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Dynamique des transformations birationnelles des variétés hyperkähleriennes: feuilletages et fibrations invariantes

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Introduction

Ce mémoire se situe à l'interface entre géométrie algébrique complexe et systèmes dynamiques. Dans la suite M désignera une variété complexe lisse projective (ou, plus généralement, compacte kählerienne), et $f\colon M \to M$ une application holomorphe (ou, plus généralement, méromorphe) dominante ; on note $\mathrm{Aut}(M)$ (resp. $\mathrm{Bir}(M)$) le groupe d'automorphismes biholomorphes (resp. de transformations biméromorphes) de M.

Les questions autour de f auxquelles je tente de répondre sont de nature dynamique : existence et densité des points périodiques, densité (ou Zariski-densité) de l'orbite d'un point très général, calcul de l'entropie topologique, existence de structures invariantes par f (fibrations, feuilletages, structures symplectiques etc.). Nous verrons que certaines de ces questions peuvent être étudiées par des méthodes de géométrie algébrique (calculs en cohomologie, notions de positivité, schémas de Hilbert des sous-variétés ...), d'analyse complexe (géométrie kählerienne, courants positifs fermés, résolution des singularités ...) et d'analyse p-adique. Des outils de systèmes dynamiques (théorie de Pesin sur les variétés stables/instables, développements récents liés à la construction de points périodiques hyperboliques...) et de géométrie différentielle (notamment la théorie des (G, X)-structures et des structures géométrique au sens de Gromov) jouent également un rôle important.

L'étude ci-présente se concentre sur les variétés compactes kähleriennes dont la première classe de Chern est nulle, et plus particulièrement sur les variétés symplectiques holomorphes irréductibles (voir la Définition 2.2).

0.1 Définitions, motivation et résultats connus

0.1.1 Entropie topologique, degrés dynamiques

Pour décrire la géométrie d'une variété M il faut passer par la compréhension de ses sous-variétés, et donc par l'analyse de ses espaces de (co)homologie. Il se trouve que l'action induite par une application holomorphe $f\colon M \to M$ sur la cohomologie de M fournit des informations sur sa dynamique.
Pour la définition suivante on renvoie à [DS04, CCLG10].

Définition 1.7. *Soit M une variété compacte kählerienne et $f\colon M \to M$ une application holomorphe. Le* degré dynamique *d'ordre p de f est le rayon spectral (i.e. le maximum des modules des valeurs propres) de l'application linéaire induite*

$$f^*_{p,p}\colon H^{p,p}(M,\mathbb{R}) \to H^{p,p}(M,\mathbb{R}).$$

Plus généralement, si $f\colon M \dashrightarrow M$ est une application méromorphe dominante, on définit

$$\lambda_p(f) = \lim_{n\to+\infty} \frac{\log ||(f^n)^*_{p,p}||}{n},$$

où $||\cdot||$ est une norme sur l'espace $\mathrm{End}_{\mathbb{R}}(H^{p,p}(M,\mathbb{R}))$.

Le nombre $\lambda_p(f)$ est un réel ≥ 1 qui (au moins dans le cas où M est projective) décrit la croissance exponentielle des volumes des sous-variétés de M "génériques" de codimension p par image inverse des itérés de f (voir [Gue04]). Par exemple, si $d = \dim M$, $\lambda_d(f)$ est le degré topologique de f.
Les degrés dynamiques sont des invariants birationnels [DS05c, DS04]. De plus, les inégalités de Teissier et Khovanskii (voir [Gro90]) permettent de montrer que l'application $p \mapsto \log \lambda_p(f)$ est concave sur l'ensemble $\{0, 1, \ldots, \dim M\}$; en particulier, il existe un $\lambda_p(f) > 1$ si et seulement si $\lambda_1(f) > 1$ si et seulement si tous les $\lambda_p(f)$ sauf au plus $\lambda_0(f)$ et $\lambda_{\dim X}(f)$ sont > 1.

A une application continue $f\colon M \to M$ d'un espace compact M dans lui-même on peut associer son *entropie topologique* $h_{top}(f)$, un nombre positif ou nul, éventuellement infini, qui donne une mesure du chaos créé par f ; plus précisément, $h_{top}(f)$ mesure à quelle vitesse exponentielle deux point proches sont éloignés par les itérés de f (voir [AKM65] pour la définition originale, [Bow71] pour une définition métrique plus simple à calculer). Dans beaucoup de cas, les propriétés dynamiques des transformations changent drastiquement selon que l'entropie est nulle ou non. Le calcul de l'entropie topologique est compliqué même pour des applications très simples, et nécessite souvent des arguments ad hoc. Cependant, grâce au théorème fondateur suivant dû à Gromov et Yomdin [Gro90], le calcul s'avère simple dans le cadre de transformations holomorphes de variétés compactes kähleriennes.

Théorème 1.10. *Si $f\colon M \to M$ est une application holomorphe dominante d'une variété compacte kählerienne M de dimension d dans elle-même, alors*

$$h_{top}(f) = \max\{\log(\lambda_p(f));\ p = 0, \ldots, d\}.$$

En particulier, par concavité on a $h_{top}(f) > 0$ si et seulement si $\lambda_1(f) > 1$.

Dans le cas d'une application méromorphe on peut adapter la définition d'entropie topologique, mais dans le Théorème 1.10 seule l'inégalité $\leq$ reste valable [DS05c].

0.1.2 Automorphismes des surfaces

On restreint maintenant notre attention aux automorphismes (ou plus généralement aux applications birationnelles) de variétés compactes kähleriennes.
Le cas des courbes n'est pas dynamiquement intéressant : en effet dans ce cas les applications birationnelles sont des automorphismes, et les seuls automorphismes d'ordre infini apparaissent sur $\mathbb{P}^1$ et sur les courbes elliptiques, et sont bien compris.
En revanche sur les surfaces on dispose de beaucoup d'exemples dynamiquement intéressants, et dans ce contexte le lien entre action en cohomologie et dynamique est particulièrement profond.

Soit $f\colon S \to S$ un automorphisme d'une surface compacte kählerienne. D'abord, on a des restrictions sur l'action en cohomologie de f qui découlent directement de l'invariance du produit d'intersection sur $H^{1,1}(S,\mathbb{R}) = H^{1,1}(X) \cap H^2(X,\mathbb{R})$ (qui est de signature $(1, h^{1,1}(S) - 1)$ par le théorème de l'indice de Hodge) :

- si le degré dynamique $\lambda_1(f)$ est $\neq 1$, alors il est soit un nombre quadratique soit un nombre de Salem et la norme de $(f^n)^*$ croît exponentiellement vite ;
- si $\lambda_1(f) = 1$, alors soit un itéré de f est isotope à l'identité (et en particulier f^* est d'ordre fini), soit $||(f^n)^*||$ croît quadratiquement en n.

On appelle *loxodromique, parabolique* ou *elliptique* un automorphisme f telle que la norme de $(f^n)^*$ a croissance exponentielle, quadratique ou bornée respectivement.
De plus, le fait que l'entropie soit nulle ou non gouverne la propriété suivante de décomposabilité de la dynamique :

Définition 5.1. *Soit $f\colon M \dashrightarrow M$ une transformation biméromorphe d'une variété compacte kählerienne M. On dit que f est* primitive *si elle n'admet pas de fibration méromorphe invariante non-triviale, i.e. s'il n'existe pas de carré commutatif*

$$\begin{array}{ccc} M & \overset{f}{\dashrightarrow} & M \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & \underset{g}{\dashrightarrow} & B \end{array}$$

où $\pi\colon M \dashrightarrow B$ est une application méromorphe dominante à fibres connexes et $0 < \dim B < \dim M$.

L'étude de la dynamique d'une transformation non primitive peut se décomposer en l'étude de l'action g sur la base (de dimension plus petite) et de l'action sur les fibres ; du coup elle est à priori qualitativement plus simple que celle d'une transformation primitive.
Cette intuition est confirmée, au moins dans le cas des surfaces, par le résultat suivant, énoncé par Cantat [Can14] dans cette version, mais essentiellement dû à Gizatullin, voir [Giz80, Gri16].

Théorème 1.30 (Cantat)**.** *Soit $f\colon S \to S$ un automorphisme d'une surface compacte kählerienne dont l'action en cohomologie est d'ordre infini ; f est primitif si et seulement si $h_{top}(f) > 0$ (si et seulement si $\lambda_1(f) > 1$).*

En fait l'énoncé est plus précis : si $\lambda_1(f) = 1$ (i.e. f est parabolique), alors il existe une fibration *holomorphe* $\pi\colon S \to C$ sur une courbe C et un automorphisme $g\colon C \to C$ tels que $g \circ \pi = \pi \circ f$; de plus, la fibre au-dessus d'un point général de C est une courbe elliptique. Pour une transformation birationnelle d'une surface on a un énoncé similaire [DF01].

0.1.3 Feuilletages stables et instables

Une notion localement analogue à celle de fibration mais bien plus générale est celle de *feuilletage*. Informellement, un feuilletage C^∞ (resp. holomorphe) lisse sur une variété différentielle (resp. complexe) M est donné par une décomposition locale de petits ouverts de M en l'union disjointe de sous-variétés (les feuilles locales), donnant une structure de produit local qui varie de manière lisse (resp. holomorphe) ; localement on peut penser aux feuilles comme aux fibres d'une fibration, mais la structure globale d'une feuille peut être bien plus compliquée (par exemple, elle peut être dense dans M). Plus généralement, on peut considérer des feuilletages singuliers, où l'on ne voit pas partout de structure de produit local (penser au feuilletage radial dans $\mathbb{C}^2$).
La notion de feuilletage est devenue centrale dans le cadre des systèmes dynamiques puisqu'elle apparaît naturellement dans l'étude qualitative des solutions d'équations différentielles, des difféomorphismes et des flots de type Anosov[1], et devient de plus en plus importante en géométrie algébrique.

Example 0.1. L'exemple fondamental d'un difféomorphisme Anosov est le suivant. Soit $T = \mathbb{R}^2/\mathbb{Z}^2$ le tore réel de dimension 2. Une matrice $A \in SL_2(\mathbb{Z})$ induit un difféomorphisme $f\colon T \to T$. On suppose que A est diagonalisable avec deux valeurs propres λ, λ^{-1} où $|\lambda| \neq 1$, et on note v_+, v_- des vecteur propres pour la valeur propre λ, λ^{-1} respectivement. Alors f préserve les feuilletages linéaires $\mathcal{F}_u, \mathcal{F}_s$ définis par les direction v_+, v_-, et $\mathcal{F}_u$ (resp. $\mathcal{F}_s$) est uniformément dilaté (resp. contracté) par f. Comme λ est irrationnel, les feuilles de $\mathcal{F}_u$ et $\mathcal{F}_s$ sont denses dans T.

[1] Un difféomorphisme d'une variété différentielle compacte est dit Anosov s'il préserve deux feuilletages transverses $\mathcal{F}_s, \mathcal{F}_u$ de classe C^0 (les feuilletages stable et instable respectivement), et s'il agit en contractant (resp. dilatant) uniformément $\mathcal{F}_s$ (resp. $\mathcal{F}_u$) par rapport à une certaine métrique sur la variété. On définit de manière analogue un flot Anosov.

Dans le cadre complexe, on peut répéter la construction sur un produit de deux courbes elliptiques isomorphes :

$$S = \mathbb{C}^2/_{\Lambda \times \Lambda} \cong \mathbb{C}/_{\Lambda} \times \mathbb{C}/_{\Lambda},$$

où $\Lambda = \langle 1, \tau \rangle$ est un réseau de $\mathbb{C}$. On appelle encore les feuilletages holomorphes définis par les directions v_+ et v_- les feuilletages instable et stable respectivement.

Soient S une surface complexe compacte et $f\colon S \to S$ un automorphisme d'entropie positive ; un point périodique p de période k est dit hyperbolique si Df_p^k agit en contractant une direction et en en dilatant une autre. En appliquant le itérés de f (resp. de f^{-1}) à un disque le long de la direction contractée (resp. dilatée), on arrive à définir une courbe entière immérgée $c\colon \mathbb{C} \hookrightarrow S$, qu'on appelle la variété stable (resp. instable) passant par p, et qui en général est d'image Zariski-dense ; cette variété peut également être définie comme l'ensemble des points $q \in M$ tels que $f^{Nk}(q)$ (resp. $f^{-Nk}(q)$) converge vers p lorsque $N \to +\infty$. Plus généralement, si $f\colon M \to M$ est un difféomorphisme d'une variété différentiable M et μ est une mesure de probabilité (ergodique et hyperbolique) invariante par f, la théorie de Pesin (voir [Pes76]) permet de définir des variétés stables et instables en μ-presque tout point ; dans le cas où f est un biholomorphisme d'une variété complexe, les variétés (in)stables sont des copies immergées de $\mathbb{C}^k$ pour un $k \in \mathbb{N}$ [JV02]. On peut se demander si les variétés stables/instables s'organisent en deux feuilletages holomorphes. Avec l'hypothèse additionnelle que la contraction/dilatation le long des feuilles stables/instables soit uniforme, on aurait un analogue purement holomorphe d'un difféomorphisme *Anosov* d'une variété différentielle compacte.

Un principe général est que, lorsqu'on impose qu'un objet issu de la théorie des systèmes dynamiques est holomorphe/algébrique, la situation devient suffisamment rigide pour qu'on puisse espérer classer tous les exemples. Par exemple, Cantat et Favre ont donné dans [CF03] une classification complète des triplets $(S, \mathcal{F}, f)$, où S est une surface projective complexe, $\mathcal{F}$ un feuilletage singulier sur S et $f\colon S \to S$ une transformation birationnelle avec $\lambda_1(f) > 1$ préservant $\mathcal{F}$: à équivalence birationnelle et revêtement fini près, $S = \mathbb{C}^2/\Lambda$ est un tore complexe, f est une application linéaire et $\mathcal{F}$ est un feuilletage linéaire dans la direction de l'une des deux valeurs propres de f.

0.2 Variétés dont la première classe de Chern est nulle

Dans la suite on se concentre sur les variétés compactes kähleriennes M dont la première classe de Chern $c_1(K_M)$ est nulle ; voir [Bea83, GHJ03]. L'importance de cette classe de variétés du point de vue de la dynamique des transformations birationnelles est justifiée par les remarques suivantes :

1. Si $f\colon M \dashrightarrow M$ est une transformation méromorphe dominante d'une variété compacte kählerienne, alors un itéré de f préserve une à une les fibres de la fibration de Kodaira

   $$\pi\colon M \dashrightarrow B \subset \mathbb{P}H^0(M, mK_M)^\vee,$$

   qui sont des variétés de dimension de Kodaira nulle ; voir [Uen75, NZ09]. Il convient donc d'étudier d'abord les variétés de type général et celles avec dimension de Kodaira 0 ou $-\infty$.

2. Le groupe de transformations birationnelles d'une variété de type général est fini (voir [Uen75]).

3. On dit que M est rationnellement connexe si deux points de M peuvent être joints par une chaîne de courbes rationnelles. Lorsque M n'est pas rationnellement connexe, on dispose

d'une fibration méromorphe $\rho\colon M \dashrightarrow Rat(M)$, le quotient rationnellement connexe de M (voir [Deb01, Chapter 5]), dont les fibres très générales sont rationnellement connexes et maximales pour cette propriété. Cette fibration est invariante par toute transformation méromorphe de M.

4. La fibration d'Albanese, la réduction algébrique [Uen75] et le coeur de Campana [Cam03] sont d'autres fibrations invariantes par toute transformation méromorphe dominante.

En conclusion, les variétés M dont il convient d'étudier la dynamique en premier sont celles dont la dimension de Kodaira est négative ou nulle et qui satisfont $\dim Rat(M) = \dim M$ (rationnellement connexes) ou 0. Parmi les variétés de dimension de Kodaira 0, celles dont la première classe de Chern est nulle occupent une position privilégiée ; par ailleurs, leur groupe d'automorphismes joue un rôle particulièrement important dans l'étude de leur géométrie (penser par exemple à la conjecture du cône de Kawamata-Morrison, voir [LOP16, Conjecture 3.3]).

0.2.1 Le théorème de décomposition de Beauville-Bogomolov

Grâce au théorème de structure suivant, l'étude des variétés dont la première classe de Chern est nulle se réduit en un premier temps à trois grandes classes (voir [GHJ03, Theorem 5.4 and Theorem 14.15]).

Théorème 2.6 (Beauville-Bogomolov)**.** *Soit M une variété compacte kählerienne avec $c_1(M) = 0$. Alors M admet un revêtement fini non-ramifié de la forme*

$$M' \cong T \times \prod_{i=1}^{n} X_i \times \prod_{j=1}^{m} Y_j,$$

où

- *$T = \mathbb{C}^k/\Lambda$ est un tore complexe ($\Lambda \cong \mathbb{Z}^{2k}$ est un réseau de $\mathbb{C}^k$) ;*
- *les X_i sont des variétés symplectiques holomorphes irréductibles ;*
- *les Y_j sont des variétés de Calabi-Yau au sens strict, i.e. $\pi_1(Y_j) = 0$, $K_{Y_j} \cong \mathcal{O}_{Y_j}$ et $H^0(Y_j, \Omega^p_{Y_j}) = 0$ pour $0 < p < \dim Y_j$.*

Définition 2.2. *Une variété compacte kählerienne X est dite* symplectique holomorphe *s'il existe une 2-forme holomorphe fermée nulle part dégénérée σ (i.e. σ induit un isomorphisme $T_X \cong \Omega^1_X$) ; X est dite* irréductible *si de plus $\pi_1(X) = 0$ et $H^0(X, \Omega^2_X) = \mathbb{C}\sigma$.*
Les variétés symplectiques holomorphes irréductibles sont aussi appelées hyperkähleriennes.

On remarque qu'en dimension deux les variétés symplectiques irréductibles et les variétés de Calabi-Yau coïncident avec les surfaces $K3$.

Dans la suite on va se concentrer en particulier sur les variétés symplectiques holomorphes irréductibles.

0.2.2 Variétés symplectiques holomorphes irréductibles

Ici on verra certaines propriétés fondamentales des variétés symplectiques irréductibles. On renvoie à [GHJ03, Part III] pour une introduction détaillée.

Soit X une variété hyperkählerienne ; d'abord, la condition de non-dégénérescence sur la forme symplectique σ implique que la dimension complexe de X est paire.

Exemples

À la différence des variétés de Calabi-Yau, on dispose de relativement peu d'exemples de variétés hyperkähleriennes. En effet, tous les exemples actuellement connus sont équivalent à déformation près à l'un des exemples suivants :

1. Schéma de Hilbert de n points sur une surface $K3$: si S est une surface $K3$, le schéma de Hilbert de n points sur S, $\mathrm{Hilb}^n(S) = S^{[n]}$, est une résolution minimale du produit symétrique $\mathrm{Sym}^n S$ qui paramètre les sous-schémas de S de dimension 0 et longueur n.

2. Variété de Kummer généralisée : si $T = \mathbb{C}^2/\Lambda$ est un tore complexe, la variété de Kummer généralisée de dimension $2n$ est le noyau $K_n(T)$ de la composition
$$\mathrm{Hilb}^{n+1}(T) \to \mathrm{Sym}^{n+1}(T) \to T,$$
où la deuxième flèche est le morphisme somme de $n+1$ points.

3. Deux exemples sporadiques en dimension 6 et 10 dûs à O'Grady [O'G99, O'G03].

À partir de l'Exemple 0.1 on peut produire des exemples dynamiquement intéressants d'automorphismes de variétés hyperkähleriennes.

Example 0.2 (Exemples de type Kummer). Soit E une courbe elliptique et $T = E \times E$; une matrice $A \in \mathrm{SL}_2(\mathbb{Z})$ induit un automorphisme f de T. Supposons que A ait deux valeurs propres réelles $\lambda > 1, \lambda^{-1} < 1$, et notons v_+, v_- deux vecteurs propres non nuls.
On peut voir que $\lambda_1(f) = \lambda^2 > 1$; f préserve les feuilletages linéaires $\mathcal{F}_T^+, \mathcal{F}_T^-$ définis par les directions v_+, v_- respectivement, et agit en dilatant le premier et en contractant le deuxième (voir l'Exemple 0.1).
On notera S la surface de Kummer associée à T, i.e. la surface $K3$ obtenue comme résolution minimale de $T/\pm \mathrm{id}_T$; f induit un automorphisme $f_S \colon S \to S$, avec $\lambda_1(f_S) = \lambda_1(f)$, préservant deux feuilletages singuliers induits par $\mathcal{F}_+, \mathcal{F}_-$.

1. Soit $X_n = S^{[n]}$ le schéma de Hilbert de n points sur S ; alors f induit un automorphisme f_X de X_n avec $\lambda_1(f_X) = \lambda_1(f) > 1$; $\mathcal{F}_T^+$ et $\mathcal{F}^-$ induisent deux feuilletages singuliers f_X-invariants $\mathcal{F}_X^\pm$ de dimension n sur X_n.

2. Soit $Y_n = K_n(T)$ la variété de Kummer généralisée de dimension $2n$; f induit un automorphisme de $\mathrm{Hilb}^{n+1}(T)$ préservant $K_n(T)$, et donc induit un automorphisme f_Y de Y_n ; on peut calculer que $\lambda_1(f_Y) = \lambda_1(f) > 1$; comme au cas précédent $\mathcal{F}_+$ et $\mathcal{F}_-$ induisent deux feuilletages singuliers f_Y-invariants $\mathcal{F}_Y^\pm$ de dimension n sur Y_n.

On renvoie au Chapitre 3 pour une analyse détaillée de ces deux exemples.

Forme de Beauville-Bogomolov et classification des transformations birationnelles

À l'aide de la forme symplectique on peut définir une forme bilinéaire non-dégénérée q_X sur $H^2(X, \mathbb{Z})$, la *forme de Beauville-Bogomolov*, dont la restriction à $H^{1,1}(X, \mathbb{R})$ est non-dégénérée de signature $(1, h^{1,1}(X) - 1)$. Cette propriété fournit un lien formel entre la cohomologie des surfaces et celle des variétés hyperkähleriennes ; on peut alors se demander les automorphismes des variétés hyperkähleriennes possèdent des propriétés dynamiques analogues à celles des automorphismes des surfaces.

Une autre propriété remarquable des variétés hyperkähleriennes (et, plus généralement, des variétés compacte kähleriennes dont le fibré canonique est nef) est que les applications birationnelles $X \dashrightarrow X'$ entre deux variétés de ce type induisent des isomorphismes entre deux ouverts $U \subset X, U' \subset X'$ dont les complémentaires sont de codimension ≥ 2 ; en particulier, les transformations birationnelles de X sont des *pseudo-automorphismes* (i.e. des transformations birationnelles qui ne contractent aucun diviseur). De plus on peut montrer qu'une transformation birationnelle de X induit un automorphisme linéaire de $H^2(X,\mathbb{Z})$ préservant la forme bilinéaire q. Grâce au résultat de signature énoncé plus haut, ceci donne une représentation

$$\mathrm{Bir}(X) \to O(H^{1,1}(X), q_X) \cong O(1, h^{1,1}(X) - 1);$$

comme pour les automorphismes des surfaces (voir le paragraphe 0.1.2) ; on dira alors que $f \in \mathrm{Bir}(X)$ est

- *loxodromique* si $\lambda_1(f) > 1$; dans ce cas $\lambda_1(f)$ est un nombre quadratique ou de Salem, et la norme de $(f^n)^*$ est à croissance exponentielle ;
- *parabolique* si f^* est d'ordre infini et $\lambda_1(f) = 1$; dans ce cas la norme de $(f^n)^*$ est à croissance quadratique ;
- *elliptique* si f^* est d'ordre fini ; dans ce cas, un itéré de f est isotope à l'identité, et comme X ne possède pas de champ de vecteurs holomorphe non trivial, f est d'ordre fini.

En essayant de pousser l'analogie avec les surfaces, une question naturelle est alors la suivante :

Question 0.1. *Soit $f\colon X \dashrightarrow X$ une transformation birationnelle d'une variété symplectique holomorphe irréductible X ; supposons que f est d'ordre infini. Est-ce que f est primitive si et seulement si elle est loxodromique (i.e. $\lambda_1(f) > 1$) ?*

Example 0.3. Si on répète les deux constructions de l'Exemple 0.2 en utilisant une matrice A unipotente non triviale, e.g. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, on peut construire des exemples d'automorphismes paraboliques de variétés hyperkähleriennes préservant une fibration méromorphe lagrangienne.

Dans [HKZ15], Hu, Keum et Zhang ont appliqué les résultats de [BM14] pour donner une réponse positive à une direction de la Question 0.1 dans le cas de variétés hyperkähleriennes équivalentes par déformation à $S^{[n]}$ (S une surface $K3$) ou à $K_n(T)$ (où T est un tore complexe de dimension 2).

Théorème 2.15 (Hu, Keum, Zhang)**.** *Soit X une variété symplectique irréductible de dimension $2n$ de type $K3^{[n]}$ ou Kummer généralisée et soit $f\colon X \dashrightarrow X$ une transformation birationnelle d'ordre infini ; f est parabolique si et seulement s'il existe une fibration rationnelle lagrangienne $\pi\colon X \dashrightarrow \mathbb{P}^n$ et un automorphisme $g \in \mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}(\mathbb{C})$ tels que $\pi \circ f = g \circ \pi$.*

Action en cohomologie et points périodiques hyperboliques d'un automorphisme

La cohomologie des variétés symplectiques irréductibles est encore assez mal comprise ; cependant, la sous-algèbre $SH^2(X,\mathbb{C})$ de $H^*(X,\mathbb{C})$ engendrée par $H^2(X,\mathbb{C})$ a été décrite par Verbitsky : si $\dim X = 2n$, alors

$$SH^2(X,\mathbb{C}) \cong {\mathrm{Sym}^* H^2(X,\mathbb{C})}\big/{\langle \alpha^{n+1} \mid q_X(\alpha) = 0\rangle}.$$

Ici, on quotiente par l'idéal de $H^*(X,\mathbb{C})$ engendré par les $\alpha^{n+1} \in H^{2n+2}(X,\mathbb{C})$, où $\alpha \in H^2(X,\mathbb{C})$ est tel que $q_X(\alpha) = 0$. En particulier, pour $p = 0, \ldots, n$, on a une injection $\mathrm{Sym}^p H^2(X,\mathbb{C}) \hookrightarrow H^{2p}(X,\mathbb{C})$.
Oguiso a remarqué que ceci permet de montrer, en utilisant la log-concavité des degrés dynamiques, que si $f\colon X \to X$ est un automorphisme d'une variété symplectique irréductible de dimension $2n$, alors pour tout $0 \leq p \leq n$ on a

$$\lambda_p(f) = \lambda_{2n-p}(f) = \lambda_1(f)^p.$$

En fait on peut décrire encore mieux la situation : le premier degré dynamique détermine toutes les modules des valeurs propres de f^*.

Théorème 4.4 (Bogomolov, Kamenova, Lu, Verbitsky)**.** *Soit f un automorphisme d'une variété symplectique irréductible X, et soit $\lambda = \lambda_1(f)$. Alors*

- *toutes les valeurs propres de $f^*\colon H^*(X,\mathbb{R}) \to H^*(X,\mathbb{R})$ sont de module $\lambda^{k/2}$ pour un $k \in \mathbb{Z}$;*

- *la somme des multiplicités des valeurs propres de*

$$f^*_{p+q}\colon H^{p+q}(X,\mathbb{R}) \to H^{p+q}(X,\mathbb{R})$$

  ayant module $\lambda^{\frac{p-q}{2}}$ est $\dim H^{p,q}(X)$.

Un *point périodique hyperbolique* pour un automorphisme $g\colon M \to M$ d'une variété complexe M est un point p f-périodique de période k tel que la Jacobienne de g^k au point p est un automorphisme linéaire de TM_p dont tous les valeurs propres sont de module $\neq 1$.

Le Théorème 4.4 permet d'appliquer des résultats récents de systèmes dynamiques pour montrer l'énoncé suivant.

Théorème 4.2. *Soit $f\colon X \to X$ un automorphisme loxodromique d'une variété symplectique holomorphe irréductible. Alors les points périodiques hyperboliques forment un sous-ensemble Zariski-dense de X.*

La démarche de la preuve, qu'on verra en détail dans le Chapitre 4, est la suivante. Dans [DS10, Theorem 4.2.1 et 4.4.2], Dinh et Sibony construisent une mesure de probabilité f-invariante μ, qui est ergodique, et qui ne peut pas être supportée par un diviseur. Pour appliquer ces théorèmes, il faut vérifier que

- $\lambda_p(f) \neq \lambda_{p+1}(f)$ pour $p = 0, \dots \dim X - 1$;

- le degré dynamique maximal $\lambda_q(f)$ est une valeur propre simple de $f^*_{q,q}$;

Ces deux points suivent du Théorème 4.4.
Un résultat de de Thélin [dT08] s'applique à la mesure μ construite par Dinh et Sibony, et montre qu'elle est hyperbolique, c'est-à-dire que les *exposants de Lyapunov* pour μ sont tous non-nuls. Les exposants de Lyapunov sont un équivalent global des (logarithmes des) valeurs propres de la matrice Jacobienne de f aux point fixes ; considérons pour $p \in X$

$$\beta(p) = \lim_{n\to+\infty} \frac{\log ||(Df^n)_p||}{n},$$

où $(Df^n)_p$ est la matrice Jacobienne de f^n au point p. La fonction β est bien définie sur un ensemble de mesure totale par le théorème ergodique multiplicatif d'Oseledec [Ose68], et elle est f-invariante, donc elle est presque partout égale à une constante β. On définit l'exposant de Lyapunov maximal comme β ; pour les exposants suivants, il faut considérer l'action de la matrice Jacobienne sur les espaces $\bigwedge^k TX_p$.

Par un résultat de Katok [Kat80, Lemma 4.2], les points périodiques hyperboliques sont denses dans le support de μ, donc Zariski-denses.

0.3 Primitivité des transformations loxodromiques des variétés symplectiques irréductibles

Mon premier résultat établit la direction manquante dans la Question 0.1.

Théorème A. *Soit X une variété symplectique holomorphe irréductible, et $f\colon X \dashrightarrow X$ une transformation birationnelle loxodromique (i.e. $\lambda_1(f) > 1$). Alors f est primitive.*

Ce résultat donne des conséquences intéressantes sur la dynamique de f.

0.3.1 Sous-variétés périodiques

On dit qu'une sous-variété fermée $V \subset X$ est f-périodique si elle n'est contenue dans le lieu d'indétermination d'aucun itéré de f et si sa transformée stricte par un certain itéré f^N est égale à V.
Un résultat de Cantat [Can10, Theorem B] implique que, si M est une variété complexe compacte et si le nombre d'hypersurfaces invariantes pour un pseudo-automorphisme $f\colon M \dashrightarrow M$ est supérieur à la constante $c(M) = \dim(M)+h^1(M,\Omega^1_M)$, alors f préserve une fonction méromorphe (et donc, par factorisation de Stein, une fibration méromorphe sur une courbe). Le résultat de Cantat est un analogue en dynamique discrète d'un théorème de Krasnov [Kra75], qui borne le nombre d'hypersufaces sur une variété n'admettant aucune fonction méromorphe, et d'un théorème de Jouanolou [Jou78] et Ghys [Ghy00], qui affirme qu'un feuilletage de codimension 1 admettant une infinité de feuilles compactes admet une intégrale première.
Dans le situation du Théorème A, on obtient :

Corollaire 0.4. *Si $f\colon X \dashrightarrow X$ est une transformation birationnelle loxodromique d'une variété symplectique holomorphe irréductible, alors f admet au plus $\dim X + b_2(X) - 2$ hypersurfaces périodiques.*

On remarque qu'un résultat analogue ne peut pas être vrai pour les sous-variétés de codimension supérieure. Un contre-exemple est donné par le point (1) de l'Exemple 0.2 ; si $p \in S$ est un point périodique pour f_S, alors les (images dans $S^{[n]}$ des) produits $\{p\} \times \ldots \times \{p\} \times S \times \ldots \times S$ sont des sous-variétés f-périodiques de codimension paire arbitraire, et comme il y a une infinité de points f_S-périodiques, on produit une infinité de sous-variétés périodique de n'importe quelle codimension paire. Pour retrouver les codimensions impaires, on peut remarquer que les 16 (-2)-courbes de S (les diviseurs exceptionnels de la résolution de $T/\pm\mathrm{id}$) sont périodiques ; il suffit alors de prendre les produits $\{p\} \times \ldots \times \{p\} \times C \times S \times \ldots \times S$, où C est une courbe périodique, pour trouver une infinité de sous-variétés périodiques en toute codimension impaire > 1.

0.3.2 Zariski-densité des orbites

Une question dynamique naturelle est celle de décrire les adhérences des orbites génériques d'un automorphisme ou d'une application birationnelle.
Dans le cadre algébrique ou analytique, on peut également considérer la topologie de Zariski, et se demander si l'orbite typique est Zariski-dense ou pas ; dans le cas contraire, toute orbite est contenue dans une certaine sous-variété analytique. Dans [AC08, Theorem 4.1], Amérik et Campana montrent que ces sous-variétés peuvent être vues comme les fibres d'une application méromorphe dominante :

Théorème 5.8 (Amerik, Campana)**.** *Soit M une variété compacte kählerienne et $f\colon M \dashrightarrow M$ un endomorphisme méromorphe dominant. Alors il existe une application méromorphe dominante*

$g\colon X \dashrightarrow B$ sur une variété compacte kählerienne B telle que la fibre X_b au-dessus d'un point très général $b \in B$ est l'adhérence de Zariski de l'orbite d'un point très général de X_b.

Ici, "très général" signifie "en dehors d'une réunion dénombrable de sous-variétés analytiques de codimension positive".

En particulier, en combinant ce résultat avec le Théorème A on obtient :

Corollaire 0.5. *Si $f\colon X \dashrightarrow X$ est une application birationnelle loxodromique d'une variété symplectique holomorphe irréductible, l'orbite d'un point très général est Zariski-dense.*

En fait, dans la preuve du Théorème A je montre le Corollaire 0.5 comme lemme.

0.3.3 Éléments de la preuve du Théorème A

Les techniques de la preuve sont inspirées principalement par [AC08].

On procède par l'absurde en supposant qu'une transformation birationnelle loxodromique $f\colon X \dashrightarrow X$ d'une variété symplectique holomorphe irréductible préserve une fibration méromorphe non-triviale $\pi\colon X \dashrightarrow B$. Dans ce cas, on montre facilement que B et X sont des variétés projectives.

Le lemme clé est que la fibre F au-dessus d'un point général de B est une variété de type général (i.e. telle que $\kappa(F) = \dim F$). Une modification mineure des preuves dans [AC08] permet de se ramener à montrer que la restriction de la forme de Beauville-Bogomolov q_X à $\pi^* NS(B)$ (où $NS(B)$ dénote le groupe de Néron-Severi de B) n'est pas identiquement nulle ; ceci dcoule des informations sur l'action de f en cohomologie, à savoir le fait que f ne préserve aucune droite q_X-isotrope définie sur $\mathbb{Z}$.

Une fois montré que la fibre au-dessus d'un point général de B est de type général, on résout le lieu d'indétermination de π (on notera encore X le nouveau modèle birationnel de X) et on considère la fibration de Iitaka relative (voir [Tsu10, Uen75])

$$\Phi\colon X \dashrightarrow Y;$$

informellement, cela permet de mettre les fibrations d'Iitaka des fibres de π en famille. Ici, comme les fibres générales de π sont de type général, Φ est birationnelle sur son image.
À l'aide de cet outil technique on arrive à montrer que les fibres au-dessus d'un ouvert de Zariski dense de B sont birationnellement équivalentes ; comme le groupe de transformations birationnelles d'une variété de type général est fini, l'identification birationnelle entre fibres proche est unique, et ceci permet de définir des "multi-sections méromorphes" de π, qui sont f-invariantes. Ceci contredit la Zariski-densité des orbites, et conclut la preuve.

0.4 Transformations birationnelles préservant une fibration

Soit M une variété projective lisse et soit $f\colon M \dashrightarrow M$ une transformation birationnelle. On suppose ici que f n'est pas primitive, c'est-à-dire qu'il existe une fibration méromorphe non-triviale $\pi\colon M \dashrightarrow B$ et une transformation birationnelle $g\colon B \dashrightarrow B$ telles que $\pi \circ f = g \circ \pi$. On a vu que c'est le cas pour f un automorphisme (resp. une transformation birationnelle) parabolique d'une surface (resp. d'une variété symplectique holomorphe irréductible de type $K3^{[n]}$ ou Kummer généralisée).
La dynamique de f est alors à priori plus simple que dans le cas primitif ; pour la comprendre il faut d'abord étudier la dynamique de g, et ensuite l'action de f sur les fibres de g.

Le théorème suivant a été motivé par le Théorème 2.15 dans le cas de X variété hyperkählerienne ; il se trouve que la preuve fonctionne avec des hypothèses plus faibles.

Théorème B. *Soit X une variété projective lisse dont le fibré canonique est trivial ou effectif, et soit $f: X \dashrightarrow X$ une transformation birationnelle. S'il existe une fibration méromorphe $\pi: X \dashrightarrow \mathbb{P}^n$ et un automorphisme $g \in \mathrm{PGL}_{n+1}(\mathbb{C})$ tels que $\pi \circ f = g \circ \pi$, alors g est d'ordre fini.*

On peut alors combiner les Théorèmes 2.15, A et B pour caractériser les transformations birationnelles avec des orbites Zariski-denses :

Corollaire 6.1. *Soit X une variété symplectique holomorphe irréductible projective de type $K3^{[n]}$ ou Kummer généralisée. Une transformation birationnelle $f: X \dashrightarrow X$ admet des orbites Zariski-denses si et seulement si $\lambda_1(f) > 1$.*

Les arguments de la preuve peuvent être facilement réadaptés pour affaiblir les hypothèse du Théorème B : il suffit de supposer que la dimension de Kodaira de X est non-négative et que $f: X \dashrightarrow X$ est une transformation birationnelle préservant une fibration méromorphe $\pi: X \dashrightarrow B$ telle que l'action $g: B \dashrightarrow B$ sur la base est un pseudo-automorphisme préservant un fibré en droites grand (big en anglais) ; voir la Remarque 6.2.

0.4.1 Éléments de la preuve

L'hypothèse sur l'effectivité du fibré canonique permet de définir une forme volume $\omega = \Omega \wedge \overline{\Omega}$ (où Ω est une section canonique) f-invariante sur X ; par push-forward on peut alors trouver une mesure sur B qui est g-invariante. En considérant une forme de Jordan de g, on montre facilement à l'aide de la mesure invariante qu'il existe des coordonnées homogènes $Y_o, \ldots Y_n$ de $\mathbb{P}^n$ telles que

$$g[Y_0 : Y_1 : \ldots : Y_n] = [Y_0 : \alpha_1 Y_1 : \ldots : \alpha_n Y_n],$$

où les α_i sont de module 1.

L'idée de la preuve est de changer de corps de base pour pouvoir réappliquer le même argument de mesure invariante, en s'inspirant de la stratégie adoptée par Tits pour montrer l'alternative de Tits dans le cas des groupes linéaires (voir [Tit72]). Pour cela, on définit d'abord le "corps des coefficients" k : une extension finiment engendrée (mais peut-être transcendante) de $\mathbb{Q}$ qui contient tous les coefficients des équations définissant X, f, π, ω et g.

Le lemme fondamental est le suivant :

Lemme 6.18 (Tits). *Soit k une extension de type fini de $\mathbb{Q}$ et soit $\alpha \in k$ un élément qui n'est pas une racine de l'unité. Alors il existe un corps local K et un plongement $\rho: k \hookrightarrow K$ tels que $|\rho(\alpha)|_K \neq 1$.*

On rappelle qu'un *corps local* est un corps muni d'une norme multiplicative telle que la topologie induite est localement compacte. Les corps locaux en caractéristique 0 sont $\mathbb{R}$, $\mathbb{C}$ et les *corps p-adiques* (i.e. les extensions finies de $\mathbb{Q}_p$).

Supposons que l'un des α_i, disons α_1, ne soit pas une racine de l'unité, et appliquons le Lemme 6.18 à k et α_1 : il existe un plongement $\rho: k \hookrightarrow K$ dans un corps local K tel que $|\rho(\alpha_1)| \neq 1$. En appliquant ρ à toutes les équations concernées (i.e. en changeant de corps de base au sens de la géométrie algébrique) on retrouve une application birationnelle $f^\rho: X^\rho \dashrightarrow X^\rho$, préservant une fibration $\pi^\rho: X^\rho \dashrightarrow \mathbb{P}^n$ et une forme volume ω^ρ ; l'action induite par f^ρ sur $\mathbb{P}^n$ est donnée par $g^\rho[Y_0 : Y_1 : \ldots : Y_n] = [Y_0 : \alpha_1^\rho Y_1 : \ldots : \alpha_n^\rho Y_n]$. Or, si $K = \mathbb{R}$ ou $\mathbb{C}$, le même argument de théorie de la mesure donne une contradiction ; l'existence d'une théorie de l'intégration p-adique permet de conclure de la même façon dans le cas des corps p-adiques.

0.4.2 Fibrations sur les variétés de Calabi-Yau de dimension 3

Les Théorèmes 1.30 et 2.15 motivent la conjecture suivante (très optimiste).

Question 0.2. *Soit f un automorphisme d'une variété projective (ou compacte kählerienne) M dont l'action sur $H^2(M, \mathbb{Z})$ est d'ordre infini et tel que $\lambda_1(f) = 1$. Est-ce que f préserve une fibration (méromorphe) non-triviale $\pi\colon M \dashrightarrow B$?*

Le premier cas à tester est celui de la dimension 3, et parmi les variétés de dimension 3, l'une des classes les mieux étudiées dans cette direction est celle des variétés de Calabi-Yau.

D'un côté Oguiso a classifié les fibrations holomorphes sur ces variétés dans [Ogu93] : si X est une variété de Calabi-Yau de dimension 3 et $\pi\colon X \to B$ est une fibration holomorphe, alors

1. ou bien $B = \mathbb{P}^1$ et les fibres générales sont des surfaces $K3$;
2. ou bien $B = \mathbb{P}^1$ et les fibres générales sont des surfaces abéliennes ;
3. ou bien B est une surface et les fibres générales sont des courbes elliptiques.

De l'autre côté, les travaux de Wilson [Wil89, Wil98] permettent de construire des fibrations elliptiques sur une variété de Calabi-Yau à partir d'un diviseur D nef satisfaisant certaines propriétés numériques.

J'ai essayé d'analyser le cas où f préserve une fibration holomorphe. On a deux résultats très partiels dans cette direction :

- si f préserve une fibration, alors f préserve une fibration en tores de dimension 1 ou 2 ;
- par contre, il existe effectivement des exemples d'autormorphismes qui préservent une fibration en tores de dimension 2 mais aucune fibration en tores de dimension 1.

0.5 Autres structures invariantes

Soit X une variété symplectique irréductible et $f\colon X \dashrightarrow X$ une transformation loxodromique. Le Théorème A assure que f ne peut pas préserver de fibration méromorphe ; cependant, on voit dans les exemples que d'autres structures moins rigides peuvent être préservées.

Dans ce qui suit on verra quelles limitations existent sur les possibles structures préservées, et on posera les bases pour une étude systématique de telles structures.

0.5.1 Paires de feuilletages préservés

Comme on l'a dit, la notion de feuilletage généralise celle de fibration ; on peut alors se demander si une transformation loxodromique $f\colon X \dashrightarrow X$ d'une variété hyperkählerienne peut préserver un feuilletage. Un sous-espace tangent à X $V \subset T_pX$ est dit *isotrope* (resp. Lagrangien) si $\sigma_p|_V = 0$ (resp. si $\dim V = \dim X/2$ et V est isotrope).
Rappelons que, dans les deux exemples fondamentaux (schéma des points de Hilbert d'une surface de Kummer et variété de Kummer généralisée), on construit deux feuilletages de dimension $\dim X/2$, génériquement transverses et Lagrangiens. Le théorème suivant montre que c'est la seule situation possible pour deux feuilletages (ou même distributions de sous-espace du tangent) f-invariants.

Théorème C. *Soit $f\colon X \dashrightarrow X$ une transformation birationnelle loxodromique d'une variété symplectique holomorphe irréductible projective de dimension $2n$. On suppose que f préserve deux distributions génériquement transverses non-triviales $\mathcal{F}_1, \mathcal{F}_2$ (i.e. en un point général $p \in X$, $T_pX = T_p\mathcal{F}_1 \oplus T_p\mathcal{F}_2$) ; alors $\mathcal{F}_1$ et $\mathcal{F}_2$ sont des distributions de dimension n Lagrangiennes.*

L'idée de la preuve est assez simple : on remarque d'abord qu'il suffit de montrer que les deux feuilletages sont isotropes. Supposons par l'absurde que ce n'est pas le cas, par exemple que $\mathcal{F}_1$ n'est pas isotrope ; à partir d'une forme symplectique σ, on construit par projection une nouvelle 2-forme :

$$\sigma'_p(v,w) := \sigma_p(\pi_1(v), \pi_1(w)),$$

où σ est une forme symplectique et $\pi_1 \colon T_pX \to T_p\mathcal{F}_1$ est la projection par rapport à la décomposition $T_pX = T_p\mathcal{F}_1 \oplus T_p\mathcal{F}_2$. A priori la forme σ' est définie seulement sur l'ouvert de Zariski $U = X \setminus E \subset X$ où $\mathcal{F}_1$ et $\mathcal{F}_2$ sont transverses.
La forme $\sigma' \in H^0(U, \Omega^2_U)$ n'est pas symplectique car $(\sigma')^n = 0$, et elle est non-triviale car $\mathcal{F}_1$ n'est pas isotrope. Il ne reste qu'à montrer que σ' s'étend en une forme sur X tout entier pour trouver une contradiction avec l'irréductibilité de X.
Soit

$$E = E_1 \cup \ldots \cup E_k$$

la décomposition du diviseur $E = X \setminus U$. Par le principe d'Hartogs, il suffit de montrer que σ' s'étend aux points généraux de chaque E_i.

Contraction du lieu exceptionnel

Fixons une composante irréductible E_i de E. Le premier pas pour étendre σ' est de contracter E_i sur une sous-variété de codimension supérieure.

Lemme 7.3. *Soit $D \subset X$ un diviseur irréductible préservé par une transformation birationnelle loxodromique $f \colon X \dashrightarrow X$ d'une variété symplectique irréductible projective. Alors il existe une variété symplectique irréductible projective (lisse) X', une application birationnelle $\phi \colon X \dashrightarrow X'$ (qui est donc un automorphisme en codimension 1) et un morphisme birationnel $\pi \colon X' \to Y$ sur une variété projective normale Y tel que le lieu exceptionnel de π est la transformée stricte $\phi_* D$.*

Ce lemme suit facilement d'un résultat de Druel [Dru11, Proposition 1.4] et des informations sur l'action de f sur $H^{1,1}(X, \mathbb{R})$. La preuve de Druel est basée sur le programme des modèles minimaux : si D est un diviseur négatif (au sens de la décomposition de Zariski divisorielle, voir [Bou04]), alors on peut changer de modèle birationnel et le contracter.
Par [Bou04, Theorem 4.5, Proposition 3.11], D est négatif si et seulement si $q_X(D) < 0$, où q_X dénote la forme de Beauville-Bogomolov sur $H^{1,1}(X, \mathbb{R})$; dans notre cas, $q_X(D) < 0$ suit de la structure des transformations loxodromiques.

Singularités symplectiques

Pour ce paragraphe on renvoie au survey [Fu06]. Quitte à remplacer f par un itéré, on peut supposer que toutes les composantes de E sont préservées par f ; on applique alors le Lemme 7.3 à une composante E_i de E, et on obtient une contraction de (la transformée stricte de) E_i sur un modèle birationnel X' de X :

$$\pi \colon X' \to Y.$$

Les singularités de Y sont particulièrement gentilles :

Définition 7.8. *Une variété normale Y est dite symplectique singulière s'il existe une 2-forme symplectique σ sur la partie lisse de Y telle que, si $\pi \colon Z \to Y$ est une résolution des singularités, $\pi^*\sigma$ s'étend en une 2-forme sur Z.*

Les singularités symplectiques des surfaces ne sont rien d'autre que les singularités rationnelles (ou "du Val") : localement, elles sont biholomorphes à $\mathbb{C}^2/G$ où G est un sous-groupe fini de $SL_2(\mathbb{C})$.

Or, le π construit au Lemme 7.3 est ce qui s'appelle une *résolution symplectique* : le pull-back de la forme symplectique sur la partie lisse de Y s'étend en une forme symplectique de X' (a priori l'extension pourrait être dégénérée le long du diviseur exceptionnel). Dans ce cadre, Kaledin a montré dans [Kal06] que π est *semi-petite* : pour toute sous-variété $F \subset X'$, on a $2 \cdot \operatorname{codim} F \geq \operatorname{codim} \pi(F)$. En particulier, $\operatorname{codim}(\pi(\phi_* E_i)) = 2$.

Les singularités symplectiques sont canoniques; donc, on peut appliquer le résultat suivant (voir [Rei80, Corollary 1.14]).

Proposition 7.11 (Reid)**.** *Soit Y une variété projective normale de dimension d dont les singularités sont canoniques, et soit Z une composante de codimension 2 dans SingY. Alors au voisinage d'un point général de Z, Y est biholomorphe à*

$$\mathbb{C}^{d-2} \times {}^{\mathbb{C}^2}\!\diagup_{G},$$

où $G \subset \mathrm{SL}_2(\mathbb{C})$ est un groupe fini.

Ceci permet de montrer que toute forme holomorphe définie en dehors de E_i s'étend au point général de E_i, donc, par Hartogs, à E_i tout entier.

La preuve du Théorème C est alors complète : on a montré que, pour toute composante E_i de E, σ' s'étend à E_i, donc σ' définit une forme globale sur X, ce qui contredit l'irréductibilité de X.

Le résultat d'extension des formes permet aussi de donner des contraintes aux feuilletages préservés par une transformation loxodromique.

0.5.2 (G, X)-structures

Le concept de "structure géométrique" est très général, et il existe plusieurs façon de le rendre rigoureux. Un premier exemple est celui des (G, X)-structures.

Définition 9.3. *Soit G un groupe agissant sur une variété différentielle X ; on suppose que l'action de G est analytique, i.e. que si deux éléments de G coincident sur un ouvert non vide, alors ils coincident.*
Une (G, X)-structure sur une variété différentielle M est la donné d'un atlas sur M à valeurs dans X tel que les changements de carte soient des restrictions d'éléments de G.

On peut penser par exemple aux métriques à courbure sectionnelle constante (structure euclidienne, sphérique ou hyperbolique), et aux structures affines ou projectives. En remplaçant "différentielle" par "holomorphe" on obtient la définition de (G, X)-structure holomorphe.

Si M est une (G, X)-variété et $\widetilde{M}$ est son revêtement universel, à l'aide de l'atlas à valeurs dans X on peut construire une application développante

$$\mathrm{dev}\colon \widetilde{M} \to X$$

et une représentation d'holonomie

$$\mathrm{hol}\colon \pi_1(M) \to G$$

qui satisfont $\mathrm{hol}(\gamma)(\mathrm{dev}(m)) = \mathrm{dev}(\gamma \cdot m)$ pour tout $\gamma \in \pi_1(M)$, $m \in M$.

Remarque 0.6. Dans le cadre des structures holomorphes, on remarque que, si X est une variété complexe affine et M est une variété complexe compacte simplement connexe, alors M n'admet pas de (G, X)-structure. En effet, si c'était le cas, l'application développante serait constante par principe du maximum, ce qui est absurde.
En particulier, une structure affine sur une variété symplectique irréductible doit dégénérer le long d'un diviseur.

(G, X)-orbifolds

Un *orbifold* est une variété singulière qui est localement modelée sur des voisinage de l'origine dans $\mathbb{R}^n/\Gamma$, où $\Gamma \subset \mathrm{SL}_n(\mathbb{R})$ est un groupe fini ; de façon analogue, un orbifold complexe est une variété complexe singulière qui est localement modelée sur des voisinages de l'origine dans $\mathbb{C}^n/\Gamma$, où $\Gamma \subset \mathrm{SL}_n(\mathbb{C})$ est un groupe fini.
Comme on l'a vu dans la Proposition 7.11, les singularités symplectiques de codimension 2 sont des singularités orbifoldes au point général.

Un (G, X)-orbifold est un orbifold muni d'un atlas à valeurs dans X/Γ, où $\Gamma \subset G$ est un sous-groupe fini. On peut généraliser la notion d'application développante et de représentation d'holonomie au cadre des (G, X)-orbifolds.

Une application : structures affines sur les surfaces $K3$

Soit X une variété symplectique holomorphe projective, et supposons qu'une transformation birationnelle loxodromique $f\colon X \dashrightarrow X$ préserve une $(G, \mathcal{X})$-structure holomorphe sur X, où $\mathcal{X}$ est une variété affine. Comme on a vu, la structure doit forcément dégénérer le long d'un diviseur E, qui est donc f-invariant.
Quitte à changer de modèle birationnel, on peut alors contracter E sur une variété symplectique singulière Y ; au point général de $\mathrm{Sing Y}$, Y est naturellement munie d'une structure de $(G, \mathcal{X})$-orbifold. Dans le cas des surfaces, $\mathrm{Sing Y}$ est une union finie de points isolés, ce qui permet de montrer l'énoncé suivant :

Théorème 9.11. *Soit $f\colon X \to X$ un automorphisme loxodromique d'une surface $K3$ et supposons que f préserve une structure affine définie sur un ouvert de Zariski $U \subset X$ non vide. Alors $X = K(T)$ est la surface de Kummer associée à un tore complexe T de dimension 2, et f est construit à partir d'un automorphisme loxodromique linéaire de T.*

Ceci complète la classification des structures géométriques naturelles invariantes sur les surfaces $K3$.

0.5.3 Structures géométriques rigides au sens de Gromov

Une possible généralisation des (G, X)-structures vient des structures géométriques au sens de Gromov. Pour donner une définition précise, on notera $R^r(M) \to M$ le fibré des r-jets de cartes de la variété M ; il s'agit d'un fibré principal dont le groupe de structure $D^r(\mathbb{R}^n)$ (n étant la dimension de M) est le groupe des r-jets de difféomorphismes de $\mathbb{R}^n$ fixant l'origine ; voir [Ehr53, Ehr54, Ben97, KMS93].
La définition suivante coïncide avec le concept de A-structure introduit par Gromov (voir [Gro88, DG91, Ben97]).

Définition 9.14. *Soit Z une variété quasi-projective sur $\mathbb{R}$ munie d'une action algébrique de $D^r(\mathbb{R}^n)$; une* structure géométrique *de type Z sur la variété M est une application lisse équivariante $\phi\colon R^r(M) \to Z$, i.e. $\phi(s \cdot g) = g^{-1} \cdot \phi(s)$, $\forall s \in R^r(M)$ et $\forall g \in D^r(\mathbb{R}^n)$.
On note $Is^{loc}(\phi)$ le pseudo-groupe des isométries locales de ϕ, c'est-à-dire les germes de difféomorphismes entre voisinages de deux points de M préservant la structure.*

Par exemple, pour définir un champ de vecteurs on prend $r = 1$ et $Z = \mathbb{R}^n$ avec l'action linéaire de $D^1(\mathbb{R}^n) = \mathrm{GL}_n(\mathbb{R})$. D'autres exemples sont les formes différentielles, les feuilletages, les métriques riemanniennes (ou holomorphes), les connexions, les structures projectives...
On définit de manière analogue une structure géométrique holomorphe ou méromorphe.

Par rapport aux (G, X)-structures, les structures géométriques au sens de Gromov permettent de se débarasser de toute hypothèse de symétrie à priori de la structure ; en revanche, on perd l'application développante, qui est susceptible de fournir des informations sur la variété de départ, et la possibilité d'étendre la structure aux singularités orbifoldes (qui, dans le cadre symplectique irréductible, apparaissent après avoir contracté le diviseur exceptionnel).
Dans certains cas, il est possible de montrer qu'une structure au sens de Gromov définit en fait une (G, X)-structure.

Rigidité et théorème de l'orbite ouverte-dense de Gromov

Une structure géométrique est dite *rigide* s'il existe un entier $r_0 \geq 0$ tel que, pour tout $r \geq r_0$, les r-jets de difféomorphismes qui préservent la structure sont déterminés par leur r_0-jet ; en particulier, si ϕ est rigide, ses isométries locales sont déterminées par un jet fini. Par exemple, l'existence de coordonnées exponentielles pour une métrique (pseudo)riemannienne (resp. holomorphe) montre qu'elle définit une structure rigide (les isométries locales sont déterminées par leur différentielle) ; par contre, un feuilletage lisee (resp. holomorphe) définit une structure non-rigide.

Le théorème suivant est un analogue méromorphe du Théorème de l'orbite ouverte-dense de Gromov [Gro88, Theorem 3.3] :

Théorème 9.21 (Dumitrescu [Dum11]). *Soit M une variété complexe connexe et soit ϕ une structure géométrique méromorphe presque-rigide (i.e. rigide en dehors d'une sous-variété stricte). Alors il existe un sous-ensemble analytique nulle part dense $S \subset M$ tel que $M \setminus S$ est $Is^{loc}(\phi)$-invariant et les $Is^{loc}(\phi)$-orbites de $M \setminus S$ sont les fibres d'une fibration holomorphe $\pi \colon M \setminus S \to B$.*
En particulier, si $Is^{loc}(\phi)$ admet une orbite Zariski-dense, alors ϕ est localement homogène sur un ouvert de Zariski non-vide.

Comme les transformations loxodromiques des variétés symplectiques irréductibles possèdent des orbites denses, on obtient le corollaire suivant.

Corollaire 9.22. *Soit $f \colon X \dashrightarrow X$ une transformation birationnelle loxodromique d'une variété symplectique irréductible. Si f préserve une structure géométrique méromorphe rigide ϕ, alors ϕ est localement homogène sur un ouvert de Zariski dense $U \subset X$.*

Application : deux feuilletages Lagrangiens préservés

Voyons comment appliquer ces notions à l'étude des transformations loxodromiques d'une variété symplectique holomorphe X préservant deux feuilletages (ou distributions) Lagrangiens génériquement transverse $\mathcal{F}_1, \mathcal{F}_2$.

D'abord, la donnée de $\mathcal{F}_1$ et $\mathcal{F}_2$ permet de définir une métrique méromorphe par

$$q(v) := \sigma(v_1, v_2), \qquad v \in T_pX$$

où $v = v_1 + v_2$ est la décomposition de v le par rapport à la décomposition $T_pX = T_p\mathcal{F}_1 \oplus T_p\mathcal{F}_2$ (aux points p où $\mathcal{F}_1$ et $\mathcal{F}_2$ sont transverses). Comme $f^*\sigma = \xi\sigma$ pour un $\xi \in \mathbb{C}^*$, f préserve la métrique q à un facteur multiplicatif près ; autrement dit, f agit par isométrie de la structure conforme (holomorphe) donnée par q. Or, les structures conformes sont rigides, ce qui permet de montrer le résultat suivant.

Théorème 9.1. *Soit X une variété symplectique irréductible, avec forme symplectique σ, et soient $\mathcal{F}_1, \mathcal{F}_2$ deux distributions Lagrangiennes génériquement transverses. Soit $f \colon X \dashrightarrow X$ une transformation loxodromique préservant $\mathcal{F}_1$ et $\mathcal{F}_2$.*

Alors la structure donnée par $\mathcal{F}_1, \mathcal{F}_2$ et σ modulo un facteur multiplicatif est rigide et localement homogène sur un ouvert de Zariski dense de X.

