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Groupes kleiniens birationnels en dimension deux

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Je tiens à remercier
J'adresse également toute ma reconnaissance à ....
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INTRODUCTION

0.1 Groupes kleiniens birationnels

0.1.1 Définitions

Soient Y une variété projective complexe lisse et $U \subset Y$ un ouvert en topologie usuelle. Soit $\Gamma \subset \mathrm{Bir}(Y)$ un groupe infini de transformations birationnelles. Nous imposons les conditions suivantes sur Γ:

1. Les points d'indétermination de Γ sont disjoints de U et Γ préserve U, c.-à-d. tout élément de Γ induit un difféomorphisme holomorphe de U;
2. L'action de Γ sur U est libre, proprement discontinue et cocompacte.

Autrement dit, U s'identifie à un revêtement galoisien de la variété complexe compacte $X = U/\Gamma$ et Γ s'identifie au groupe de revêtement. Quand U est simplement connexe, Γ est isomorphe au groupe fondamental de X; dans le cas général Γ est un groupe quotient de $\pi_1(X)$. On appelle la donnée de (Y, Γ, U, X) un *groupe kleinien birationnel*. Lorsqu'il n'y a pas d'ambiguïté (cf. la discussion sur l'ensemble limite dans Section 0.1.2), on appellera Γ un groupe kleinien birationnel tout court.

On dit que deux groupes kleiniens birationnels (Y, Γ, U, X) et (Y', Γ', U', X') sont *géométriquement conjugués* s'il existe une application birationnelle $\varphi : Y \dashrightarrow Y'$ qui est biholomorphic sur U telle que $U' = \varphi(U)$ et $\Gamma' = \varphi\Gamma\varphi^{-1}$. Les variétés quotient X et X' sont biholomorphes. Si (Y, Γ, U, X) est un groupe kleinien birationnel et si Γ' est un sous-groupe d'indice fini de Γ, alors (Y, Γ', U, X') est aussi un groupe kleinien birationnel où X' est un revêtement fini de X.

Pour autant que l'auteur sache, la notion de groupes kleiniens birationnels est considérée par Shing-Tung Yau and Fedor Bogomolov en premier, et puis par Serge Cantat.

0.1.2 Comparaison avec les groupes kleiniens classiques

De nos jours un *groupe kleinien* signifie un sous-groupe discret arbitraire de $\mathrm{PGL}_2(\mathbf{C})$. Dans ce texte nous les appellerons *groupes kleiniens classiques* pour éviter la confusion. La théorie des groupes kleiniens classiques est un domaine riche et actif; nous renvoyons les lecteurs à [Mas88] et [Ser05] pour une introduction aux groupes kleiniens classiques.

Domaine de discontnuité et ensemble limite. Si $\Gamma \subset \mathrm{PGL}_2(\mathbf{C})$ est un groupe kleinien classique, alors il existe un unique sous-ensemble maximal $\Omega_\Gamma \subset \mathbb{P}^1$ ouvert, Γ-invariant, non nécessairement connexe, sur lequel Γ agit de manière discontinue; il est appelé le *domaine de discontnuité* de Γ. Son complément $\mathbb{P}^1 \backslash \Omega_\Gamma$ est appelé *l'ensemble limite* de Γ.

En dimension supérieure la notion de domaine de discontnuité et celle d'ensemble limite ne sont pas bien définies. Nous renvoyons les lecteurs à [Kul78] et [CNS13] pour des discussions à ce propos. Même pour un groupe d'automorphismes d'une variété projective complexe lisse, un ouvert invariant sur lequel le groupe agit de manière discontinue et maximal parmi les ouverts ayant cette propriété n'est pas nécessairement unique, voir [CNS13] pour des exemples. C'est pourquoi nous définissons un groupe kleinien birationnel comme un quadruple (Y,Γ,U,X): Y,Γ,U déterminent X mais a priori U n'est pas uniquement déterminé par $\Gamma \subset \mathrm{Bir}(Y)$.

Groupe de fonction cocompact. Soit Γ un groupe kleinien classique de type fini et sans torsion. Théorème de finitude d'Ahlfors affirme que Ω_Γ/Γ est une union finie de surfaces de Riemann de type fini, c.-à-d. courbes quasi-projectives lisses (il y a une lacune dans la preuve originale d'Ahlfors dans [Ahl64], voir par exemple [Sul85] pour une preuve rigoureuse). Si Γ est un groupe kleinien classique de type fini et sans torsion tel que Ω_Γ possède une composante connexe invariante Ω_Γ^0, alors Γ est appelé un *groupe de fonction*; dans ce cas Ω_Γ^0/Γ est une courbe quasi-projective lisse connexe.

