mathbb{C})$ consisting only of elliptic elements:

Theorem 6.1.3. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a subgroup of elliptic elements. Then we are in one of the following cases:*

(1) *G is not a torsion group and G contains a finitely generated subgroup that does not preserve any fibration. In this case, G is conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$, of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ or of $\mathrm{Aut}(S_6)$, where S_6 is the del Pezzo surface of degree 6.*

(2) *G is a torsion group.*

(3) *G preserves a rational fibration and is therefore conjugate to a subgroup of the de Jonquières group $\mathcal{J} \simeq \mathrm{PGL}_2(\mathbb{C}(t)) \rtimes \mathrm{PGL}_2(\mathbb{C})$, which is the subgroup of $\mathrm{Cr}_2(\mathbb{C})$ that preserves a given rational fibration.*

(4) *Every finitely generated subgroup of G preserves a rational fibration and there exists a torsion subgroup $G_0 \subset G$ and an exact sequence*

$$1 \to G_0 \to G \to \mathrm{PGL}_2(\mathbb{C}).$$

In the case, where G is a torsion subgroup of $\mathrm{Cr}_2(\mathbb{C})$, we can say more:

Theorem 6.1.4. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a torsion subgroup. Then G can be embedded into $\mathrm{GL}_N(\mathbb{C})$ for some $N \leq 48$ and $N \leq 24$ if G is infinite.*

Remark 6.1.1. There exists an $N \in \mathbb{Z}_+$ such that every finite subgroup of $\mathrm{Cr}_2(\mathbb{C})$ can be embedded into $\mathrm{GL}_N(\mathbb{C})$ (Proposition 6.2.2). In this text we will give the rough upper bound $N \leq 48$, but we expect that this bound can be lowered significantly. We will see in particular, that every finite subgroup of $\mathrm{Cr}_2(\mathbb{C})$ can be either embedded into $\mathrm{GL}_{36}(\mathbb{C})$ or it is isomorphic to an extension of $\mathcal{S}_4$ by the Klein four-group.

Let us give some consequences of Theorem 6.1.3. It has been proven by Schur that every torsion subgroup of $\mathrm{GL}_n(\mathbb{C})$ is abelian up to finite index. More precisely:

Theorem 6.1.5 (Schur, see [CR62, p.258]). *Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a torsion subgroup. Then G contains an abelian subgroup of index at most*

$$(\sqrt{8n}+1)^{2n^2} - (\sqrt{8n}-1)^{2n^2}.$$

So from Theorem 6.1.4 (2) and Theorem 6.1.5 we can deduce directly that the same property holds for torsion subgroups of $\mathrm{Cr}_2(\mathbb{C})$:

Corollary 6.1.6. *Every torsion subgroup G of $\mathrm{Cr}_2(\mathbb{C})$ is finite or isomorphic to a subgroup of $\mathrm{GL}_N(\mathbb{C})$ for some $N \leq 24$. In particular, G contains an abelian subgroup of index at most*

$$(96\sqrt{2}+1)^{2592} - (96\sqrt{2}-1)^{2592} < 10^{5537}.$$

Recall that the derived series of a group G is the series of groups defined by

- $G^{(0)} \coloneqq G$

- $G^{(i+1)} \coloneqq [G^{(i)}, G^{(i)}]$,

where the *commutator subgroup* $[H, H]$ of a group H is the subgroup generated by all elements of the form $aba^{-1}b^{-1}$, $a, b \in H$.

A group G is solvable if and only if its derived series terminates at the identity element after finitely many steps. The derived length of G is the smallest k such that $G^{(k)} = \{\mathrm{id}\}$.

Another result that can be deduced from Theorem 6.1.3 concerns the Tits alternative. In [Tit72], Tits showed the following:

Theorem 6.1.7 ([Tit72])**.** *Let* **k** *be a field of characteristic zero and* $n \in \mathbb{Z}_+$. *Then every subgroup* G *of* $\mathrm{GL}_n(\mathbf{k})$ *is either virtually solvable or contains a non-abelian free subgroup.*

This result has lead to the following definitions:

Definition 6.1.3.

(1) A group G satisfies *Tits' alternative* if every subgroup of G is either virtually solvable or contains a non-abelian free subgroup.

(2) A group G satisfies *Tits' alternative for finitely generated subgroups* if every finitely generated subgroup of G either is virtually solvable or contains a non-abelian free subgroup.

Theorem 6.1.7 states that linear groups over fields of characteristic zero satisfy Tits' alternative. Linear groups over fields of positive characteristic only satisfy Tits' alternative for finitely generated subgroups ([Tit72]). Other well-known examples of groups that satisfy Tits' alternative include mapping class groups of surfaces ([Iva84]), $\mathrm{Out}(\mathbb{F}_n)$, the outer automorphism group of the free group of finite rank n ([BFH00]) or hyperbolic groups in the sense of Gromov ([Gro87]).

In his PhD thesis Lamy showed Tits' alternative for subgroups of $\mathrm{Aut}(\mathbb{A}_{\mathbb{C}}^2)$:

Theorem 6.1.8 ([Lam01b])**.** $\mathrm{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ *satisfies Tits' alternative.*

The proof of Theorem 6.1.8 relies on the amalgamated product structure of $\mathrm{Aut}(\mathbb{A}_{\mathbb{C}}^2)$ that is given by the Theorem of Jung and van der Kulk (see for example [Lam02]) and Bass-Serre theory (see [Ser77]).

Cantat established Tits' alternative for finitely generated subgroups of $\mathrm{Cr}_2(\mathbb{C})$:

Theorem 6.1.9 ([Can11a])**.** $\mathrm{Cr}_2(\mathbb{C})$ *satisfies Tits' alternative for finitely generated subgroups.*

Theorem 6.1.9 is part of a series of profound results about the group structure of the plane Cremona group that Cantat deduces from the action by isometries on the Picard-Manin space $\mathbb{H}^\infty$ by $\mathrm{Cr}_2(\mathbb{C})$ ([Can11a]). The main obstacle to generalize Theorem [Can11a] to arbitrary subgroups was caused by unbounded groups of elliptic elements that do not preserve any fibration. At this point, Theorem 6.1.3 steps in. It turns out that it yields the techniques to complete the result:

Theorem 6.1.10. $Cr_2(\mathbb{C})$ *satisfies Tits' alternative.*

In [Dés15], Déserti gives a description of solvable subgroups of $Cr_2(\mathbb{C})$. We complement her result with the following observation:

Theorem 6.1.11. *There exists a constant $K \leq 35$ such that the derived length of solvable subgroups of $Cr_2(\mathbb{C})$ is $\leq K$.*

Remark 6.1.2. Also here, the upper bound 35 is only a rough estimate and we expect that it can be lowered by a detailed examination of the derived length of finite subgroups of $Cr_2(\mathbb{C})$.

For each $n \in \mathbb{Z}_+$ there exists an $N \in \mathbb{Z}_+$ such that every solvable subgroup of $GL_n(\mathbb{C})$ has derived length $\leq N$. This result seems to go back to Zassenhaus, see for example [New72], where explicit bounds are computed. The following bound is attributed to Huppert:

Theorem 6.1.12 ([New72, p.1]). *Let $G \subset GL_n(\mathbb{C})$ be a solvable subgroup. The derived length of G is $\leq \min\{2n, 1 + 7\log_2(n)\}$.*

In [FP16], Furter and Poloni show that the maximal derived length of a solvable subgroup of $Aut(\mathbb{A}^2_{\mathbb{C}})$ is 5 (and that this bound is optimal). Theorem 6.1.11 can be deduced from Theorem 6.1.3. However, we will give a direct and easier proof.

6.2   Maximal algebraic subgroups

In this section we recall some results about algebraic subgroups of $Cr_2(\mathbb{C})$. In [Bla09], Blanc classified all maximal algebraic subgroups of $Cr_2(\mathbb{C})$ (see [Ume82b], [Enr93] for the case of maximal connected algebraic subgroups). There are 11 classes of maximal algebraic subgroups. We summarize them in Theorem 6.2.1. We will recall in Section 6.2.1 and Section 6.2.2 the notions of del Pezzo surfaces and conic bundles.

Theorem 6.2.1 ([Bla09]). *Every algebraic subgroup of $Cr_2(\mathbb{C})$ is contained in a maximal algebraic subgroup. The maximal algebraic subgroups of $Cr_2(\mathbb{C})$ are conjugate to one of the following groups:*

(1) $Aut(\mathbb{P}^2) \simeq PGL_3(\mathbb{C})$

(2) $Aut(\mathbb{P}^1 \times \mathbb{P}^1) \simeq (PGL_2(\mathbb{C}))^2 \rtimes \mathbb{Z}/2\mathbb{Z}$

(3) $Aut(S_6) \simeq (\mathbb{C}^*)^2 \rtimes (\mathcal{S}_3 \times \mathbb{Z}/2\mathbb{Z})$, *where S_6 is the del Pezzo surface of degree 6.*

(4) $Aut(\mathbb{F}_n) \simeq \mathbb{C}[x,y]_n \rtimes GL_2(\mathbb{C})/\mu_n$, *where $n \geq 2$ and $\mathbb{F}_n$ is the n-th Hirzebruch surface and $\mu_n \subset GL_2(\mathbb{C})$ is the subgroup of n-torsion elements in the center of $GL_2(\mathbb{C})$.*

(5) $Aut(S, \pi)$, *where $\pi \colon S \to \mathbb{P}^1$ is an exceptional conic bundle.*

(6)-(10) $Aut(S)$, *where S is a del Pezzo surface of degree 5, 4, 3, 2 or 1. In this case, $Aut(S)$ is finite.*

(11) $\mathrm{Aut}(S, \pi)$, *where* (S, π) *is a* $(\mathbb{Z}/2\mathbb{Z})^2$-*conic bundle and* S *is not a del Pezzo surface. There exists an exact sequence*

$$1 \to V \to \mathrm{Aut}(S, \pi) \to H_V \to 1,$$

where $V \simeq (\mathbb{Z}/2\mathbb{Z})^2$ *and* $H_V \subset \mathrm{PGL}_2(\mathbb{C})$ *is a finite subgroup.*

In [Bla09] one finds a more detailed description of the groups above as well as a classification of the conjugacy classes of the maximal algebraic subgroups.

The aim of this section is to deduce from Theorem 6.2.1 the following Proposition, which implies that there exists a positive integer N such that every bounded subgroup of $\mathrm{Cr}_2(\mathbb{C})$ that is not contained in the automorphism group of a Hirzebruch surface can be embedded into $\mathrm{GL}_N(\mathbb{C})$:

Proposition 6.2.2. *There exists an* $N \in \mathbb{Z}_+$ *such that for every bounded subgroup* $G \subset \mathrm{Cr}_2(\mathbb{C})$ *there exists a group-homomorphism* $\varphi\colon G \to \mathrm{GL}_N(\mathbb{C})$ *such that* $\ker(\varphi)$ *is isomorphic to* $\mathbb{C}^n$ *for some* n. *The constant* N *can be chosen to be* ≤ 48.

We will prove Proposition 6.2.2 in Section 6.2.3.

6.2.1   Automorphism groups of del Pezzo surfaces

Recall that a *del Pezzo surface* is a smooth projective surface whose anticanonical divisor class is ample. The degree of a del Pezzo surface S is the self-intersection number of its canonical class. It is well-known that the degree of a del Pezzo surface is a positive integer ≤ 9. A del Pezzo surface is isomorphic to either $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ or to the blow-up S of r general points in $\mathbb{P}^2$. Here, general means that S does not contain a curve of self-intersection ≤ -2. In this last case, the degree of S is exactly $9 - r$. There exists a unique isomorphism class of del Pezzo surfaces of degree $5, 6, 7$ and 9, two isomorphism classes of del Pezzo surfaces of degree 8 and infinitely many isomorphism classes of del Pezzo surfaces of degree $1, 2, 3$ or 4 (see for example [Dol12, Chapter 8] for proofs and references for these and other results on del Pezzo surfaces). Automorphism groups of del Pezzo surfaces are always algebraic subgroups of $\mathrm{Cr}_2(\mathbb{C})$ ([Bla09, Proposition 2.2.6]) and they are finite if and only if the degree of the corresponding surface is ≤ 5.

If the degree of a del Pezzo surface S is 5, then $\mathrm{Aut}(S) = \mathcal{S}_5$. A precise description of automorphism groups of del Pezzo surfaces of degree ≤ 4 can be found in the tables in [DI09, Section 10]. This description yields in particular the following:

Theorem 6.2.3 ([DI09]). *If the automorphism group of a del Pezzo surface is finite, then it has order at most* 648.

From Theorem 6.2.3 it follows directly that every automorphism group of a del Pezzo surface that is finite can be embedded into $\mathrm{GL}_{648}(\mathbb{C})$. However, this bound can be improved significantly:

Lemma 6.2.4. *If the automorphism group of a del Pezzo surface is finite, then it can be embedded into* $\mathrm{GL}_8(\mathbb{C})$.

Proof. Let S be a del Pezzo surface such that $\mathrm{Aut}(S)$ is finite. This implies that S is of degree $d \leq 5$, hence S is isomorphic to the blow-up of $r = 9 - d$ general points $p_1, \ldots, p_r$ in $\mathbb{P}^2$, where $4 \leq r \leq 8$. The Néron-Severi space $\mathrm{NS}(S) \otimes \mathbb{R}$ is therefore of dimension $r + 1$ and has a basis $[E_0], [E_{p_1}], \ldots, [E_{p_r}]$, where $[E_0]$ is the pullback of the class of a line and the $[E_{p_i}]$ are the classes of the exceptional lines E_{p_i} corresponding to the points p_i. Since the self-intersection of a class $[E_{p_i}]$ equals -1, the exceptional line E_{p_i} is the only representative of $[E_{p_i}]$ on S. If $f \in \mathrm{Aut}(S)$ acts as the identity on $\mathrm{NS}(S) \otimes \mathbb{R}$, we obtain that f preserves the exceptional lines E_{p_i} for all i. Therefore, f induces an automorphism on $\mathbb{P}^2$ that fixes the points p_i. But since the p_i are in general position and $r \geq 4$, the automorphism f is the identity. Thus, the action of $\mathrm{Aut}(S)$ on $\mathrm{NS}(S) \otimes \mathbb{R}$ is faithful and we obtain a faithful representation $\mathrm{Aut}(S) \to \mathrm{GL}_{r+1}(\mathbb{C})$. As every element $f \in \mathrm{Aut}(S)$ fixes the canonical divisor K_S, the one-dimensional subspace $\mathbb{R} \cdot K_S$ in $\mathrm{NS}(S) \otimes \mathbb{R}$ is fixed. We project to the orthogonal complement of K_S in $\mathrm{NS}(S) \otimes \mathbb{R}$ and obtain a faithful representation of $\mathrm{Aut}(S)$ into $\mathrm{GL}_r(\mathbb{C})$. $\qquad \square$

A del Pezzo surface of degree 6 is isomorphic to the surface

$$S_6 = \{((x : y : z), (a : b : c)) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid ax = by = cz\},$$

which is the blow-up of $\mathbb{P}^2$ in three general points. The group $\mathrm{Aut}(S_6)$ is isomorphic to $(\mathbb{C}^*)^2 \rtimes (\mathcal{S}_3 \times \mathbb{Z}/2\mathbb{Z})$, where $\mathcal{S}_3$ acts by permuting the coordinates of the two factors simultanously, $\mathbb{Z}/2\mathbb{Z}$ exchanges the two factors and $d \in (\mathbb{C}^*)^2$ acts by sending $((x : y : z), (a : b : c))$ to $(d(x : y : z), d^{-1}(a : b : c))$ $(d(x : y : z)$ is the standard action on $\mathbb{P}^2)$. In other words, $\mathrm{Aut}(S_6)$ is conjugate to the subgroup $(\mathcal{S}_3 \times \mathbb{Z}/2\mathbb{Z}) \ltimes D_2 \subset \mathrm{GL}_2(Z) \ltimes D_2$.

Lemma 6.2.5. *The group* $\mathrm{Aut}(S_6)$ *can be embedded into* $\mathrm{GL}_6(\mathbb{C})$.

Proof. Consider the rational map $f \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^6$ given by

$$[x : y : z] \longmapsto [x^2y : x^2z : y^2x : y^2z : z^2x : z^2y : xyz].$$

Then, the rational action of $(\mathcal{S}_3 \times \mathbb{Z}/2\mathbb{Z}) \ltimes D_2$ on $f(\mathbb{P}^2)$ extends to a regular action on $\mathbb{P}^6$ that preserves the affine space given by $x_6 \neq 0$. This yields an embedding of $(\mathcal{S}_3 \times \mathbb{Z}/2\mathbb{Z}) \ltimes D_2$ into $\mathrm{GL}_6(\mathbb{C})$.

$\qquad \square$

Lemma 6.2.6. *Let* $G \subset \mathrm{Cr}_2(\mathbb{C})$ *be a subgroup that is conjugate to an automorphism group of a del Pezzo surface. Then* G *can be embedded into* $\mathrm{GL}_N(\mathbb{C})$ *for some* $N \leq 8$.

Proof. We prove that $\mathrm{Aut}(S)$ can be embedded into $\mathrm{GL}_N(\mathbb{C})$ for all del Pezzo surfaces S. The following results are proven in [Bla09, Section 3]. If S is a del Pezzo surface of degree 9 then S is isomorphic to $\mathbb{P}^2$, so $\mathrm{Aut}(S) = \mathrm{PGL}_3(\mathbb{C}) \subset \mathrm{GL}_8(\mathbb{C})$. This corresponds to case (1) of Theorem 6.2.1. If the degree of S is 8 then S is either isomorphic to $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ or to $\mathbb{F}_1$. The automorphism group $\mathrm{Aut}(\mathbb{F}_1)$ is not a maximal algebraic subgroup of $\mathrm{Cr}_2(\mathbb{C})$ and $\mathrm{Aut}(\mathbb{F}_0) = (\mathrm{PGL}_2(\mathbb{C}))^2 \rtimes \mathbb{Z}/2\mathbb{Z} \subset \mathrm{GL}_6(\mathbb{C})$. In the case that S is a del Pezzo surface of degree 7, $\mathrm{Aut}(S)$ is conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$. A del Pezzo surface of degree 6 can be embedded into $\mathrm{GL}_6(\mathbb{C})$, by Lemma 6.2.5. If the degree of a del Pezzo surface S is ≤ 5 then $\mathrm{Aut}(S)$ is finite and the claim follows from Lemma 6.2.4. $\qquad \square$

6.2.2 Automorphism groups of rational fibrations

In this section we consider the cases (4), (5) and (11) of Theorem 6.2.1.

Automorphism groups of Hirzebruch surfaces

In case (4) of Theorem 6.2.1, the semidirect product gives a natural homomorphism $\varphi\colon \mathrm{Aut}(\mathbb{F}_n) \to \mathrm{GL}_2(\mathbb{C})/\mu_n$ whose kernel is isomorphic to $\mathbb{C}^n$. The following lemma shows, that $\mathrm{GL}_2(\mathbb{C})/\mu_n$ can be embedded into $\mathrm{GL}_4(\mathbb{C})$ for all $n \geq 2$:

Lemma 6.2.7. *If $n \geq 2$ is even, the group $\mathrm{GL}_2(\mathbb{C})/\mu_n$ is isomorphic as an algebraic group to $\mathrm{PGL}_2(\mathbb{C}) \times \mathbb{C}^*$. If n is odd, then $\mathrm{GL}_2(\mathbb{C})/\mu_n$ is isomorphic as an algebraic group to $\mathrm{GL}_2(\mathbb{C})$.*

Proof. Assume that $n \geq 2$ is even, so $n = 2k$ for some $k \in \mathbb{Z}_+$. Consider the algebraic group homomorphism $\varphi_e\colon \mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C}) \times \mathbb{C}^*$ given by $A \mapsto ([A], \det(A)^k)$, where $[A]$ denotes the class of A modulo the center of $\mathrm{GL}_2(\mathbb{C})$. The kernel of φ_e consists of the scalar matrices of the form $a \cdot \mathrm{id}$ such that $\det(a)^k = 1$; hence, $\ker(\varphi_e) = \mu_n$. Let $(M, c) \in \mathrm{PGL}_2(\mathbb{C}) \times \mathbb{C}^*$. We choose a representative $A \in \mathrm{SL}_2(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{C})$ of the class of M and a n-th root of c in $\mathbb{C}$, which we denote by d. Then $\varphi_e(d \cdot A) = (M, c)$. So φ_e is surjective and the claim follows.

Assume that n is odd; so $n = 2k + 1$ for some $k \in \mathbb{Z}_+$. Consider the algebraic group homomorphism $\varphi_o\colon \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$ given by $A \mapsto \det(A)^k A$. Let $B \in \mathrm{GL}_2(\mathbb{C})$ and $c \in \mathbb{C}$ a n-th root of $\det(B)^k$. Then

$$\varphi_o(c^{-1} \cdot B) = c^{-2k-1} \det(B)^k \cdot B = B,$$

hence φ_o is surjective. Moreover, $\ker(\varphi_o) = \mu_n$, and the claim follows. $\square$

Automorphism groups of exceptional fibrations

In a next step, we consider the case $\mathrm{Aut}(S, \pi)$, where $\pi\colon S \to \mathbb{P}^1$ is an exceptional fibration, i.e. case (5) of Theorem 6.2.1. An *exceptional fibration S* is by definition a conic bundle with singular fibers above $2n$ points in $\mathbb{P}^1$ and with two sections s_1 and s_2 of self-intersection $-n$, where n is an integer ≥ 2 (see [Bla09]).

We recall a construction from representation theory that is usually called the *induced representation*:

Lemma 6.2.8. *Let G be a group and $H \subset G$ a subgroup of index n. If H can be embedded into $\mathrm{GL}_m(\mathbb{C})$ then G can be embedded into $\mathrm{GL}_{mn}(\mathbb{C})$.*

Proof. Let $\pi\colon G \to G/H$ be the quotient map and let V be a vector space of dimension m on which H acts faithfully. We define the mn-dimensional vector space W as the direct sum of n copies of V indexed by the elements G/H, i.e.:

$$W := \bigoplus_{k \in G/H} V_k.$$

We fix a set of representatives $g_1, \ldots, g_n \in G$ of the left cosets of H in G. Then every $g \in G$ can be written in a unique way as $g = g_i h$, where $h \in H$. Let $g \in G$

and $v \in V_l$; then g defines an automorphism of W in the following way: g permutes the summands of W by sending V_l to $V_{g(l)}$ and the restriction of g to V_l sends a vector $v \in V_l$ to the vector $h(v) \in V_{g(l)}$. This induces a faithful action of G on W. □

The proof of the following lemma can be found in [Bla09, proof of Lemma 4.3.3]. We briefly recall the arguments of the proof.

Lemma 6.2.9. *Let* $\pi \colon S \to \mathbb{P}^1$ *be an exceptional fibration. There exists a split exact sequence*

$$1 \to H \to \mathrm{Aut}(S, \pi) \to \mathbb{Z}/2\,\mathbb{Z} \to 1,$$

where H *is isomorphic to a subgroup of* $\mathrm{PGL}_2(\mathbb{C}) \times \mathbb{C}^* \subset \mathrm{GL}_4(\mathbb{C})$.