Si on suppose que X soit projective, on peut montrer qu'un itéré de f préserve σ ; dans la proposition précédente on peut alors enlever la partie "modulo un facteur multiplicatif".

Recherche de modèles locaux

Le Théorème 9.1 suggère que la structure donnée par $\mathcal{F}_1, \mathcal{F}_2$ et σ modulo $\mathbb{C}^*$ pourrait en fait être décrite comme $(G, \mathcal{X})$-structure. On verra qu'il existe en effet un candidat naturel pour le modèle $(G, \mathcal{X})$, mais qu'il reste quelque chose à vérifier pour obtenir une $(G, \mathcal{X})$-structure.

Pour une structure rigide ϕ sur une variété M et $p \in M$, on dénote $\mathcal{G}_p$ l'algèbre de Lie des germes de champs de Killing pour ϕ ; autrement dit, un champs de vecteurs local v autour de p est dans $\mathcal{G}_p$ si et seulement si les flots le long de v agissent par isométries locales de ϕ. Comme la structure est rigide, $\mathcal{G}_p$ est de dimension finie : en effet, un champ de Killing est déterminé par un jet fini.
On note $\mathcal{I}_p \subset \mathcal{G}_p$ le sous-espace des champs de Killing s'annulant en p, et par G_p (resp. I_p) l'unique groupe de Lie connexe et simplement connexe ayant pour algèbre de Lie $\mathcal{G}_p$ (resp. $\mathcal{I}_p$).

Supposons maintenant que ϕ soit localement homogène. Alors $G_p = G$ et $I_p = I$ sont indépendants du point choisi, et le candidat naturel de modèle local pour la structure ϕ est la variété homogène G/I avec l'action de G.

Il reste à montrer que I est un sous-groupe de Lie *fermé* de G ; ceci est toujours vrai en dimension ≤ 4 (mais pas en général). On obtient donc :

Corollaire 0.7. *Soit X une variété symplectique holomorphe de dimension 2 ou 4 et soit $f\colon X \dashrightarrow X$ une transformation birationnelle loxodromique.*
Alors toute structure géométrique méromorphe rigide préservée par f est décrite par une certaine $(G, \mathcal{X})$-structure sur un ouvert de Zariski dense $U \subset X$.

Ces énoncés ne permettent donc pas pour l'instant de classer toutes les structures de Gromov rigides invariantes par une transformation birationnelle loxodromique, mais ils contraignent fortement les structures possibles.
La stratégie pour attaquer le problème est maintenant en place : il faut d'abord décrire les possibles modèles $(G/I, G)$; la partie divisorielle du diviseur exceptionnel $E = X \setminus U$ peut être contractée à l'aide du Lemme 7.3 ; dans les cas où la variété modèle G/I est affine, ceci permet d'appliquer la théorie des $(G, \mathcal{X})$-orbifolds comme on l'a fait pour les structures affines sur les surfaces $K3$.
La conjecture suivante semble accessible.

Conjecture 0.3. *Soit X une variété symplectique holomorphe irréductible projective de dimension 4, et soit $f\colon X \dashrightarrow X$ une transformation birationnelle loxodromique. Si f préserve une paire de feuilletage Lagrangiens génériquement transverses ou une structure affine (définie sur un ouvert de Zariski dense), alors (X, f) est de type Kummer (voir l'Exemple 0.2).*

Part I

Preliminaries

In Part I I set the fundamental notions which will be used in the rest of the thesis, and I give a survey of the known results.

In Chapter 1 I talk about dynamical degrees and topological entropy, holomorphic (singular) foliations and the dynamics of automorphisms of surfaces.
The part about entropy just is a motivation to introduce the dynamical degrees, but is not necessary for what follows; on the contrary, dynamical degrees (and their relative counterpart) will play a central role throughout the rest of the thesis.
Then I introduce holomorphic foliations; the material of this section is quite classical, except maybe the description of foliations in terms of pluri-forms (§1.2.2), which plays a (minor) role in Chapter 4.
Finally, the dynamics of automorphisms of surfaces is explicitly used only in Chapter 8 and §9.1.3; however, because of the close relationship between surfaces and irreducible symplectic manifolds, the results on the first are natural candidates to be extended to the latter.

In Chapter 2 I introduce the main object of my research: irreducible symplectic manifolds.
I first introduce the definitions and classical results: the Beauville-Bogomolov decomposition theorem, the Beauville-Bogomolov quadratic form (which establishes a link with surfaces), and the known examples.
Then, I move on to a more dynamical point of view, classifying the birational transformations of such a manifold as elliptic, parabolic or loxodromic as in the surface case. Finally, I state some relatively recent results about the non-primitivity of parabolic transformations and the computation of the dynamical degrees of an automorphism.

In Chapter 3 I describe in detail two fundamental examples of loxodromic automorphisms on irreducible symplectic manifolds.
In the first two sections I recall some basic results about the construction of Kummer surfaces and the Hilbert scheme of n points on a surface.
Then I move on to the two examples: the Hilbert scheme of n points on a Kummer surface, and the generalized Kummer variety of a two-dimensional torus. In both cases, loxodromic linear automorphisms of the torus induce loxodromic automorphisms of the irreducible symplectic manifold. Furthermore, the stable and unstable foliations on the torus induce invariant foliations on the symplectic manifolds, which will be analyzed in detail.
As for the part about automorphisms of surfaces, this chapter is not essential to understand the rest of the thesis, although it provides a good understanding of the situation in two concrete cases.

Finally, in Chapter 4 I prove the Zariski-density of hyperbolic periodic points of a loxodromic automorphism of an irreducible symplectic manifold.
The first step of the proof is to understand the action in cohomology of such an automorphism f: a recent result of Bogomolov, Kamenova, Lu and Verbitsky, of which we give a detailed proof, gives the complete description of the (moduli of) eigenvalues of f^*.
Then one can apply recent results in dynamical systems to construct an invariant probability measure μ, which is ergodic and doesn't charge positive codimensional subvarieties; to conclude, one shows that μ is hyperbolic (i.e. the Lyapounov exponents are all different from zero), and conclude by a result of Katok that the hyperbolic periodic points are Zariski-dense in the support of μ, hence in the manifold.
This result is not needed in the following chapters.

Chapter 1

Dynamics of endomorphisms of compact complex manifolds

1.1 Dynamical degrees and topological entropy

1.1.1 Topological and measure-theoretic entropy

For this section we refer to [HK02, §I.2.5 and §I.3.7]. Let M be a compact topological space, and let $f\colon M \to M$ be a continuous map.

The topological entropy of f is a non-negative number, possibly infinite, which gives a measure of the chaos created by f and its iterates. More accurately, it measures the exponential growth of orbit segments $p, f(p), \ldots, f^N(p)$ which can be distinguished with arbitrarily fine but finite precision.

A strictly related concept is the one of measure-theoretic entropy, which depends on the choice of an f-invariant measure μ on M. Intuitively, the entropy of f with respect to a probability measure μ measures how much the dynamics of f are deterministic; in other words, how much the knowledge of the past trajectory of a point (with arbitrarily fine but finite precision) determines its future trajectory.

One key difference between the two definitions is the behavior with respect to the disjoint union of two dynamical systems: the topological entropy is then the maximum of the topological entropies of the two systems, whereas the measure-theoretic entropy is the average of the two measure-theoretic entropies, weighted by their measures (see [HK02, §I.4.4]. Therefore, topological entropy measures the maximal dynamical complexity, whereas measure-theoretic entropy gives the average dynamical complexity.

Topological entropy: original definition

Let us first describe the original definition of topological entropy, due to Adler, Konheim, and McAndrew; see[AKM65].

If $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{V} = \{V_j\}_{j\in J}$ are covers of M by open sets, we will denote by

$$\mathcal{U} \vee \mathcal{V} = \{U_i \cap V_j\}_{i\in I, j\in J}$$

their minimal common refinement.
Furthermore, if $f\colon M \to M$ is a continuous map, denote by

$$f^{-1}\mathcal{U} = \{f^{-1}(U_i)\}_{i\in I}$$

the pull-back of the cover $\mathcal{U}$.

The topological entropy $H(\mathcal{U})$ of an open cover $\mathcal{U}$ is the logarithm of the minimal cardinality of a sub-cover of $\mathcal{U}$; it is a finite number because M is compact.
By using the fact that $H(\mathcal{U} \vee \mathcal{V}) \leq H(\mathcal{U}) + H(\mathcal{V})$, one shows that the following limit exists for every open cover $\mathcal{U}$ of M:

$$h(f,\mathcal{U}) = \lim_{n\to+\infty} \frac{1}{n} H\left(\mathcal{U} \vee f^{-1}\mathcal{U} \vee \ldots \vee f^{-(n-1)}\mathcal{U}\right)$$

Definition 1.1. *The* topological entropy *of f is*

$$h_{top}(f) := \sup_{\mathcal{U}} h(f,\mathcal{U}),$$

where the supremum is taken among all open covers of M.

This definition requires very little assumptions about the dynamical system, but since one needs to consider all possible open covers at once it is often very complicated to use it to compute the topological entropy, even for simple dynamical systems.

Topological entropy: metric definition

In the case where M is a compact metrizable space, Bowen and Dinaburg proposed an equivalent definition of topological entropy, which is much easier to compute in practice; see [Bow71]. Let us fix a metric d on M which induces the topology of M.

Define

$$d_k(p,q) = \max_{i=0,\ldots,k-1} d(f^i(p), f^i(q)),$$

measuring the distance between two orbit segments of length k; for $\epsilon > 0$ and $k \in \mathbb{N}$, let

$$N(f,\epsilon,k) = \max\{N \mid \exists p_1,\ldots,p_N \in M,\ d_k(p_i,p_j) > \epsilon \text{ for } i,j = 1,\ldots,N\}$$

be the maximal number of k-segments of orbits which can be distinguished at the scale ϵ.

Then, the topological entropy of f is equal to the limit

$$\lim_{\epsilon\to 0^+} \limsup_{k\to+\infty} \frac{1}{k} \log N(f,\epsilon,k).$$

In particular, the limit is independent on the chosen metric.

Example 1.2. Let us see some concrete computations of topological entropy.

- Let $T = \mathbb{R}^N/\mathbb{Z}^N$ be the N-dimensional real torus and let $f\colon t \mapsto t+\alpha$ be a translation. The topology on the torus is induced by the euclidean distance; since such distance is preserved by f, the number of ϵ-separated k-orbits does not depend on k. Therefore, $h_{top}(f) = 0$.
- Let $A = \{1,\ldots,m\}$ be a finite alphabet endowed with the discrete topology, $M = A^{\mathbb{N}}$ be the set of infinite words with letters in A with the product topology, and let

  $$\begin{aligned} f\colon M &\to M \\ (a_i)_{i\in\mathbb{N}} &\mapsto (a'_i = a_{i+1})_{i\in\mathbb{N}} \end{aligned}$$

  be the (one-sided) shift map. The topology on M is induced by the distance

  $$d(\mathbf{a},\mathbf{b}) = m^{-n} \quad \text{where } n = \min\{i \in \mathbb{N} \mid a_i \neq b_i\}.$$

Picking $\epsilon = m^{-n}$, $n \in \mathbb{N}$, two k-segments of orbits $\mathbf{a}, f(\mathbf{a}), \ldots, f^{k-1}(\mathbf{a})$ and $\mathbf{b}, f(\mathbf{b}), \ldots, f^{k-1}(\mathbf{b})$ are ϵ-separated if and only if $b_i \neq a_i$ for some $0 \leq i \leq n+k-1$; hence

$$N(f, m^{-n}, k) = m^{n+k},$$

and

$$h_{top}(f) = \lim_{n\to+\infty} \limsup_{k\to+\infty} \frac{1}{k} \log m^{n+k} = \log m.$$

- Let $M = S^1 = \mathbb{R}/\mathbb{Z}$ be the circle and let $f\colon \alpha \mapsto d\cdot\alpha$ be the multiplication by d. If

$$\alpha = 0, a_1 a_2 \ldots \qquad a_i \in \{0, 1, \ldots, d-1\}$$

is the representation of $\alpha \in [0,1)$ in base d (with the convention that infinite sequences of digits $d-1$ should not appear), then the representation of $d\cdot\alpha$ in base d is

$$d\cdot\alpha = a_1, a_2 a_3 \ldots \equiv 0, a_2 a_3 \ldots \quad mod\ \mathbb{Z}.$$

Therefore $f\colon S^1 \to S^1$ can be interpreted as an invariant subset of the shift on the alphabet $A = \{0, \ldots, d-1\}$; reasoning as in the previous example, one can show that $h_{top}(f) = \log d$.

- Let

$$\begin{aligned} f\colon \mathbb{P}^1(\mathbb{C}) &\to \mathbb{P}^1(\mathbb{C}) \\ z &\mapsto z^d \end{aligned}$$

Then $S^1 = \{z \mid |z| = 1\}$ is an f-invariant set, and all orbits except those of 0 and ∞ tend to the set S^1. Therefore, intuitively, the number of ϵ-separated orbits of f is not significantly different from the number of ϵ-separated orbits of $f|_{S^1}$; this can made be precise, and shows that

$$h_{top}(f) = \log d.$$

- Let $A \in SL_N(\mathbb{Z})$; then A induces an automorphism f of the N-dimensional real torus $\mathbb{R}^N/\mathbb{Z}^N$. One can show that, if $\lambda_1, \ldots, \lambda_N \in \mathbb{C}^*$ are the eigenvalues of A and

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_k| > 1 \geq |\lambda_{k+1}| \geq \ldots \geq |\lambda_N|,$$

then $h_{top}(f) = \log|\lambda_1| + \ldots + \log|\lambda_k|$.
Using dynamical degrees, we will show an analogous result for the automorphism defined by A on the product E^N of N elliptic curves $E = \mathbb{C}/\Lambda$ (see Example 1.11).

Measure-theoretic entropy

Now suppose that f preserves a Borel probability measure μ; one can always find such a measure by considering the sequence of discrete measures

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(p)}$$

for some $p \in M$, and then taking an accumulation point for the weak-* topology on the set of Borel measures; see [KB37].

If $\xi = \{C_i\}_{i \in I}$ is a measurable partition of M, we define the entropy of ξ as

$$H_\mu(\xi) := -\sum_{i \in I} \mu(C) \log \mu(C) \geq 0,$$

with the convention $0 \, \log 0 = 0$.
The entropy of f with respect to the partition ξ is

$$h_\mu(f, \xi) := \lim_{n \to +\infty} \frac{1}{n} H_\mu \left(\xi \vee f^{-1}\xi \vee \ldots \vee f^{-(n-1)}\xi \right),$$

where $\xi \vee \xi'$ denotes the joint partition, i.e. the partition whose elements are the $C \cap C'$, $C \in \xi$, $C' \in \xi'$.

The *Kolmogorov entropy* (sometimes called the Kolmogorov-Sinai entropy) of f is defined as

$$h_\mu(f) := \sup_\xi h_\mu(f, \xi),$$

where the supremum is taken over all the measurable partitions of M satisfying $H_\mu(\xi) < +\infty$. However, in most cases the supremum is actually a maximum, and it is attained by ξ if it is a generator (see [HK02, Corollary 3.7.10]): roughly said, this means that the smallest σ-algebra containing $\xi, f^{-1}\xi, f^{-2}\xi, \ldots$ gives arbitrarily fine approximations of any measurable set.

The following theorem establishes a link between the concepts of topological entropy and measure-theoretic entropy; see [HK02, Theorem 4.4].

Theorem 1.3. *Let $\mathcal{M}(f)$ be the set of f-invariant Borel probability measures on M. Then*

$$h_{top}(f) = \sup_{\mu \in \mathcal{M}(f)} h_\mu(f).$$

We call *measure of maximal entropy* for f an f-preserved probability measure μ such that $h_\mu(f) = h_{top}(f)$.

Ergodicity

Roughly speaking, a dynamical system is ergodic if it is minimal with respect to some invariant probability measure.

Definition 1.4. *Let $f \colon M \to M$ be a continuous map of a topological space, and let μ be an f-invariant Borel probability measure. We say that μ is* ergodic *if every f-invariant measurable set has measure 0 or 1.*

Equivalently, μ is ergodic if every f-invariant measurable function $\phi \colon M \to \mathbb{R}$ is μ-almost everywhere equal to a constant.

Proposition 1.5. *Let M be a compact topological space and let $f \colon M \to M$ be a continuous map. If f admits a unique measure μ of maximal entropy, then μ is ergodic.*

For a proof, see Proposition 4.3.16(ii) in [HK02]. □

1.1.2 The calculus of currents

Before giving the complete definition of dynamical degrees, we need some background on the calculus of current; we refer to the review of Demailly [Dem97].

Let M be a manifold of dimension n; we denote by $\mathcal{D}^p(M)$ the space of smooth differential forms of degree p on M. We define a topology on $\mathcal{D}^p(M)$: let $\Omega \subset X$ be a coordinate open set, so that locally a form $\alpha \in \mathcal{D}^p(X)$ can be written as $\alpha = \sum_{|I|=p} \alpha_I dx_I$. For every $s \in \mathbb{N}$ and for every compact $K \subset \Omega$, we define the seminorm

$$|\alpha|_{\Omega,K,s} = \sup_{x \in K} \max_{|I|=p, |J| \le s} |D^J \alpha_I|,$$

where we denote by D^J the differential operator

$$D^J = \left(\frac{\partial}{\partial x_1}\right)^{J_1} \circ \ldots \circ \left(\frac{\partial}{\partial x_1}\right)^{J_1}.$$

On $\mathcal{D}^p(M)$ we fix the topology defined by the seminorms $|\cdot|_{\Omega,K,s}$ as Ω, K and s vary.

Roughly said, a current is a form with distribution coefficients: if X is a complex manifold of dimension d, a *current* T of bidegree (s,s) is a continuous linear form on $\mathcal{D}^{d-s,d-s}(X)$ (the space of smooth differential forms of bidegree $(d-s,d-s)$ endowed with the restriction of the topology of $\mathcal{D}^{2d-2s}(X)$).

A $(d-s,d-s)$-form α is *weakly positive* if at every point $p \in X$

$$\alpha \wedge i\beta_1 \wedge \overline{\beta_1} \wedge \ldots \wedge i\beta_s \wedge \overline{\beta_s} \ge 0$$

for every choice of holomorphic one-forms $\beta_j \in \Omega^1_{X,p}$ at the point p. The current T is called *(strongly) positive* if $\langle T, \alpha \rangle \ge 0$ for every weakly positive $(d-s,d-s)$-form α.
The current T is *closed* (resp. exact) if $\langle T, \alpha \rangle = 0$ for every exact (resp. closed) $(d-s,d-s)$-form α.

Example 1.6.
- A differential form is a special case of a current.
- If $Z \subset X$ is an analytic subvariety of codimension p, the integration current along Z, denoted $[Z]$, is a positive closed current of bidegree (p,p).

Some aspects of the calculus on differential forms (exterior differential, direct image, inverse image by a submersion, tensor product of currents on two manifolds, wedge product of a current with a form) extend naturally to currents; the most delicate operations are the pull-back of a current by holomorphic maps which are not submersions and the definition of the wedge product of two currents.

1.1.3 Definition of dynamical degrees and theorem of Gromov-Yomdin

Let $f\colon X \dashrightarrow Y$ be a dominant meromorphic map between compact Kähler manifolds; the map f is then holomorphic outside its indeterminacy locus $\mathcal{I} \subset X$, which has codimension at least 2. The closure Γ of its graph over $X \setminus \mathcal{I}$ is an irreducible analytic subset of dimension $d = \dim X$ in $X \times Y$. Let π_X, π_Y denote the restrictions to Γ of the projections from $X \times Y$ to X and to Y respectively; then π_X induces a biholomorphism $\pi_X^{-1}(X \setminus \mathcal{I}) \cong X \setminus \mathcal{I}$ and we can identify f with $\pi_Y \circ \pi_X^{-1}$.
Let α be a smooth (p,q)-form on Y; we define the pull-back of α by f as the (p,q)-current on X

$$f^*\alpha := (\pi_X)_*(\pi_Y^*\alpha).$$

It is not difficult to see that if α is closed (resp. positive), then so is $f^*\alpha$, so that f induces a linear morphism between the Hodge cohomology groups. This definition of pull-back coincides with the usual one when f is holomorphic.

From now on, let M denote a d-dimensional compact Kähler manifold. Recall the Hodge decomposition for the cohomology of M:

$$H^k(M,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M), \qquad \overline{H^{p,q}(M)} = H^{q,p}(M),$$

where $H^{p,q}(M)$ is the set of classes of forms of bidegree (p,q) on M. The space $H^{p,p}(M)$ has a real structure, i.e. $H^{p,p}(M,\mathbb{R}) := H^{p,p}(M) \cap H^{2p}(M,\mathbb{R})$ satisfies

$$H^{p,p}(M) \cong H^{p,p}(M,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

By the above discussion, a dominant meromorphic map $f\colon M \dashrightarrow M$ induces linear maps

$$f^*_{p,q}\colon H^{p,q}(M) \to H^{p,q}(M).$$

If $p=q$, $f^*_{p,p}$ preserves the real structure of $H^{p,p}(M)$.

Definition 1.7. *The p-th dynamical degree of a dominant meromorphic map $f\colon M \dashrightarrow M$ is defined as*

$$\lambda_p(f) = \limsup_{n\to+\infty} \| (f^n)^*_{p,p} \|^{\frac{1}{n}},$$

where $\|\cdot\|$ is any matrix norm on the space $\mathcal{L}(H^{p,p}(M,\mathbb{R}))$ of linear maps of $H^{p,p}(M,\mathbb{R})$ into itself.

Using to the above definition of pull-back, one can prove that

$$\lambda_p(f) = \lim_{n\to+\infty} \left(\int_M (f^n)^*\omega^p \wedge \omega^{d-p} \right)^{\frac{1}{n}} \tag{1.1}$$

for any Kähler form ω; see [DS04], [CCLG10] for details.

The p-th dynamical degree measures the exponential growth of the volume of $f^{-n}(V)$ for subvarieties $V \subset M$ of codimension p [Gue04].

Remark 1.8. By definition $\lambda_0(f) = 1$; $\lambda_d(f)$ coincides with the topological degree of f: it is equal to the number of points in a generic fibre of f.

Remark 1.9. Let f be holomorphic automorphism. Then we have $(f^n)^* = (f^*)^n$, so that $\lambda_p(f)$ is the maximal modulus of eigenvalues of the linear map $f^*_{p,p}$; since f^* also preserves the positive cone $\mathcal{K}_p \subset H^{p,p}(M,\mathbb{R})$, a theorem of Birkhoff [Bir67] implies that $\lambda_p(f)$ is a positive real eigenvalue of f^*_p. In particular, $\lambda_p(f)$ is an algebraic integer.
Furthermore, it can be showed that $\lambda_p(f)$ is the spectral radius of $f^*_{2p}\colon H^{2p}(M,\mathbb{R}) \to H^{2p}(M,\mathbb{R})$ (see for example [LB14b, Lemma 2.2.4]); therefore, all the conjugates of $\lambda_p(f)$ over $\mathbb{Q}$ have modulus $\leq \lambda_p(f)$.

It should be noted however that in the meromorphic setting we have in general $(f^n)^* \neq (f^*)^n$.

The main interest in the definition of dynamical degrees lies in the following theorem by Yomdin [Yom87] and Gromov [Gro90].

Theorem 1.10 (Yomdin-Gromov)**.** *Let $f\colon M \to M$ be a dominant self-map of a compact Kähler manifold of dimension d; then the topological entropy of f is given by*

$$h_{top}(f) = \max_{p=0,\dots,d} \log \lambda_p(f).$$

It is also possible to give a definition of topological entropy in the meromorphic context (see [DS08]), but in this situation we only have

$$h_{top}(f) \leq \max_{p=0,\dots,d} \log \lambda_p(f).$$

Example 1.11. Let us see some examples of computations of topological entropy by dynamical degrees.

- Let $p \in \mathbb{C}[X]$ be a polynomial of degree d and let

$$f\colon \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$$
$$z \mapsto p(z)$$

  Then $\lambda_0(f) = 1$ and $\lambda_1(f)$ is the topological degree of f, hence $\lambda_1(f) = d$. Therefore, by Theorem 1.10 $h_{top}(f) = \log d$; in particular, as we already showed by a direct computation in Example 1.2, this is true for $p(X) = X^d$.

- Let $f \in \mathrm{PGL}_{n+1}(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^n(\mathbb{C}))$ be an automorphism of $\mathbb{P}^n(\mathbb{C})$; then, since f sends hyperplanes to hyperplanes, f preserves a Kähler class $\omega = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$, thus $\lambda_1(f) = 1$. Analogously, $\lambda_p(f) = 1$ for all $p = 0, \dots, n$ (this also follows from log-concavity, see Proposition 1.14), thus $h_{top}(f) = 0$.
  More generally, if a projective manifold has Picard number $\rho(X) = 1$, then all automorphisms of X have zero entropy.

- Let $X = E \times E$, where $E = \mathbb{C}/\Lambda$ is an elliptic curve; a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ defines an automorphism of $\mathbb{C}^2$ which induces an automorphism $f\colon X \to X$. Fixing global linear coordinates x, y on the two factors, we have

$$H^{1,0}(X) = \mathrm{Span}(dx, dy), \qquad H^{0,1} = \mathrm{Span}(d\bar{x}, d\bar{y}),$$

  and f^* acts on $H^{1,0}(X, \mathbb{C})$ with basis dx, dy (resp. on $H^{0,1}(X, \mathbb{C})$ with basis $d\bar{x}, d\bar{y}$) as the matrix A^t (resp. $\overline{A^t} = A^t$). In particular, the eigenvalues of $f^*_{1,0}$ and $f^*_{0,1}$ are exactly the same as A, say λ_1, λ_2.
  Since

$$H^{1,1}(X) = \mathrm{Span}(dx \wedge d\bar{x}, dx \wedge d\bar{y}, dy \wedge d\bar{x}, dy \wedge d\bar{y}) \cong H^{1,0}(X) \otimes H^{0,1}(X),$$

  f^* acts on $H^{1,1}(X)$ with the above basis as the matrix $A^t \otimes A^t$; in particular, the eigenvalues of $f^*_{1,1}$ are exactly $\lambda_1^2, \lambda_1\lambda_2$ (with multiplicity 2) and λ_2^2. Therefore, $\lambda_1(f) = \lambda^2$ where λ is the spectral radius of A, and $h_{top}(f) = \log \lambda_1(f) = 2 \log \lambda$.

  This example can be generalized to any dimension. See also §1.3.3 for more details.

1.1.4 Properties of dynamical degrees

We give here some fundamental properties of dynamical degrees.

Invariance by bimeromorphic or generically finite maps

One of the central properties of dynamical degrees, which allows to define them in the meromorphic case, is the fact that they are bimeromorphic invariants: if $\phi\colon X \dashrightarrow Y$ is a bimeromorphic map and $f\colon X \dashrightarrow X$, $g\colon Y \dashrightarrow Y$ are dominant meromorphic maps such that $\phi \circ f = g \circ \phi$, then $\lambda_p(f) = \lambda_p(g)$ for $p = 0, \dots, \dim X = \dim Y$; see [DS04].
More generally, one can show the following result:

Proposition 1.12. *Let $\phi\colon X \dashrightarrow Y$ be a generically finite dominant meromorphic map between compact Kähler manifolds and let $f\colon X \dashrightarrow X$, $g\colon Y \dashrightarrow Y$ be dominant meromorphic maps such that $\phi \circ f = g \circ \phi$. Then*

$$\lambda_p(f) = \lambda_p(g) \qquad \textit{for } p = 0, \dots, \dim X = \dim Y.$$

One can show the result directly using the characterization 1.1 of dynamical degrees by the pull-back of a Kähler form; the result follows as well as an easy corollary of Theorem 1.16. □

Dynamical degrees of the inverse

For bimeromorphic maps, one can deduce the dynamical degrees of the inverse from the ones of the original map.

Lemma 1.13. *If $f\colon M \dashrightarrow M$ is bimeromorphic and $d = \dim M$, we have*

$$\lambda_p(f) = \lambda_{d-p}(f^{-1}).$$

Proof. Let ω be a Kähler form on M. For f biregular we have

$$\int_M (f^n)^*\omega^p \wedge \omega^{d-p} = \int_M (f^{-n})^*(f^n)^*\omega^p \wedge (f^{-n})^*\omega^{d-p} = \int_M \omega^p \wedge (f^{-n})^*\omega^{d-p},$$

which proves the equality by taking the limit.
If f is only bimeromorphic, for all n we can find two dense open subsets $U_n, V_n \subset M$ such that f^n induces an isomorphism $U_n \cong V_n$; by the definition of pull-back the measures $(f^n)^*\omega^p \wedge \omega^{d-p}$ and $\omega^p \wedge (f^{-n})^*\omega^{d-p}$ have no mass on any proper closed analytic subset, so that

$$\int_M (f^n)^*\omega^p \wedge \omega^{d-p} = \int_{U_n} (f^n)^*\omega^p \wedge \omega^{d-p} =$$

$$\int_{V_n} \omega^p \wedge (f^{-n})^*\omega^{d-p} = \int_M \omega^p \wedge (f^{-n})^*\omega^{d-p},$$

which proves the equality in the bimeromorphic case as well. □

Log-concavity

Dynamical degrees enjoy a log-concavity property (see [Kho79],[Tei79], [Gro90], [CCLG10]).

Proposition 1.14. *If $f\colon M \dashrightarrow M$ is a meromorphic dominant map, the sequence $p \mapsto \log \lambda_p(f)$ is concave on the set $\{0, 1, \dots, d\}$; in other words*

$$\lambda_p(f)^2 \geq \lambda_{p-1}(f)\lambda_{p+1}(f) \qquad \textit{for } p = 1, \dots, d-1.$$

As a consequence we have $\lambda_p \geq 1$ for all $p = 0, \dots, d$; furthermore, there exist $0 \leq p \leq p + q \leq d$ such that

$$1 = \lambda_0(f) < \cdots < \lambda_p(f) = \lambda_{p+1}(f) = \cdots = \lambda_{p+q}(f) > \cdots > \lambda_d(f). \tag{1.2}$$

In particular, the following are equivalent:

- $\lambda_1(f) > 1$;
- there exists $0 \leq p \leq \dim M$ such that $\lambda_p(f) > 1$;
- for f holomorphic, the topological entropy of f is strictly positive (see Theorem 1.10).

1.1.5 Relative setting

Dinh, Nguyên and Truong have studied the behaviour of dynamical degrees in the relative setting ([DN11] and [DNT12]). Throughout this paragraph we denote by $f\colon M \dashrightarrow M$ a meromorphic transformation of a compact Kähler manifold M of dimension d, by $\pi\colon M \dashrightarrow B$ a meromorphic fibration onto a compact Kähler manifold B of dimension k and by $g\colon B \dashrightarrow B$ a meromorphic transformation such that

$$g \circ \pi = \pi \circ f.$$

The p-th *relative dynamical degree* of f is defined as

$$\lambda_p(f|\pi) = \limsup_{n\to+\infty} \left(\int_M (f^n)^* \omega_M^p \wedge \pi^* \omega_B^k \wedge \omega_M^{d-p-k} \right)^{\frac{1}{n}},$$

where ω_M and ω_B are arbitrary Kähler forms on M and B respectively. In particular $\lambda_p(f|\pi) = 0$ for $p > d-k$.

Roughly speaking, $\lambda_p(f|\pi)$ gives the exponential growth of $(f^n)^*$ acting on the subspace of classes in $H^{p+k,p+k}(M,\mathbb{R})$ that can be supported on a generic fibre of π; if M is projective, it also represents the growth of the volume of $f^n(V)$ for subvarieties $V \subset \pi^{-1}(b)$ of a general fibre of π having codimension p in the fibre.

Remark 1.15. If $F = g^{-1}(b)$ is a regular, f-invariant, non-multiple fibre, then $\lambda_p(f|\pi) = \lambda_p(f_{|F})$ for all p (see [DN11]).

The following theorem is due to Dinh, Nguyên and Truong [DN11].

Theorem 1.16. *Let M be a compact Kähler manifold, $f\colon M \dashrightarrow M$ a dominant meromorphic endomorphism, $\pi\colon M \dashrightarrow B$ a meromorphic fibration and $g\colon B \dashrightarrow B$ a dominant meromorphic endomorphism such that $\pi \circ f = g \circ \pi$. Then for all $p = 0, \ldots \dim(M)$*

$$\lambda_p(f) = \max_{q+r=p} \lambda_q(f|\pi)\lambda_r(g).$$

Properties

Many of the properties of absolute dynamical degrees have natural analogues in the relative case.

Proposition 1.17. *Let M be a compact Kähler manifold, $f\colon M \dashrightarrow M$ a meromorphic transformation, $\pi\colon M \dashrightarrow B$ a meromorphic fibration and $g\colon B \dashrightarrow B$ a meromorphic transformation such that $\pi \circ f = g \circ \pi$; let $d = \dim M - \dim B$ be the dimension of the general fibres of π. Then*

1. *the $\lambda_p(f|\pi)$ are bimeromorphic invariants: if $\phi\colon M \dashrightarrow M'$ and $\psi\colon B \dashrightarrow B'$ are bimeromorphic maps, then*

$$\lambda_p(\phi \circ f \circ \phi^{-1}|\psi \circ \pi \circ \phi^{-1}) = \lambda_p(f|\pi) \qquad \textit{for } p = 0, \ldots, d;$$

2. *if f is invertible, $\lambda_p(f|\pi) = \lambda_{d-p}(f^{-1}|\pi)$;*

3. *the map $p \mapsto \log \lambda_p(f|\pi)$ is concave on the set $\{0, \ldots, d\}$.*

The proof of (2) is analogous to the proof of Lemma 1.13; for point (1) and (3) see [DN11] and [DNT12].

1.2 Holomorphic foliations

Informally, a holomorphic (smooth) *foliation* $\mathcal{F}$ of dimension p on a complex manifold M of dimension n is a local decomposition of M into disjoint submanifolds of dimension p. More accurately, it is the data of local charts U_i on M isomorphic to $V_i \times W_i$, with $V_i \subset \mathbb{C}^p$ and $W_i \subset \mathbb{C}^{n-p}$, and such that the horizontal slices $V_i \times \{w_i\}$ (the *plaques* of $\mathcal{F}$) are compatible with the changes of coordinates: if $\phi_i \colon U_i \xrightarrow{\sim} V_i \times W_i$ and $\phi_j \colon U_j \xrightarrow{\sim} V_j \times W_j$ are two such charts, then the change of coordinates

$$\phi_{ij} = \phi_j \circ \phi_i^{-1} \colon \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is of the form

$$\phi_{ij}(v, w) = (f_{ij}(v, w), g_{ij}(w)).$$

The plaques glue together from chart to chart to form the *leaves* of $\mathcal{F}$, which in general are not closed in M.

We will be generally interested in *singular foliations*: informally, we will allow a singular locus $\mathrm{Sing}\mathcal{F}$ where the local decomposition is not defined. We will only consider foliations in normal form, so that $\operatorname{codim} \mathrm{Sing}\mathcal{F} \geq 2$.

Example 1.18.

- The trivial examples are the ones where $\mathcal{F}$ has dimension or codimension equal to 0: in the first case every point is a leaf of $\mathcal{F}$, in the second case $\mathcal{F}$ has a unique leaf which coincides with the ambient manifold.

- The radial foliation on $\mathbb{C}^2$, whose leaves are

  $$L_{[v]} = \mathbb{C}^* v \qquad [v] \in \mathbb{P}^1(\mathbb{C})$$

  is a holomorphic one-dimensional foliation with a singularity at the origin.
  More generally, a vector field X determines a one-dimensional foliation whose leaves are the orbits of the flow Φ_X, and whose singularities are the zeros of X.

- Let $T = \mathbb{C}^2/\Gamma$ be a two-dimensional complex torus, and let $\pi \colon \mathbb{C}^2 \to T$ be the universal cover. For any fixed direction $[v] \in \mathbb{P}^1(\mathbb{C}) = \mathbb{P}(\mathbb{C}^2)$, the images of the affine lines with direction $[v]$ decompose T into the leaves of a one-dimensional smooth foliation $\mathcal{F}$; the leaves of $\mathcal{F}$ are either all compact (if the direction $[v]$ is rational with respect to the lattice Λ) or all Zariski-dense.
  More generally, a *linear foliation* on a torus $T = \mathbb{C}^n/\Lambda$ with universal cover $\pi \colon \mathbb{C}^n \to T$ is given by the decomposition of T into the images of all affine subspaces of $\mathbb{C}^n$ obtained as translations of a fixed linear subspace $V \subset \mathbb{C}^n$.

- Let M be a compact complex manifold and let $\pi \colon M \to B$ be a submersion. Then there exists a foliation $\mathcal{F}$ whose leaves are the fibres of π.
  More generally, let $\pi \colon M \dashrightarrow B$ be a meromorphic fibration, i.e. a dominant meromorphic map with connected fibres; then there exists a foliation $\mathcal{F}$ such that, in the open set $U \subset M$ such that $\pi|_U$ is a submersion, its leaves coincide with the fibres of π. The leaves of $\mathcal{F}$ are locally closed.

  Conversely, if $\mathcal{F}$ is a foliation all of whose leaves are algebraic (i.e. the dimension of their Zariski-closure is equal to the dimension of $\mathcal{F}$), then $\mathcal{F}$ is a meromorphic fibration in the sense we just explained (see for example [AD13, Lemma 3.2]).

- Let $\pi \colon X \dashrightarrow Y$ be a dominant meromorphic map, and let $\mathcal{F}$ be a foliation on Y; then the pull-back of leaves of $\mathcal{F}$ induce a foliation $\pi^*\mathcal{F}$ on X, which has the same codimension as $\mathcal{F}$.

- If $\mathcal{F}_1, \mathcal{F}_2$ are two foliations on X, the intersection $\mathcal{F}_1 \cap \mathcal{F}_2$ is a foliation; the local plaques are defined as the intersection of plaques of $\mathcal{F}_1$ and plaques of $\mathcal{F}_2$.
  The span of $\mathcal{F}_1, \mathcal{F}_2$, associating to each point $p \in X$ the span of the tangent spaces of $\mathcal{F}_1$ and $\mathcal{F}_2$, is a (meromorphic) *distribution* of tangent subspaces, which in general is not a foliation.
- If $\mathcal{F}_1$, $\mathcal{F}_2$ are foliations on X_1, X_2 respectively, the product of leaves defines a foliation $\mathcal{F}_1 \times \mathcal{F}_2$ on $X_1 \times X_2$.

1.2.1 Foliations as subsheaves of the tangent sheaf

A foliation is determined by its tangent space at each point. The formal definition is as follows: a (singular) foliation $\mathcal{F}$ on a complex manifold M is determined by a coherent subsheaf $T\mathcal{F}$ of TM (the *tangent sheaf* of $\mathcal{F}$) such that

1. $T\mathcal{F}$ is involutive (closed under the Lie bracket); and
2. the quotient $TM/T\mathcal{F}$ is torsion free.

The dimension of $\mathcal{F}$ is the generic rank of $T\mathcal{F}$, and the singular set of $\mathcal{F}$ is the singular set of the sheaf $TM/T\mathcal{F}$. A foliation $\mathcal{F}$ is smooth if, and only if, both $T\mathcal{F}$ and $TM/T\mathcal{F}$ are locally free sheaves.

1.2.2 Foliations and differential forms

Let $\mathcal{F}$ be a foliation of codimension q and define the normal sheaf of $\mathcal{F}$ as

$$N\mathcal{F} := \left({}^{TM}\!\big/_{T\mathcal{F}}\right)^{**}.$$

Denote by $N^*\mathcal{F} := (N\mathcal{F})^*$ the conormal sheaf of $\mathcal{F}$. From the inclusion $N^*\mathcal{F} \hookrightarrow \Omega^1_M$ we deduce a morphism $\det N^*\mathcal{F} \to \Omega^q_M$. If we set

$$\mathcal{L} = (\det N^*\mathcal{F})^* = \det(N\mathcal{F}),$$

where the determinant of a coherent sheaf $\mathcal{S}$ of generic rank r is defined as the line bundle $\det \mathcal{S} := (\bigwedge^r \mathcal{S})^{**}$, we get a q-form

$$\omega \in H^0(M, \Omega^q_M \otimes \mathcal{L})$$

which defines the foliation $\mathcal{F}$ in the sense that $T\mathcal{F}$ can be recovered as the kernel of the sheaf morphism

$$\begin{aligned} TM &\longrightarrow \Omega^{q-1}_M \otimes \mathcal{L} \\ v &\longmapsto i_v\omega\,. \end{aligned}$$

The singular locus of $\mathcal{F}$ is exactly the zero locus of ω.

This abstract construction has a clear interpretation: locally, the foliation is defined as the kernel of a local q-form ω_i, which is well-defined up to multiplication by a holomorphic function without zeros. We pick an atlas of open sets U_i where $\mathcal{F}$ is defined by ω_i, so that, on $U_i \cap U_j$,

$$\omega_i = h_{ij}\omega_j$$

for some holomorphic function h_{ij} without zeros. The line bundle $\mathcal{L}$ is then defined by the transition functions h_{ij}.

Foliations and meromorphic differential forms

Now, let s be any non-trivial meromorphic section of $\mathcal{L}$; then $\alpha := \omega/s$ is a meromorphic q-form which defines $\mathcal{F}$ in the sense that, away from its zeros and poles, $T\mathcal{F}$ is exactly the kernel of the contraction

$$\begin{aligned} TM &\longrightarrow \Omega_M^{q-1} \\ v &\longmapsto i_v\alpha\,. \end{aligned}$$

The zeros in codimension one (resp. the poles) of α are exactly the poles (resp. the zeros) of s.

Conversely, suppose that a meromorphic q-form α defines a foliation $\mathcal{F}$ as above, and let $D = div(\alpha)$ be the divisor of zeros in codimension one minus poles of α. Denoting by s the meromorphic section of $\mathcal{L} := \mathcal{O}_M(-D)$ having $-D$ as divisor of zeros minus poles, the form

$$\omega := s \cdot \alpha \in H^0(M, \Omega_M^q \otimes \mathcal{L})$$

is holomorphic and has no zeros in codimension one; furthermore, as it is a multiple of α, it still defines the foliation $\mathcal{F}$. This shows that

$$\mathcal{L} = \det(N\mathcal{F}).$$

Foliations and pluri-forms

The above construction allows to express any foliation in terms of a meromorphic differential form. It can be desirable for such a form α not to have zeros in codimension one; this is possible if and only if the line bundle $\mathcal{L} = \det(N\mathcal{F})$ admits non-trivial holomorphic sections. However, if some multiple $m\mathcal{L} = \mathcal{L}^{\otimes m}$ of $\mathcal{L}$ admits non-trivial sections, we can bypass the problem as follows.

A *pluri-form* is a section of the bundle $(\Omega^q)^{\otimes m}$ for some $q \geq 0, m \geq 1$; let $\omega \in H^0(M, \Omega_M^q \otimes \mathcal{L})$ be a form defining $\mathcal{F}$, and let

$$\omega^{\otimes m} \in H^0(M, (\Omega_M^q)^{\otimes m} \otimes m\mathcal{L}).$$

We can recover ω modulo m-th roots of unity, hence $\mathcal{F}$, from $\omega^{\otimes m}$; we will say that $\omega^{\otimes m}$ is a pluri-form with values in $m\mathcal{L}$ which defines $\mathcal{F}$.
Now, if $m\mathcal{L}$ admits a non-trivial holomorphic section s, the meromorphic pluri-form $\alpha := \omega^{\otimes m}/s$ has no zeros in codimension one and defines $\mathcal{F}$ (in the sense that, outside the poles of α, $T\mathcal{F}$ is the kernel of the contraction morphism $TM \to (\Omega_M^q)^{\otimes(m-1)} \otimes \Omega_M^{q-1}$).

1.3 Dynamics of automorphisms of surfaces

From now on, we will focus on the dynamics of automorphisms (or birational transformations) of compact Kähler manifolds. In this section, we will summarize the known results for automorphisms of surfaces; we refer to [Can14].

Throughout this section, we will denote by

$$f\colon S \to S$$

an automorphism of a compact Kähler surface S.

Remark 1.19. The one-dimensional case is not dynamically interesting. In this case, birational transformations coincide with automorphisms. If C is a curve of genus $g \geq 2$, then by Hurwitz's automorphisms theorem (see [Mir95, Theorem 3.9]) the group $\mathrm{Aut}(C)$ has cardinality at most

$84(g-1)$; therefore, the only interesting dynamics appear on $\mathbb{P}^1(\mathbb{C})$ and on elliptic curves. These two cases can be studied explicitly: in particular, $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbb{C})$; and if E is an elliptic curve, the group of translations has finite index in $\mathrm{Aut}(E)$.
Summarizing, if $f\colon C \to C$ is an automorphism of a curve C, then there exists an iterate f^N of f such that

$$f^N = \begin{cases} \text{homography} & \text{if } C = \mathbb{P}^1(\mathbb{C}) \\ \text{translation} & \text{if } C \text{ is an elliptic curve} \\ \mathrm{id}_C & \text{if } g(C) \geq 2 \end{cases}$$

1.3.1 Hodge index theorem

By Theorem 1.10, the topological entropy of an automorphism is uniquely determined by its action on the cohomology; therefore, it is natural to classify automorphisms of surfaces by their action on cohomology.

Recall that the intersection product is the pairing

$$q_S\colon H^{1,1}(S,\mathbb{R}) \times H^{1,1}(S,\mathbb{R}) \to \mathbb{R}$$
$$(\alpha,\beta) \mapsto \int_S \alpha\wedge\beta$$

The *Neron-Severi group* of S is defined as

$$\mathrm{NS(S)} = \mathrm{H}^{1,1}(\mathrm{X},\mathbb{R}) \cap \mathrm{H}^2(\mathrm{S},\mathbb{Z})/\mathrm{torsion};$$

by Lefschetz's theorem on $(1,1)$ classes, it coincides with the group of numerical classes of divisors. The restriction of q_S to $\mathrm{NS}_{\mathbb{R}}(\mathrm{S}) = \mathrm{NS(S)} \otimes_{\mathbb{Z}} \mathbb{R}$ has a clear geometric interpretation: if D, D' are effective divisors without common components, then $q_S(c_1(D), c_1(D'))$ is the number of points of $D \cap D'$ (counted with multiplicities).

An automorphism $f\colon S \to S$ induces by pull-back linear automorphisms

$$f^*_{1,1}\colon H^{1,1}(S,\mathbb{R}) \to H^{1,1}(S,\mathbb{R}), \qquad f^*_{\mathrm{NS}}\colon \mathrm{NS}_{\mathbb{R}}(\mathrm{S}) \to \mathrm{NS}_{\mathbb{R}}(\mathrm{S}).$$

It is clear by the definition that q_S is f^*-invariant, i.e.

$$q_S(f^*\alpha, f^*\beta) = q_S(\alpha,\beta) \qquad \forall \alpha,\beta \in H^{1,1}(S,\mathbb{R}).$$

A *hyperbolic vector space* is a real vector space V of dimension $N+1$ endowed with a non-degenerate quadratic form q of signature $(1,N)$: in other words, there exists an orthogonal basis $e_0, e_1, \ldots, e_N$ of V such that

$$q_S(e_0) = 1, \qquad q_S(e_i) = -1 \quad i = 1,\ldots N.$$

Theorem 1.20 (Hodge index theorem)**.** *Let S be a compact Kähler surface; then the two vector spaces $H^{1,1}(S,\mathbb{R})$ and $\mathrm{NS}_{\mathbb{R}}(\mathrm{S})$, endowed with the intersection product, are hyperbolic.*

Therefore, after choosing a good basis for $H^{1,1}(S,\mathbb{R})$ we obtain a group morphism

$$\mathrm{Aut}(S) \to O(1,N),$$

where $N = h^{1,1}(S) - 1$ and $O(1,N)$ denotes the group of linear automorphisms of $\mathbb{R}^{N+1}$ preserving the standard quadratic form of signature $(1,N)$:

$$q(x_0, x_1, \ldots, x_N) = x_0^2 - (x_1^2 + \ldots + x_N^2).$$

Analogously, the choice of a good basis of $\mathrm{NS}_{\mathbb{R}}(\mathrm{S})$ induces a group morphism

$$\mathrm{Aut}(S) \to O(1, \rho(S) - 1),$$

where $\rho(S) = \dim \mathrm{NS}_{\mathbb{R}}(\mathrm{S})$ denotes the Picard number.

1.3.2 Automorphisms of hyperbolic spaces

In this paragraph we classify the automorphisms of a hyperbolic vector space (V, q) of dimension d which preserve q. Fix any norm $\|\cdot\|$ on the space $\mathcal{L}(V)$ of linear endomorphisms of V.

Definition 1.21. *Let $\phi \in O(V, q)$. We say that ϕ is*

- loxodromic *(or* hyperbolic*) if it admits an eigenvalue of modulus strictly greater than* 1*;*
- parabolic *if all its eigenvalues have modulus* 1 *and* $\| \phi^n \|$ *is not bounded as* $n \to +\infty$*;*
- elliptic *if all its eigenvalues have modulus* 1 *and* $\| \phi^n \|$ *is bounded as* $n \to +\infty$.

In each of the cases above, simple linear algebra arguments allow to further describe the situation.
Let

$$\mathcal{C}_{\geq 0} = \{v \in V \mid q(v) \geq 0\},$$

$$\mathcal{C}_0 = \{v \in V \mid q(v) = 0\}.$$

$\mathcal{C}_0$ is called the *isotropic cone* for q.
Remark that an automorphism $\phi \in O(V, q)$ preserves the cone $\mathcal{C}_{\geq 0}$, hence acts on the set of lines $\mathbb{P}\mathcal{C}_{\geq 0}$; as this set is homeomorphic to the $(d-1)$-dimensional closed ball, by Brouwer's fixed point theorem ϕ preserves at least one line contained in $\mathcal{C}_{\geq 0}$.
We denote by $O^+(V, q) \subset O(V, q)$ the index two subgroup of automorphisms preserving the two connected components of $\mathcal{C}_{\geq 0} \setminus \{0\}$. If $\phi \in O(V, q)$, then $\phi^2 \in O^+(V, q)$.

For the following result see fo example [Gri16].

Theorem 1.22. *Let $\phi \in O^+(V, q)$.*

- *If ϕ is loxodromic, then it has exactly one eigenvalue λ with modulus > 1 and exactly one eigenvalue with modulus < 1; these eigenvalues are real, simple and they are the inverse of each another; their eigenspaces $\mathbb{R}v_+, \mathbb{R}v_-$ are contained in $\mathcal{C}_0$ and they are the only ϕ-invariant lines in $\mathcal{C}_{\geq 0}$; ϕ is semi-simple, and in particular the norm of iterates of ϕ grows exponentially.*
  If furthermore ϕ preserves a lattice $\Gamma \subset V$, then $\mathbb{R}v_+$ and $\mathbb{R}v_-$ are irrational with respect to Γ; in this case λ is an algebraic integer whose conjugates over $\mathbb{Q}$ are λ^{-1} and complex numbers of modulus 1, i.e. λ is a quadratic or Salem number.

- *If ϕ is parabolic, then it fixes exactly one line $\mathbb{R}v$ of $\mathcal{C}_{\geq 0}$, which is contained in $\mathcal{C}_0$; the norm of iterates of ϕ grows quadratically.*
  If furthermore ϕ preserves a lattice $\Gamma \subset V$, then $\mathbb{R}v$ is rational with respect to Γ; all the eigenvalues of ϕ are roots of unity, and some iterate of ϕ has Jordan form

$$\begin{pmatrix} 1 & 1 & 0 & \mathbf{0} \\ 0 & 1 & 1 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{d-3} \end{pmatrix}.$$

- *If ϕ is elliptic, then it fixes exactly one line $\mathbb{R}v$ in the interior of $\mathcal{C}_{\geq 0}$; since q is negative definite on $v^\perp$, ϕ acts as a rotation on $v^\perp$.*
  If furthermore ϕ preserves a lattice, then it has finite order.

Application to automorphisms of surfaces

An automorphism $f\colon S \to S$ of a compact Kähler surface S is called loxodromic, parabolic or elliptic if $f^*_{1,1}$ is loxodromic, parabolic or elliptic respectively; using the fact that f^*_{NS} preserves the lattice $\mathrm{NS(S)}/\{\mathrm{torsion}\}$, one can show that f is loxodromic/parabolic/elliptic if and only if f^*_{NS} is.

Denote again by $\mathcal{C}_0, \mathcal{C}_{\geq 0}$ the isotropic and positive cones for the intersection form on $H^{1,1}(S,\mathbb{R})$. Remark that, since f^* sends Kähler classes to Kähler classes, if preserves the two components of $\mathcal{C}_{\geq 0} \setminus \{0\}$.
The component of the interior of $\mathcal{C}_{\geq 0}$ containing the ample cone coincides with the big cone, as defined in [Bou04].

From Theorem 1.22, we deduce the following properties.

Corollary 1.23. *Let $f\colon S \to S$ be an automorphism of a compact Kähler surface S.*

- *If f is loxodromic (or, equivalently, $\lambda_1(f) > 1$), then $f^*_{1,1}$ is semi-simple has exactly one eigenvalue with modulus > 1 and exactly one eigenvalue with modulus < 1; these eigenvalues are equal to $\lambda := \lambda_1(f)$ and λ^{-1}, and they are simple; the norm of $(f^n)^*_{1,1}$ grows as $c\,\lambda^n$. The eigenspaces $\mathbb{R}v_+, \mathbb{R}v_-$ of λ and λ^{-1} are contained in $\mathcal{C}_0$ and they are the only $f^*_{1,1}$-invariant lines in $\mathcal{C}_{\geq 0}$; they are not defined over $\mathbb{Q}$. The dynamical degree $\lambda_1(f)$ is a quadratic or Salem number (i.e. an algebraic integer λ which is not a quadratic integer and whose only conjugates over $\mathbb{Q}$ are λ^{-1} and complex numbers with modulus 1).*

- *If f is parabolic, then $f^*_{1,1}$ fixes exactly one line $\mathbb{R}v$ of $\mathcal{C}_{\geq 0}$, which is contained in $\mathcal{C}_0$ and defined over $\mathbb{Q}$; the norm of iterates of $f^*_{1,1}$ grows quadratically. All the eigenvalues of $f^*_{1,1}$ are roots of unity, and some iterate of $f^*_{1,1}$ has Jordan form*
$$\begin{pmatrix} 1 & 1 & 0 & \mathbf{0} \\ 0 & 1 & 1 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{N-3} \end{pmatrix} \qquad N = \dim H^{1,1}(S);$$
in particular, the norm of $(f^n)^*$ grows like n^2 as $n \to +\infty$.

- *If f is elliptic, then $f^*_{1,1}$ fixes exactly one line $\mathbb{R}v$ in the interior of $\mathcal{C}_{\geq 0}$; since q is negative definite on $v^\perp$, $f^*_{1,1}$ acts as a rotation on $v^\perp$. Furthermore, $f^*_{1,1}$ has finite order.*

Remark that, if f is homotopic to the identity, then its action on cohomology is trivial. Conversely, if f acts trivially on cohomology, then some of its iterates is homotopic to the identity. More precisely:

Theorem 1.24 (Fujiki, Liebermann [Fuj78, Lie78])**.** *Let M be a compact Kähler manifold. If $[\kappa]$ is a Kähler class on M, the connected component of the identity $\mathrm{Aut}(X)^0$ has finite index in the group of automorphisms of M fixing $[\kappa]$.*

This implies that a surface automorphism is elliptic if and only if one of its iterates is homotopic to the identity.

1.3.3 Linear dynamics on two-dimensional tori

One of the fundamental examples of dynamics on surfaces is provided by linear automorphisms of two-dimensional complex tori: on the one hand, the description of automorphisms is particularly explicit; on the other hand, many of the general phenomena can be observed in this case. Furthermore, one can construct manifolds with interesting dynamics starting from an automorphism of a torus (see for example Chapter 3); some results also allow to describe a dynamical system in terms of such a linear automorphism (see Corollary 1.32).

Let $T = \mathbb{C}^n/\Lambda$ be an n-dimensional complex torus; any affine automorphism $\tilde{f}$ of $\mathbb{C}^n$ such that $\tilde{f}(\Lambda) = \Lambda$ induces an automorphism $f\colon T \to T$.

Lemma 1.25. *Let $f\colon T \dashrightarrow T$ be a birational transformation of an n-dimensional complex torus T; then f is an automorphism, and it is induced by an affine automorphism of $\mathbb{C}^n$.*

Proof. If f were not biregular, then by [KM98, Corollary 1.4] T would contain a rational curve. But a torus does not contain any rational curve: otherwise we would have a non-constant map $\mathbb{P}^1 \to T$, which would lift to a non-constant map $\mathbb{P}^1 \to \mathbb{C}^n$, a contradiction. Therefore f is an automorphism.

The tangent bundle of T is trivial; fixing a trivialization, one can see the Jacobian of f at a point $p \in T$ as a matrix in $\mathrm{GL}_n(\mathbb{C}) \subset \mathcal{M}_{n\times n}(\mathbb{C}) \cong \mathbb{C}^{n^2}$. Since T is compact, the map $p \mapsto Df_p$ is constant; this means exactly that f is induced by an affine automorphism of $\mathbb{C}^n$. □

From now on, assume that $\dim T = 2$ and that $T = E \times E$ for some elliptic curve $E = \mathbb{C}/\Lambda$. Any matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ preserves $\Lambda \times \Lambda$, hence induces an automorphism $f = f_A\colon T \to T$.

As we have seen in Example 1.11, if α is the spectral radius of A, then $\lambda_1(f_A) = \alpha^2$.

Parabolic case

Take
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$
which induces the automorphism of T
$$f = f_A\colon (x, y) \mapsto (x + y, y).$$
Since A has infinite order, none of the iterates of f is homotopic to the identity (i.e. a translation); furthermore $\lambda_1(f) = 1$. Thus f is parabolic.

Since
$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$
by the computations in Example 1.11 we have
$$\| (f^n)^*_{1,1} \| = \| A^n \|^2 \sim n^2,$$
as stated in Theorem 1.22.

The fibre of the projection $\pi\colon T \to E$ on the second factor are f-equivariant, and f acts by translations on the fibres of π. As the translations on fibres depend on the base point, it can be shown that the points $e \in E$ such that the restriction of f to $\pi^{-1}(e)$ has infinite (resp. finite) order is a full-measure (resp. dense) subset of E. In particular, f-periodic points are dense in T.

Remark 1.26. f does not preserve any other foliation beside the one whose leaves are the fibres of π; in particular π is the only equivariant fibration.
Indeed, for every periodic point p, of period n, the line $L \subset T_p\mathcal{F}$ which is tangent to the foliation must be invariant by $D_p f^n$. This differential is given by the matrix A^n, and preserves a unique line, which is tangent to the fibration. Thus, $\mathcal{F}$ must be tangent to the fibration at all periodic points. Since these points are Zariski dense, $\mathcal{F}$ coincides with the invariant fibration.

Remark 1.27. We will see in Theorem 1.30 that all parabolic automorphisms of surfaces admit exactly one equivariant fibration, and that its general fibres are elliptic curves.

Loxodromic case

Take

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

which induces the automorphism of T

$$f = f_A \colon (x, y) \mapsto (2x + y, x + y).$$

Since the spectral radius of A is equal to $\alpha = (3+\sqrt{5})/2$, $\lambda_1(f) = \alpha^2 > 1$. Thus f is loxodromic.

Since

$$\| A^n \| \sim c\alpha^n,$$

by the computations in Example 1.11 we have

$$\| (f^n)^*_{1,1} \| = \| A^n \|^2 \sim c\lambda_1(f)^n,$$

as stated in Theorem 1.22.