Par conséquent notre notion de groupe kleinien birationnel est une généralisation directe de groupe de fonction cocompact. Puisqu'il n'y a pas encore de théorie générale en dimension supérieure, nous préférons les appeler "groupes kleiniens birationnels" plutôt que "groupes de fonction birationnels cocompacts" que nous trouvons peu clair pour ceux qui ne connaissent pas la terminologie classique.

0.1.3 Comparaison avec les groupes kleiniens complexes projectifs

Si un groupe kleinien birationnel (Y,Γ,U,X) satisfait que $Y = \mathbb{P}^n$ et $\Gamma \subset \mathrm{PGL}_{n+1}(\mathbf{C})$, alors nous l'appellerons *un groupe kleinien complexe projectif*. Les groupes kleiniens complexes projectifs sont classifieés en dimension deux (cf. [KO80], [Kli98], [MY93], [CS14]), et en dimension supérieure à deux si la variété quotient est projective (cf. [JR15]). Dans ces classifications déjà obtenues la situation est très rigide, en particulier quand le quotient est projectif; il y a peu d'exemples et ils n'ont pas de déformation riche comme les groupes kleiniens classiques. Nous renvoyons les lecteurs à Chapitre 3 pour une présentation de ces résultats. En général les groupes kleiniens complexes projectifs en dimension ≥ 3 à quotients non nécessairement projectifs sont encore très mystérieux. Nous renvoyons les lecteurs à la référence récente [CNS13] où les auteurs utilisent la terminologie "groupe kleinien complexe".

Pour $n \geq 2$ le groupe de transformations birationnelles de $\mathbb{P}^n$ est beaucoup plus grand que $\mathrm{PGL}_n(\mathbf{C})$: il faut un nombre infini de paramètres pour le décrire. On insiste sur le fait que l'action d'un groupe de transformations birationnelles sur une variété n'est pas une vraie action de groupe au sens ensembliste: il se peut qu'une transformation birationnelle n'est pas définie en certains points, ou n'est pas localement un biholomorphisme en certains points où elle est définie. Néanmoins dans notre situation l'action de Γ sur U est bien sûr une action de groupe bien définie au sens usuel. Données ci-dessus sont quelques raisons pour lesquelles les groupes kleiniens birationnels semblent être beaucoup plus généraux et compliqués que les groupes kleiniens complexes projectifs.

0.2 Uniformisation

Le théorème d'uniformisation de Koebe-Poincaré affirme que toute surface de Riemann simplement connexe est biholomorphe soit à $\mathbb{P}^1$, soit à $\mathbf{C}$, soit à $\mathbb{D}$. Il joue un rôle omniprésent dans l'étude des groupes kleiniens classiques. Les revêtements universels des variétés projectives de dimension ≥ 2 peuvent être très compliqués, sans parler des varíés complexes compactes en général. Une manière raisonnable de les étudier consiste à ajouter des hypothèses de nature algébrique.

Les trois éléments clés d'uniformisation sont le groupe fondamental, le revêtement universel et l'action du groupe de revêtement sur le revêtement universel. Il y a de différentes manières pour spécifier ces données. Par exemple dire qu'une surface de Riemann est revêtue par le disque unité avec groupe de revêtement inclus dans $\mathrm{PSL}_2(\mathbf{R})$ ne dit rien sur les propriétés arithmétiques des coefficients des matrices. Nous nous intéressons ici à des hypothès de nature algébrico-géométrique. Donnés ci-dessous sont trois résultats de ce type dont le premier fait une hopothèse algébrique sur le groupe fondamental, le second fait une une hypothèse algébrico-géométrique sur le revêtement universel, et le troisième fait des hypothèses sur le groupe fondamental et le revêtement universel. Le premier théorme cité ci-dessous a été démontré pour les variétés projectives complexes lisses par Eyssidieux-Katzarkov-Pantev-Ramachandran [KR98], [Eys04], [Eys+12], et a été généralisé aux variétés Kählériennes compactes par Campana-Claudon-Eyssidieux [CCE15]:

Théorème 0.2.1 (Conjecture de Shafarevich linéaire) *Si le groupe fondamental d'une variété Kählérienne compacte Y admet une représentation linéaire fidèle, alors le revêtement universel de Y est holomorphiquement convexe.*

Théorème 0.2.2 (Claudon-Höring-Kollár [CHK13]) *Soient Y une variété projective normale sur $\mathbf{C}$ et $\tilde{Y}$ son revêtement universel. La conjecture d'abondance implique que les assertions suivantes sont équivalentes:*

1. *$\tilde{Y}$ est biholomorphe à une variété quasi-projective.*
2. *$\tilde{Y}$ est biholomorphe à un produit $\mathbf{C}^m \times F$ où $m \geq 0$ et F est une variété projective simplement connexe.*
3. *Il existe un revêtement galoisien fini Y' de Y qui est un fibré au-dessus d'une variété abélienne à fibres simplement connexes.*