Proof. There exists a birational morphism $\eta_0 \colon S \to \mathbb{P}^1 \times \mathbb{P}^1$ of conic bundles that is the blow-up of $2n$ points for some $n \geq 1$; n of them lie on one line l_1 of self-intersection 0 and the other n on another line l_2 of self intersection 0 such that l_1 and l_2 do not intersect (see [Bla09, Lemma 4.3.1]). Let s_1 and s_2 in S be the strict transforms of l_1 and l_2 under η. Hence, s_1 and s_2 are of self-intersection $-n \leq -2$. The group $\mathrm{Aut}(S, \pi)$ then acts on the set $\{s_1, s_2\}$, since s_1 and s_2 are the unique curves of self-intersection $-n$. This gives an exact sequence

$$1 \to H \to \mathrm{Aut}(S, \pi) \to W \to 1,$$

where $W \subset \mathbb{Z}/2\,\mathbb{Z}$ and H preserves each of the sections s_1 and s_2. Therefore, H is conjugate by η_0 to a subgroup of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ that preserves the structure of a conic bundle, so H is conjugate to a subgroup of $\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})$. In fact, since H preserves the two lines l_1, and l_2, it is contained in $\mathrm{PGL}_2(\mathbb{C}) \times \mathbb{C}^*$. It remains to show that W is isomorphic to $\mathbb{Z}/2\,\mathbb{Z}$ and that it can be lifted to $\mathrm{Aut}(S, \pi)$. We may assume that the $2n$ points blown up by η_0 are of the form $\{(p_i, [0 : 1])\}_{i=1}^n$ and $\{(p_i, [1 : 0])\}_{i=n+1}^{2n}$ for some $p_i \in \mathbb{P}^1$. For each $i = 1, \ldots, 2n$, let $m_i \in \mathbb{C}[x_0, x_1]$ be a homogeneous form of degree 1 that vanishes on p_i. Consider the birational involution of $\mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\tau \colon ([x_0 : x_1], [y_0 : y_1]) \dashrightarrow \left([x_0 : x_1], [y_1 \prod_{i=1}^n m_i(x_0, x_1) : y_1 \prod_{i=n+1}^{2n} m_i(x_0, x_1)] \right).$$

The base-points of τ are exactly the $2n$ points blown up by η_0, and $\eta_0^{-1} \tau \eta_0$ is in $\mathrm{Aut}(S, \pi)$ and it exchanges the two sections s_1 and s_2. Hence $W = \langle \eta_0^{-1} \tau \eta_0 \rangle$, which proves the claim. □

Lemma 6.2.8 and Lemma 6.2.9 directly imply the following:

Corollary 6.2.10. *Let* $\pi \colon S \to \mathbb{P}^1$ *be an exceptional fibration. Then* $\mathrm{Aut}(S, \pi)$ *can be embedded into* $\mathrm{GL}_8(\mathbb{C})$.

$(\mathbb{Z}/2\mathbb{Z})^2$-conic bundles

In this section we treat case (11) of Theorem 6.2.1.

Lemma 6.2.11. *Let G be a group such that there exists a short exact sequence*

$$1 \to V \to G \xrightarrow{\pi} H_V \to 1,$$

where $V \simeq (\mathbb{Z}/2\mathbb{Z})^2$ and $H_V \in \{\mathcal{A}_5, \mathcal{S}_4, \mathcal{A}_4\}$ is a finite subgroup. Then G can be embedded into $\mathrm{GL}_{48}(\mathbb{C})$. Moreover, either G is an extension of $\mathcal{S}_4$ by V, or G can be embedded into $\mathrm{GL}_{24}(\mathbb{C})$.

Proof. First we assume that $H_V \simeq A_5$. Recall that a *stem extension* of a group H is a central extension $1 \to K \to C \to H \to 1$, such that K is contained in the commutator subgroup of C. If the group H is perfect, there exists a largest stem extension, unique up to isomorphism. It is called the *universal perfect central extension*. The central group K of the universal perfect central extension is isomorphic to the *Schur multiplier* of H. The Schur multiplier of A_5 is $\mathbb{Z}/2\mathbb{Z}$ and the universal central extension is $\mathrm{SL}_2(\mathbb{F}_5)$ (see [Sch04], [Sch07] and [HH92] for these results). Let $H := [G, G]$ be the commutator subgroup of G. There exist hence two possibilities. Either $V \cap H = \mathrm{id}$ or $V \cap H = \mathbb{Z}/2\mathbb{Z}$. Since A_5 is perfect there exists also a surjective group homomorphism $\varphi \colon H \to A_5$ defined on the generators of H by $\varphi(aba^{-1}b^{-1}) = \pi(a)\pi(b)\pi(a)^{-1}\pi(b)^{-1}$. Clearly, π is indeed a group homomorphism. If $V \cap H = \mathrm{id}$, the homomorphism φ is for cardinality reasons an isomorphism, i.e. $H \simeq A_5$. Hence we obtain a split exact sequence $1 \to H \to G \to V \to 1$ and since V is central, we have $G \simeq A_5 \times V$. If $V \cap H = \mathbb{Z}/2\mathbb{Z}$, then $H \simeq \mathrm{SL}_2(\mathbb{F}_5)$ and we conclude similarly that $G \simeq \mathrm{SL}_2(\mathbb{F}_5) \times \mathbb{Z}/2\mathbb{Z}$. Recall that we can embed A_5 into $\mathrm{GL}_3(\mathbb{C})$, so if $G \simeq A_5 \times V$, then G can be embedded into $\mathrm{GL}_5(\mathbb{C})$. Since $\mathrm{SL}_2(\mathbb{F}_5)$ is the universal perfect central extension, the projective representation of A_5 into $\mathrm{PGL}_2(\mathbb{C})$ lifts to a representation of $\mathrm{SL}_2(\mathbb{F}_5)$ into $\mathrm{GL}_2(\mathbb{C})$ and so $\mathrm{SL}_2(\mathbb{F}_5) \times \mathbb{Z}/2\mathbb{Z}$ can be embedded into $\mathrm{GL}_4(\mathbb{C})$.

If H_V is isomorphic to $\mathcal{A}_4$, the group G can be embedded into $\mathrm{GL}_{24}(\mathbb{C})$, by Lemma 6.2.8 and analogously, if H_V is isomorphic to $\mathcal{S}_4$, then G can be embedded into $\mathrm{GL}_{48}(\mathbb{C})$. $\qquad\square$

Our goal is to prove the following group theoretical lemma:

Lemma 6.2.12. *Let G be a group such that there exists a short exact sequence*

$$1 \to V \to G \xrightarrow{\pi} H_V \to 1,$$

where $V \simeq (\mathbb{Z}/2\mathbb{Z})^2$ and $H_V \subset \mathrm{PGL}_2(\mathbb{C})$ is a finite subgroup. Then G can be embedded into $\mathrm{GL}_{48}(\mathbb{C})$. Moreover, either G is an extension of $\mathcal{S}_4$ by V, or G can be embedded into $\mathrm{GL}_{36}(\mathbb{C})$.

Recall that the finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ are $\mathcal{A}_5, \mathcal{S}_4, \mathcal{A}_4, \mathbb{Z}/n\mathbb{Z}$ and D_{2n} for all $n \in \mathbb{Z}_+$ (see [Bea10]) and that the subgroups of a dihedral group D_{2n} are either cyclic or dihedral.

Proof of Lemma 6.2.12. If H_V is isomorphic to $\mathcal{A}_5, \mathcal{S}_4$ or $\mathcal{A}_4$, the claim follows from Lemma 6.2.11. It remains to consider the cases $H_V = \mathbb{Z}/n\mathbb{Z}$ and $H_V = D_{2n}$ for $n \in \mathbb{Z}_+$. Let $g \in H_V$ and $\tilde{g} \in G$ be an element of $\pi^{-1}(g)$. Conjugation by $\tilde{g}$ induces an automorphism of V and, since V is abelian, this automorphism does not depend on the choice of $\tilde{g}$. This yields an action of H_V on V. The automorphism group of V is isomorphic to $\mathcal{S}_3$ and we thus obtain a homomorphism $\varphi \colon H_V \to \mathcal{S}_3$, whose image we denote by H. This yields a short exact sequence

$$1 \to G' \to G \xrightarrow{\varphi \circ \pi} H \to 1,$$

where $G' = \pi^{-1}(\ker(\varphi))$ and V is contained in the center of G'. We claim that G' can be embedded into $\mathrm{GL}_6(\mathbb{C})$. From this the statement of the Lemma follows by Lemma 6.2.8.

So it remains to show the claim. Consider the short exact sequence

$$1 \to V \to G' \xrightarrow{\pi|_{G'}} \ker(\varphi) \to 1.$$

As a subgroup of $\mathbb{Z}/n\mathbb{Z}$ or D_{2n}, the group $\ker(\varphi)$ is either cyclic or dihedral. We first assume that $\ker(\varphi) = \mathbb{Z}/l\mathbb{Z}$ for some $l \in \mathbb{Z}_+$. In this case G' is abelian and contains either an element of order l or of order $2l$, so by the structure theorem for finite abelian groups, G' is either isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z}$ or to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2l\mathbb{Z}$. In both cases, G' can be embedded into $\mathrm{GL}_3(\mathbb{C})$.

If $\ker(\varphi) = D_{2n}$ for some $n \in \mathbb{Z}_+$, let $\psi \colon \ker(\varphi) \to \mathbb{Z}/2\mathbb{Z}$ be the natural projection given by the semi-direct product structure of D_{2n}. We thus obtain a short exact sequence

$$1 \to G'' \to G' \xrightarrow{\psi \circ \varphi \circ \pi} \mathbb{Z}/2\mathbb{Z} \to 1,$$

where $G'' = (\varphi \circ \pi|_{G'})^{-1}(\ker(\psi))$. By Lemma 6.2.8, it is enough to show that G'' can be embedded into $\mathrm{GL}_3(\mathbb{C})$. We consider the following short exact sequence

$$1 \to V \to G'' \xrightarrow{\varphi \circ \pi|_{G''}} \ker(\psi) \to 1.$$

Again, V is contained in the center of G'' and since $\ker(\psi) = \mathbb{Z}/n\mathbb{Z}$ we can apply the same argument as above to conclude that G'' can be embedded into $\mathrm{GL}_3(\mathbb{C})$. $\square$

6.2.3   On the structure of bounded subgroups

We are finally able to prove Proposition 6.2.2:

Proof of Proposition 6.2.2. It is enough to prove Proposition 6.2.2 for maximal algebraic subgroups, so we need to consider the cases (1)-(11) from Theorem 6.2.1.

The statement follows directly for the groups that appear in the cases (1)-(2). For case (4), we observe that by Lemma 6.2.7, $\mathrm{GL}_2(\mathbb{C})/\mu_n$ can be embedded into $\mathrm{GL}_4(\mathbb{C})$, so the natural projection given by the semi-direct product structure induces a homomorphism to $\mathrm{GL}_4(\mathbb{C})$ with kernel isomorphic to $\mathbb{C}^n$. In case (5), the statement follows from Lemma 6.2.9. In the cases (3) as well as (6)-(10) we use Lemma 6.2.6. In case (11) the statement follows from Lemma 6.2.12. $\square$

In fact,the only obstruction, why we have to choose $N \leq 48$ instead of $N \leq 36$ in Proposition 6.2.2, are the finite groups G of the form

$$1 \to V \to G \xrightarrow{\pi} H_V \to 1,$$

where $V = (2\,\mathbb{Z})^2$ and $H_V \simeq \mathcal{S}_4$ that appear as automorphism groups of exceptional bundles in case (11) of Theorem 6.2.1 (see Lemma 6.2.12). This observation allows us to state the following more precise formulation of Proposition 6.2.2:

Proposition 6.2.13. *There exists an $N \in \mathbb{Z}_+$ such that for every bounded subgroup $G \subset \mathrm{Cr}_2(\mathbb{C})$, either G is an extension of $\mathcal{S}_4$ by $(\mathbb{Z}/2\,\mathbb{Z})^2$, or there exists a group-homomorphism $\varphi \colon G \to \mathrm{GL}_N(\mathbb{C})$ such that $\ker(\varphi)$ is isomorphic to $\mathbb{C}^n$ for some n. The constant N can be chosen to be ≤ 36.*

6.3   Finitely generated groups of elliptic elements

In [Can11a], Cantat developed the action of the plane Cremona group on the Picard-Manin space $\mathbb{H}^\infty$ to prove various group theoretical results about $\mathrm{Cr}_2(\mathbb{C})$. In particular, he shows the following result:

Theorem 6.3.1 ([Can11a, Proposition 6.14])**.** *Let Γ be a finitely generated subgroup of elliptic elements. Then either Γ is bounded or Γ preserves a rational fibration, i.e. Γ is conjugate to a subgroup of $\mathcal{J}$.*

Example 6.1.2 shows that the condition that Γ is finitely generated in Theorem 6.3.1 is necessary. It is an open question, that has been asked in [Can11a] and [Fav10], whether there exist finitely generated groups of elliptic elements that are not bounded.

Lemma 6.3.2. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a group of elliptic elements. Then one of the following is true:*

(a) *G preserves a fibration and is therefore conjugate to a subgroup of the de Jonquières group $\mathcal{J}$ or to a subgroup of $\mathrm{Aut}(X)$, where X is a Halphen surface.*

(b) *Every finitely generated subgroup of G is bounded.*

Moreover, if G fixes a point $p \in \partial\,\mathbb{H}^\infty$ that does not correspond to the class of a rational fibration $\pi \colon \mathbb{P} \dashrightarrow \mathbb{P}$, then we ar in case (b).

Proof. By Theorem 5.2.5, G fixes a point $p \in \mathbb{H}^\infty \cup \partial\,\mathbb{H}^\infty$. If $p \in \mathbb{H}^\infty$, then G is bounded and we are done. If $p \in \partial\,\mathbb{H}^\infty$, then either p corresponds to the class of a general fiber of some fibration $\pi \colon Y \to \mathbb{P}^1$, where Y is a rational surface. In this case, G preserves this fibration and is therefore conjugate to a subgroup of $\mathcal{J}$ (if the fibration is rational) or to a subgroup of $\mathrm{Aut}(X)$, where X is a Halphen surface (if the fibration consists of curves of genus 1). Or p does not correspond to the class of a fibration. Let us prove that in this case (b) holds. Let $\Gamma \subset G$ be any finitely generated subgroup. By Theorem 6.3.1, Γ is either bounded or it preserves a rational fibration. In the first case we are done. In the second, it follows that Γ fixes a point $q \in \partial\,\mathbb{H}^\infty$ that corresponds to the class of the rational fibration that is preserved by Γ. Hence $q \neq p$ and G therefore fixes the geodesic line through p and q. In particular, G fixes a point in $\mathbb{H}^\infty$ and is therefore bounded, by Lemma 5.4.3. $\square$

The Burnside problem asks whether a finitely generated torsion group is finite. In general it has a negative answer. However, there are some important classes of groups in which the Burnside property holds. Most prominently, every finitely generated torsion subgroup of a linear group is finite by results of Burnside and Schur (see [CR62]). The following theorem, which can be deduced from the Tits alternative for finitely generated groups, asserts that the same is true for finitely generated torsion subgroups of $\mathrm{Cr}_2(\mathbb{C})$.

Theorem 6.3.3 (Theorem 7.7, [Can11a]). *Every finitely generated torsion subgroup of* $\mathrm{Cr}_2(\mathbb{C})$ *is finite.*

Theorem 6.3.1 and 6.3.3 are crucial for the proof of Theorem 6.1.3.

6.4   The compactness theorem

The compactness theorem is a well known result from model theory. It states that a set of first order sentences has a model if and only if each of its finite subsets has a model. The countable version of the theorem has been proven by Gödel in 1930, the general version is due to Malcev ([Mal40b]). We recall the original version as stated by Malcev:

Definition 6.4.1. Let $\{x_i\}_{i \in I}$ be a set of variables. A *condition* is an expression of the form $F(x_{i_1}, \ldots, x_{i_k}) = 0$ or an expression of the form $F_1(x_{i_1}, \ldots, x_{i_k}) \neq 0 \vee F_2(x_{i_k}, \ldots, x_{i_k}) \neq 0 \vee \cdots \vee F_l(x_{i_1}, \ldots, x_{i_k}) \neq 0$, where F and the F_i are polynomials with integer coefficients.
   A *mixed system* is a set of conditions.

Definition 6.4.2. A mixed system S is *compatible* if there exists a field $\mathbf{k}$ which contains values $\{z_i\}_{i \in I}$ that satisfy S.

Theorem 6.4.1 (Mal'cev, [Mal40b]). *If every finite subset of a mixed system S is compatible, then S is compatible.*

Malcev used Theorem 6.4.1 to deduce that if for a given group G every finitely generated subgroup can be embedded into $\mathrm{GL}_n(\mathbf{k})$ for some field $\mathbf{k}$ then there exists a field $\mathbf{k}'$ such that G can be embedded into $\mathrm{GL}_n(\mathbf{k}')$. Note that a priori nothing can be said about the structure of the field $\mathbf{k}'$.

6.5   Proof of Theorem 6.1.3

The aim of this section is to prove Theorem 6.1.3. We start with some technical lemmas.

Lemma 6.5.1. *Let A be a group acting by automorphisms on $\mathbb{P}^1 \times \mathbb{P}^1$. If A has a fixed point then it is birationally conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$.*

Proof. Assume that A acts on $\mathbb{P}^1 \times \mathbb{P}^1$ with a fixed point p. We consider $\mathbb{P}^1 \times \mathbb{P}^1$ embedded as a smooth quadric Q in $\mathbb{P}^3$. All the automorphisms of Q are induced by the elements in $\mathrm{Aut}(\mathbb{P}^3) = \mathrm{PGL}_4(\mathbb{C})$ that preserve the non-degenerate quadratic

form that defines Q. Hence the action of A on Q extends to an action on $\mathbb{P}^3$. Let $\pi\colon Q \dashrightarrow \mathbb{P}^2$ be the birational transformation given by stereographic projection from p. Since A fixes p, it induces a regular action on $\mathbb{P}^2$ that is π-equivariant. Hence, π conjugates A to a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$. $\hfill\square$

Lemma 6.5.2. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a group of elliptic elements that is not a torsion group and contains a finitely generated subgroup Γ that does not preserve any fibration. Then G is conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ or $\mathrm{Aut}(S_6)$.*

Proof. As G is not a torsion group, we may assume that Γ contains an element of infinite order. By Theorem 6.3.1, Γ is bounded and therefore contained in a maximal algebraic subgroup of $\mathrm{Cr}_2(\mathbb{C})$. By Theorem 6.2.1, the only infinite maximal algebraic subgroups of $\mathrm{Cr}_2(\mathbb{C})$ that do not preserve any rational fibrations are the groups $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ and $\mathrm{Aut}(S_6)$, where S_6 is the del Pezzo surface of degree 6. Therefore, every finitely generated subgroup of G that does contain Γ is conjugate to a subgroup of either $\mathrm{Aut}(\mathbb{P}^2)$, $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ or $\mathrm{Aut}(S_6)$. By adding elements to a given finitely generated group, we observe that there exists $H \in \{\mathrm{Aut}(\mathbb{P}^2), \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1), \mathrm{Aut}(S_6)\}$ such that every finitely generated subgroup of G is conjugate to a subgroup of H.

We first consider the case $H = \mathrm{Aut}(S_6)$. The del Pezzo surface S_6 is the blow-up of 3 general points p_1, p_2, p_3 in $\mathbb{P}^2$. We may assume that $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$, $p_3 = [0 : 0 : 1]$. Denote by l_1, l_2, l_3 the strict transforms in S_6 of the three coordinate lines in $\mathbb{P}^2$ and let e_1, e_2, e_3 be the exceptional divisors. The automorphism group of S_6 is $D_2 \rtimes (\mathcal{S}_3 \times \mathbb{Z}/2\,\mathbb{Z})$, where the action of $D_2 \rtimes \mathcal{S}_3$ is induced by the action of $D_2 \rtimes \mathcal{S}_3$ on $\mathbb{P}^2$ preserving the set of points $\{p_1, p_2, p_3\}$. The pullback of the standard quadratic involution σ yields an automorphism of S_6, which corresponds to the factor $\mathbb{Z}/2\,\mathbb{Z}$ in the automorphism group ([Dol12], Chapter 8.4.2)). Let $\Gamma \subset G$ be a finitely generated infinite subgroup that does not preserve any fibration. We may assume $\Gamma \subset \mathrm{Aut}(S_6)$. We denote by $\overline{\Gamma} \subset \mathrm{Aut}(S_6)$ the Zariski-closure of Γ. Since Γ is infinite, the subgroup $D' := \Gamma \cap D_2 \subset \mathrm{Aut}(S_6)$ is infinite. Moreover, D' is a normal subgroup in Γ, which implies in particular that Γ preserves the closures of the orbits of D'. As Γ does not preserve any fibration and since D' is infinite, we obtain that D' has a dense orbit and therefore that $D_2 = \overline{D'} \subset \overline{\Gamma}$. Note that D_2 is the connected component of the identity of $\mathrm{Aut}(S_6)$. This implies that every chain of strict inclusions of closed algebraic groups in $\mathrm{Aut}(S_6)$

$$\overline{\Gamma} = G_1 \subset G_2 \subset \cdots \subset G_k$$

has length $k \le 12 = |\mathcal{S}_3 \times \mathbb{Z}/2\,\mathbb{Z}|$. Assume that there exists an element $g_1 \in G$ that is not contained in $\overline{\Gamma}$. In this case, there exists a $f_1 \in \mathrm{Cr}_2(\mathbb{C})$ such that $\overline{f_1 \langle \Gamma, g_1 \rangle f_1^{-1}}$ is a subgroup of $\mathrm{Aut}(S_6)$ and we obtain a strict inclusion $\overline{f_1 \Gamma f_1^{-1}} \subset \overline{f_1 \langle \Gamma, g_1 \rangle f_1^{-1}}$. If $G \subset \overline{\langle \Gamma, g_1 \rangle}$, we are done, otherwise we continue inductively until we obtain a chain of inclusions of algebraic subgroups of $\mathrm{Aut}(S_6)$

$$\overline{f_r \Gamma f_r^{-1}} \subset \overline{f_r \langle \Gamma, g_1 \rangle f_r^{-1}} \subset \cdots \subset \overline{f_r \langle \Gamma, g_1, \dots, g_r \rangle f_r^{-1}},$$

such that $G \subset \overline{\langle \Gamma, g_1, \dots, g_r \rangle}$. Such an r exists and it is at most 12 by the above remark. In particular, $f_r G f_r^{-1} = \overline{f_r \langle \Gamma, g_1, \dots, g_r \rangle f_r^{-1}}$ and is therefore contained in $\mathrm{Aut}(S_6)$.