Let v_+, v_- be the eigenvectors of A with eigenvalue α, α^{-1} respectively, and let $\mathcal{F}^+, \mathcal{F}^-$ be the linear foliations on T defined by the directions of v_+ and v_- (see Example 1.18). The foliations $\mathcal{F}^+, \mathcal{F}^-$ are preserved by f, and are called the *unstable and stable foliation* respectively, in analogy with the case of Anosov diffeomorphisms of (real) compact surfaces (see [HK02]); f acts by expanding (respectively contracting) the leaves of $\mathcal{F}^+$ (resp. $\mathcal{F}^-$) by a constant factor α (resp. α^{-1}). Remark that the leaves of $\mathcal{F}^+$ and $\mathcal{F}^-$ are dense in T.

The same proof as in Remark 1.26 shows that $\mathcal{F}^+$ and $\mathcal{F}^-$ are the only f-invariant foliations; in particular, f doesn't admit any equivariant fibration.

It can be shown that, for p in a full-measure subset of T, the f-orbit of p is dense in T, and that periodic points are dense in T.

Remark 1.28. We will see in Theorem 1.30 and Theorem 1.31 that loxodromic automorphisms of surfaces never admit equivariant fibrations and that their periodic points are Zariski-dense; however, the existence of invariant foliations is specific to tori and surfaces constructed starting from tori (see Corollary 1.32).

1.3.4 Invariant fibrations

As we have seen, the action on cohomology of a dominant endomorphism $f \colon M \to M$ of a compact Kähler manifold M determines its topological entropy. In the case of surfaces, one can deduce even more dynamical properties from the cohomological action; as we will see, the dynamical behavior changes drastically depending on the entropy being zero or strictly positive.

Definition 1.29. *Let $f \colon M \to M$ be an endomorphism of a compact Kähler manifold; we say that a fibration $\pi \colon M \to B$ (i.e. a surjective map with connected fibres) is* f-equivariant *if there exists an endomorphism $g \colon B \to B$ such that $\pi \circ f = g \circ \pi$, i.e. the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \pi\downarrow & & \downarrow\pi \\ B & \xrightarrow{g} & B \end{array}.$$

The dynamics of an automorphism admitting a non-trivial equivariant fibration can be analyzed by studying the action g on the base first, and then the action of f on fibres; intuitively, this means that the dynamics of f is not very chaotic.

An *elliptic fibration* is a fibration whose general fibres are elliptic curves.

The following theorem was stated and proved in the present form by Cantat [Can01], and follows from a result of Gizatullin (see [Giz80], or [Gri16] for a survey); see also [DF01] for the birational case.

Theorem 1.30. *Let S be a compact Kähler surface and let f be an automorphism of S.*

1. *If f is parabolic, there is an f-equivariant elliptic fibration $\pi\colon S \to C$. If F is a fibre of the fibration, its class $[F]$ is contained in the unique isotropic line which is fixed by $f^*_{1,1}$; in particular, this line intersects $\mathrm{NS(S)} \setminus \{0\}$, and f admits exactly one equivariant fibration. Moreover, if the induced automorphism $g\colon C \to C$ does not have finite order, then S is isomorphic to a torus $\mathbb{C}^2/\Lambda$.*

2. *Conversely, if a non-elliptic automorphism of a surface $f\colon S \to S$ admits an equivariant non-trivial fibration $\pi\colon S \to C$, then f is parabolic. In particular, the fibration π is elliptic, and is the only equivariant fibration.*

In other words, a non-elliptic automorphism of a surface admits an equivariant fibration if and only if its topological entropy is zero.

1.3.5 The loxodromic case: periodic points and invariant foliations

Let $f\colon S \to S$ be an automorphism of a compact Kähler surface S. In Theorem 1.30 we have seen that, if f is loxodromic (if and only if $\lambda_1(f) > 1$, if and only if $h_{top}(f) > 0$), then the dynamics of f is too chaotic for f to admit an equivariant fibration. Let us see more results in this direction; throughout this section, $f\colon S \to S$ denotes a loxodromic automorphism of a compact Kähler surface.

Zariski-density of hyperbolic periodic points

Let p be a periodic point of the automorphism f and let k be its period. One says that p is a hyperbolic (or saddle) periodic point if one eigenvalue of the Jacobian matrix $D(f^k)_p$ has modulus > 1 and the other has modulus < 1.
Since f has topological entropy $\log(\lambda_1(f))$ and S has dimension 2, one can apply a result due to Katok [Kat80] (stated for real surfaces, but still valid for complex surfaces) to prove the following result.

Theorem 1.31 (Cantat [Can14])**.** *Let f be a loxodromic automorphism of a compact Kähler surface. The set of saddle periodic points of f is Zariski dense in S.*
Furthermore, the number $N(f,k)$ of saddle periodic points of f of period at most k grows like $\lambda_1(f)^k$.

The same result holds for isolated periodic points in place of saddle periodic points. In Chapter 4, we will present all the elements which are necessary for the proof in the case of surfaces, and we will prove an analogous result for automorphisms of irreducible symplectic manifolds (see Theorem 4.2).

Classification of invariant foliations

The property of preserving a foliation is weaker than admitting an equivariant fibration, but nonetheless it forces some rigidity on the dynamics of f. It turns out that, if a loxodromic automorphism of a projective surface preserves a foliation, then it is induced by a loxodromic automorphism of a two-dimensional torus.

Theorem 1.32 (Cantat, Favre [CF03]). *Let $f\colon S \to S$ be a loxodromic automorphism of a projective surface S which preserves a (singular) foliation $\mathcal{F}$. Then there exist a birational morphism $\pi\colon S \to \hat{S}$ onto a singular surface $\hat{S}$, an automorphism $\hat{f}$ of $\hat{S}$, a finite map $\eta\colon T \to \hat{S}$ from a two-dimensional complex torus onto $\hat{S}$, and a loxodromic automorphism $f_T\colon T \to T$ such that*

$$\pi \circ f = \hat{f} \circ \pi, \qquad \eta \circ f_T = \hat{f} \circ \eta.$$

In other words, the following diagram commutes

$$\begin{tikzcd} f \circlearrowright S \arrow[dr, "\pi"'] & & T \circlearrowleft f_T \arrow[dl, "\eta"] \\ & \hat{f} \circlearrowright \hat{S} & \end{tikzcd} .$$

The foliation $\mathcal{F}$ is induced by pull-back of the stable or unstable foliation of f_T; in particular, f preserves an additional foliation which is generically transverse to $\mathcal{F}$.

Theorem 1.32 has a particularly neat interpretation when S is a projective $K3$ surface, see §3.1.1.

Chapter 2

Holomorphic irreducible symplectic manifolds

2.1 Definition and examples

We give here the basic notions and properties of irreducible holomorphic symplectic manifolds (see [GHJ03, Part III], [Mar11] for details).

Definition 2.1. *A $K3$* surface *is a compact Kähler surface such that $K_S \cong \mathcal{O}_S$ and $H^1(S, \mathbb{Q}) = 0$.*

$K3$ surfaces play an important role in the classification theory of compact surfaces: together with two-dimensional complex tori, they are the only surfaces with trivial canonical bundle. Irreducible symplectic manifolds are one of the possible generalizations in higher dimension of $K3$ surfaces.

Definition 2.2. *A compact Kähler manifold X is called* symplectic *if it admits a symplectic form σ;* irreducible symplectic *(or* hyperkähler*) if furthermore it is simply connected and the space of holomorphic two-forms on X is spanned by σ.*

Here, a two-form $\sigma \in H^0(X, \Omega_X^2)$ is called symplectic if it is nowhere degenerate, i.e. the contraction $v \mapsto i_v\sigma = \sigma(v, \cdot)$ induces an isomorphism $TX \xrightarrow{\sim} \Omega_X^1$.

Remark 2.3. Because of the non-degeneracy of σ, one can easily prove that a holomorphic symplectic manifold has even complex dimension $2n$, and that σ^n is nowhere zero.

Caution! Some authors refer to possibly reducible symplectic manifolds as hyperkähler manifolds. Her I stick to the terminology of [GHJ03].

Throughout this section X denotes an irreducible holomorphic symplectic manifold of dimension $2n$ and σ a non-degenerate holomorphic two-form on X.

Darboux's theorem

It is not hard to see that, if σ is a symplectic form on X and $\dim X = 2n$, then for all $p \in X$ there exist linear coordinates $x_1, y_1, \ldots, x_n, y_n$ of T_pX such that, in these coordinates,

$$\sigma_p = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

This representation can be actually made local thanks to the following theorem.

Theorem 2.4 (Darboux, see [AG01])**.** *A symplectic form has locally the form*

$$\sigma = \sum_{i=1}^{n} dx_i \wedge dy_i$$

for a suitable choice of local coordinates $x_1, y_1, \ldots, x_n, y_n$.

A submanifold $V \subset X$ of an irreducible symplectic manifold is called *isotropic* if, denoting by $i \colon V \hookrightarrow X$ the embedding morphism, $i^*\sigma = 0$; by non-degeneracy, this implies that $\dim V \leq \dim X/2$. An isotropic submanifold is called *Lagrangian* if its dimension is $\dim X/2$.
Here is a relative version of Darboux theorem.

Theorem 2.5. *Let $V \subset X$ be a Lagrangian submanifold of a symplectic manifold (X, σ); then around every point of V there exist local coordinates $x_1, y_1, \ldots, x_n, y_n$ such that, in these coordinates,*

$$\sigma = \sum_{i=1}^{n} dx_i \wedge dy_i, \qquad V = \{x_1 = \ldots = x_n = 0\}.$$

Examples

Here is a list of the known examples of such manifolds which are not deformation equivalent; see Chapter 3 for a detailed discussion on the first two families.

1. Let S be a $K3$ surface. Then the Hilbert scheme $S^{[n]} = Hilb^n(S)$ (or the Douady space), parametrizing 0-dimensional subschemes of S of length n, is a $2n$-dimensional irreducible holomorphic symplectic manifold.

2. Let T be a complex torus of dimension 2, let $\phi \colon Hilb^{n+1}(T) \to \mathrm{Sym}^{n+1}(T)$ be the natural morphism and let $s \colon \mathrm{Sym}^{n+1}(T) \to T$ be the sum morphism. Then the kernel $K_n(T)$ of the composition $s \circ \phi$ is an irreducible holomorphic symplectic manifold of dimension $2n$, which is called a *generalized Kummer variety*.

3. O'Grady has found two sporadic examples of irreducible holomorphic symplectic manifolds of dimension 6 and 10 [O'G99, O'G03].

An irreducible holomorphic symplectic manifold is said of type $K3^{[n]}$ (respectively of type generalized Kummer) if it is deformation equivalent to $Hilb^n(S)$ for some $K3$ surface S (respectively to $K_{n-1}(T)$ for some two-dimensional complex torus T).

2.2 The Beauville-Bogomolov decomposition theorem

Irreducible symplectic manifolds play a central role in the classification of compact Kähler manifolds with trivial Chern class; for the next theorem see [GHJ03, Theorem I.5.4, II.14.15].

Theorem 2.6 (Beauville-Bogomolov)**.** *Let X be a compact Kähler manifold with trivial Chern class. Then there exists a finite étale cover $X' \to X$ such that*

$$X' \cong T \times \prod_{i=1}^{k} Y_i \times \prod_{j=1}^{h} Z_j,$$

where

- *T is a complex torus;*
- *the Y_i are Calabi-Yau manifolds in the strict sense: $\pi_1(Y_i) = 0$, $K_{Y_i} \cong \mathcal{O}_{Y_i}$ and $h^0(\Omega^p_{Y_i}) = 0$ for $0 < p < \dim Y_i$;*
- *the Z_j are irreducible symplectic manifolds.*

2.3 The Beauville-Bogomolov form

We can define a natural quadratic form on the second cohomology $H^2(X, \mathbb{R})$ which enjoys similar properties to the intersection form on compact surfaces; for details and proofs see [GHJ03, Part III].

Definition 2.7. *Let σ be a holomorphic two-form such that $\int(\sigma\bar{\sigma})^n = 1$. The* Beauville-Bogomolov quadratic form *q_X on $H^2(X, \mathbb{R})$ is defined by*

$$q_X(\alpha) = \frac{n}{2}\int_X \alpha^2(\sigma\bar{\sigma})^{n-1} + (1-n)\left(\int_X \alpha\sigma^n\bar{\sigma}^{n-1}\right)\left(\int_X \alpha\sigma^{n-1}\bar{\sigma}^n\right).$$

Proposition 2.8. [GHJ03, Proposition III.23.14] *There exists a positive multiple of q_X which is defined over $\mathbb{Z}$ and primitive.*

In what follows, we will denote by q_X the positive multiple of the Beauville-Bogomolov form which is defined over $\mathbb{Z}$ and primitive.

The Beauville-Bogomolov form satisfies two important properties: first the Beauville relation, saying that there exists a constant $c > 0$ such that

$$q_X(\alpha)^n = c\int_X \alpha^{2n} \qquad \text{for all } \alpha \in H^2(X, \mathbb{R}).$$

Second, the next Proposition describes completely the signature of the form.

Proposition 2.9. *The Beauville-Bogomolov form has signature $(3, b_2(X) - 3)$ on $H^2(X, \mathbb{R})$. More precisely, the decomposition $H^2(X, \mathbb{R}) = H^{1,1}(X, \mathbb{R}) \oplus \big(H^{2,0}(X) \oplus H^{0,2}(X)\big)_{\mathbb{R}}$ is orthogonal with respect to q_X, and q_X has signature $(1, h^{1,1}(X) - 1)$ on $H^{1,1}(X, \mathbb{R})$ and is positive definite on $\big(H^{2,0}(X) \oplus H^{0,2}(X)\big)_{\mathbb{R}}$.*

We denote by

$$\mathcal{C}_{\geq 0} = \{v \in H^{1,1}(X, \mathbb{R}) \,|\, q_X(v) \geq 0\} \subset H^{1,1}(X, \mathbb{R}),$$

$$\mathcal{C}_0 = \{v \in H^{1,1}(X, \mathbb{R}) \,|\, q_X(v) = 0\} \subset H^{1,1}(X, \mathbb{R})$$

the positive cone and the isotropic cone of q_X.

Remark 2.10. For a divisor $D \in Div(X)$, we define $q_X(D) := q_X(c_1(\mathcal{O}_X(D)))$.
If D is effective and the complete linear system $|D|$ does not have any fixed component, then $q_X(D) \geq 0$. Indeed, let D' be an effective divisor linearly equivalent to D and with no components in common with D. Up to a positive constant,

$$q_X(D) = \int_{D\cap D'} (\sigma\bar{\sigma})^{n-1},$$

where each irreducible component of the intersection $D \cap D'$ is counted with its multiplicity. The integral on the right hand side is non-negative because σ is a holomorphic form.
If furthermore D is ample, then by the Beauville relation $q_X(D) > 0$.

2.4 Bimeromorphic maps between irreducible holomorphic symplectic manifolds

A bimeromorphic map $f\colon M \dashrightarrow M'$ between complex manifolds is an isomorphism in codimension 1 if there exist dense open subsets $U \subset M$ and $U' \subset M'$ such that

1. $\operatorname{codim}(M \setminus U) \geq 2, \operatorname{codim}(M' \setminus U') \geq 2$;
2. f induces an isomorphism $U \cong U'$.

A *pseudo-automorphism* of a complex manifold is a bimeromorphic transformation which is an isomorphism in codimension 1.

Proposition 2.11 (Proposition III.21.6 and III.25.14 in [GHJ03]). *Let $f\colon X \dashrightarrow X'$ be a bimeromorphic map between irreducible holomorphic symplectic manifolds. Then f is an isomorphism in codimension 1 and induces a linear isomorphism*

$$f^*\colon H^2(X',\mathbb{Z}) \xrightarrow{\sim} H^2(X,\mathbb{Z})$$

which preserves the Beauville-Bogomolov form.
In particular, the group of birational transformations of an irreducible holomorphic symplectic manifold X coincides with its group of pseudo-automorphisms and acts by isometries on $H^2(X,\mathbb{Z})$.

Remark 2.12 ([GHJ03], §III.21.3). In fact the proposition holds in a greater generality: if $f\colon X \dashrightarrow X'$ is a birational map between manifolds X, X' with nef canonical bundle, then f is an isomorphism in codimension 1. In this case one can show that for a resolution of the indeterminacy locus of f (see §5.1) $X \leftarrow Z \rightarrow X'$ the set of exceptional divisors is the same for both projections; in particular, f induces an isomorphism $H^2(X,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$.

Classification of birational transformations

Let $f\colon X \dashrightarrow X$ be a birational transformation of an irreducible symplectic manifold. By Proposition 2.11, $f^*_{1,1}\colon H^{1,1}(X,\mathbb{R}) \rightarrow H^{1,1}(X,\mathbb{R})$ is a linear automorphism which preserves the Beauville-Bogomolov form. Therefore, we obtain a group homomorphism

$$\operatorname{Bir}(X) \to O(H^{1,1}(X,\mathbb{R}), q_X) \cong O(1, h^{1,1}(X)-1),$$

and we can classify birational transformations of X as loxodromic, parabolic or elliptic depending on their action on $H^{1,1}(X,\mathbb{R})$ (see §1.3.2): $f \in \operatorname{Bir}(X)$ is

- *loxodromic* (or *hyperbolic*) if $f^*_{1,1}$ admits an eigenvalue of modulus strictly greater than 1 (i.e. $\lambda_1(f) > 1$);
- *parabolic* if all the eigenvalues of $f^*_{1,1}$ have modulus 1 and $\parallel (f^n)^*_{1,1} \parallel$ is not bounded as $n \to +\infty$;
- *elliptic* if all the eigenvalues of $f^*_{1,1}$ have modulus 1 and $\parallel (f^n)^*_{1,1} \parallel$ is bounded as $n \to +\infty$.

Remark 2.13. Since $H^{0,2}(X)$ has dimension 1, $f^*\sigma = \xi\sigma$ for some $\xi \in \mathbb{C}^*$; furthermore, f preserves the total volume

$$\operatorname{vol}(X) := \int_X (\sigma \wedge \bar{\sigma})^n,$$

so that $|\xi| = 1$.

The action of f on cohomology preserves the lattice $H^2(X,\mathbb{Z})/\text{torsion}$ of $H^2(X,\mathbb{R})$; although $H^2(X,\mathbb{R})$ is not a hyperbolic space, thanks to the above Remark one can prove that the properties stated in Theorem 1.22 under the assumption that $\phi \in O(V,q)$ preserves a lattice are true for $f^*_{1,1}$.

Proposition 2.14. *Let $f\colon X \dashrightarrow X$ be a birational transformation of an irreducible symplectic manifold X.*

- *If f is loxodromic (or, equivalently, $\lambda_1(f) > 1$), then $f^*_{1,1}$ is semi-simple has exactly one eigenvalue with modulus > 1 and exactly one eigenvalue with modulus < 1; these eigenvalues are equal to $\lambda = \lambda_1(f)$ and λ^{-1} respectively and they are simple; in particular the norm of $(f^n)^*_{1,1}$ grow as $c\lambda^n$. The eigenspaces $\mathbb{R}v_+, \mathbb{R}v_-$ of λ and λ^{-1} are contained in $\mathcal{C}_0$ and they are the only $f^*_{1,1}$-invariant lines in $\mathcal{C}_{\geq 0}$; they are not defined over $\mathbb{Q}$. The first dynamical degree $\lambda_1(f)$ is a quadratic or Salem number.*

- *If f is parabolic, then $f^*_{1,1}$ fixes exactly one line $\mathbb{R}v$ of $\mathcal{C}_{\geq 0}$, which is contained in $\mathcal{C}_0$ and defined over $\mathbb{Q}$; the norm of iterates of $f^*_{1,1}$ grows quadratically. All the eigenvalues of $f^*_{1,1}$ are roots of unity, and some iterate of $f^*_{1,1}$ has Jordan form*

$$\begin{pmatrix} 1 & 1 & 0 & \mathbf{0} \\ 0 & 1 & 1 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{N-3} \end{pmatrix} \qquad N = \dim H^{1,1}(X)$$

- *If f is elliptic, then $f^*_{1,1}$ fixes exactly one line $\mathbb{R}v$ in the interior of $\mathcal{C}_{\geq 0}$; since q is negative definite on $v^\perp$, $f^*_{1,1}$ acts as a rotation on $v^\perp$. Furthermore, f has finite order.*

2.5 The parabolic case

Recall that, in the case of surfaces, an automorphism with action of infinite order on the cohomology is parabolic if and only if it preserves an elliptic fibration (Theorem 1.30).

We could expect the situation to be similar in the irreducible holomorphic symplectic context; indeed, inTheorem A (Chapter 5) I prove that loxodromic transformation s of irreducible symplectic manifolds do not preserve any meromorphic fibration.

Hu, Keum and Zhang have proved a partial analogue of the other direction of Theorem 1.30:

Theorem 2.15 ([HKZ15])**.** *Let X be a $2n$-dimensional projective irreducible holomorphic symplectic manifold of type $K3^{[n]}$ or of type generalized Kummer and let $f \in Bir(X)$ be a bimeromorphic transformation which is not elliptic; f is parabolic if and only if it admits a rational Lagrangian equivariant fibration $\pi\colon X \dashrightarrow \mathbb{P}^n$ such that the induced transformation on $\mathbb{P}^n$ is biregular, i.e. there exists $g \in Aut(\mathbb{P}^n)$ such that $\pi \circ f = g \circ \pi$. More accurately, there exists a birational irreducible symplectic model X' of X and a holomorphic Lagrangian fibration $X' \to \mathbb{P}^n$ whose fibres are preserved by the induced transformation $f' \in \mathrm{Bir}(X')$.*

The hard direction is to exhibit an equivariant fibration for a parabolic transformation; this is a consequence of the Lagrangian conjecture (or SYZ conjecture), which is a hyperkähler version of the abundance conjecture. It was proved in the case of manifolds of type $K3^{[n]}$ or generalized Kummer by Bayer and Macrì in [BM14]; see also [Mat13, Corollary 1.1].

Conjecture 2.1 (Lagrangian conjecture)**.** *Let X be a projective irreducible symplectic manifold and let L be a line bundle on X such that $q_X(L) = 0$. Then, after replacing X by an irreducible symplectic birational model, $\pm L$ is semi-ample (i.e. some multiple $L^{\otimes m}$ of L is base-point-free).*

If the conjecture is verified for some pair (X, L), then, after replacing X by an irreducible symplectic birational model, some multiple of L defines a regular fibration $\pi\colon X \to B$ and $0 < \dim B < \dim X$. Then, by Matsushita's results [Mat01, Mat00] all fibres are Lagrangian (possibly singular), and the smooth fibres are complex tori of dimension $n = \dim X/2$; furthermore, if B is smooth then $B = \mathbb{P}^n$ [Hwa08]. See [LOP16] for a survey about the abundance conjecture in its different versions.

2.6 Cohomology of irreducible symplectic manifolds

Although the second cohomology group of irreducible symplectic manifolds is fairly well-understood, the rest of the cohomology is still quite mysterious. We will list here some of the results in this direction.

2.6.1 Holomorphic forms

Proposition 2.16 ([Bea83], Proposition 4.3)**.** *The only holomorphic forms on an irreducible symplectic manifold are the multiple of the symplectic forms and of its powers. In other words, if σ is the symplectic form on X,*

$$H^0(X, \Omega^p_X) = \begin{cases} \mathbb{C}\sigma^{p/2} & \textit{if } p \textit{ is even} \\ 0 & \textit{if } p \textit{ is odd} \end{cases}$$

Example 2.17. Let us study the Hodge diamond of an irreducible symplectic fourfold. By Proposition 2.16, the Hodge numbers $h^{p,0}$ are 0 for p odd and 1 for p even. Poincaré duality and the relation $h^{p,q} = h^{q,p}$ imply that the Hodge diamond is of the form

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 1 & & a & & 1 & & \\
 & 0 & & b & & b & & 0 & \\
1 & & c & & d & & c & & 1 \\
 & 0 & & b & & b & & 0 & \\
 & & 1 & & a & & 1 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}.$$

Corollary 2.18. *Let $\mathcal{F}$ be a non-trivial foliation on an irreducible symplectic manifold X. Then* $\det N\mathcal{F} \not\cong \mathcal{O}_X$.

Proof. If $\det N\mathcal{F}$ were trivial, $\mathcal{F}$ would be defined by a global holomorphic form ω. By Proposition 2.16, $\omega = c\sigma^k$ for some $0 \le k \le \dim X/2$; but such a form does not define a $2k$-codimensional foliation because it has no kernel, which is a contradiction. □

Some additional restrictions on Betti and Hodge numbers of irreducible symplectic manifolds are listed in Example 4.7.

2.6.2 Dynamical degrees of automorphisms

The following Proposition by Verbitsky [Ver96] completely describes the part of cohomology which is generated by $H^2(X, \mathbb{C})$.

Proposition 2.19. *Let X be an irreducible holomorphic symplectic manifold of dimension $2n$ and let $SH^2(X,\mathbb{C}) \subset H^*(X,\mathbb{C})$ be the subalgebra generated by $H^2(X,\mathbb{C})$. Then we have an isomorphism*

$$SH^2(X,\mathbb{C}) \cong \operatorname{Sym}^* H^2(X,\mathbb{C})/\langle \alpha^{n+1} | q_X(\alpha) = 0\rangle$$

.

As a corollary, one can describe all the dynamical degrees (and thus the topological entropy) of an automorphism in terms of its first dynamical degree.

Corollary 2.20 (Oguiso [Ogu09]). *Let $f\colon X \to X$ be an automorphism of an irreducible symplectic manifold of dimension $2n$. Then for $p = 0, 1, \ldots, n$*

$$\lambda_p(f) = \lambda_{2n-p}(f) = \lambda_1(f)^p.$$

In particular f has topological entropy

$$h_{top}(f) = n \log \lambda_1(f).$$

Proof. By Proposition 2.19 the cup-product induces an injection

$$\operatorname{Sym}^p H^2(X,\mathbb{C}) \hookrightarrow H^{2p}(X,\mathbb{C})$$

for $p = 1, \ldots, n$.
Let $v_1 \in H^{1,1}(X,\mathbb{R})$ be an eigenvector for the eigenvalue $\lambda = \lambda_1(f)$. Then, because of the injection $\operatorname{Sym}^p H^2(X,\mathbb{C}) \hookrightarrow H^{2p}(X,\mathbb{C})$, $v_p := v_1^p \in H^{p,p}(X,\mathbb{R})$ is a non-zero class for $p = 1, \ldots, n$ and $f^*v_p = (f^*v_1)^p = \lambda^p v_p$. This implies that $\lambda_p(f) \geq \lambda_1(f)^p$, and we must have equality by log-concavity (Proposition 1.14). This proves the result for $p = 0, 1, \ldots, n$.
Now by Lemma 1.13 we have $\lambda_{2n-p}(f) = \lambda_p(f^{-1})$. Applying what we have just proved to f^{-1} we obtain

$$\lambda_{2n-1}(f) = \lambda_1(f^{-1}) = \lambda_n(f^{-1})^{1/n} = \lambda_n(f)^{1/n} = \lambda_1(f)$$

and thus, for $p = 0, \ldots, n$,

$$\lambda_{2n-p}(f) = \lambda_p(f^{-1}) = \lambda_1(f^{-1})^p = \lambda_1(f)^p,$$

which concludes the proof. □

A more precise description of the action on cohomology of an automorphism has been later proven by Verbitsky (see Theorem 4.4).

Chapter 3

Examples of loxodromic automorphisms

In this chapter we will describe the fundamental examples of loxodromic automorphisms of irreducible symplectic manifolds; we refer to [Bea83] for more details. We expect that if a loxodromic transformation preserves some "rigid enough" algebraic or holomorphic structure, then it automatically belongs to one of the two fundamental examples.

For both constructions, the starting point is a loxodromic linear automorphism $f_T \colon T \to T$ of a two-dimensional torus T; for instance we can take $T = E \times E$ where $E = \mathbb{C}/\Lambda$ is an elliptic curve, and $f_T \colon T \to T$ given in linear coordinates by an invertible matrix with integer coefficients $M \in \mathrm{SL}_2(\mathbb{Z})$ and such that $|\operatorname{Tr} M| > 2$ (see §1.3.3 and [Can14]).
In general $T = \mathbb{C}^2/\Lambda$ for a lattice Λ of $\mathbb{C}^2$, and f_T is described by a matrix M preserving Λ and whose eigenvalues α, β satisfy

$$|\alpha| > 1 > |\beta| = \frac{1}{|\alpha|}.$$

Recall that

- $\lambda_1(f_T) = |\alpha|^2$;
- f_T preserves two linear transverse foliations corresponding to the two eigenvalues of M: the stable foliation $\mathcal{F}_T^-$ (corresponding to β) and the unstable foliation $\mathcal{F}_T^+$ (corresponding to α);
- there exists a measurable subset $T_0 \subset T$ of total Lebesgue measure such that the f_T-orbits of points of T_0 are dense in T for the euclidean topology (hence in particular for the Zariski topology);
- the f_T-periodic points are dense in T for the euclidean topology (hence in particular for the Zariski topology).

The following lemma will be used later.

Lemma 3.1. *Let T be a two-dimensional torus and let $f_T \colon T \to T$ be a loxodromic automorphism. Denote by $f = (f_T, \ldots, f_T) \colon T^n \to T^n$ the induced automorphism of $A := T^n$, and suppose that f preserves a singular distribution $\mathcal{F}$ on A; then $\mathcal{F}$ is a linear foliation. Furthermore, denoting*

$$\mathcal{F}^+ = \mathcal{F}_T^+ \times \ldots \times \mathcal{F}_T^+, \qquad \mathcal{F}^- = \mathcal{F}_T^- \times \ldots \times \mathcal{F}_T^-,$$

the tangent bundle $\mathcal{F}$ decomposes as $T\mathcal{F} = (T\mathcal{F} \cap T\mathcal{F}^+) \oplus (T\mathcal{F} \cap T\mathcal{F}^-)$.

Proof. Fix global linear coordinates x_i, y_i on the i-th factor T such that in such coordinates f_T is given by the matrix

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \qquad |\alpha| = \sqrt{\lambda_1(f_T)}$$

Let $\mathcal{F}^+ = \mathcal{F}_T^+ \times \ldots \times \mathcal{F}_T^+$, $\mathcal{F}^- = \mathcal{F}_T^- \times \ldots \times \mathcal{F}_T^-$; $\mathcal{F}^+$ and $\mathcal{F}^-$ are f-invariant linear foliations, and

$$T\mathcal{F}^+ = \operatorname{Span}\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right), \qquad T\mathcal{F}^- = \operatorname{Span}\left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right).$$

Then

$$T\mathcal{F} = (T\mathcal{F} \cap T\mathcal{F}^+) \oplus (T\mathcal{F} \cap T\mathcal{F}^-);$$

indeed, if $p \in T^n$ is an N-periodic point, then the differential Df_p^N acts on $T_p\mathcal{F}^+$ (resp. on $T_p\mathcal{F}^-$) as α^N id (resp. as β^N id). Therefore,

$$T_p\mathcal{F} = (T_p\mathcal{F} \cap T_p\mathcal{F}^+) \oplus (T_p\mathcal{F} \cap T_p\mathcal{F}^-),$$

and by density of f-periodic points the decomposition holds at every point of T^n.
If we show that $\mathcal{F} \cap \mathcal{F}^+$ and $\mathcal{F} \cap \mathcal{F}^-$ are both linear, then $\mathcal{F}$ is also linear. Thus from now on we will assume that $\mathcal{F} \subset \mathcal{F}^+$.

Since the tangent sheaf of X is trivial, we may choose an identification $T_pX \cong \mathbb{C}^{2n}$ not depending on the point; the foliation $\mathcal{F}^+$ being linear, the tangent spaces $T_p\mathcal{F}^+$ are identified with some subspace $\mathbb{C}^n \subset \mathbb{C}^{2n}$ not depending on p.
A singular sub-distribution $\mathcal{F}$ of dimension k of $\mathcal{F}^+$ is then given by a meromorphic function

$$\Phi \colon X \dashrightarrow \operatorname{Gr}(k, T_p\mathcal{F}^+) \subset \operatorname{Gr}(k, T_pX),$$

where $\operatorname{Gr}(k, V)$ is the Grassmannian variety of k-planes in a vector space V. Remark that

$$\Phi \circ f(p) = Df_p \circ \Phi,$$

where Df_p is the linear action of the differential of f at p on $\operatorname{Gr}(k, T_pX)$; since $Df_p|_{T_p\mathcal{F}^+} = \alpha\, \mathrm{id}_{T_p\mathcal{F}^+}$, we have

$$\Phi \circ f = \Phi.$$

The automorphism f admits dense orbits, therefore Φ is constant, which proves the claim. □

Before describing the examples, we will need some background about Kummer surfaces and the Hilbert scheme of points on a variety.

3.1 Kummer surfaces

Let T be a two dimensional complex torus and let

$$\begin{aligned} \theta \colon T &\to T \\ x &\mapsto -x \end{aligned}$$

be the involution sending an element $x \in T$ to its inverse with respect to the group law on T. The fixed points of θ are the 16 points of order two of T (i.e. $x \in T$ satisfying $2x = 0$).
The quotient

$$\hat{S} = {}^{T}\!/_{\langle\theta\rangle}$$

is a singular variety whose singularities are exactly the images of the fixed points for θ. Locally around these points, θ acts as $-\mathrm{id}_{\mathbb{C}^2}\colon \mathbb{C}^2 \to \mathbb{C}^2$; therefore the resulting singularity is locally biholomorphic to the cone

$$\{x^2 + y^2 + z^2 = 0\} \subset \mathbb{C}^3,$$

and can be resolved with a single blow-up (see for example [BPVdV84]). This is the simplest example of du Val (or rational Gorenstein) surface singularities.

Definition 3.2. *The* Kummer surface $S = K(T)$ *associated to T is the minimal resolution of $\hat{S}$.*

In order to construct such a resolution, consider the blow-up $\pi\colon \widetilde{T} \to T$ of T at the sixteen two-torsion points $p_1, \dots, p_{16}$. Since θ fixes the set $\{p_1, \dots, p_{16}\}$ and has invertible differential at every point, it induces an involution $\tilde{\theta}$ which is locally conjugated to $(x, y) \mapsto (-x, y)$; the quotient map $\widetilde{T} \to \widetilde{T}/\langle\tilde{\theta}\rangle$ is then locally conjugated to $(x, y) \mapsto (x^2, y)$ (see the proof of Proposition 3.4 for more details). Therefore the quotient $\widetilde{T}/\langle\tilde{\theta}\rangle$ is smooth and fits into a commutative diagram

$$\begin{array}{ccc} \widetilde{T} & \xrightarrow{\quad\pi\quad} & T \\ {\scriptstyle\tilde{\nu}}\downarrow & & \downarrow{\scriptstyle\nu} \\ S = \widetilde{T}/\langle\tilde{\theta}\rangle & \xrightarrow{\pi_S} & \hat{S} = T/\langle\theta\rangle \end{array} .$$

The surface S is the Kummer surface associated to T.

Remark 3.3. In some contexts, the singular surface $\hat{S}$ is still called a Kummer surface; we prefer to restrict our attention to smooth varieties.

Proposition 3.4. *A Kummer surface $S = K(T)$ is a $K3$ surface.*

This result is well-known, but we give the idea of the proof in order to introduce the notation and the kind of computations which will be used in the rest of the chapter.

Proof. Let x, y be global linear coordinates on T and let $\omega_T = dx \wedge dy$ be a section of K_T. Then, keeping the same notation as above, the blow-up π can be expressed in local coordinates x_1, z on $\widetilde{T}$ as

$$\pi\colon (x_1, z) \mapsto (x_1, x_1 z),$$

where $\{x = 0\}$ corresponds to the (local) π-exceptional divisor. Hence, in such local coordinates,

$$\pi^* dx \wedge dy = dx_1 \wedge d(x_1 z) = x_1 dx_1 \wedge dz;$$

the involution $\tilde{\theta}$ becomes

$$\tilde{\theta}(x_1, z) = (-x_1, z),$$

thus the quotient map $\tilde{\nu}$ can be locally written as

$$\tilde{\nu}(x_1, z) = (x_1^2, z)$$

for some local coordinates x_2, y_2 on S. Remark that

$$x_1 dx_1 \wedge dz = \frac{1}{2}\tilde{\nu}^*(dx_2 \wedge dy_2),$$

so that ω_T induces a canonical section without zeros on S. Hence $K_S \cong \mathcal{O}_S$.

In order to show that $H^1(S, \mathbb{Q}) = 0$, remark that $H^1(\widetilde{T}, \mathbb{Q}) = \pi^* H^1(T, \mathbb{Q})$, and that the only θ-invariant element of $H^1(T, \mathbb{Q})$ is 0; hence $H^1(S, \mathbb{Q}) = 0$. This concludes the proof. □

Since $K_S \cong \mathcal{O}_S$, by adjunction the exceptional rational curves $E_1, \dots, E_{16}$ arising from the resolution of singularities of $\hat{S}$ are (-2)-curves. This can also be proven directly using the same computations in local coordinates as in the proof of Proposition 3.4.

3.1.1 Linear automorphisms and invariant foliations

A linear automorphism $f_T\colon T \to T$ commutes to θ and in particular fixes the set of two-torsion points; therefore, f_T induces an automorphism $f_S\colon S \to S$.
Since the rational map

$$\hat{\pi} = \tilde{\nu} \circ \pi^{-1}\colon T \dashrightarrow S$$

is generically finite and $f_S \circ \hat{\pi} = \hat{\pi} \circ f_T$, by Proposition 1.12 we have

$$\lambda_1(f_S) = \lambda_1(f_T).$$

Now, suppose that f_T is loxodromic (hence so is f_S). The (linear) vector fields v_+, v_- which define the unstable and stable foliation for f_T on T are not θ-invariant; however, the directions they define are, so that f_S preserves two foliations $\mathcal{F}_S^{\pm}$ induced by $\mathcal{F}_T^{\pm}$. A simple computation in local coordinates shows that, once we put $\mathcal{F}_S^{\pm}$ in reduced form, the exceptional divisors are the closure of a leaf for both foliations; $\mathcal{F}_S^{+}$ (resp. $\mathcal{F}_S^{-}$) has exactly one singular point along each E_i, corresponding to the direction of $\mathcal{F}_T^{+}$ (resp. $\mathcal{F}_T^{-}$).

The linear foliations $\mathcal{F}_T^{\pm}$ are defined by global (linear) forms, say dx, dy, on T; these forms are not θ-invariant, but the pluri-forms

$$(dx)^2, (dy)^2 \in H^0(T, (\Omega_T^1)^{\otimes 2})$$

are, and therefore define meromorphic pluri-forms on S. More accurately, choosing local coordinates as in the proof of Proposition 3.4, we have

$$\pi^*(dx)^2 = (dx_1)^2 = \frac{1}{4}\tilde{\nu}^*\left(\frac{1}{x_2}(dx_2)^2\right).$$

Remark that the simple pole $\{x_2 = 0\}$ corresponds to the exceptional divisor $E = E_1 \cup \ldots \cup E_{16}$.

Proposition 3.5. *Let $S = K(T)$ be the Kummer surface associated to a torus T and let $E = E_1 \cup \ldots \cup E_{16} \in Div(S)$ be the exceptional divisor of the map $S \to T/\pm \mathrm{id}_T$. The stable and unstable foliation $\mathcal{F}_S^{\pm}$ are defined by global pluri-forms with values in E*

$$\omega_{\pm} \in H^0(S, (\Omega_S^1)^{\otimes 2} \otimes E).$$

In other words, $N\mathcal{F}_S^{\pm} = \frac{E}{2} = \frac{1}{2}(E_1 + \ldots + E_{16})$ as $\mathbb{Q}$-line bundles.

Proof. By the above discussion, the foliation $\mathcal{F}_S^{+}$ is defined by a meromorphic pluri-form α which in local coordinates as above is written

$$\alpha = \frac{1}{4x_2}(dx_2)^2.$$

Thus α has simple poles along E; let s be a section of $\mathcal{O}(E)$ with simple zeros along E. Then $\beta = s\alpha$ is a regular pluri-form with values in E defining $\mathcal{F}_S^{+}$ as claimed.

The foliation $\mathcal{F}_S^{+}$ can as well be defined by a holomorphic form

$$\omega \in H^0(S, \Omega_S^1 \otimes \mathcal{L}),$$

where $\mathcal{L} = N\mathcal{F}_S^{+}$; therefore,$\beta$ and $(\omega)^2$ must be proportional to one another at each point: there exists a meromorphic function $\xi\colon S \dashrightarrow \mathbb{C}$ such that

$$(\omega)^2 = \xi \cdot \beta.$$

Since the poles of ξ are contained in the zeros of β, which are in codimension 2, ξ is holomorphic, thus constant. Thus, $(\omega)^2 = \xi\beta$, whence $N\mathcal{F}_S^{+} = E/2$.

The proof for $\mathcal{F}_S^{-}$ is identical. $\square$

The dynamics of automorphisms of $K3$ surfaces has been studied in [Can01]. In the case of $K3$ surfaces, Theorem 1.32 implies the following corollary.

Corollary 3.6 (Cantat, Favre [CF03])**.** *Let $f\colon S \to S$ be a loxodromic automorphism of a projective $K3$ surface S which preserves a (singular) foliation $\mathcal{F}$. Then $S = K(T)$ is the Kummer surface associated to a two-dimensional complex torus T, f comes from a linear automorphism f_T of T, and $\mathcal{F}$ is either the stable or the unstable foliation of f.*

3.1.2 Affine structure outside the exceptional divisor

Let $S = K(T)$ be a Kummer surface, let $E = E_1 \cup \ldots \cup E_{16}$ be the exceptional divisor of $S \to T/\pm \mathrm{id}_T$ and let $p \in S \setminus E$. We can fix local coordinates x, y at a neighborhood U of p, induced by linear coordinates on the torus T; if $V \subset S \setminus E$ is another open set and x', y' are other local coordinates induced by the same linear coordinates on T, then the change of coordinates is an affine transformation.

We have thus defined an *affine structure* on $S \setminus E$: that is, an atlas of charts with values in $\mathbb{C}^2$ and affine changes of coordinates. More precisely, the change of coordinates are affine, and their linear parts are matrices of type

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \qquad \alpha, \beta \in \mathbb{C}^*.$$

Let $f_T\colon T \to T$ be a loxodromic automorphism, with stable/unstable foliations $\mathcal{F}_T^{\pm}$, and let $f_S\colon S \to S$ be the induced loxodromic automorphism of S, with stable/unstable foliations $\mathcal{F}_S^{\pm}$. In the above construction, one can pick linear coordinates x, y on T in the directions of $\mathcal{F}_T^{+}$ and $\mathcal{F}_T^{-}$; therefore, the resulting affine coordinates on $S \setminus E$ are such that the stable/unstable foliations are exactly the coordinate foliations.

3.2 The Hilbert scheme of points on a surface

Let S be a surface and let

$$\pi\colon S^n \to S^{(n)} = {S^n}\big/{\mathfrak{S}_n}$$

be the natural projection from S^n onto the n-th symmetric product of S; the smooth locus of $S^{(n)}$ parametrizes subsets of n points of S, but as two or more of these points tend to coincide, the variety becomes singular. The *Douady space* or *Hilbert scheme* of n points on S

$$Hilb^n(S) = S^{[n]}$$

is a smooth variety which parametrizes zero-dimensional (possibly non-reduced) analytic subspaces of S of length n; therefore, one has a natural morphism, called the Hilbert-Chow morphism,

$$\rho\colon S^{[n]} \to S^{(n)}$$

which associates to a subscheme the associated 0-cycle. At least in the projective case, the construction of the Hilbert scheme of points is a special case of a much more general scheme parametrizing the subschemes of a given variety; see [Gro95].

For $1 \leq i < j \leq n$ let

$$\Delta_{ij} = \{(x_1, \ldots, x_n) \in S^n \mid x_i = x_j\} \subset S^n,$$

and let

$$\Delta = \Delta(S^n) = \bigcup_{1\le i<j\le n} \Delta_{ij}, \qquad D = \pi(\Delta) \subset S^{(n)}, \qquad E = \rho^* D \subset S^{[n]}.$$

Also let

$$\Delta_0 = \Delta \setminus \left(\bigcup_{i,j,k,l} \Delta_{ij} \cap \Delta_{kl} \right), \qquad D_0 = \pi(\Delta_0), \qquad E_0 = \rho^* D_0,$$

$$S_0^n = (S^n \setminus \Delta) \cup \Delta_0, \qquad S_0^{(n)} = (S^{(n)} \setminus D) \cup D_0, \qquad S_0^{[n]} = (S^{[n]} \setminus E) \cup E_0.$$

We summarize here some of the properties of the Hilbert scheme of n points on a surface (see [Bea83]).

1. $S^{[n]}$ is smooth (this is specific to the surface case: in general Hilbert schemes of points are not smooth);
2. ρ is a birational morphism: hence $S^{[n]}$ is a resolution of singularities of $S^{(n)}$;
3. $S^{[n]} \setminus S_0^{[n]}$ has codimension at least 2 in $S^{[n]}$;
4. the restriction $\rho\colon S_0^{[n]} \to S_0^{(n)}$ identifies with the blow-up of $S_0^{(n)}$ along D_0;
5. $\mathfrak{S}_n$ acts naturally on the blow-up $\widetilde{S_0^n}$ of S_0^n along Δ_0, and $S_0^{[n]}$ identifies naturally with the quotient $S_0^n/\mathfrak{S}_n$.
6. we have therefore a commutative diagram

$$\begin{array}{ccc} \widetilde{S_0^n} & \xrightarrow{\tilde\eta} & S_0^{[n]} \\ {\scriptstyle \tilde\rho=\mathrm{Bl}_{\Delta_0}}\big\downarrow & & \big\downarrow{\scriptstyle \rho=\mathrm{Bl}_{D_0}} \\ S_0^n & \xrightarrow{\eta} & S_0^{(n)} \end{array},$$

   where $\mathrm{Bl}_\bullet$ denotes the blow-up along a subvariety and $\eta, \tilde\eta$ are quotients with respect to the actions of $\mathfrak{S}_n$.

Let

$$f_S\colon S \to S$$

be an automorphism of S; then the natural action of f_S on zero-dimensional subspaces of S of length n induces an automorphism

$$f_S^{[n]}\colon S^{[n]} \to S^{[n]}.$$

Suppose that S is compact Kähler and let $f_n = (f, \dots, f)\colon S^n \to S^n$ be the automorphism of S^n induced by f. By picking a Kähler form ω on S and $\pi_1^*\omega + \ldots + \pi_n^*\omega$ on S^n, one shows that $\lambda_1(f_n) = \lambda_1(f)$; hence, since the meromorphic map $S^n \dashrightarrow S^{[n]}$ induced by the quotient by $\mathfrak{S}_n$ is generically finite and induces $f^{[n]}$, by Proposition 1.12

$$\lambda_1(f^{[n]}) = \lambda_1(f).$$

3.3 First example: the Hilbert scheme of points on a Kummer surface

Let S be a $K3$ surface, and let $X = S^{[n]}$ be the Hilbert scheme of n points on S; then X is an irreducible symplectic manifold (see [Bea83]). The symplectic structure is induced by

$$\pi_1^*\sigma_S + \ldots + \pi_n^*\sigma_S \in H^0(S^n, \Omega^2_{S^n}),$$

where $\pi_i \colon S^n \to S$ is the projection onto the i-th factor and σ_S is a symplectic form on S.

As we have seen in §3.2, if

$$f_S \colon S \to S$$

is an automorphism of S, then the induced automorphism

$$f_X = f_S^{[n]} \colon X \to X$$

satisfies $\lambda_1(f_X) = \lambda_1(f_S)$.

Suppose that f_S is loxodromic and preserves a foliation; if S is projective, by Corollary 3.6, S is the Kummer surface associated to a complex torus T and f_S comes from a linear automorphism $f_T \colon T \to T$. From now on, we assume that S is the Kummer surface associated to a torus T, and that f_S comes from a linear loxodromic automorphism $f_T \colon T \to T$.

Denoting by $\mathcal{F}_S^{\pm}$ the stable/unstable foliation for f_S, the product foliations

$$\mathcal{F}^+_{S^n} = \mathcal{F}^+_S \times \ldots \times \mathcal{F}^+_S, \qquad \mathcal{F}^-_{S^n} = \mathcal{F}^-_S \times \ldots \times \mathcal{F}^-_S$$

are $\mathfrak{S}_n$-invariant, and therefore define two (singular) foliations $\mathcal{F}_X^{\pm}$ on X, which we will call the stable/unstable foliation for f_X.
We will take $\mathcal{F}_X^+, \mathcal{F}_X^-$ in reduced form, so that

$$\operatorname{codim} \operatorname{Sing}(\mathcal{F}^{\pm}_{\mathrm{X}}) \geq 2.$$

Lemma 3.7. *The foliations $\mathcal{F}_X^+$ and $\mathcal{F}_X^-$ are generically transverse n-dimensional Lagrangian foliations which are f_X-invariant.*

Proof. We can check the statement at a general point of X; in this case, X is locally biholomorphic to $S^{(n)}$, thus to S^n; therefore we can check the claims by replacing $\mathcal{F}_X^{\pm}$ by $\mathcal{F}^{\pm}_{S^n}$ and f_X by f_n. By construction, the foliations $\mathcal{F}^{\pm}_{S^n}$ are generically transverse n-dimensional foliations, and they are f_n-invariant; furthermore, they are Lagrangian with respect to the symplectic form $\pi_1^*\sigma_S + \ldots + \pi_n^*\sigma_S$, which induces the symplectic structure on $S^{[n]}$. □

3.3.1 Structure of $\mathcal{F}_X^{\pm}$

In this paragraph we give a more detailed description of the stable/unstable foliations $\mathcal{F}_X^{\pm}$.

Singular locus

Let us try to explicitly describe the singular locus of $\mathcal{F}_X^{\pm}$. First of all, it is clear that the strict transform of

$$\operatorname{Sing}(\mathcal{F}^+_{\mathrm{S}}) \times \mathrm{S} \times \ldots \times \mathrm{S} \subset \mathrm{S}^{\mathrm{n}}$$

by the natural map $S^n \dashrightarrow S^{[n]} = X$ is contained in the singular locus of $\mathcal{F}_X^+$. These are the only singularities of $\mathcal{F}_X^+$ outside the exceptional divisor E_X (which was denoted as E in §3.2).

Now let us deal with the exceptional divisor E_X; recall that we have defined $E_0 \subset E_X$ such that $\operatorname{codim}_X(E_X \setminus E_0) \geq 2$, and that $X_0 = X \setminus (E_X \setminus E_0)$ identifies with the quotient $\widetilde{S_0^n}/\mathfrak{S}_n$, where $\widetilde{S_0^n} \to S_0^n$ is the blow-up of $S_0^n = S^n \setminus (\Delta \setminus \Delta_0)$ along Δ_0.

Lemma 3.8. *The exceptional divisor $E_0 \subset X$ contains the leaves of $\mathcal{F}_X^{\pm}$ which pass through its points. Furthermore, each exceptional fibre of the birational map*

$$X_0 \to S_0^{(n)}$$

intersects the singular locus of $\mathcal{F}_X^+$ (resp. of $\mathcal{F}_X^-$) in exactly one point.

Proof. Consider linear coordinates x, y on T along the stable/unstable foliations. Take a point $p = (p_1, \ldots, p_n)$ of $\Delta_0 \subset S^n$; the coordinates x, y induce local coordinates $x_1, y_1, \ldots, x_n, y_n$ on a neighborhood $U \subset S^n$ of p such that the foliations $\mathcal{F}_{S^n}^{\pm}$ are locally defined by the forms

$$\omega_+ = dx_1 \wedge \ldots \wedge dx_n, \qquad \omega_- = dy_1 \wedge \ldots \wedge dy_n$$

respectively; assuming by symmetry that $p \in \Delta_{12}$, locally we have

$$\Delta = \Delta_{12} = \{x_1 = x_2, y_1 = y_2\}.$$

Let us apply the linear change of coordinates

$$(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) \mapsto (x_1 - x_2, y_1 - y_2, x_1 + x_2, y_1 + y_2, x_3, y_3, \ldots, x_n, y_n)$$

so that, in the new coordinates (which we still denote $x_1, y_1, \ldots x_n, y_n$),

$$\omega_+ = dx_1 \wedge \ldots \wedge dx_n, \qquad \omega_- = dy_1 \wedge \ldots \wedge dy_n,$$
$$\Delta = \{x_1 = y_1 = 0\}.$$

Now, the blow-up Bl of U along $\Delta \cap U$ can be trivialized by two charts; in each of them, one can write Bl in coordinates as

$$\mathrm{Bl}\colon (x, z, x_2, y_2, \ldots x_n, y_n) \mapsto (x, zx, x_2, y_2, \ldots, x_n, y_n),$$
$$\mathrm{Bl}\colon (t, y, x_2, y_2, \ldots x_n, y_n) \mapsto (ty, y, x_2, y_2, \ldots, x_n, y_n)$$

respectively, with change of coordinates $t = 1/z$, $y = zx$. In these two charts, one can write $\mathrm{Bl}^* \omega_+$ as

$$\mathrm{Bl}^* \omega_+ = dx \wedge dx_2 \wedge \ldots \wedge dx_n = d(ty) \wedge dx_2 \wedge \ldots \wedge dx_n.$$

Remark that the exceptional divisor $\widetilde{\Delta}_0 = \mathrm{Bl}^{-1}(\Delta_0)$ is expressed in local coordinates by $\{x = 0\}$ ($\{y = 0\}$ respectively). This shows that

- the exceptional divisor $\widetilde{\Delta}$ contains the leaves of $\mathrm{Bl}^* \mathcal{F}_{S^n}^+$ which pass through its points;
- the singular locus $\{t = y = 0\}$ of $\mathrm{Bl}^* \mathcal{F}_{S^n}^+$ intersects each exceptional fibre of Bl in exactly one point.

Now, since Δ_0 locally coincides with Δ_{12}, the action of $\mathfrak{S}_n$ is locally just the involution swapping the first two factors of S^n; in coordinates, this is the involution

$$\theta\colon (x_1, y_1, x_2, y_2 \ldots, x_n, y_n) \mapsto (-x_1, -y_1, x_2, y_2, \ldots, x_n, y_n),$$

so that the induced action on $\widetilde{S_0^n}$ is, in coordinates,

$$\tilde{\theta}\colon (x, z, x_2, y_2, \ldots x_n, y_n) \mapsto (-x, z, x_2, y_2, \ldots, x_n, y_n).$$

Therefore, the quotient of $\widetilde{S_0^n}$ by the action of $\mathfrak{S}_n$ (which locally coincides with X) can be locally written as

$$\tilde{\rho}\colon (x, z, x_2, y_2, \ldots x_n, y_n) \mapsto (x^2, z, x_2, y_2, \ldots, x_n, y_n).$$

for some local coordinates $x', z', x_2', \ldots$ on $S_0^{[n]}$.

In particular, $\tilde{\theta}$ acts as the identity on the exceptional divisor $\widetilde{\Delta}_0$, therefore the claim about $\mathcal{F}_X^+$ follows from the analysis of $\mathrm{Bl}^* \mathcal{F}_{S^n}^+$. □

Computation of $\det N\mathcal{F}_X^{\pm}$

As we have seen in §1.2.2, finding a meromorphic (multi-)form defining a foliation $\mathcal{F}$ is equivalent to finding the line bundle $\mathcal{L} = \det N\mathcal{F}$. We can find such a form (resp. pluri-form) without zeros in codimension one if and only if $\mathcal{L}$ (resp. a multiple of $\mathcal{L}$) admits non-trivial holomorphic sections.

Let $E_S = E_1 + \ldots + E_{16} \subset S$ be the exceptional divisor over the singularities of $T/\pm \mathrm{id}_T$, and consider the divisor

$$E_{S^n} = \pi_1^* E + \ldots + \pi_n^* E.$$

Such divisor is clearly $\mathfrak{S}_n$-invariant, and thus defines a divisor $E_{S^{(n)}} \subset S^{(n)}$; denote by $E' \subset X$ its strict transform through the resolution $X \to S^{(n)}$.

Proposition 3.9. *Let $E_X \subset X$ be the exceptional divisor of the natural map $X = S^{[n]} \to S^{(n)}$; then*

$$\det(N\mathcal{F}_X^+) = \det(N\mathcal{F}_X^-) = \frac{1}{2}(E_X + E').$$

Proof. We will deal with the unstable foliation $\mathcal{F}_X^+$, but the same reasoning can be applied to the stable foliation.

Recall that, by Proposition 3.5, the unstable foliation $\mathcal{F}_S^+$ is defined by a global pluri-form with values in E

$$\beta_S \in H^0(S, (\Omega_S^1)^{\otimes 2} \otimes E_S).$$

In other words, $N\mathcal{F}_S^+ = \frac{E_S}{2} = \frac{1}{2}(E_1 + \ldots + E_{16})$. By dividing β_S by a section of E_S, we obtain a meromorphic pluri-form α_S with simple a pole along E_S.

Now take the meromorphic pluri-form on S^n defining $\mathcal{F}_{S^n}^+$

$$\alpha_{S^n} = \pi_1^* \alpha_S \wedge \ldots \wedge \pi_n^* \alpha_S;$$

this pluri-form has simple poles along $E_{S^n} := \pi_1^* E_S \cup \ldots \cup \pi_n^* E_S$ and is holomorphic and different from zero elsewhere.

Furthermore, α_{S^n} is $\mathfrak{S}_n$-invariant; thus it defines a pluri-form α on the smooth locus of $S^{(n)}$, which is isomorphic to $X \setminus E_X$. The only poles of α are along E', and α defines $\mathcal{F}_X^+$.

In order to conclude, we only need to show that α extends to a meromorphic form to the whole X, and that the extension has a simple pole along E_X. This can be checked along the open subset $E_0 \subset E_X$.

Let p be a point in $\Delta_0 \subset S^n$; recall that, at a neighborhood of p, one can suppose that $\Delta = \Delta_0 = \Delta_{12}$. Using the same notation as in the proof of Lemma 3.8, we have

$$\mathrm{Bl}^* \alpha_{S^n} = (dx \wedge \ldots \wedge dx_n)^2 = \tilde{\rho}^* \left(\frac{1}{4x'} (dx' \wedge dx_2' \wedge \ldots \wedge dx_n')^2 \right).$$

This means that α locally extends to the meromorphic pluri-form on X

$$\frac{1}{4x'} (dx' \wedge dx_2' \wedge \ldots \wedge dx_n')^2,$$

which has a simple pole along $\tilde{\rho}(\mathrm{Bl}^{-1}(E)) = E_X$. This concludes the proof. □

3.3.2 Uniqueness of the invariant foliations

Proposition 3.10. *Let T be a two-dimensional complex torus, $S = K(T)$ the associated Kummer surface, $X = S^{[n]}$ the Hilbert scheme of n points on S, and $f_X \colon X \to X$ a loxodromic automorphism of X induced by a linear automorphism f_T of the torus T.*
Then the stable and unstable foliations $\mathcal{F}_X^{\pm}$ are the only non-trivial f_X-invariant singular distributions on X.

Proof. Suppose that f_X preserves a non-trivial singular distribution $\mathcal{G}_X$. The natural dominant meromorphic map $T \dashrightarrow S$ induces a dominant meromorphic map $T^n \dashrightarrow S^n$; let π denote the composition

$$\pi \colon T^n \dashrightarrow S^n \dashrightarrow S^{[n]},$$

where $S^n \dashrightarrow S^{[n]}$ is the natural (dominant) meromorphic map. Then the pull-back $\mathcal{G} := \pi^*\mathcal{G}_X$ is a distribution on T^n which is invariant under the action of

1. the involutions $\theta_i \colon T^n \to T^n$ acting as $-\operatorname{id}_T$ on the i-th factors and as id_T on the other factors;
2. the symmetric group $\mathfrak{S}_n$ acting by permutation of the factors of T^n;
3. the linear automorphism $f_{T^n} = (f_T, \ldots, f_T) \colon T^n \to T^n$.