Théorème 0.2.3 (Kollár-Pardon [KP12]) *Soit Γ un groupe qui agit librement, proprement discontinûment et de manière cocompacte sur un domaine borné symétrique M de dimension ≥ 2. Soit Y une variété projective complexe lisse, $\pi_1(Y) \to \Gamma$ un homomorphisme surjectif et $\tilde{Y}_\Gamma$ le revêtement galoisien correspondant. Les assertions suivantes s'équivalent:*

1. *$\tilde{Y}_\Gamma$ est biholomorphe à un sous-ensemble semi-algébrique d'une variété projective.*
2. *$\tilde{Y}_\Gamma$ est biholomorphe à $M \times F$ où $m \geq 0$ et F est une variété projective.*
3. *Y est un fibré au-dessus de M/Γ.*

Le souhait principal de Kollár-Pardon [KP12] est de généraliser Théoréme 0.2.2 aux domaines symétriques bornés; ils conjecturent dans [KP12] que les seuls sous-ensembles semi-algébriques des variétés projectives qui revêtent des variétés projectives normales sont biholomorphes à des produits de la forme $\mathbf{C}^m \times M \times F$ où M est un domaine borné symétrique et F est une variété projective simplement connexe. Par plongement de Borel un domaine borné symétrique M se plonge comme un sous-ensemble semi-algébrique dans un espace symétrique hermitien compact dual $\hat{M}$ qui est une variété projective complexe lisse. En plus tout auto-biholomorphisme de M est la restriction d'un automorphisme de $\hat{M}$. Autrement dit, les domaines bornés symétriques donnent lieu à des groupes kleiniens birationnels, qui sont des groupes d'automorphismes. Ainsi l'étude des groupes kleiniens birationnels consiste à mettre, dans le problème d'uniformisation, une hypothèse algébrico-géométrique sur l'action du groupe de revêtement, en essayant de généraliser ce qui se passe pour les domaines bornés symétriques.

Nous disposons de très peu d'exemples de domaines dans des varétés projectives de dimension > 1 qui sont des revêtements infinis des variétés projectives. Soit $m \geq 2$ un entier. Wong [Won77] et Rosay [Ros79] ont prouvé que si un domaine borné à bord lisse dans $\mathbf{C}^m$ revête une variété projective alors il est biholomorphe à la boule unité euclidienne. Frankel [Fra89] a prouvé que les seuls domaines bornés convexes dans $\mathbf{C}^m$ qui revêtent des variétés projectives sont les domaines bornés symétriques. Ces résultats suggèrent que les domaines qui revêtent des variétés projectives sont compliqués et difficiles à décrire lorsqu'ils ne sont pas des domaines bornés symétriques.

À part les $\mathbf{C}^m$s et les domaines bornés symétriques, les seuls exemples de domaines revêtant des variétés projectives de dimension > 1 que l'auteur connaît sont les revêtements universels des fibrations de Kodaira et leurs variantes. Une fibration de Kodaira est une surface projective complexe lisse avec une fibration submersive non-isotriviale au-dessus d'une surface de Riemann hyperbolique à fibres des surfaces de Riemann hyperboliques. Il suit des travaux fondamentaux de Bers sur la structure analytique complexe des espaces de Teichmüller (cf. [Ber60]) que le revêtement universel d'une fibration de Kodaira est une fibration submersive au-dessus du disque unité à fibres toutes biholomorphes aux disques, et qu'il est biholomorphe à un domaine borné dans $\mathbf{C}^2$ (voir [Gri71] pour des variantes en dimension supérieure). Nous verrons qu'il n'existe pas de groupe kleinien birationnel (Y,Γ,U,X) tel que X soit une fibration de Kodaira (cf. Chapitre 5 Section 5.7). Nous verrons aussi qu'un problème majeur non résolu dans notre classification de groupes kleiniens biationnels en dimension deux concerne un type de domaines, s'ils existent, qui sont analogues aux domaines qui revêtent des fibraitons de Kodaira: ces domaines sont aussi des graphes de mouvements holomorphes; le rôle des fibrations de Kodaira est remplacé par certains feuilletages; les espaces de Teichmüller de dimension finie sont remplacés par les espaces de Teichmüller de dimension infinie (voir Chapitre 7).

0.3 Résultats principaux

Nous observons d'abord qu'une variété doit avoir un groupe de transformations birationnelles suffisamment grand pour admettre un groupe kleinien birationnel.

Théorème 0.3.1 *Il n'existe pas de groupe kleinien birationnel agissant sur une variété projective complexe lisse de dimension* ≥ 0.