Unfortunately, we can't apply the above argument to the cases, where $H = \mathrm{Aut}(\mathbb{P}^2)$ or $H = \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$, since it relied on the fact that $\mathrm{Aut}(S_6)$ has dimension 2.

Now let $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $X = \mathbb{P}^2$. First we assume that there exists a finitely generated infinite subgroup $\Gamma \subset G$ that does not preserve any fibration and has no fixed point on X. We may assume that $\Gamma \subset \mathrm{Aut}(X)$. Let $\overline{\Gamma}^0$ be the connected component of the identity of the closed subgroup $\overline{\Gamma} \subset \mathrm{Aut}(X)$. Since Γ is infinite, $\overline{\Gamma}^0$ contains an element of infinite order. It has therefore orbits of positive dimension. It is moreover a normal subgroup in $\overline{\Gamma}$, so all elements in Γ preserve the closures of its orbits. Since Γ does not preserve any fibration, $\overline{\Gamma}^0$ has therefore an orbit of dimension 2. Assume that there is an element $g \in G$ that is not contained in Γ. Then there exists a $f \in \mathrm{Bir}(X)$ such that $f\langle \Gamma, g \rangle f^{-1}$ is contained in $\mathrm{Aut}(X)$. Note that f conjugates $\Gamma \subset \mathrm{Aut}(X)$ to the subgroup $f\Gamma f^{-1} \subset \mathrm{Aut}(X)$. Therefore, we obtain two different actions ρ_1 and ρ_2 of $\overline{\Gamma}^0$ on X, which are conjugate by f. The first one is given by

$$\rho_1 \colon \overline{\Gamma}^0 \times X \to X, \quad (g, x) \mapsto g(x)$$

and the second by

$$\rho_2 \colon \overline{\Gamma}^0 \times X \to X, \quad (g, x) \mapsto fgf^{-1}(x).$$

The exceptional divisor of f, i.e. the union of irreducible curves contracted by f, and the set of indeterminacy points of f in X are therefore both $\overline{\Gamma}^0$-invariant subsets of X, with respect to ρ_1. And the exceptional divisor of f^{-1} and the set of indeterminacy points of f^{-1} in X are both $\overline{\Gamma}^0$-invariant subsets of X, with respect to ρ_2.

If the two dimensional orbit of $\overline{\Gamma}^0$ under ρ_1 is closed we thus obtain that the exceptional locus of f as well as the set of indeterminacy points are empty, hence f is an automorphism. If $\overline{\Gamma}^0$ does not act transitively on X under ρ_1, its two dimensional orbit U is open in X. Hence the curve $C = X \setminus U$ is $\overline{\Gamma}^0$-invariant under ρ_1. Since $\overline{\Gamma}^0$ does not have any fixed points on X under ρ_1, the curve C has to be smooth. Moreover, $\overline{\Gamma} \subset \mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}$ does not preserve any fibration, by assumption. Hence, C cannot consist of more than one fiber of the same fibration. In particular, C is irreducible. The number of irreducible components of the exceptional divisor of f is larger or equal to the number of indeterminacy points of f^{-1}, thus f^{-1} has at most one indeterminacy point, which has to be a fixed point under ρ_2. But $\overline{\Gamma}^0$ has no fixed points under ρ_1 and thus also not under ρ_2, so f^{-1} has no indeterminacy points. Similarly one shows that f has no indeterminacy points. It follows that f is an automorphism of X and therefore that $g \in \mathrm{Aut}(X)$ and since g was chosen arbitrarily we obtain $G \subset \mathrm{Aut}(X)$.

If $X = \mathbb{P}^2$ and every finitely generated subgroup Γ of G has a fixed point p_Γ, this implies that every finitely generated subgroup Γ fixes a rational fibration, namely the pencil of lines through p_Γ. This does not appear, by assumption.

If $X = \mathbb{P}^1 \times \mathbb{P}^1$ and every finitely generated subgroup of G has a fixed point, we obtain by Lemma 6.5.1 that every finitely generated subgroup of G is conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$, so this case reduces to the previous case.

□

Proof of Theorem 6.1.4. If G is finite, the theorem is covered by Proposition 6.2.2. Hence we may assume G to be infinite.

We will now generalize Proposition 6.2.13 with the help of the compactness theorem to arbitrary torsion subgroups. Let $N \in \mathbb{Z}$ be an integer such that every large enough finite subgroup of $\mathrm{Cr}_2(\mathbb{C})$ can be embedded into $\mathrm{GL}_N(\mathbb{C})$, e.g. $N = 36$ (Proposition 6.2.13). To every element $g \in G$ we associate a $N \times N$ matrix of variables (x_{ij}^g). We now construct a mixed system S consisting of the following conditions:

(1) the equations given by the matrix product $(x_{ij}^f)(x_{ij}^g) = (x_{ij}^h)$ for all $f, g, h \in G$ such that $fg = h$;

(2) for all $g \in G \setminus \{\mathrm{id}\}$ the conditions $(\bigvee_i x_{ii}^g - 1 \neq 0) \vee (\bigvee_{i \neq j} x_{ij}^g \neq 0)$;

(3) $x_{ii}^{\mathrm{id}} - 1 = 0$ and $x_{ij}^{\mathrm{id}} = 0$ for all $1 \leq i, j \leq N$, $i \neq j$;

(4) $p \neq 0$ for all primes $p \in \mathbb{Z}_+$.

First we show that S is compatible. For this, it suffices to show that every finite subset of S is compatible, by Theorem 6.4.1. Let $c_1, \ldots, c_n \in S$ be finitely many conditions. There are only finitely many variables x_{ij}^g that appear in $c_1, \ldots, c_n$. Denote by $\{g_1, \ldots, g_l\} \subset G$ the finite set of all elements g in G such that for some $1 \leq i, j \leq N$ the variable x_{ij}^g appears in one of the conditions $c_1, \ldots, c_n$. Consider the finitely generated subgroup $\Gamma = \langle g_1, \ldots, g_l \rangle \subset G$. By Theorem 6.3.3, Γ is finite and has therefore a faithful representation to $\mathrm{GL}_N(\mathbb{C})$, where $N = 48$ (Proposition 6.2.2). More precisely, since G is infinite, we may assume that Γ contains strictly more than 96 elements, so by Proposition 6.2.13, we can choose N to be ≤ 36. The existence of such a faithful representation implies in particular that $\mathbb{C}$ contains values that satisy the finite set of conditions $c_1, \ldots, c_n$, i.e it is compatible.

Hence there exists a field $\mathbf{k}$ which contains values z_{ij}^g for all $i, j \in \{1, \ldots, N\}$, and all $g \in G$ that satisfy conditions (1) to (4). First we note that $\mathrm{char}(\mathbf{k}) = 0$ because of the conditions (4). Since $G \subset \mathrm{Cr}_2(\mathbb{C})$, it has at most the cardinality of the continuum, so in particular, the values $\{z_{ij}^g\}$ are contained in a subfield $\mathbf{k}'$ of $\mathbf{k}$ with the same cardinality as $\mathbb{C}$, which implies that $\mathbf{k}'$ can be embedded into $\mathbb{C}$ as a subfield. So without loss of generality we may assume that $\mathbf{k} = \mathbb{C}$. Consider now the map $\varphi \colon G \to \mathrm{GL}_N(\mathbb{C})$ given by $g \mapsto (z_{ij}^g)_{i,j}$. It is well defined, since the conditions (3) imply $\varphi(\mathrm{id}) = \mathrm{id}$ and by the conditions (1) the image of every element of G is an invertible matrix. The conditions (1) furthermore ensure that φ is a group automorphism and the conditions (2) ensure that it is injective. □

Proof of Theorem 6.1.3. Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a group of elliptic elements. We distinguish 3 different cases.

Case 1. If G is a torsion group, there is nothing to prove.

Case 2. When G is not a torsion group and contains a finitely generated subgroup Γ that does not preserve any fibration the theorem is covered by Lemma 6.5.2.

Case 3. Now, assume that for every finite set of elements $\gamma_1, \ldots, \gamma_n \in G$ there exists a rational fibration $\pi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ such that the group $\Gamma \coloneqq \langle \gamma_1, \ldots, \gamma_n \rangle$ permutes the fibers of π. In other words, every subgroup $\Gamma \subset G$ generated by elements of infinite order $\gamma_1, \ldots, \gamma_n$, is conjugate to a subgroup of the de Jonquières subgroup $\mathcal{J} = \mathrm{PGL}_2(\mathbb{C}(t)) \rtimes \mathrm{PGL}_2(\mathbb{C})$. Let

$$\varphi \colon \mathcal{J} = \mathrm{PGL}_2(\mathbb{C}(t)) \rtimes \mathrm{PGL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$$

be the natural projection. We consider two subcases:

Case 3a. Assume that every finitely generated subgroup of G can be conjugated to a subgroup of $\mathcal{J}$, but that there exist elements $\gamma_1, \ldots, \gamma_n \in G$ such that the finitely generated subgroup $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle \subset G$ satisfies the following property: whenever we conjugate Γ to a subgroup of $\mathcal{J}$, there exists a generator of infinite order γ_i that is contained in the kernel of φ.

WBy changing the order of the generators, we may assume that there exists a $k \in \{1, \ldots, n\}$ such that the generators $\gamma_1, \ldots, \gamma_k$ of Γ are of infinite order and such that the generators $\gamma_{k+1}, \ldots, \gamma_n$ are torsion elements. For every finite set of elements $\{\delta_1, \ldots, \delta_l\} \subset \Gamma$ the group $\langle \gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_l \rangle$ preserves a fibration $\pi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ whose fibers are γ_i-invariant for some $i \in \{1, \ldots, k\}$, by hypothesis of Case 3a. By construction of $\gamma_1, \ldots, \gamma_n$, there exists a $j \in \{1, \ldots, k\}$ such that for every finitely generated subgroup Δ of G there exists a γ_j-invariant fibration $\pi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ that is preserved by Δ. The general fibers of π are exactly the orbit closures of γ_j. We conclude that every element of G preserves the orbit closures of γ_j. Therefore, G is conjugate to a subgroup of $\mathcal{J}$. This proves Theorem 6.1.3 for the subcase (3a).

Case 3b. We assume that for every finite subset $\gamma_1, \ldots, \gamma_n$ of G the group $\Gamma \coloneqq \langle \gamma_1, \ldots, \gamma_n \rangle$ of G can be conjugated to a subgroup of $\mathcal{J}$ in such a way that the kernel of φ does not contain any element of the set $\{\gamma_1, \ldots, \gamma_n\}$ that has infinite order.

Again, as in case (2), we use the compactness theorem. To every element $g \in G$ we associate a 3×3 matrix of variables (x_{ij}^g). We consider $\mathrm{PGL}_2(\mathbb{C})$ as a closed subgroup of $\mathrm{GL}_3(\mathbb{C})$ (e.g. through the embedding given by the adjoint representation). Let $f_1, \ldots, f_m \in \mathbb{C}[x_{ij}]_{1 \leq i, j \leq 3}$ be the polynomials such that $\mathrm{PGL}_2(\mathbb{C}) \subset \mathrm{GL}_3(\mathbb{C})$ is exactly the zero set of $f_1, \ldots, f_m$ in $\mathrm{GL}_3(\mathbb{C})$. We construct a mixed system S consisting of the following conditions:

(1) the equations given by $(x_{ij}^f)(x_{ij}^g) = (x_{ij}^h)$ for all $f, g, h \in G$ such that $fg = h$;

(2) for all $g \in G$ of infinite order the conditions

$$(\bigvee_i x_{ii}^g - 1 \neq 0) \vee (\bigvee_{i \neq j} x_{ij}^g \neq 0);$$

(3) $f_1(x_{ij}^g) = \cdots = f_m(x_{ij}^g) = 0$ for all $g \in G \setminus \{\mathrm{id}\}$;

(4) $x_{ii}^{\mathrm{id}} = 1$ and $x_{ij}^{\mathrm{id}} = 0$ for all $1 \leq i, j \leq 3$, $i \neq j$;

(5) $p \neq 0$ for all primes $p \in \mathbb{Z}_+$.

We first show that S is compatible. For this, it suffices to show that every finite subset of S is compatible, by Theorem 6.4.1. Let $c_1, \ldots, c_n \in S$ be finitely many conditions. There are only finitely many variables x_{ij}^g that appear in $c_1, \ldots, c_n$. Denote by $\{g_1, \ldots, g_l\} \subset G$ the finite set of all elements g in G such that for some $1 \leq i, j \leq N$ the variable x_{ij}^g appears in one of the conditions $c_1, \ldots, c_n$. Consider the finitely generated group $\Gamma = \langle g_1, \ldots, g_l \rangle$. By assumption, Γ is conjugate to a subgroup of $\mathcal{J}$ such that $\ker(\varphi)$ does not contain any element from $\{g_1, \ldots, g_l\}$ that has infinite order. Therefore, the projection φ allows us to associate to each element $g \in \{g_1, \ldots, g_l\}$ a 3×3-matrix whose variables satisfy conditions (1) to (4), i.e. the conditions $c_1, \ldots, c_n$ are compatible.

Let now $\mathbf{k}$ be a field containing values that satisfy the conditions of S. By the same argument as in the proof of Theorem 6.1.4, we may assume that $\mathbf{k} = \mathbb{C}$. Consider the map $\psi \colon G \to \mathrm{GL}_3(\mathbb{C})$. It is well defined, since the conditions (4) imply $\varphi(\mathrm{id}) = \mathrm{id}$ and by the conditions (1) the image of every element of G is an invertible matrix. All elements in $\psi(G)$ are contained in $\mathrm{PGL}_2(\mathbb{C})$ since they satisfy conditions (4). From the conditions (1) we obtain that ψ is a group homomorphism and the conditions (2) ensure that no element of infinite order is contained in the kernel of ψ. Hence, there exists a homomorphism $\psi \colon G \to \mathrm{PGL}_2(\mathbb{C})$ such that $\ker(\psi)$ is a torsion subgroup. This finishes the proof of Theorem 6.1.3. $\qquad \square$

6.6 Tits' alternative

In this section we prove Theorem 6.1.10. Consider a subgroup $G \subset \mathrm{Cr}_2(\mathbb{C})$. First we consider the case, where G contains a loxodromic element, then the case, where G contains a parabolic but no loxodromic element, and then the case, where G is a group of elliptic elements. The first two cases are treated analogously as in the proof of the Tits alternative for finitely generated subgroups in [Can11a], whereas in the last case the Tits alternative can be deduced from Theorem 6.1.3.

6.6.1 The loxodromic case

We first start with some preparation.

Lemma 6.6.1. *Let $g \in \mathrm{Cr}_2(\mathbb{C})$ be an algebraic element that fixes two different rational fibrations. Then g is of finite order.*

Proof. Assume that the two rational fibrations fixed by g are given by the rational maps $\pi_1, \pi_2 \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. We thus obtain a dominant g-invariant rational map $\pi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ given by $\pi = (\pi_1, \pi_2)$. Since π is of finite degree, it is locally a finite cover, hence g must be of finite order. $\qquad \square$

Lemma 6.6.2. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be an algebraic subgroup of dimension ≥ 9. Then G preserves a unique rational fibration.*

Proof. It follows from Theorem 6.2.1 that G is conjugate to a subgroup of $\mathrm{Aut}(\mathbb{F}_n)$ for some Hirzebruch surface $\mathbb{F}_n$, $n \geq 2$, and therefore that G preserves a rational fibration $\pi \colon \mathbb{F}_n \to \mathbb{P}^1$. As G permutes the fibers of π, we obtain a homomorphism $\varphi \colon G \to \mathrm{PGL}_2(\mathbb{C})$ with a kernel of dimension ≥ 6. Assume that there exists a

second rational fibration $\pi' \colon \mathbb{F}_n \dashrightarrow \mathbb{P}^1$ that is preserved by G. We obtain a second homomorphism $\varphi' \colon G \to \mathrm{PGL}_2(\mathbb{C})$. The restriction of π' to $\ker(\pi)$ has a positive dimensional kernel. Hence, the intersection $\ker(\varphi) \cap \ker(\varphi')$ is positive dimensional. In particular, $\ker(\varphi) \cap \ker(\varphi')$ contains an element of infinite order. By Lemma 6.6.1, the rational maps π and π' define the same fibration. $\qquad\square$

The following result is a generalization of Theorem 5.5.4:

Theorem 6.6.3. *Let N be a subgroup of $\mathrm{Cr}_2(\mathbb{C})$ containing at least one loxodromic element. Assume that there exists a short exact sequence*

$$1 \to A \to N \to B \to 1,$$

where A is an infinite group of elliptic elements. Then N is conjugate to a subgroup of $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$.

Proof. By Theorem 5.2.5, A fixes a point in $p \in \partial \mathbb{H}^\infty \cup \mathbb{H}^\infty$. If $p \in \mathbb{H}^\infty$, then A is bounded and we are in the case of Theorem 5.5.4. So we may assume that $p \in \partial \mathbb{H}^\infty$ and that p is the only fixed point of A in $\partial \mathbb{H}^\infty$, since otherwise A fixes the geodesic between p and q and again, A would be bounded.

Let $f \in N$ be a loxodromic element. Since f normalizes A, we obtain that f fixes p. Being loxodromic, f does not preserve any fibration and hence, p does not correspond to the class of a fibration. In particular, every finitely generated group of elliptic elements that fixes p is bounded, by Lemma 6.3.2. Denote by $G \subset \mathrm{Cr}_2(\mathbb{C})$ the subgroup of elements that fix p. Let L be the one-dimensional subspace of $\mathrm{Z}(\mathbb{P}^2)$ that corresponds to p. Since G fixes p, its linear action on $\mathrm{Z}(\mathbb{P}^2)$ acts on L by automorphisms preserving the orientation. This yields a group homomorphism $\rho \colon G \to \mathbb{R}_+^*$. The kernel of ρ consists of elliptic elements, since loxodromic elements do not fix any vector in $\mathrm{Z}(\mathbb{P}^2)$, and G does not contain any parabolic elements, as p does not correspond to the class of a fibration. Moreover, all elliptic elements in G are contained in $\ker(\rho)$, since 1 is the only eigenvalue of a transformation of $\mathrm{Z}(\mathbb{P}^2)$ that is induced by an elliptic element (see [Can15, Section 4.1.3]). Let $f \in G$ be loxodromic; let us show that no power of f is tight. Assume the contrary, i.e. that f^n is tight in G for some $n \in \mathbb{Z}$. By Theorem 5.6.2, all elements except the identity in the normal subgroup generated by f^n are loxodromic. In particular, all the elements of the form $gf^ng^{-1}f^{-n}$ are loxodromic, where $g \in G$ is an element that does not commute with f^n (such elements exist by Corollary 5.2.3, since we assumed $N \subset G$ to be infinite). But $\rho(gf^ng^{-1}f^{-n}) = \mathrm{id}$ and hence $gf^ng^{-1}f^{-n}$ is elliptic - a contradiction. Hence, no power of f is tight, which implies, by Theorem 5.6.4, that there exists a $h \in \mathrm{Cr}_2(\mathbb{C})$ and an algebraic subgroup $B \subset G$ such that hfh^{-1} is monomial and $hBh^{-1} = D_2$.

Let $\Gamma \subset \ker(\rho)$ be a finitely generated subgroup. Since Γ is bounded, the Zariski-closure $\overline{\Gamma}$ of Γ is an algebraic subgroup of G. Let

$$d := \sup\{\dim(\overline{\Gamma}) \mid \Gamma \subset \ker(\rho) \text{ finitely generated }\}.$$

First, assume that d is finite. Since $\ker(\rho)$ contains a subgroup that is conjugate to D_2, we have $d \geq 2$. Let $\Gamma \subset \ker(\rho)$ be a finitely generated subgroup such that $\dim(\overline{\Gamma}) = d$ and denote by $\overline{\Gamma}^0$ the connected component of the identity in

the algebraic group $\overline{\Gamma}$. Let $f \in N$ be any element. Note that $f\overline{\Gamma}^0 f^{-1}$ is again an algebraic subgroup and that $\langle \overline{\Gamma}^0, f\overline{\Gamma}^0 f^{-1} \rangle$ is contained in the Zariski-closure of the finitely generated group $\langle \Gamma, f\Gamma f^{-1} \rangle$. By [Hum75, Chapter 7.5], $\langle \overline{\Gamma}^0, f\overline{\Gamma}^0 f^{-1} \rangle$ is closed and connected. Since it is of dimension $\leq d$ and contains $\overline{\Gamma}^0$ it equals $\overline{\Gamma}^0$, i.e. f normalizes $\overline{\Gamma}^0$. By Theorem 5.5.4 and Lemma 5.5.1, there exists a $g \in \mathrm{Cr}_2(\mathbb{C})$ such that $g\Gamma g^{-1}$ is a dense subgroup of D_2 and gfg^{-1} is monomial. Since the normalizer of D_2 is $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$, we obtain that $gGg^{-1} \subset \mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$ and hence in particular $gNg^{-1} \subset \mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$.