Condition 1 (resp. 2) is needed to ensure that $\mathcal{G}$ passes to the quotient $T^n \to \hat{S}^n \cong T^n/\langle \theta_1, \ldots, \theta_n \rangle$ (resp. $S^n \to S^{(n)} \cong S^n/\mathfrak{S}_n$); condition 3 ensures that $\mathcal{G}_X$ is f_X-invariant.

Fix global linear coordinates x_i, y_i on each factor of T^n such that the unstable/stable foliations on T^n have tangent space (independent of the point)

$$T\mathcal{F}^+_{T^n} = \operatorname{Span}\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right), \qquad T\mathcal{F}^-_{T^n} = \operatorname{Span}\left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right).$$

By condition 3 and Lemma 3.1, $\mathcal{G}$ is a linear foliation; in other words, identifying the tangent bundle of T^n with $T^n \times \mathbb{C}^{2n}$, the foliation $\mathcal{G}$ corresponds to a linear subspace $W \subset \mathbb{C}^{2n}$ independent of the point. Denoting again by θ_i (resp. $s \in \mathfrak{S}_n$) the action of θ_i (resp. s) on the tangent space $\mathbb{C}^{2n}$, we have

1. $\theta_i(W) = W$ for $i = 1, \ldots, n$;
2. $s(W) = W$ for all $s \in \mathfrak{S}_n$.

Since the θ_i commute with one another, condition 1 implies that

$$W = \operatorname{Span}\left(\frac{\partial}{\partial x_{i_1}}, \ldots, \frac{\partial}{\partial x_{i_m}}, \frac{\partial}{\partial y_{i_{m+1}}}, \ldots, \frac{\partial}{\partial y_{i_k}}\right) \quad \text{for some } i_1, \ldots, i_k \in \{1, \ldots, n\}.$$

By condition 2, we then have that either all or none of the x_i appear, and the same for the y_i. Since $\mathcal{G}$ is non-trivial, either $\mathcal{G} = \mathcal{F}^+_{T^n}$ or $\mathcal{G} = \mathcal{F}^-_{T^n}$; hence either $\mathcal{G}_X = \mathcal{F}^+_X$ or $\mathcal{G}_X = \mathcal{F}^-_X$. $\square$

3.4 Second example: generalized Kummer variety

Let T be a two-dimensional torus, and let $T^{[n+1]}$ be the Hilbert scheme of $n+1$ points on T; then $T^{[n+1]}$ is a symplectic manifold, with symplectic structure induced by

$$\sigma_{n+1} := \pi_1^*\sigma_T + \ldots + \pi_{n+1}^*\sigma_T \in H^0(T^{n+1}, \Omega^2_{T^{n+1}}),$$

where $\pi_i \colon T^{n+1} \to T$ is the projection onto the i-th factor and σ_T is a symplectic form on T. However, $T^{[n+1]}$ is not simply connected, therefore the symplectic structure is not unique. In order to overcome this problem, we consider the composition

$$\Sigma : T^{[n+1]} \xrightarrow{\rho} T^{(n+1)} \xrightarrow{s} T,$$

where ρ is the natural morphism and s is the sum morphism, associating to a set of $n+1$ points on T their sum (with respect to the abelian group law on T).

We define $Y = K_n(T)$, the *generalized Kummer variety of* T, as the fibre of Σ over $0 \in T$. Then Y is an irreducible symplectic manifold, with symplectic structure induced by σ_{n+1} (see [Bea83]).

Remark 3.11. The morphism Σ is an isotrivial fibration (i.e. all its fibres are isomorphic). Indeed, if we denote by $\mu_{n+1}\colon T \to T$ the morphism $t \mapsto (n+1)t$, we have a commutative diagram

$$\begin{tikzcd} T \times_T T^{[n+1]} \cong T \times K_n(T) \arrow[r] \arrow[d] & T^{[n+1]} \arrow[d, "\Sigma"] \\ T \arrow[r, "\mu_{n+1}"] & T \end{tikzcd}$$

Now let us add an automorphism to the picture: as we have seen in §3.2, if

$$f_T\colon T \to T$$

is an automorphism of T, then the induced automorphism

$$f_T^{[n+1]}\colon T^{[n+1]} \to T^{[n+1]}$$

satisfies $\lambda_1(f_T^{[n+1]}) = \lambda_1(f_T)$. Since $K_n(T)$ is $f^{[n+1]}$-invariant, we can define an automorphism

$$f_Y = f_T^{[n+1]}|_Y\colon Y \to Y.$$

Assume that f_T is loxodromic and denote by $\mathcal{F}_T^{\pm}$ the stable/unstable foliation for f_T.

Lemma 3.12. *There exists a generically finite meromorphic map*

$$\pi\colon T^n \dashrightarrow Y$$

such that

1. $\pi \circ f_n = f_Y \circ \pi$, *where* $f_n = (f_T, \dots, f_T)\colon T^n \to T^n$*; in particular* $\lambda_1(f_Y) = \lambda_1(f_T)$*;*
2. π *is birationally equivalent to the quotient map*
$$q\colon T^n \to T^n/\mathfrak{S}_{n+1},$$
where the action of $\mathfrak{S}_{n+1}$ *on* T^n *is the restriction of the action by permutation of coordinates on* T^{n+1} *to*
$$\{(t_1, \dots, t_{n+1}) \mid t_1 + \ldots + t_{n+1} = 0\} \cong T^n \subset T^{n+1};$$
more accurately, the restrictions of π *and* q *to the Zariski-open subset* $T^n \setminus \Delta(T^{n+1})$ *are biregularly conjugate;*
3. *the product foliations* $\mathcal{F}_{T^n}^+ = \mathcal{F}_T^+ \times \ldots \times \mathcal{F}_T^+$ *and* $\mathcal{F}_{T^n}^- = \mathcal{F}_T^- \times \ldots \times \mathcal{F}_T^-$ *induce* f_Y*-invariant foliations* $\mathcal{F}_Y^+$ *and* $\mathcal{F}_Y^-$ *on* Y*;*
4. *the foliations* $\mathcal{F}_Y^{\pm}$ *are generically transverse* f_Y*-invariant Lagrangian foliations.*

Proof. The sum morphism

$$\Sigma_0\colon T^{n+1} \to T$$
$$(t_1, \dots, t_{n+1}) \mapsto t_1 + \ldots + t_{n+1}$$

is equal to the composition

$$T^{n+1} \to T^{n+1}/\mathfrak{S}_{n+1} = T^{(n+1)} \overset{\rho^{-1}}{\dashrightarrow} T^{[n+1]} \xrightarrow{\Sigma} T.$$

Therefore, $Y = K_n(T)$ is the strict transform of $\Sigma_0^{-1}(0)$ through the natural map

$$\eta \colon T^{n+1} \dashrightarrow T^{[n+1]};$$

let $\eta_0 = \eta|_{\Sigma^{-1}(0)}$.
Denote by $f_{n+1} \colon T^{n+1} \to T^{n+1}$ the morphism $(f, f, \dots, f)$ acting as f on each coordinate; since $f_{n+1} \circ \eta = \eta \circ f^{[n+1]}$,

$$(\eta \circ f_{n+1})|_{\Sigma_0^{-1}(0)} = f_Y \circ \eta_0.$$

Now, let $\nu \colon T^{n+1} \to T^n$ be the projection on the first n coordinates. The restriction $\nu_0 = \nu|_{\Sigma_0^{-1}(0)}$ is an isomorphism, and $f_n \circ \nu_0 = \nu_0 \circ f_{n+1}$; this means that the composition

$$\pi \colon T^n \xrightarrow{\nu_0^{-1}} \Sigma^{-1}(0) \overset{\eta_0}{\dashrightarrow} Y$$

is generically finite.

(1) We have

$$\pi \circ f_n = \eta_0 \circ \nu_0^{-1} \circ f_n = \eta_0 \circ f_{n+1} \circ \nu_0^{-1} = (\eta \circ f_{n+1})|_{\Sigma^{-1}(0)} \circ \nu_0^{-1} =$$

$$f_Y \circ \eta_0 \circ \nu_0^{-1} = f_Y \circ \pi.$$

In particular, $\lambda_1(f_Y) = \lambda_1(f_n)$ by Proposition 1.12; since $\lambda_1(f_n) = \lambda_1(f_T)$, this concludes the proof of the first statement.

(2) We have $\pi = \eta_0 \circ \nu_0^{-1}$; since ν_0 is an isomorphism, π is birationally equivalent to η_0. Since $\eta \colon T^{n+1} \dashrightarrow T^{[n+1]}$ is birationally equivalent to the quotient morphism $T^{n+1} \to T^{n+1}/\mathfrak{S}_{n+1}$, η_0 (hence π) is birationally equivalent to the restriction

$$q \colon \Sigma^{-1}(0) \cong T^n \to {T^n}\big/{\mathfrak{S}_{n+1}}.$$

Furthermore, all the involved meromorphic maps (and their inverses if they are bimeromorphic) become holomorphic once we restrict to the Zariski-open subset $\Sigma^{-1}(0) \cap \Delta(T^{n+1})$; therefore, the restrictions of q and π to this subset are biregularly conjugate. This proves the second claim.

(3) By (2), the meromorphic map π is birationally equivalent to the quotient map

$$q \colon T^n \to {T^n}\big/{\mathfrak{S}_{n+1}},$$

where the action of $\mathfrak{S}_{n+1}$ on $T^n \cong \{\Sigma_0 = 0\} \subset T^{n+1}$ is the restriction of the action on T^{n+1} by permutation of coordinates.
Let us prove that the foliations $\mathcal{F}^{\pm}_{T^n}$ are $\mathfrak{S}_{n+1}$-invariant: we can check this for transpositions in $\mathfrak{S}_{n+1}$. Take $\theta = (12) \in \mathfrak{S}_{n+1}$, and fix global (linear) coordinates x_i, y_i on the i-th factor in the direction of $\mathcal{F}^+_T, \mathcal{F}^-_T$. Then the restriction of the coordinates $x_1, y_1, \dots, x_n, y_n$ to T^n defines global linear coordinates on $T^n = \Sigma_0^{-1}(0) \subset T^{n+1}$ such that the linear foliations $\mathcal{F}^{\pm}_{T^n}$ are defined by

$$T\mathcal{F}^+_{T^n} = \operatorname{Span}\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right), \qquad T\mathcal{F}^-_{T^n} = \operatorname{Span}\left(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\right).$$

Both spaces are clearly θ-invariant, thus $\mathcal{F}^+_{T^n}$ and $\mathcal{F}^-_{T^n}$ are $\mathfrak{S}_{n+1}$-invariant; this implies that the projection q defines foliations on the smooth locus of $T^n/\mathfrak{S}_{n+1}$, hence on Y. This proves the third claim.

(4) Since $\pi \circ f_n = f_Y \circ \pi$, the foliations $\mathcal{F}^{\pm}_Y$ are f_Y-invariant.
The other other claims can be checked locally around generic points: transversality follows form

transversality of $\mathcal{F}^{\pm}_{T^n}$; and since $\mathcal{F}^{\pm}_{T^n}$ are Lagrangian with respect to the symplectic form $\sigma_{n+1}|_{\Sigma_0^{-1}(0)}$ inducing the symplectic structure on Y, $\mathcal{F}^{\pm}_Y$ are Lagrangian with respect to the symplectic structure on Y. This concludes the proof. $\square$

We will call $\mathcal{F}^{\pm}_Y$ the stable/unstable foliation of f_Y. We will take $\mathcal{F}^{+}_Y, \mathcal{F}^{-}_Y$ in reduced form, so that

$$\operatorname{codim}\operatorname{Sing}(\mathcal{F}^{\pm}_{Y}) \geq 2.$$

3.4.1 Structure of $\mathcal{F}^{\pm}_Y$

In this paragraph we give a more detailed description of the stable/unstable foliations $\mathcal{F}^{\pm}_Y$.

Singular locus

Let us try to explicitly describe the singular locus of $\mathcal{F}^{\pm}_Y$. Let

$$\Delta = \Delta(T^{n+1}) \subset T^{n+1}, \quad D \subset T^{(n+1)} \quad E_{n+1} = E(T^{[n+1]}) \subset T^{[n+1]}$$

be defined as in §3.2. Denote by $E_Y \subset Y$ the divisor defined by the scheme-theoretic intersection

$$E_Y = E_{n+1} \cap Y.$$

By Lemma 3.12, the natural map $T^{[n+1]} \dashrightarrow T^{(n+1)}$ induces an isomorphism

$$Y \setminus E_Y \cong {}^{(\Sigma_0^{-1}(0) \setminus \Delta)}\!\diagup_{\mathfrak{S}_{n+1}} \subset {}^{T^n}\!\diagup_{\mathfrak{S}_{n+1}},$$

where as usual we have identified $\Sigma_0^{-1}(0)$ with T^n. Furthermore, the restriction to $Y \setminus E_Y$ of the foliations $\mathcal{F}^{\pm}_Y$ are induced by the linear foliations $\mathcal{F}^{\pm}_{T^n}$ on T^n, and therefore are smooth.

Now let us deal with the divisor E_Y. Let

$$Y_0 = Y \cap T_0^{[n+1]},$$

where $T_0^{[n+1]}$ is defined as in §3.2; then Y_0 is naturally identified with the blow-up

$$\operatorname{Bl}_{s^{-1}(0)\cap D_0}\left(s^{-1}(0) \cap T_0^{(n+1)}\right).$$

Remark 3.13. With the notation of §3.2, the action of T on $T^{[n+1]}$ by translations on factors preserves the open set $T_0^{[n+1]}$. Let

$$Y_0 := Y \setminus \left(E(T^{[n+1]}) \setminus E_0(T^{[n+1]})\right) = Y \cap T_0^{[n+1]};$$

then

$$\operatorname{codim}_Y(Y \setminus Y_0) \geq 2.$$

Indeed, the action of T on $T^{[n+1]}$ is transitive on the set of fibres of $\Sigma\colon T^{[n+1]} \to T$; if we had $\operatorname{codim}_Y(Y \setminus Y_0) \leq 1$, then also $\operatorname{codim}_{T^{[n+1]}}(T^{[n+1]} \setminus T_0^{[n+1]}) \leq 1$, a contradiction.

At this point, the proof of the following lemma is completely analogous to that of Lemma 3.8.

Lemma 3.14. *The exceptional locus $E_Y \cap Y_0$ contains all the leaves which pass through its points. Furthermore, each exceptional fibre of the birational map*

$$Y_0 \to {}^{T^n}\!\diagup_{\mathfrak{S}_{n+1}}$$

intersects the singular locus of $\mathcal{F}^{+}_Y$ (resp. of $\mathcal{F}^{-}_Y$) in exactly one point.

$\square$

Computation of $\det N\mathcal{F}_Y^{\pm}$

Proposition 3.15. *Let $E_Y \subset Y$ be the exceptional divisor of the natural map $Y = \Sigma^{-1}(0) \to s^{-1}(0) \subset T^{(n+1)}$); then*

$$\det(N\mathcal{F}_Y^{+}) = \det(N\mathcal{F}_Y^{-}) = \frac{1}{2}E_Y.$$

Proof. We will deal with the unstable foliation $\mathcal{F}_Y^{+}$, but the same reasoning can be applied to the stable foliation.

Fix the same notation as in the proof of Lemma 3.8: consider coordinates x_i, y_i on each T factor of T^{n+1} along the stable/unstable foliations. On $T^n \cong \{t_1 + \ldots + t_{n+1} = 0\} \subset T^{n+1}$, the unstable foliation is defined by the restriction of (for example) $dx_1 \wedge \ldots \wedge dx_n$. Such form is not $\mathfrak{S}_{n+1}$-invariant, but the pluri-form

$$(dx_1 \wedge \ldots \wedge dx_n)^2 \in H^0(T^n, (\Omega^n_{T^n})^{\otimes 2})$$

is, and therefore defines a pluri-form α on the smooth locus of $\Sigma_0^{-1}(0) \subset T^{(n+1)}$. This form has no zeros and no poles.

In order to conclude, we need to show that α extends meromorphically to the whole Y, and that the extension has simple poles along E_Y. By Remark 3.13, we only need to check this statement along Y_0; now we conclude exactly as the proof of Proposition 3.9. □

3.4.2 Uniqueness of the invariant foliations

Proposition 3.16. *Let T be a two-dimensional complex torus, $Y = K_n(T)$ the generalized Kummer variety of dimension $2n$ of T, and $f_Y\colon Y \to Y$ a loxodromic automorphism of Y induced by a loxodromic linear automorphism f_T of the torus T.*
Then the stable and unstable foliations $\mathcal{F}_Y^{\pm}$ are the only non-trivial f_Y-invariant singular distributions on Y.

Proof. Suppose that f_Y preserves a non-trivial singular distribution $\mathcal{G}_Y$. Let

$$\pi\colon T^n \dashrightarrow Y$$

be a generically finite map as in Lemma 3.12. Then the pull-back $\mathcal{G} := \pi^*\mathcal{G}_Y$ is a foliation on T^n which is invariant under the action of

1. the symmetric group $\mathfrak{S}_{n+1}$ acting by restricting the permutation of factors of T^{n+1} to $\{t_1 + \ldots + t_{n+1}\} \cong T^n$;
2. the linear automorphism $f_{T^n} = (f_T, \ldots, f_T)\colon T^n \to T^n$.

Condition 1 is needed to ensure that $\mathcal{G}$ passes to the quotient $T^n \to T^n/\mathfrak{S}_{n+1}$ (recall that by Lemma 3.12 such quotient is birationally equivalent to $\pi\colon T^n \dashrightarrow Y$); condition 2 ensures that $\mathcal{G}_Y$ is f_Y-invariant.

Fix global linear coordinates x_i, y_i on each factor of T^n such that the unstable/stable foliations on T^n have tangent space (independent of the point)

$$T\mathcal{F}^{+}_{T^n} = \mathrm{Span}\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right), \qquad T\mathcal{F}^{-}_{T^n} = \mathrm{Span}\left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right).$$

By condition 2 and Lemma 3.1, $\mathcal{G}$ is a linear foliation; in other words, identifying the tangent bundle of T^n with $T^n \times \mathbb{C}^{2n}$, the foliation $\mathcal{G}$ corresponds to a linear subspace $W \subset \mathbb{C}^{2n}$ independent of the point. Denoting again by $s \in \mathfrak{S}_n$ the action of s on $\mathbb{C}^{2n}$, we have

$$s(W) = W \qquad \text{for all } s \in \mathfrak{S}_n.$$

Remark that the action of $\mathfrak{S}_{n+1}$ on $\mathbb{C}^{2n}$ is the direct sum of the action on $T\mathcal{F}^{+}_{T^n}$ and on $T\mathcal{F}^{-}_{T^n}$; both these linear representations of $\mathfrak{S}_{n+1}$ are isomorphic to the "standard representation", i.e. the restriction of the action by permutation of coordinates on $\mathbb{C}^{n+1}$ to the invariant subspace $\{(x_1, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \,|\, x_1 + \ldots + x_{n+1} = 0\}$. This representation of $\mathfrak{S}_{n+1}$ is irreducible (i.e. $\mathfrak{S}_{n+1}$ does not preserve any proper subspace): in order to show this, we may compute explicitly that the space $\mathrm{End}_{\mathfrak{S}_{n+1}}(\mathbb{C}^{n+1})$ of endomorphisms of $\mathbb{C}^{n+1}$ commuting with the action of $\mathfrak{S}_{n+1}$ is equal to the set of matrices

$$\{(a_{ij})_{i,j=1,\ldots,n+1} \,|\, a_{ii} = a_{jj} \; \forall i, j = 1, \ldots n+1 \text{ and } a_{ij} = a_{kl} \text{ for all } i \neq j,\; k \neq l\}$$

and thus has dimension 2; hence the representation on $\mathbb{C}^{n+1}$ decomposes in exactly two irreducible representations by Schur's lemma, namely the one-dimensional representation $\mathrm{Span}(e_1 + \ldots + e_{n+1})$ and the standard representation.
Therefore, since $\mathcal{G}$ is non-trivial, it is either equal to $T\mathcal{F}^{+}_{T^n}$ or to $T\mathcal{F}^{-}_{T^n}$; upon passing to quotient, this concludes the proof. □

3.5 Further examples

The following examples show that not all interesting dynamics come from linear maps on tori.

Example 3.17. In [Ogu16a] the author gives an explicit example of a $K3$ surface S such that $\mathrm{Aut}(S)$ is finite and such that $X = S^{[2]}$ admits a loxodromic automorphism $F\colon X \to X$. Since the group of automorphisms of S is finite, F cannot be written as $f^{[2]}$ for some $f \in \mathrm{Aut}(S)$.

Example 3.18. Amerik and Verbitsky have proved in [AV16] that, as long as $b_2(X) \geq 5$ (a condition which is respected by all known families of irreducible holomorphic symplectic manifolds), there exists a deformation of X admitting a loxodromic automorphism.

Chapter 4

Hyperbolic points for loxodromic automorphisms

Definition 4.1. *Let M be a manifold and $g\colon M \to M$ a diffeomorphism; a g-periodic point $p \in M$ of period k is said to be* hyperbolic *(or* saddle*) if the Jacobian $(Df^k)_p$ is a linear automorphism of T_pM all of whose eigenvalues have modulus $\neq 1$.*

The goal of this section is to prove the following theorem:

Theorem 4.2. *Let X be an irreducible symplectic manifold and let $f\colon X \to X$ be a loxodromic automorphism of X. Then the hyperbolic periodic points of f are Zariski-dense in X.*

The result follows from Theorem 4.4 by Bogomolov, Kamenova, Lu and Verbitsky and recent results in dynamical systems [Kat80, dT08, DS10] by Dinh and Sibony and de Thélin. The proof shows something stronger: the hyperbolic periodic points are dense in a non-pluripolar subset of X.

Example 4.3. Let $f\colon S \to S$ be a loxodromic automorhism of a $K3$ surface, let $X = S^{[n]}$ be the Hilbert scheme of n points on S and let $f^{[n]}\colon X \to X$ be the induced automorphism on the Hilbert scheme of n points of S. Since the hyperbolic periodic points of f are Zariski-dense in S, the hyperbolic periodic points of $f^{[n]}$ are Zariski-dense in X: indeed, if $p_1, \ldots, p_n \in S$ are distinct hyperbolic points for f, then the reduced scheme $p_1 + \ldots + p_n \in X$ is a hyperbolic point for $f^{[n]}$.

In [McM02], MacMullen gives an example of a loxodromic automorphism f of a (non-projective) $K3$ surface S which admits a Siegel disk, i.e. an embedded complex disk on which f acts by an irrational rotation; in particular the periodic points of f are not dense in the Euclidean topology, and therefore neither are the periodic points of $f^{[n]}$.

4.1 Action on cohomology of loxodromic automorphisms

We are going to need a precise description of the action on cohomology of an automorphism of an irreducible symplectic manifold. The following result is proven in [BKLV]; however, the statement in the original article is slightly different, and the proof is quite hermetic, which is why I chose to rewrite it in the rest of this Section.

Theorem 4.4 ([BKLV], Theorem 6.3)**.** *Let $f\colon X \to X$ be a loxodromic automorphism of an irreducible symplectic manifold X of dimension $2n$. For $0 \leq p \leq 2n$ let $\lambda_p(f)$ be the dynamical degrees of f. Then, denoting $\lambda = \lambda_1(f)$,*

- *all the eigenvalues of* $f^*\colon H^*(X,\mathbb{R}) \to H^*(X,\mathbb{R})$ *have modulus* $\lambda^{k/2}$ *for some* $k \in \mathbb{Z}$;
- *the sum of the multiplicities of eigenvalues of*

$$f^*_{p+q}\colon H^{p+q}(X,\mathbb{R}) \to H^{p+q}(X,\mathbb{R})$$

  having modulus $\lambda^{\frac{p-q}{2}}$ *is* $\dim H^{p,q}(X)$.

As a consequence

1. *for* $p = 1,\dots,n$, *the maximal eigenvalue of* f^* *on* $H^{2p}(X,\mathbb{R})$ *(resp. on* $H^{2p-1}(X,\mathbb{R})$*) is equal to* λ^p *(resp. has modulus* $< \lambda^{\frac{2p-1}{2}}$*);*
2. *for* $p = 0,1,\dots,n$,
$$\lambda_p(f) = \lambda_{2n-p}(f) = \lambda^p;$$
3. $\lambda_p(f)$ *is a simple eigenvalue of* $f^*_{p,p}\colon H^{p,p}(X) \to H^{p,p}(X)$;
4. *if* α *is an eigenvalue of* $f^*|_{H^{2p}(X,\mathbb{R})}$ *which is distinct from* $\lambda_p(f)$, *then*
$$|\alpha| \leq \frac{\lambda_p(f)}{\lambda}.$$

We have already seen the proof of the statement about dynamical degrees (Corollary 2.20); in the rest of the section we will prove the rest of the statement.

Remark 4.5. The second main statement of Theorem 4.4 could be slightly misleading: although the sum of multiplicities of eigenvectors of f^*_{p+q} having modulus $\lambda^{(p-q)/2}$ is equal to $\dim H^{p,q}(X)$, such eigenvalues do not necessarily appear as eigenvalues of $f^*_{p,q}\colon H^{p,q}(X) \to H^{p,q}(X)$. For example, $\lambda_1(f)$ and $\lambda_1(f)^{-1}$ are the only eigenvalues of f_2^* having modulus different from 1, but they appear as eigenvalues of the restriction $f^*_{1,1} = f^*|_{H^{1,1}(X)}$, and not as eigenvectors of $f^*_{2,0}$ or $f^*_{0,2}$.

Remark 4.6. The assumption that f is loxodromic is actually useless: indeed, if an automorphism f satisfies $\lambda_1(f) = 1$, all the eigenvalues of f^* have modulus 1 (see [LB14b, Lemma 2.2.4]).

Example 4.7. Let X be an irreducible symplectic fourfold. Then the Hodge diamond of X is of the form

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 1 & & a & & 1 & & \\
 & 0 & & b & & b & & 0 & \\
1 & & c & & d & & c & & 1 \\
 & 0 & & b & & b & & 0 & \\
 & & 1 & & a & & 1 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

Hence, for example, $f_4^*\colon H^4(X,\mathbb{R}) \to H^4(X,\mathbb{R})$ has exactly 1 (resp. $c, d, c, 1$) eigenvalue(s) with modulus λ^2 (resp. $\lambda, 1, \lambda^{-1}, \lambda^{-2}$), taking multiplicities into account. Moreover, λ^2 and λ^{-2} are eigenvalues of $f^*|_{H^{2,2}(X)}$; thus, $\lambda^{\pm 2}$ is not an eigenvalue of $f^*|_{H^{3,1}(X)}$.
Furthermore, the Betti numbers of irreducible symplectic manifolds must respect some constraints.

- Wakakuwa [Wak58]: the odd Betti numbers must be divisible by four, so that, with the above notation, b is even.

- Salamon [Sal96]: if $\dim X = 2n$, then

$$2\sum_{i=1}^{2n}(-1)^i(3i^2-n)b_{2n-i}(X) = nb_{2n}(X);$$

  in dimension 4, this leads to $b_3 + b_4 = 10b_2 + 46$, i.e. $2b + 2c + d = 10a + 64$.

- Guan [Gua01]: if $\dim X = 4$, then either $3 \leq b_2(X) \leq 8$ or $b_2(X) = 23$; thus, either $1 \leq a \leq 6$ or $a = 21$. Guan also gives restrictions to $b_3(X)$ depending on $b_2(X)$.

4.1.1 The orthogonal group $O(p,q)$

Real algebraic groups and their lattices

Let us recall some properties of real algebraic groups and their lattices; we refer to [Hum75, Bor91, Rag07].

By an *algebraic group* we mean a smooth algebraic variety over a field k (in our case $k = \mathbb{R}$ or $\mathbb{Q}$) with a group structure such that the group multiplication and the inverse are algebraic morphisms .
We will be mostly interested in *linear algebraic groups*, i.e. algebraic groups which are isomorphic to a subgroup of some GL_N defined by polynomial equations. In the case we will be interested in, the base field is either $\mathbb{R}$ or $\mathbb{Q}$, and by an abuse of notation we will often identify an algebraic group G with its real points $G(\mathbb{R})$.
Let G be a linear algebraic group defined over $\mathbb{Q}$; in other words, G can be realized as a subgroup of $\mathrm{GL}_N(\mathbb{R})$ defined by equations with rational coefficients. Then, the set $G(\mathbb{Q})$ of matrices of G with rational coefficients forms a subgroup of G; an *arithmetic subgroup* is a subgroup $\Gamma \leq G(\mathbb{Q})$ which is commensurable with the subgroup $G(\mathbb{Z}) := G(\mathbb{Q}) \cap \mathrm{GL}_N(\mathbb{Z})$ (i.e. $G(\mathbb{Z}) \cap \Gamma$ has finite index in both $G(\mathbb{Z})$ and Γ). Although the set $G(\mathbb{Z})$ depends on the chosen embedding $G(\mathbb{R}) \hookrightarrow \mathrm{GL}_N(\mathbb{R})$, the definition of an arithmetic subgroup doesn't.

A real algebraic group determines a *real Lie group*, i.e. a manifold with a group structure such that the group multiplication and the inverse are diffeomorphisms. If G is a Lie group, its *Lie algebra* is the tangent space at the origin $\mathfrak{g} = T_eG$ endowed with the Lie bracket $[\cdot,\cdot]$ (see [Hum75, Chapter III]); in the case of a linear algebraic group $G \subset \mathrm{GL}_N(\mathbb{R})$, the Lie bracket of two elements of $\mathfrak{g} \subset T_{Id}\mathrm{GL}_N(\mathbb{R}) = \mathcal{M}_n(\mathbb{R})$ is simply the matrix commutator:

$$[x,y] = xy - yx.$$

A *lattice* of a Lie group G is a discrete subgroup $\Gamma \leq G$ such that the quotient space G/Γ has a G-invariant finite Borel measure.

A real Lie group G is called *simple* (resp. *semisimple*) if its Lie algebra $\mathfrak{g}$ is simple (resp. semisimple), i.e. if it has no non-trivial ideals (resp. if it is isomorphic to a direct sum of simple Lie algebras). Equivalently, a Lie group is simple if it has no non-trivial connected normal immersed subgroup.

A connected (real) algebraic group is semisimple if it has no Zariski-connected normal commutative subgroup other than the identity; here, "Zariski-connected" means "connected for the Zariski topology" (a connected algebraic set is Zariski-connected, but the converse might not be true, see Example 4.10). By [Mil06, Theorem 14.1], a real algebraic group G is semisimple if and only if it is semisimple as a Lie group.

The following theorem links the notion of lattice with that of arithmetic group; see [Rag07, Theorem 13.28].

Theorem 4.8. *Let G be a semisimple algebraic group defined over $\mathbb{Q}$ and let $\Gamma < G$ be an arithmetic subgroup. Then Γ is a lattice.*

We say that a real semisimple algebraic group G does not have compact factors if there is no surjective morphism $G \twoheadrightarrow K$ onto a compact (in the euclidean topology) real algebraic group K of positive dimension.

Theorem 4.9 (Borel density theorem, [Bor60])**.** *Let G be a connected semisimple real algebraic group without compact factors and let $\Gamma \leq G$ be a lattice. Then Γ is Zariski-dense in G.*

Example 4.10. For $p, q \geq 0$, let $G = O(p,q)$ be the subgroup of element of $\mathrm{GL}_{p+q}(\mathbb{R})$ preserving a fixed quadratic form b of signature (p,q); the Zariski-connected component of the identity is

$$SO(p,q) = O(p,q) \cap SL_{p+q}(\mathbb{R}).$$

The Lie algebra $\mathfrak{so}(p,q)$ of G is simple (see [FH91], or [Hel01] for a proof); in particular, $SO(p,q)$ is semisimple as a real algebraic group, and the only surjective algebraic group morphisms $SO(p,q) \twoheadrightarrow G$ over $\mathbb{R}$ onto a positive-dimensional group G are quotients by finite groups.

From the point of view of euclidean topology, we have two separate cases:

- if $p = 0$ or $q = 0$, then $SO(p,q) \cong SO_{p+q}(\mathbb{R})$ is a connected and compact Lie group;
- if p and q are both strictly positive, G has exactly four connected components, which are non-compact. More accurately, $SO(p,q)$ has two connected components: assuming that $p \leq q$, a component of $SO(p,q)$ fixes the two connected components of the hyperboloid $\{v \in \mathbb{R}^{p+q} \,|\, b(v) = 1\}$, whereas the other one swaps them. The connected component of the identity is usually denoted by $SO^+(p,q)$.

In particular, if p and q are both strictly positive, then $SO(p,q)$ does not have compact factors; therefore, by Theorem 4.8 and Theorem 4.9, the subgroup of integral points $SO(p,q) \cap GL_{p+q}(\mathbb{Z})$ is a lattice of, and is Zariski-dense in, $SO(p,q)$.

Application to irreducible symplectic manifolds

We are going to apply these results in the context of irreducible symplectic manifolds. Let X be an irreducible symplectic manifold and let q_X denote the Beauville-Bogomolov form on $H^2(X,\mathbb{R})$; recall that $H^2(X,\mathbb{R})$ has an integral structure determined by the isomorphism

$$H^2(X,\mathbb{R}) \cong H^2(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

By Proposition 2.9, the group

$$G := O(H^2(X,\mathbb{R}), q_X) \leq \mathrm{GL}(H^2(X,\mathbb{R}))$$

of linear automorphisms of $H^2(X,\mathbb{R})$ preserving q_X is a linear algebraic group isomorphic to the orthogonal group $O(3, b_2(X) - 3)$; furthermore, since q_X is defined over $\mathbb{Z}$, G is defined over $\mathbb{Q}$. Denote by

$$G_0 = SO(H^2(X,\mathbb{R}), q_X)$$

the Zariski-connected component of the identity of G, and by $SO(H^2(X,\mathbb{Z}), q_X)$ the set $G_0(\mathbb{Z}) = G_0 \cap \mathrm{SL}(H^2(X,\mathbb{Z}))$ of integral points of G_0.

Proposition 4.11. *Let X be an irreducible symplectic manifold with $b_2(X) > 3$. Then $SO(H^2(X,\mathbb{Z}), q_X)$ is a lattice of $SO(H^2(X,\mathbb{R}), q_X)$, and it is a Zariski-dense subset.*

Proof. By Theorem 4.8 the subgroup $G_0(\mathbb{Z}) = G_0 \cap \mathrm{SL}_{b_2(X)}(\mathbb{Z})$ is a lattice.
The semisimple group G_0 is its own only factor because it is simple; furthermore, G_0 is non-compact because $b_2 > 3$ (see Example 4.10). Hence, by Theorem 4.9, $G_0(\mathbb{Z})$ is Zariski-dense in G_0. □

Remark 4.12. If an irreducible symplectic manifold X admits a loxodromic birational transformation $f\colon X \dashrightarrow X$, then $b_2(X) > 3$. Indeed, $\lambda_1(f)$ and $\lambda_1(f)^{-1}$ are distinct eigenvalues of $f^*_{1,1}\colon H^{1,1}(X) \to H^{1,1}(X)$; thus $b_2(X) = h^{1,1}(X) + 2 \geq 4$.

4.1.2 Proof of Theorem 4.4

In this section we prove Theorem 4.4. Let $f\colon X \to X$ be an automorphism of an irreducible symplectic manifold X of dimension $2n$; let $\lambda = \lambda_1(f) > 1$.

The Hodge decomposition defines a standard multiplicative action of $\mathbb{C}^*$ on $H^*(X, \mathbb{C})$: $t \in \mathbb{C}^*$ acts on $H^{p,q}(X)$ by t^{p-q}. The idea is to compare this action to the action of f^*.

The strategy of the proof in [BKLV] goes as follows:

1. first we define a linear algebraic subgroup G of $\mathrm{GL}(H^*(X, \mathbb{R}))$ which contains f^* and such that the complexification $G_{\mathbb{C}}$ contains the standard multiplicative action of $\mathbb{C}^*$ on $H^*(X, \mathbb{C})$;
2. then we replace f^* by $\gamma \in G$ whose eigenvalues on $H^2(X, \mathbb{R})$ are $\lambda_1(f), \lambda_1(f)^{-1}, 1, \ldots, 1$, and prove that the absolute values of eigenvalues of f^* and γ on $H^*(X, \mathbb{C})$ are the same;
3. finally we prove that γ and $\sqrt{\lambda} \in \mathbb{C}^*$ are conjugated in $G_{\mathbb{C}}$, and in particular have the same eigenvalues on $H^*(X, \mathbb{C})$.

The main ingredient of the proof is a theorem of Verbitsky on the structure of the cohomology algebra $H^*(X, \mathbb{C})$ ([Ver13, Theorem 3.5]).

Remark that if we prove the claim for an iterate of f, the claim for f follows immediately; therefore, from now on we will allow ourselves to replace f by one of its iterates.

Step 1

Let G denote the group of automorphisms of the graded $\mathbb{R}$-algebra $H^*(X, \mathbb{R})$ preserving the even Chern classes $c_{2k}(X)$; in other words, a linear automorphism $g \in \mathrm{GL}(H^*(X, \mathbb{R}))$ is in G if and only if

- $g(c_{2k}(X)) = c_{2k}(X)$ for $k = 0, \ldots n$;
- g preserves the subspaces $H^p(X, \mathbb{R})$ for $p = 0, \ldots, 4n$;
- $g(\alpha \wedge \beta) = g(\alpha) \wedge g(\beta)$ for all $\alpha, \beta \in H^*(X, \mathbb{R})$.

Lemma 4.13. *The group G defined above is an algebraic subgroup of $\mathrm{GL}(H^*(X, \mathbb{R}))$ which is defined over $\mathbb{Q}$.*

Proof. All the conditions are imposed by polynomials equations on the coefficients of elements of $\mathrm{GL}(H^*(X, \mathbb{R}))$, therefore G is an algebraic subgroup of $\mathrm{GL}(H^*(X, \mathbb{R}))$.
The condition $g(c_{2k}(X)) = c_{2k}(X)$ for $k = 0, \ldots n$ is expressed by equations with integer coefficients because $c_{2k}(X) \in H^{4k}(X, \mathbb{Z})$.
The same is true for the two other conditions, as the subspaces $H^p(X, \mathbb{R}) \subset H^*(X, \mathbb{R})$ and the wedge-product are defined over $\mathbb{Q}$.
Therefore G is defined over $\mathbb{Q}$ as claimed. □

For every automorphism $f\colon X \to X$, $f^* \in G$: indeed, the Chern classes are f^*-invariant, f^* preserves the degree of forms, and $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$; thus

$$f^* \in G(\mathbb{Z}).$$

Now consider the standard multiplicative action of $\mathbb{C}^*$ on $H^*(X,\mathbb{C})$, with $t \in \mathbb{C}^*$ acting on $H^{p,q}(X)$ as t^{p-q}; t preserves the Chern classes (because $c_{2k}(X) \in H^{2k,2k}(X)$) and the degree of cohomology classes, and $t(\alpha \wedge \beta) = t\alpha \wedge t\beta$ for all $\alpha, \beta \in H^*(X,\mathbb{C})$. Therefore, this realizes $\mathbb{C}^*$ as a one-parameter subgroup of the complexification $G_{\mathbb{C}} \subset \mathrm{GL}(H^*(X,\mathbb{C}))$

$$\rho\colon \mathbb{C}^* \to G_{\mathbb{C}}.$$

Now let

$$\phi\colon G \to \mathrm{GL}(H^2(X,\mathbb{R}))$$

be the restriction to $H^2(X,\mathbb{R})$, and let $G_2 := \operatorname{Im}\phi$. Let $\Gamma = G(\mathbb{Z}) = G \cap \mathrm{GL}(H^*(X,\mathbb{Z}))$ and $\Gamma_2 = G_2(\mathbb{Z}) = G_2 \cap \mathrm{GL}(H^2(X,\mathbb{Z}))$; Γ (resp. Γ_2) is a discrete subgroup of G (resp. of G_2).

By [Ver13, Theorem 3.5], we have

- $G_2 \subset O(H^2(X,\mathbb{R}), q_X)$, where q_X denotes, as usual, the Beauville-Bogomolov form;
- ϕ has finite kernel;
- Γ_2 is an arithmetic subgroup of $O(H^2(X,\mathbb{R}), q_X)$; in other words, Γ_2 has finite index in $O(H^2(X,\mathbb{Z}), q_X)$.

Let $SO(H^2(X,\mathbb{R}), q_X)$ be the Zariski-connected component of the identity of $O(H^2(X,\mathbb{R}), q_X)$; by Proposition 4.11 and Remark 4.12, $SO(H^2(X,\mathbb{Z}), q_X)$ is a lattice of, and is Zariski-dense in, $SO(H^2(X,\mathbb{R}), q_X)$. Hence, the Zariski-closure of Γ_2 in G_2 contains $SO(H^2(X,\mathbb{R}), q_X)$, and a fortiori so does G_2.

Furthermore, since ϕ has finite kernel, G has the same Lie algebra as

$$O(H^2(X,\mathbb{R}), q_X) \cong O(3, b_2(X) - 3);$$

in particular, $\phi(G)$ contains the connected component of the identity of $O(H^2(X,\mathbb{R}), q_X)$.

Step 2

Proposition 2.9 and 2.14 imply that, if $N = b_2(X) - 4$ and σ is the class of a symplectic form on X, there exists a base

$$\mathcal{B} = \{v_+, v_-, \operatorname{Re}\sigma, \operatorname{Im}\sigma, v_1, \ldots v_N\}$$

of $H^2(X,\mathbb{R})$ such that

- the decomposition

$$H^2(X,\mathbb{R}) = (\mathbb{R}v_+ \oplus \mathbb{R}v_-) \oplus \mathbb{R}\operatorname{Re}\sigma \oplus \mathbb{R}\operatorname{Im}\sigma \oplus \mathbb{R}v_1 \oplus \ldots \oplus \mathbb{R}v_N$$

is q_X-orthogonal; more precisely, the matrix of q_X with respect to $\mathcal{B}$ is

$$Q_X = \begin{pmatrix} 0 & 1 & \mathbf{0} & \mathbf{0} \\ 1 & 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -I_{N+2} \end{pmatrix};$$

- possibly after replacing f by f^2, the matrix of $f_2^*\colon H^2(X,\mathbb{R}) \to H^2(X,\mathbb{R})$ with respect to $\mathcal{B}$ is

$$A = \begin{pmatrix} \lambda & & & & & & \\ & \lambda^{-1} & & & & & \\ & & R_{\theta_1} & & & \mathbf{0} & \\ & & & R_{\theta_2} & & & \\ & \mathbf{0} & & & \ddots & & \\ & & & & & R_{\theta_k} & \\ & & & & & & I_h \end{pmatrix},$$

where the R_{θ_i} are 2×2 rotation matrices.

Denoting

$$D = \begin{pmatrix} \lambda & & \\ & \lambda^{-1} & \\ & & I_{N+2} \end{pmatrix}, \quad R = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & R_{\theta_1} & & & \mathbf{0} & \\ & & & R_{\theta_2} & & & \\ & \mathbf{0} & & & \ddots & & \\ & & & & & R_{\theta_k} & \\ & & & & & & I_h \end{pmatrix},$$

we have

$$A = D \cdot R = R \cdot D.$$

Furthermore, D preserves the bilinear form q_X, so that it defines an element $\gamma_2 \in O(H^2(X,\mathbb{R}), q_X)$; by replacing λ with $\lambda(s)$ and making it converge to 1, one shows that γ_2 belongs to the connected component of the identity of $O(H^2(X,\mathbb{R}), q_X)$, therefore to $\phi(G)$. We choose an element γ of G such that $\gamma|_{H^2(X,\mathbb{R})} = \gamma_2$; we can choose such a γ in the connected component of the identity.

Now let us prove that the absolute values of the eigenvalues of γ and f^* are the same. First, remark that f^* and γ commute: indeed, after possibly replacing f by an iterate, they both belong to the connected component of the identity of the real Lie group G, thus we can write $f^* = \exp X_1, \gamma = \exp X_2$ for some $X_1, X_2 \in \mathfrak{g}$. Since the Lie algebra $\mathfrak{g}_2$ of G_2 is isomorphic to $\mathfrak{g}$, and since f_2^* and γ_2 commute, we have $[X_1, X_2] = 0$, which implies that f^* and γ commute as claimed.
Now let $\beta = f^* \circ \gamma^{-1}$, so that $\phi(\beta) = \beta_2$ acts as the matrix R on the basis $\mathcal{B}$ of $H^2(X,\mathbb{R})$; β_2 belongs to the subgroup

$$H_2 = \{\mathrm{id}_{\mathbb{R}^2}\} \times SO_2(\mathbb{R}) \times \ldots \times SO_2(\mathbb{R}) \times \{\mathrm{id}_{\mathbb{R}^h}\} \subset G_2,$$

where each $SO_2(\mathbb{R})$ acts by rotations on the corresponding pair of coordinates in the basis $\mathcal{B}$; H_2 is compact, hence $H = \phi^{-1}(H_2)$ is compact too. Therefore the eigenvalues of elements of H have modulus 1, thus so does β; since f^*, γ and β commute, this implies that the eigenvalues of f^* and γ have the same moduli.

Step 3

Now we show that $\rho(\sqrt{\lambda})$ is conjugate to the complexification $\gamma_{\mathbb{C}}$ of γ in $G_{\mathbb{C}}$. Taking $y \in \mathfrak{g}_{\mathbb{C}}$ such that $\rho(\sqrt{\lambda}) = \exp y$, this is the same as showing that $x = ad_z(y)$ for some $z \in \mathfrak{g}_{\mathbb{C}}$.
The following two bases of $H^2(X,\mathbb{C})$ diagonalize $\gamma_{\mathbb{C}}$ and $\rho(\sqrt{\lambda})$ respectively:

$$\mathcal{B} = \{v_+, v_-, \operatorname{Re}\sigma, \operatorname{Im}\sigma, v_1, \ldots v_N\} \quad \text{as in Step 2},$$

$$\mathcal{B}' = \{c\sigma, c'\bar{\sigma}, u, u', w_1, \dots, w_N\},$$

where c, c' are constants such that $q_X(c\sigma, c'\bar{\sigma}) = 1$ and $\{u, u', w_1, \dots, w_N\}$ is a q_X-orthogonal base of $H^{1,1}(X)$ such that

$$q_X(u) = q_X(u') = 1, \qquad q_X(w_1) = q_X(w_2) = \dots = q_X(w_N) = -1;$$

recall that, as we are now working with the complexification of q_X, the signature does no longer mean anything.
Let $g \in GL(H^2(X,\mathbb{C})$ be the basis change from $\mathcal{B}$ to $\mathcal{B}'$. Since the matrix defining q_X in both bases is

$$Q_X = \begin{pmatrix} 0 & 1 & \mathbf{0} & \mathbf{0} \\ 1 & 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -I_{N+2} \end{pmatrix},$$

g is an element of $O(H^2(X,\mathbb{C}))$; up to changing the sign of an element of $\mathcal{B}'$, we can suppose that $g \in SO(H^2(X,\mathbb{C}))$.
Furthermore, g conjugates $\gamma_{\mathbb{C}}$ and $\rho(\sqrt{\lambda})$. We have seen that the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$ is isomorphic to $\mathfrak{so}_{\mathbb{C}}(3, b_2(X) - 3) \cong \mathfrak{so}_{\mathbb{C}}(b_2(X))$; therefore, at the level of Lie algebras this says exactly that x and y are conjugated by adjoint action of an element of $\mathfrak{g}_{\mathbb{C}}$.

Now, if $\varphi\colon G \to \mathrm{Aut}(V)$ is a finite dimensional representation and $g, h \in G$, then $\varphi(g)$ and $\varphi(hgh^{-1})$ have the same eigenvalues; therefore, the (absolute values of the) eigenvalues of f^* are the same as $\rho(\sqrt{\lambda})$. This shows that

- as all the eigenvalues of $\rho(\sqrt{\lambda})$ are powers of $\sqrt{\lambda}$, the same is true for the moduli of eigenvalues of f^*;

- more accurately, the moduli of eigenvectors of f_k^* are exactly the eigenvectors of $\rho(\sqrt{\lambda})$ on $H^k(X,\mathbb{C})$, i.e. $\lambda^{k/2}$ (with multiplicity $\dim H^{k,0}(X)$), $\lambda^{(k-2)/2}$ (with multiplicity $\dim H^{k-1,1}(X)$) etc.

This concludes the proof of the two main assertions of Theorem 4.4.

Remark that the dominant eigenvalues $\lambda^{p/2}$ of $\rho(\sqrt{\lambda})$ on $H^p(X,\mathbb{C})$ appear only in $H^{p,0}(X)$ (if this space is non-trivial); to conclude, one just needs to recall (see Proposition 2.16) that for an irreducible symplectic manifold

$$\dim H^{p,0}(X) = \dim H^0(X, \Omega_X^p) = \begin{cases} 1 & \text{if } p \text{ is even} \\ 0 & \text{if } p \text{ is odd} \end{cases}$$

This completes the proof.

4.2 Lyapounov exponents

In this section we define the Lyapounov exponents, which are a global equivalent of the logarithms of the eigenvalues of the Jacobian matrix at a fixed point; see [Kat80].

Let $g\colon M \to M$ be a diffeomorphism of a compact d-dimensional manifold. Fix a Riemannian metric on M. One defines the first Lyapounov exponent at a point $p \in M$ as the limit (if it exists)

$$\chi_1(g,p) = \lim_{n\to+\infty} \frac{\log \| (Dg^n)_p \|}{n},$$

where $(Dg^n)_p \colon T_pM \to T_{g^n(p)}M$ denotes the differential of g^n at the point p and $\|\cdot\|$ is the norm on $\mathcal{L}(T_pM, T_{g^n(p)}M)$ induced by the Riemannian metric. Then one defines recursively

$$\chi_k(g,p) = \lim_{n\to+\infty} \frac{\log \| \bigwedge^k (Dg^n)_p \|}{n} - (\chi_1(g,p) + \ldots + \chi_{k-1}(g,p)),$$

where $\bigwedge^k (Dg^n)_p \colon \bigwedge^k T_pM \to \bigwedge^k T_{g^n(p)}M$ denotes the transformation induced by $(Dg^n)_p$ between the k-th exterior powers of the tangent spaces and $\|\cdot\|$ is a norm on $\mathcal{L}(\bigwedge^k T_pM, \bigwedge^k T_{g(p)}M)$ induced by the Riemannian metric. We then have

$$\chi_1(g,p) \geq \chi_2(g,p) \geq \ldots \geq \chi_d(g,p).$$

This definition is independent of the chosen Riemannian metric.

Example 4.14.

- If p is a fixed point for g, then the Lyapounov exponents are just the logarithms of the eigenvalues of Dg_p.
- Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional real torus, and let $g \colon \mathbb{T}^n \to \mathbb{T}^n$ be the linear automorphism induced by a matrix $A \in \mathrm{SL}_N(\mathbb{Z})$. Then, if we consider the Lebesgue measure, the Lyapounov exponents are the logarithms of the eigenvalues of A.

If μ is a g-invariant probability measure on M, then Oseledets theorem [Ose68] implies that the limit above exists for μ-almost every $p \in M$. If furthermore μ is ergodic, the g-invariant functions $p \mapsto \chi_k(g,p)$ are μ-almost everywhere equal to some constants $\chi_k(g)$; we can then give the following definition:

Definition 4.15. *Let $g \colon M \to M$ be a diffeomorphism of a compact manifold M of dimension d and let μ be a g-invariant ergodic probability measure; the* Lyapounov exponents *of g are the constants $\chi_1(g), \ldots, \chi_d(g)$.*

Remark 4.16. The Lyapounov exponents depend on the chosen measure: for example, if p is a fixed (respectively, N-periodic) point of g, the Lyapounov exponents for the Dirac measure $\mu = \delta_p$ (resp. $\mu = (\delta_p + \delta_{g(p)} + \ldots + \delta_{g^{N-1}(p)})/N$) are the logarithms of the eigenvalues of Dg_p (resp. $1/N$ times the eigenvalues of Dg_p^N).

Definition 4.17. *Let $g \colon M \to M$ be a diffeomorphism of a compact variety M; a g-invariant ergodic probability measure is said to be* hyperbolic *if all the Lyapounov exponents with respect to μ are non-zero.*

4.3 Construction of an invariant measure of maximal entropy

In this subsection we summarize Dinh and Sibony's results about the construction of Green currents and Green measures; in what follows (X,ω) denotes a compact Kähler manifold of dimension d.

The goal is to construct an ergodic f-invariant measure μ of maximal entropy satisfying some regularity condition (namely that μ doesn't charge any positive codimensional analytic subset).

4.3.1 Green currents and Dinh-Sibony measure

Let (X,ω) be a compact Kähler manifold and let $f \colon X \to X$ be a positive entropy automorphism of X. Recall that, by log-concavity (Proposition 1.14), there exist $0 \leq m \leq m' \leq d$ such that the dynamical degrees λ_i of f satisfy

$$1 = \lambda_0 < \lambda_1 < \ldots < \lambda_{m-1} < \lambda_m = \lambda_{m+1} = \ldots = \lambda_{m'} > \lambda_{m'+1} > \ldots > \lambda_d = 1.$$

In [DS05b], Dinh and Sibony construct positive closed currents $T_s = T_s(f)$ of bidegree (s, s) for $s \leq m$, called the *Green currents* for f, satisfying

$$f^* T_s = \lambda_s(f) T_s.$$

This generalizes the results for Hénon maps of $\mathbb{C}^2$ and for algebraically stable meromorphic self-maps of $\mathbb{P}^k$ (see [DS05a]).

Now assume that all the consecutive dynamical degrees are distinct, i.e. that $m = m'$; we denote by T^+ the Green current $T_m(f)$. Recall that, by Lemma 1.13, $\lambda_p(f^{-1}) = \lambda_{d-p}(f)$, thus repeating the construction for f^{-1} produces a Green current $T_{d-m}(f^{-1})$, which we denote by T^-. One can try constructing an f-invariant measure by taking the wedge-product *Dinh-Sibony measure*

$$\mu = T^+ \wedge T^-.$$

One needs to check that

- the wedge-product actually exists: this is a consequence of T^+ and T^- having *continuous super-potentials* (denoted "PC" in [DS05b]; see also [DS10, Section 3]);
- the construction doesn't lead to a trivial measure: this is the content of the following theorem.

Theorem 4.18 ([DS10])**.** *Let $f\colon X \to X$ be an automorphism of a compact Kähler manifold X. Assume that*

- *all the consecutive dynamical degrees of f are distinct (i.e., with the above notations, $m = m'$);*
- *$\lambda_m(f)$ is a simple eigenvalue of $H^{m,m}(X, \mathbb{R})$.*

Then one can choose Green currents T^+, T^- such that $\mu := T^+ \wedge T^-$ is an f-invariant probability measure; μ is ergodic and of maximal entropy. All quasi-p.s.h. functions are μ-integrable; in particular, μ does not charge any positive-codimensional analytic subvariety (i.e. such subvarieties have measure zero). If furthermore all the dominant eigenvalues of $f^*_{p,p}$ are equal to $\lambda_p(f)$, then μ is mixing (see [HK02]).

Definition 4.19. *We call a measure as in Theorem 4.18 a* Dinh-Sibony measure *for f.*

The following theorem, which is a special case of [dT08, Corollary 3], shows that a Dinh-Sibony measure is hyperbolic.

Theorem 4.20. *Let $f\colon X \to X$ be a dominant holomorphic endomorphism of a d-dimensional compact Kähler manifold preserving an ergodic probability measure μ; if μ is a measure of maximal entropy and the dynamical degrees λ_i of f satisfy*

$$1 = \lambda_0 < \lambda_1 < \ldots < \lambda_{m-1} < \lambda_m > \lambda_{m+1} > \ldots > \lambda_d,$$

then μ is hyperbolic. More precisely, the Lyapounov exponents χ_i of f satisfy

$$\chi_1 \geq \ldots \geq \chi_m \geq \frac{1}{2} \log \frac{\lambda_m}{\lambda_{m-1}} > 0 > \frac{1}{2} \log \frac{\lambda_{m+1}}{\lambda_m} \geq \chi_{m+1} \geq \ldots \geq \chi_d.$$

Example 4.21. If X is an irreducible symplectic manifold of dimension $2n$ and $f\colon X \to X$ is an automorphism, then by Theorem 4.4, f satisfies the hypothesis of Theorem 4.18 and 4.20; hence we can construct a Dinh-Sibony measure μ for f which is hyperbolic. More precisely, since $\lambda_m(f) = \lambda_n(f) = \lambda_1(f)^n$ and $\lambda_{m-1}(f) = \lambda_{m+1}(f) = \lambda_1(f)^{n-1}$, the Lyapounov exponents of f with respect to μ satisfy

$$\chi_1 \geq \ldots \geq \chi_n \geq \frac{1}{2}\log\lambda_1(f) > 0 > -\frac{1}{2}\log\lambda_1(f) \geq \chi_{n+1} \geq \ldots \geq \chi_{2n}.$$

4.3.2 Proof of Theorem 4.2

Let $f\colon X \to X$ be a loxodromic automorphism of an irreducible symplectic manifold X of dimension $2n$. By Theorem 4.4, the dynamical degrees λ_i of f satisfy

$$1 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n > \lambda_{n+1} > \ldots > \lambda_{2n} = 1.$$

By Theorem 4.18 and 4.20, we can then construct a Dinh-Sibony measure μ, which is an f-invariant, ergodic and hyperbolic probablity measure with maximal entropy. By a result of Katok [Kat80, Lemma 4.2], the closure of the set of hyperbolic periodic points contains $Supp(\mu)$; by Theorem 4.18 μ cannot charge a divisor, hence a fortiori it cannot be supported on a divisor. This implies that hyperbolic periodic points are Zariski dense in X, which concludes the proof.

Part II

Invariant fibrations

Part II constitutes the bulk of this work: therein, I analyse the existence of equivariant fibrations, and give some constraints on the induced action on the base when one exists.

In Chapter 5 I prove that loxodromic transformations of irreducible symplectic manifolds are primitive, i.e. they do not preserve any meromorphic fibration; as a corollary, I obtain a bound on the number of periodic hypersurfaces and I show that the orbits of very general points are Zariski-dense. The proof uses the definition of dynamical degrees and all the content of Chapter 2.

In Chapter 6 I prove that, if a transformation of a projective manifold whose canonical bundle admits sections preserves a fibration onto $\mathbb{P}^n$ and induces an automorphism on the base, then the action on the base has finite order. The main ingredient of the proof is an argument of p-adic integration, and this part is self-contained.
In the final part of the chapter, I give some partial results about automorphisms of Calabi-Yau threefolds preserving a fibration; again, no preliminaries are needed, except for the notion of dynamical degrees.

Chapter 5

Primitivity of loxodromic transformations of irreducible symplectic manifolds

The material of this chapter constitutes the bulk of an Arxiv preprint [LB], to appear in International Research Mathematical Notices in summer 2017.