Now assume that $d = \infty$. Let $\Gamma \subset \ker(\rho)$ be a finitely generated subgroup such that $\dim(\overline{\Gamma}) \geq 9$. By Lemma 6.6.2, $\overline{\Gamma}$ preserves a unique rational fibration given by a rational map $\pi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. Let $g \in \ker(\rho)$ be any element. The algebraic group $\overline{\langle \Gamma, g \rangle}$ preserves again a rational fibration and since it contains $\overline{\Gamma}$, this fibration is again given by π. It follows that $\ker(\rho)$ preserves a rational fibration, which implies that $\ker(\rho)$ is bounded and we can apply Theorem 5.5.4 to conclude that $gGg^{-1} \subset \mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$ and hence in particular $gNg^{-1} \subset \mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$. $\qquad\square$

From Theorem 6.6.3 we can in particular draw the following results:

Lemma 6.6.4. *Let $f, g \in \mathrm{Cr}_2(\mathbb{C})$ be two loxodromic elements such that $\mathrm{Ax}(f) \neq \mathrm{Ax}(g)$. Then either f and g do not have a common fixed point on $\partial \mathbb{H}^\infty$, or $\langle f, g \rangle$ is conjugate to a subgroup of $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$.*

Proof. Assume that f and g have a common fixed point $p \in \partial \mathbb{H}^\infty$. Let L be the one-dimensional subspace of $\mathrm{Z}(\mathbb{P}^2)$ that corresponds to p. Since $\langle f, g \rangle$ fixes p, its linear action on $\mathrm{Z}(\mathbb{P}^2)$ acts on L by automorphisms preserving the orientation. As in the proof of Theorem 6.6.3, this yields a group homomorphism $\rho \colon \langle f, g \rangle \to \mathbb{R}_+^*$ whose kernel consists of elliptic elements. Assume that f^n is tight for some n. By Theorem 5.6.2, all elements except the identity in the normal subgroup generated by f^n are loxodromic. In particular, $gf^n g^{-1} f^{-n}$ is loxodromic. But $\rho(gf^n g^{-1} f^{-n}) = \mathrm{id}$ and hence $gf^n g^{-1} f^{-n}$ is elliptic - a contradiction. Hence, no power of f is tight, which implies, by Theorem 5.6.4, that there exists a $h \in \mathrm{Cr}_2(\mathbb{C})$ and a bounded subgroup $\Delta_2 \subset \langle f, g \rangle$ such that hfh^{-1} is monomial and $h\Delta_2 h^{-1}$ is a dense subgroup of D_2. In particular, $\ker(\rho)$ is infinite as $\Delta_2 \subset \ker(\rho)$. The statement now follows from Theorem 6.6.3. $\qquad\square$

Lemma 6.6.5. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a subgroup containing loxodromic elements f and g such that $\mathrm{Ax}(f) \neq \mathrm{Ax}(g)$. Then either there exist loxodromic elements $h_1, h_2 \in G$ such that h_1 and h_2 do not have a common fixed point on $\partial \mathbb{H}^\infty$, or G is conjugate to a subgroup of $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$.*

Proof. If f and g do not have a common fixed point, we are done. So we may assume that f and g have a common fixed point $p \in \partial \mathbb{H}^\infty$ and that $\langle f, g \rangle$ is conjugate to a subgroup of $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$, by Lemma 6.6.4. If all elements in G fix p, we obtain an exact sequence

$$1 \to \ker(\rho) \to G \to H \to 1$$

and we are done by Theorem 6.6.3, since all elements in $\ker(\rho)$ are elliptic. Otherwise, there exists a loxodromic element $h \in G$ that does not fix p. If h does not

have a common fixed point in $\partial\mathbb{H}^\infty$ with either f or g we are done. Otherwise, h has a common fixed point q_1 with f and a common fixed point q_2 with g and $q_1 \neq q_2$ (otherwise, f and g would have the same axis). There exists an algebraic subgroup $B \subset \mathrm{Cr}_2(\mathbb{C})$ that is conjugate to D_2 and that fixes p, q_1 and q_2. We may assume that $B = D_2$. Hence, h normalizes a subgroup of elliptic elements containing D_2. By Theorem 6.6.3, there exists an element $a \in \mathrm{Cr}_2(\mathbb{C})$ such that $a\langle D_2, h \rangle a^{-1} \subset \mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$. In particular, a conjugates D_2 to a subgroup of $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$. This implies $aD_2a^{-1} = D_2$ and therefore $a \in \mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$. Hence $h \in \mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$. $\qquad\square$

The main tool to prove the Tits alternative for subgroups of $\mathrm{Cr}_2(\mathbb{C})$ containing loxodromic elements is the so-called ping-pong Lemma:

Lemma 6.6.6 (Ping-pong Lemma, [dlH00, II.B.]). *Let S be a set and f_1, f_2 two bijections of S. Assume that there exist subsets S_1 and S_2 of S such that $f_1^n(S_1) \subset S_2$ and $f_2^n(S_2) \subset S_1$ for all $n \in \mathbb{Z}, n \neq 1$. Then f_1 and f_2 generate a non-abelian free group.*

Lemma 6.6.7. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a subgroup that contains a loxodromic element. Then one of the following is true:*

(1) *G is conjugate to a subgroup of $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$,*

(2) *G contains a subgroup G^0 of index at most two that is isomorphic to $\mathbb{Z} \ltimes H$, where H is a finite group,*

(3) *G contains a non-abelian free subgroup.*

Proof. Let $f \in G$ be a loxodromic element. We consider three cases.

Case 1. First we assume that all elements in G preserve the axis $\mathrm{Ax}(f)$ of f. There is a subgroup $G^0 \subset G$ of index at most 2 such that G^0 preserves the orientation of the axis. Hence every element $g \in G^0$ translates the points on $\mathrm{Ax}(f)$ by a constant $c_g \in \mathbb{R}$. This yields a group homomorphism $\pi \colon G^0 \to \mathbb{R}$, whose kernel is a bounded group. By Theorem 5.5.4, $\ker(\pi)$ is either finite or G^0 is conjugate to a subgroup of $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$. If G^0 is conjugate to a subgroup of $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$, then so is G. The image of π is discrete by the gap property (Theorem 5.2.2) and therefore isomorphic to $\mathbb{Z}$. Hence, if $\ker(\pi)$ is finite, we are in case (2).

Case 2. Now assume that there is an element $g \in G$ that does not preserve $\mathrm{Ax}(f)$. By Lemma 6.6.5, either G is conjugate to a subgroup of $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$ or G contains two loxodromic elements h_1, h_2 that do not have a common fixed point in $\partial\mathbb{H}^\infty$. In the first case we are done, in the second one we apply the ping-pong Lemma by considering the action of h_1 and h_2 on the border $\partial\mathbb{H}^\infty$ and chosing as subsets S_1 and S_2 small enough neighborhoods of the fixed points of f and g on $\partial\mathbb{H}^\infty$. More precisely, denote by α^+ the attracting fixed point of f in $\partial\mathbb{H}^\infty$ and by α^- its repulsive fixed point. Similarly, we denote by β^+ and β^- the attractive and repulsive fixed point of g on $\partial\mathbb{H}^\infty$ respectively. Let S_1^+ be a small neighborhood of α^+ and S_1^- a small neighborhood of α^- in $\partial\mathbb{H}^\infty$. Similarly, let S_2^+ be a small neighborhood of β^+ and S_2^- a small neighborhood of β^-. We assume that S_1^+, S_1^-, S_2^+, and S_2^- are pairwise disjoint. Let $S_1 := S_1^+ \cup S_1^-$ and $S_2 := S_2^+ \cup S_2^-$.

There exist positive integers n_1, n_2, n_3, n_4 satisfying $f^{n_1}(S_2) \subset S_1^+$, $f^{-n_2}(S_2) \subset S_1^-$, $g^{n_3}(S_1) \subset S_1^+$ and $g^{-n_3}(S_2) \subset S_1$. Define $n := \max\{n_1, n_2, n_3, n_4\}$. As $f(S_1^+) \subset S_1^+$ and $f^{-1}(S_1^-) \subset S_1^-$ as well as $g(S_2^+) \subset S_2^+$ and $g^{-1}(S_2^-) \subset S_2^-$, we obtain that $f^m(S_2) \subset S_1$ and $f^{-m}(S_2) \subset S_1$ as well as $g^m(S_1) \subset S_2$ and $g^{-m}(S_1) \subset S_2$ for all $m \geq n$. The two maps f^n and g^n together with the sets S_1, S_2 therefore satisfy all the conditions from the Ping-Pong Lemma and we obtain that f^n and g^n generate a non-abelian free subgroup of G. $\qquad\square$

6.6.2 The parabolic case

Recall that a subgroup of $\mathrm{Cr}_2(\mathbb{C})$ that contains no loxodromic element, but a parabolic element always preserves a rational or elliptic fibration (Lemma 5.3.4). From the structure theorems about these groups we will deduce that subgroups of this type satisfy Tits' alternative.

Theorem 6.6.8 ([Can12a], Proposition 6.3). *Assume that we have a short exact sequence of groups*

$$1 \to G_1 \to H \to G_2 \to 1.$$

If G_1 and G_2 satisfy Tits' alternative then H satisfies Tits' alternative.

Remark 6.6.1. In the published version of the paper [Can12a] there is a gap in the proof of Theorem 6.6.8. However, in the version of the paper on the website of the author [Can11b] this gap has been filled and the proof is complete.

Remark 6.6.2. Theorem 6.6.8 implies in particular, that the group $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$ satisfies Tits' alternative.

Lemma 6.6.9. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a subgroup that contains a parabolic element but no loxodromic element. Then G satisfies Tits' alternative.*

Proof. By Lemma 5.3.4, G is either conjugate to a subgroup of $\mathcal{J}$ or to a subgroup of $\mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ is the automorphism group of a Halphen surface. In the first case, the Tits alternative follows from Theorem 6.6.8 and the Tits alternative for linear groups in characteristic zero, since $J \simeq \mathrm{PGL}_2(\mathbb{C}) \ltimes \mathrm{PGL}_2(\mathbb{C}(t))$. In the second case, G is solvable since the automorphism group of a Halphen surface is virtually abelian. $\qquad\square$

6.6.3 Proof of Theorem 6.1.10

We are now ready to prove Tits' alternative for the Cremona group:

Proof of Theorem 6.1.7. Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a subgroup. If G contains a loxodromic element, then, by Lemma 6.6.7, G is either conjugate to a subgroup of $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$, in which case the Tits alternative holds (see Remark 6.6.2), G is cyclic up to finite index, or G contains a non-abelian free subgroup. Therefore, the Tits alternative holds for groups containing loxodromic elements.

For the case that G contains a parabolic but no loxodromic element, Tits' alternative is proven in Lemma 6.6.9.

Assume that all elements in G are elliptic. We thus are in one of the cases of Theorem 6.1.3. All the groups that satisy case (1) or case (2) are linear and therefore satisfy the Tits alternative. If G is a group from case (3) or case (4) the Tits alternative follows from Theorem 6.6.8. $\qquad\square$

6.7 Derived length

In this section we will prove Theorem 6.1.11. Our starting point is the following version of a Theorem by Déserti ([Dés15]), which can be deduced from the properties of the action of $\mathrm{Cr}_2(\mathbb{C})$ on $\mathbb{H}^\infty$. We will briefly recall its proof.

Theorem 6.7.1. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a solvable subgroup, then one of the following is true:*

(1) *G is a subgroup of elliptic elements, and hence isomorphic to a solvable subgroup of one of the groups from Theorem 6.1.3.*

(2) *G is conjugate to a subgroup of $\mathcal{J} \simeq \mathrm{PGL}_2(\mathbb{C}) \ltimes \mathrm{PGL}_2(\mathbb{C}(t))$.*

(3) *G is conjugate to a subgroup of the automorphism group of a Halphen surface.*

(4) *G is conjugate to a subgroup of $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$.*

(5) *There is a loxodromic element $f \in \mathrm{Cr}_2(\mathbb{C})$ and a finite group $H \subset \mathrm{Cr}_2(\mathbb{C})$ such that $G = \langle f \rangle \ltimes H$.*

Proof. Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a solvable subgroup.
Case 1. G contains a loxodromic element. In this case the statement follows directly from Lemma 6.6.7.
Case 2. G does not contain a loxodromic element, but G contains a parabolic element. In this case the statement of the theorem follows from Lemma 5.3.4.
Case 3. G is a group of elliptic elements and as such isomorphic to a subgroup of one of the groups from Theorem 6.1.3. $\qquad\square$

Lemma 6.7.2. *Let G be a group. If there exists an N such that every finitely generated subgroup of G has derived length $\leq N$, then G is solvable and has derived length $\leq N$.*

Proof. We will show that $G^{(N)} = \{\mathrm{id}\}$. Let $g \in G^{(N)}$. Then there exists a finitely generated group H such that $g \in H^{(N)}$. By assumption, $H^{(N)} = \{\mathrm{id}\}$. $\qquad\square$

We also recall the following well known Lemma about solvable subgroups:

Lemma 6.7.3. *Assume that there is an exact sequence of groups*

$$1 \to H_1 \to G \to H_2 \to 1.$$

Then G is solvable if and only if H_1 and H_2 are solvable. Moreover, the derived length of G is at most the sum of the solvable lengths of H_1 and of H_2.

Lemma 6.7.4. *There exists a constant $K \leq 35$ such that every finitely generated solvable subgroup of elliptic elements of $\mathrm{Cr}_2(\mathbb{C})$ has derived length $\leq K$ and every finite solvable subgroup has derived length $\leq K - 1$.*

Proof. Let $\Gamma \subset \mathrm{Cr}_2(\mathbb{C})$ be a finitely generated subgroup of elliptic elements. By Theorem 6.3.1, Γ is either contained in a maximal algebraic subgroup or it is conjugate to a subgroup of $\mathcal{J} = \mathrm{PGL}_2(\mathbb{C}(t)) \rtimes \mathrm{PGL}_2(\mathbb{C})$.

In the first case, by Proposition 6.2.13, either Γ is an extension of $\mathcal{S}_4$ by $(\mathbb{Z}/2\mathbb{Z}^2)$ in which case the derived length is ≤ 4. Or there exists a homomorphism $\varphi \colon \Gamma \to \mathrm{GL}_N(\mathbb{C})$ for $N \leq 24$ such that $\ker(\varphi)$ is isomorphic to $\mathbb{C}^n$ for some n. By Theorem 6.1.12 there exists a constant $K \leq 1 + 7\log_2(24) \leq 34$ such that every solvable subgroup of $\mathrm{GL}_N(\mathbb{C})$ has derived length $\leq K$. It follows in particular that the derived length of Γ is $\leq K + 1 \leq 35$, and that it is ≤ 34, if Γ is finite.

As for the second case, we note that, again by Theorem 6.1.12, every solvable subgroup of $\mathrm{PGL}_2(\mathbb{C})$ and $\mathrm{PGL}_2(\mathbb{C}(t))$ has derived length ≤ 6, since $\mathrm{PGL}_2(\mathbb{C})$ can be embedded into $\mathrm{GL}_3(\mathbb{C})$. By Lemma 6.7.3, every solvable subgroup of $\mathrm{PGL}_2(\mathbb{C}(t)) \rtimes \mathrm{PGL}_2(\mathbb{C})$ has therefore derived length ≤ 12. This proves the Lemma for the second case. $\square$

Proof of Theorem 6.1.11. Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a solvable subgroup.

Case 1. First we assume that G contains a loxodromic element. By Theorem 6.7.1, G is either isomorphic to a subgroup of $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$ or to a group of the form $\mathbb{Z} \ltimes H$, where $H \subset \mathrm{Cr}_2(\mathbb{C})$ is finite. Every solvable subgroup of $\mathrm{GL}_2(\mathbb{Z})$ has derived lentgh ≤ 4, hence in the first case, G has derived length ≤ 5. In the second case, G is solvable if and only if H is solvable and the derived length of G is one larger than the derived length of H. By Lemma 6.7.4, the derived length of H is ≤ 34.

Case 2. In a next step we consider the case where G does not contain a loxodromic element, but a parabolic element. In this case, G is either isomorphic to a subgroup of the de Jonquières group $\mathcal{J}$ or to a subgroup of the automorphism group of a Halphen surface. If G is isomorphic to a subgroup of the de Jonquières group there exists a short exact sequence

$$1 \to H_1 \to G \to H_2 \to 1,$$

where $H_1 \subset \mathrm{PGL}_2(\mathbb{C}(t))$ and $H_2 \subset \mathrm{PGL}_2(\mathbb{C})$. Since G is solvable, H_1 and H_2 are solvable. By Theorem 6.1.12, the derived length of H_1 and H_2 is bounded by 6 and therefore, by Lemma 6.7.3, the derived length of G is ≤ 12. If G is isomorphic to a subgroup of the automorphism group of a Halphen surface X, then there exists, by Theorem 5.3.3, a homomorphism $\rho \colon \mathrm{Aut}(X) \to H$, where $H \subset \mathrm{PGL}_2(\mathbb{C})$ is a finite group, and $\ker(\rho)$ is an extension of an abelian group by a cyclic group of order dividing 24. Hence the derived length of G is ≤ 8.

Case 3. Finally, we consider the case where G is a group of elliptic elements. In this case the Lemma follows from Lemma 6.7.2 and Lemma 6.7.4. $\square$

Chapter 7

Simple subgroups of the plane Cremona group

7.1 Introduction and results

It had been a long-standing open question, whether the plane Cremona group is simple as a group until Cantat and Lamy showed in 2012 that it is not. The main idea to prove this result was to use the action of $\mathrm{Cr}_2(\mathbb{C})$ on the Picard-Manin space $\mathbb{H}^\infty$ by isometries and use techniques from small cancellation theory. In this chapter, we refine these techniques with the aim of classifying all simple subgroups of $\mathrm{Cr}_2(\mathbb{C})$. However, during the work I have encountered a problem that I was not able to solve. We formulate it in the following conjecture, which will be discussed in more detail in Section 7.2.3.

Conjecture 7.1.1. *Let $f \in \mathrm{Cr}_2(\mathbb{C})$ be a loxodromic element, $p \in \mathbb{P}^2$ a point that is not contained in any of the coordinate lines of $\mathbb{P}^2$, k a positive integer. Then the constructible set*

$$\{d(fd)^k(p) \mid d \in D_2 \text{ such that } p \notin \mathrm{Ind}((fd)^l) \text{ for all } 1 \le l \le k\}$$

is open and dense in $\mathbb{P}^2$.

The main result of this chapter will be:

Theorem 7.1.2. *Assume that Conjecture 7.1.1 holds.*
Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a simple group. Then:

(1) *G does not contain any loxodromic element.*

(2) *If G contains a parabolic element, then G fixes a rational fibration, i.e. there exists a G-invariant rational map $\pi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ with rational fibers. In that case, G is isomorphic to a subgroup of $\mathrm{PGL}_2(\mathbb{C}(t))$.*

(3) *If all elements in G are elliptic then either G is a simple subgroup of an algebraic subgroup of $\mathrm{Cr}_2(\mathbb{C})$ or it is conjugate to a subgroup of the de Jonquières group $\mathcal{J} \simeq \mathrm{PGL}_2(\mathbb{C}) \ltimes \mathrm{PGL}_2(\mathbb{C}(t))$.*

Remark 7.1.1. Only part (1) of Theorem 7.1.2 depends on Conjecture 7.1.1. If we assume that the simple subgroup G does not contain any loxodromic element, Part (2) and (3) can be proven without assuming Conjecture 7.1.1 (Lemma 7.3.1 and 7.3.2).

As for the case (3) of Theorem 7.1.2, one observes that a simple subgroup of $\mathrm{PGL}_2(\mathbb{C}) \ltimes \mathrm{PGL}_2(\mathbb{C}(t))$ is abstractly isomorphic to a simple subgroup of $\mathrm{PGL}_2(\mathbb{C})$. However, we are not able to give a description of the conjugacy classes of simple subgroups of $\mathrm{PGL}_2(\mathbb{C}) \ltimes \mathrm{PGL}_2(\mathbb{C}(t))$.

From Theorem 7.1.2 one can deduce the following observation by looking at the classification of maximal algebraic subgroups of $\mathrm{Cr}_2(\mathbb{C})$ (see Theorem 6.2.1):

Corollary 7.1.3. *Assume that Conjecture 7.1.1 holds.*

A simple group G can be embedded into $\mathrm{Cr}_2(\mathbb{C})$ if and only if G is isomorphic to a subgroup of $\mathrm{PGL}_3(\mathbb{C})$.

Corollary 7.1.3 naturally leads to the following question, which is, to our knowledge, an open problem:

Question 7.1.1. What are the simple subgroups of $\mathrm{PGL}_3(\mathbb{C})$ and $\mathrm{PGL}_2(\mathbb{C})$?

Obvious classes of simple subgroups of $\mathrm{PGL}_2(\mathbb{C})$ are subgroups of the form $\mathrm{PSL}_2(\mathbf{k})$, where $\mathbf{k} \subset \mathbb{C}$ is a subfield, or finite simple subgroups. It is unclear, whether there exist other examples:

Question 7.1.2. What are the simple subgroups of $\mathrm{PSL}_2(\mathbb{Q})$? Does $\mathrm{PSL}_2(\mathbb{Q})$ contain a proper infinite simple subgroup?

7.1.1 Finitely generated simple subgroups

A group G satisfies the *property of Malcev*, if every finitely generated subgroup $\Gamma \subset G$ is *residually finite*, i.e. for every element $g \in \Gamma$ there exists a finite group H and a homomorphism $\varphi \colon \Gamma \to H$ such that g is not contained in the kernel of φ. Malcev showed that linear groups satisfy this property ([Mal40a]). Other groups that fulfill the property of Malcev include automorphism groups of algebraic varieties over fields of characteristic zero. In [Can11a], Cantat asked whether the plane Cremona group has the property of Malcev, a question that is still open.

Note that finitely generated simple subgroups of groups with the property of Malcev are always finite. We will prove the following theorem for the plane Cremona group:

Theorem 7.1.4. *Every finitely generated simple subgroup of $\mathrm{Cr}_2(\mathbb{C})$ is finite.*

Note that in order to prove Theorem 7.1.4 we do not need Conjecture 7.1.1.

From the classification of finite subgroups of $\mathrm{Cr}_2(\mathbb{C})$ (see [DI09]) we obtain:

Corollary 7.1.5. *A finitely generated simple subgroup of $\mathrm{Cr}_2(\mathbb{C})$ is isomorphic to*

$$\mathbb{Z}/p\mathbb{Z}, \text{ for some prime } p, \quad \mathcal{A}_5, \quad \mathcal{A}_6, \quad \mathrm{PSL}_2(7).$$

The conjugacy classes of these finite groups are classified in [DI09].

7.2 The loxodromic case

One of the main steps towards Theorem 7.1.2 is the following:

Theorem 7.2.1. *Assume that Conjecture 7.1.1 holds.*
A simple subgroup $G \subset \mathrm{Cr}_2(\mathbb{C})$ does not contain any loxodromic element.

The starting point to prove Theorem 7.2.1 is Lemma 5.6.5. It states that all loxodromic elements in a simple group G are of monomial type and that, up to conjugation, G contains a dense subgroup $\Delta_2 \subset D_2$ for each of its loxodromic elements. Our strategy is to show that if G contains a loxodromic element, these conditions imply that G is conjugate to a subgroup of $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$ and from this we will deduce a contradiction to the simplicity of G.

In Section 7.2.1 we first prove a result about the degrees of elements that conjugate loxodromic elements to monomial elements. In Section 7.2.2 we take a closer look at the dynamical behavior of exceptional curves and base-points. This will allow us to prove Theorem 7.2.1 in Section 7.2.5.

7.2.1 Degree bounds

One of the numerous corollaries that Blanc and Cantat draw from the gap property of the dynamical degrees of elements in $\mathrm{Cr}_2(\mathbb{C})$ (Theorem 5.2.2) in [BC16], is the following result:

Theorem 7.2.2 (Cantat, Blanc, [BC16, Corollary 1.5]). *Two loxodromic elements $f, g \in \mathrm{Cr}_2(k)$ of degree $\leq d$ are conjugate if and only if they are conjugate by an element of degree $\leq (2d)^{57}$.*

Another important result we will use is that the dynamical degree is a lower semi-continuous function on $\mathrm{Cr}_2(\mathbb{C})$ with respect to the Zariski topology:

Theorem 7.2.3 ([Xie15, Theorem 4.3]). *The dynamical degree is a lower semi-continuous function. More precisely, let $A \subset \mathrm{Cr}_2(\mathbb{C})$ be a family of birational transformations parametrized by an algebraic variety A. Then for all $\lambda \in \mathbb{R}$, the set $\{f \in A \mid \lambda(f) > \lambda\}$ is open in A.*

Theorem 7.2.4 ([Xie15, Theorem 1.6]). *Let k be an algebraically closed field and $d \geq 2$ an integer. Denote by $\mathrm{Cr}_2(k)_d$ the space of Cremona transformations of degree d. Then for any $\lambda < d$, the set*

$$U_\lambda = \{f \in \mathrm{Cr}_2(k)_d \mid \lambda_1(f) > \lambda\}$$

is open and Zariski-dense in $\mathrm{Cr}_2(k)_d$.

Lemma 7.2.5. *Let $A \in \mathrm{SL}_2(\mathbb{Z})$ be a loxodromic element. Then A is conjugate in $\mathrm{GL}_2(\mathbb{Z})$ to a matrix B such that all entries of B are non-negative, or all entries of B are non-positive.*

Proof. Let L_A be the linear transformation of $\mathbb{R}^2$ corresponding to A. Since it is loxodromic, it has two different eigenspaces $E_{\pm\lambda}$ and $E_{\pm 1/\lambda}$ with eigenvalues $\pm\lambda > 1$ and $\pm 1/\lambda$ respectively. Fix two eigenvectors $e_1 \in E_{\pm\lambda}$ and $e_2 \in E_{\pm 1/\lambda}$. Let

u and v be vectors in $\mathbb{Z}^2 \subset \mathbb{R}^2$ that form a basis of $\mathbb{Z}^2$ and such that $v = \alpha_1 e_1 + \alpha_2 e_2$, $u = \beta_1 e_1 - \beta_2 e_2$ with $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$. With respect to the basis (u, v), the images of u and v under L_A are either both in the first quadrant (if the eigenvalues of A are λ and $1/\lambda$) or in the third quadrant (if the eigenvalues of A are $-\lambda$ and $-1/\lambda$). Therefore, all the entries of the matrix of L_A with respect to the basis (u, v) are either non-negative or non-positive. $\qquad\square$

Lemma 7.2.6. *Let $A, B \in \mathrm{GL}_2(\mathbb{Z})$ be two loxodromic elements such that $\mathrm{tr}(A) = \mathrm{tr}(B)$ and $\det(A) = \det(B)$. Then A is conjugate to B if and only if A^2 is conjugate to B^2.*

Proof. If A is conjugate to B, then A^2 is conjugate to B^2. On the other hand, assume that there exists a $C \in \mathrm{GL}_2(\mathbb{Z})$ such that $CA^2C^{-1} = B^2$. Since A is loxodromic, $\mathrm{tr}(A) \neq 0$. By the Theorem of Cayley-Hamilton, $A = 1/\mathrm{tr}(A)(A^2 + \det(A))$ and, since $\mathrm{tr}(A) = \mathrm{tr}(B)$ and $\det(A) = \det(B)$ by assumption, $B = 1/\mathrm{tr}(A)(B^2 + \det(A))$. Therefore, $CAC^{-1} = 1/\mathrm{tr}(A)(CA^2C^{-1} + \det(A)) = C(1/\mathrm{tr}(A)(B^2 + \det(A)))C^{-1} = B$. $\qquad\square$

Lemma 7.2.7. *For an integer $n \in \mathbb{Z}$ there exist only finitely many conjugacy classes of loxodromic elements in $\mathrm{GL}_2(\mathbb{Z})$ with trace n.*

Proof. We first show the claim for elements in $\mathrm{SL}_2(\mathbb{Z})$. By Lemma 7.2.5, every conjugacy class of loxodromic elements in $\mathrm{SL}_2(\mathbb{Z})$ contains an element B such that all entries of B are non-negative or all entries of B are non-positive. We show that there exist only finitely many elements of that form with trace n. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$, such that a, b, c and d are either all non-negative or all non-positive. First assume that $\mathrm{tr}(B) = n > 0$, hence we have $a, d, b, c \geq 0$. Moreover, there are only finitely many values $a, d \in \mathbb{Z}_{\geq 0}$ that satisfy $a + d = n$. The condition $ad - bc = \pm 1$ furthermore implies that there are only finitely many values for a, b, c and d. If we assume $n < 0$, we proceed analogously. If the trace of a matrix $A \in \mathrm{GL}_2(\mathbb{Z})$ is zero, then A has finite order and is therefore not loxodromic.

Now we consider the case of loxodromic elements in $\mathrm{GL}_2(\mathbb{Z})$ of determinant -1. Let $B \in \mathrm{GL}_2(\mathbb{Z})$ be a loxodromic element of trace n and of determinant -1 and denote by $\lambda(B) > 1$ its spectral radius. We have $\lambda - 1 < |\mathrm{tr}(B)| < \lambda + 1$ and hence, $|\mathrm{tr}(B^2)| < \lambda^2 + 1 \leq |\mathrm{tr}(B)^2| + 2|\mathrm{tr}(B)|$, in particular, there are only finitely many values that $\mathrm{tr}(B^2)$ can take. Since $B^2 \in \mathrm{SL}_2(\mathbb{Z})$, this implies that there exist only finitely many conjugacy classes to which B^2 can belong. By Lemma 7.2.6, there exists a one-to-one correspondence between the conjugacy classes of elements in $\mathrm{GL}_2(\mathbb{Z})$ of trace n and determinant -1 and the conjugacy classes of the squares of such elements. Hence, there exist only finitely many conjugacy classes to which B can be belong. This finishes the proof. $\qquad\square$

Lemma 7.2.8. *Let $\lambda > 0$ and $g \in \mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$. If $\lambda_1(g) \leq \lambda$ then g is conjugate in $\mathrm{GL}_2(\mathbb{Z})$ to an element of degree $\leq C(\lambda)$, where the constant $C(\lambda)$ only depends on λ.*

Proof. We first show the claim for elements in $\mathrm{GL}_2(\mathbb{Z})$. The dynamical degree $\lambda_1(g)$ is the spectral radius of g, i.e. the absolute value of the matrix g that is strictly

larger than 1. The condition $\lambda_1(g) \leq \lambda$ implies that $|\operatorname{tr}(g)| = |\lambda_1(g) + \lambda_1(g)^{-1}| \leq \lambda + 1$. So $\operatorname{tr}(g)$ is contained in the finite set of integers between $-(\lambda+1)$ and $(\lambda+1)$. By Lemma 7.2.7, there exist only finitely many conjugacy classes in $\operatorname{GL}_2(\mathbb{Z})$ to which g can belong. Denote by $f_1, \dots, f_n$ representants of these classes. We set $C(\lambda) = \max\{\deg(f_1), \dots, \deg(f_n)\}$.

Let $g = md$, where $m \in \operatorname{GL}_2(\mathbb{Z})$ and $d \in D_2$, and let $n \in \operatorname{GL}_2(\mathbb{Z})$ be the element that conjugates m to an element in $\operatorname{GL}_2(\mathbb{Z})$ of degree $\leq C(\lambda)$. Since $\deg(nmdn^{-1}) = \deg(nmn^{-1})$, we are done. $\qquad\square$

Finally, we are able to prove the main result of this section:

Lemma 7.2.9. *Let $g \in \operatorname{Cr}_2(k)$ be a loxodromic element of monomial type. Then there exists an $m \in \operatorname{GL}_2(\mathbb{Z}) \ltimes D_2$ and a constant K depending only on $d := \deg(g)$, such that g is conjugate to m by an element of degree $\leq K$.*

Proof. Note that $\lambda_1(g) \leq d$. So, by Lemma 7.2.8, there exists a constant $C(d)$ such that g is conjugate to an $m \in W_2 \ltimes D_2$ of degree $\leq C(d)$. By Theorem 7.2.2, g can be conjugated to m by an element of degree $\leq K$, where $K = (2r)^{57}$ for $r = \max\{d, C(d)\}$. $\qquad\square$

7.2.2 Base-points and toric boundaries

Let S be a smooth projective surface with a given regular D_2-action that has an open orbit $U \subset S$. The fixed points of this action are called *toric points*, the algebraic set $\partial S := S \setminus U$ is called the *toric boundary*. In what follows, we consider $\mathbb{P}^2$ equipped with the standard action of D_2, or blow-ups $\pi \colon S \to \mathbb{P}^2$ of finitely many toric points with the pull-back of the standard action of D_2 on $\mathbb{P}^2$. In this case, ∂S is always a curve whose irreducible components are lines with self-intersection ≤ 1.

A *toric point* in the bubble space $\mathcal{B}(\mathbb{P}^2)$ is a point of the form (p, S, π), where $\pi \colon S \to \mathbb{P}^2$ is the blow-up of finitely many toric points and $p \in S$ is a toric point. If a toric point $q_1 \in \mathcal{B}(\mathbb{P}^2)$ lies above a point $q_2 \in \mathcal{B}(\mathbb{P}^2)$, then q_2 is toric as well.

Let S be a projective surface, $f \in \operatorname{Bir}(S)$ and assume that f contracts a curve $C \subset S$. If $f(C) = p \in S$ we say that f *contracts C to p*. We extend this notion to infinitely near points. Consider a point in the bubble space $\mathcal{B}(S)$ with a representative (p, T, π). Let $\tilde{f} \in \operatorname{Bir}(T)$ be given by $\tilde{f} := \pi^{-1} f \pi$ and denote by $\tilde{C}$ the strict transform of C under π. We say that f *contracts C to p* if $\tilde{f}(\tilde{C}) = p$. If p lies above a point q in $\mathcal{B}(S)$ and f contracts a curve $C \subset S$ to p, then f also contracts C to q.

Definition 7.2.1. Let S be a projective surface and $f \in \operatorname{Bir}(S)$. We denote by $E(f)$ the number of irreducible components of the exceptional divisor of f and by $I(f)$ the number of indeterminacy points of f.

Remark 7.2.1. For $f \in \operatorname{Cr}_2$, the numbers $E(f)$ and $I(f)$ can be bounded by a constant depending only on the degree of f. Let g be another Cremona transformation. Then $E(fg) \leq E(f) + E(g)$ and $I(fg) \leq I(f) + I(g)$.

Lemma 7.2.10. *Let S be a rational projective surface, $f \in \operatorname{Bir}(S)$ of monomial type and $\pi \colon S \dashrightarrow \mathbb{P}^2$ a birational transformation. Then $I(f^n)$ and $E(f^n)$ are*

uniformly bounded for all n by a constant K only depending on π and the degree of $\pi f \pi^{-1}$.

Proof. The birational transformation $\pi \colon S \dashrightarrow \mathbb{P}^2$ only contracts finitely many irreducible curves and has only finitely many base-points. So $I(f^n)$ and $E(f^n)$ are uniformly bounded for all n if and only if $I(\pi f \pi^{-1})$ and $E(\pi f \pi^{-1})$ are uniformly bounded. It is therefore enough to consider the case $f \in \mathrm{Bir}(\mathbb{P}^2)$.

By Lemma 7.2.9, there exists a $g \in \mathrm{Cr}_2(\mathbb{C})$ of degree $\leq C$, where C only depends on $\deg(f)$, such that $gfg^{-1} = m \in \mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$. We have $E(m^n) \leq 3$ and $I(m^n) \leq 3$ for all n. By Remark 7.2.1, $E(g)$ and $I(g)$ are bounded by a constant K' depending only on $\deg(g)$ and hence only on $\deg(f)$. Therefore, $E(f^n) = E(gm^ng^{-1}) \leq 2K' + 3$ and $I(f^n) = I(gm^ng^{-1}) \leq 2K' + 3$. We thus set $K := 2K' + 3$. $\qquad\square$

7.2.3   A conditional lemma

Let $f \in \mathrm{Cr}_2(\mathbb{C})$ be a birational transformation, $k \in \mathbb{Z}_+$ a positive integer and $p \in \mathbb{P}^2$ a point that is not contained in any of the coordinate lines, i.e. all the coordinates of p are non zero. We consider the following rational map

$$\varphi_{k,p}^f \colon D_2 \dashrightarrow \mathbb{P}^2, \quad d \mapsto d(fd)^k(p).$$

We note that $\varphi_{k,p}^f$ does not need to be well defined. For instance, if $\varphi_{k,p}^f$ is a well defined rational map and $\varphi_{k,p}^f(d) = q$ is an indeterminacy point of f for a general $d \in D_2$, then $\varphi_{k+1,p}^f$ is not defined anymore.

However, if $\varphi_{k,p}^f$ is a well defined rational dominant map, then $\varphi_{k+1,p}^f$ is a well defined rational map as well (although not necessarily dominant).

Example 7.2.11. Consider the standard quadratic involution $\sigma := [yz : xz : xy] \in \mathrm{Cr}_2(\mathbb{C})$. Since $\sigma d = d^{-1}\sigma$, we obtain that $\varphi_{2k,p}^\sigma(d) = \sigma(p)$ for all positive integers k and all $p \in U$. In particular, $\varphi_{k,p}^\sigma$ is well defined for all k. Moreover, if k is even, then $\varphi_{k,p}^\sigma$ is constant, and if k is odd then $\varphi_{k,p}^\sigma$ is dominant.

Example 7.2.12. Consider the birational map of the form $f = (x^{-1}, p(x,y)) \in \mathrm{Cr}_2(\mathbb{C})$ defined with respect to local coordinates (x,y) for some $p(x,y) \in \mathbb{C}(x,y)$. For a $d \in D_2$ given by $d = (d_1x, d_2y)$, one calculates $dfd = (x^{-1}, d_2p(d_1x, d_2y))$. Let $p = (a,b) \in U$. Then, if $\varphi_{2k,p}^f(D_2)$ is well defined, the image $\varphi_{2k,p}^f(D_2)$ is contained in the line $x = a$, for all positive integers k.

However, the maps in Example 7.2.11 and 7.2.12 are rather particular and both of them preserve rational fibrations. We expect that similar phenomena do not occur if f is a loxodromic element, i.e. that for a loxodromic element $f \in \mathrm{Cr}_2(\mathbb{C})$, the rational map $\varphi_{k,p}^f$ is well defined and dominant for all positive integers k. This leads to Conjecture 7.1.1 stated in the introduction:

Conjecture 7.1.1: *Let $f \in \mathrm{Cr}_2(\mathbb{C})$ be a loxodromic element, $p \in \mathbb{P}^2$ a point that is not contained in any of the coordinate lines of $\mathbb{P}^2$, k a positive integer. Then the constructible set*

$$\{d(fd)^k(p) \mid d \in D_2 \text{ such that } p \notin \mathrm{Ind}((fd)^l) \text{ for all } 1 \leq l \leq k\}$$

is open and dense in $\mathbb{P}^2$.

A positive answer to Conjecture 7.1.1 implies in particular the following result, which we need in order to prove Theorem 7.1.2.

Conditional Lemma 7.2.13. *Assume that Conjecture 7.1.1 holds.*

Let $\pi \colon S \to \mathbb{P}^2$ *be the blow-up of finitely many toric points,* $f \in \mathrm{Bir}(S)$ *and* $N \in \mathbb{Z}_+$. *Assume that* f *contracts a curve* $C \subset S$ *that is not contained in the toric boundary of* S. *Then, there exists an open dense subset* $U \subset D_2 \times D_2$ *such that for all* $(d_1, d_2) \in U$ *the following condition is satisfied:*

(A) *For all* $k \leq N$ *the curve* $d_2^{-1}(C)$ *is not contained in the exceptional locus of* $(d_1 f d_2)^{-k}$, *i.e.* $(d_1 f d_2)^{-k}(d_2^{-1}(C))$ *is a curve for all* $k \leq N$. *Moreover,* $(d_1 f d_2)^{-k}(d_2^{-1}(C))$ *is not contained in the toric boundary* ∂S *for all* $k \leq N$ *and* $d_1 f d_2$ *is loxodromic.*

In particular, $(d_1 f d_2)^k$ *contracts a curve that is not contained in* ∂S.

Proof. It is enough to show the claim for $S = \mathbb{P}^2$. Indeed, the push-forward of $d_1 f d_2$ under π satisfies condition (A) on $\mathbb{P}^2$ if and only if $d_1 f d_2$ staisfies condition (A) on S. So we may and do assume that $S = \mathbb{P}^2$.

Denote by $\mathrm{Exc}(f^{-1})$ the exceptional locus of f^{-1}. Let $p \in C$ be a non-toric point and $1 \leq l \leq k$. The set

$$V_l := \{d \in D_2 \mid (d(f^{-1}d)^l(p) \notin \mathrm{Exc}(f^{-1} \cup \partial \mathbb{P}^2)\} \subset D_2$$

is open in D_2 and since the set

$$\{d(f^{-1}d)^l(p) \mid d \in D_2, d(f^{-1}d)^l(p) \notin \mathrm{Ind}(d(f^{-1}d)) \cup \cdots \cup \mathrm{Ind}(d(f^{-1}d)^l)\}$$

is dense in $\mathbb{P}^2$ by Conjecture 7.1.1, the set V_l is not empty and therefore dense. Define now the open dense subset $V := \bigcap_{l=1}^{k} V_k \subset D_2$. Hence, for all $d \in V$, and $1 \leq l \leq k$ we have that $d(f^{-1}d)^l(C)$ is not contained in the exceptional locus of f^{-1} nor in $\partial \mathbb{P}^2$.

Consider now the map $m \colon D_2 \times D_2 \to D_2$ given by $(d_1, d_2) \mapsto d_1^{-1} d_2^{-1}$ and define the open dense subset $U_1 := m^{-1}(V) \subset D_2 \times D_2$. Let $(d_1, d_2) \in U_1$. Therefore, we have

$$(d_1^{-1} d_2^{-1})(f^{-1}(d_1^{-1} d_2^{-1}))^l(C) = d_1^{-1}(d_1 f d_2)^{-l} d_2^{-1}(C) \not\subset \mathrm{Exc}(f^{-1}) \cup \partial \mathbb{P}^2,$$

which is equivalent to $(d_1 f d_2)^{-l}(d_2^{-1}(C))$ not being contained in $\mathrm{Exc}((d_1 f d_2)^{-1}) \cup \partial \mathbb{P}^2$ for all $1 \leq l \leq k$.

By Theorem 7.2.3, there exists an open set $U_2 \subset D_2 \times D_2$ such that $d_1 f d_2$ is loxodromic for all $(d_1, d_2) \in U_2$. Since $(\mathrm{id}, \mathrm{id})$ is contained in U_2, it is non-empty and therefore dense. Define $U := U_1 \cap U_2$. By construction, all elements (d_1, d_2) in U satisfy property (A). $\qquad\square$

7.2.4   An application of Conjecture 7.1.1

Assume that a group $G \subset \mathrm{Cr}_2(\mathbb{C})$ contains a loxodromic element f that contracts a curve $C \subset \mathbb{P}^2$ that is not contained in the toric boundary. In this section, we

want to show that it is possible to find diagonal automorphisms $d_1, d_2 \in D_2$ such that an iterate of $d_1 f d_2$ contracts many curves and is therefore not of monomial type, by Lemma 7.2.10. Unfortunately, our arguments depend on Lemma 7.2.13 and therefore on Conjecture 7.1.1. The next lemmas will be rather technical, so let us briefly sketch the rough strategy. Let $f \in \mathrm{Cr}_2(\mathbb{C})$ be a loxodromic element that contracts a curve C that is not contained in the toric boundary. We consider the following cases:

- In Lemma 7.2.14, we consider the case, where $f^n(C)$ is a toric point for all $n \geq 1$.

- In Lemma 7.2.15 we consider the case, where $f(C)$ is a point not contained in the toric boundary.

- The situation is more complicated, if $f(C)$ is contained in the toric boundary. In Lemma 7.2.17 we treat the special case, where $f(C)$ is contracted to a point on a line L of the toric boundary and the restriction of f to L is a birational self-map of L.

- In Lemma 7.2.19 finally, we show the general case. The main step there is to replace f with a loxodromic transformation of the form $df d^{-1} f^{-1}$ (Lemma 7.2.18 asserts that this map is still loxodromic), which in the main case fullfills the assumptions of Lemma 7.2.14.

- In Lemma 7.2.21 we will use all these results to see that, after possibly blowing up some toric points, loxodromic elements in a simple subgroup of $\mathrm{Cr}_2(\mathbb{C})$ only contract curves that are contained in the toric boundary, which will be the key result to prove Theorem 7.2.1.

Lemma 7.2.14. *Assume that Conjecture 7.1.1 holds.*

Let $\pi\colon S \to \mathbb{P}^2$ be a blow-up of finitely many toric points, $f \in \mathrm{Bir}(S)$ a loxodromic element and $\Delta_2 \subset D_2$ a dense subgroup. Assume that there exists a curve $C \subset S$ that is not contained in the toric boundary and that is contracted by f^n to a toric point for all $n \in \mathbb{Z}_+$. Then, the group $\langle f, \Delta_2 \rangle$ contains an element that is not of monomial type.

Proof. By Lemma 7.2.10, there exists a constant K only depending on $\deg(f)$ with the following property: for each $g \in \mathrm{Bir}(S)$ of degree at most $\deg(f)$, if there exists an $n \in \mathbb{Z}_+$ such that g^n contracts $K + 1$ different curves on S, then g is not of monomial type.

By Lemma 7.2.13, there exists an open dense subset $U \subset D_2 \times D_2$ such that for all $k \leq K + 1$ the curve C is not contained in the exceptional locus of $(d_1 f d_2)^{-k}$, i.e. $(d_1 f d_2)^{-k}(C)$ is a curve for all $k \leq K + 1$. Moreover, $(d_1 f d_2)^{-k}(C)$ is not contained in the toric boundary ∂S for all $k \leq K + 1$. Let $(d_1, d_2) \in U \cap \Delta_2 \times \Delta_2$. Then $(d_1 f d_2)^{-1}(d_2^{-1}(C)), \ldots, (d_1 f d_2)^{-K-1}(d_2^{-1}(C))$ are $K + 1$ different curves. By assumption, $f^n(C)$ is a toric point for all $n \in \mathbb{Z}_+$; in particular, $(d_1 f d_2)^n(d_2^{-1}(C))$ is a toric point for all $n \in \mathbb{Z}_+$. Hence, $(d_1 f d_2)^{K+1}$ contracts $K + 1$ different curves and therefore, $d_1 f d_2$ is not of monomial type. $\square$

Lemma 7.2.15. *Assume that Conjecture 7.1.1 holds.*

Let $\pi\colon S \to \mathbb{P}^2$ be a blow-up of finitely many toric points and $f \in \mathrm{Bir}(S)$. If f contracts a curve $C \subset S$ that is not contained in the toric boundary ∂S of S onto a point $p \in S$ that is not contained in ∂S, then there exists an open dense subset $U \subset D_2 \times D_2$ such that $d_1 f d_2$ is not of monomial type.

Proof. We first show that, for a given $N \in \mathbb{Z}_+$, there exists an open dense subset $V \subset D_2 \times D_2$ such that all $(d_1, d_2) \in V$ satisfy the condition:

(B) For all $0 < k \le N$, the point $(d_1 f d_2)^k(d_2^{-1}(C))$ is not a base point of $d_1 f d_2$.