As discussed in §1.3.4, if an automorphism admits an equivariant fibration, its dynamics can be studied on smaller dimensional varieties (the base and the fibres of the fibration), which is a priori simpler than the general case.
We work in the meromorphic setting (see [HKZ15]):

Definition 5.1. *Let $g\colon M \dashrightarrow M$ be a bimeromorphic transformation of a compact Kähler manifold. A meromorphic fibration $\pi\colon M \dashrightarrow B$ (i.e. a dominant meormorphic map with connected fibres) onto a compact Kähler manifold B such that $0 < \dim B < \dim X$ is called g-*equivariant *if there exists a bimeromorphic transformation $h\colon B \dashrightarrow B$ such that $\pi \circ g = h \circ \pi$, i.e. if the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, dashed, "g"] \arrow[d, dashed, "\pi"] & M \arrow[d, dashed, "\pi"] \\ B \arrow[r, dashed, "h"] & B \end{tikzcd}$$

The transformation g is said to be primitive *if it admits no equivariant fibration.*

Theorem A. *Let X be an irreducible holomorphic symplectic manifold and let $f\colon X \dashrightarrow X$ be a loxodromic bimeromorphic transformation. Then*

1. *f is primitive;*
2. *f admits at most $\dim(X) + b_2(X) - 2$ periodic hypersurfaces;*
3. *the f-orbits of very general points of X are Zariski-dense.*

Here a hypersurface $H \subset X$ is said to be f-periodic if its strict transform $(f^n)^*H$ by some iterate of f is equal to H.

Remark 5.2. Assertion (2) follows from assertion (1) and [Can10, Theorem B]; assertion (3) follows from assertion (1) and [AC08, Theorem 4.1], but is proven here as a lemma (Lemma 5.10).

Theorem A has been applied by Oguiso to construct primitive automorphisms of irreducible holomorphic symplectic manifolds (see [Ogu16b] and §3.5).

Remark 5.3. As discussed above, the primitivity of automorphisms with first dynamical degree > 1 is known for surfaces. Theorem A extends the reults to irreducible holomorphic symplectic manifolds (which bear a formal resemblance to surfaces through the Beauville-Bogomolov form (see §2.3).
To the best of my knowledge, these are the only big classes for which the question has been completely answered (apart from complex tori, where the dynamics becomes essentially linear). The first open case is that of threefolds (see [Les15] for some partial results); Calabi-Yau threefolds should be the most approachable case (see [OT15, Ogu16b] for existence results).

The proof of Theorem A is by contradiction: throughout this chapter, $f\colon X \dashrightarrow X$ denotes a loxodromic bimeromorphic transformation of an irreducible holomorphic symplectic manifold X, $\pi\colon X \dashrightarrow B$ a meromorphic equivariant fibration onto a Kähler manifold B such that $0 < \dim B < \dim X$ and $g\colon B \dashrightarrow B$ the induced transformation of the base.

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, dashed, "\pi"] & X \arrow[d, dashed, "\pi"] \\ B \arrow[r, dashed, "g"] & B \end{tikzcd}$$

Throughout this chapter we denote by $\mathrm{NS}(\cdot) \subset \mathrm{H}^{1,1}(\cdot, \mathbb{R})$ the Neron-Severi group with real coefficients. We will adopt the usual vocabulary of algebraic geometry: "general" means "in a Zariski-open dense set"; "very general" means "in a countable intersection of Zariski-open dense subsets".

5.1 Meromorphic fibrations on irreducible holomorphic symplectic manifolds

We collect here some useful facts about the fibration π. The results and proofs in this Section are largely inspired by Amerik and Campana [AC08].

Remark 5.4. Let $\pi\colon X \dashrightarrow B$ be a dominant meromorphic map. If B is Kähler, then it is projective.
Indeed, if B weren't projective, by Kodaira's projectivity criterion and Hodge decomposition

$$H^2(B, \mathbb{C}) = H^{2,0}(B) \oplus H^{1,1}(B) \oplus H^{0,2}(B),$$

we would have $H^{2,0}(B) \neq \{0\}$, meaning that B carries a non-trivial holomorphic 2-form σ_B. Since the indeterminacy locus of π has codimension at least 2, the pull-back $\pi^*\sigma_B$ could then be extended to a global non-trivial 2-form on X which is not a multiple of σ, contradicting the irreducibility of X.

Here we use the same conventions as in [AC08]: let $\eta\colon \tilde{X} \to X$ be a resolution of the indeterminacy locus of π (see [Uen75]), and let $\nu\colon \tilde{X} \to B$ be the induced holomorphic fibration, whose general fibres are bimeromorphic to those of π.

$$\begin{tikzcd} \tilde{X} \arrow[d, "\eta"] \arrow[dr, "\nu"] & \\ X \arrow[r, dashed, "\pi"] & B \end{tikzcd}$$

The pull-back π^*D of an effective divisor $D \in Div(B)$ is defined as

$$\pi^*D = \eta_*\nu^*D,$$

where η_* is the pushforward as cycles. The pull-back induces linear morphisms $Pic(B) \to Pic(X)$ and $NS(B) \to NS(X)$, and is compatible with the pull-back of smooth forms defined in §1.1.3.

Remark 5.5. Let $\pi\colon X \dashrightarrow B$ be a dominant meromorphic map onto a projective manifold, and let$L = \pi^* H$ where $H \in Pic(B)$ is an ample class; then L is not numerically trivial and $q_X(L) \geq 0$.
Indeed, since X is simply connected, it bears no holomorphic 1-forms, so that the numerical class of a divisor determines its linear class. The pull-back of the complete linear system $|H|$ is a linear system $U \subset |L|$, whose associated Iitaka map is exactly π. In particular, L is effective, hence not numerically trivial; furthermore L has no fixed component, and by Remark 2.10 we have $q_X(L) \geq 0$.

The following Lemma is essentially proven in [AC08].

Lemma 5.6. *The restriction of the Beauville-Bogomolov form to the pull-back $\pi^* NS(B)$ is not identically zero if and only if general fibres of ν are of general type. If this is the case, then X is projective.*

Proof. Remark first that, by [Moi67] if there exists a big line bundle on a compact Kähler manifold X, then X is projective.

Suppose that general fibres of ν are of general type. Let H be an ample divisor on B and let $L = \pi^* H$. By [AC08, Theorem 2.3] we have $\kappa(X, L) = \dim(B) + \kappa(F)$, where F is a general fibre of ν; we conclude that L is big (and in particular X is projective). We can thus write $L = A + E$ for an ample divisor A and an effective divisor E on X. Now, if q denotes the Beauville-Bogomolov form, we have

$$q(L) = q(L, A) + q(L, E) \geq q(L, A) = q(A, A) + q(A, E) \geq q(A, A) > 0,$$

where the first and second inequalities are consequences of L and A being without fixed components and the last one follows directly from Remark 5.5. This proves the "if" direction.

Now assume that the restriction of q_X to $\pi^* NS(B)$ is not identically zero. Since ample classes generate $NS(B)$, there exists an ample line bundle $H \in Pic(B)$ such that, denoting $L = \pi^* H$, $q(L) \neq 0$; furthermore, L is without fixed components, so that $q(L) > 0$ by Remark 5.5. It follows by [Bou04][Theorem 4.3.i] that L is big (thus X is projective), and so is $\eta^* L$ since η is a birational morphism. Therefore, the restriction $\eta^* L|_F$ to a generic fibre of ν is also big (see [Laz04][Corollary 2.2.11]).

By the definition of L we have

$$\eta^* L = \nu^* H + \sum a_i E_i \qquad \text{for some } a_i \in \mathbb{Z},$$

where the sum runs over all the irreducible components of the exceptional divisor of η. The adjunction formula leads to

$$K_F = K_{\tilde{X}}|_F + \det N^*_{F/\tilde{X}} = K_{\tilde{X}}|_F$$

since the conormal bundle $N^*_{F/\tilde{X}}$ is trivial. Since η is a sequence of blow-ups, we have

$$K_{\tilde{X}} = \sum e_i E_i|_F \qquad \text{for some } e_i > 0.$$

This implies that, for some $m > 0$, the divisor $mK_F - \eta^* L|_F$ is effective because $\nu^* H|_F$ is trivial. Thus

$$\kappa(F) \geq \kappa(F, \eta^* L|_F) = \dim(F),$$

meaning that F is of general type. This proves the "only if" direction. $\square$

Corollary 5.7. *If a generic fibre of ν is not of general type, then $\pi^* NS(B) \subset H^{1,1}(X, \mathbb{R})$ is a line contained in the isotropic cone $\mathcal{C}_0$.*

Proof. By Lemma 5.6, $\pi^* NS(B)$ is contained in the isotropic cone. The pull-back L of an ample line bundle on B is effective and non-trivial, so that its numerical class is also non-trivial; thus $\pi^* NS(B)$ cannot be trivial. To conclude it suffices to remark that $\pi^* NS(B)$ is a linear subspace of $H^{1,1}(X, \mathbb{R})$, and the only non-trivial subspaces contained in the isotropic cone are lines. □

5.2 Density of orbits

The following theorem was proven in [AC08].

Theorem 5.8. *Let X be a compact Kähler manifold and let $f: X \dashrightarrow X$ be a dominant meromorphic endomorphism. Then there exists a dominant meromorphic map $\pi: X \dashrightarrow B$ onto a compact Kähler manifold B such that*

1. *$\pi \circ f = \pi$;*
2. *the Zariski-closure of a very general fibre X_b of π is the Zariski closure of the f-orbit of a very general point of X_b.*

Lemma 5.9. *Let $\phi: X \dashrightarrow Y$, $\psi: Y \dashrightarrow Z$ be meromorphic maps between compact complex manifolds. If ϕ is an isomorphism in codimension 1, then for all $D \in Div(Z)$*

$$(\psi \circ \phi)^* D = \phi^* \psi^* D.$$

Proof. In order to prove the result, we give a characterization of the pull-back of a divisor by a rational map which is equivalent to the one defined in §5.1. Let $f: X_1 \dashrightarrow X_2$ be a rational map between projective manifolds and let $U \subset X_1$ be the maximal open subset of X_1 such that f is well-defined on U; in particular $\operatorname{codim} X_1 \setminus U \geq 2$. Then for all divisors $D \in Div(X_2)$

$$f^* D = \overline{f|_U^* D}.$$

Indeed, we can construct a resolution of indeterminacies of f

$$\begin{tikzcd} \tilde{X} \arrow[d, "\eta"] \arrow[dr, "\tilde{f}"] & \\ X_1 \arrow[r, dashed, "f"] & X_2 \end{tikzcd}$$

by successive blow-ups of smooth subvarieties of $X_1 \setminus U$; in particular the exceptional locus of η is contained in $\eta^{-1}(X_1 \setminus U)$, and since f is identified with $\tilde{f}$ on $U \cong \eta^{-1} U$ the claim follows.

Now let $U \subset X$, $V \subset Y$ two open sets such that ϕ induces an isomorphism $U \cong V$ and such that $\operatorname{codim}(X \setminus U) \geq 2, \operatorname{codim}(Y \setminus V) \geq 2$.Up to shrinking V to some other open subset whose complement has codimension at least 2, we can suppose that ψ is regular on V; therefore the composition $\psi \circ \phi$ is regular on U, and since the complement of U has codimension ≥ 2 in X, for all $D \in Div(Z)$ we have

$$(\psi \circ \phi)^* D = \overline{(\psi \circ \phi)|_U^* D} = \overline{\phi|_U^* (\psi|_V^* D)} = \overline{\phi|_U^* (\psi^* D)} = \phi^* \psi^* D,$$

where the third equality follows again from the fact that the complement of V has codimension at least 2 in Y. This proves the claim. □

Let us prove point (3) of Theorem A.

Lemma 5.10. *Let $f\colon X \dashrightarrow X$ be a bimeromorphic loxodromic transformation of an irreducible holomorphic symplectic manifold. Then the very general orbit of f is Zariski-dense.*

Proof. Suppose by contradiction that the very general orbits of f are not Zariski-dense; then by Theorem 5.8 we can construct a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, dashed, "\pi"] & X \arrow[d, dashed, "\pi"] \\ B \arrow[r, "\mathrm{id}_B"] & B \end{tikzcd}$$

where π is a meromorphic map whose very general fibre X_b coincides with the Zariski-closure of the f-orbit of a very general point of X_b, and $\dim B > 0$. Remark 5.4 applies in the case where the fibres are not connected; therefore the base B is projective.

Let $V = \pi^* NS_{\mathbb{R}}(B) \subset H^{1,1}(X, \mathbb{R})$; then $f^*|_V = \mathrm{id}_V$: indeed, for $v \in NS(B)$,

$$f^*\pi^*v = (\pi \circ f)^*v = (\mathrm{id}_B \circ \pi)^*v = \pi^*v,$$

where the first equality follows from Lemma 5.9.

By Theorem 2.14, no non-zero vector of $\mathcal{C}_{\geq 0}$ (the positive cone for the Beauville-Bogomolov form) is preserved by f^*; hence $V \cap \mathcal{C}_{\geq 0} = \{0\}$, i.e. the Beauville-Bogomolov form is negative definite on V.

Now, let $v \in NS(B)$ be an ample class and let $w = \pi^*v \in V$. By Remark 5.5 $w \neq 0$ and $q_X(w) \geq 0$. This proves the claim by contradiction. □

5.3 A key lemma

The following key lemma, together with Proposition 5.14, implies Theorem A.

Lemma 5.11 (Key lemma). *Let X be an irreducible holomorphic symplectic manifold, $f\colon X \dashrightarrow X$ a loxodromic bimeromorphic transformation and $\pi\colon X \dashrightarrow B$ a meromorphic f-equivariant fibration onto a compact Kähler manifold. Then X is projective and the general fibre of π is of general type.*

Proof. Let $g\colon B \dashrightarrow B$ be a bimeromorphic transformation such that $g \circ \pi = \pi \circ f$.

Let us define

$$V := Span\,\{(h \circ \pi)^* NS_{\mathbb{R}}(B) \,|\, h \in \mathrm{Bir}(B)\} \subset NS_{\mathbb{R}}(X).$$

The linear subspace V is clearly defined over $\mathbb{Q}$. Since the pull-back by π of an ample class is numerically non-trivial, we also have $V \neq \{0\}$.
Furthermore, V is f^*-invariant: if $v = (h \circ \pi)^*w$ for some $w \in NS(B)$ and for some birational transformation $h\colon B \dashrightarrow B$, then

$$f^*v = f^*(h \circ \pi)^*w = (h \circ \pi \circ f)^*w = (h \circ g \circ \pi)^*w = (\tilde{h} \circ \pi)^*w,$$

where $\tilde{h} = h \circ g\colon B \dashrightarrow B$ is a birational transformation and the second equality follows from Lemma 5.9.

Now suppose that the general fibre of π is not of general type; we are first going to show that V is contained in the isotropic cone $\mathcal{C}_0 = \{v \in H^{1,1}(X, \mathbb{R}) | q_X(v) = 0\}$. The general fibre of the meromorphic fibration $h \circ \pi$ is bimeromorphic to that of π. By Lemma 5.6 we know that $(h \circ \pi)^* NS_{\mathbb{R}}(B)$ is contained in the isotropic cone for all birational transformations $h\colon B \dashrightarrow B$.

We just need to show that for all birational transformations of B onto itself h_i, h_j and for all $w_i, w_j \in NS_{\mathbb{R}}(B)$ we have

$$q_X((h_i \circ \pi)^* w_i, (h_j \circ \pi)^* w_j) = 0.$$

Let $h = h_j \circ h_i^{-1}$, and let $\rho\colon \tilde{B} \to B$ be a resolution of the indeterminacy locus of h; denote by $\tilde{h}\colon \tilde{B} \to B$ the induced holomorphic transformation, and let $\tilde{\pi} = \rho^{-1} \circ h_i \circ \pi\colon X \dashrightarrow \tilde{B}$; $\tilde{\pi}$ is a meromorphic fibration onto the birational model $\tilde{B}$, whose general fibre is bimeromorphic to that of π. Finally, let $\eta\colon \tilde{X} \to X$ be a resolution of singularities of $\tilde{\pi}$ and let $\nu\colon \tilde{X} \to B$ be the induced holomorphic map.

$$\begin{tikzcd}
\tilde{X} \arrow[dr, "\nu"] \arrow[dd, "\eta"'] & & \\
& \tilde{B} \arrow[d, "\rho"] \arrow[dr, "\tilde{h}"] & \\
X \arrow[ur, dashed, "\tilde{\pi}"] \arrow[r, dashed, "h_i \circ \pi"] & B \arrow[r, dashed, "h"] & B
\end{tikzcd}$$

Now it is clear that $\eta\colon \tilde{X} \to X$ is a resolution of indeterminacies of both $h_i \circ \pi$ and $h_j \circ \pi = h \circ h_i \circ \pi$. Therefore

$$(h_i \circ \pi)^* w_i = \eta_* \nu^* \rho^* w_i = \tilde{\pi}^* \rho^* w_i \in \tilde{\pi}^* NS(\tilde{B})$$

and

$$(h_j \circ \pi)^* w_j = \eta_* \nu^* \tilde{h}^* w_j = \tilde{\pi}^* \tilde{h}^* w_j \in \tilde{\pi}^* NS(\tilde{B}).$$

Since the fibres of $\tilde{\pi}$ are not of general type, it suffices to apply Lemma 5.6 to the fibration $\tilde{\pi}\colon X \dashrightarrow \tilde{B}$ to conclude that $q_X((h_i \circ \pi)^* w_i, (h_j \circ \pi)^* w_j) = 0$. This proves that V is contained in the isotropic cone.

Now the only non trivial vector subspaces of $NS_{\mathbb{R}}(X)$ contained in the isotropic cone are lines; by Theorem 2.14, V is then an f^*-invariant line contained in the isotropic cone and not defined over $\mathbb{Q}$. But this contradicts the definition of V. We have thus proved that the general fibre of π is of general type.

In order to prove that X is projective it suffices to apply the last part of Lemma 5.6.

□

By [Uen75, Corollary 14.3] we know that the group of birational transformations of a variety of general type is finite. Therefore, we expect the dynamics of f on the fibres to be simple.

5.4 Relative Iitaka fibration

Before giving the proof of Theorem A, let us recall the basic results about the relative Iitaka fibration. We will follow the approach of [Tsu10] with some elements from [Uen75]. See also [GD71], [Gro61].

Let X be a smooth projective variety, and suppose that some multiple of K_X has a non trivial section. Recall that, for $m > 0$ divisible enough, the rational map

$$\begin{aligned}
\phi_{|mK_X|}\colon X &\dashrightarrow \mathbb{P}H^0(X, mK_X)^* \\
p &\mapsto \{s \in \mathbb{P}H^0(X, mK_X) | s(p) = 0\}
\end{aligned}$$

has connected fibres. Moreover the rational maps $\phi_{|mK_X|}$ eventually stabilize to a rational fibration onto its image, which we call *canonical fibration* of X. See for example [Laz04].

Remark 5.12. If $f\colon X \dashrightarrow X$ is a bimeromorphic transformation of X, the pull-back of forms induces a linear automorphism $f^*\colon H^0(X, mK_X) \to H^0(X, mK_X)$. For example, for $m = 1$ a section $\sigma \in H^0(X, K_X)$ is a holomorphic d-form ($d = \dim X$); f is defined on an open set $U \subset X$ such that $X \setminus U$ has codimension at least 2. Therefore by Hartogs theorem the pull-back $f|_U^*\sigma$ can be extended to X. It is easy to see that the construction is invertible and induces a linear automorphism of $\mathbb{P}H^0(X, mK_X)^*$ which commutes with the Iitaka fibration:

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, dashed, "\phi_{|mK_X|}"] & X \arrow[d, dashed, "\phi_{|mK_X|}"] \\ \mathbb{P}H^0(X, mK_X)^* \arrow[r, "\tilde{f}"] & \mathbb{P}H^0(X, mK_X)^* \end{tikzcd}$$

The above construction can be generalized to the relative setting: let $\pi\colon X \to B$ be a regular fibration onto a smooth projective variety B, and let $K_{X/B} = K_X \otimes \pi^* K_B^{-1}$ be the relative canonical bundle.

For some fixed positive integer $m > 0$ (divisible enough), let $\mathcal{S} = \pi_*(mK_{X/B})^*$. $\mathcal{S}$ is a coherent sheaf over B; therefore one can construct (generalizing the construction of the projective bundle associated to a vector bundle, see [Uen75] for details) the algebraic projective fibre space

$$\eta\colon \mathbb{P}(\mathcal{S}) \to B$$

associated to $\mathcal{S}$, which is a projective scheme (a priori neither reduced nor irreducible) over B; denote by $Y = \mathbb{P}(\mathcal{S})_{red} \to B$ the reduced structure of $\mathbb{P}(\mathcal{S})$. Its fibre Y_b over a general point $b \in B$ is canonically isomorphic to $\mathbb{P}H^0(X_b, mK_{X_b})^*$. The Iitaka morphisms

$$\phi_b\colon X_b \dashrightarrow \mathbb{P}H^0(X_b, mK_{X_b})^*$$

induce a rational map $\phi\colon X \dashrightarrow Y$ over B.

The *relative Iitaka fibration* of X with respect to π is

$$\begin{aligned} \phi\colon X &\dashrightarrow Y \\ x \in X_b &\mapsto \left[\{s \in H^0(X_b; mK_{X_b}) | s(x) = 0\}\right] \in Y_b. \end{aligned}$$

It can be shown that, for m divisible enough:

- the fibres of ϕ are connected;
- the image by ϕ of a generic fibre $X_b = \pi^{-1}(b)$ of π is contained inside the fibre $\eta^{-1}(b)$ of the natural projection $\eta\colon Y \to B$;
- the restriction of ϕ to a generic fibre X_b is birationally equivalent to the canonical fibration of X_b.

Remark 5.13. The construction in Remark 5.12 can also be generalized to the relative setting: let $f\colon X \dashrightarrow X$ and $g\colon B \dashrightarrow B$ be birational transformations such that $\pi \circ f = g \circ \pi$. For a generic $b \in B$ define

$$\begin{aligned} \tilde{f}|_{Y_b}\colon \mathbb{P}H^0(X_b, mK_{X_b})^* &\dashrightarrow \mathbb{P}H^0(X_{g(b)}, mK_{X_{g(b)}})^* \\ [s^*] &\mapsto \left\{[s] \in \mathbb{P}H^0(X_{g(b)}, mK_{X_{g(b)}}) | s^*(f^*s) = 0\right\}. \end{aligned}$$

These are well defined linear automorphisms because, for a fibre X_b of π not contained in the indeterminacy locus of f, the restriction $f\colon X_b \dashrightarrow X_{g(b)}$ is a birational map, and thus induces a linear isomorphism

$$f^*\colon H^0(X_{g(b)}, mK_{X_{g(b)}}) \to H^0(X_b, mK_{X_b}).$$

Furthermore the $\tilde{f}_{X_b}$ can be glued to a birational transformation $\tilde{f}\colon Y \dashrightarrow Y$ such that $\eta \circ \tilde{f} = g \circ \eta$.

Now suppose general fibres of π are of general type. Since the restriction of ϕ to a generic fibre of g is birational onto its image and the images of fibres are disjoint, ϕ itself must be birational onto its image; denote by Z the closure of the image of ϕ and let $f_Z = \phi \circ f \circ \phi^{-1} \colon Z \dashrightarrow Z$ be the birational transformation induced by f.
By the above Remark, f_Z is the restriction of the birational transformation $\tilde{f} \colon Y \dashrightarrow Y$. In particular f_Z induces an isomorphism between generic fibres of $\eta|_Z$.

$$\begin{tikzcd}
X \arrow[loop above, "f"] \arrow[r, dashed, "\phi"] \arrow[d, dashed, "\pi"] & Z \arrow[loop above, "f_Z"] \arrow[r, hook] \arrow[dl, dashed] & Y \arrow[loop above, "\tilde{f}"] \arrow[dll, dashed, "\eta"] \\
B \arrow[loop left, "g"] & &
\end{tikzcd}$$

5.5 Proof of Theorem A

Proposition 5.14. *Let X, B be projective manifolds, $f \colon X \dashrightarrow X$ and $g \colon B \dashrightarrow B$ birational transformations and $\pi \colon X \to B$ a holomorphic fibration such that $\pi \circ f = g \circ \pi$.*

$$\begin{tikzcd}
X \arrow[r, dashed, "f"] \arrow[d, "\pi"] & X \arrow[d, "\pi"] \\
B \arrow[r, dashed, "g"] & B
\end{tikzcd}$$

Assume that a smooth fibre of π is of general type (if and only if general fibres are of general type) and that g has a Zariski-dense orbit. Then

1. *all the fibres over a non-empty Zariski open subset of B are birationally equivalent;*

2. *the orbits of f are not Zariski-dense; more precisely, the Zariski-closures of very general orbits have dimension* $\dim B$.

Proof. Denote as before

$$\phi \colon X \dashrightarrow Y$$

the relative Iitaka fibration. Since we are only interested in the birational type of fibres of $\pi \colon X \to B$ and in the Zariski-density of orbits of the birational transformation f, we can and will identify X with its birational model $\overline{\phi(X)}$. Let $F = \pi^{-1}(b_0)$ be the fibre of π over a point b_0 whose orbit is Zariski-dense in B; let us fix a dense open subset $U \subset B$ such that the restriction $\pi \colon X_U = \pi^{-1}(U) \to U$ is smooth.

(1) Consider the relative isomorphism functor $\mathrm{Isom}_U(X_U, F \times U)$: for any U-scheme S, $\mathrm{Isom}_U(X_U, F \times U)(S)$ is by definition the set of isomorphisms over S between $X_U \times_U S$ and $(F \times U) \times_U S = F \times S$.

Since $\pi \colon X_U \to U$ is flat, the functor $\mathrm{Isom}_U(X_U, F \times U)$ can be represented by a U-scheme

$$\mathfrak{I} = \mathfrak{Isom}_U(X_U, F \times U);$$

see [ACG11, §9.7], [Gro95, §4]. In other words, for every U-scheme S, there is a canonical bijection between $\mathrm{Isom}_U(X_U, F \times U)(S)$ and the set $Hom_U(S, \mathfrak{I})$ of morphisms of U-schemes between S and $\mathfrak{I}$; in particular, for every closed point $b \in U$, the closed points of the fibre $\mathfrak{I}_b$ parametrize the set of isomorphisms $X_b \cong F$. We are going to prove that the image of the natural morphism $\phi \colon \mathfrak{I} \to U$ contains a dense open set $U' \subset U$, so that all the fibres of $\pi \colon X_{U'} \to U'$ are isomorphic.

One can realize $\mathfrak{I}$ as an open subset of the Hilbert scheme $\mathfrak{Hilb}_U(X_U \times_U (U \times F))$ by identifying a morphism $X_b \to F$ with its graph in $X_b \times F$. Therefore,

$$\mathfrak{I} = \coprod_{P \in \mathbb{Q}[\lambda]} \mathfrak{I}^P,$$

where the fibre $\mathfrak{I}_b^P$ is the (a priori non irreducible and non reduced) quasi-projective scheme of (graphs of) isomorphisms $X_b \xrightarrow{\sim} F$ having fixed Hilbert polynomial $P(\lambda)$; such polynomials are calculated with respect to the restriction to the fibre $X_b \times F$ of a fixed line bundle L on $X_U \times_U (U \times F)$ relatively very ample over U. We shall fix

$$L = H_Y|_{X_U} \boxtimes_U H_F,$$

where H_Y is a very ample line bundle on Y and H_F is a very ample line bundle on F.

Now, the pull-back of forms by f induces a linear isomorphism

$$\tilde{f}_b \colon \mathbb{P}H^0(X_b, mK_{X_b})^* \xrightarrow{\sim} \mathbb{P}H^0(X_{g(b)}, mK_{X_{g(b)}})^*$$

between fibres of $\eta \colon Y \xrightarrow{\sim} B$, which restricts to an isomorphism $X_b \to X_{g(b)}$; under the canonical identification of fibres of η with $\mathbb{P}^N$, $H_Y|_{Y_b} \cong \mathcal{O}_{\mathbb{P}^N}(d)$ (meaning that the section $H_Y|_{Y_b}$ has degree d) for some $d > 0$ independent of the fibre. Under the identification, the action of $\tilde{f}_b$ is linear, so that $\tilde{f}_b^*(H_Y|_{Y_{g(b)}})$ also has degree d on $\mathbb{P}^N$. In particular we have

$$\tilde{f}_b^*(H_Y|_{X_{g(b)}}) = H_Y|_{X_b}.$$

Now take any isomorphism $X_{b_0} \xrightarrow{\sim} F$, which we identify with its graph $\Gamma \subset X_{b_0} \times F$; the image of Γ by the isomorphism $\tilde{f}_{b_0} \times \mathrm{id}_F \colon X_{b_0} \times F \xrightarrow{\sim} X_{g(b_0)} \times F$ is the graph Γ' of an isomorphism $X_{g(b_0)} \xrightarrow{\sim} F$. Furthermore, since $(\tilde{f}_{b_0} \times \mathrm{id}_F)^*(L|_{X_{g(b_0)} \times F}) = L|_{X_{b_0} \times F}$, Γ' has the same Hilbert polynomial as Γ. Iterating this reasoning we find that for some $P \in \mathbb{Q}[\lambda]$ the image of the natural morphism $\psi \colon \mathfrak{I}^P \to B$ is Zariski-dense.

By Chevalley's theorem [Har95, Theorem 3.16], we also know that $\psi(\mathfrak{I}^P)$ is constructible; since every constructible Zariski-dense subset of an irreducible scheme contains a dense open set [Har95, Proof of Theorem 3.16], we have $X_b \cong F$ for all b in an open dense subset of B. This concludes the proof of the first assertion.

(2) As all the fibres over a dense open subset $U \subset B$ are birationally equivalent to a fixed manifold F of general type, by [BBGvB16, Theorem 1.1], after maybe shrinking U, there exists a finite cover $\nu \colon U' \to U$ such that the fibre product $U' \times_U X_U$ is birationally equivalent to $U' \times F$ over U.

Denote by ψ the composition

$$\psi \colon U' \times F \overset{\sim_{bir}}{\dashrightarrow} U' \times_U X_U \to X_U \hookrightarrow X,$$

which is a rational map over U. Let $G = \mathrm{Bir}(F)$; then G is a finite group, and thus for every $y \in F$ the subvariety

$$U' \times G \cdot y \subset U' \times F$$

has dimension $\dim B$.

We claim that the subvarieties

$$W_y = \overline{\psi(U' \times G \cdot y)}^{Zar} \subset X$$

are f-invariant. Indeed, for a general $b \in B$ (such that the fibre X_b is not contained in the indeterminacy locus of f) the transformation f induces a birational map $f_b \colon X_b \dashrightarrow X_{g(b)}$. After the

(non-unique) birational identifications $X_b \sim_{bir} F$ and $X_{g(b)} \sim_{bir} F$, f_b corresponds to an element of $\mathrm{Bir}(F)$, which is only well-defined up to composition with another element of $\mathrm{Bir}(F)$; for a general $x \in X_b$, corresponding to $y \in F \sim X_b$, the image $f(x)$ corresponds to an element of $G \cdot y \subset F \sim X_{g(b)}$.
This shows that the varieties W_y are f-invariant.

Remark that the irreducible components of the W_y passing through general points of X have dimension $\dim B$. Since the very general orbits of g are Zariski-dense, the W_y are finite unions of Zariski-closures of orbits of f; furthermore, Theorem 5.8 implies that the general W_y are smooth and have constant dimension. This proves that the Zariski-closure of a very general orbit of f has dimension $\dim B$, which concludes the proof. □

5.6 Invariant subvarieties

Let X be a compact complex manifold. If $f\colon X \to X$ is an automorphism, we say that a subvariety $W \subset X$ is *invariant* if $f(W) = W$, or, equivalently, if $f^{-1}(W) = W$. We say that $W \subset X$ is *periodic* if it is invariant for some positive iterate f^n of f.
Now let $f\colon X \dashrightarrow X$ be a pseudo-automorphism of X (i.e. a bimeromorphic transformation which is an isomorphism in codimension 1, see §2.4). We say that a hypersurface $W \subset X$ is invariant if the strict transform f^*W of W is equal to W (as a set); since f and f^{-1} don't contract any hypersurface, this is equivalent to $f(W) = W$ (here $f(W)$ denotes the analytic closure of $f|_U(W \cap U)$, where $U \subset X$ is the maximal open set where f is well defined). We say that a hypersurface is periodic if it is invariant for some positive iterate of f.

The following Theorem is a special case of [Can10, Theorem B].

Theorem 5.15. *Let $f\colon X \dashrightarrow X$ be a pseudo-automorphism of a compact complex manifold X. If f admits at least $\dim(X)+b_2(X)-1$ invariant hypersurfaces, then it preserves a non-constant meromorphic function.*

Proof of Theorem A, point (2). Let $f\colon X \dashrightarrow X$ be a loxodromic bimeromorphic transformation of an irreducible holomorphic symplectic manifold X (which is a pseudo-automorphism by Proposition 2.11).
Suppose that f admits more than $\dim(X) + b_2(X) - 2$ periodic hypersurfaces; then some iterate f^N of f satisfies the hypothesis of Theorem 5.15. Therefore f^N preserves a non-constant meromorphic function $\pi\colon X \dashrightarrow \mathbb{P}^1$, and, up to considering the Stein factorization of (a resolution of indeterminacies of) π, we can assume that π is an f^N-equivariant fibration onto a curve. As f^N is loxodromic, this contradicts point (1) of Theorem A. □

The following example shows that we cannot hope to obtain an analogue of point (2) of Theorem A for higher codimensional subvarieties.

Example 5.16. Let $f\colon S \to S$ be a loxodromic automorphism of a $K3$ surface S, and let $X = Hilb^n(S)$. Then X is an irreducible holomorphic symplectic manifold and f induces a loxodromic automorphism f_n of X. By point (2) of Theorem A, f_n admits only a finite number of invariant hypersurfaces. However f admits infinitely many periodic points by Theorem 1.31; if x is a periodic point in S, then (the image in X of) $\{x\}^p \times S^{n-p}$ is a periodic subvariety of codimension $2p$.
If furthermore f admits an invariant curve C (it is the case for the automorphisms of Kummer surfaces induced by automorphisms of tori, which fix the sixteen exceptional (-2)-curves, see §3.1), then (the image in X of) $C \times \{x\}^p \times S^{n-p-1}$ is a periodic subvariety of codimension $2p+1$.

Thus we have showed the following Proposition.

Proposition 5.17. *For all integers $1 < p \leq 2n$, there exist a $2n$-dimensional projective irreducible holomorphic symplectic manifold X and a loxodromic automorphism $f\colon X \to X$ admitting infinitely many periodic subvarieties of codimension p.*

5.7 Appendix: An alternative approach to Theorem A

In this section we describe a different approach to the proof of Theorem A which doesn't require Proposition 5.14. The result we obtain is actually slightly weaker than Theorem A; however this approach allows to prove point (2) and (3), as well as point (1) for automorphisms.

Lemma 5.18. *Let X be a smooth projective variety, $f\colon X \dashrightarrow X$ a birational transformation of X, $\pi\colon X \dashrightarrow B$ a rational f-equivariant fibration onto a smooth projective variety B. If a fibre of π is of general type (if and only if general fibres of π are of general type), then all the relative dynamical degrees $\lambda_p(f|\pi)$ are equal to 1 (for $p = 0, \ldots, \dim(X) - \dim(B)$).*

Proof. Since the Kodaira dimension and the relative dynamical degrees are bimeromorphic invariants (Proposition 1.17), up to considering a resolution of the indeterminacy locus of π, we can suppose that π is regular.
Let

$$\begin{tikzcd} \phi\colon X \arrow[r] \arrow[d, "\pi"] & Y := \mathbb{P}(\pi_* K_{X/B}^{\otimes m}) \arrow[dl, "\eta"] \\ B & \end{tikzcd}$$

be the relative Iitaka fibration. Since general fibres of π are of general type, ϕ is birational onto its image; therefore, denoting $Z \subset Y$ the closure of $\phi(X)$, the claim is equivalent to $\lambda_p(f_Z|\eta_Z) = 1$, where η_Z denotes the restriction of η to Z and $f_Z = \phi \circ f \circ \phi^{-1}\colon Z \dashrightarrow Z$.
The construction of Remark 5.13 provides a birational transformation $\tilde{f}\colon Y \dashrightarrow Y$ extending f_Z.

$$\begin{tikzcd} X \arrow[loop above, dashed, "f"] \arrow[r, dashed, "\phi"] \arrow[d, "\pi"'] & Z \arrow[loop above, dashed, "f_Z"] \arrow[r, hook] \arrow[dl] & Y \arrow[loop above, dashed, "\tilde{f}"] \arrow[dll, "\eta"] \\ B \arrow[loop left, dashed, "g"] & & \end{tikzcd}$$

Now we will prove that if $\lambda_p(\tilde{f}|\eta) = 1$ then $\lambda_p(f_Z|\eta_Z) = 1$. Let $H_Y \in Pic(Y)$ and $H_B \in Pic(B)$ be ample classes; therefore $H_Y|_Z$ is an ample class on Z. Recall that $2n = dim(X)$, and let $\mathcal{K}_{2n} \subset H^{2n,2n}(Y, \mathbb{R})$ be the closure of the cone of classes of positive $(2n, 2n)$-currents; then the map

$$H^{2n,2n}(Y, \mathbb{R}) \to \mathbb{R}$$
$$\alpha \mapsto \int_Y \alpha \wedge c_1(H_Y)^{\dim(Y)-\dim(X)} = \alpha \cdot H_Y^{\dim(Y)-\dim(X)}$$

is linear and strictly positive (except on 0) on the closed positive cone $\mathcal{K}_{2n}$. Since $\alpha \mapsto \alpha \cdot [Z]$ is linear too, we can define

$$M = \max_{\alpha \in \mathcal{K}_{2n} \setminus \{0\}} \frac{\alpha \cdot [Z]}{\alpha \cdot H_Y^{\dim(Y)-\dim(X)}} \geq 0.$$

Now

$$\lambda_p(f_Z|\eta_Z) = \lim_{n\to+\infty}\left((\tilde{f^n})^*H_Y^p\cdot\eta^*H_B^{\dim(B)}\cdot H_Y^{2n-p-\dim(B)}\cdot[Z]\right)^{\frac{1}{n}} \leq$$

$$\lim_{n\to+\infty}\left(M(\tilde{f^n})^*H_Y^p\cdot\eta^*H_B^{\dim(B)}\cdot H_Y^{\dim(Y)-p-\dim(B)}\right)^{\frac{1}{n}} =$$

$$\lim_{n\to+\infty}\left((\tilde{f^n})^*H_Y^p\cdot\eta^*H_B^{\dim(B)}\cdot H_Y^{\dim(Y)-p-\dim(B)}\right)^{\frac{1}{n}} = \lambda_p(\tilde{f}|\eta) = 1,$$

and since all relative dynamical degrees are ≥ 1 we have $\lambda_p(f_Z|\eta_Z) = 1$.

Now all is left to prove is that $\lambda_p(\tilde{f}|\eta) = 1$. There exists $k > 0$ such that $\eta^*H_B^{\dim(B)} \equiv_{num} k[F]$, where $[F]$ is the numerical class of a fibre F of η. We have

$$\lambda_p(\tilde{f}|\eta) = \lim_{n\to+\infty}\left((\tilde{f^n})^*H_Y^p\cdot\eta^*H_B^{\dim(B)}\cdot H_Y^{\dim(Y)-p-\dim(B)}\right)^{\frac{1}{n}} =$$

$$\lim_{n\to+\infty}\left((\tilde{f^n})^*H_Y^p\cdot k[F]\cdot H_Y^{\dim(Y)-p-\dim(B)}\right)^{\frac{1}{n}} =$$

$$\lim_{n\to+\infty}\left(\left((\tilde{f^n})^*H_Y\right)|_F^p\cdot H_Y|_F^{\dim(Y)-p-\dim(B)}\right)^{\frac{1}{n}}.$$

For each fibre we have a canonical identification $F \cong \mathbb{P}^N$, and by this identification $H_Y|_F \cong \mathcal{O}_{\mathbb{P}^N}(d)$, meaning that the hyperplane section $H_Y|_F$ is defined by an equation of degree d. Under the identification, the action of $\tilde{f}$ from one fibre to another is linear, so that $\left((\tilde{f^n})^*H_Y\right)|_F$ is also defined by an equation of degree d on $\mathbb{P}^N$. This means that

$$\lambda_p(\tilde{f}|\eta) = \lim_{n\to+\infty}(d^{\dim(F)})^{\frac{1}{n}} = 1$$

as we wanted to show. This concludes the proof. □

The following Proposition is a weaker version of point (1) of Theorem A.

Proposition 5.19. *Let $f\colon X \dashrightarrow X$ be a loxodromic transformation of an irreducible holomorphic symplectic manifold X of dimension $2n$, and let*

$$1 = \lambda_0(f) < \cdots < \lambda_{p_0}(f) = \cdots = \lambda_{p_0+k}(f) > \cdots > \lambda_{2n}(f) = 1$$

be its dynamical degrees.
If $\pi\colon X \dashrightarrow B$ is an f-equivariant meromorphic fibration, then $\dim(B) \geq 2n-k$. In particular, if f is an automorphism (or, more generally, if all the consecutive dynamical degrees of f are distinct), then it is primitive.

Proof. Let $g\colon B \dashrightarrow B$ be a birational transformation such that $g\circ\pi = \pi\circ f$.

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, dashed, "\pi"] & X \arrow[d, dashed, "\pi"] \\ B \arrow[r, dashed, "g"] & B \end{tikzcd}$$

We know by Lemma 5.11 that a generic fibre of π is of general type; by Lemma 5.18 this implies that all the relative dynamical degrees $\lambda_p(f|\pi)$ are equal to 1. By Theorem 1.16 we then have

$$\lambda_p(f) = \max_{p-\dim(F)\leq q\leq p}\lambda_q(g),$$

where $\dim(F) = \dim(X) - \dim(B)$ is the dimension of a generic fibre.
Let $q \in \{0, 1, \ldots, \dim(B)\}$ be such that $\lambda_q(g)$ is maximal. Then

$$\lambda_q(f) = \lambda_{q+1}(f) = \cdots = \lambda_{q+\dim(F)}(f) = \lambda_q(g),$$

meaning that $k \geq \dim(F) = 2n - \dim(B)$. This concludes the proof. $\square$

Remark 5.20. Since in the Theorem we have $k \leq 2n - 1$, the base of an equivariant fibration cannot be a curve. Therefore Proposition 5.19 implies point (2) of Theorem A.
Furthermore, we have $g \neq \mathrm{id}_B$, otherwise $\lambda_1(f) = 1$ by Theorem 1.16; thus, thanks to Theorem 5.8, Proposition 5.19 implies point (3) of Theorem A.

Chapter 6

Preserved fibrations: action on the base

In this chapter we deal with imprimitive birational transformations (see Definition 5.1). In other words, we consider transformations $f\colon X \dashrightarrow X$ such that there exists a meromorphic fibration $\pi\colon X \dashrightarrow B$ and a birational transformation $g\colon B \dashrightarrow B$ such that $\pi \circ f = g \circ \pi$.
By Theorem 2.15, this is the case when X is an irreducible symplectic manifold of type $K3^{[n]}$ or generalized Kummer and f is a parabolic transformation; in this case, $B = \mathbb{P}^n$ and g is an automorphism.

As we have seen, the study of the dynamics of an imprimitive transformation reduces to the study of the dynamics on the base and of the action on the fibres. The following result describes the action on the base in a slightly more general setting than Theorem 2.15.

Theorem B. *Let X be a projective manifold with trivial or effective canonical bundle and let $f\colon X \dashrightarrow X$ be a birational transformation. Suppose that there exist a meromorphic fibration $\pi\colon X \dashrightarrow \mathbb{P}^n$ and an element $g \in \mathrm{Aut}(\mathbb{P}^n) = PGL_{n+1}(\mathbb{C})$ such that $g \circ \pi = \pi \circ f$. Then g has finite order.*

In the context of Theorem 2.15, we get a characterization of transformations with Zariski-dense orbits:

Corollary 6.1. *Let f be a birational transformation of an irreducible symplectic manifold X of type $K3^{[n]}$ or generalized Kummer; then f admits a Zariski-dense orbit if and only if f is loxodromic (i.e. the first dynamical degree $\lambda_1(f)$ is > 1).*

Proof. By Theorem A, if f is loxodromic then the very general orbit of f is Zariski-dense.

Now, suppose that f is not loxodromic. If f is elliptic, then it has finite order by Proposition 2.14, and in particular its orbits are not Zariski-dense.
If f is parabolic, then by Theorem 2.15 f preserves a fibration π onto $\mathbb{P}^n$, and the induced action on the base is biregular; by Theorem B, the action on the base has finite order, thus any orbit is contained in a finite number of fibres of π, and in particular is not Zariski-dense. $\square$

Remark 6.2. As it was also remarked by one of the referees, the assumption of Theorem B on K_X being trivial or effective can be weakened to the Kodaira dimension $\kappa(X)$ of X being non-negative. Indeed, if this is the case, some positive multiple mK_X of the canonical bundle admits a non-zero section Ω; to such a section one can associate a volume form ω by taking the m-th root of a well-chosen multiple of $\Omega \wedge \bar{\Omega}$. Analogously, if X is defined over a p-adic field, then we can define a measure $\sqrt[m]{|\Omega|}$. The proof goes exactly as the one of Theorem B.

Furthermore, one can weaken the assumptions on the base of the preserved fibration: it is enough to suppose that f preserves a fibration $\pi\colon X \dashrightarrow B$ and that the action $g\colon B \dashrightarrow B$ on the base is a pseudo-automorphism which preserves a big line bundle L. Indeed, after applying the

Iitaka map $\Phi\colon B \hookrightarrow \mathbb{P}^N$ with respect to L, which is birational onto its image, g is identified with the restriction to $\Phi(B)$ of some linear automorphism $h \in \mathrm{PGL}_{N+1}(\mathbb{C})$, and the same measure-theoretic argument as in the proof of Theorem B applies.

Hence, we have following:

Theorem 6.3. *Let X be a projective manifold and let $f\colon X \dashrightarrow X$ be a birational transformation. Suppose that there exist a meromorphic fibration $\pi\colon X \dashrightarrow B$ onto a projective manifold B and a pseudo-automorphism $f_B\colon B \dashrightarrow B$ such that $f_B \circ \pi = \pi \circ f$. Assume that*

1. *the Kodaira dimension $\kappa(X)$ is non-negative;*

2. *f_B preserves a big line bundle L.*

Then f_B has finite order.

The complete proof of this result will appear in a later article.

6.1 Elements of p-adic integration

In this section we give an introduction to p-adic integration; see [CLNS14], [Pop11, Chapter 3] and [Igu00].

6.1.1 p-adic and local fields

We remind that, for a prime number p, the p-adic norm on $\mathbb{Q}$ is defined as

$$\left|p^n \cdot \frac{a}{b}\right| = p^{-n} \qquad p \nmid a, \quad p \nmid b.$$

We denote $\mathbb{Q}_p$ the metric completion of $(\mathbb{Q}, |\cdot|_p)$; every element of $\mathbb{Q}_p$ can be uniquely written as a Laurent series

$$a = \sum_{n=n_0}^{+\infty} a_n p^n \qquad a_i \in \{0, 1, \ldots, p-1\}.$$

Denote by $\mathbb{Z}_p$ the closed unit ball in $\mathbb{Q}_p$; it is an integrally closed local subring of $\mathbb{Q}_p$ with maximal ideal $p\mathbb{Z}_p$ and residue field $\mathbb{F}_p$; its field of fractions is $\mathbb{Q}_p$, and it is a compact, closed and open subset of $\mathbb{Q}_p$.

A *p-adic field* is a finite extension K of $\mathbb{Q}_p$ for some prime p; on K there exists a unique absolute value $|\cdot|_K$ extending $|\cdot|_p$. We denote by $\mathcal{O}_K$ the integral closure of $\mathbb{Z}_p$ in K.

A *local field* is a field K with a absolute value $|\cdot|\colon K \to \mathbb{R}_{\geq 0}$ such that K with the induced topology is locally compact; we say that two local fields $(K_1, |\cdot|_1), (K_2, |\cdot|_2)$ are equivalent if there exists a field isomorphism $\phi\colon K_1 \to K_2$ such that $|\phi(x)|_2 = |x|_1^\alpha$ for some $\alpha \in \mathbb{R}_{>0}$.

Theorem 6.4. *Up to equivalence, local fields of characteristic* 0 *are* $\mathbb{R}$ *and* $\mathbb{C}$ *endowed with the usual absolute values (archimedean case) and p-adic fields (non-archimedean case).*

In what follows, K denotes a p-adic field.

Remark 6.5. $\mathcal{O}_K$ is a local ring, with maximal ideal $\mathfrak{m}_K$; the residue field is

$$\mathcal{O}_K/\mathfrak{m}_K \cong \mathbb{F}_q \qquad q = p^r \text{ for some } r > 0.$$

In fact $r = [K : \mathbb{Q}_p]/v_K(p)$, where v_K denotes the valuation on $\mathbb{Q}_p$ such that $\mathcal{O}_K = \{v_K(x) \leq 1\}$ is equal to the closed unit ball in K.

6.1.2 Measure on K

On a locally compact topological group G there exists a positive measure μ, unique up to scalar multiplication, called the *Haar measure of* G such that:

- any continuous function $f\colon G \to \mathbb{C}$ with compact support is μ-integrable (i.e. μ is locally finite);
- μ is G-invariant to the left.

Other important properties of the Haar measure are as follows: every Borel subset of G is measurable; $\mu(A) > 0$ for every nonempty open subset of G.

We consider $G = (\mathbb{Q}_p, +)$, and take on it the Haar measure μ normalized so that

$$\mu(\mathbb{Z}_p) = 1.$$

Example 6.6. For $m \geq 0$, $\mu(p^m\mathbb{Z}_p) = p^{-m}$.
Indeed, we have

$$\mathbb{Z}_p = \coprod_{i=0}^{p^m-1} (p^m\mathbb{Z}_p + i)$$

and by translation invariance $\mu(p^m\mathbb{Z}_p + i) = \mu(p^m\mathbb{Z}_p)$. Therefore

$$1 = \mu(\mathbb{Z}_p) = p^m\mu(p^m\mathbb{Z}_p),$$

which proves the claim.

Example 6.7. Since $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus p\mathbb{Z}_p$, $\mu(\mathbb{Z}_p^*) = 1 - \frac{1}{p}$.

A useful observation for calculating integrals is that, as in every measured space, if $f\colon \mathbb{Z}_p \to \mathbb{C}$ has its image contained in a countable set $\{a_i\}_{i\in\mathbb{N}}$, then

$$\int_{\mathbb{Z}_p} f(x)d\mu(x) = \sum_{i\in\mathbb{N}} a_i\mu\{f(x) = a_i\}.$$

Example 6.8. Let $s \in \mathbb{R}_{\geq 0}, d \in \mathbb{N}$; then

$$I = \int_{\mathbb{Z}_p} |x^d|^s d\mu(x) = \frac{p-1}{p-p^{-ds}}.$$

Indeed,

$$|x^d|^s = \frac{1}{p^{kds}} \qquad \text{for } x \in p^k\mathbb{Z}_p \setminus p^{k+1}\mathbb{Z}_p,$$

so that

$$I = \sum_{k=0}^{+\infty} \frac{1}{p^{kds}}\mu(p^k\mathbb{Z}_p \setminus p^{k+1}\mathbb{Z}_p) = \sum_{k=0}^{+\infty} \frac{1}{p^{kds}}\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right) =$$

$$\frac{p-1}{p}\sum_{k=0}^{+\infty} \frac{1}{p^{k(ds+1)}} = \frac{p-1}{p-p^{-ds}}$$

More generally, on a p-adic field K we consider the Haar measure μ such that

$$\mu(\mathcal{O}_K) = 1.$$

Remark 6.9. As in the examples above, one can show that

- $\mu(\mathfrak{m}_K^k) = q^{-k}$;
- $\mu(\mathcal{O}_K^*) = 1 - \frac{1}{q}$;
- $$\int_{\mathcal{O}_K} |x^d|^s d\mu = \frac{q-1}{q-q^{-ds}}.$$

6.1.3 Integration on K-analytic manifolds

K-analytic manifolds

Let K be a p-adic field with norm $|\cdot|$. For any open subset $U \subset K^n$, a function $f\colon U \to K$ is said to be K-analytic if locally around each point it is given by a convergent power series. Similarly, we call $f = (f_1, \dots, f_m)\colon U \to K^m$ a K-analytic map if all the f_i are analytic.

As in the real and complex context, we define a K-analytic manifold of dimension n as a Hausdorff topological space locally modelled on open subsets of K^n and with K-analytic change of charts.

Example 6.10.

1. Every open subset $U \subset K^n$ is a K-analytic manifold of dimension n; in particular, the set $\mathcal{O}_K^n \subset K^n$ is a K-analytic manifold.
2. The projective space $\mathbb{P}_K^n$ over K is a K-analytic manifold.
3. The set of K-points of every smooth algebraic variety over K is a K-analytic manifold; in order to see this one needs a K-analytic version of the implicit function theorem (see [CLNS14, §1.6.4]).
4. The blow-up of a K-analytic manifold along a submanifold of codimension ≥ 2 is a K-analytic manifold.
   Let us describe explicitly the blow-up $\pi\colon X \to K^2$ of K^2 at the origin. As in the complex case, one can cover X with two charts isomorphic to K^2, in which π is written as
   $$\pi\colon (x,z) \mapsto (x, xz) \in K^2, \qquad \pi\colon (w,y) \mapsto (yw, y) \in K^2,$$
   and with change of coordinates $w = 1/z$, $y = xz$.
   Since K is totally discontinuous, we can cover $Y := \pi^{-1}(\mathcal{O}_K^2)$ with two disjoint open charts
   $$U = \{(x,z) \mid |x| \leq 1, |z| \leq 1\} \qquad V = \{(w,y) \mid |w| < 1, |y| \leq 1\},$$
   by remarking that
   $$\pi(U) = \{(s,t) \in K^2 \mid |t| \leq |s| \leq 1\}, \quad \pi(V) = \{(s,t) \in K^2 \mid |s| < |t| \leq 1\}.$$

Compact K-analytic manifolds are completely classified.

Proposition 6.11 (Serre 1965). *Let X be a K-analytic compact manifold, and let q be the cardinality of the residue field $\mathcal{O}_K/\mathfrak{m}_K$. Then there exists a unique $m \in \{1, \dots, q-1\}$ such that X is K-diffeomorphic to a disjoint union of m unit polydisks.*

Differential forms and integration

Differential forms are defined in the usual way via charts: on a chart with coordinates $x_1, \dots, x_n$, a differential form of degree k can be written as

$$\alpha = \sum_{|I|=k} f_I(x_1, \dots, x_n) dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

with $f_I\colon U \to K$ functions on U; if the f_I are K-analytic we say that the form is analytic.
Now take a maximal degree analytic differential form ω; let $\phi\colon U \to K^n$ be a local chart, defining local coordinates $x_1, \dots, x_n$. In these coordinates we can write

$$\phi_*\omega = f(x_1, \dots, x_n) dx_1 \wedge \dots \wedge dx_n.$$

Then one can define a Borel measure $|\omega|$ on U as follows: for any open subset $A \subset U$, we set

$$|\omega|(A) = \int_{\phi(A)} |f(x)|_K \, d\mu,$$

where μ is the usual normalized Haar measure on $\phi(U) \subset K^n$.
To define a Borel measure $|\omega|$ on the whole manifold X, one uses partitions of unity exactly as in the real case. The only thing to check is that $|\omega|$ transforms precisely like differential forms when changing coordinates, which is a consequence of the following K-analytic version of the change of variables formula.

Theorem 6.12 (Change of variables formula)**.** *Let U be an open subset of K^n and let $\phi\colon U \to K^n$ be an injective K-analytic map whose Jacobian matrix J_ϕ is invertible on U. Then for every measurable positive (resp. integrable) function $f\colon \phi(U) \to \mathbb{R}$*

$$\int_{\phi(U)} f(y) d\mu(y) = \int_U f(\phi(x)) \, |\det J_\phi(x)|_K \, d\mu(x).$$

Example 6.13. Integration of differential forms on K-analytic manifolds sometimes allows to simplify the computation of integrals on open subsets of K^n. For example, consider

$$I = \int_{\mathcal{O}_K^2} |f(x,y)|^s |dx \wedge dy|, \quad \text{where } f(x,y) = x^a y^b (x-y)^c.$$

Let $\pi\colon X \to \mathcal{O}_K^2$ be the blow-up of $\mathcal{O}_K^2$ at the origin; then

$$I = \int_X |\pi^* f|^s |\pi^*(dx \wedge dy)|,$$

and we can decompose the integral over X into the integral over U plus the integral over V, as defined in Example 6.10. It turns out that, by using the K-analytic version of Fubini's theorem, the integrals over U and V can be computed separately in the two variables, thus allowing to obtain the value of I; see [Pop11, Pages 12-13].

Relationship with rational points over finite fields

The theory of p-adic integration is linked to the computation of the number of rational points of a scheme over a finite field; the content of this paragraph will not be needed in the following sections.

As a first result, we can state the following lemma (see [Pop11, Lemma 4.6]).

Lemma 6.14. *Let K be a p-adic field with residue field $\mathcal{O}_K/\mathfrak{m}_K \cong \mathbb{F}_q$ and let $f \in \mathcal{O}_K[X_1, \ldots, X_n]$. Then, for any integer $m \geq 0$,*

$$|\{x \in (\mathcal{O}_K/\mathfrak{m}_K^m)^n \mid f(x) \equiv 0 \, (mod\, \mathfrak{m}_K^m)\}| = q^{mn} \cdot \mu \left\{ x \in \mathcal{O}_K^n \mid |f(x)| \leq \frac{1}{q^m} \right\}$$

A global analogue of Lemma 6.14 is the following result.

Theorem 6.15 (Weil [Wei82])**.** *Let X be a smooth scheme of dimension n defined over $S = \operatorname{Spec} \mathcal{O}_K$, and let $\omega \in \Gamma(X, \Omega^n_{X/S})$ be a nowhere vanishing n-form on X defined over S. Then $X(\mathcal{O}_K)$ is a K-analytic manifold and*

$$\int_{X(\mathcal{O}_K)} |\omega| = \frac{|X(\mathbb{F}_q)|}{q^n}$$

6.2 Proof of Theorem B

In this section we give the proof of Theorem B. The strategy of the proof goes as follows:

1. Find an f-invariant volume form ω on X (for X irreducible symplectic, $\omega = (\sigma \wedge \bar{\sigma})^n$, where $2n = \dim X$ and σ is a symplectic form).

2. The push-forward of ω by π defines a g-invariant measure vol on $\mathbb{P}^n$ not charging positive codimensional subvarieties; using this it is not hard to put $g \in PGL_{n+1}(\mathbb{C})$ in diagonal form with only complex numbers of modulus 1 on the diagonal.

3. Define the field of coefficients k: roughly speaking, a finitely generated (but not necessarily finite) extension of $\mathbb{Q}$ over which X, f, the volume form and all the relevant maps are defined.

4. Apply a key lemma: if one of the coefficients α of g is not a root of unity, there exists an embedding $k \hookrightarrow K$ into a local field K such that $|\rho(\alpha)| \neq 1$. Then the same integration argument as in point (2) leads to a contradiction.
   A similar idea appears in the proof of Tits alternative for linear groups, see [Tit72].