Condition (B) is an open condition. There is an open dense subset $W \subset (D_2 \times D_2)^N$ such that for all $0 < k \le N - 1$ every tuple $((d_{1_1}, d_{1_2}), \ldots, (d_{n_1}, d_{n_2})) \in W$ satisfies that $((d_{k_2} f d_{k_1}) \cdots (d_{2_2} f d_{2_1})(d_{1_2} f d_{1_1}))(d_{1_2}^{-1}(C))$ is not a base point of $d_{(k+1)_1} f d_{(k+1)_2}$. We identify $D_2 \times D_2$ with the diagonal D in $(D_2 \times D_2)^N$ and note that all the elements in the intersection $W \cap D$, which is an open set, satisfy condition (B). It therefore remains to show that $V \cap D$ is non-empty, i.e. that there exists at least one pair $(d_1, d_2) \in D_2 \times D_2$ that satisfies condition (B). Let $d_2 \in D_2$ be arbitrary and let $q \in d_2^{-1}(C)$ be a point that is not a base point of $f d_2$, that is not p and that is not contained in the toric boundary of S. Let $d_1 \in D_2$ be the element such that $d_1(p) = q$. It follows that $(d_1 f d_2)^k(d_2^{-1}(C)) = q$ for all $k \ge 0$; in particular, $(d_1 f d_2)^k(C)$ is not a base-point of $(d_1 f d_2)$, and hence (d_1, d_2) satisfies condition (B).

Let $(d_1, d_2) \in D_2 \times D_2$. By Lemma 7.2.10 there exists a $K \in \mathbb{Z}_+$ depending only on $\deg(f) = \deg(d_1 f d_2)$ such that if $E((d_1 f d_2)^n) > K$ for some $n \in \mathbb{Z}_+$, then $d_1 f d_2$ is not of monomial type. Let $V' \subset D_2 \times D_2$ be an open dense subset such that $(d_1 f d_2)^{-k}$ does not contract C and the strict transform $(d_1 f d_2)^{-k}(C)$ is a curve that is not contained in the toric boundary of S for all $k \le K + 1$. Such a V' exists by Lemma 7.2.13. Let $V \subset D_2 \times D_2$ be an open dense subset such that for all $0 < k \le K + 1$, the point $(d_1 f d_2)^k(C)$ is not a base point of $d_1 f d_2$ for all $(d_1, d_2) \in V$. Define the open dense set $U := V \cap V' \subset D_2 \times D_2$. By construction, for every $(d_1, d_2) \in U$ the transformation $(d_1 f d_2)^{N+1}$ contracts the $K + 1$ different curves $d_2^{-1}(C), (d_1 f d_2)^{-1}(d_2^{-1}(C)), \ldots, (d_1 f d_2)^{-N}(d_2^{-1}(C))$. Hence, $E((d_1 f d_2)^n) > K$, and we obtain that $d_1 f d_2$ is not of monomial type. $\qquad\square$

Lemma 7.2.16. *Let $p \in \mathbb{P}^1 \setminus \{[0:1], [1:0]\}$ and $g \in \mathrm{PGL}_2(\mathbb{C})$. Then one of the following is true:*

(1) *the set $\{d(gd)^n(p) \mid d \in D_1\}$ is open and dense in $\mathbb{P}^1$ for all $n \in \mathbb{Z}_{\ge 0}$,*

(2) *the set $\{d(g^2 d)^n(p) \mid d \in D_1\}$ is open and dense in $\mathbb{P}^1$ for all $n \in \mathbb{Z}_{\ge 0}$.*

Proof. Assume that $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$, where $r, s, t, u \in \mathbb{C}$, $p = [1:c]$, where $c \ne 0$ and let $d = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. Let $n \in \mathbb{Z}_+$ be any positive integer. We first calculate $gd = \begin{pmatrix} rx & sy \\ tx & uy \end{pmatrix}$ and see by induction that, for all $n \ge 1$,

$$(gd)^n = \begin{pmatrix} r^n x^n + y p_1(x, y) & y p_2(x, y) \\ x p_3(x, y) & u^n y^n + x p_4(x, y) \end{pmatrix},$$

where $p_1, p_2, p_3, p_4 \in \mathbb{C}[x, y]$ are homogeneous polynomials of degree $n - 1$. Hence,

$$d(gd)^n = \begin{pmatrix} r^n x^{n+1} + xyp_1(x, y) & xyp_2(x, y) \\ xyp_3(x, y) & u^n y^{n+1} + xyp_4(x, y) \end{pmatrix}.$$

Therefore, $d(gd)^n(p)$ is given by

$$[r^n x^{n+1} + xyp_1(x, y) + c(xyp_2(x, y)) : xyp_3(x, y) + c(u^n y^{n+1} + xyp_4(x, y))].$$

This defines a morphism from $\mathbb{P}^1$ to $\mathbb{P}^1$. If it is not constant, the image of the open subset of points $[x : y]$, $x \neq 0, y \neq 0$, which is exactly the set $\{d(gd)^n(p) \mid d \in D_1\}$, is open and dense in $\mathbb{P}^1$. If it is constant for some $n \in \mathbb{Z}_+$, i.e. $d(gd)^n(p)$ does not depend on d and thus the above expression is independent of $[x : y]$, then $r = u = 0$. If this is the case, then $g^2 \in D_1$, which implies in particular that $d(g^2 d)^n(p) = d^{n+1} g^2(p)$ and hence that the set $\{d(g^2 d)^n(p) \mid d \in D_1\}$ is open and dense in $\mathbb{P}^1$ for all $n \in \mathbb{Z}_+$.

For $n = 0$ the sets $\{d(gd)^n(p) \mid d \in D_1\}$ and $\{d(g^2 d)^n(p) \mid d \in D_1\}$ are always open and dense. $\qquad \square$

Lemma 7.2.17. *Assume that Conjecture 7.1.1 holds.*

Let $\pi \colon S \to \mathbb{P}^2$ be a blow-up of finitely many toric points, $f \in \mathrm{Bir}(S)$ and $\Delta_2 \subset D_2$ a dense subset. Assume that f contracts a curve $C \subset S$ that is not contained in the toric boundary of S, onto a non-toric point $p \in S$ that is contained in a line $L \subset \partial S$. If f restricts to a birational transformation of L to itself, then the subgroup $\langle \Delta_2, f \rangle \subset \mathrm{Cr}_2(\mathbb{C})$ contains a loxodromic element that is not of monomial type.

Proof. By Lemma 7.2.13, there exists an open dense $U \subset D_2 \times D_2$ such that for all $(d_1, d_2) \in U$,

(1) $d_1 f d_2$ is loxodromic,

(2) $(d_1 f d_2)^{-2}$ does not contract $d_2^{-1}(C)$,

(3) $(d_1 f d_2)^{-2}(C)$ is a curve that is not contained in the toric boundary of S.

Since $\Delta_2 \subset D_2$ is dense, the intersection $\Delta_2 \times \Delta_2$ with U is non-empty, i.e. there exist $e_1, e_2 \in \Delta_2$ that satisfy conditions (1) to (3). We fix e_1 and e_2 and define $\tilde{f} := e_1 f e_2 \in \langle \Delta_2, f \rangle$ and $\tilde{C} := e_2^{-1}(C)$. The birational transformation $\tilde{f}$ contracts the curve $\tilde{C}$, which is not contained in ∂S, to the point $p' := \tilde{f}(\tilde{C}) \in L$. Let $C' := \tilde{f}^{-1}(\tilde{C})$. The birational transformation $\tilde{f}^2$ contracts the curve C', which is not contained in ∂S, to the point $p' \in L$.

Let K be the integer from Lemma 7.2.10 satisfying that for every loxodromic element of monomial type $h \in \mathrm{Cr}_2(\mathbb{C})$ of degree $\leq \deg(\tilde{f})^2$, the number $E(h^n)$ is $\leq K$ for all $n \in \mathbb{Z}$.

Elements of D_2 restrict to automorphisms of L. In fact, by choosing homogeneous coordinates of L such that the two toric points on L correspond to $[1 : 0]$ and $[0 : 1]$ respectively, we obtain a surjective homomorphism of algebraic groups $\varphi_L \colon D_2 \to D_1$. Denote by $g \in \mathrm{PGL}_2(\mathbb{C})$ the transformation of L induced by $\tilde{f}$. By Lemma 7.2.16, we are always in one of the following two cases:

Case 1: The set $\{d(gd)^n(p') \mid d \in D_1\}$ is open and dense in $\mathbb{P}^1$ for all $n \in \mathbb{Z}_{\geq 0}$. Let $p_1, \ldots, p_r$ be the indeterminacy points of $\tilde{f}$ that lie on L. There exists an open dense set $V \subset D_1$ such that $\{p_1, \ldots, p_r\}$ are not contained in $\{d(gd)^n(p) \mid d \in V\}$ for all $0 \leq n \leq K + 1$. Consider the morphism $\psi_L : D_2 \times D_2 \to D_1$ defined by $\psi_L(d_1, d_2) := \varphi_L(d_2)\varphi_L(d_1)$. Since ψ_L is surjective, the set $W := \psi_L^{-1}(V)$ is open and dense in $D_2 \times D_2$. Hence for all $(d_1, d_2) \in W$ we have:

(1) $d_2 d_1 (\tilde{f} d_2 d_1)^n(p') = d_2(d_1 \tilde{f} d_2)^n d_1(p')$ is not a base-point of $\tilde{f}$ for all $0 \leq n \leq K + 1$ which is equivalent to $(d_1 \tilde{f} d_2)^n d_1(p')$ not being a base-point of $d_1 \tilde{f} d_2$ for all $0 \leq n \leq K + 1$.

By Lemma 7.2.13, there exists an open dense $U \subset D_2 \times D_2$ such that for all $(d_1, d_2) \in U$:

(2) $d_1 \tilde{f} d_2$ is loxodromic,

(3) $(d_1 \tilde{f} d_2)^{-k}$ does not contract $d_2^{-1}(C)$ for all $0 \leq k \leq K + 1$,

(4) $(d_1 \tilde{f} d_2)^{-k}(C)$ is a curve that is not contained in the toric boundary of S for all $0 \leq k \leq K + 1$.

The intersection $W \cap U \subset D_2 \times D_2$ is open and dense, hence the intersection $\Delta_2 \cap U \cap W$ is not empty. Let $(d_1, d_2) \in \Delta_2 \cap U \cap W$. The birational transformation $d_1 \tilde{f} d_2 \in \langle \Delta_2, f \rangle$ is loxodromic by condition (2), it contracts the curve $d_2^{-1}(\tilde{C})$ to the point $d_1(p') \in L$ and condition (1) ensures that $(d_1 f d_2)^k$ contracts $d_2^{-1}(\tilde{C})$ for all $1 \leq k \leq K + 1$. It follows that $(d_1 \tilde{f} d_2)^{K+1}$ contracts the $K + 1$ different curves $d_2^{-1}(\tilde{C}), (d_1 \tilde{f} d_2)^{-1}(d_2^{-1}(\tilde{C})), \ldots, (d_1 \tilde{f} d_2)^{-K}(d_2^{-1}(\tilde{C}))$, which are not contained in the toric boundary. By Lemma 7.2.10 and the way we have chosen K, it follows that $d_1 \tilde{f} d_2$ is loxodromic but not of monomial type.

Case 2: The set $\{d(g^2 d)^k(p') \mid d \in D_1\}$ is open and dense in $\mathbb{P}^1$ for all $k \in \mathbb{Z}_{\geq 0}$. In this case we proceed analogously as in case 1 by replacing $\tilde{f}$ by $\tilde{f}^2$ and $\tilde{C}$ by $\tilde{f}^{-1}(\tilde{C})$ and show that there exist $d_1, d_2 \in \Delta_2$ such that $(d_1 \tilde{f}^2 d_2)^{K+1}$ contracts $K + 1$ different curves and is therefore not of monomial type. $\square$

Lemma 7.2.18. *Assume that Conjecture 7.1.1 holds.*

Let $f \in \mathrm{Cr}_2(\mathbb{C})$ be a loxodromic element that is not contained in $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$. There exists an $n \in \mathbb{Z}_+$ and an open dense set $V \subset D_2$ such that $f^n d^{-1} f^{-n} d$ is loxodromic.

Proof. Let $\alpha^+ \in \partial \mathbb{H}^\infty$ be the attracting fixed point of the isometry induced by f on $\mathbb{H}^\infty$, and let $\alpha^- \in \partial \mathbb{H}^\infty$ be its repulsive fixed point. The axis $\mathrm{Ax}(f)$ is the geodesic line between α^+ and α^-. We claim that there exists an open dense subset $U \subset D_2$ of elements that fix neither α^+ nor α^-. Denote by $G \subset \mathrm{Cr}_2(\mathbb{C})$ the subgroup of all elements that fix α^+. As in the proof of Theorem 6.6.3, denote by $\rho : G \to \mathbb{R}_{>0}$ the group homomorphism given by the action of G on the one-dimensional subspace $L \subset \mathrm{Z}(\mathbb{P}^2)$ that corresponds to the point α^+. The kernel of ρ is a bounded subgroup, which is normalised by f. If $\ker(\rho)$ is infinite, there exists, by Theorem 6.6.3, an element $h \in \mathrm{Cr}_2(\mathbb{C})$, such that $hGh^{-1} \subset \mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$. As f is not in $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$, the transformation h is not in $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$ and therefore,

by Lemma 5.5.2, $h^{-1}D_2h \cap D_2$ is a proper closed subset. In particular, there exists an open dense set $U_1 \subset D_2$ that is not contained in G. If G is finite, the existence of such an open dense $U_1 \subset D_2$ follows trivially. With the same argument, we obtain an open dense set $U_2 \subset D_2$ that does not fix α^-. Define $U := U_1 \cap U_2$. This proves the claim.

Let $d \in U$ be arbitrary. Denote by $\beta^+ \in \partial \mathbb{H}^\infty$ the attracting fixed point of the loxodromic isometry $d^{-1}f^{-1}d$ and by $\beta^- \in \partial \mathbb{H}^\infty$ its repulsive fixed point. By the above observation, $\alpha^+, \alpha^-, \beta^+$ and β^- are pairwise disjoint. Let S_1^+ be a small neighborhood of α^+ in $\partial \mathbb{H}^\infty$ and S_1^- a small neighborhood of α^-. Similarly, let S_2^+ be a small neighborhood of β^+ and S_2^- a small neighborhood of β^-. We may assume that S_1^+, S_1^-, S_2^+ and S_2^- are pairwise disjoint. Since β^+ is attractive, there exists an $n_1 \in \mathbb{Z}_+$ such that $d^{-1}f^{-n_1}d(S_1^+) \subset S_2^+$. Similarly, let $n_2 \in \mathbb{Z}_+$ be such that $f^{n_2}(S_2^+) \subset S_1^+$ is a proper subset. For $n := \max\{n_1, n_2\}$, we obtain that $f^n d^{-1} f^{-n} d(S_1^+)$ is a proper subset of S_1^+. Analogously, if we choose n large enough, $(f^n d^{-1} f^{-n} d)^{-1}(S_2^-)$ is a proper subset of S_2^-. Thus, $f^n d^{-1} f^{-n} d$ has an attractive fixed point in S_1^+ and a repulsive fixed point in S_2^-. In particular, $f^n d^{-1} f^{-n} d$ is loxodromic.

Consider the family of birational transformations $\{f^n d^{-1} f^{-n} d \mid d \in D_2\}$. It contains one element of dynamical degree $\lambda > 1$. By Theorem 7.2.3, the dynamical degree is a lower semi-continuous function. Hence, there exists an open dense subset $V \subset D_2$ such that the dynamical degree of $f^n d^{-1} f^{-n} d$ is > 1 for all $d \in V$, which is equivalent to $f^n d^{-1} f^{-n} d$ being loxodromic. $\qquad \square$

Lemma 7.2.19. *Assume that Conjecture 7.1.1 holds.*

Let $\pi \colon S \to \mathbb{P}^2$ be a blow up of finitely many toric points and $f \in \mathrm{Bir}(S)$ a loxodromic element. Let $\Delta_2 \subset D_2$ be a dense subgroup. If f contracts a curve $C \subset S$ that is not contained in the toric boundary to a point in the bubble space $\mathcal{B}(S)$ that is not toric, then the group $\langle f, \Delta_2 \rangle$ contains a loxodromic element that is not of monomial type.

Proof. By Lemma 7.2.13, there exists for each $n \in \mathbb{Z}_+$ an open dense subset $V_n \subset D_2 \times D_2$ such that for all $k \leq n$ and all $(d_1, d_2) \in V_n$, the transformation $(d_1 f d_2)^k$ is loxodromic and contracts a curve that is not contained in the toric boundary of S. Since the field of complex numbers is uncountable, $V := \bigcap_{n \in \mathbb{Z}_+} U_n$ is a very general non-empty set, i.e. the complement of countably many proper closed subsets. Let $(d_1, d_2) \in V$. Since f is not loxodromic, there exists, by Lemma 7.2.18, an $N \in \mathbb{Z}_+$ and an open dense set $U \subset D_2$ such that $d(d_1 f d_2)^n d^{-1} (d_1 f d_2)^{-n}$ is loxodromic for all $d \in U$. By Theorem 7.2.3, there exists therefore an open dense set $W \subset (D_2)^3$ such that $d(d_1 f d_2)^n d^{-1} (d_1 f d_2)^{-n}$ is loxodromic for all $(d, d_1, d_2) \in W$.

Since $\Delta_2 \subset D_2$ is dense in D_2, there exist $(d_1, d_2) \in \Delta_2$ and $U' \subset D_2$ open dense such that $(U', d_1, d_2) \subset W$. By the choice of (d_1, d_2), we obtain that $(d_1 f d_2)^{-n+1}(C)$ is still a curve not contained in the toric boundary. For an open dense set $U'' \subset D_2$, the curve $d(d_1 f d_2)^{-n+1}(C)$ is not contained in the exceptional divisor of $(d_1 f d_2)^n$ and the curve $\tilde{C} := (d_1 f d_2)^n d(d_1 f d_2)^{-n+1}(C)$ is not contained in ∂S. Define the open dense subset $U := U' \cap U''$. Define $g := (d_1 f d_2)^n$. Hence, for all $d \in U$, the loxodromic map $dgd^{-1}g^{-1}$ contracts the curve $\tilde{C}$ to the non-toric point $d_\bullet(p) \in \mathcal{B}(S)$.

Our goal is now to find a $d \in U \cap \Delta_2$ such that we can construct on the basis of $dgd^{-1}g^{-1}$ an element in $\langle f, \Delta_2 \rangle$ that is not of monomial type. Let K be the constant from Lemma 7.2.10, such that for all loxodromic elements $h \in \mathrm{Cr}_2(\mathbb{C})$ of monomial type of degree $\leq \deg(g)^2$ the number of irreducible curves contracted by h^k is bounded by K for all $k \in \mathbb{Z}$. By Lemma 7.2.13, there exists for each $d \in U$ an open dense set $V_d \subset D_2 \times D_2$ such that $(d_1 dgd^{-1}g^{-1}d_2)^{-l}(d_2^{-1}(\tilde{C}))$ is a curve not contained in ∂S for all $l \leq K+1$ and such that $(d_1 dgd^{-1}g^{-1}d_2)^{-l}$ is loxodromic. This gives an open dense subset $\tilde{W} \subset (D_2)^3$ such that all $(d, d_1, d_2) \in \tilde{W}$ satisfy the prescribed properties.

Claim. one of the following possibilities occur:

(1) There exist $d, d_1, d_2 \in \Delta_2$ such that $(d_1 dgd^{-1}g^{-1}d_2)^l(d_2^{-1}(\tilde{C}))$ is a point for all $l \leq K+1$, i.e. $(d_1 dgd^{-1}g^{-1}d_2)^l$ contracts the curve C for all $l \leq K+1$.

(2) There exist $d, d_1, d_2 \in \Delta_2$ and a $k \leq K+1$ such that

$$(d_1 dgd^{-1}g^{-1}d_2)^k(d_2^{-1}(\tilde{C})) \in S$$

is a point not contained in the toric boundary.

(3) There exist a blow-up of finitely many toric points $\pi \colon S' \to S$ and elements $d, d_1, d_2 \in \Delta_2$, as well as a $k \leq K+1$, such that the pull-back of $(d_1 dgd^{-1}g^{-1}d_2)^l$ on S' contracts the strict transform of $d_2^{-1}(\tilde{C})$ to a non-toric point $p \in S'$ that is contained in a line $L \subset \partial S'$ and the pull-back of $(d_1 dgd^{-1}g^{-1}d_2)^k$ restricts to a birational transformation of L to itself.

If our claim holds, we are done. Indeed, if we are in situation (1), the map $(d_1 dgd^{-1}g^{-1}d_2)^{K+1}$ contracts $K+1$ distinct curves, hence $(d_1 dgd^{-1}g^{-1}d_2)$ is not of monomial type, by Lemma 7.2.10, as $\deg(d_1 dgd^{-1}g^{-1}d_2) \leq \deg(g)^2$. Whereas in the second and third situation, we are in the situation of Lemma 7.2.15 or Lemma 7.2.17 respectively.

The proof of the claim requires multiple steps and various case-distinctions. We will prove it by induction. More precisely, we show by induction over N that either we are in situation (2) or (3), or there exists a blow-up of finitely many toric points $\pi_N \colon S_N \to S$ and an open dense set $W_N \subset (D_2)^3$ such that $(d_1 dgd^{-1}g^{-1}d_2)^l(d_2^{-1}(\tilde{C}))$ is a non-toric point in a line $L \subset \partial S_N$ for all $l \leq N$ and for all $(d, d_1, d_2) \in W_N$. The claim then follows with $N = K+1$ and some $(d, d_1, d_2) \in (\Delta_2)^3 \cap W_{K+1}$.

If $\pi \colon S' \to S$ is a given blow-up of toric points, $\mathrm{Bir}(S)$ is isomorphic to $\mathrm{Bir}(S')$ through the isomorphism $f \mapsto \pi^{-1} f \pi$. In order to simplify notation, in what follows we will always identify, by abuse of notation, $\mathrm{Bir}(S)$ directly with $\mathrm{Bir}(S')$ without mentioning the isomorphism. We will also identify the curve $\tilde{C}$ with its strict transform under π_N.

First, we consider $N = 1$. We set $W_0 = \tilde{W}$. If $(d_1 dgd^{-1}g^{-1}d_2)(d_2^{-1}(\tilde{C}))$ is not contained in ∂S for one triple $(d, d_1, d_2) \in W_0$, it is not contained in ∂S for all triples (d, d_1, d_2) in an open dense subset $W_0' \subset W_0$ and we are in situation (2). Otherwise, since $(d_1 dgd^{-1}g^{-1}d_2)(d_2^{-1}(\tilde{C}))$ is not toric for all $(d, d_1, d_2) \in W_0$, we blow-up finitely many toric points $\pi_1 \colon S_1 \to S$ until the point $d_1 dgd^{-1}g^{-1}d_2(d_2^{-1}(\tilde{C}))$ is a

proper non-toric point in a line $L \subset \partial S_1$ for one triple $(d, d_1, d_2) \in W_0$ and hence for all triples (d, d_1, d_2) in an open dense subset $W_1 \subset W_0$.