6.2.1 Invariant volume on X

Remark that, given a holomorphic n-form Ω (n being the dimension of X), the pull-back $f^*\Omega$ is defined outside the indeterminacy locus of f; the latter being of codimension ≥ 2, by Hartogs principle we can extend $f^*\Omega$ to an n-form on the whole X. This action determines a linear automorphism

$$f^* \colon H^0(X, K_X) \to H^0(X, K_X).$$

The complex vector space $H^0(X, K_X)$ has finite dimension, thus there exists an eigenvector $\Omega \in H^0(X, K_X) \setminus \{0\}$:

$$f^*\Omega = \xi\Omega. \tag{6.1}$$

Furthermore, since the integration measure $\Omega \wedge \overline{\Omega}$ doesn't charge positive codimensional analytic subvarieties and since f is birational, f preserves the (finite) total measure:

$$\int_X f^*(\Omega \wedge \overline{\Omega}) = \int_X \Omega \wedge \overline{\Omega};$$

thus $|\xi| = 1$, and in particular the volume form $\omega = \Omega \wedge \overline{\Omega}$ is f-invariant.
The push-forward by π induces a measure vol on $\mathbb{P}^n$:

$$\mathrm{vol}(A) := \int_{\pi^{-1}(A)} \omega,$$

where $\pi^{-1}(A)$ denotes the inverse image of A by the restriction of π to the maximal open subset $U \subset X$ where π is well-defined; the measure vol is g-invariant.

6.2.2 A first reduction of g

In a given system of homogeneous coordinates on $\mathbb{P}^n$, an automorphism $g \in \mathrm{Aut}(\mathbb{P}^n) = PGL_{n+1}(\mathbb{C})$ is represented by a matrix M acting linearly on such coordinates; M is well-defined up to scalar multiplication. We will say that g is semi-simple if M is; in this case there exist homogeneous coordinates $Y_0, \ldots, Y_n$ such that the action of g on these coordinates can be written

$$g([Y_0 : \ldots : Y_n]) = \begin{bmatrix} 1 & & & \\ & \alpha_1 & & \\ & & \ddots & \\ & & & \alpha_n \end{bmatrix} \underline{Y} = [Y_0 : \alpha_1 Y_1 : \ldots : \alpha_n Y_n].$$

By an abuse of terminology, we will call the α_i the eigenvalues of g; they are not well-defined, but the property that they are all of modulus 1 is.

Lemma 6.16. *Let g be an automorphism of $\mathbb{P}^n$ which preserves a finite measure* vol *which doesn't charge positive-codimensional subvarieties; then g is semi-simple and its eigenvalues have all modulus* 1.

Proof. Let us prove first that g is semi-simple. If this were not the case, the Jordan form of g (which is well-defined up to scalar multiplication) would have a non-trivial Jordan block, say of dimension $k \geq 2$. It turns out that the computations are clearer if we consider the lower triangular Jordan form. In some good homogeneous coordinates $Y_0, \ldots Y_n$ of $\mathbb{P}^n$, after rescaling the coefficients of g we can write

$$g(\underline{Y}) = \begin{bmatrix} 1 & & & \mathbf{0} & & & \\ 1 & 1 & & & & & \\ & \ddots & \ddots & & & \mathbf{0} & \\ \mathbf{0} & & 1 & 1 & & & \\ & & & & \alpha_k & & \mathbf{0} \\ & \mathbf{0} & & & & \ddots & \\ & & & & \bigstar & & \alpha_n \end{bmatrix} \underline{Y}.$$

Take the affine chart $\{Y_0 \neq 0\} \cong \mathbb{C}^n$ with the induced affine coordinates $y_i = Y_i / Y_0$. In these coordinates we can write

$$g(y_1, \ldots, y_n) = (y_1 + 1, y_2 + y_1, \ldots)$$

and thus

$$g^N(y_1, \ldots, y_n) = (y_1 + N, \ldots).$$

Let $A = \{(y_1, \ldots, y_n) \in \mathbb{C}^n \,|\, 0 \leq \mathrm{Re}(y_1) < 1\}$; then we have

$$\mathbb{C}^n = \coprod_{N \in \mathbb{Z}} g^N(A)$$

and

$$\mathrm{vol}(\mathbb{P}^n) = \mathrm{vol}(\mathbb{C}^n) = \sum_{N \in \mathbb{Z}} \mathrm{vol}(g^N(A)) = \sum_{N \in \mathbb{Z}} \mathrm{vol}(A) = 0 \text{ or } +\infty,$$

which is a contradiction with the finiteness of vol.
This shows that g is diagonalizable.

Next we show that, up to rescaling, in good homogeneous coordinates one can write

$$g(\underline{Y}) = \begin{bmatrix} 1 & & & \\ & \alpha_1 & & \\ & & \ddots & \\ & & & \alpha_n \end{bmatrix} \underline{Y} = [Y_0 : \alpha_1 Y_1 : \ldots : \alpha_n Y_n]$$

with $|\alpha_i| = 1$. Suppose by contradiction that $|\alpha_1| \neq 1$ (for example $|\alpha_1| > 1$), and define

$$A = \{(y_1, \ldots, y_n) \,|\, 1 \leq |y_1| < |\alpha_1|\} \subset \{Y_0 \neq 0\} \cong \mathbb{C}^n.$$

The same argument as above leads to a contradiction.

□

6.2.3 The field of coefficients

A key idea of the proof will be to define the "smallest" extension k of $\mathbb{Q}$ over which X and all the relevant applications are defined, and to embed k in a local field in such a way as to obtain a contradiction.

Let us fix a cover of X by affine charts $U_1, \ldots, U_m$ trivializing the canonical bundle. Each of these U_i is isomorphic to the zero locus of some polynomials $p_{i,1}, \ldots, p_{i,n_i}$ in an affine space $\mathbb{C}^{N_i}$; fix some rational functions $g_{i,j} \colon \mathbb{C}^{N_i} \dashrightarrow \mathbb{C}^{N_j}$ giving the changes of coordinates from U_i to U_j. Denote $\phi_{i,j} \colon U_i \cap U_j \to \mathbb{C}^*$ the change of charts for the canonical bundle; such functions are algebraic, therefore they are given by some rational functions $h_{i,j}$ on $\mathbb{C}^{N_i}$ (or, equivalently, $\mathbb{C}^{N_j}$). Let $f \colon \mathbb{C}^{N_i} \dashrightarrow \mathbb{C}^{N_j}$ (resp. $\Omega_i \colon \mathbb{C}^{N_i} \dashrightarrow \mathbb{C}$,) be some rational functions defining f (resp. Ω). Finally, fix homogeneous coordinates on $\mathbb{P}^n$ diagonalizing g (see §6.2.2), and some rational maps $\pi_i \colon \mathbb{C}^{N_i} \dashrightarrow \mathbb{C}^{n+1}$ defining π_{U_i} upon passing to quotient.
We define the *field of coefficients* $k = k_\Omega$ as the extension of $\mathbb{Q}$ generated by all the coefficients appearing in the $p_{i,k}, f_{i,j}, g_{i,j}, h_{i,j}, \Omega_i, \pi_i$ and by $\alpha_1, \ldots, \alpha_n$; this is a finitely generated (but not necessarily finite) extension of $\mathbb{Q}$ over which X is defined.

Change of base field

Let $\rho \colon k \hookrightarrow K$ be an embedding of k into a local field K; since $\mathbb{R}$ is naturally embedded in $\mathbb{C}$, we may and will assume that K is either $\mathbb{C}$ or a p-adic field. We can now apply a base change in the sense of algebraic geometry to recover a smooth projective scheme over K and all the relevant functions.

Here are the details of the construction: the polynomials $p^\rho_{i,k} = \rho(p_{i,k})$ define affine varieties X^ρ_i of K^{N_i}; the rational functions $g^\rho_{i,j}$ allow to glue the X^ρ_i-s into an algebraic variety X^ρ over K. This variety is actually smooth since smoothness is a local condition which is algebraic in the coefficients of the $p_{i,k}$. Furthermore, by applying ρ to all the relevant rational functions, we can recover a birational transformation $f^\rho \colon X^\rho \dashrightarrow X^\rho$ and a canonical section $\Omega^\rho \in H^0(X^\rho, \mathcal{K}_{X^\rho})$. Remark that we can suppose that X^ρ is projective: indeed, $X \subset \mathbb{P}^N(\mathbb{C})$ is the zero locus of some homogeneous polynomials $P_1, \ldots, P_k \in \mathbb{C}[Y_0, \ldots, Y_N]$, and, up to adding the affine open subsets $X_i = X \cap \{Y_i \neq 0\}$ to the above constructions, it is easy to see that $X^\rho \subset \mathbb{P}^N(K)$ is the zero locus of $P^\rho_1, \ldots, P^\rho_k$. Furthermore, applying ρ to the equations of π defines a meromorphic fibration $\pi^\rho \colon X^\rho \dashrightarrow \mathbb{P}^n_K$, and, denoting $g^\rho \colon \mathbb{P}^n_K \to \mathbb{P}^n_K$ the automorphism given by $g^\rho[Y_0 : \ldots : Y_n] = [Y_0 : \alpha^\rho_1 Y_1 : \ldots : \alpha^\rho_n Y_n]$, we have $\pi^\rho \circ f^\rho = g^\rho \circ \pi^\rho$:

$$\begin{tikzcd} X^\rho \arrow[r, dashed, "f^\rho"] \arrow[d, dashed, "\pi^\rho"] & X^\rho \arrow[d, dashed, "\pi^\rho"] \\ \mathbb{P}^n_K \arrow[r, "g^\rho"] & \mathbb{P}^n_K \end{tikzcd}$$

We will denote by $|\Omega^\rho|$ the measure on X associated to Ω^ρ: this has been defined in Section 6.1.3 in the non-archimedean case, while if $K = \mathbb{C}$ it is defined as the measure of integration of $\omega^\rho = \Omega^\rho \wedge \overline{\Omega^\rho}$. In both cases, $|\Omega^\rho|$ doesn't charge positive codimensional analytic subvarieties.

Remark 6.17. At this stage we can already prove that the α_i are algebraic numbers all of whose conjugates over $\mathbb{Q}$ have modulus 1. Indeed suppose that this is not the case, say for α_1; by a standard argument in Galois theory (see for example [Lan02]), one can find an embedding $\rho\colon k \hookrightarrow \mathbb{C}$ such that $|\rho(\alpha_1)| \neq 1$. Now, g^ρ preserves the measure vol^ρ induced on $\mathbb{P}^n$ by ω^ρ

$$\mathrm{vol}^\rho(A) := \int_{(\pi^\rho)^{-1}(A)} \omega^\rho,$$

and Lemma 6.16 leads to a contradiction.

If we somehow knew that the α_i are algebraic integers, we could conclude by a lemma of Kronecker (see [Kro57]) that they are roots of unity. However, this is in general not true for algebraic numbers: for example,

$$\alpha = \frac{3+4i}{5}$$

has only $\bar{\alpha}$ as a conjugate over $\mathbb{Q}$, and they both have modulus 1, but they are not roots of unity. In order to exclude this case we will have to use the p-adic argument.

6.2.4 Key lemma and conclusion

In his original proof of the Tits alternative for linear groups [Tit72], Tits proved and used (much like we do in this context) the following simple but crucial lemma:

Lemma 6.18 (Key lemma)**.** *Let k be a finitely generated extension of $\mathbb{Q}$ and let $\alpha \in k$ be an element which is not a root of unity. Then there exist a local field K (with norm $|\cdot|$) and an embedding $\rho\colon k \hookrightarrow K$ such that $|\rho(\alpha)| > 1$.*

Example 6.19. Take

$$\alpha = \frac{3+4i}{5},$$

and define $k = \mathbb{Q}(\alpha)$. Then there exists $\rho\colon k \hookrightarrow \mathbb{Q}_5$ such that $|\rho(\alpha)| = 5$.
This can be seen directly by picking a root $\beta \in \mathbb{Z}_5$ of X^2+1 with $\overline{\beta} = 2 \in \mathbb{Z}/5\mathbb{Z}$ (such a β exists by Hensel's lemma), and then sending α to $(3+4\beta)/5 \in \mathbb{Q}_p$. Since $\overline{3+4\beta} = 1 \in \mathbb{Z}/5\mathbb{Z}$, one finds

$$\left|\frac{3+4\beta}{5}\right| = \left|\frac{1}{5}\right| = 5.$$

We can now show that the factor ξ appearing in 6.1 is actually a root of unity; this follows from the classical result that f induces a linear map on cohomology preserving the integral structure (see for example [NU73]) and from the fact that all the conjugates of ξ over $\mathbb{Q}$ also have modulus 1 (the method for the proof being the same as the one explained in Remark 6.17), but to the best of my knowledge the present proof using Lemma 6.18 is original.

Lemma 6.20. *Let $f\colon X \dashrightarrow X$ be a birational transformation of a projective manifold X and let $\Omega \in H^0(X, K_X)$ be a canonical form such that*

$$f^*\Omega = \xi\Omega \qquad \textit{for some } \xi \in \mathbb{C}^*.$$

Then some iterate f^N of f preserves Ω.

Proof. We need to show that ξ is a root of unity. Suppose by contradiction that this is not the case, and define the field of coefficients $k = k_\Omega$; since $f^*\Omega = \xi\Omega$, $\xi \in k$. Applying Lemma 6.18, one finds an embedding $\rho\colon k \hookrightarrow K$ into a local field K such that $|\rho(\xi)| \neq 1$.

The measure $|\Omega^\rho|$ on X^ρ doesn't charge positive codimensional analytic subvarieties and $f^\rho\colon X^\rho \dashrightarrow X^\rho$ is a birational map, therefore the (finite) total measure is preserved by f^ρ:

$$|(f^\rho)^*\Omega^\rho|(X^\rho) = |\Omega^\rho|(X^\rho).$$

On the other hand $|(f^\rho)^*\Omega^\rho| = |\xi^\rho\Omega^\rho| = |\xi^\rho| \cdot |\Omega^\rho|$ and $|\xi^\rho| \neq 1$, a contradiction. $\square$

Proof of Theorem B. Suppose by contradiction that one of the eigenvalues, say α_1, is not a root of unity.

We replace f by an iterate f^N preserving Ω, and define the field of coefficients k. Now, since $f^*\Omega = \Omega$, we have $(f^\rho)^*(\Omega^\rho) = \Omega^\rho$, and in particular g^ρ preserves the measure vol^ρ on $\mathbb{P}^n_K$ induced by the push-forward of $|\Omega|$:

$$\mathrm{vol}^\rho(A) := |\Omega^\rho| \left((\pi^\rho)^{-1}(A)\right).$$

The measure vol^ρ is non-trivial, finite, and doesn't charge positive codimensional analytic subvarieties of $\mathbb{P}^n_K$, thus we can conclude just as in the proof of Lemma 6.16.

Denote $A := \{[1 : Y_1 : \ldots : Y_n] \in K^n \,|\, 1 \leq |Y_1| < |\alpha_1|\} \subset \{Y_0 \neq 0\} \cong K^n$ if $|\rho(\alpha_1)| > 1$ (respectively $A := \{[1 : Y_1 : \ldots : Y_n] \in K^n \,|\, |\alpha_1^\rho| \leq |Y_1| < 1\} \subset \{Y_0 \neq 0\} \cong K^n$ if $|\rho(\alpha_1)| < 1$); then, since vol^ρ doesn't charge positive codimension analytic subsets, we have

$$|\omega^\rho|(X^\rho) = \mathrm{vol}^\rho(\mathbb{P}^n_K) = \mathrm{vol}^\rho(K^n) =$$

$$\sum_{N\in\mathbb{Z}} \mathrm{vol}^\rho((g^\rho)^N(A)) = \sum_{N\in\mathbb{Z}} \mathrm{vol}^\rho(A) = 0 \text{ or } +\infty,$$

a contradiction. $\square$

6.3 On automorphisms of Calabi-Yau threefolds preserving a fibration

In this section we treat the case of automorphisms of Calabi-Yau threefolds preserving a holomorphic fibration.

Here a compact Kähler manifold X is called a Calabi-Yau manifold if X is simply connected, $K_X = \mathcal{O}_X$ and $H^0(X, \Omega^p_X) = 0$ for $0 < p < \dim X$. Remark that, since $H^2(X, \mathbb{C})$ coincides with $H^{1,1}(X)$, X is automatically projective by Kodaira's criterion.

As usual, a (non-trivial) holomorphic (resp. meromorphic) fibration on X is a surjective morphism $X \to B$ (resp. dominant meromorphic map $X \dashrightarrow B$) with connected fibres and such that $0 < \dim B < \dim X$.

Proposition 6.21 (Oguiso [Ogu93]). *Non-trivial holomorphic fibrations $\pi\colon X \to B$ on a Calabi-Yau threefold X are of the following types:*

1. *$B = \mathbb{P}^1$, a general fibre is a $K3$ surface;*
2. *$B = \mathbb{P}^1$, a general fibre is an abelian surface;*
3. *B is a rational surface, a general fibre is an elliptic curve.*

In analogy with the case of surfaces (see Theorem 1.30), we may wonder if every zero entropy automorphism of a Calabi-Yau threefold whose action in cohomology has infinite order (i.e. f^* is virtually unipotent of infinite order) preserves an elliptic fibration.

In Theorem 6.22 we show that such a conjecture is too optimistic, and that, at least in the case of

birational transformations, we must at least allow fibrations in abelian surfaces to hope to obtain a positive answer.
However, in Theorem 6.25 we show that in some cases such an equivariant elliptic fibration can be produced from other existing equivariant fibrations.

The question remains: does every automorphism of a Calabi-Yau threefold with virtually unipotent action in cohomology preserves a fibration whose general fibres are either elliptic curves or abelian surfaces? This question is linked to the abundance conjecture for Calabi Yau threefolds, see [LOP16]; see also Conjecture 2.1 for the hyperkähler version.

Conjecture 6.1 (Abundance conjecture for Calabi-Yau threefolds)**.** *Let L be a nef line bundle on a Calabi-Yau threefold X; then $\kappa(X, L) \geq 0$ (i.e. some multiple of L admits non-trivial sections).*

If the conjecture is true, then by [Miy87, Miy88] one also obtains that L is semi-ample (i.e. some multiple of L is base-point free, and thus defines a fibration $X \to B$).

6.3.1 Automorphisms not admitting equivariant elliptic fibrations

Theorem 6.22. *Let X be a Calabi-Yau threefold, and let $f\colon X \dashrightarrow X$ be a birational transformation of infinite order with $\lambda_1(f) = 1$. Suppose that there exists an f-equivariant fibration $\pi\colon X \to \mathbb{P}^1$ whose general fibres are* simple *abelian surfaces. Then f doesn't preserve any other meromorphic fibration; in particular f doesn't preserve an elliptic fibration.*

Lemma 6.23. *Let A be an abelian surface.*

1. *If $\pi\colon A \to C$ is a holomorphic fibration onto a curve, then C is an elliptic curve and A is isogenous to $C \times F$, where F is any fibre of π.*

2. *If $\pi\colon A \dashrightarrow C$ is a meromorphic fibration onto a curve which is equivariant with respect to a translation $\tau\colon A \to A$ with infinite order, then π is holomorphic; therefore, C is an elliptic curve and A is isogenous to $C \times F$ where F is any fibre of π.*

Proof. (1) First let $\pi\colon A \to C$ be a holomorphic fibration onto a curve; for $a \in A$, denote by F_a the fibre of π passing through a. We will show first that

$$F_{a+b} = F_a + b. \tag{6.1}$$

Since $a + b \in F_{a+b} \cap (F_a + b)$, if F_{a+b} and $F_a + b$ had no common components we would have

$$0 < F_{a+b} \cdot (F_a + b) = F_{a+b} \cdot F_b = f^2 = 0,$$

where f denotes the numerical class of a general fibre of π, a contradiction. This shows that for general $a, b \in A$ (so that the fibres F_a, F_b are irreducible), 6.1 holds. Now let

$$F = D_1 \cup \ldots \cup D_k$$

be the decomposition of a fibre F into its irreducible components and for $i = 1, \ldots, k$ let $a_i \in D_i \setminus \bigcup_{j\neq i} D_j$. Then the above computation show that, for a general a, $F_a + (a_i - a) = D_i$; in other words, the irreducible components of F are all translations of F_a, hence they are all disjoint. This show that $k = 1$ and thus 6.1 holds for any $a, b \in A$.
Therefore, the abelian group law on A induces an abelian group law on C which makes π into a morphism of abelian varieties. In particular C is an elliptic curve, A is isogenous to $C \times \ker(\pi) = C \times F_0$, and π identifies with the projection onto the first factor. This shows the first claim.

(2) Now let $\tau\colon A \to A$ be a translation with infinite order, and suppose that τ admits an equivariant meromorphic fibration $\pi\colon A \dashrightarrow C$ onto a curve. The indeterminacy locus $\mathcal{I} = \mathcal{I}(\pi)$

is a τ-invariant finite set (because $\mathcal{I}$ is algebraic and $\operatorname{codim}\mathcal{I} \geq 2$); if $\mathcal{I} \neq 0$, then some iterate of τ would fix a point $p \in \mathcal{I}$. But non-trivial translations do not fix any point, thus $\tau^N = \mathrm{id}_A$ for some $N > 1$, contradicting the fact that τ has infinite order.
This shows that the indeterminacy locus of π is empty, i.e. π is holomorphic; thus, by point 1, C is an elliptic curve, $A \cong C \times F$, and π identifies with the projection onto the first factor. This concludes the proof of the second claim. □

Proof of Theorem 6.22. Let $f\colon X \dashrightarrow X$ be an infinite order birational transformation of a Calabi-Yau threefold, and suppose that there exists an f-equivariant fibration $\pi\colon X \to \mathbb{P}^1$ whose fibre $A = \pi^{-1}(t)$ over a general point $t \in \mathbb{P}^1$ is a simple abelian surface. By Theorem B, the action of f on $\mathbb{P}^1$ has finite order; therefore, after replacing f by an iterate, we may suppose that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "\mathrm{id}_{\mathbb{P}^1}"] & \mathbb{P}^1 \end{tikzcd}$$

Let us prove first that some iterate of f acts as a translation on general fibres of π. Since $\lambda_1(f) = 1$, $\lambda_2(f) = 1$ by log-concavity; therefore, by Theorem 1.16 the relative dynamical degree $\lambda_1(f|\pi)$ is equal to 1. By Remark 1.15, $\lambda_1(f|\pi) = \lambda_1(f|_A)$. This means that $f_A := f|_A$ is either elliptic or parabolic; if f_A were parabolic, by Theorem 1.30 f_A would preserve an elliptic fibration on A, contradicting its simplicity. This shows that f_A is elliptic; therefore one of its iterates is isotopic to the identity, thus a translation. Therefore, some iterate of f acts as a translation on general fibres of π, and from now on we will replace f by such an iterate.
Remark that, for a very general fibre, such a translation has infinite order: if this were not the case, then $\mathbb{P}^1$ would be the union of the Zariski-closed sets

$$Y_N = \{p \in \mathbb{P}^1 \mid f^N|_{\pi^{-1}(p)} = \mathrm{id}_{\pi^{-1}(p)}\};$$

thus $Y_N = \mathbb{P}^1$ for some N, i.e. f has finite order, contradicting the hypothesis.
Remark also that, if $f_A\colon A \to A$ is a translation of infinite order and A is simple, then all of its orbits are Zariski-dense: otherwise by Theorem 5.8 the closures of orbits would induce an f_A-invariant meromorphic map $A \dashrightarrow C$, hence, after Stein factorization, an f_A-invariant meromorphic fibration. By Lemma 6.23, this would imply that A is isogenous to a product of elliptic curves, contradicting its simplicity.

Now suppose by contradiction that f admits another equivariant fibration $\eta\colon X \dashrightarrow B$, i.e. that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, dashed, "\eta"'] & X \arrow[d, dashed, "\eta"] \\ B \arrow[r, dashed, "g"] & B \end{tikzcd}$$

Up to replacing B by a resolution of singularities, we may assume that B is a smooth projective variety.

If $\dim B = 1$, then the general fibres of η are of dimension 2; the restriction of η to a general fibre A of π defines (maybe after Stein factorization) an f_A-equivariant meromorphic fibration. By Lemma 6.23, if $A = \pi^{-1}(t)$ for very general $t \in \mathbb{P}^1$ (so that f_A has infinite order), such a fibration induces an isomorphism $A \cong E \times F$ for some elliptic curves E, F, contradicting the simplicity of A.

Suppose then that $\dim B = 2$, so that general fibres of η have dimension 1. If the fibres of η are contained in the fibres of π, then the restriction of η to a very general fibre A of π would

give an f-equivariant meromorphic fibration, and we would find a contradiction as in the case of $\dim B = 1$.
Thus we can assume that the fibres of π and η are generically transverse: this means that, if $p \in X$ is a general point and A_p and F_p denote the fibres of π and η respectively passing through p, then

$$T_pX = T_pA_p \oplus T_pF_p.$$

In particular, for a general fibre A of π, the restriction of η to A is dominant.

Since $\lambda_1(f) = 1$, by Theorem 1.16 $\lambda_1(g) = 1$ too; therefore, according to [DF01, Theorem 0.2], two cases are possible:

- either g admits an equivariant meromorphic fibration $\nu\colon B \dashrightarrow C$ onto a curve C;
- or, up to a change of birational model, g is an automorphism one of whose iterates is isotopic to the identity (i.e. $g^N \in \mathrm{Aut}^0(B)$ for some $N > 0$).

In the first case, the restriction of $\nu \circ \eta$ to a general fibre A of π would produce (after Stein factorization) an f_A-invariant fibration, contradicting the simplicity of A.
Therefore, after replacing f by an iterate, we may assume that g is isotopic to the identity.

Assume first that $g = \mathrm{id}_B$; denoting by A a very general fibre of π (so that f_A has infinite order) and by F a general fibre of η, this implies that f_A preserves the finite set $A \cap F$. Hence, some iterate of f_A fixes a point of A, contradicting the fact that f_A is a translation of infinite order.

Now suppose that $g \neq \mathrm{id}_B$; since g is isotopic to the identity, this implies that there exists a non-trivial holomorphic vector field v on B. Now, if $A = \pi^{-1}(t)$ is a very general fibre of π, the restriction $\eta_A = \eta|_A\colon A \dashrightarrow B$ is a dominant meromorphic map, and the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f_A"] \arrow[d, dashed, "\eta_A"'] & A \arrow[d, dashed, "\eta_A"] \\ B \arrow[r, "g"] & B \end{tikzcd}$$

In particular, the g-orbit of general points of B are Zariski-dense.
Remark that η_A is actually holomorphic: indeed, the indeterminacy locus is an f_A-invariant Zariski-closed proper subset of A, thus it is empty. The same is true for the exceptional locus of η_A (i.e. the locus of points where η_A is not a local diffeomorphism onto its image); therefore, η_A is an étale cover, and in particular B is an abelian variety.
Let $\alpha \in H^0(B, \Omega^1_B)$ be a non-trivial holomorphic 1-form on B; the pull-back $\pi^*\alpha$ extends by Hartogs principle to a non-trivial holomorphic 1-form on X; but, since X is simply connected, $H^0(X, \Omega_1) \cong H^{1,0}(X) = 0$, which leads to a contradiction. This concludes the proof of the claim. $\square$

Example 6.24. We can construct an example of such a situation using the results of Schoen in [Sch86]. The author constructs explicitly a Calabi-Yau threefold X as a well-chosen small resolution of the singular quintic

$$\hat{X} = \{x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5x_0x_1x_2x_3x_4 = 0\} \subset \mathbb{P}^5,$$

and shows that there exists a holomorphic fibration $\pi\colon X \to \mathbb{P}^1$ such that

1. the fibre of π over a general point of $\mathbb{P}^1$ is a simple abelian surface;
2. the Mordell-Weil group of π has rank at least two.

Let $\pi\colon Y \to B$ be a fibration whose general fibres are abelian surfaces and which adimits a section $s_0\colon B \to Y$. The *Mordell-Weil group* $Y(B)$ of π is defined formally as the set of B-rational points of Y (considered as a B-scheme); it is a group as it coincides with the $\mathbb{C}(B)$-points of the generic fibre.
Concretely, it is the group of birational trasformations of Y which fix every fibre of π and act on smooth simple fibres as a translation: fixing the section s_0 as the zero for the group law, a section $s\colon B \to Y$ acts on a fibre $A = \pi^{-1}(b)$ as the translation by $s(b) - s_0(b)$. By Mordell-Weil theorem, the Mordell-Weil group is always a finitely generated abelian group.
Property 2 means that we can embed $\mathbb{Z}^2$ in $X(\mathbb{P}^1)$; take an element of infinite order in $X(\mathbb{P}^1)$, and consider it as a birational transformation $f\colon X \dashrightarrow X$ with infinite order. The fibration π is f-equivariant by construction; therefore, f doesn't admit other equivariant meromorphic fibrations (and in particular it doesn't preserve any elliptic fibration).

6.3.2 Existence of equivariant elliptic fibrations

Theorem 6.25. *Let X be a Calabi-Yau threefold and let $f\colon X \to X$ be an automorphism with zero topological entropy (i.e. $\lambda_1(f) = 1$) and infinite action on cohomology. If one of the following conditions is satisfied*

1. *f preserves at least two distinct holomorphic fibrations;*
2. *or f preserves a fibration whose general fibres are $K3$ surfaces;*

then f preserves an elliptic fibration.

Before giving the proof, it is useful to recall that, since $H^1(X, \mathcal{O}_X) = 0$ and $H^2(X, \mathcal{O}_X) = 0$, the exponential sequence induces an isomorphism

$$\mathrm{Pic(X)} \xrightarrow{\sim} \mathrm{H^2(X, \mathbb{Z})}.$$

In other words, the isomorphism class of a line bundle is uniquely determined by its numerical class; conversely, every cohomology class $\alpha \in H^2(X, \mathbb{Z})$ is equal to $c_1(\mathcal{L})$ for some line bundle $\mathcal{L}$.

Proof. 1) Let $\pi_1\colon X \to B_1, \pi_2\colon X \to B_2$ be two f-equivariant fibrations. By Proposition 6.21, we may assume that $B_1 = B_2 = \mathbb{P}^1$, otherwise one of the two fibrations is already elliptic.
Consider the product map

$$(\pi_1, \pi_2)\colon X \to \mathbb{P}^1 \times \mathbb{P}^1.$$

Let F_1, F_2 be the fibres of π_1, π_2 respectively passing through a general point $p \in X$; since $F_1 \cap F_2$ is a curve, denoting by $[F_1], [F_2] \in H^2(X, \mathbb{Z})$ the numerical classes of general fibres of π_1 and π_2 respectively, $[F_1] \cdot [F_2] \neq 0 \in H^4(X, \mathbb{Z})$. This means that generic fibres of π_1 and generic fibres of π_2 have non-empty intersection, therefore (π_1, π_2) is surjective; thus, after Stein factorization, (π_1, π_2) defines a fibration $\pi\colon X \to B$ onto a surface, which is then an elliptic fibration by Proposition 6.21. This concludes the proof of point 1.

2) Since $\lambda_1(f) = 1$, the eigenvalues of $f_2^*\colon H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ have modulus 1; furthermore, they are algebraic integers and their conjugates over $\mathbb{Q}$ are also eigenvalues of f_2^* because f_2^* preserves the integral structure of $H^2(X, \mathbb{Z})$. By a lemma of Kronecker (see [Kro57]), an algebraic integers all of whose conjugates have modulus 1 is a root of unity. Thus, up to replacing f by an iterate, we may suppose that 1 is the only eigenvalue of f_2^*.

We use now [LB14a, Theorem 2.1]: *if X is a compact Kähler threefold and $f\colon X \to X$ is an automorphism such that the only eigenvalue of f_2^* is 1, then f_2^* admits a unique non-trivial*

Jordan block, whose dimension is either 3 *or* 5. In other words, in a suitable basis of $H^2(X,\mathbb{R}) = H^{1,1}(X,\mathbb{R})$ (which we may pick in $H^2(X,\mathbb{Z})$),

$$f_2^* = \begin{pmatrix} J & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & J_1 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & J_h \end{pmatrix},$$

where

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and the J_i are Jordan blocks of smaller dimension.
Let $k+1$ ($k = 2$ or 4) be the dimension of the maximal Jordan block, and let $v_k, v_{k-1}, \dots, v_0$ be a basis as above. Then $v_k \wedge v_k = 0 \in H^4(X,\mathbb{Z})$: if this were not the case, since

$$(f^n)^*(v_0) = c_k n^k v_k + c_{k-1} n^{k-1} + \dots$$

for explicit constants $c_i > 0$, we would have

$$(f^n)^*(v_0 \wedge v_0) = (f^n)^* v_0 \wedge (f^n)^* v_0 \sim c_k^2 n^{2k} v_k \wedge v_k,$$

so that $\| (f^n)_4^* \|$ would grow at least like n^{2k} as n tends to $+\infty$. On the other hand, by Poincaré duality we obtain

$$\| (f^n)_4^* \| \sim \| (f^{-n})_2^* \| \sim cn^k \qquad \text{as } n \to +\infty,$$

a contradiction.
Similarly,

$$v_k \wedge c_2(X) = 0 \in H^6(X,\mathbb{Z});$$

indeed, if $v_k \wedge c_2(X) \in H^6(X,\mathbb{Z}) \setminus \{0\}$, then

$$(f^n)^*(v_0 \wedge c_2(X)) = (f^n)^* v_0 \wedge (f^n)^* c_2(X) = (f^n)^* v_0 \wedge c_2(X) \sim cn^k (v_k \cdot c_2(X)),$$

contradicting the fact that $(f^n)_6^* = \mathrm{id}_{H^6(X,\mathbb{R})}$.

Now let $\pi\colon X \to \mathbb{P}^1$ be an f-equivariant fibration whose general fibres are $K3$ surfaces; let F be a general fibre of π, and let

$$v = c_1(\mathcal{O}_X(F)) \in H^2(X,\mathbb{Z}), \qquad w = v + v_k \in H^2(X,\mathbb{Z}).$$

Then $v \wedge c_2(X) = c_2(F) = 24 > 0$, so that

$$w \wedge c_2(X) > 0.$$

Since $v \wedge v = [F] \cdot [F] = 0 \in H^4(X,\mathbb{Z})$ and $v_k \wedge v_k = 0$,

$$w^3 := w \wedge w \wedge w = 0 \in H^6(X,\mathbb{Z}).$$

Finally, if $H \cap X$ is a generic hyperplane section of $X \subset \mathbb{P}^N$, by Hodge index theorem the quadratic form

$$(\alpha,\beta) \in H^{1,1}(X,\mathbb{R}) \times H^{1,1}(X,\mathbb{R}) \mapsto \int_{H \cap X} \alpha \wedge \beta$$

has signature $(1, h^{1,1}(X) - 1)$; in particular if α and β are isotropic, then either they are proportional or $\alpha \wedge \beta \neq 0$. Applying this to $\alpha = v, \beta = v_k$ we obtain that

$$w \wedge w = 2v \wedge v_k \neq 0 \in H^4(X, \mathbb{Z}).$$

Summarizing, we have shown that

$$w^2 \neq 0, \qquad w^3 = 0, \qquad w \wedge c_2(X) \neq 0; \tag{6.2}$$

furthermore, since for any ample class $H \in Pic(X) \cong H^2(X, \mathbb{Z})$ we have

$$\lim_{n \to +\infty} \frac{(f^n)^* H}{n^k} = cv_k \qquad \text{for some } c \in \mathbb{Q},$$

v_k is a nef class. Since v is the pull-back of an ample class, it is also nef, which implies that w is nef too.

By [Wil89, Proposition 3.2], a nef class w satisfying 6.2 is (the numerical class of) a semi-ample line bundle $\mathcal{L}$, and the complete linear system $|\mathcal{L}|$ induces an elliptic fibration $\eta\colon X \to S$. Since w is f-invariant, so is $\mathcal{L}$, thus the fibration η' is f-equivariant. This concludes the proof of point 2. $\square$

Using Oguiso's classification of fibrations, we obtain the following corollary.

Corollary 6.26. *Let X be a Calabi-Yau threefold and let $f\colon X \to X$ be an automorphism with zero topological entropy and infinite action on cohomology. If f admits an equivariant fibration, then it admits an equivariant fibration whose general fibres are elliptic curves or abelian surfaces.*

Part III

Other invariant structures

In Part III I analyze further examples of preserved structures (mainly foliations/distributions and affine structures) on symplectic manifolds, and I give the elements for the possible future developments of my research.

In Chapter 7 I deal with pairs of generically transverse foliations (or distributions) on a projective irreducible symplectic manifold. The main result in this direction is that, if such a pair is preserved by a loxodromic automorphism, then both foliations are Lagrangian.
The proof consists of two steps: first, using results coming from the Minimal Model Program, we manage to contract the exceptional divisor (on which the foliations are not transverse) onto a normal projective variety; then I introduce the notion of symplectic singularity, and I use a known result about the local form of such a singularity to show that holomorphic forms defined outside the exceptional divisor extend to it holomorphically.
In the last part of the chapter, I give some additional constraints on foliations which are preserved by a loxodromic transformation.
This chapter requires the notion of dynamical degrees and the content of Chapter 2.

In Chapter 8 I classify foliations on projective reducible symplectic fourfolds which are invariant by "doubly loxodromic" birational transformations (see Theorem 8.8. The classification is rather zoologic, and the interest of it is mainly to show that one can avoid the assumption on irreducibility. The proof uses heavily the known results on surfaces summarized in Chapter 1.

In Chapter 9 I give a frame to treat more general geometric structures which are preserved by a loxodromic transformation.
First I introduce (G, X)-manifolds and their singular counterpart, (G, X)-orbifolds. As an application, I classify (degenerate) affine structures on $K3$ surfaces which are preserved by a loxodromic automorphism.
Then, in order to include structures which a priori do not enjoy the symmetry assumptions of (G, X)-structures, I speak about Gromov's A-structures: this frame allows to treat e.g. riemannian or holomorphic metrics, forms, foliations, affine structures, conformal structures etc. in a unified way. We will restrict in general to rigid structures in order to apply the result on the Zariski-density of orbits: as a consequence, I show that invariant rigid structures are locally homogeneous on a Zariski-open dense subset.
As I explain in the last Section, the hope is then to find a local model for the rigid structure, so that we go back to the theory of (G, X)-manifolds; this would allow to use the notion of developing map, and to extend structures to the singularities appearing upon contraction of the exceptional locus. Finding a local model is always possible in low dimension, and this should allow, for example, to extend the classification of invariant affine structures to the case of irreducible symplectic fourfolds.
This chapter is mostly self-contained, although I repeatedly make use of the main result of Chapter 5 and of the contraction result of Chapter 7.

In Chapter 10 I give a summary of the main results of Chapter 5, 6, 7 and 9, and I list the open questions and conjectures which are still to be answered.

Chapter 7

Pairs of invariant foliations

In this chapter we analyse the geometric structures that can be preserved by a loxodromic birational transformation $f\colon X \dashrightarrow X$ of an irreducible symplectic manifold. As we have seen in Theorem A, f cannot preserve any (meromorphic) fibration. However, the exemples coming from tori (see Chapter 3) show that f could preserve other structures, like a pair of generically transverse foliations.

The main result of this chapter is the following:

Theorem C. *Let $f\colon X \dashrightarrow X$ be a loxodromic birational transformation of a projective irreducible symplectic manifold X of dimension $2n$. Suppose that f preserves two generically transverse non-trivial distributions $\mathcal{F}_1, \mathcal{F}_2$ (i.e. at a general point $p \in X$, $T_pX = T_p\mathcal{F}_1 \oplus T_p\mathcal{F}_2$); then $\mathcal{F}_1$ and $\mathcal{F}_2$ are both n-dimensional Lagrangian distributions.*

Remark 7.1. Two generically transverse distributions $\mathcal{F}_1, \mathcal{F}_2$ in an irreducible symplectic manifold (e.g. two generically transverse foliations) cannot be everywhere transverse.
Indeed, it this were the case, they would define a decomposition of the tangent bundle

$$TX = T\mathcal{F}_1 \oplus T\mathcal{F}_2. \tag{7.1}$$

As $c_1(X) = 0$, by the proof of Yau of the Calabi conjecture we deduce that X admits a Kähler-Einstein metric (see [GHJ03, §I.5]); by [Bea00a, Theorem A], since X is simply connected, the decomposition 7.1 induces then a decomposition of X: $X \cong X_1 \times X_2$.
But since X is irreducible symplectic, it doesn't admit a non-trivial product decomposition: indeed the pull-back of forms induces an isomorphism

$$H^0(X, \Omega_X^2) = H^{2,0}(X) \cong \bigoplus_{i=0,1,2} H^{i,0}(X_1) \otimes H^{2-i,0}(X_2).$$

Since the form σ generating $H^0(X, \Omega_X^2)$ is non-degenerate, if X_1 and X_2 are both non-trivial then $\sigma = \pi_1^*\alpha_1 \wedge \pi^*\alpha_2$ for some 1-forms $\alpha_i \in H^0(X_i, \Omega_{X_i}^1)$. By the Beauville-Bogomolov decomposition theorem (Theorem 2.6) this implies that X_1 and X_2 have a non-trivial torus factor, and in particular they are not simply connected. This shows that X is not simply connected either, which is a contradiction.

The idea of the proof is to consider, at a general point $p \in X$, one of the two-forms induced by the symplectic form σ and the decomposition of T_pX along the two distributions. Such a form σ' is not defined along the tangency divisor E of the two distributions; a result of Druel [Dru11] allows, after a change of (irreducible symplectic) birational model of X, to contract a component E_i of E. Then, by analyzing the singularities that appear upon contraction, one shows that σ' extends to a general point of E_i, then by Hartogs principle on the whole X. Since σ' is not symplectic, it must be zero; thus both distributions are σ-isotropic, hence Lagrangian.

As a corollary, we manage to exclude that some more rigid structures are preserved:

Corollary 7.2. *A loxodromic birational transformation of a projective irreducible symplectic manifold X cannot preserve $k \geq 3$ generically transverse singular distributions $\mathcal{F}_1, \ldots, \mathcal{F}_k$ (i.e. at a general point $p \in X$, $T_pX = T_p\mathcal{F}_1 \oplus \ldots \oplus T_p\mathcal{F}_k$).*

Proof. Let $2n = \dim X$. By a simple combinatorial argument, there exists a partition of $\{1, \ldots, k\}$ into two non-empty sets A_1, A_2 such that

$$\sum_{i \in A_1} \dim \mathcal{F}_i \neq n;$$

let $\mathcal{G}_1$ (resp. $\mathcal{G}_2$) the singular distribution obtained as the span of the $T\mathcal{F}_i$ for $i \in A_1$ (resp. $i \in A_2$). Then $\mathcal{G}_1, \mathcal{G}_2$ are f-invariant generically transverse non-trivial distributions, and since they do not have dimension n they are not Lagrangian; this contradicts Theorem C. □

7.1 Contraction of exceptional divisors

The goal of this section is to prove the following lemma:

Lemma 7.3. *Let $f\colon X \dashrightarrow X$ be a loxodromic birational transformation of a projective irreducible symplectic manifold preserving a prime divisor E. Then there exists an irreducible symplectic manifold X', a birational map $\phi\colon X \dashrightarrow X'$ and a birational morphism $\pi\colon X' \to Y$ onto a normal projective variety Y such that the exceptional locus of π is the strict transform ϕ_*E of E.*

This proof is essentially an application of a result of Druel (see Proposition 7.7). In what follows, we will explain how he combined results in the Minimal Model Program with a generalization in higher dimension of the Zariski decomposition of divisors on surfaces (see [Nak04] and [Bou04]), in order to show that a prime divisor which is "negative" (in the sense of the Zariski decomposition) is contractible, up to change of birational model; since in the irreducible symplectic case negativity can be checked via the Beauville-Bogomolov form, the conclusion follows from the informations about the action on cohomology of f.

7.1.1 Divisorial Zariski decomposition

In his pioneering work [Zar62], Zariski shows that an effective $\mathbb{Q}$-divisor D on a surface can be uniquely decomposed into a sum

$$D = P + N,$$

where P (the positive part) is a nef $\mathbb{Q}$-divisor and $N = \sum a_j D_j$ (the negative part) is an effective $\mathbb{Q}$-divisor such that the Gram matrix $(D_i \cdot D_j)$ is negative definite, and $P \cdot N = 0$.

Extending algebro-geometric concepts of positivity to currents and using currents with minimal singularities, Boucksom describes in [Bou04] a similar decomposition of real pseudo-effective $(1,1)$-cohomology class α on a compact complex manifold X:

$$\alpha = Z(\alpha) + N(\alpha);$$

the positive part is in general not nef, but just "nef in codimension 1". In the case of surfaces, one recovers the usual Zariski decomposition.

Boucksom then proceeds giving a detailed description of the situation on irreducible symplectic manifolds, which, as one might suspect from the formal resemblance with surfaces, is particularly easy to describe. In particular, as in the surface case, negativity of a divisor can be checked from the Gram matrix with respect to the Beauville-Bogomolov form.

Theorem 7.4. *A family $E_1, \ldots, E_k$ of prime divisors on an irreducible symplectic manifold is exceptional in the sense of Boucksom (see [Bou04, Definition 3.10]) if and only if the Gram matrix $(q_X(E_i, E_j))_{i,j=1,\ldots,k}$ is negative definite (where q_X denotes the Beauville-Bogomolov form).*

Proposition 7.5. *If an effective divisor $E = \sum E_i$ is exceptional (i.e. the E_i form an exceptional family), then $N(E) = E$.*

7.1.2 The Minimal Model Program on irreducible symplectic manifolds

For this subsection we refer to [Dru11] and the references therein.

Singularities of pairs

Let X be a normal projective variety over $\mathbb{C}$ and let Δ be a Weil $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. A *resolution of singularities* of the pair (X, Δ) is a birational morphism $\pi\colon \widetilde{X} \to X$ such that $\widetilde{X}$ is smooth and the strict transform $\widetilde{\Delta}$ of Δ union the exceptional locus of π has normal crossings. Then

$$K_{\widetilde{X}} + \widetilde{\Delta} = \pi^*(K_X + \Delta) + \sum_F a_F(X, \Delta) F$$

where one sums over the set of π-exceptional prime divisors. The pair (X, Δ) is said to be *klt* (Kawamata log-terminal) if $a_F(X, \Delta) > -1$. See [Kol97] for more details.

Directed log-MMP

For this paragraph, we refer to [BCHM10, Kal06].

Given a klt pair (X, Δ), a nef (resp. minimal) model of (X, Δ) is a klt pair (X', Δ') with a birational map $\phi\colon X \dashrightarrow X'$ such that

1. Δ' is the strict transform of $\phi_*\Delta$;
2. ϕ is $(K_X + \Delta)$-negative (resp. $(K_X + \Delta)$-strictly negative);
3. $K_{X'} + \Delta'$ is nef.

The MMP algorithm goes like this:

1. we contract some $(K_X + \Delta)$-negative curves using the (log-)contraction theorem by a morphism $c\colon X \to X'$ onto a projective normal variety X';
2. if $\dim X' < \dim X$, then c is a Mori fibration, and the algorithm terminates;
3. if X' has good singularities (if and only if the exceptional locus of c has codimension 1, i.e. c is a *divisorial contraction*), we replace X by X' and we start again;
4. if X' has bad singularities (if and only if the exceptional locus of c has codimension > 1, i.e. c is a *small contraction*), then we build the *flip* $c^+\colon X^+ \dashrightarrow X'$, and we replace X by X^+.

Remark that if X is $\mathbb{Q}$-factorial and $\phi\colon X \to X'$ is a divisorial contraction, then the Picard numbers of X and X' satisfy $\rho(X') < \rho(X)$; therefore one can have only a finite number of divisorial contraction. In particular, in order to show that the algorithm always terminates, one has to show that there is no infinite sequence of flips.

There exists a directed version of the MMP algorithm, where the contractions at point (1) are not chosen randomly. We fix a klt pair $(X, \Delta) = (X_0, \Delta_0)$ with X $\mathbb{Q}$-factorial and H an effective $\mathbb{R}$-divisor such that $K_X + \Delta + H$ is nef and $(X, \Delta + H)$ is klt; fix $t_0 = 1$.
The MMP for the pair (X, Δ) directed by H produces birational maps

$$\phi_i\colon (X_i, \Delta_i, H_i) \dashrightarrow (X_{i+1}, \Delta_{i+1}, H_{i+1})$$

and a decreasing sequence of real numbers $0 \leq t_i \leq 1$ such that $K_{X_i} + \Delta_i + t_i H_i$ and $(X_i, \Delta_i + t_i H_i)$ is klt.

Link with the Zariski decomposition

Even if we don't know if the directed MMP terminates, the existence of contractions and flips allows to prove a number of useful results. Using an almost termination result contained in [BCHM10], Druel proved the following:

Theorem 7.6 ([Dru11], Theorem 3.3). *Let (X, Δ) be a klt pair, with X $\mathbb{Q}$-factorial, and let H be an ample $\mathbb{Q}$-divisor on X such that $(X, \Delta + H)$ is klt. We suppose that $K_X + \Delta$ is pseudo-effective. Consider an MMP directed by H for the pair (X, Δ):*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow X_3 \dashrightarrow \dots$$

where each $\phi_i\colon X_i \dashrightarrow X_{i+1}$ is a birational map.
Then, for $i >> 0$, $K_{X_i} + \Delta_i \in \overline{Mob}(X_i)$ (the closure of the mobile cone of X_i). Furthermore the prime divisors contracted by $\phi_{i-1} \circ \dots \circ \phi_0$ are the irreducible components of the negative part $N(K_X + \Delta)$.

As a Corollary, Druel show the following result:

Proposition 7.7 ([Dru11], Proposition 1.4). *Let $E \subset X$ be a prime divisor on an irreducible symplectic manifold X such that $N(E) = E$ (i.e. $q(E) > 0$); then there exists an irreducible symplectic manifold X', a birational map $\phi\colon X \dashrightarrow X'$ and a birational morphism $\pi\colon X' \to Y$ onto a normal projective variety Y such that the exceptional locus of π is the strict transform $\phi_* E$.*

Proof. For $\epsilon > 0$ small enough, the pair $(X, \epsilon E)$ is klt; let H be an ample effective $\mathbb{Q}$-divisor on X such that the pair $(X, \epsilon E + H)$ is still klt.

Now we run an MMP for the pair $(X, \epsilon E)$ directed by H; with the same notations as paragraph 7.1.2, we know by Theorem 7.6 that, for $i >> 0$, the only prime divisor contracted by $\phi_{i-1} \circ \dots \circ \phi_1\colon X \dashrightarrow X_i$ is the negative part $N(K_X + \epsilon E) = N(\epsilon E) = \epsilon E$. Let i_0 be the (unique) index such that $\phi_{i_0} = c_{i_0}$ is the divisorial contraction of (the strict transform of) E.
We set $\phi := \phi_0 \circ \dots \circ \phi_{i_0 - 1}\colon X \dashrightarrow X' = X_{i_0}$, $Y = X_{i_0+1}$, $\pi = \phi_{i_0} = c_{i_0}\colon X' \to Y$. By definition, π is a birational morphism whose exceptional locus is the strict transform $\phi_* E$. For $i \leq i_0 - 1$, ϕ_i is a flip; therefore, the singularities of X_{i+1} are terminal, therefore, thanks to [Nam06, Corollary 1], X_{i+1} is smooth. This shows that X' is a manifold; since ϕ is an isomorphism in codimension 1, a symplectic form on X induces a symplectic form on X', which concludes the proof. $\square$

Proof of Lemma 7.3. Let X be an irreducible symplectic manifold, and let $f\colon X \dashrightarrow X$ be a loxodromic birational transformation. By Proposition 7.7, we just need to show that a prime divisor E which is preserved by f^* is negative, in the sense that $N(E) = E$. By Proposition 7.5, it is enough to show that $q_X(E) < 0$, where q_X is the Beauville-Bogomolov form on $H^2(X, \mathbb{Z})$.

Since E is effective and non-trivial, $c_1(E) \neq 0$. As we have seen in Proposition 2.14, the only integral non-negative class preserved by $f^*\colon H^{1,1}(X, \mathbb{R}) \to H^{1,1}(X, \mathbb{R})$ is the null class, hence $q_X(E) < 0$. This concludes the proof. □

7.2 Singular symplectic varieties

As we have seen in Lemma 7.3, a prime divisor $E \in Div(X)$ which is preserved by a birational loxodromic transformation of an irreducible symplectic manifold can be contracted, after possibly replacing X by another birationally equivalent irreducible symplectic manifold X', giving a birational morphism $\pi\colon X' \to Y$ onto a normal variety Y.
The singularities of Y are particularly well-behaved: Y is called a symplectic variety, and π is a symplectic resolution. In this section we recall the definition and some fundamental properties of symplectic varieties and symplectic resolutions; we refer to the survey [Fu06].

The definition of symplectic singularity was introduced by Beauville in [Bea00b].

Definition 7.8. *Let Y be a normal variety; we say that Y is a* symplectic variety *(or that the singularities of Y are symplectic) if there exists a holomorphic symplectic form σ_{sm} on the smooth locus of Y such that for every resolution of singularities $\pi\colon \widetilde{Y} \to Y$ the pull-back of σ_{sm} extends to a holomorphic form σ on $\widetilde{Y}$.*
If furthermore the induced form on $\widetilde{Y}$ is everywhere symplectic, $\pi\colon \widetilde{Y} \to Y$ is called a symplectic resolution.

Remark 7.9. It is enough to check that σ_{sm} extends to a global holomorphic form on *some* resolution of singularities: indeed, if $\pi\colon X_1 \to X_2$ is a birational morphism between smooth varieties, a meromorphic form α on X_2 is holomorphic if and only if $\pi^*\alpha$ is holomorphic (see [Fu06, Nam01]).

7.2.1 Local structure

As shown in [Bea00b], symplectic singularities are automatically rational Gorenstein (see [Ish14, Chapter 6] for definitions); in dimension two, such singularities coincide with du Val singularities (or "rational double points"): they are locally biholomorphic to $\mathbb{C}^2/G$, where G is a finite subgroup of $SL_2(\mathbb{C})$ (see [Ish14, §7.5] or [BPVdV84]).
For normal Gorenstein varieties, having rational singularities is the same as having canonical singularities [Ish14, Corollary 6.2.15]. In this case, one can use the following theorem to describe the local structure of singularities:

Theorem 7.10 ([Rei80], Theorem 1.13)**.** *Let Y be a normal projective variety; if Y has canonical singularities, then so does its general hyperplane section.*

This leads to the following corollary, see [Rei80, Corollary 1.14]; see also [GKKP11, Proposition 9.4] for a more general and rigorous proof.

Corollary 7.11. *Let Y be an n-dimensional normal projective variety with canonical singularities (e.g. a projective symplectic variety), and let $Z \subset Y$ be a codimension 2 component of the singular locus* SingY*. Then around general points of Z, Y is locally biholomorphic to*

$$\mathbb{C}^{n-2} \times {\mathbb{C}^2}/{G},$$

where $G \subset \mathrm{SL}_2(\mathbb{C})$ is a finite group, so that $\mathbb{C}^2/G$ is the germ of a surface with du Val singularities.

Proof. Let $Y \subset \mathbb{P}^N$ be an embedding and let S be the intersection of Y with $n-2$ general hyperplanes $H_1, \dots H_{n-2}$; by Theorem 7.10, S is a surface with du Val singularities. Furthermore, $S \cap Z \neq \emptyset$ and around a point $p \in S \cap Z$, S is locally biholomorphic to $\mathbb{C}^2/G$ for some finite $G \leq \mathrm{SL}_2(\mathbb{C})$.

Now let the hyperplanes H_i vary holomorphically; du Val singularities have no moduli, meaning that small deformations of germs of du Val singularities are trivial. Hence, varying the hyperplanes holomorphically leads to a local product structure as claimed. □

7.2.2 Semi-smallness and stratification

We say that a (birational) morphism $\pi\colon X \to Y$ is *semi-small* if

$$\operatorname{codim}\{y \in Y,\, \dim \pi^{-1}(y) \geq k\} \geq 2k,$$

or, equivalently, for every subvariety $V \subset X$,

$$\operatorname{codim}(\pi(V)) \leq 2 \cdot \operatorname{codim}(V).$$

The semi-smallness of symplectic resolutions was proven partially by Wierzba [Wie03], Namikawa [Nam01] and Hu-Yau [BHL03], then in full generality by Kaledin [Kal06, Lemma 2.11]:

Theorem 7.12. *Symplectic resolutions are semi-small.*

This implies for example that, if $\pi\colon X \to Y$ is a symplectic resolution and $E \subset X$ is a π-exceptional divisor, then $\operatorname{codim} \pi(E) = 2$; in particular, at a general point of $\pi(E)$, Y is locally biholomorphic to $\mathbb{C}^{2n-2} \times \mathbb{C}^2/G$, where $2n = \dim X$ and $G \subset \mathrm{SL}_2(\mathbb{C})$ is a finite group, so that $\mathbb{C}^2/G$ is the germ of a du Val singularity.

Concerning the global structure of the singular locus of a symplectic variety, Kaledin proved that symplectic singularities appear with a natural stratification, such that (roughly said) each stratum is also a symplectic variety:

Theorem 7.13 ([Kal06], Theorem 2.3)**.** *Let Y be a symplectic variety. Then there exists a stratification*

$$Y = \overline{Y_0} \supset \overline{Y_1} \supset \dots$$

such that

1. *Y_i (resp. $\overline{Y_{i+1}}$) is the smooth (resp. singular) locus of $\overline{Y_i}$;*
2. *the normalization of every irreducible component of $\overline{Y_i}$ is a symplectic variety (and in particular has even dimension).*

Example 7.14. If $\dim Y = 4$, then we can write

$$Y = Y_0 \sqcup Y_1 \sqcup Y_2,$$

where Y_0 is the smooth locus of Y, Y_1 is the smooth locus of $\mathrm{Sing(Y)}$, and Y_2 is the singular locus of $\overline{Y_1}$; the irreducible components of Y_1 are either isolated points or surfaces whose normalization is symplectic, i.e. normal with du Val singularities, whereas the irreducible components of Y_2 (if there are any) are isolated points.

Suppose furthermore that Y is obtained by contracting some prime divisor E on an irreducible symplectic manifold (see Lemma 7.3)

$$\pi \colon X \to Y.$$

Then, π is a symplectic resolution, and by semi-smallness, $\operatorname{codim} \pi(E) = 2$, so that $\operatorname{codim} Y_1 = 2$. Locally around general points of Y_1, Y is biholomorphic to $\mathbb{C}^2 \times S$ for some germ of singular symplectic surface $S \cong \mathbb{C}^2/G$ for some finite subgroup $G \subset \mathrm{SL}_2(\mathbb{C})$; Y_1 corresponds to $\{0\} \times S$.

7.2.3 Extension of forms

Symplectic resolutions enjoy a good property of extension of differential forms. The following proposition is actually true in a much greater generality: Y can be replaced by a normal variety with klt singularities and X by any resolution of singularities of Y (see [Nam01] for the case of 2-forms and symplectic singularities, [GKKP11] for the general result).

Proposition 7.15. *Let Y be a singular symplectic variety and let $\pi \colon X \to Y$ be a symplectic resolution. Then every holomorphic differential p-form α defined on the smooth locus of Y induces a holomorphic differential form on the whole X.*

Proof. Remark first that, by Hartogs principle, $\pi^*\alpha$ extends automatically across every component of the exceptional locus of π which has codimension ≥ 2; therefore, we may assume that π contracts a divisor E. Denote by $\beta \in H^0(X \setminus E, \Omega^p_X)$ the pull-back of α by $\pi|_{X \setminus E}$; we need to show that β extends to a form on X.

By Hartogs principle, it is enough to check that β extends to a general point of E. Furthermore, by semi-smallness of π (Theorem 7.12), the image of each component of E has codimension 2; hence, by Corollary 7.11, around a general point p of $\pi(E)$, Y is locally biholomorphic to

$$\mathbb{C}^2/G \times \mathbb{C}^{2n-2},$$

where $2n = \dim X$ and $G \leq \mathrm{SL}_2(\mathbb{C})$ is a finite group. Locally, the singular locus corresponds to $\{0\} \times \mathbb{C}^{2n-2}$.