Assume now the induction hypothesis to hold for N. We assume that there exists a blow-up of finitely many toric points $\pi_N \colon S_N \to S$ and an open dense set $W_N \subset (D_2)^3$ such that $(d_1 d g d^{-1} g^{-1} d_2)^l (d_2^{-1}(\tilde{C}))$ is a non-toric point in a line $L \subset \partial S_N$ for all $l \leq N$ and all $d, d_1, d_2 \in W_N$. Our goal is to show that either we are in situation (2) or (3), or there exists a blow-up of finitely many toric points $\pi \colon S_{N+1} \to S_N$ and an open dense set $W_{N+1} \subset W_N$ such that $(d_1 d g d^{-1} g^{-1} d_2)(d_1 d g d^{-1} g^{-1} d_2)^N (d_2^{-1}(\tilde{C}))$ is a a proper non-toric point in a line $L \subset \partial S_{N+1}$ for all $(d, d_1, d_2) \in W_{N+1}$. Let $p := (d_1 d g d^{-1} g^{-1} d_2)^N (d_2^{-1}(\tilde{C}))$. We distinguish various cases:

Case 1: Assume that there exists a triple $(d, d_1, d_2) \in W_N$ such that $d_2(p)$ is not a base-point of g^{-1}, $d^{-1} g^{-1} d_2(p)$ is not a base-point of g and $d_1 d g d^{-1} g^{-1} d_2(p)$ is not contained in ∂S_N. It follows that this property holds for an open dense subset $W_N' \subset W_N$ and we are in situation (2).

Case 2: Assume that p lies on a line $L \subset \partial S_N$ and that the restriction of g^{-1} to L induces a birational map from L to the strict transform $g^{-1}(L)$. If $g^{-1}(L)$ is not contained in ∂S_N, we can choose $d, d_1, d_2 \in W_1 \cap (\Delta_2)^3$ in such a way that $(d_1 d g d^{-1} g^{-1} d_2)(p)$ is a point in S_N that is not contained in ∂S_N and we are in situation (2). If $g^{-1}(L)$ is contained in ∂S_1, then $g^{-1}(L) = L'$, where $L' \subset \partial S_N$ is a line; moreover, $g(L') = L$. This implies that $d_1 d^{-1} g^{-1} d_2$ and therefore $(d_1 d^{-1} g^{-1} d_2)^N$ induces a birational transformation from L to itself for all $(d, d_1, d_2) \in W_N \cap (\Delta_2)^3$ and we are in situation (3).

Case 3. Assume that p lies on a line $L \subset \partial S_N$ and that L is contracted by g^{-1} to a non-toric point $q \in S_N$. If q is not contained in ∂S_N, we choose $(d, d_1, d_2) \in W_N \cap (\Delta_2)^3$ in such a way that $d^{-1} g^{-1} d_2(p)$ is not an indeterminacy point of g, and in such a way that $(d_1 d g d^{-1} g^{-1} d_2)(q) \in S_N$ is not contained in ∂S_N. Thus, we are in situation (2). If q is contained in a line $L' \subset \partial S_N$, we define $V_2 \subset (D_2)^3$ to be the open dense subset of elements (d, d_1, d_2) such that $d^{-1} g^{-1} d_2(p)$ is not a base-point of g. Since we can choose d_1 and d_2 arbitrarily, V_2 is of the form $O \times D_2 \times D_2$, where $O \subset D_2$ is open and dense. Hence, $W_N' := W_N \cap V_2$ is open and dense in $(D_2)^3$. We consider subcases:

(a) The restriction of g to L' induces a birational map to the strict transform $g(L')$. If $g(L')$ is not contained in ∂S_N, for all (d, d_1, d_2) in an open dense subset of W_N', the point $d_1 d g d^{-1} g^{-1} d_2(p)$ is not contained in ∂S_N and we are in situation (2). If $g(L')$ is a line L'' that is contained in ∂S_N, then $g^{-1}(L'') = L'$. Choose $d, d_1, d_2 \in W_N' \cap (\Delta_2)^3$ such that:

  - $d^{-1}(q) \in L'$ is not an indeterminacy point of g,
  - $d^{-1}(q) \in L'$ is not mapped to a toric point by g,
  - $d_2 d_1 d g d^{-1}(q) \in L''$ is not an indeterminacy point of g^{-1},
  - $d_2 d_1 d g d^{-1}(q) \in L''$ is not mapped by g^{-1} to a toric point,
  - $d^{-1} g^{-1} d_2 d_1 d g d^{-1}(q) \in L''$ is not an indeterminacy point of g,
  - $d^{-1} g^{-1} d_2 d_1 d g d^{-1}(q) \in L''$ is not mapped by g to a toric point.

It follows that $(d_1 d g d^{-1} g^{-1} d_2)^{N+2} (d_2^{-1}(\tilde{C}))$ is a non-toric point contained

in the line $L'' \subset \partial S_N$ and the restriction of $(d_1 dg d^{-1}g^{-1}d_2)^{N+2}$ to L'' is a birational transformation to itself, hence we are in situation (3).

(b) The map g contracts L' to a non-toric point. If $g(L')$ is not contained in ∂S_N, then, for all $(d, d_1, d_2) \in W'_N$, the point $d_1 dg d^{-1}g^{-1}d_2(p)$ is not contained in ∂S_N and we are in situation (2). If $g(L')$ is contained in a line $L'' \subset \partial S_N$, we set $S_{N+1} = S_N$ and $W_{N+1} := W'_N$.

(c) If g maps L' to a toric point, we blow up toric points $\pi \colon S'_N \to S_N$ until $g(L')$ is either a line conained in $\partial S'_N$, or $g(L')$ is a non-toric point in $\partial S'_N$. In the first case, we proceed as in (a), in the second case we proceed as in (b), by considering S'_N instead of S_N.

Case 4. Finally, assume that p lies on a line $L \subset \partial S_N$ and that L is contracted by g^{-1} to a toric point $q \in S_N$. In this case, we blow up toric points $\pi \colon S'_N \to S_N$ until $g^{-1}(L)$ is either a line $L' \subset \partial S'_N$, or $g^{-1}(L)$ is a non-toric point in $\partial S'_N$. By considering S'_N instead of S_N, we proceed as in case 2 or as in case 3, respectively. $\qquad\square$

A birational transformation f of a projective surface S is called *algebraically stable* if there exists no point p in the bubble space $\mathcal{B}(S)$ such that p is a basepoint of f^i and of f^{-j} for integers $i, j > 0$. In [DF01] it is shown that for a given $f \in \mathrm{Bir}(S)$ there exists a blow-up $\pi \colon S' \to S$ such that $\pi^{-1}f\pi \in \mathrm{Bir}(S')$ is algebraically stable. In the following lemma we prove a similar statement, but here we only consider toric points.

Lemma 7.2.20. *Let $f \in \mathrm{Cr}_2(\mathbb{C})$. There exists a blow-up of finitely many toric points $\pi \colon S \to \mathbb{P}^2$ such that $\hat{f} := \pi^{-1}f\pi \in \mathrm{Bir}(S)$ satisfies the following property: every toric point $p \in \mathcal{B}(S)$ is either not a base-point of $\hat{f}^i$ for all $i > 0$ or p is not a base-point of $\hat{f}^i$ for all $i < 0$.*

Proof. Let K be the set of all base-points $p \in \mathcal{B}(\mathbb{P}^2)$ such that p is a base-point of f^i and f^{-j} for some $i, j > 0$. We claim that K is finite. Let $p \in K$ be arbitrary and $i, j > 0$ such that p is a base-point of f^i and of f^{-j} and $|i + j|$ is minimal among all such pairs i, j. We denote by $f_\bullet$ the self-map of the bubble space induced by f (see Section 5.2.1). The point $q = f_\bullet^{i-1}(p)$ is a base-point of f. In that way, we associate to every point in K a base-point of f. There are exactly $|i - j - 1|$ points in K that are associated to the base-point q, namely $f_\bullet^k(p)$ for $-j < k < i$. Since f has only finitely many base-points, this implies that K is finite. Let $T \subset K$ be the subset of all points in K that are toric. Note that if $p \in T$ is not a proper point and p lies above a point $q \in \mathcal{B}(\mathbb{P}^2)$, then q is in T as well. Hence, T is a *cluster*, i.e. a finite subset of points in $\mathcal{B}(S)$ that can be blown up. Let $\pi \colon S \to \mathbb{P}^2$ be the blow-up of all the points in T. The birational transformation $\hat{f}$ may have more base points than f, but if $p \in \mathcal{B}(S)$ is a base-point of $\hat{f}^i$ and of $\hat{f}^{-j}$ for some $i, j > 0$, then p is not toric. $\qquad\square$

Lemma 7.2.21. *Assume that Conjecture 7.1.1 holds.*

Let $f \in \mathrm{Cr}_2(\mathbb{C})$ be a loxodromic element and $\Delta_2 \subset D_2$ a dense subgroup. If all loxodromic elements in $\langle f, \Delta_2 \rangle$ are of monomial type, then there exists a blow-up of

finitely many toric points $\pi\colon S \to \mathbb{P}^2$ such that $\hat{f}^m := \pi^{-1} f^m \pi$ only contract curves contained in the toric boundary for all $m \in \mathbb{Z}$.

Proof. Let $\pi\colon S \to \mathbb{P}^2$ be the blow-up of finitely many toric points that is given by Lemma 7.2.20. Assume that $\hat{f}$ contracts a curve $C \subset S$ that is not contained in ∂S to a point p in the bubble space $\mathcal{B}(S)$. If p is non-toric, we obtain, by Lemma 7.2.19, that $\langle \hat{f}, \Delta_2 \rangle$ contains an element that is not of monomial type, which contradicts our assumption. So p has to be toric. Since p is a base-point of $\hat{f}^{-1}$, it follows that p is not a base-point of $\hat{f}^n$ for any $n \in \mathbb{Z}_+$, by the construction of $\pi\colon S \to \mathbb{P}^2$. In other words, $\hat{f}^n$ contracts C to a point, for all $n \in \mathbb{Z}_+$. Again by Lemma 7.2.19, $\hat{f}^n$ does not contract C to a non-toric point for any $n \in \mathbb{Z}_+$. Hence, $\hat{f}^n$ contracts C to a toric point for all $n \in \mathbb{Z}_+$. But in this case, Lemma 7.2.14 implies that $\langle \hat{f}, \Delta_2 \rangle$ contains an element that is not of monomial type, which contradicts our assumption. Therefore, all curves that are contracted by f are contained in ∂S.

If we replace $\hat{f}$ by $\hat{f}^m$ for $m \in \mathbb{Z}$ the proof works analogously. $\square$

7.2.5   Proof of Theorem 7.2.1

Definition 7.2.2. Let C be a curve on a smooth projective surface with irreducible components $C_1, \ldots, C_n$ and simple normal crossings.

C is called a *zigzag* if the C_i are smooth, rational and if for all i we have $C_i \cdot C_{i+1} = 1$, whereas $C_i \cdot C_j = 0$ for $j \neq i, i+1$.

C is called a *loop* if it has simple normal crossings and one of the following conditions are satisfied:

- If $n = 1$ then C has exactly one singular point.

- If $n = 2$ then the C_i are smooth and $C_1 \cdot C_2 = 2$.

- If $n \geq 3$ then the C_i are smooth, $C_{i-1} \cdot C_i = C_i \cdot C_{i+1} = 1$ for $1 < i < n$, $C_1 \cdot C_n = 1$ and $C_i \cdot C_j = 0$ in all the other cases where $i \neq j$.

In [Giz71a] and [Giz71b], Gizatullin considered boundaries of embeddings of completions of algebraic surfaces. We need the following result:

Theorem 7.2.22 ([Giz71a, Proposition 5])**.** *Let S be a smooth projective surface, X any algebraic surface and $\iota\colon X \to S$ an open embedding. Let $\overline{\partial}S := S \setminus \iota(X)$. Assume that $\overline{\partial}S$ is the finite union of smooth irreducible rational curves E_i of self-intersection $\neq -1$, and that $\overline{\partial}S$ is a simple normal crossing divisor. Then one of the following is true:*

(1) *$\overline{\partial}S$ is empty.*

(2) *$\overline{\partial}S$ is the disjoint union of finitely many smooth rational curves of self-intersection 0.*

(3) *$\overline{\partial}S$ is a zigzag.*

(4) *$\overline{\partial}S$ is a loop.*

If we assume in addition that the boundary curve consists of curves that are contracted by birational transformations, we obtain further restrictions. In [GD75], [GD77], Gizatullin and Danilov investigated this situation in detail and developed various techniques to prove a series of important results on automorphisms of affine surfaces.

The following lemma is well known. Here, as well as in our proof of Proposition 7.2.28 we follow [CC17, Section 6].

Lemma 7.2.23. *Let S be a smooth rational projective surface, X any algebraic surface and $\iota\colon X \to S$ an open embedding. Let $\overline{\partial}S := S \setminus \iota(X)$. Assume that $\overline{\partial}S$ is the finite union of smooth disjoint irreducible rational curves $E_1, \ldots, E_k$, $k \geq 1$, of self-intersection 0, and that $\overline{\partial}S$ is a simple normal crossing divisor. Furthermore, assume that there exists a birational transformation $f \in \mathrm{Bir}(S)$ such that f induces an automorphism on X and such that each E_i is contracted by f^n for some $n \in \mathbb{Z}$. Then f preserves a rational fibration.*

Proof. Riemann-Roch for surfaces (see for example [Har77, Theorem V.1.6]) states that
$$h^0(S, E_1) - h^1(S, E_1) + h^2(S, E_1) = \frac{1}{2}(E_1^2 - K_S \cdot E_1) + \chi(\mathcal{O}_S).$$

Since S is rational, one has $\chi(\mathcal{O}_S) = 1$. The genus formula states
$$g(E_1) = 1 + \frac{1}{2}(E_1^2 + E_1 \cdot K_S).$$

As $g(E_1) = 0$ and $E_1^2 = 0$, we obtain $E_1 \cdot K_S = -2$, in particular $(K_S - E_1)E_1 = -2$. Since E_1 is nef, its intersection with any effective divisor is non-negative. Therefore, $K_S - E_1$ is not effective and $h^0(S, K_S - E_1) = 0$, which implies, by Serre duality, that $h^2(S, E_1) = 0$. By combining all these results we get:
$$h^0(S, E_1) = h^1(S, E_1) + \frac{1}{2}(0 + 2) + 1 \geq 2.$$

Let s_1, s_2 be to linearly independent elements in $H^0(S, E_1)$. In that way we obtain a birational map $\pi\colon S \dashrightarrow \mathbb{P}^1$ given by $x \mapsto [s_1(x) : s_2(x)]$. The birational transformation π is base-point free, since E_1 has self-intersection 0 and so the curves cut out by s_1 or s_2 do not intersect. The curve E_1 is contracted by π. The curves E_i for $i \neq 1$ do not intersect E_1, so $\pi(E_i) \subset \mathbb{P}^1 \setminus \{\pi(E_1)\}$. They are therefore contracted as well. Let now F be a fiber under π of a point in $\mathbb{P}^1 \setminus \{\pi(E_1), \ldots, \pi(E_n)\}$. Since F is linearly equivalent to E_1, it is rational. We have that $F \subset X$. Since $f(F)$ is again a complete rational curve, its image $\pi(f(F))$ is a point and $f(F) \subset \mathbb{P} \setminus \{\pi(E_1)\}$ is again a fiber. Hence f preserves a rational fibration. $\qquad\square$

The following result and proof can be found in [GD75, Proposition 6.4].

Lemma 7.2.24. *Let S be a smooth rational projective surface, X any algebraic surface and $\iota\colon X \to S$ an open embedding. Let $\overline{\partial}S := S \setminus \iota(X)$. Assume that $\overline{\partial}S$ is the finite union of smooth irreducible rational curves $E_1, \ldots, E_k$, $k \geq 1$ forming a loop or a zig-zag, and that $\overline{\partial}S$ is a simple normal crossing divisor. Furthermore, assume that there exists a birational transformation $f \in \mathrm{Bir}(X)$ such that f induces an automorphism on X and such that each E_i is contracted by f^n for some $n \in \mathbb{Z}$. Then X is the blow-up of an affine surface.*

Proof. If $\overline{\partial}S$ contains (-1)-curves, we contract them; this does not change X and $\overline{\partial}S$ remains a zig-zag or a loop respectively. Without loss of generality, we may thus assume that $\overline{\partial}S$ does not contain any (-1)-curves.

We claim that there exists an effective divisor D on S with support $\overline{\partial}S$ satisfying $D \cdot E_i > 0$ for all $1 \leq i \leq k$. For all $1 \leq i \leq k$, there exists an $n \in \mathbb{Z}$ such that f^n contracts E_i and the curves in $\overline{\partial}S$ are the only curves contracted by iterates of f. Hence, there exists at least one component E_r of $\overline{\partial}S$ with self-intersection ≥ 0.

If $\overline{\partial}S$ is a zig-zag, we may assume that the curves E_i are ordered in such a way that $E_i \cdot E_{i+1} = 1$ for $1 \leq i \leq k-1$, $E_i \cdot E_j = 0$ for $j \notin \{i-1, i, i+1\}$ and $E_i \cdot E_i = d_i$, where $d_r \geq 0$. Choose now integers $a_1, \ldots, a_k > 0$ such that $a_{i+1} > -d_i a_i^2$ for all $1 \leq i < r$ and $a_{i-1} > -d_i a_i^2$ for all $r < i \leq k$, for example by setting $a_1 = 1$ and $a_k = 1$ and the other a_i accordingly. If we set

$$D := \sum_{i=1}^{k} a_i E_i,$$

then $D \cdot E_i = a_{i-1} + a_i^2 d_i + a_{i+1}$ for $1 < i < k$ and $D \cdot E_1 = a_1 d_1^2 + a_2$, $D \cdot E_k = a_{k-1} + a_k^2 d_k$. The conditions on the a_i ensure that $D \cdot E_i > 0$ for $1 \leq i \leq k$.

If $\overline{\partial}S$ is a loop, we may assume that the E_i are ordered in such a way that $E_i \cdot E_{i+1} = 1$ for all $1 \leq i \leq k$, where we identify E_{k+1} with E_1, and $d_i = E_i \cdot E_i$. Again, there exists an i such that $d_i \geq 0$. Without loss of generality, we assume $i = k$. Let $a_1 = 1$ and then choose integers $a_2, \ldots, a_k$ such that $a_{i+1} > -d_i a_i^2$. Again, let $D := \sum_{i=1}^{k} a_i E_i$. Then, $D \cdot E_i = a_{i-1} + d_i a_i^2 + a_{i+1} \geq a_{i-1} + d_i a_i^2 > 0$ for $1 \leq i < k$ and $D \cdot E_k = a_{k-1} + d_k a_k^2 + 1 > 0$ since $d_k \geq 0$. This proves the claim.

Since $D \cdot E_i > 0$ for all $1 \leq i \leq k$, the restriction of D to $\overline{\partial}S$ has positive degree on each component, hence $D|_{\overline{\partial}S}$ is an ample divisor on $\overline{\partial}S$. Let $m \in \mathbb{Z}_+$ be large enough such that $mD|_{\overline{\partial}S}$ is very ample and such that $H^1(\overline{\partial}S, mD|_{\overline{\partial}S}) = 0$ (such an m exists by the Serre vanishing theorem (see [Har77, Theorem III.5.2])). The short exact sequence

$$0 \to \mathcal{O}_S((m-1)D) \to \mathcal{O}_S(mD) \to \mathcal{O}_S(mD)|_D \to 0.$$

yields a long exact sequence

$$0 \to H^0(S, (m-1)D) \to H^0(S, mD) \to H^0(D, mD|_D)$$

$$\to H^1(S, (m-1)D) \xrightarrow{\phi_m} H^1(S, mD) \to H^1(D, mD) = 0.$$

The homomorphism ϕ_m is surjective and $h^1(S, mD)$ decreases as m increases. So for m large enough, ϕ_m is an isomorphism and we obtain an exact sequence

$$0 \to H^0(S, (m-1)D) \to H^0(S, mD) \xrightarrow{\alpha} H^0(D, mD|_D) \to 0.$$

Let $s'_1, \ldots, s'_N$ be a basis of $H^0(D, mD|_D)$, let $s_1, \ldots, s_N \in H^0(S, mD)$ be preimages of $s'_1, \ldots, s'_N$ under α and let $s_0 \in H^0(S, mD)$ be a section that cuts out D. Since $mD|_D$ is very ample, $s_1, \ldots, s_N$ do not vanish all together at any point in $\overline{\partial}S$, so $s_0, s_1, \ldots, s_N$ never vanish all together at any point in S. So the rational map $\varphi_{mD} \colon S \dashrightarrow \mathbb{P}^n$ defined by $H^0(S, mD)$ is base-point free, hence a morphism.

The divisor D is nef since $D \cdot \Gamma > 0$ for every curve $\Gamma \subset \overline{\partial}S$, by construction, and $D \cdot \Gamma \geq 0$ for all other curves Γ in S. Moreover, D is big, since $D \cdot D > 0$. This implies that φ_{mD} is a birational morphism. The restriction of φ_{mD} to $\overline{\partial}S$ is an embedding and $\varphi_{mD}(\overline{\partial}S)$ is a hyperplane section. Therefore, the restriction of φ_{mD} to X is a birational morphism from X to an affine surface. Hence, X is the blow-up of an affine surface. $\qquad \square$

Combining Proposition 7.2.22 with Lemma 7.2.23 and Lemma 7.2.24 yields the following:

Proposition 7.2.25. *Let S be a smooth rational projective surface, X any algebraic surface and $\iota \colon X \to S$ an open embedding. Let $\overline{\partial}S := S \setminus \iota(X)$. Assume that $\overline{\partial}S$ is the finite union of smooth irreducible rational curves $E_1, \ldots, E_k$ of self-intersection $\neq -1$, and that $\overline{\partial}S$ is a simple normal crossing divisor. Furthermore, assume that there exists a birational transformation $f \in \mathrm{Bir}(S)$ such that f induces an automorphism on X and such that each E_i is contracted by f^n for some $n \in \mathbb{Z}$. Then one of the following is true:*

(1) *$\overline{\partial}S$ is empty.*

(2) *$\overline{\partial}S$ is the disjoint union of finitely many smooth rational curves of self-intersection 0.*

(3) *$\overline{\partial}S$ is a zigzag.*

(4) *$\overline{\partial}S$ is a loop.*

Moreover, in case (2), f preserves a rational fibration and in case (3) and (4) the surface X is a blow-up of an affine surface. $\qquad \square$

An affine toric surface is a normal affine surface that admits a regular faithful D_2-action. Affine toric surfaces and their automorphism groups are well understood (see for example [KPZ15],[Ful93]). We will briefly recall some well-known facts. Denote by $\mathbb{A}^1_* := \mathbb{A}^1 \setminus \{0\}$ the affine line deprived of one of its points.