There are two natural notions of desingularization of the germ of singularity (Y, p): on the one hand the universal orbifold cover (see §9.1.2), i.e. the natural projection

$$\eta \colon \mathbb{C}^n \to \mathbb{C}^2/G \times \mathbb{C}^{2n-2};$$

on the other hand, the algebro-geometric minimal resolution

$$\pi_0 \colon X_0 = S \times \mathbb{C}^{2n-2} \to \mathbb{C}^2/G \times \mathbb{C}^{2n-2},$$

where $\pi_S \colon S \to \mathbb{C}^2/G$ is the minimal resolution of the singularity $\mathbb{C}^2/G$, which can be realized as a sequence of blow-ups of points.

The resolution π_0 is minimal in the sense that every resolution of the singularity (Y, p) factors through π_0; in particular so does the germ of π around $\pi^{-1}(p)$. If we show that α induces a global form β_0 on X_0, then one can extend β by pull-back of β_0, which shows the result. Thus, from now on we will suppose that $\pi = \pi_0$ is a (local) minimal resolution of the singularity (Y, p).
Also, from now on we will denote again by Y the germ of Y at p, by $\eta \colon \widetilde{Y} \to Y$ the germ of universal orbifold cover, and by $\pi \colon X \to Y$ the algebro-geometric minimal resolution.

Consider the fibre product

$$\begin{tikzcd} \widetilde{X} = \widetilde{Y} \times_Y X \arrow[r, "\tilde{\eta}"] \arrow[d, "\tilde{\pi}"'] & X \arrow[d, "\pi"] \\ \widetilde{Y} \cong \mathbb{C}^n \arrow[r, "\eta"] & Y \cong {}^{\mathbb{C}^2}\!/_{G} \times \mathbb{C}^{n-2} \end{tikzcd} .$$

Concretely, $\pi\colon X \to Y$ is a composition

$$X = Y_k \xrightarrow{\pi_k} Y_{k-1} \xrightarrow{\pi_{k-1}} \ldots \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} Y_0 = Y,$$

where $Y_i \cong S_i \times \mathbb{C}^{2n-2}$ and $\pi_i\colon S_i \times \mathbb{C}^{2n-2} \to S_{i-1} \times \mathbb{C}^{2n-2}$ is the blow-up of Y_{i-1} along $\{p_{i-1}\} \times \mathbb{C}^{2n-2}$ for some $p_i \in S_{i-1}$.
Then $\tilde{\pi}\colon \widetilde{X} \to \widetilde{Y}$ can be obtained as the composition

$$\widetilde{X} = \widetilde{Y}_k \xrightarrow{\tilde{\pi}_k} \widetilde{Y}_{k-1} \xrightarrow{\tilde{\pi}_{k-1}} \ldots \xrightarrow{\tilde{\pi}_2} \widetilde{Y}_1 \xrightarrow{\tilde{\pi}_1} \widetilde{Y}_0 = \widetilde{Y},$$

where each $\tilde{\pi}_i$ is the blow up of $\widetilde{Y}_{i-1} \cong \widetilde{S}_i \times \mathbb{C}^{2n-2}$ along $\{\widetilde{p}_{i-1}\} \times \mathbb{C}^{2n-2}$, where $\tilde{p}_{i-1}$ is the only point of the cover $\widetilde{S}_{i-1} \to S_{i-1}$ over p_{i-1}.
Then $\widetilde{X}$ is a germ of manifold; furthermore, since G preserves the centers of blow-ups, $\widetilde{X}$ inherits an action of G, and X identifies with $\widetilde{X}/G$ via $\tilde{\eta}$.

Now, $\eta^*\alpha$ is a G-invariant form on $\widetilde{Y} \setminus \eta^{-1}(\mathrm{SingY}) \cong (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}^{2\mathrm{n}-2}$; by Hartogs principle, it can be extended to a G-invariant form on the whole $\widetilde{Y}$. The pull-back $\tilde{\pi}^*(\eta^*\alpha)$ is then a G-invariant form on $\widetilde{X}$, and thus it defines a form $\eta_*\tilde{\pi}^*(\eta^*\alpha)$ on the whole X. It is easy to check that $\eta_*\tilde{\pi}^*(\eta^*\alpha)$ agrees with β outside E, so that β can be holomorphically extended to general points of E. By the above discussion, this concludes the proof. $\square$

Proposition 7.16. *Let $f\colon X \dashrightarrow X$ be a loxodromic bimeromorphic transformation of an irreducible symplectic manifold X of dimension $2n$.*
Let α be a holomorphic differential form defined on the complement of a divisor E, and suppose that E is f-invariant. Then α extends to the whole X.

Proof. Remark that, by Hartogs principle, we just need to show that α extends to a general point of E. Let

$$E = E_1 \cup \ldots \cup E_k$$

be the decomposition of E into irreducible components. Up to replacing f by some iterate, we can suppose that $f^*E_i = E_i$ for $i = 1, \ldots, k$.

Fix $i \in \{1, \ldots, k\}$; by Proposition 7.7, there exists an irreducible symplectic manifold X', a birational map $\phi\colon X \dashrightarrow X'$ and a birational morphism $\pi\colon X' \to Y$ onto a normal projective variety Y such that the exceptional locus of π is the strict transform $E_i' = \phi_* E_i$; set $E_Y := \pi(\phi_*(E_1 \cup \ldots \cup E_{i-1} \cup E_{i+1} \cup \ldots \cup E_k))$.
Since ϕ is an isomorphism in codimension 1, $(\phi^{-1})^*\alpha$ can be extended to $X' \setminus \phi_*(E)$; let V be a dense affine subset of $Y \setminus E_Y$ such that $V \cap \phi_*(E_i) \neq \emptyset$, and let $U := \pi^{-1}(V)$.
By Proposition 7.16, the form $(\phi^{-1})^*\alpha$ extends to $\phi_*(E_i) \cap U$; thus (again because ϕ is an isomorphism in codimension 1), α extends to the general point of E_i.

As this construction can be realized for every component E_i, α can be extended at every point of X but at most an analytic subset of codimension 2, hence on all X by Hartogs principle. This concludes the proof. $\square$

7.3 Proof of Theorem C

Suppose by contradiction that $f\colon X \dashrightarrow X$ is a loxodromic birational transformations of a projective irreducible symplectic manifold of dimension $2n$ preserving two generically transverse distributions $\mathcal{F}_1, \mathcal{F}_2$ which are not both Lagrangian.

Remark that, if $\mathcal{F}_1, \mathcal{F}_2$ are both isotropic for σ at a general point $p \in X$ (i.e. the restriction of σ_p to $\mathcal{F}_1(p)$ or $\mathcal{F}_2(p)$ is identically zero), then they are both Lagrangian by a dimensional argument; therefore we can suppose that $\mathcal{F}_1$ is not isotropic for σ.

At a general point of X, $\mathcal{F}_1$ and $\mathcal{F}_2$ are transverse; therefore, at such a point p we can define a holomorphic two-form σ' by

$$\sigma'(v, w) := \sigma(\pi_1 v, \pi_1 w) \qquad \forall v, w \in T_pX,$$

where $\pi_1\colon T_pX \to T_p\mathcal{F}_1$ denotes the projection with respect to the decomposition $T_pX = T_p\mathcal{F}_1 \oplus T_p\mathcal{F}_2$. More explicitly, if $x_1, \ldots, x_p, y_{p+1}, \ldots y_{2n}$ are linear coordinates of T_pX such that $T_p\mathcal{F}_1$ (resp. $T_p\mathcal{F}_2$) is defined by $\{y_{p+1} = \ldots = y_{2n} = 0\}$ (resp. by $\{x_1 = \ldots = x_p = 0\}$), and if $\sigma_p = \sum a_{ii'} dx_i \wedge dx_i' + \sum b_{ij} dx_i \wedge dy_j + \sum c_{jj'} dy_j \wedge dy_{j'}$, then $\sigma'_p = \sum a_{ii'} dx_i \wedge dx_{i'}$.
Since the decomposition $T_pX = T_p\mathcal{F}_1 \oplus T_p\mathcal{F}_2$ depends holomorphically on p, we define this way a holomorphic two-form on $X \setminus E$, where $E = E_1 \cup \ldots \cup E_k$ denotes the tangency divisor of the pair $(\mathcal{F}_1, \mathcal{F}_2)$.
Since E is f-invariant, by Proposition 7.16, σ' extends to a holomorphic form on X.

Now, since $\dim \mathcal{F}_1 < 2n$, we have $(\sigma')^{n-1} = 0$; on the other hand by construction $\sigma' \neq 0$, so that σ' cannot be a multiple of σ, contradicting the minimality of X. This concludes the proof. $\square$

7.4 Further results

Let $f\colon X \dashrightarrow X$ be a loxodromic transformation of an irreducible symplectic manifold and suppose that f preserves a foliation $\mathcal{F}$ of codimension p.

Recall that $\mathcal{F}$ is defined by a holomorphic p-form

$$\omega \in H^0(X, \Omega_X^p \otimes \mathcal{L}), \qquad \mathcal{L} := \det N\mathcal{F}$$

with zeros in codimension at least 2. If s is a meromorphic section of $\mathcal{L}$, then ω/s is a meromorphic p-form which defines $\mathcal{F}$. One could wonder if f might preserve a meromorphic form defining an invariant foliation. The answer is no, because of the following corollary of Proposition 7.16.

Corollary 7.17. *Let $f\colon X \dashrightarrow X$ be a loxodromic bimeromorphic transformation of an irreducible symplectic manifold X of dimension $2n$.*
Suppose that there exists a meromorphic p-form α on X such that $f^*\alpha = \lambda\alpha$ for some $\lambda \in \mathbb{C}^*$. Then α is a holomorphic form; in other words

$$\alpha = \begin{cases} c\sigma^{p/2} & \text{for some } c \in \mathbb{C}^* \text{ if } p \text{ is even} \\ 0 & \text{if } p \text{ is odd} \end{cases}$$

In particular, α does not define a non-trivial foliation.

Proof. The divisor D of poles of α is f-invariant; suppose by contradiction that $D \neq \emptyset$. Then by Proposition 7.16, α extends holomorphically to D, which contradicts the definition. This shows that α is a holomorphic form, hence a power of the symplectic form σ by Proposition 2.16.

Since σ is non-degenerate, for $k = 1, \ldots, n$ and for any local vector field v on X,

$$i_v \sigma^k \neq 0.$$

This means that powers of σ do not define non-trivial foliations. $\square$

This results applies for example in the case where $\mathcal{L}$ has some holomorphic section.

Corollary 7.18. *Let $f\colon X \dashrightarrow X$ be a loxodromic bimeromorphic transformation of an irreducible symplectic manifold X of dimension $2n$.*
If f preserves a foliation $\mathcal{F}$, then $\mathcal{L} = \det(N\mathcal{F})$ does not have any non-trivial holomorphic section.

Proof. Suppose by contradiction that $H^0(X, \mathcal{L}) \neq \{0\}$. Since $\mathcal{F}$ is f-invariant, f acts linearly by pull-back on the finite dimensional vector space $H^0(X, \mathcal{L})$. This is clear if f is an automorphism; if f is only birational, let U be the definition set of f. Then $\mathrm{codim}(X \setminus U) \geq 2$; the pull-back f^*s of a section $s \in H^0(X, \mathcal{L})$ is well-defined on U, and can then be extended by Hartogs principle to the whole X. Pick an eigenvector s of the linear automorphism

$$f^*\colon H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}).$$

Next, let

$$V = \{\omega \in H^0(X, \Omega^p_X \otimes \mathcal{L}) \,|\, \text{for all local vector fields } v \in T\mathcal{F},\ i_v\omega = 0\}.$$

Then V is a linear subspace of the finite dimensional vector space $H^0(X, \Omega^p_X \otimes \mathcal{L})$, hence it is finite dimensional itself. Again, f acts linearly by pull-back on V; pick any eigenvector $\omega \in V$.

Now, the meromorphic form ω/s defines $\mathcal{F}$ and is almost f-invariant:

$$f^*\left(\frac{\omega}{s}\right) = \lambda\frac{\omega}{s} \qquad \text{for some } \lambda \in \mathbb{C}^*.$$

This contradicts Corollary 7.17. $\square$

Chapter 8

Invariant foliations on reducible symplectic fourfolds

In this chapter we deal with the dynamics of birational transformations of projective *reducible* symplectic manifolds in dimension 4. As we will see, after taking a finite étale cover, such varieties are isomorphic to the product of two surfaces (Lemma 8.2), and their birational transformations are actually automorphisms, which come from automorphisms of the two factors (Lemma 8.3):

$$f = (f_1, f_2)\colon S_1 \times S_2 \to S_1 \times S_2.$$

In Theorem 8.8 we assume that both the automorphisms f_1, f_2 are loxodromic, and we give a complete description of invariant foliations: non-trivial example arise from linear dynamics on tori, in the following sense.

Let $f\colon S \to S$ be an automorphism of a surface S; we will say that the pair (S, f) is *of Kummer type* if

- either S is a torus and f is a linear automorphism;
- or S is a (smooth) Kummer surface equipped with an automorphism induced by a linear one on a torus (see §3.1).

These are the only examples of loxodromic automorphisms of projective surfaces with trivial Chern class preserving a foliation (see Corollary 3.6).

Remark 8.1. Let $f = (f_1, f_2)\colon S_1 \times S_2 \to S_1 \times S_2$ be the product of two automorphisms as above. Then, by Theorem 1.16, $\lambda_1(f) = \max\{\lambda_1(f_1), \lambda_1(f_2)\}$; therefore, the natural dynamical hypothesis would be that at least one of the f_i is loxodromic. In this case, it should be possible to classify invariant foliations using the same methods as in the proof of Theorem 8.8.

8.1 Classification of reducible symplectic fourfolds

Let X_0 be a projective symplectic manifold; in other words, there exists a symplectic two-form $\sigma_0 \in H^0(X_0, \Omega^2_{X_0})$, but a priori there could exist other holomorphic two-forms on X_0.
Remark that, if $\dim X_0 = 2n$, then σ_0^n trivializes the canonical bundle KX_0; hence, by the Beauville-Bogomolov decomposition theorem (Theorem 2.6), there exists an étale finite cover of $\eta\colon X \to X_0$ of the form

$$X = T \times \prod_{i=1}^{k} Y_i \times \prod_{j=1}^{h} Z_j,$$

where T is a torus, the Y_i are Calabi-Yau manifolds in the strict sense, and the Z_j are irreducible symplectic manifolds. The pull-back $\eta^*\sigma$ of a symplectic form on X_0 defines a symplectic form on X.

From now on, we will suppose that $\dim X_0 = 4$ and that X_0 is not irreducible symplectic. Remarking that in dimension two Calabi-Yau manifolds and irreducible symplectic manifolds coincide with $K3$ surfaces, the following four cases are a priori possible:

1. $X \cong \mathbb{C}^4/\Lambda$ is a complex torus;
2. $X = T \times S$, where T is a two-dimensional complex torus and S is a $K3$ surface;
3. $X = S_1 \times S_2$, where S_1 and S_2 are $K3$ surfaces;
4. $X = E \times Y$, where E is an elliptic curve and Y is a three dimensional Calabi-Yau manifold.

Lemma 8.2. *A reducible symplectic fourfold is a finite étale quotient of a torus or of a product $S_1 \times S_2$, where S_1 is a $K3$ surface and S_2 is either a two-dimensional torus or a $K3$ surface.*

Proof. Let X be a finite étale cover as in Beauville-Bogomolov decomposition theorem. By the above discussion, we only need to exclude the case $X = E \times Y$, where E is an elliptic curve and Y is a three-dimensional Calabi-Yau manifold.
Suppose by contradiction that such a product $E \times Y$ is symplectic, and let σ be a symplectic form; also denote by v_0 a vector field on E trivializing TE and by v the induced vector field on $E \times Y$. The contraction $i_v\sigma = \sigma(v, \cdot)$ is then a 1-form on X; we will prove that $i_v\sigma = 0$, contradicting symplecticity.

Remark that, if $j\colon \{e\} \times Y \hookrightarrow E \times Y$ is the embedding of a fibre of the first projection $\pi_E\colon X \to E$, then $j^*(i_v\sigma)$ is a 1-form on Y, hence it is equal to 0 because $H^{1,0}(Y) = 0$.

Now, if $u \in T_pX$ is any vector at a point $p = (p_1, p_2) \in X$, one can decompose u along the two projections:

$$u = u_E + u_Y \qquad u_E \in T_{p_1}E, \quad u_Y \in T_{p_2}Y.$$

Then

$$i_v\sigma(u) = i_v\sigma(u_E + u_Y) = \sigma(v, u_E) + j^*(i_v\sigma)(u_Y) = 0$$

because v and u_E are proportional. □

Now let $f_0\colon X_0 \dashrightarrow X_0$ be a birational transformation; by Remark 2.12, f_0 is an isomorphism in codimension one: there exist Zariski-open sets $U, V \subset X_0$ such that

- $\operatorname{codim}(X_0 \setminus U), \operatorname{codim}(X_0 \setminus V) \geq 2$; in particular the fundamental groups $\pi_1(X, p)$, $\pi_1(U, p)$ and $\pi_1(V, p)$ are canonically isomorphic for $p \in U \cap V$;
- f_0 induces an isomorphism $U \xrightarrow{\sim} V$.

Lemma 8.3. *Let $f_0\colon X_0 \dashrightarrow X_0$ be a birational transformation of a reducible symplectic manifold X_0, and let X be a finite étale cover as in Beauville-Bogomolov decomposition theorem.*
Then, after possibly replacing X by a finite étale cover and f_0 by its iterate f_0^2, f_0 induces an automorphism $f\colon X \to X$. Furthermore, if $X = S_1 \times S_2$, where S_1 and S_2 are either $K3$ surfaces or two-dimensional tori, then $f = (f_1, f_2)$ for some automorphisms $f_1 \in \operatorname{Aut}(S_1)$, $f_2 \in \operatorname{Aut}(S_2)$.

Proof. Since f_0 is a pseudo-automorphism, by Lemma 11.2, there exists a finite étale cover $X' \to X$ such that f_0 lifts to a pseudo-automorphism $f\colon X' \dashrightarrow X'$; since $K3$ surfaces are simply connected, the finite cover $X' \to X$ is either a finite étale cover of a four-dimensional torus, or $X' = T' \times S \to T \times S$, where $T' \to T$ is a finite étale cover of a torus and S is a $K3$ surface. In particular, X' is still a four-dimensional torus or a product of tori and $K3$ surfaces. From now on, assume that $X = X'$.

Now let us prove that f is an automorphism. If X is a four-dimensional torus, then by Lemma 1.25 all birational transformations of X are automorphisms; thus we can suppose that

$$X = S_1 \times S_2$$

where S_1 is a $K3$ surface and S_2 is either a $K3$ surface or a two-dimensional torus.
By Kunneth formula

$$H^2(X,\mathbb{C}) \cong \bigoplus_{i=0}^{2} H^i(S_1,\mathbb{C}) \otimes H^{2-i}(S_2,\mathbb{C}) \cong H^2(S_1,\mathbb{C}) \oplus H^2(S_2,\mathbb{C})$$

because $H^1(S_1,\mathbb{C}) = 0$. By writing the terms of the Hodge decomposition and identifying $H^{p,0}(Y)$ with $H^0(Y,\Omega^p_Y)$ for $Y = X, S_1, S_2$, we obtain

$$H^0(X,\Omega^2_X) \cong H^0(S_1,\Omega^2_{S_1}) \oplus H^0(S_2,\Omega^2_{S_2}).$$

In other words, if σ_1 (resp. σ_2) is a symplectic form on S_1 (resp. S_2), then

$$H^0(X,\Omega^2_X) = \mathbb{C}\sigma_1 \oplus \mathbb{C}\sigma_2;$$

here, by abuse of notation, we denote again by σ_i the pull-back $\pi_i^*\sigma_i$ by the canonical projection $\pi_i \colon X \to S_i$.

Now $f^*\sigma_1$ can be extended to a holomorphic two-form $a\sigma_1 + b\sigma_2$ on X; since $(f^*\sigma_1)^2 = 2ab\,\sigma_1 \wedge \sigma_2 = 0$, we must have $f^*\sigma_1 \in \mathbb{C}\sigma_i$ for $i = 1$ or 2. After replacing f by f^2, we can suppose that $f^*\sigma_1 \in \mathbb{C}\sigma_1$; then, $f^*\sigma_2 \in \mathbb{C}\sigma_2$.

Pick local coordinates x_1, y_1 on S_1 (resp. x_2, y_2 on S_2) such that $\sigma_i = dx_i \wedge dy_i$ for $i = 1, 2$; such coordinates exist by Darboux theorem (Theorem 2.4). In such coordinates, we can write

$$f(x_1,y_1,x_2,y_2) = (g_1(\mathbf{x},\mathbf{y}), h_1(\mathbf{x},\mathbf{y}), g_2(\mathbf{x},\mathbf{y}), h_2(\mathbf{x},\mathbf{y})).$$

Now

$$f^*\sigma_1 = f^*(dx_1 \wedge dy_1) = dg_1 \wedge dh_1 = c\,dx_1 \wedge dy_1,$$

thus g_1 and h_1 only depend on x_1, y_1; in the same way one shows that g_2 and h_2 only depend on x_2, y_2. This means that

$$f = (f_1, f_2) \qquad \text{for some } f_1 \in \mathrm{Bir}(S_1), f_2 \in \mathrm{Bir}(S_2).$$

We conclude by recalling that, since S_1, S_2 are $K3$ surfaces or two-dimensional tori, $\mathrm{Bir}(S_i) = \mathrm{Aut}(S_i)$ □

From now on we will suppose that $X = S_1 \times S_2$, where S_1 and S_2 are either a $K3$ surfaces or two-dimensional tori, and that

$$f = (f_1, f_2) \colon X \to X$$

is an automorphism such that $f_1 \colon S_1 \to S_1, f_2 \colon S_2 \to S_2$ are both loxodromic.
Recall that, by Corollary 3.6, if f_i preserves a foliation on S_i and S_i is projective, then S_i is a torus (resp. a Kummer surface) and f_i is (resp. is induced by) a loxodromic linear automorphism. Also recall that the hyperbolic periodic points for f_i are Zariski-dense in S_i.

Lemma 8.4. *Let $f \colon X \to X$ be as above and let $\phi \colon X \dashrightarrow \mathbb{C}$ be a meromorphic function such that $\phi \circ f = c\phi$ for some $c \in \mathbb{C}^*$. Then ϕ is constant.*

Proof. Suppose by contradiction that ϕ is not constant; the connected components of the fibres of ϕ are then the leaves of an f-invariant codimension one foliation $\mathcal{F}$. Let $p \in S_1$ be a periodic point; then the intersection of leaves of $\mathcal{F}$ with $\{p\} \times S_2$ defines an f_2-periodic foliation $\mathcal{F}_2$ on S_2. Furthermore, since the leaves of $\mathcal{F}$ are compact, then so are the leaves of $\mathcal{F}_2$; therefore, by Ghys-Jouanolou's Theorem (see [Jou78, Ghy00]), the leaves of $\mathcal{F}_2$ are the fibres of an f_2-equivariant fibration $\pi\colon S_2 \dashrightarrow C$. By Theorem 1.30 π is trivial; in other words, $\{p\} \times S_2$ is contained in a leaf of $\mathcal{F}_2$, i.e. ϕ is constantly equal to α on $\{p\} \times S_2$.

In the same way one shows that ϕ is constantly equal to α_q on each $S_1 \times \{q\}$ for all f_1-periodic points $q \in S_1$; since for $(p, q) \in S_1 \times \{q\} \cap \{p\} \times S_2$, the constant α_q does not actually depend on q. Therefore ϕ is constant on the subset $\{f_1 - \text{periodic points}\} \times S_2$, which is Zarsiki-dense by Theorem 1.30. Hence ϕ is constant as claimed. □

8.2 Invariant foliations: general case and special examples

From now on assume that X is projective. By Lemma 8.2 and 8.3, the dynamics of birational transformations of reducible symplectic fourfolds can be reduced to linear dynamics on a torus or to the dynamics of a product automorphism

$$f = (f_1, f_2)\colon S^1 \times S^2 \to S^1 \times S^2,$$

where S_i is either a $K3$ surface or a complex torus and $f_i \in \mathrm{Aut}(S_i)$ for $i = 1, 2$. In what follows we will restrict to the case where f_1 and f_2 are both loxodromic, and we will give a complete classification of foliations preserved by such an automorphism. Remark that the foliations whose leaves are the fibres of one of the two natural projections $\pi_i\colon S_1 \times S_2 \to S_i$, $i = 1, 2$, are automatically f-invariant.

Let us first describe some easy examples coming from invariant foliations on one of the factors S_i.

Example 8.5. Let $f = (f_1, f_2)\colon S_1 \times S_2 \to S_1 \times S_2$ be as above. Suppose that f_1, f_2 are both loxodromic, and that f_1 preserves a foliation on S_1; in this case, by Corollary 3.6, (S_1, f_1) is of Kummer type and f_1 preserves exactly two foliations $\mathcal{F}_1^+$ and $\mathcal{F}_1^-$.
Then the following are f-invariant foliations on X:

- the one-dimensional foliations $\mathcal{F} = \mathcal{F}_1^{\pm} \times \mathcal{P}_2$, where $\mathcal{P}_2$ denotes the foliations by points on S_2;
- if (S_2, f_2) is also of Kummer type, the two-dimensional foliations $\mathcal{F} = \mathcal{F}_1^{\epsilon_1} \times \mathcal{F}_2^{\epsilon_2}$, where $\epsilon_i \in \{+, -\}$ and $\mathcal{F}_2^{\pm}$ are the stable/unstable foliations for f_2;
- the three-dimensional foliations $\mathcal{F} = \mathcal{F}_1^{\pm} \times \{S_2\}$, where $\{S_2\}$ denotes the single leaf foliation on S_2.

Before describing some additional examples, let us fix a notation which we will use in the rest of this chapter.

Remark 8.6. Kummer examples admit affine structures at the general point such that the invariant foliations locally correspond to coordinate foliations: such a structure is actually global for a torus, and is described in §3.1 for Kummer surfaces.
If $g\colon S \to S$ is a Kummer example, then in affine local coordinates f is linear and, after maybe replacing it by f^2, it is given by the matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

where $\lambda = \pm\sqrt{\lambda_1(f)}$.

In the special case where both factors admit invariant foliations and $\lambda_1(f_1) = \lambda_1(f_2)$, we get some additional examples.

Example 8.7. Let $f = (f_1, f_2)\colon S^1 \times S^2 \to S^1 \times S^2$ be as above. Suppose that f_1, f_2 are both loxodromic, and that both f_i preserve a foliation on S_i; in this case, by Corollary 3.6, (S_i, f_i) is of Kummer type and f_i preserves exactly two foliations $\mathcal{F}_i^+$ and $\mathcal{F}_i^-$.

Assume that $\lambda_1(f_1) = \lambda_1(f_2) = \lambda^2$. We will describe some families of f-invariant foliations.

Consider affine structures at the general point of each factor S_i, as described in Remark 8.6: there exist local coordinates x_i, y_i of $S_i \cong \mathbb{C}^2/\Lambda_i$ around the general point for which f is linear and, up to replacing f by f^2,

$$f(x_1, y_1, x_2, y_2) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}.$$

Then the following families of foliations are f-invariant:

- in dimension 1:

$$T\mathcal{F}_t^x = \mathrm{Span}\left(t_1 \frac{\partial}{\partial x_1} + t_2 \frac{\partial}{\partial x_2}\right), \quad T\mathcal{F}_t^y = \mathrm{Span}\left(t_1 \frac{\partial}{\partial y_1} + t_2 \frac{\partial}{\partial y_2}\right),$$
$$t = [t_1 : t_2] \in \mathbb{P}^1.$$

- in dimension 2:

$$T\mathcal{F}_{t,s} = \mathrm{Span}\left(t_1 \frac{\partial}{\partial x_1} + t_2 \frac{\partial}{\partial x_2}; s_1 \frac{\partial}{\partial y_1} + s_2 \frac{\partial}{\partial y_2}\right),$$
$$(t, s) = ([t_1 : t_2], [s_1 : s_2]) \in \mathbb{P}^1 \times \mathbb{P}^1.$$

- in dimension 3:

$$\widehat{\mathcal{F}}_t^x = \{t_1 x_1 + t_2 x_2 = 0\}, \quad \widehat{\mathcal{F}}_t^y = \{t_1 y_1 + t_2 y_2 = 0\},$$
$$t = [t_1 : t_2] \in \mathbb{P}^1.$$

Theorem 8.8. *Let X_0 be a reducible symplectic manifold of dimension 4 which is not a torus, and let*

$$X = S_1 \times S_2 \to X_0$$

be a finite cover as in Beauville-Bogomolov decomposition theorem; then, by Lemma 8.2, we can suppose that S_1 and S_2 are either $K3$ surfaces of two-dimensional complex tori (and at least one of them is a $K3$ surface).
Let $f_0\colon X_0 \dashrightarrow X_0$ be a birational transformation; by Lemma 8.3, up to replacing X by a finite étale cover and f by f^2, we can suppose that there exist $f_1 \in \mathrm{Aut}(S_1)$, $f_2 \in \mathrm{Aut}(S_2)$ such that

$$f = (f_1, f_2)\colon S_1 \times S_2 \to S_1 \times S_2.$$

Suppose that f_1 and f_2 are both loxodromic and that f_0 preserves a singular foliation $\mathcal{F}_0$ on X_0; denote by $\mathcal{F}$ the induced foliation on X and suppose that its leaves don't coincide with the fibres of one of the two natural projections $\pi_1\colon X \to S_1$, $\pi_2\colon X \to S_2$.

Then

1. *(S_1, f_1) or (S_2, f_2) is of Kummer type;*

2. *if furthermore one supposes that the two dynamical degrees $\lambda_1(f_1)$, $\lambda_1(f_2)$ are distinct or that (S_1, f_1) and (S_2, f_2) are not both of Kummer type, then the foliation comes from the stable or unstable foliation of either f_1 or f_2, as described in Example 8.5;*

3. *If both factors are of Kummer type and $\lambda_1(f_1) = \lambda_2(f_2)$, then either $\mathcal{F}$ is as above or $\mathcal{F}$ belongs to one of the families described in Example 8.7.*

We will split the proof as follows: the case of foliations of dimension 1 or 3 is treated in Lemma 8.9 and Lemma 8.10; the case of foliations of dimension 2 is treated in Proposition 8.11.

Recall that, by Theorem 1.31, the hyperbolic periodic points of f_i are Zariski-dense in S_i; therefore, the hyperbolic periodic points of f are Zariski-dense in X.

8.3 Invariant foliations of dimension 1 and 3

Throughout this section, let $X = S_1 \times S_2$, where S_1 is a $K3$ surface and S_2 is either a $K3$ surface or a two-dimensional torus. Let

$$f = (f_1, f_2) \colon X \to X$$

be an automorphism such that $f_1 \colon S_1 \to S_1$, $f_2 \colon S_2 \to S_2$ are both loxodromic.

Lemma 8.9. *Suppose that f preserves a foliation $\mathcal{F}$ of dimension 1 (resp. 3); then*

1. *(S_1, f_1) or (S_2, f_2) is of Kummer type;*

2. *the projection of $\mathcal{F}$ on (resp. the intersection of $\mathcal{F}$ with) the fibres of the two canonical projections is either trivial or the stable/unstable foliation; furthermore, if one is stable, then the other cannot be unstable and vice-versa;*

3. *if only one of the two factors is of Kummer type or if $\lambda_1(f_1) \neq \lambda_1(f_2)$, then $\mathcal{F}$ comes from the stable or unstable foliation of a Kummer or torus factor, as described in Exaxmple 8.5.*

Proof. Suppose first that $\dim \mathcal{F} = 1$. Let $p \in S_1$ be a hyperbolic periodic point for f_1 such that $\{p\} \times S_2$ is not contained in the singular locus of $\mathcal{F}$ (such a point exists by Zariski-density). Consider the projection $\mathcal{F}_{2,p}$ of $\mathcal{F}$ on $\{p\} \times S_2$ with respect to the decomposition

$$TX = \pi_1^* TS_1 \oplus \pi_2^* TS_2;$$

in other words, for $q \in S_2$ define

$$T\mathcal{F}_{2,p}(q) = (D\pi_1)_{(p,q)} T\mathcal{F}(p, q).$$

We have two cases:

- Either $\mathcal{F}_{2,p}$ is a non-trivial foliation, which is preserved by some iterate of f_2; then (S_2, f_2) is of Kummer type;

- or $\mathcal{F}_{2,p}$ is the foliation by points; if this happens for all hyperbolic periodic points of f_1, then the leaves of $\mathcal{F}$ are contained in the fibres of π_2. We can then repeat this construction with hyperbolic points for f_2 without obtaining trivial foliations, so that (S_1, f_1) is of Kummer type.

This shows that, if $\dim \mathcal{F} = 1$, then one of the two factors is of Kummer type.

Now let $\dim \mathcal{F} = 3$. Again, let $p \in S_1$ be a hyperbolic periodic point for f_1 such that $\{p\} \times S_2$ is not contained in the singular locus of $\mathcal{F}$. Consider the intersection $\mathcal{F}_{2,p}$ of $\mathcal{F}$ with $\{p\} \times S_2$; we have two cases.

- Either $\mathcal{F}_{2,p}$ is a non-trivial foliation, which is preserved by some iterate of f_2; then (S_2, f_2) is of Kummer type.

- Or $\mathcal{F}_{2,p}$ is the single leaf foliation; if this happens for all hyperbolic periodic points for f_1, then the leaves of $\mathcal{F}$ are unions of fibres of π_1. We can then repeat the construction with hyperbolic points for f_2 without obtaining trivial foliations, so that (S_1, f_1) is of Kummer type.

This shows that, if $\dim \mathcal{F} = 3$, then one of the two factors is of Kummer type.

Now suppose that only one of the two factors, say (S_1, f_1), is of Kummer type; then the above constructions involving periodic points for f_2 lead to trivial foliations on the fibres of π_1 (if $\dim \mathcal{F} = 1$) or π_2 (if $\dim \mathcal{F} = 3$). This means that only the cases described in Theorem 8.8 can occur.

Finally, suppose that both factors are of Kummer type and that $\lambda_1(f_1) \neq \lambda_1(f_2)$. Consider on both factors an affine structure at the general point as in Remark 8.6: there exist local coordinates x_i, y_i of S_i such that in these coordinates f is linear and, up to replacing f by f^2,

$$f(x_1, y_1, x_2, y_2) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1^{-1} & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_2^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix},$$

where $\alpha_i = \lambda_1(f_i)$.

Let $p = (p_1, p_2) \in S_1 \times S_2$ be a hyperbolic N-periodic point for f. Such points are dense thanks to the density of hyperbolic points for f_1 and f_2. The linear space $T_p\mathcal{F}$ is Df_p^N-invariant; but since all the eigenvalues of Df_p^N are distinct, the only invariant one-dimensional (resp. three-dimensional) subspaces of T_pX are $\mathrm{Span}(\partial/\partial x_i)$, $\mathrm{Span}(\partial/\partial y_i)$ (resp. $\{x_i = 0\}$, $\{y_i = 0\}$). By density, $\mathcal{F}$ is then a linear foliation on the torus $S_1 \times S_2$ coming from a foliation on one of the factors (as in Example 8.5); this concludes the proof. □

Lemma 8.10. *Suppose that (S_1, f_1) and (S_2, f_2) are both of Kummer type and that $\lambda_1(f_1) = \lambda_2(f_2) = \lambda$. Then the foliations of dimension one and three preserved by f are exactly the four families described in Example 8.7.*

Proof. If a factor S_i is a Kummer surface, we replace it by the torus which defines it; then we can suppose that both factors are complex tori, and that f_1 and f_2 are loxodromic linear automorphisms. Let us fix global coordinates x_1, y_1, x_2, y_2 that diagonalize $f = (f_1, f_2)$.

Suppose first that $\dim \mathcal{F} = 1$; then $T\mathcal{F}$ is locally generated by a vector field

$$v = a_1(x,y)\frac{\partial}{\partial x_1} + b_1(x,y)\frac{\partial}{\partial y_1} + a_2(x,y)\frac{\partial}{\partial x_2} + b_2(x,y)\frac{\partial}{\partial y_2}.$$

By Lemma 8.9, the projection of $\mathcal{F}$ on the fibres of the two canonical projections is either trivial or the stable/unstable foliation; furthermore, if one is stable, then the other cannot be unstable and vice-versa. This implies that either $a_1 \equiv a_2 \equiv 0$ or $b_1 \equiv b_2 \equiv 0$; by symmetry, suppose that $b_1 \equiv b_2 \equiv 0$, so that $\mathcal{F}$ projects to the unstable or trivial foliation on each fibre of the natural projections. The vector field v is only well-defined modulo multiplication by (local) invertible functions, but the ratio a_1/a_2 is well defined and induces a meromorphic function

$$\begin{aligned} \phi\colon X &\dashrightarrow \mathbb{P}^1 \\ p=(p_1,p_2) &\mapsto \mathbb{P}(T_p\mathcal{F}) \subset \mathbb{P}(T_{p_1}\mathcal{F}_1^+ \oplus T_{p_2}\mathcal{F}_2^+) \cong \mathbb{P}^1 \end{aligned}$$

which is f-invariant. By Lemma 8.4, ϕ is constant; thus we can suppose that v is globally defined and linear, which concludes the proof for the case $\dim \mathcal{F} = 1$.

Now let $\dim \mathcal{F} = 3$; the proof is essentially the same as for $\dim \mathcal{F} = 1$, replacing the language of vector fields with the one of forms: $\mathcal{F}$ is locally defined as the kernel of a one-form

$$\omega = a_1(x,y)dx_1 + b_1(x,y)dy_1 + a_2(x,y)dx_2 + b_2(x,y)dy_2.$$

By Lemma 8.9, the intersection of $\mathcal{F}$ with the fibres of the two canonical projections is either trivial or the stable/unstable foliation; furthermore, if the intersection with the fibres of a projection is stable, (resp. unstable), then the intersection with the fibres of the other is trivial or stable (resp. trivial or unstable). This implies that either $a_1 \equiv a_2 \equiv 0$ or $b_1 \equiv b_2 \equiv 0$; by symmetry, suppose that $b_1 \equiv b_2 \equiv 0$. The form ω is only well-defined modulo multiplication by (local) invertible functions, but the ratio a_1/a_2 is well defined and induces a meromorphic function

$$\phi\colon X \dashrightarrow \mathbb{P}^1,$$

which is f-invariant. By Lemma 8.4, ϕ is constant; thus we can suppose that ω is globally defined and linear, which concludes the proof for the case $\dim \mathcal{F} = 3$. □

8.4 Invariant foliations of dimension 2

Throughout this section, let $X = S_1 \times S_2$, where each S_i is either a $K3$ surface or a two-dimensional torus (but not both are tori), and let

$$f = (f_1, f_2)\colon X \to X$$

be an automorphism such that $f_1\colon S_1 \to S_1$, $f_2\colon S_2 \to S_2$ are both loxodromic.
We suppose that

$(*)$ f preserves a foliation $\mathcal{F}$ of dimension 2 whose leaves do not coincide with the fibres of one of the two natural fibrations $\pi_1\colon X \to S_1$, $\pi_2\colon X \to S_2$.

The goal of this section is to prove the following result.

Proposition 8.11. *Both factors (S_1, f_1) and (S_2, f_2) are of Kummer type, and $\mathcal{F}$ comes essentially from a linear foliation on a torus. More accurately:*

- *if $\lambda_1(f_1) \neq \lambda_1(f_2)$, then $\mathcal{F}$ is the product of the stable or unstable foliation on S_1 by the stable or unstable foliation on S_2;*
- *if $\lambda_1(f_1) = \lambda_1(f_2)$, then $\mathcal{F}$ belongs to the family $\{\mathcal{F}_{t,s}\}_{t,s\in\mathbb{P}^1}$ described in Example 8.7.*

We say that $\mathcal{F}$ is *transverse* to π_1 (resp. π_2) if $T_p\mathcal{F} + T_pS_2 = T_pX$ (resp. if $T_p\mathcal{F} + T_pS_1 = T_pX$).

Lemma 8.12. *Suppose that $\mathcal{F}$ is nowhere transverse to π_1 or π_2. Then both factors are of Kummer type, and $\mathcal{F} = \mathcal{F}_1^{\epsilon_1} \times \mathcal{F}_2^{\epsilon_2}$, where $\mathcal{F}_i^{\pm}$ denotes the stable and unstable foliations for f_i on S_i and $\epsilon_i \in \{+,-\}$.*

Proof. Let $p_1 \in S_1$ be a general N-periodic point for f_1; since the leaves of $\mathcal{F}$ do not coincide with the fibres of π_1, the intersection of $\mathcal{F}$ with the fibre $\{p_1\} \times S_2 \cong S_2$ defines an f_2-periodic foliation on S_2. Therefore, S_2 is of Kummer type and $\mathcal{F} \cap (\{p_1\} \times S_2)$ is either the stable or the unstable foliation for f_2; symmetrically, one shows that also S_1 is of Kummer type.

Thus, $\mathcal{F} \cap (\{p_1\} \times S_2)$ is equal to the stable (or unstable) foliation for f_2 for all general f_1-periodic points p_1, thus for all $p_1 \in S_1$ by density; by symmetry, $\mathcal{F} \cap (S_1 \times \{p_2\})$ is equal to the stable (or unstable) foliation for all $p_2 \in S_2$. This shows that $\mathcal{F}$ is a product of stable or unstable foliations, which concludes the proof. □

Linear maps defined by $\mathcal{F}$

From now on we suppose that $\mathcal{F}$ is transverse to one of the two fibrations, say π_1, at general points of X.

If $p = (p_1, p_2) \in X$ is such that $\mathcal{F}$ is transverse to π_1, then $T_p\mathcal{F}$ can be interpreted as the graph of a linear map

$$\phi_p \colon T_{p_1}S_1 \to T_{p_2}S_2;$$

more explicitly, $\phi_p = \eta_p \circ \nu_p$, where $\nu_p \colon T_{p_1}S_1 \to T_p\mathcal{F}$ is the restriction to $T_{p_1}S_1$ of the projection with respect to the decomposition $T_pX = T_p\mathcal{F}_1 \oplus T_{p_2}S_2$ and $\eta_p \colon T_p\mathcal{F} \to T_{p_2}S_2$ is the restriction to $T_p\mathcal{F}$ of the projection with respect to the decomposition $T_pX = T_{p_1}S_1 \oplus T_{p_2}S_2$. Remark that ϕ_p is an isomorphism if and only if $\mathcal{F}$ is also transverse to π_2.
Of course, the same construction can be carried out if $\mathcal{F}$ is transverse to π_2 at p; we obtain then a linear map

$$\psi_p \colon T_{p_2}S_2 \to T_{p_1}S_1,$$

which coincides with ϕ_p^{-1} if $\mathcal{F}$ is transverse to both π_1 and π_2.

Let σ_1 (resp. σ_2) be a symplectic form on S_1 (resp. S_2); the isomorphisms ϕ_p act on forms by pull-back, and thus we can define a two-form $\phi_p^*(\sigma_2(p_2)) \in \bigwedge^2 T^*_{p_1}S_1 = \Omega^2_{S_1,p_1}$. Since $\dim \Omega^2_{S_1,p_1} = 1$, there exists a constant $\xi(p) \in \mathbb{C}$ (the "determinant" of ϕ_p) such that

$$\phi_p^*(\sigma_2(p_2)) = \xi(p)\sigma_1(p_1).$$

Lemma 8.13. *Suppose that $\mathcal{F}$ is transverse to π_1 at some point of X. Then $p \mapsto \xi(p)$ is a constant function; in particular,*

- *if $\xi \equiv 0$, then $\mathcal{F}$ is nowhere transverse to π_2;*
- *if $\xi \equiv c \neq 0$, then $\mathcal{F}$ is transverse to both π_1 and π_2 outside a hypersurface D, and to neither π_1 or π_2 along D.*

Proof. Since the foliation $\mathcal{F}$ is holomorphic, the function

$$p \mapsto \xi(p)$$

is meromorphic on X. Let $\chi_1, \chi_2 \in \mathbb{C}^*$ such that $f_1^*\sigma_1 = \chi_1\sigma_1$, $f_2^*\sigma_2 = \chi_2\sigma_2$. Then a linear algebra computation shows that

$$\xi \circ f = \frac{\chi_2}{\chi_1}\xi.$$

Thus, by Lemma 8.4, ξ is constant.

Assume that $\xi \equiv 0$. If $\mathcal{F}$ and π_2 were transverse at some point, then they would be transverse at the general point of X. Since $\mathcal{F}$ and π_1 are transverse at the general point of X, that would imply that $\mathcal{F}$ is transverse to both natural projections at some point p; hence $\xi(p) \neq 0$, contradicting $\xi \equiv 0$.

Now suppose that $\xi \equiv c \neq 0$. If $\mathcal{F}$ were transverse to exactly one of the two projections π_i at a point $p \in X$, then exactly one among ϕ_p and ψ_p would be well-defined at p, implying that $\xi(p) = 0$ or $\xi(p) = \infty$. This contradicts the assumption $\xi \equiv c \neq 0$.
In order to conclude, we only need to prove that the tangency locus between $\mathcal{F}$ and π_1 (which coincides with the tangency locus between $\mathcal{F}$ and π_2) is a hypersurface. This can be seen easily in local coordinates: in a small enough neighborhood of each point of X, the fibres of π_1 (resp. the leaves of $\mathcal{F}$) are defined by the kernel of a 2-form ω_1 (resp. $\omega_{\mathcal{F}}$); the tangency locus is then

$$D = \{p \in X \mid (\omega_1 \wedge \omega_{\mathcal{F}})(p) = 0\},$$

which is the zero locus of a differential form of maximal degree. Hence, D is a hypersurface; this concludes the proof. □

First case: $\xi = 0$

Suppose that $\xi = 0$; then $\mathcal{F}$ is nowhere transverse to π_2.

Lemma 8.14. *If $\xi = 0$, then both factors are of Kummer type.*

Proof. Let $p = (p_1, p_2)$ be a point where $\mathcal{F}$ is transverse to π_1; since $\mathcal{F}$ is nowhere transverse to π_2, the image of $\phi_p \colon T_{p_1}S_1 \to T_{p_2}S_2$ has dimension 0 or 1. If $\dim \operatorname{Im} \phi_p = 0$ for general p, then the leaves of $\mathcal{F}$ coincide with the fibres of π_2, a contradiction. Therefore, for general p, $\operatorname{Im} \phi_p$ has dimension 1. Fix a general N-periodic point $p_1 \in S_1$ for f_1 (recall that hyperbolic periodic points are Zariski-dense in S_1); then the meromorphic distribution of lines $p_2 \mapsto \operatorname{Im} \phi_{(p_1,p_2)}$ on $\{p_1\} \times S_2$, which is automatically integrable, defines an f_2-periodic foliation on S_2. Thus, by Corollary 3.6, the factor (S_2, f_2) is of Kummer type.

Now let $p_2 \in S_2$ be a general N-periodic point for f_2. Since ϕ_p has rank 1 at the general point of X, the map

$$p_1 \in S_1 \mapsto \ker(\phi_{(p_1,p_2)}) \subset T_{p_1}S_1$$

defines a foliation on S_1 which is invariant by some iterate of f_1. Hence, by Corollary 3.6, the factor (S_1, f_1) is also of Kummer type. □

Second case: $\xi \neq 0$

Suppose that $\xi \neq 0$; by Lemma 8.13, $\mathcal{F}$ is transverse to π_1 at p if and only if it is transverse to π_2 at p. Let

$$D = \{p = (p_1, p_2) \in X \mid \dim(T_p\mathcal{F} + T_{p_2}S_2) \leq 3\}$$

be the divisor of tangency between $\mathcal{F}$ and π_1; by the above discussion, D is also equal to the divisor of tangency between $\mathcal{F}$ and π_2:

$$D = \{p = (p_1, p_2) \in X \mid \dim(T_p\mathcal{F} + T_{p_1}S_1) \leq 3\}.$$

Let us deal with the case where $D = \emptyset$ first.

Definition 8.15. *Let $\mathcal{F}$ be a smooth foliation on a manifold M; the* intrinsic topology *on a leaf L of $\mathcal{F}$ is the one induced by the distance*

$$d_L(x, y) = \inf\{l(\gamma) \mid \gamma : [0, 1] \to L \text{ continuous }, \gamma(0) = x, \gamma(1) = y\},$$

where l is the length of γ with respect to any fixed Riemannian metric on M.

Corollary 8.16. *If $\xi \neq 0$, then the tangency divisor D is non-trivial.*

Proof. Assume without loss of generality that S_1 is a $K3$ surface, and suppose by contradiction that $D = \emptyset$. Then the natural composition

$$T\mathcal{F} \hookrightarrow TX \to \pi_1^* TS_1$$

is a sheaf isomorphism. Now

$$\mathrm{Hom}(T\mathcal{F}, TX) \cong \mathrm{Hom}(\pi_1^* TS_1, TX) \cong$$
$$\mathrm{Hom}(\pi_1^* TS_1, \pi_1^* TS_1) \oplus \mathrm{Hom}(\pi_1^* TS_1, \pi_2^* TS_2)$$

and

$$\mathrm{Hom}(\pi_1^* TS_1, \pi_2^* TS_2)|_{S_1 \times \{p\}} \cong \mathrm{Hom}_{S_1}(TS_1, \mathcal{O}_{S_1}^{\oplus 2}) \cong H^0(S_1, \Omega_{S_1}^{\oplus 2}) = 0$$

because S_1 is a $K3$ surface. Therefore $T\mathcal{F} \cong T_{X/S_2}$, meaning that the leaves of $\mathcal{F}$ coincide with the fibres of π_2. This however contradicts the assumption that $\mathcal{F}$ is non-trivial. $\square$

Lemma 8.17. *If $\xi \neq 0$, then both factors are of Kummer type.*

Proof. By Corollary 8.16, the tangency divisor D is nonempty; remark that D is f-invariant. Consider the intersection $D \cap (\{p_1\} \times S_2)$ as p_1 runs through all the periodic points for f_1 (which are Zariski-dense in S_1). If $\{p_1\} \times S_2 \subset D$ for all f_1-periodic points $p_1 \in S_1$, then D would be Zariski-dense in X, a contradiction. Therefore, some $\{p_1\} \times S_2$ is not contained in D. In this case $(\{p_1\} \times S_2) \cap D$ defines an f_2-periodic divisor D_2 on S_2; since there is a finite number of such divisors, by Zariski-density of the f_1-periodic points we get that there is an f_2-periodic (possibly empty) divisor $D_2 \subset S_2$ such that

$$(\{p_1\} \times S_2) \cap D = \{p_1\} \times D_2 \qquad \forall p_1 \in S_1 \text{ such that } \{p_1\} \times S_2 \not\subseteq D.$$

By symmetry, there exists an f_1-periodic divisor D_1 on S_1 such that

$$(S_1 \times \{p_2\}) \cap D = D_1 \times \{p_2\} \qquad \forall p_2 \in S_2 \text{ such that } S_1 \times \{p_2\} \not\subseteq D.$$

In other words

$$D = (D_1 \times S_2) \cup (S_1 \times D_2).$$

Since D is nonempty, we can suppose without loss of generality that $D_1 \neq \emptyset$; also, after replacing f by some iterate, we can suppose that every component of D_1 is f_1-invariant. Now, loxodromic transformations of tori do not admit invariant divisors; hence, S_1 is a $K3$ surface, and D_1 is a disjoint union of smooth rational curves: indeed, each component of D has negative self-intersection by Corollary 1.23, and by adjunction formula negative curves on a $K3$ surface are smooth rational curves with self-intersection -2. Since all automorphisms of $\mathbb{P}^1$ admit a fixed point, there exists a point $p_1 \in D_1$ which is fixed by f_1.

Consider the intersection $\mathcal{F}_{2,p_1} = (\{p_1\} \times S_2) \cap \mathcal{F}$: since $\{p_1\} \times S_2 \subset D$, $\mathcal{F}_{2,p_1}$ is not the foliation by points; however, if $\mathcal{F}_{2,p_1}$ were the single leaf foliation, then $\{p_1\} \times S_2$ would be a leaf of $\mathcal{F}$, hence it would be transverse to π_2, again contradicting the fact that $\{p_1\} \times S_2 \subset D$. Thus $\mathcal{F}_{2,p_1}$ is a one-dimensional foliation on $\{p_1\} \times S_2 \cong S_2$, which is f_2-invariant. This shows that the factor (S_2, f_2) is of Kummer type.

Now fix an f_2-hyperbolic periodic point p_2 such that $S_1 \times \{p_2\} \not\subseteq D$, and let $v_+ \in T_{p_2}S_2$ be a non-zero vector in the unstable direction; at a general point $p \in S_1 \times \{p_2\}$ the linear map

$$\phi_p \colon T_{p_1}S_1 \to T_{p_2}S_2$$

is well-defined and invertible. Thus the map

$$p_1 \in S_1 \mapsto \phi_p^{-1}(v_+) \in T_{p_1}S_1$$

defines a meromorphic vector field v' on S_1 which is f_1-invariant. The trajectories along v' define an f_1-invariant foliation on S_1; thus the factor (S_1, f_1) is also of Kummer type. $\square$

Proof of Proposition 8.11

We prove here Proposition 8.11. Let $X = S_1 \times S_2$, where S_1 and S_2 are either $K3$ surfaces or two-dimensional tori (but at least one of them is a $K3$ surface). Let

$$f = (f_1, f_2)\colon X \to X$$

be an automorphism such that $f_1\colon S_1 \to S_1$ and $f_2\colon S_2 \to S_2$ are both loxodromic.
We suppose that f preserves a foliation $\mathcal{F}$ of dimension 2 whose leaves do not coincide with the fibres of one of the two natural fibrations $\pi_1\colon X \to S_1$, $\pi_2\colon X \to S_2$.

By Lemma 8.12, if $\mathcal{F}$ is never transverse to π_1 or π_2, then both factors are linear and $\mathcal{F}$ is the product of the stable or unstable foliation on each factor.

Let us suppose now that $\mathcal{F}$ is transverse to π_1 or π_2 at a general point of X (suppose by symmetry that $\mathcal{F}$ is transverse to π_1). Then we can define the determinant function $\xi\colon X \dashrightarrow \mathbb{C}$ as in §8.4, which is a meromorphic f-equivariant function, hence constant by Lemma 8.4. By Lemma 8.14 and 8.17, in any case both factors are of Kummer type.

Now, by replacing a Kummer surface by the torus which defines it, we can suppose that both factors are tori and that each f_i is a linear automorphism of a torus. Therefore, there exist global (linear) coordinates x_1, y_1 on S_1 and x_2, y_2 on S_2 such that in these coordinates f is given by the matrix

$$A = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1^{-1} & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_2^{-1} \end{pmatrix},$$

where $\alpha_i = \lambda_1(f_i)$.

If $\alpha_1 \neq \alpha_2$, then for all $p \in X$

$$T_p\mathcal{F} = \mathrm{Span}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial w_2}\right) \qquad \text{for some } z, w \in \{x, y\}:$$

indeed, if p is a periodic point, $T_p\mathcal{F}$ is A^N-invariant for some $N > 0$, hence $T_p\mathcal{F}$ is the span of some vectors of the basis; by density of periodic points, this is true for all points of p, and the vectors of the basis spanning $T_p\mathcal{F}$ cannot depend on the chosen point. Since leaves of $\mathcal{F}$ do not coincide with the fibres of one of the two natural projections, one must choose a vector from the basis of $T_{p_1}S_1$ and one from the basis of $T_{p_2}S_2$. This means that

$$\mathcal{F} = \mathcal{F}_1^{\epsilon_1} \times \mathcal{F}_2^{\epsilon_2}, \qquad \epsilon_1, \epsilon_2 \in \{+, -\}.$$

Now suppose that $\alpha_1 = \alpha_2$; then, since TX is trivial, the correspondence $p \mapsto T_p\mathcal{F} \subset T_pX$ defines a meromorphic map

$$\Phi\colon X \dashrightarrow \mathrm{Gr}(2, 4),$$

where $\mathrm{Gr}(2, 4)$ denotes the Grassmannian of two-planes in $\mathbb{C}^4$. If p is a periodic point, then

$$T_p\mathcal{F} = \mathrm{Span}\left(t_1\frac{\partial}{\partial x_1} + t_2\frac{\partial}{\partial x_2}, s_1\frac{\partial}{\partial y_1} + s_2\frac{\partial}{\partial y_2}\right) \qquad \text{for some } t_1, t_2, s_1, s_2 \in \mathbb{C};$$

since such points are dense in X, the same is true for all points of X where Φ is defined. In other words

$$\mathrm{Im}\,\Phi \subset \mathrm{Gr}(1, 2)_x \times \mathrm{Gr}(1, 2)_y \cong \mathbb{P}^1 \times \mathbb{P}^1,$$

where $\mathrm{Gr}(1, 2)_x$ (resp. $\mathrm{Gr}(1, 2)_y$) is the Grassmannian of lines in $\mathrm{Span}(\partial_{x_1}, \partial_{x_2})$ (resp. in $\mathrm{Span}(\partial_{y_1}, \partial_{y_2})$).

Furthermore, Φ is f-equivariant, with f acting on $\mathrm{Gr}(2,4)$ as the differential Df (which does not depend on the point). Since Df acts as the identity on $\mathrm{Gr}(1,2)_x \times \mathrm{Gr}(1,2)_y$, by projection on either of the two factors we obtain an f-invariant meromorphic function $X \dashrightarrow \mathbb{P}^1$, which is constant by Lemma 8.4.
Therefore, there exist global constants $t = [t_1 : t_2] \in \mathbb{P}^1, s = [s_1 : s_2] \in \mathbb{P}^1$ such that

$$T_p\mathcal{F} = \mathrm{Span}\left(t_1\frac{\partial}{\partial x_1} + t_2\frac{\partial}{\partial x_2}, s_1\frac{\partial}{\partial y_1} + s_2\frac{\partial}{\partial y_2}\right) \qquad \forall p \in X;$$

thus $\mathcal{F} = \mathcal{F}_{t,s}$ as described in Example 8.7.

This concludes the proof of the Proposition. $\square$

Chapter 9

Invariant geometric structures

Informally, a geometric structure on a differential variety M is an additional geometric information attached to M; one can for example think about vector fields, differential forms, foliations, Riemannian metrics, connections, projective structures...
Around this vague idea, several rigorous and fruitful mathematical theories have been developed. Here we will restrict our attention to (G, X)-manifolds in the sense of Ehresmann and to a generalization of this concept provided by Gromov's definition of A-structures.

The goal of this section is to introduce the vocabulary and the precise definition of a geometric structure, and to prove the following homogeneity result.

Theorem 9.1. *Let (X, σ) be an irreducible symplectic manifold and let $f\colon X \dashrightarrow X$ be a loxodromic birational transformation preserving two generically transverse Lagrangian distributions $\mathcal{F}_1, \mathcal{F}_2$; let Φ (resp. $[\Phi]$) be the geometric structure defined by the symplectic form σ (resp. by the symplectic form σ modulo multiplication by an element of $\mathbb{C}^*$) and the two distributions $\mathcal{F}_1, \mathcal{F}_2$. Then there exists a closed subvariety $S' \subset X$ with positive codimension (the "exceptional locus" of $[\Phi]$) such that*

- *$[\Phi]$ is locally homogeneous on $X \setminus S'$;*
- *the local isometries of $[\Phi]$ on $X \setminus S'$ are uniquely determined by their Jacobian and their second order partial derivatives.*

If furthermore X is projective, there exists a closed subvariety $S \subset X$ with positive codimension (the "exceptional locus" of Φ) such that

- *Φ is locally homogeneous on $X \setminus S$;*
- *the local isometries of Φ on $X \setminus S$ are uniquely determined by their Jacobian.*

Remark 9.2. We will see that the structures Φ and $[\Phi]$ degenerate along a non-empty divisor; therefore, the exceptional loci of Theorem 9.1 satisfy $\operatorname{codim} S = \operatorname{codim} S' = 1$.