Lemma 7.2.26. *Let X be a smooth affine toric surface that admits a loxodromic automorphism whose dynamical degree is not an integer. Then X is isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1_*$.*

Proof. By the combinatorial classification of toric varieties, smooth affine surfaces correspond bijectively to the cones in a two dimensional real vector space that are generated by a part of a given underlying basis ([Ful93]). There are exactly three such cones: the zero cone, which corresponds to toric surface $\mathbb{A}^1_* \times \mathbb{A}^1_*$, the half line given by a single basis vector, which corresponds to $\mathbb{A}^1_* \times \mathbb{A}^1$ and the cone generated by two basis vectors, which corresponds to $\mathbb{A}^2$. Hence every smooth affine toric surface is isomorphic to $\mathbb{A}^2$, $\mathbb{A}^1_* \times \mathbb{A}^1$, or to $\mathbb{A}^1_* \times \mathbb{A}^1_*$. The dynamical degree of an element in $\mathrm{Aut}(\mathbb{A}^2)$ is always an integer. The $\mathbb{A}^1$-fibration is given by the invertible functions on $\mathbb{A}^1_* \times \mathbb{A}^1$, so automorphisms of $\mathbb{A}^1_* \times \mathbb{A}^1$ preserve this $\mathbb{A}^1$-fibration. This implies that their dynamical degree is 1. Since dynamical degrees are invariant under conjugacy, X can only be isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1_*$ whose automorphism group is $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$. $\qquad \square$

Lemma 7.2.27. *Let* $f \in \mathrm{Cr}_2(\mathbb{C})$ *be loxodromic and of monomial type. Then* f *is not conjugate to an automorphism of a projective surface.*

Proof. A point $p \in \mathbb{P}^2$ is called a *persistent* base-point of f if there exists an integer N such that p is a base-point of f^n for all $n \geq N$, but p is not a base-point of f^{-n} for any $n \geq N$. Blanc and Déserti showed that the number of persistent base-points is invariant under conjugation and that a birational transformation is conjugate to an automorphism of a smooth projective surface if and only if it has no persistent base-points ([BD15, Proposition 3.3]).

Since f is loxodromic and of monomial type, we can conjugate, by Lemma 7.2.5, either f, f^2 or f^4 to an element in $\mathrm{GL}_2(\mathbb{Z})$ of the form $m = (x^a y^b, x^c y^d)$, where $a, b, c, d \geq 0$. We observe that the point $(0, 0)$ is not a base-point of m^n for all positive n. On the other hand, for all positive n, m^n contracts a coordinate line to $(0, 0)$, so $(0, 0)$ is a base-point of m^{-n} for all positive n. Hence, $(0, 0)$ is a persistent base-point of m^{-1}. By the above mentioned result of Blanc and Déserti, the birational transformation m, and therefore also f, can not be conjugated to an automorphism of a projective surface. $\qquad\square$

Proposition 7.2.28. *Let* $\pi \colon S \to \mathbb{P}^2$ *be a blow-up of finitely many toric points and let* $f \in \mathrm{Cr}_2(k)$ *be a loxodromic element of monomial type such that* $\hat{f}^i := \pi^{-1} f^i \pi$ *only contract curves of the toric boundary for all* $i \in \mathbb{Z}$. *Then* f *is monomial.*

Proof. Let $\overline{\partial}S$ be the finite union of lines that belong to the toric boundary of S and that are contracted by f^k for some $k \in \mathbb{Z}$. Define $Y := S \setminus \overline{\partial}S$. If there is an indeterminacy point $p \in Y$ of f, let $C \subset \overline{\partial}S$ be the union of all the curves contracted by f^{-1} to p. Assume that there is an irreducible component E of $\overline{\partial}S$ that intersects C but does not belong to C. Then $f^{-1}(E)$ passes through p. Since E is contained in $\overline{\partial}S$, there exists a $k \in \mathbb{Z}$ such that f^k contracts E. Therefore, f^{k+1} contracts $f^{-1}(E)$. But as p lies in Y, no curve that is contracted by f^k for any $k \in \mathbb{Z}$ passes through p. It follows that such an E does not exist and that C is a connected component of $\overline{\partial}S$. Therefore, there exists an open neighborhood U of p and an open neighborhood V of C such that f induces an isomorphism from $U \setminus p$ to $V \setminus C$. This implies the existence of a birational morphism $\pi_1 \colon S \to S_1$ that contracts C to a point $\pi(C)$ on a smooth projective variety S_1 and induces an isomorphism from $S \setminus C$ to $S \setminus \pi(C)$.

We continue this process finitely many times, until we obtain a contraction morphism $\pi \colon S \to S_m$, where S_m is a smooth surface with boundary $\overline{\partial}S_m = \pi_m(\overline{\partial}S)$ such that there are no indeterminacy points of f left in $S_m \setminus \overline{\partial}S_m$. In a next step we repeat the same procedure for f^{-1} instead of f. In that way we obtain a birational morphism $\pi_n \colon S \to S_n$ that contracts finitely many components of $\overline{\partial}S$, where S_m is smooth and projective such that $Y = S_n \setminus \overline{\partial}S_n$ does not contain any base-points of $\tilde{f} = \pi f \pi^{-1}$ and $\tilde{f}^{-1}$. We observe that $\tilde{f}$ and $\tilde{f}^{-1}$ leave Y invariant. Indeed, $\tilde{f}(\overline{\partial}S_n) \subset \overline{\partial}S_n$ and $\tilde{f}^{-1}(\overline{\partial}S_n) \subset \overline{\partial}S_n$ and both, $\tilde{f}$ and $\tilde{f}^{-1}$ are local isomorphisms around every point $p \in Y$. Hence f is an automorphism of Y. Since we only contracted curves that belong to the boundary, we can also push forward the D_2-action on S to S_n. We may and do also assume that $\overline{\partial}S_n$ contains no (-1)-curves (otherwise we contract them as well).

The curve $\overline{\partial} S_n$ has simple normal crossings and all its components are smooth rational curves, none of them has self intersection (-1). So we find ourselves in one of the four possibilities of Proposition 7.2.25. We note that (1) is not possible, since f is loxodromic and of monomial type and therefore not conjugate to an automorphism of a smooth projective surface, by Lemma 7.2.27. In case (2), f has to preserve a rational fibration, which is not possible since f is loxodromic. Thus, we are either in case (3) or case (4) and obtain therefore that Y is the blow-up of an affine toric surface, i.e. there exists a blow-up morphism $\rho\colon Y \to X$, where X is affine. The exceptional divisor of ρ consists of projective curves and since X is affine, they are the only projective curves on Y and are therefore preserved by $\tilde{f}$ and by D_2. Hence $\tilde{f}$ induces an automorphism on X and we have a D_2-action on X. Since f is loxodromic and of monomial type, its dynamical degree is a quadratic integer. By Lemma 7.2.26, the only affine toric surface with a loxodromic automorphism of non-integral dynamical degree is $\mathbb{A}^1_* \times \mathbb{A}^1_*$. Hence, $\tilde{f}$ and therefore f normalize D_2, which implies that f is monomial. $\qquad\square$

Proof of Theorem 7.2.1. Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a simple subgroup and assume that G contains a loxodromic element g. By Lemma 5.6.5, all loxodromic elements in G are of monomial type and G contains a subgroup Δ_2 that is normalized by g and there exists an element $f \in \mathrm{Cr}_2(\mathbb{C})$ that conjugates Δ_2 to a dense subgroup of D_2 and $\langle g, \Delta_2 \rangle$ to a subgroup of $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$. We may thus assume that g is monomial and $\Delta_2 \subset D_2$ is a dense subgroup. Assume that there exists another loxodromic element $f \in G$. By Lemma 7.2.21, there exists a blow-up of finitely many toric points $\pi\colon S \to \mathbb{P}^2$ such that f only contracts curves in the toric boundary ∂S. Proposition 7.2.28 implies that, in fact, f is monomial. Hence, all loxodromic elements of G are contained in $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$. Let $h \in G$ be an arbitrary element. Since hgh^{-1} is loxodromic, it is monomial. By Lemma 5.5.3, h is contained in $\mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$ as well. Hence $G \subset \mathrm{GL}_2(\mathbb{Z}) \ltimes D_2$ and we obtain a non-trivial homomorphism $\varphi\colon G \to \mathrm{GL}_2(\mathbb{Z})$ whose kernel contains Δ_2 - a contradiction to G being simple. Therefore, G does not contain any loxodromic element. $\qquad\square$

7.3 The parabolic and elliptic case

Lemma 7.3.1. *let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a simple subgroup that contains no loxodromic element, but a parabolic element. Then G is conjugate to a subgroup of the de Jonquières group and G is isomorphic to a subgroup of $\mathrm{PGL}_2(\mathbb{C})$.*

Proof. By Lemma 5.3.4, we know that G is either conjugate to a subgroup of the automorphism group of a Halphen surface or to a subgroup of the de Jonquières subgroup $\mathcal{J}$. By Theorem 5.3.3, automorphism groups of Halphen surfaces are finite extensions of abelian subgroups. It follows that the automorphism group of a Halphen surface does not contain infinite simple subgroups. Therefore, G is conjugate to a subgroup of $\mathcal{J}$. Let

$$1 \to \mathrm{PGL}_2(\mathbb{C}(t)) \to \mathcal{J} \to \mathrm{PGL}_2(\mathbb{C}) \to 1$$

be the short exact sequence from the semi-direct product structure of $\mathcal{J}$. Since G is simple, it is either contained in the kernel or the image of φ. In both cases it is

isomorphic to a subgroup of $\mathrm{PGL}_2(\mathbb{C})$. $\square$

To treat groups of elliptic elements we will apply similar techniques as in the proof of Theorem 6.6.3.

Lemma 7.3.2. *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a simple subgroup of elliptic elements. Then either G is a subgroup of an algebraic group in $\mathrm{Cr}_2(\mathbb{C})$ or G is conjugate to a subgroup of the de Jonquières group $\mathcal{J}$.*

Proof. By Lemma 6.3.2, G preserves either a fibration or every finitely generated subgroup of G is bounded. In the first case, G either preserves a rational fibration and is conjugate to a subgroup of $\mathcal{J}$, or G preserves a fibration of genus 1 curves and is conjugate to a subgroup of $\mathrm{Aut}(X)$, where X is a Halphen surface. In the latter case, Theorem 5.3.3 implies that G is finite.

Now assume that every finitely generated subgroup of G is bounded. If G is a torsion group, G is finite by Theorem 6.1.4. We thus may assume that G contains an element of infinite order. Define

$$d := \sup\{\dim(\overline{\Gamma}) \mid \Gamma \subset G \text{ finitely generated }\}.$$

First assume that d is finite. Since G contains an element of infinite order, $d \geq 1$. Let $\Gamma \subset G$ be a finitely generated subgroup such that $\dim(\overline{\Gamma}) = d$ and denote by $\overline{\Gamma}^0$ the neutral component of the algebraic group $\overline{\Gamma}$. Let $f \in G$ be an arbitrary element. Then $f\overline{\Gamma}^0 f^{-1}$ is again an algebraic subgroup and $\langle \overline{\Gamma}^0, f\overline{\Gamma}^0 f^{-1} \rangle$ is contained in the Zariski-closure of the finitely generated group $\langle \Gamma, f\Gamma f^{-1} \rangle$. Therefore, $\langle \overline{\Gamma}^0, f\overline{\Gamma}^0 f^{-1} \rangle$ is closed and connected (see [Hum75, Chapter 7.5]). Since it is of dimension $\leq d$ and contains $\overline{\Gamma}^0$ it equals $\overline{\Gamma}^0$, i.e. f normalizes $\overline{\Gamma}^0$. Since G is simple, $f \in \overline{\Gamma}^0$, hence $G \subset \overline{\Gamma}^0$. In particular, G is a bounded subgroup and as such is a subgroup of an algebraic group.

Now assume that $d = \infty$. Let $\Gamma \subset G$ be a finitely generated subgroup such that $\dim(\overline{\Gamma}) \geq 9$. By Lemma 6.6.2, $\overline{\Gamma}$ preserves a unique rational fibration given by a rational map $\pi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. Let $g \in G$ be an arbitrary element. The algebraic group $\overline{\langle \Gamma, g \rangle}$ preserves again a rational fibration and since it contains $\overline{\Gamma}$, this fibration is again given by π. It follows that G preserves a rational fibration and thus that G is conjugate to a subgroup of $\mathcal{J}$. $\square$

Proof of Theorem 7.1.2. The first statement of the Theorem is proven in Theorem 7.2.1, the second statement of Theorem 7.1.2 is proven in Lemma 7.3.1 and the third statement in Lemma 7.3.2. $\square$

7.4   The case of finitely generated subgroups

If we assume that Conjecture 7.1.1 holds, Theorem 7.1.4 follows directly from Theorem 7.1.2 and the property of Malcev for linear groups. In this section we give a short direct proof that relies on different techniques than the proof of Theorem 7.1.2 and in particular does not rely on Conjecture 7.1.1. The main advantage, when working with a finitely generated group, is that we can find a good reduction modulo p. We start by explaining this construction and will then apply it in a second step to our problem.

7.4.1 Reduction modulo p

The following lemma is due to Lech and Bell ([Lec53], [Bel06]). It can be seen as an analogue of the Lefschetz principle and is used in various contexts of number theory, dynamics and algebra:

Lemma 7.4.1 ([Bel06, Lemma 3.1]). *Let K be a finitely generated extension of $\mathbb{Q}$ and let S be a finite subset of K. Then there exist infinitely many primes p so that K embeds into $\mathbb{Q}_p$; moreover, we can chose this embedding in such a way that S is mapped into $\mathbb{Z}_p$.*

The following proposition shows how Lemma 7.4.1 can be applied to obtain information about the structure of subgroups of $\mathrm{Cr}_2(\mathbb{C})$. A similar statement has already been proven and applied by de Cornulier in order to show that the Cremona group is sofic ([Cor13], see also [Can12b]).

Proposition 7.4.2. *Let $\Gamma \subset \mathrm{Cr}_n(\mathbb{C})$ be a finitely generated subgroup. Then there exist infinitely many primes p such that there exists a non-trivial group-homomorphism $\varphi\colon \Gamma \to \mathrm{Cr}_n(\mathbb{F}_p)$ that satisfies $\deg(\varphi(f)) \leq \deg(f)$ for each $f \in \Gamma$.*

Proof. Let $g_1, \ldots, g_l \in \Gamma$ be a symmetric set of generators. We may assume that $g_i \neq \mathrm{id}$ for all i. Fix homogeneous polynomials $G_{ij} \in \mathbb{C}[x_0, \ldots, x_n]$ such that $g_i = [G_{i0} : \cdots : G_{in}]$ and define the endomorphisms $G_i := (G_{i0}, \ldots, G_{in}) \in \mathrm{End}(\mathbb{A}^{n+1})$. Assume that $g_i^{-1} = g_j$ and let

$$F_i := G_i \circ G_j = (F_{i0}, \ldots, F_{in}) \in \mathbb{A}^{n+1}.$$

Note that $g_i \circ g_j = [F_{i0} : \cdots : F_{in}] = [x_0 : \cdots : x_n]$, i.e. $F_{ij} = P_i x_j$ for some homogeneous polynomial $P_i \in \mathbb{C}[x_0, \ldots, x_n]$.

Let T be the finite set of all non-zero coefficients that appear in the polynomials G_{ij}, the F_{ij} or the polynomials $G_{1i}G_{2j} - G_{1j}G_{2i}$ and let $S = T \cup T^{-1}$. By Lemma 7.4.1, we can consider the field $K = \mathbb{Q}(S)$ as a subfield of the field $\mathbb{Q}_p$ in such a way that S is contained in $\mathbb{Z}_p \setminus p\,\mathbb{Z}_p$. Hence we may consider all our polynomials G_{ij}, F_{ij} and $G_{1i}G_{2j} - G_{1j}G_{2i}$ to be elements of $\mathbb{Z}_p[x_0, \ldots, x_n]$. By reducing the coefficients modulo p, we obtain a ring homomorphism $\psi\colon \mathbb{Z}_p[x_0, \ldots, x_n] \to \mathbb{F}_p[x_0, \ldots, x_n]$. Define the rational maps

$$\varphi(g_i) = [\psi(G_{i0}) : \cdots : \psi(G_{in})].$$

Note that

$$\varphi(g_i) \circ \varphi(g_i^{-1}) = [\psi(F_{i1}) : \cdots : \psi(F_{in})] = [\psi(P_i)x_0 : \cdots : \psi(P_i)x_n] = \mathrm{id},$$

so $\varphi(g_i)$ is a birational transformation of $\mathbb{P}^2_{\mathbb{F}_p}$. Assume that $g_{i_1} g_{i_2} \cdots g_{i_l} = \mathrm{id}$ for some $1 \leq i_1, \ldots, i_l \leq k$. Then, $G_{i_1} \circ \cdots \circ G_{i_l} = (Qx_0, \ldots, Qx_n)$ for some homogeneous polynomial Q. It follows that $\varphi(g_{i_1})\varphi(g_{i_2}) \cdots \varphi(g_{i_l}) = [\psi(Q)x_0 : \psi(Q)x_1 : \cdots : \psi(Q)x_n] = \mathrm{id}$. Therefore, the map φ can be extended to a homomorphism of groups $\varphi\colon \Gamma \to \mathrm{Cr}_2(\mathbb{F}_p)$. By construction, at least one of the polynomials $\psi(G_{1i})\psi(G_{2j}) - \psi(G_{1j})\psi(G_{2i})$ is not zero and hence $\varphi(g_1) \neq \varphi(g_2)$; in particular, φ is not trivial.

Let $g = g_{i_1} g_{i_2} \cdots g_{i_l} \in \Gamma$. Then $g = [H_0 : H_1 : \cdots : H_n]$, where $(H_0, \ldots, H_n) = G_{i_1} \circ \cdots \circ G_{i_l}$. We then have $\varphi(g) = [\psi(H_0) : \psi(H_1) : \cdots : \psi(H_n)]$. This shows that $\deg(\varphi(g)) \leq \deg(g)$. $\qquad\square$

Together with Theorem 5.6.4 we obtain the following result:

Theorem 7.4.3. *Let $\Gamma \subset \mathrm{Cr}_2(\mathbb{C})$ be a finitely generated subgroup. If Γ contains a loxodromic element, then Γ is not simple.*

Proof. Let $f \in \Gamma$ be a loxodromic element. If there exists a n such that f^n is tight in Γ, the group Γ is not simple by Theorem 5.6.2 and we are done. If no power of f is tight, it follows from Theorem 5.6.4 that Γ contains an infinite subgroup Δ_2 that is normalized by f and that is conjugate to a subgroup of D_2. The group Δ_2 being conjugate to a subgroup of D_2 implies in particular that the degrees of the elements in Δ_2 are uniformly bounded by an integer K. By Proposition 7.4.2, there exists a prime p and a non-trivial group homomorphism $\varphi \colon \Gamma \to \mathrm{Cr}_n(\mathbb{F}_p)$ that satisfies $\deg(\varphi(f)) \leq \deg(f)$. In $\mathrm{Cr}_n(\mathbb{F}_p)$ there exist only finitely many elements of degree $\leq K$, hence the image $\varphi(\Delta_2)$ is finite. It follows that φ has a proper kernel and therefore that Γ is not simple. $\square$

We are now able to prove Theorem 7.1.4 using the same strategy as in the proof of Theorem 7.1.2.

Lemma 7.4.4. *Let $\Gamma \subset \mathrm{PGL}_2(\mathbb{C}(t)) \rtimes \mathrm{PGL}_2(\mathbb{C})$ be a finitely generated simple subgroup. Then Γ is finite.*

Proof. Since Γ is simple, it is either isomorphic to a subgroup of $\mathrm{PGL}_2(\mathbb{C})$ or to a subgroup of $\mathrm{PGL}_2(\mathbb{C}(t))$, in particular Γ is linear. Since linear groups satisfy the property of Malcev, finitely generated simple linear groups are finite. $\square$

Proof of Theorem 7.1.4. Let $\Gamma \subset \mathrm{Cr}_2(\mathbb{C})$ be a finitely generated simple subgroup. By Theorem 7.4.3, Γ does not contain any loxodromic element. If Γ contains a parabolic element, then Γ is conjugate to a subgroup of the de Jonquières group $\mathcal{J} \simeq \mathrm{PGL}_2(\mathbb{C}(t)) \rtimes \mathrm{PGL}_2(\mathbb{C})$ or to a subgroup of the automorphism group $\mathrm{Aut}(X)$ of a Halphen surface X. This last case is not possible, since $\mathrm{Aut}(X)$ is abelian up to finite index, by Theorem 5.3.3. If Γ is a subgroup of $\mathcal{J}$, the claim follows with Lemma 7.4.4. If all elements in Γ are elliptic, Γ is, by Theorem 6.3.1, either conjugate to a subgroup of $\mathcal{J}$ or to a subgroup of an algebraic group. In the first case, Γ is finite by Lemma 7.4.4. As for the second case we recall that algebraic subgroups are always linear. Hence Γ is linear and therefore finite, since linear groups satisfy the property of Malcev. $\square$

Chapter 8

Summary of open questions

In this chapter we will briefly recall the most important and interesting questions that came up during the work on this thesis that I was not able to answer and that are, to my knowledge, open. The aim is to point out, what is missing from this thesis and also, to sketch a starting point for future research projects.

With regards to homomorphisms between Cremona groups the question, how the example of Gizatullin could be generalized to other projective representations of $\mathrm{PGL}_3(\mathbb{C})$ is still open:

Question 8.0.1 (Section 3.3)**.** Which projective representations of complex Lie groups from $\mathrm{PGL}_3(\mathbb{C})$ to $\mathrm{PGL}_{n+1}(\mathbb{C})$ can be extended to homomorphisms of Cremona groups from $\mathrm{Cr}_2(\mathbb{C})$ to $\mathrm{Cr}_n(\mathbb{C})$?

I did not succeed in refining the techniques used to prove Theorem 4.1.4 in order to answer the following question positively, nor to construct examples with degree growth that is not polynomial:

Question 8.0.2 (Section 4.1.3)**.** Does there exist a birational transformation $f \in \mathrm{Aut}(\mathbb{A}^d)$ such that $\deg(f^n) \sim \sqrt{n}$? Does there exist an $f \in \mathrm{Aut}(\mathbb{A}^d)$ such that $\deg(f^n) \sim \exp(\sqrt{n})$?

It seems that it should be possible to refine Theorem 6.1.3 in order to exclude case (4):

Question 8.0.3 (Section 6.1)**.** Are all subgroups of elliptic elements in $\mathrm{Cr}_2(\mathbb{C})$ either bounded or conjugate to a subgroup of the de Jonquères group?

The following question is not directly related to Cremona groups, but is compelling on its own:

Question 8.0.4 (Section 7.1)**.** What are the simple subgroups of $\mathrm{SL}_2(\mathbb{Q})$? Or, more generally, what are the simple subgroups of $\mathrm{SL}_n(\mathbb{C})$?

Obviously, it would be important to prove Conjecture 7.1.1 in order to classify all simple subgroups of $\mathrm{Cr}_2(\mathbb{C})$:

Question 8.0.5 (Section 7.2.3)**.** Let $f \in \mathrm{Cr}_2(\mathbb{C})$ be a loxodromic element, $p \in \mathbb{P}^2$ a point that is not contained in any of the coordinate lines of $\mathbb{P}^2$ and k a positive integer. Is the set

$$\{d(fd)^k(p) \mid d \in D_2 \text{ such that } p \neq \mathrm{Ind}((fd)^l) \text{ for all } 1 \leq l \leq k\}$$

always dense in $\mathbb{P}^2$?

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