Furthermore, as an application of (G, X)-orbifolds, we will classify loxodromic automorphisms of $K3$ surfaces which preserve a (degenerate) affine structure; see Theorem 9.11.

9.1 (G, X)-manifolds and (G, X)-orbifolds

For this paragraph we refer to [Thu97, Chapter 3], [Rat06, Chapter 8].

According to Klein's Erlangen program, geometry is the study of the properties of a space which are invariant under a group of transformations. Hence, a geometry in Klein's sense is a pair (G, X) where X is a manifold and G is a (Lie) group acting transitively on X; for example, euclidean geometry is defined by the pair $(\mathbb{R}^n, \mathrm{Isom}(\mathbb{R}^n))$, where $\mathrm{Isom}(\mathbb{R}^n)$ denotes the group of isometries of $\mathbb{R}^n$, whereas spherical geometry is defined by $(\mathbb{S}^n, SO_{n+1})$.

The study of manifolds which are locally modelled on such a geometry was started by Ehresmann [Ehr36]; see the introduction of [Gol88] and [Rat06, §8.7] for more historical remarks.

Definition 9.3. *Let X be a manifold and let $G \leq \mathrm{Diff}(X)$ be a group acting analytically, i.e. if two elements of G coincide on a non-empty open subset, they are the same.*
A (G, X)-manifold is a manifold admitting an atlas with values in X and change of charts in G: more explicitly, M is a (X, G)-manifold if and only if there exists an open cover $\{U_i\}_{i\in I}$ of M and homeomorphisms $\phi_i \colon U_i \to V_i \subset X$ such that $(\phi_j \circ \phi_i^{-1}) \colon \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is the restriction of an element of G for all $i, j \in I$.

Example 9.4. The following are all examples of (G, X)-structures.

- A euclidean (resp. spherical, resp. hyperbolic) structure is a $(\mathrm{Isom}(\mathbb{R}^n), \mathbb{R}^n)$ (resp. $(SO_{n+1}(\mathbb{R}), \mathbb{S}^n)$, resp. $(SO(1, n), \mathbb{H}^n)$) structure.
- An affine structure is a $(\mathrm{Aff}(\mathbb{R}^n), \mathbb{R}^n)$-structure; a projective structure is a $(\mathrm{PGL}_{n+1}(\mathbb{R}), \mathbb{P}^n_{\mathbb{R}})$-structure.

As Example 9.4 shows, in most cases we will require G to act transitively on X, which forces the structure to be locally homogeneous(see §9.2.2 for a precise definition). Structures which are not a priori locally homogeneous do not fit well in the theory of (G, X)-structures: it is the case for general Riemannian or holomorphic metrics.
Even if the structure we consider is locally homogeneous, it may occur that the changes of charts do not naturally sit in a global group acting on a model manifold X. For this reason, it is sometimes more convenient to allow G to be a pseudo-group of germs of diffeomorphisms of X (see [Thu97]). This weaker definition allows to construct more interesting examples; however, some of the key features of the theory of (G, X)-manifolds, namely the construction of a developing map and of a holonomy representation, do not generalize to this case.

Example 9.5. If we allow G to be a pseudo-group of germs of diffeomorphims, we obtain the following additional examples.

- A manifold with a smooth (real) foliation of dimension k is a $(G, \mathbb{R}^n)$-manifold, where G is the pseudo-group of local diffeomorphisms of $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ of the form $(\mathbf{x}, \mathbf{y}) \mapsto (f(\mathbf{x}, \mathbf{y}), g(\mathbf{y}))$.
- A vector field without zeros is a $(G, \mathbb{R}^n)$-structure, where G is the pseudo-group of germs of diffeomorphisms of $\mathbb{R}^n$ preserving the vector field $\partial/\partial x_1$.
- A manifold with a volume form is a $(G, \mathbb{R}^n)$-manifold, where G is the pseudo-group of germs of diffeomorphisms preserving the Lebesgue measure on $\mathbb{R}^n$.
- A complex structure on a manifold M is a $(G, \mathbb{C}^n)$-structure ($n = \dim_{\mathbb{R}} M/2$), where G is the pseudo-group of germs of biholomorphic maps between open sets of $\mathbb{C}^n$.
- Recall that, by Darboux Theorem (Theorem 2.4), if σ is a holomorphic symplectic form on a complex manifold M, locally at each point there are local coordinates $p_1, \ldots, p_n, q_1, \ldots, q_n$ such that, in these coordinates
$$\sigma = \sum_{i=1}^{n} dp_i \wedge dq_i;$$

hence, a holomorphic symplectic structure is a $(G, \mathbb{C}^{2n})$-structure, where G is the pseudo-group of local biholomorphisms of open subsets of $\mathbb{C}^{2n}$ preserving the standard symplectic form.

9.1.1 Developing map and holonomy

In this section, denote by M a (G, X)-manifold; let $\pi\colon \widetilde{M} \to M$ be the universal cover of M. Remark that on $\widetilde{M}$ one can define a (G, X)-structure induced by that of M.

Intuitively, the developing map "unrolls" $\widetilde{M}$ over X using the local charts. The local construction goes as follows: let $\phi_i\colon U_i \to X$ and $\phi_j\colon U_j \to X$ be two local charts of $\widetilde{M}$ with $U_i \cap U_j \neq \emptyset$ and $U_i \cap U_j$ connected, and let g be the element of G whose restriction to $\phi_i(U_i \cap U_j)$ is equal to $\phi_j \circ \phi_i^{-1}$. Then the two maps ϕ_i and $g^{-1} \circ \phi_j$ glue together to a map $U_i \cup U_j \to X$. The construction can be iterated; one can check (see [Rat06, §8.4]) that the only inconsistencies that can arise come from the fundamental group of $\widetilde{M}$, which is trivial. We define in this way a *developing map*

$$\operatorname{dev}\colon \widetilde{M} \to X.$$

Remark 9.6. A developing map only depends on the choice of a coordinate chart as "base point" (U_i in the local construction above); hence, two developing maps only differ by composition with an element of G. With this in mind, by abuse of notation we will often talk about *the* developing map.

Fix a base point $p \in M$ and one of his lifts $q \in \widetilde{M}$. An element γ of $\pi_1(M, p)$ defines a cover isomorphism $\gamma\colon \widetilde{M} \to \widetilde{M}$; as $\operatorname{dev} \circ\gamma$ is another developing map, there is a unique element $g_\gamma \in G$ such that $\operatorname{dev} \circ\gamma = g_\gamma \circ \operatorname{dev}$. The resulting *holonomy representation*

$$\operatorname{hol}\colon \pi_1(M, p) \to G$$

is a group homomorphism such that $\operatorname{dev} \circ\gamma = \operatorname{hol}(\gamma) \circ \operatorname{dev}$ for all $\gamma \in \pi_1(M, p)$.

Application to holomorphic (G, X)-structures

Let X be a holomorphic manifold and G a group acting by biholomorphic maps on X; a holomorphic (G, X) structure on a complex manifold M is a holomorphic atlas on M with values in X such that the changes of charts are restrictions of elements of G.

Proposition 9.7. *Let M be a compact simply connected complex manifold; then M doesn't admit any globally defined holomorphic (G, X) structure such that X is a Stein (i.e. closed complex affine) manifold.*

Proof. Suppose by contradiction that such a structure exists. Then, since M is simply connected, we get a developing map

$$\operatorname{dev}\colon M \to X,$$

which is holomorphic.
Since M is compact and $X \subset \mathbb{C}^N$ is affine, by the maximum principle dev is constant. However, dev is defined to be a local biholomorphism, which leads to a contradiction. □

The most important example of a (G, X)-structure with X affine is probably that of an affine structure. We obtain the following corollary.

Corollary 9.8. *Let M be a simply connected compact complex manifold; then M does not admit any globally defined affine structure.*

9.1.2 (G, X)-orbifolds

The definition of (G, X)-structure, developing map and holonomy can be extended to orbifolds; for what follows we refer to [TM79, Chapter 13] and [Rat06].

An *orbifold* O is a Hausdorff topological space which is locally modelled on open sets of $\mathbb{R}^n$ modulo the action of a finite group; similarly, a (G, X)-orbifold is an orbifold which is locally modelled on open sets of X modulo finite subgroups of G.

An orbifold cover is a map between orbifolds $O_1 \to O$ which locally looks like a quotient map $U/H' \to U/H$ with $U \subset \mathbb{R}^n$ and $H' \leq H$. For an orbifold O, one defines the *universal orbifold cover* $\pi\colon \widetilde{O} \to O$ as the unique connected orbifold cover of O such that any other connected orbifold cover $\pi'\colon O' \to O$ factors π (see [TM79, Proposition 13.2.4] for more details). The universal orbifold cover always exists, but it need not be a manifold; it coincides with the usual universal cover in the case of manifolds.

Example 9.9. If Γ is a group acting properly discontinuously on a manifold M, then M/Γ has a natural orbifold structure (see [TM79, Proposition 13.2.1]); the universal orbifold cover of M/Γ is the usual universal cover of M.

The *orbifold fundamental group* $\pi_1^{orb}(O)$ is by definition the group of deck transformations of $\widetilde{O}$.

Theorem 9.10 ([TM79], Proposition 13.3.2)**.** *If O is a (G, X)-orbifold, then*

- *the universal orbifold cover $\widetilde{O}$ is a simply connected (G, X)-manifold;*
- *there exists a developing map*
$$\mathrm{dev}\colon \widetilde{O} \to X;$$
- *there exists a holonomy representation*
$$\mathrm{hol}\colon \pi_1^{orb}(O) \to G$$
such that $\mathrm{dev} \circ \gamma = \mathrm{hol}(\gamma) \circ \mathrm{dev}$ *for all* $\gamma \in \pi_1^{orb}(O)$*.*

The proof is essentially the construction of the developing map as in the manifold case; while doing so, one also recovers an orbifold cover which is a manifold, and this implies that the universal orbifold cover is also a manifold.

9.1.3 Application: preserved affine structures on $K3$ surfaces

Recall that, by Corollary 9.8, a $K3$ surface X does not admit any globally defined affine structure. However, we have shown in §3.1.2 that a Kummer surface $X = K(T)$ admits an affine structure on the complement U of the 16 exceptional curves appearing from resolution of $T/\pm\mathrm{id}_T$; if f_T is a loxodromic automorphism of the torus T, the induced automorphism of X preserves U and the affine structure.

Now let X be any $K3$ surface, and suppose that there exists an affine structure on a Zariski-open dense subset $U \subset X$; let $E = X \setminus U$. Remark that, by Hartogs principle, the affine structure can be extended to components of E having codimension ≥ 2 (i.e. isolated points); therefore E is a divisor of X.
We say that an automorphism $f\colon X \to X$ preserves the above affine structure if $f(U) = U$ and $f|_U$ acts locally on affine coordinate charts as an affine transformation.

Theorem 9.11. *Let $f\colon X \to X$ be a loxodromic automorphism of a $K3$ surface and suppose that f preserves an affine structure defined on a maximal Zariski-open dense subset $U \subset X$. Then $X = K(T)$ is the Kummer surface associated to a two-dimensional complex torus T, and f is constructed from a loxodromic linear automorphism $f_T\colon T \to T$ (see §3.1.1).*

Proof. Since $f(U)=U$, f preserves the divisor E; after maybe replacing f by an iterate, we may suppose that every irreducible component E_i of E is f-invariant.
Since f is loxodromic, by Theorem 1.22 the restriction of the intersection form to the subspace of $H^{1,1}(X,\mathbb{R})$ generated by the classes of the E_i is negative definite. Remark that, as $H^1(X,\mathcal{O}_X)=0$, the linear equivalence class of a divisor $D\in Div(X)$ is uniquely determined by its numerical class in $H^{1,1}(X,\mathbb{R})$; in particular the intersection matrix $(q_X(E_i,E_j))$ is negative definite: if this were not the case, some linear combination $D=\sum_i a_iE_i$ of the E_i (which we may take with integer coefficients) would have trivial numerical class, implying that $\mathcal{O}_X(D)=0$. Therefore, there would exist a meromorphic function $\phi\colon X\dashrightarrow\mathbb{C}$ such that the divisor $div(\phi)$ of zeros minus poles is exactly D. Since D is f-invariant, $\phi\circ f=\xi\phi$ for some $\xi\in\mathbb{C}^*$; after taking the Stein factorization of ϕ, we get an f-equivariant meromorphic fibration, contradicting the fact that f is loxodromic (for example by Theorem A, or Theorem 1.16).

By Grauert-Mumford theorem (see [BPVdV84, Theorem III.2.1]), there exists a birational morphism

$$\pi\colon X\to Y$$

whose exceptional set is exactly E. More explicitly, $\pi|_{X\setminus E}$ is an isomorphism onto its image, and every connected component of E is mapped onto a single point.

The singular (complex analytic) variety Y is a two-dimensional symplectic variety, and π is a symplectic resolution (see §7.2); thus we may apply Corollary 7.11 to describe the singular locus of Y. The situation on surfaces is actually a classical subject and can be treated more explicitly: since X is a $K3$ surface, all irreducible components E_i of E are smooth rational (-2)-curves by adjunction; therefore, by [BPVdV84, §III.2.ii], the dual graph of each connected component of E is a Dynkin A_n $(n\geq 1)$, D_n $(n\geq 4)$, E_6,E_7 or E_8. In particular, the image of each connected component is a *du Val singularity*: around the images of connected components of E, Y is locally biholomorphic to

$$\mathbb{C}^2\big/_{G}\qquad G\subset SL_2(\mathbb{C})\text{ finite subgroup.}$$

In particular Y admits an orbifold structure; let us show that the affine structure on $X\setminus E$ induces an affine orbifold structure on Y. Let U be a simply connected neighborhood of a singular point $p\in Y$ which is biholomorphic to a neighborhood V of 0 in $\mathbb{C}^2/G$, and let $\{U_i\}$ be an atlas of $U\setminus\{p\}$ by affine charts. The orbifold universal cover $\widetilde{U}\to U$ of U can be identified with the restriction of the quotient map $\rho\colon\mathbb{C}^2\to\mathbb{C}^2/G$ to $\widetilde{V}=\rho^{-1}(V)$. The affine charts U_i define by pull-back affine charts on $\widetilde{V}\setminus\{0\}$; since $\widetilde{V}\setminus\{0\}$ is simply connected, these affine charts glue together to a global affine chart $\widetilde{V}\setminus\{0\}\to\mathbb{C}^2$ (the local developing map). By Hartogs principle, such chart extends to $\{0\}$, defining an orbifold affine structure at p. This proves that on Y we can define an affine orbifold structure.

Therefore, if we denote by $\rho\colon\widetilde{Y}\to Y$ the universal orbifold cover of Y, by Theorem 9.10 $\widetilde{Y}$ is a manifold and we can define a developing map

$$\mathrm{dev}\colon\widetilde{Y}\to\mathbb{C}^2$$

and a holonomy representation

$$\mathrm{hol}\colon\Gamma:=\pi_1^{orb}(Y)\to\mathrm{Aff}(\mathbb{C}^2).$$

As $\mathrm{Aff}(\mathbb{C}^2)$ can be embedded in $\mathrm{GL}_3(\mathbb{C})$, by Selberg's lemma (see [Rat06, §7.6 Corollary 4]) the image $H:=\mathrm{hol}(\Gamma)$ admits a finite index subgroup $H_0\subset H$ without torsion. Let $\Gamma_0=\mathrm{hol}^{-1}(H_0)\subset\Gamma$.
Let us prove that Γ_0 acts without fixed points on $\widetilde{Y}$. Suppose by contradiction that $\tilde{p}\in\rho^{-1}(p)\subset\widetilde{Y}$ is a fixed point for some non-trivial $\gamma\in\Gamma_0$; let us fix an affine orbifold chart U at a neighborhood

of p which identifies with a neighborhood of 0 in $\mathbb{C}^2/G$, and let $\widetilde{U}$ be the connected component of $\rho^{-1}(U)$ which contains $\tilde{p}$; then the restriction $\rho|_{\widetilde{U}}\colon \widetilde{U} \to U$ identifies with the restriction of the natural projection $\mathbb{C}^2 \to \mathbb{C}^2/G$. Since $Y \cong \widetilde{Y}/\Gamma$ and $\gamma(\tilde{p}) = \tilde{p}$, the action of γ on $\widetilde{U}$ identifies with an element of the finite group $G \subset \mathrm{SL}_2(\mathbb{C})$. In particular $\gamma|_{\widetilde{U}}$ has finite order, and so does $\mathrm{hol}(\gamma) \in H_0$, contradicting the definition of H_0.

Now consider the quotient $Y_0 := \widetilde{Y}/\Gamma_0$; since Γ_0 acts without fixed points, Y_0 is a manifold. Furthermore, since Γ_0 has finite index in Γ, the natural map

$$\rho_0\colon Y_0 = {}^{\widetilde{Y}}\!\diagup_{\Gamma_0} \to Y \cong {}^{\widetilde{Y}}\!\diagup_{\Gamma} \cong {}^{Y_0}\!\diagup\!\left({}^{\Gamma}\!\diagup_{\Gamma_0}\right)$$

has finite fibres; in particular, Y_0 is compact.

Since the automorphism f fixes E, it induces an automorphism $f_Y\colon Y \to Y$. Let U be the smooth locus of Y (which is in natural bijection with $X \setminus E$), and let $U_0 = \rho_0^{-1}(U)$. By Proposition 11.2, there exists a finite étale cover $\nu\colon U_1 \to U_0$ such that the restriction $f|_U$ lifts to an automorphism $f_1\colon U_1 \to U_1$. The finite cover ν corresponds to a finite index subgroup Γ_1 of $\pi_1(U_0)$; since Y_0 is smooth and $\mathrm{codim}(Y_0 \setminus U_0) = 2$, $\pi_1(U_0)$ is canonically isomorphic to $\pi_1(Y_0) = \Gamma_0$, and the group $\Gamma_1 \subset \Gamma_0$ defines a finite étale cover

$$\nu\colon Y_1 \to Y_0$$

which extends $\nu\colon U_1 \to U_0$. Then f_1 is a pseudo-automorphism of the surface Y_1, thus an automorphism (because f_1 and f_1^{-1} do not contract any curve). Therefore, we have a commutative diagram

$$\begin{tikzcd} Y_1 \arrow[r, "f_1"] \arrow[d, "\rho 1"'] & Y_1 \arrow[d, "\rho_1"] \\ Y \arrow[r, "f_Y"] & Y \end{tikzcd}, \qquad \phi_1 := \rho_0 \circ \nu$$

The fundamental group of Y_1 is infinite: indeed, otherwise Γ would be finite, thus $\widetilde{Y}$ would be compact and the developing map would be constant, a contradiction.
Furthermore, since f has positive entropy, so does f_Y and, by the properties of topological entropy (see [HK02, Page 36]), so does f_1.

By [Can99, Proposition 1], the only surfaces with infinite fundamental group admitting positive entropy automorphisms are two-dimensional complex tori and their blow-ups. If Y_1 contained a rational curve $C \cong \mathbb{P}^1$, then its universal cover $\widetilde{Y}$ would also contain a $\mathbb{P}^1$; its image by dev would be a point of $\mathbb{C}^2$, contradicting the fact that dev is a local diffeomorphism. This shows that Y_1 is a minimal surface, thus a torus.

This shows that the pair (X, f) is of "generalized Kummer type": X is obtained as a resolution of Y_1/G, where Y_1 is a torus and $G \subset \mathrm{Aut}(Y_1)$ is a finite group, and $f\colon X \to X$ comes from a loxodromic automorphism of Y_1. As shown in [CF03] (see also [CF05]), if (X, f) is of generalized Kummer type, f is loxodromic and X is a $K3$ surface, then X is a Kummer surface (in the usual sense). This concludes the proof. □

9.2 Gromov's A-structures

As we have seen, the notion of (G, X)-manifold often presumes a property of local homogeneity for the structure one is trying to describe. In order to allow more general geometric structures, we are going to introduce the notion of Gromov's A-structures (which we will simply call geometric structures). We refer to the reviews [Ben97, DG91, Dum14] or to the original article of Gromov [Gro88].

9.2.1 Jets and principal frame bundles

If M, N are two manifolds, the *r-jet* at p of the germ at p of a C^∞ map $f\colon (M,p) \to (N,q)$, which we will denote by $J^r_p f$, is its class of equivalence modulo the relation "having the same Taylor development up to the r-th order". We denote by $D^r(\mathbb{R}^n)$ (or simply D^r if the dimension is clear from the context) the group of r-jets of local diffeomorphisms $(\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$; it is a real algebraic group (see §4.1.1). Remark that for all $r \geq 1$, there exists a canonical surjection

$$D^r(\mathbb{R}^n) \twoheadrightarrow D^{r-1}(\mathbb{R}^n).$$

Example 9.12.
- $D^1(\mathbb{R}^n) = \mathrm{GL}_n(\mathbb{R})$;
- $D^r(\mathbb{R})$ is the group of r-jets of local diffeomorphisms $(\mathbb{R},0) \to (\mathbb{R},0)$; hence, $D^r(\mathbb{R}) \cong \{\sum_{i\geq 1} a_i x^i \mid a_1 \neq 0\}/(x^{r+1})$, with group law given by the composition;
- more generally, the elements of $D^r(\mathbb{R}^n)$ are given by n-tuples of polynomials in n variables modulo terms of total degree $\geq r+1$; the local invertibility condition is $\det(Df_0) \neq 0$ for all $f \in D^r(\mathbb{R}^n)$ (where Df_0 denotes the Jacobian matrix of f at 0).

For $p \in M$ and $n, r \in \mathbb{N}$, denote by $J^{n,r}M_p$ the set of r-jets of germs of C^∞ maps $(\mathbb{R}^n,0) \to (M,p)$. The disjoint union of the $J^{n,r}(M)_p$ for all $p \in M$ can be given the structure of a bundle $J^{n,r}M \to M$; the group $D^r(\mathbb{R}^n)$ acts on $J^{n,r}M$ by right multiplication. For example, the bundle $J^{1,1}M$ is just the tangent bundle TM.
A C^∞ map $f\colon M \to N$ induces a map between the jet bundles $J^{n,r}f\colon J^{n,r}M \to J^{n,r}N$ given by $J^{n,r}f(\psi) = J^r f \circ \psi$.
There exists a canonical injection

$$i_{r,s}\colon J^{n,r+s}M \to J^{n,r}(J^{n,s}M)$$

which is defined as follows: an element $\psi \in J^{n,r+s}M$ is the $(r+s)$-jet at 0 of the germ of a map $\tilde{\psi}(\mathbb{R}^n,0) \to (M,p)$. Then the image of ψ is defined as the s-jet of the map $(\mathbb{R}^n,0) \to J^r M$ which sends $x \in \mathbb{R}^n$ to the r-th jet at 0 of the map $y \mapsto \tilde{\psi}(x+y)$. This definition does not depend on the choice of $\tilde{\psi}$.

Now take $n = \dim M$; the *r-th principal frame bundle* $R^r(M)$ is the open subset $J^{n,r}$ corresponding to jets of germs of diffeomorphisms $(\mathbb{R}^n,0) \to (M,p)$. The action of $D^r(\mathbb{R}^n)$ on $J^{n,r}M$ preserves $R^r(M)$ and acts simply transitively on its fibres. In other words, the r-th principal frame bundle is a principal bundle with structure group $R^r(\mathbb{R}^n)$.

Example 9.13.
- $R^0(M)$ is just the trivial bundle of dimension 0 on M.
- Recall that $D^1(\mathbb{R}^n) = GL_n(\mathbb{R})$; hence, the first principal frame bundle is what is usually called the frame bundle:

$$R^1(M) = \{(p,\phi) \mid p \in M,\ \phi \text{ linear isomorphism } : \mathbb{R}^n \xrightarrow{\sim} T_pM\}$$

  with the natural projection onto M.

9.2.2 Geometric structures in the Gromov sense

The following definition coincides with the concept of A-structure introduced by Gromov (where A stands for "algebraic"; see[Gro88, DG91, Ben97]).

Definition 9.14. *Let Z be a smooth quasi-projective variety over $\mathbb{R}$ equipped with an algebraic action of the algebraic group $D^r(\mathbb{R}^n)$; a* geometric structure *of type Z and order r on a manifold*

M of dimension n is a smooth equivariant map $g\colon R^r(M) \to Z$, i.e. $g(\psi f) = f^{-1} \cdot g(\psi)$, $\forall \psi \in R^r(M)$ and $\forall f \in D^r(\mathbb{R}^n)$.
If M and Z are complex manifolds, we define in a similar way a holomorphic (or meromorphic) geometric structure.

Remark 9.15. If $\phi_i\colon U_i \subset M \xrightarrow{\sim} V_i \subset \mathbb{R}^n$ is a chart of M, the r-jet of ϕ_i defines a local trivialization of the principal frame bundle $R^r(M)|_{U_i} \cong V_i \times D^r(\mathbb{R}^n)$; a geometric structure of type Z is defined by smooth maps $g_i\colon V_i \to Z$ such that, denoting $\phi_{ij} = \phi_i \circ \phi_j^{-1}\colon \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ the change of chart,

$$g_i(\phi_{ij}x) = (J^r_x \phi_{ij}) \cdot g_j(x) \qquad \forall x \in \phi_j(U_i \cap U_j).$$

Example 9.16.

- *Pseudo-Riemannian structures*: take

$$r = 1, \qquad Z = \{\text{non-degenerate quadratic forms } q \text{ on } \mathbb{R}^n\};$$

  Z is a Zariski-open subset of the set of symmetric matrices, with the action of $D^1(\mathbb{R}^n) \cong \mathrm{GL}_n(\mathbb{R})$ given by $\phi \cdot q = q \circ \phi^{-1}$. A geometric structure of type Z on a manifold M is then a pseudo-Riemannian metric on M (Riemannian if the image of g is included in the connected component of Z corresponding to positive definite quadratic forms). To see this, take a point of $(p, \phi) \in R^1(M)$; then ϕ is a linear isomorphism between the tangent space T_pM and $\mathbb{R}^n$, which corresponds to the choice of a basis $\mathcal{B}$ of T_pX. The image $g(p, \phi) \in Z$ gives then the pseudo-Riemannian metric with respect to $\mathcal{B}$; the equivariancy of g ensures that the definition of the metric does not depend on the chosen basis.
  If we take instead $Z_{\mathbb{C}} = \{\text{non-degenerate quadratic forms } q \text{ on } \mathbb{C}^n\}$, then a holomorphic structure of type $Z_{\mathbb{C}}$ on a complex manifold M is a *holomorphic metric* (remark that in the complex case the signature is not defined).

- *Conformal structures*: intuitively, a conformal structure at a point $p \in M$ is a measure of angles in T_pM. More accurately, a conformal structure on M is the datum at each point of M of a Riemannian (or pseudo-Riemannian) metric modulo multiplication by strictly positive real numbers. Therefore, it can be expressed as a structure with

$$r = 1, \qquad Z = \{\text{non-degenerate quadratic forms } q \text{ on } \mathbb{R}^n\}/\mathbb{R}^+$$

  with the action of $\mathrm{GL}_n(\mathbb{R})$ given by $\phi \cdot [q] = [q \circ \phi^{-1}]$. Replacing Z by

$$Z_{\mathbb{C}} = \{\text{non-degenerate quadratic forms } q \text{ on } \mathbb{C}^n\}/\mathbb{C}^*,$$

  one obtains holomorphic conformal structures.

- *Tensors*: take

$$r = 1, \qquad Z = \mathbb{R}^n \quad \text{with the natural action of } \mathrm{GL}_n(\mathbb{R}).$$

  Then a structure of type Z is a vector field; in a similar way one can define a structure corresponding to a differential form (or more generally a tensor).
  One can impose additional conditions to a form: for example, a closed k-form is a structure

$$r = 2, \qquad Z = \{\text{1-jets at 0 of closed } k\text{-forms on } \mathbb{R}^n\}.$$

  Since the differential of a form only involves the first derivatives of its coefficients, it makes sense to talk about a closed 1-jet, and for the same reason $D^2(\mathbb{R}^n)$ acts naturally on Z. Since the conditions on the coefficients are polynomial, Z is an algebraic subvariety of $\{\text{1-jets at 0 of } k\text{-forms on } \mathbb{R}^n\}$.

- *Distributions*: take

$$r = 1, \qquad Z = \mathrm{Gr}(k, \mathbb{R}^n) = \{V \subset \mathbb{R}^n \text{ linear subspace of dimension } k\}$$

with the natural action of $\mathrm{GL}_n(\mathbb{R})$; then a Z-structure on a manifold M is a distribution of k-planes, i.e. a sub-vector bundle $T\mathcal{F} \subset TM$ of dimension k. Alternatively, one can adopt the dual point of view and describe a distribution by its kernel (modulo the multiplicative action of $\mathbb{R}^*$):

$$r = 1, \qquad Z' = \mathbb{P}\{\alpha = \alpha_1 \wedge \ldots \wedge \alpha_{n-k} \mid \alpha_i \in (\mathbb{R}^n)^*\}$$

with the adjoint action of $\mathrm{GL}_n(\mathbb{R})$.

- *Foliations*: a foliation is a distribution $T\mathcal{F}$ satisfying the integrability condition $[X_1, X_2] \in T\mathcal{F}$ for all local vector fields $X_1, X_2 \in T\mathcal{F}$;

$$r = 2, \quad Z = \{\text{1-jets at 0 of integrable distributions of } k\text{-planes in } \mathbb{R}^n\}.$$

By this we mean the following: a distribution of k-planes on $\mathbb{R}^n$ is a differentiable map $\phi\colon \mathbb{R}^n \to \mathrm{Gr}(k, \mathbb{R}^n)$, and we take its 1-jet. The integrability condition only involves the 1-jets of two vector fields contained in $T\mathcal{F}$, hence it makes sense to say that the 1-jet of a distribution is integrable; one can check it on vector fields whose value at 0 forms a base of $T\mathcal{F}_0$, therefore the conditions imposed on the coefficients are polynomial and Z is an algebraic subvariety of $\{\text{1-jets at 0 of distributions of } k\text{-planes in } \mathbb{R}^n\}$. Again since the Lie brackets only depend on the 1-jet of vector fields, one can show that $D^2(\mathbb{R}^n)$ acts algebraically on Z.
Adopting the dual point of view, a foliation is locally given by (the class modulo $\mathbb{R}^*$ of) a $(n-k)$-form ω such that $\ker(\omega)$ has dimension k and which satisfies the integrability condition $d\omega \in \omega \wedge \Omega^1_M$; thus a foliation can also be given by a structure

$$r = 2, \qquad Z' = \mathbb{P}\{\text{1-jets at 0 of integrable } (n-k)\text{-forms on } \mathbb{R}^n\};$$

as before, since the integrability condition only involves first derivatives, it makes sense to talk about an integrable 1-jet; for the same reason, $D^2(\mathbb{R}^n)$ acts on Z'.

- A structure g of order r defines a structure g^s of order $r+s$ for any $s \geq 0$: it is the structure of type $J^{n,s}Z$ given by the composition

$$R^{r+s}M \to J^s(R^rM) \xrightarrow{J^{n,s}g} J^{n,s}Z$$

where the first map is the restriction of $i_{s,r}\colon J^{n,r+s}M \to J^{n,s}(J^{n,r}(M))$ to $R^{r+s}M$.

- *Superposition of structures*: if g_1, g_2 are two structures of order r_1, r_2 and type Z_1, Z_2 respectively, then the structure $g_1 \times g_2$ is a structure of order $\max\{r_1, r_2\}$ and type $Z_1 \times Z_2$, which, roughly said, is the union of the two structures g_1, g_2.

9.2.3 Isometries and rigidity

Given a structure $g\colon R^r(M) \to Z$, a *local isometry* of g is a germ of diffeomorphism $f\colon (U_1, p_1) \to (U_2, p_2)$ between neighborhoods of $p_1, p_2 \in M$ which preserves g, i.e. such that

$$g \circ J^r_{p_1} f = g \qquad \text{on } R^rU_1.$$

We denote by $\mathsf{Is}^{loc}_{p_1,p_2}(g)$ (or simply by $\mathsf{Is}^{loc}_{p_1,p_2}$ when there is no risk of confusion) the set of germs of local isometries between (neighborhoods of) p_1 and p_2; we then have a composition law

$$\mathsf{Is}^{loc}_{p_2,p_3} \times \mathsf{Is}^{loc}_{p_1,p_2} \to \mathsf{Is}^{loc}_{p_1,p_3}$$

which defines a pseudo group structure on the disjoint union of all the $\mathsf{Is}^{loc}_{p,q}$; we call the resulting subgroup $\mathsf{Is}^{loc}(g)$ the *pseudo-group of local isometries* of g.
If $\mathsf{Is}^{loc}(g)$ acts transitively on M, we say that the structure g is *locally homogeneous*.

For a given point $p \in M$, the local isometries of g fixing p form a group, the *isotropy group* of g at p.

For $s \geq 0$, an *isometric* $(r+s)$*-jet* of g is the $(r+s)$-jet of the germ of a diffeomorphism $f\colon (U_1,p_1) \to (U_2,p_2)$ which preserves g at order s, i.e. such that

$$g^s \circ J^{r+s} f = g^s \qquad \text{on } R^{r+s}_{p_1} M.$$

Denote by $\mathsf{Is}^{r+s}_{p_1,p_2}(g)$ (or $\mathsf{Is}^{r+s}_{p_1,p_2}$) the set of isometric $(r+s)$-jets from p_1 to p_2. Remark that the jets of local isometries are always isometric jets; however, an isometric jet does not necessarily extends to a local isometry.
The $(r+s)$-jet of an isometric $(r+s+1)$-jet is automatically isometric; thus we can define maps

$$\mathsf{Is}^{r+s+1}_{p_1,p_2}(g) \to \mathsf{Is}^{r+s}_{p_1,p_2}(g).$$

Definition 9.17. *A geometric structure g of order r is* rigid *at the order $r+s_0$ (or $(r+s_0)$-rigid) if for all $s \geq s_0$ and for all $x \in M$, the natural maps*

$$\mathsf{Is}^{r+s+1}_{x,x}(g) \to \mathsf{Is}^{r+s}_{x,x}(g).$$

are injective.
A meromorphic geometric structure is almost rigid *if it is rigid on a Zariski-open dense subset of M.*

Example 9.18.
- Thanks to the local bijectivity of the exponential map, isometric r-jets for (pseudo-)Riemannian structures are uniquely determined by their Jacobian; in other words, pseudo-Riemannian structures are rigid at the order 1; the same is true for holomorphic metrics.
- Conformal structures are rigid at the order 2; this is essentially a consequence of Liouville Theorem describing conformal automorphisms of $\mathbb{R}^m$. See [DG91, Kob95]; see also [Bal00] for a complete proof and [Fra03] for a shorter proof in the analytic case.

Proposition 9.19. *Let X be a symplectic $2n$-fold.*

1. *Let Φ be the meromorphic structure given by a symplectic form σ and two generically transverse singular Lagrangian distributions (resp. foliations) $\mathcal{F}_1, \mathcal{F}_2$.*
   Then Φ is a meromorphic geometric structure of order 1 (resp. 2) which is almost rigid at order 1; in other words, on a non-empty Zariski open subset of X local isometries are uniquely determined by their Jacobian.

2. *Let $[\Phi]$ be the meromorphic structure given by the class of a symplectic form σ modulo multiplication by elements of $\mathbb{C}^*$ and two generically transverse singular Lagrangian distributions (resp. foliations) $\mathcal{F}_1, \mathcal{F}_2$.*
   Then $[\Phi]$ is a meromorphic geometric structure of order 1 (resp. 2) which is almost rigid at order 2; in other words, on a non-empty Zariski open subset of X local isometries are uniquely determined by their first and second partial derivatives.

Proof. Let us prove the first assertion. Remark first that forms and distributions are structures of order 1, whereas foliations are structures of order 2; therefore Φ is of order 1 or 2 accordingly.

For a general $p \in X$, consider the map

$$q \colon v \in T_pX \mapsto \sigma(v_1, v_2),$$

where $v = v_1 + v_2$ is the decomposition of v with respect to the direct sum $T_pX = T_p\mathcal{F}_1 \oplus T_p\mathcal{F}_2$. Then q defines a holomorphic metric on a neighborhood of p, thus on the Zariski-open subset $U \subset X$ where the two foliations are transverse: indeed, for $p \in U$, there exist linear coordinates $p_1, \dots p_n, q_1, \dots, q_n$ on T_pX such that, in these coordinates,

$$\sigma_p = \sum_{i=1}^{n} dp_i \wedge dq_i,$$

$$T_p\mathcal{F}_1 = \{q_1 = \ldots = q_n = 0\}, \quad T_p\mathcal{F}_2 = \{p_1 = \ldots = p_n = 0\},$$

whence

$$q = \sum_{i=1}^{n} dp_i . dq_i .$$

Any local isometry of $\Phi|_U$ has to preserve q, thus it is uniquely determined by its Jacobian (see Example 9.18). This concludes the proof of the first assertion.

Now let us prove the second assertion. First, the class of a form modulo $\mathbb{C}^*$ is a structure of order 1 (with $Z = \mathbb{P}(\bigwedge^2(\mathbb{C}^n)^*)$ and dual action of $\mathrm{GL}_n(\mathbb{C})$); hence, as before, the superposition $[\Phi]$ has order 1 (for distributions) or 2 (for foliations).
We can repeat the above construction to obtain a holomorphic metric q, which is well-defined modulo multiplication by elements of $\mathbb{C}^*$; in other words, $[\Phi]$ induces a conformal structure on a Zariski-open dense subset $U \subset X$. Since conformal structures are rigid at the order 2 (see Example 9.18), so is $[\Phi]$. This concludes the proof of the second claim. □

9.2.4 Gromov's open-dense orbit theorem

One of the main consequences of rigidity is that the analysis of the orbits of the pseudo-group of local isometries is more handy than the general case. In the C^∞ context, this is a consequence of Gromov's celebrated open-dense orbit theorem (see [Gro88, 3.1,3.3,3.4]):

Theorem 9.20 (Gromov's open-dense orbit theorem)**.** *Let g be a rigid geometric structure on a differentiable manifold M; then, there exists a dense open set $U \subset M$ such that the orbits of the pseudogroup $\mathsf{Is}^{loc}(g)$ are closed subvarieties.*
In particular, if $\mathsf{Is}^{loc}(g)$ has a dense orbit, then g is locally homogeneous on a dense open subset.

In the case of meromorphic structures, Dumitrescu has proven the following analogue, see [Dum11, Theorem 2.1]:

Theorem 9.21. *Let M be a connected complex manifold and let g be an almost rigid geometric structure on M. Then there exists a positive codimensional analytic subset $S \subset M$ and fibration $\pi \colon M \setminus S \to B$ with differential of constant rank and such that $M \setminus S$ is Is^{loc}-invariant and the fibres of π are exactly the orbits of the action of Is^{loc} on $M \setminus S$.*
In particular, if Is^{loc} has a Zariski-dense orbit, then g is locally homogeneous outside of a nowhere dense analytic subset.

In the case of loxodromic transformations of irreducible symplectic manifolds we obtain the following:

Corollary 9.22. *Let $f\colon X \dashrightarrow X$ be a loxodromic birational transformation of an irreducible symplectic manifold X and let g be an almost rigid geometric structure which is preserved by f (i.e. f acts by local isometries on the points where it is defined). Then g is homogeneous on a Zariski-open dense subset $U \subset X$.*

Proof. By Theorem A, the f-orbits of very general points of X are Zariski-dense; hence $\mathsf{Is}^{loc}(g)$ has a Zariski-dense orbit. The claim follows from Theorem 9.21. □

We are ready to prove Theorem 9.1.

Proof of Theorem 9.1. Let X be an irreducible symplectic manifold, and let $f\colon X \dashrightarrow X$ be a loxodromic birational transformation preserving two generically transverse Lagrangian distributions $\mathcal{F}_1, \mathcal{F}_2$.

Then $f^*\sigma = \xi\sigma$ for some $\xi \in \mathbb{C}^*$, so that f preserves the structure $[\Phi]$. If furthermore X is projective, by Lemma 6.20, some iterate of f preserves $\Omega = \sigma^n$; since $f^*\sigma = \xi\sigma$ by irreducibility, some iterate of f preserves the structure Φ.

The structures Φ and $[\Phi]$ are rigid by Proposition 9.19; thus the claim follows from Corollary 9.22. □

9.2.5 Research of local models

We close this Section with a discussion on the general strategy to analyse invariant structures for loxodomic transformations.

As we have seen, geometric structures in the sense of Gromov are the natural frame to discuss locally defined structures; take such a structure g on an irreducible symplectic manifold M. We will suppose that g is rigid and preserved by a loxodromic transformation $f\colon M \dashrightarrow M$; then, by Theorem A and Theorem 9.21, g is locally homogeneous on a non-empty Zariski-open subset $U \subset M$. What one hopes in this situation is to describe one or more possible local models for g, i.e. that g is actually a $(G, \mathcal{X})$-structure as in Section 9.1.

Although this is not always the case, one can try and find such models as follows. A *Killing field* is a (local) vector field v preserving the structure g, i.e. such that the flows along v are local isometries for g. We will denote by $\mathcal{G}_p$ (resp. $\mathcal{I}_p$) the Lie algebra of germs of Killing fields at p (resp. of germs of Killing fields vanishing at p). As g is rigid, $\mathcal{G}_p$ and $\mathcal{I}_p$ are finite-dimensional: indeed, since local isometries are uniquely determined by a finite jet, then so are Killing fields. Furthermore, if the structure is locally homogeneous on U, G_p and I_p do not actually depend on the chosen point $p \in U$; from now on we will assume local homogeneity and drop the p in the notation.

Denote by G (resp. I) the simply connected Lie group having Lie algebra $\mathcal{G}$ (resp. $\mathcal{I}$). Then I is isomorphic to (the connected component of) the isotropy group $\mathsf{Is}^{loc}_{p,p}$ for any $p \in U$.

The candidate for a local model is the homogeneous space G/I; however, in order for G/I to be a manifold, one needs to check that I is a *closed* subgroup of G. If this is the case, g is actually described by a $(G, G/I)$-structure.

Example 9.23. Let g be a meromorphic metric on a compact complex manifold M; for example, if σ is a holomorphic symplectic form and $\mathcal{F}_1, \mathcal{F}_2$ are two generically transverse Lagrangian distributions, for $p \in M$ where $\mathcal{F}_1$ and $\mathcal{F}_2$ are transverse we may define

$$g_p(v) = \sigma_p(\pi_1 v, \pi_2 v) \qquad v \in T_pM,$$

where, for $i = 1, 2$, $\pi_i\colon T_pM \to T_p\mathcal{F}_i$ is the projection with respect to the decomposition

$$T_pM = T_p\mathcal{F}_1 \oplus T_p\mathcal{F}_2;$$

see the proof of Proposition 9.19 for more details.
Suppose that the structure g is rigid on a Zariski-open dense subset $U \subset M$ (e.g., M is irreducible symplectic and g is preserved by a loxodromic transformation $f\colon M \dashrightarrow M$); in this case, the Lie groups G_p and I_p defined above do no depend on the point $p \in U$.
Remark that the elements of I_p fix p and act as orthogonal transformations on T_pM; in particular $\dim I_p \leq \dim O(n, \mathbb{C}) = n(n-1)/2$, where $n = \dim M$. In the case of maximal dimension $\dim I_p = n(n-2)/2$, g has constant sectional curvature; up to rescaling, two cases are possible:

- zero sectional curvature, i.e. flat metric. In this case

$$G/I \cong \mathrm{Isom}(\mathbb{C}^n)/SO_n(\mathbb{C}) \cong \mathbb{C}^n,$$

  where $\mathrm{Isom}(\mathbb{C}^n)\mathrm{GL}_n(\mathbb{C})$ denotes the group of affine automorphisms of $\mathbb{C}^n$ preserving the standard metric; g corresponds then to a holomorphic euclidean structure, i.e. a $(\mathrm{Isom}(\mathbb{C}^n), \mathbb{C}^n)$-structure.

- sectional curvature $= 1$. In this case

$$G/I \cong O_{n+1}(\mathbb{C})/O_n(\mathbb{C}),$$

  the "holomorphic n-sphere"; g corresponds then to a holomorphic spherical structure, i.e. a $(O_{n+1}(\mathbb{C})/O_n(\mathbb{C}), O_{n+1}(\mathbb{C}))$-structure.

See [DZ09, Thu97] for more details about the models with constant sectional curvature; in [DZ09] one can find explicit computations of G and I in dimension 3 and without assumptions on the sectional curvature of g. In particular, (M, g) is locally biholomorphic to a G-invariant geometric structure on X.

By a result of Mostow [Mos50], if I_p has codimension at most 4 in G_p, then it is closed. In particular we obtain the following result (see [Dum14, Theorem 5]).

Theorem 9.24. *Let M be a complex manifold of dimension ≤ 4, and let g be a locally homogeneous rigid geometric structure on M; then g induces a $(G, G/I)$-structure on M, where G and I are defined as above. Furthermore, (M, g) is locally isomorphic to a G-invariant geometric structure on G/I.*

Let us see how this applies to the symplectic case. Let $f\colon X \dashrightarrow X$ is a loxodromic transformation of an irreducible symplectic fourfold; suppose that f preserves two generically transverse Lagrangian foliations. Then, the structure given by the symplectic structure plus the two foliations (resp. a fixed symplectic form plus the two foliations if X is projective) is rigid, and defines a (G, X)-structure on a Zariski-open dense subset of X.

Example 9.25. Let X be a symplectic fourfold, and let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ be four generically transverse one-codimensional foliations; assume that the structure Φ given by the $\mathcal{F}_i$ and a symplectic form is rigid and locally homogeneous on a Zariski-open dense subset $U \subset X$. This would happen for example if X were irreducible and Φ were preserved by a loxodromic transformation of X (although by Corollary 7.2 this cannot happen).
Then, since Φ is locally homogeneous,

$$\dim G = \dim I + 4;$$

one can show that $\dim I \leq 2$, and find all the possible local models by a case by case analysis on $\dim I$ and $\dim I^{(1)}$ (the isometric 1-jets of local automorphisms of (X, p) fixing p).

For example, letting $Q = \mathbb{P}^1 \times \mathbb{P}^1 \setminus diag \cong PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C})/\mathbb{C}^*$ (where $\mathbb{C}^*$ acts diagonally on $PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C})$), one of the possible models is

$$(G, \mathcal{X}) = (\mathrm{Isom}(\mathbb{C}^2) \times \mathbb{C}^*, \mathbb{C}^2 \times Q)$$

endowed with the symplectic form

$$\sigma = a_{12}\, dx_1 \wedge dx_2 + a_{34} \frac{dx_3 \wedge dx_4}{(x_3 - x_4)^2},$$

where $a_{12}, a_{34} \in \mathbb{C}^*$, and coordinate foliations (the leaves of $\mathcal{F}_i$ are given by $x_i = const.$).

An example of such a situation is a product $S_1 \times S_2$ of two-dimensional complex tori or Kummer surfaces equipped with a pair of linear foliations.
If instead of considering four foliations one takes a 4-tissue of dimension 1 (i.e., roughly speaking, the local data of four one-dimensional foliations), the same considerations are still true; this situation appears on the two standard examples of dimension 4 (see Chapter 3).

Chapter 10

Conclusion and open questions

The aim of this thesis project was to describe the interaction between the geometry of compact Kähler manifolds with trivial Chern class (more specifically irreducible symplectic manifolds) and the dynamics of automorphisms/birational transformations acting on such a manifold; the main focus was a description of invariant structures (fibrations, foliations, distributions, affine structures...).

As we have seen, the birational transformations of an irreducible symplectic manifold can be classified as loxodromic, parabolic or elliptic, the first having a priori a more complicated dynamics; indeed, Theorem A shows that a loxodromic transformation is imprimitive (and in particular the orbits of its very general points are Zariski-dense).

Additionally, I gave some restrictions on the foliations or distributions which can be preserved by such a transformation f: by Theorem C and Corollary 7.2, if f preserves $k > 1$ generically transverse distributions, then $k = 2$ and the two distributions are Lagrangian; preserved foliations must satisfy some additional conditions (see Corollary 7.17 and 7.18).

Concerning more general structures, by Corollary 9.22 and Theorem 9.24 almost rigid meromorphic geometric structures preserved by f are automatically locally homogeneous on a Zariski-open dense subset, and in dimension 2 or 4 they admit a local model in the sense of (G, X)-structures. By Proposition 9.19, the structure defined by two generically transverse Lagrangian distributions plus the symplectic structure is almost rigid, hence locally homogeneous.

In Chapter 6, I investigated birational transformations preserving a fibration: Theorem B treats the case of transformations of a manifold with trivial or effective canonical bundle preserving a fibration onto $\mathbb{P}^n$ and acting by an automorphism on it; in Theorem 6.22 and 6.25 I give some results on preserved fibrations on Calabi-Yau threefolds.

10.1 More on structures preserved by loxodromic transformations

Let X be an irreducible symplectic manifold and let $f\colon X \dashrightarrow X$ be a loxodromic transformation.

- A first natural development of my work would be to classify loxodromic transformations preserving a pair of Lagrangian distributions.

  Conjecture 10.1. *The only instances of a loxodromic transformation of an irreducible symplectic manifold preserving a pair of Lagrangian distributions/foliations are the two examples of Kummer type of Chapter 3.*

  The machinery to attack the problem is already in place, at least for the case of X projective of dimension 4: in this case we know that the structure defined by the two foliations and the symplectic form is locally homogeneous and modelled on some model manifold $\mathcal{X}$ on a Zariski-open dense subset $U \subset X$. To manage (the divisorial part of) the exceptional locus $E = X \setminus U$, we can contract its irreducible components using Lemma 7.3; then, at least if $\mathcal{X}$ is affine, we get a structure of $(G, \mathcal{X})$-orbifold at the general points of the singularities which appear upon contraction. This should help to describe E, and possibly to detect the two-dimensional torus from which the transformation is constructed.

- More generally, the above strategy can be applied to try and classify rigid structures preserved by f.

- Generalizing Theorem 9.11 one can try and classify loxodromic transformations which preserve an affine structure on a Zariski-open dense subset; again, the first step would be to contract the components of the exceptional set and to analyze the resulting affine orbifold.

Conjecture 10.2. *The only instances of a loxodromic transformation of an irreducible symplectic manifold preserving an affine structure (defined on a Zariski-dense open set) are the two examples of Kummer type of Chapter 3.*

- In Corollary 7.17 and 7.18, I gave some restrictions on the foliations which can be preserved by a loxodromic transformation: such a foliation cannot be defined by a global meromorphic form without zeros in codimension 1, nor can f act by multiplication by a constant on a meromorphic form defining the foliation.
In the two standard examples, we see that the preserved foliations are defined by pluriforms (more accurately squares of forms) without zeros in codimension 1, and that f acts by multiplication by a constant on such forms.

Question 10.3. *Are the standard examples the only instances of such a situation?*

10.2 More on imprimitive transformations

Let $f\colon X \dashrightarrow X$ be an imprimitive transformation of a projective (or compact Kähler) manifold X.

- It should be possible to weaken the assumptions of Theorem B; in particular, we hope that the assumptions $\rho(B) = b_2(B) = 1$ and K_B^* ample are enough to conclude. The interest of such assumption is explained by what follows: if X is an irreducible symplectic manifold and $\pi\colon X \to B$ is a holomorphic fibration onto a Kähler manifold, Huybrechts showed, following Matsushita, that $\dim B = \dim X/2$, $\rho(B) = b_2(B) = 1$ and K_B^* is ample. The main motivation for studying fibrations onto $\mathbb{P}^n$ was Theorem 2.15, which allows to construct equivariant fibrations for parabolic transformations of special classes of irreducible symplectic manifolds; hoping that a similar result holds for general irreducible symplectic manifolds (see Conjecture 2.1), it is natural to ask for the minimal assumptions that are verified for the base of an equivariant holomorphic fibration.
Remark that, by [AC08, Theorem 3.6], if $\dim X = 4$ and $\pi\colon X \dashrightarrow B$ is a meromorphic fibration whose general fibres have Kodaira dimension 0, then there exists an irreducible symplectic birational model of X on which the fibration becomes holomorphic.

- In the context of Theorem B, even knowing the action on the base, we still need to describe the action on the fibres in order to completely grasp the dynamics of the automorphism; motivated by Theorem 2.15, we may want to add the assumption that the fibres are abelian varieties over which f acts by translations. The intuition is that, as happens for surfaces, the polynomial growth of $\| (f^n)^* \|$ arises, roughly said, from a huge variation of the translation vector[1].

Conjecture 10.4. *Let $f\colon X \to X$ be a parabolic transformation of a projective irreducible symplectic manifold preserving the fibres of a Lagrangian fibration $\pi\colon M \dashrightarrow B$ (whose smooth fibres are then abelian varieties); then f acts as a periodic translation on the fibres over a dense subset of B with zero Lebesgue measure, and as a dense orbit translation on the fibres over a full Lebesgue measure subset of B.*

- We can try generalizing Theorem 1.30 to Calabi-Yau threefolds.

Conjecture 10.5. *Let $f\colon X \to X$ be an automorphism of a Calabi-Yau threefold whose action on cohomology is virtually unipotent and non-trivial. Then f admits an equivariant non-trivial fibration.*

In [Wil89, Wil98], Wilson has given some criteria to construct (elliptic) fibrations on X; see Conjecture 6.1. His results together with the bounds on the action of the automorphism in cohomology which I found in [LB14a] do not allow to conclude directly; however, I believe that with a deeper understanding of Wilson's proof, depending heavily on the Minimal Model Program, and possibly by using some dynamical systems machinery, one can prove the result.

[1]The fibration being not a priori isotrivial, one has to consider at once the variation of fibres in the moduli space of abelian varieties and the translation vector.

Chapter 11

Appendix: lift of pseudo-automorphisms to étale covers

Lemma 11.1. *Let G be a finitely generated group, $H \leq G$ a finite index subgroup and $\phi\colon G \to G$ an automorphism. Then there exists a finite index subgroup $H' \leq H$ such that $\phi(H') = H'$.*

Proof. Let $K = \bigcap_{g \in G} gHg^{-1}$ be the normal core of H. Then K is a normal, finite index subgroup of H, and we are going to show that $\phi^N(K) = K$ for some $N > 0$.
The subgroups $K, \phi(K), \phi^2(K), \ldots \leq G$ are all normal subgroups of same index i; therefore, each of them is the kernel of a surjective group morphism $\psi_j\colon G \to G_j$, where the G_j-s are groups of order i. The number of possible G_j-s is finite, and since G is finitely generated, for a fixed G_j the number of morphisms $G \to G_j$ is also finite. This implies that $\phi^m(K) = \phi^n(K)$ for some $0 \leq n < m$, thus $\phi^{m-n}(K) = K$. Now let

$$H' = \bigcap_{j=0}^{m-n-1} \phi^j(K).$$

The subgroup H' has finite index in K, hence in H; furthermore, by the above discussion it is clear that $\phi(H') \subset H'$. Since H' and $\phi(H')$ have the same index in G, we must have equality. ☐

As a consequence, we will show a property of lifting of pseudo-automorphisms to finite étale covers.

Lemma 11.2. *Let X be a complex manifold, $\nu\colon X' \to X$ be a finite étale cover, and $f\colon X \dashrightarrow X$ be a pseudo-automorphism (see §2.4).*
Then there exists a finite étale cover $\eta\colon X'' \to X'$ of X' such that f induces a pseudo-automorphism $f''\colon X'' \dashrightarrow X''$.

Proof. Without loss of generality we may suppose that X' is connected. Since f is a pseudo-automorphism, there exist open sets $U, V \subset X$ such that

1. $\operatorname{codim}(X \setminus U), \operatorname{codim}(X \setminus V) \geq 2$;
2. f induces an isomorphism $U \cong V$.

Denoting by $U' = \nu^{-1}(U), V' = \nu^{-1}(V)$ the inverse images of U and V, we want to show that, up to replacing X' by a finite étale cover, $f\colon U \to V$ lifts to a map $U' \to V'$, i.e. there exists $f'\colon U' \to V'$ such that $\nu \circ f' = f \circ \nu$.
Remark that, as ν is an étale cover, $\operatorname{codim}(X' \setminus U') = \operatorname{codim}(X \setminus U) \geq 2$ and $\operatorname{codim}(X' \setminus V') =$

$\mathrm{codim}(X \setminus V) \geq 2$; furthermore, $f' \colon U' \to V'$ is automatically an isomorphism. In other words, if a lift $f' \colon U' \to V'$ exists, then it defines a pseudo-automorphism $f' \colon X' \dashrightarrow X'$.

Fix some base point $p \in U \cap V$, let $q = f(p) \in U \cap V$, and fix $p' \in \nu^{-1}(p), q' \in \nu^{-1}(q)$. By a classical topological lemma, in order to check that $f \colon U \to V$ lifts to a morphism $f' \colon U' \to V'$, one has to check that

$$f_*(\nu_*(\pi_1(U', p'))) \subset \nu_*(\pi_1(V', q')),$$

where $f_* \colon \pi_1(U, p) \to \pi_1(V, q)$ and $\nu_* \colon \pi_1(U', p') \to \pi_1(U, p)$ (resp. $\nu_* \colon \pi_1(V', q') \to \pi_1(V, q)$) denote the maps induced by f and ν.

Remark that, if $W \subset Y$ is an analytic subset of a complex manifold Y whose complement $Y \setminus W$ has codimension ≥ 2, then $\pi_1(Y) \cong \pi_1(W)$. More accurately, if $q \in W$, the inclusion $i \colon W \hookrightarrow Y$ induces an isomorphism of fundamental groups

$$i_* \colon \pi_1(W, q) \xrightarrow{\sim} \pi_1(Y, q).$$

In our situation, we get canonical isomorphisms

$$\pi_1(X, p) \cong \pi_1(U, p) \cong \pi_1(V, p), \qquad \pi_1(X', p') \cong \pi_1(U', p') \cong \pi_1(V', p'),$$

$$\pi_1(X, q) \cong \pi_1(U, q) \cong \pi_1(V, q), \qquad \pi_1(X', q') \cong \pi_1(U', q') \cong \pi_1(V', q')$$

which commute with f_* and ν_*. Therefore, we may rephrase the condition to lift f as

$$f_*(\nu_*(\pi_1(X', p'))) \subset \nu_*(\pi_1(X', q')).$$

Let $\gamma \colon [0, 1] \to X'$ be a path relying p' to q'; conjugation of loops by γ (resp. $\pi \circ \gamma$) induces a (non-canonical) isomorphism $\pi_1(X', p') \colon \pi_1(X', q')$ (resp. $\pi_1(X, p) \cong \pi_1(X, q)$); these automorphisms commute with ν_*. Define, using these automorphisms,

$$G := \pi_1(X, p) = \pi_1(X, q), \qquad H = \nu_*(\pi_1(X', p')) = \nu_*(\pi_1(X', q')),$$

$$\phi = f_* \colon G \to G.$$

Now, G is finitely generated and $H \subset G$ is a finite index subgroup; therefore, by Lemma 11.1, some finite index subgroup H' of H is f_*-invariant. Considering the finite cover X'' of X corresponding to H' (which factors through ν), this means exactly that f lifts to $f'' \colon X'' \dashrightarrow X''$. We have showed that such a lift is automatically a pseudo-automorphism, so this concludes the proof. $\square$

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