Tous nos autres résultats sont en dimension deux. Une version courte de notre classification est:

Théorème 0.3.2 *Soit (Y,U,Γ,X) un groupe kleinien birationnel en dimension deux tel que U est simplement connexe et X est kählérienne. Alors quitte à faire une conjugaison géométrique et quitte à prendre un sous-groupe d'indice fini de Γ, nous sommes dans l'une des situations suivantes:*

1. *$Y=\mathbb{P}^2$, $U=\mathbf{C}^2$, Γ est un groupe de translations isomorphe à $\mathbf{Z}^4$ et X est un tore complexe.*
2. *$Y=\mathbb{P}^1\times\mathbb{P}^1$, U est le bidisque $\mathbb{D}\times\mathbb{D}$ et Γ est un réseau irréductible cocompact sans torsion dans* $\mathrm{PSL}_2(\mathbf{R})\times\mathrm{PSL}_2(\mathbf{R})\subset\mathrm{Aut}(\mathbb{P}^1)\times\mathrm{Aut}(\mathbb{P}^1)$.
3. *$Y=\mathbb{P}^2$, U est la boule unité euclidienne $\mathbb{B}^2$, Γ est un réseau cocompact sans torsion dans* $\mathrm{PU}(1,2)\subset\mathrm{PGL}_3(\mathbf{C})=\mathrm{Aut}(\mathbb{P}^2)$.
4. *$\Gamma=\Gamma_1\times\Gamma_2$ est un produit de deux groupes kleiniens classiques dont l'un peut être trivial, U est un produit $D_1\times D_2$, où $D_i\subset\mathbb{P}^1$ est une composante simplement connexe invariante du domaine de discontinuité de Γ_i.*
5. *$Y=\mathbb{P}^1\times\mathbb{P}^1$, $U=D_1\times\mathbf{C}$ où D_1 est une composante simplement connexe invariante du domaine de discontinuité d'un groupe kleinien classique Γ_1. La projection de $\mathbb{P}^1\times\mathbb{P}^1$ sur le premier facteur est équivariant par rapport à un homomorphisme surjectif $\Gamma\to\Gamma_1$. La surface X un fibré principal elliptique au-dessus de D_1/Γ_1. Tout élément de Γ s'écrit de la forme $(x,y)\dashrightarrow(\gamma_1(x),y+R(x))$ où $\gamma_1\in\Gamma_1$ et $R\in\mathbf{C}(x)$ n'a pas de pôles dans D_1.*
6. *$Y=\mathbb{P}^1\times\mathbb{P}^1$, $U=D_1\times\mathbb{P}^1$ where D_1 où D_1 est une composante simplement connexe invariante du domaine de discontinuité d'un groupe kleinien classique Γ_1. La projection de $\mathbb{P}^1\times\mathbb{P}^1$ sur le premier facteur est équivariant par rapport à un isomorphisme $\Gamma\to\Gamma_1$. La surface X est géométriquement réglée au-dessus de D_1/Γ_1.*
7. *$Y=\mathbb{P}^1\times\mathbb{P}^1$. U est contenu dans le sous-ensemble $\mathbb{D}\times\mathbb{P}^1$ et la projection $r:Y\to\mathbb{P}^1$ sur le premier facteur induit une fibration de U au-dessus du disque unité $\mathbb{D}\subset\mathbb{P}^1$ dont les fibres, toutes biholomorphes au disque unité, s'organisent en un mouvement holomorphe. Le bord de U dans $\mathbb{D}\times\mathbb{P}^1$ est aussi le graphe d'un mouvement holomorphe. Γ préserve la fibration r. La surface X est de type général et est munie d'un feuilletage holomorphe régulier transversalement hyperbolique dont toute feuille est dense dans X.*

Remarque 0.3.3 Les groupes kleiniens classiques qui apparaissent dans Théorme 0.3.2 sont des B-groupes. Un B-groupe est un groupe kleinien classique de type fini ayant une composante simplement connexe; donc les B-groupes sont isomorphes aux groupes de surface et sont des exemples particuliers de groupes de fonction (cf. Section 0.1.2). À part les groupes quasi-fuchsiens, il existe aussi des B-groupes plus compliqués découverts par Bers [Ber70], obtenus comme limite dégénérée de groupes quasi-fuchsiens (voir aussi [Mas70], [Abi75], [Mas88]). Bers a conjecturé que tout B-groupe est une limite algébrique de groupes quasi-fuchsiens; ceci a été démontré par Brock-Canary-Minsky (cf. [BCM12] page 4).

Remarque 0.3.4 Dans Théorme 0.3.2, Cas 2) est un cas particulier de Cas 7). L'auteur ignore si Cas 7) contient d'autres exemples que Cas 2). Nous parlerons de cette question dans Chaptitre 7. Nous nous attendons à ce que Cas 7) du Théorme 0.3.2 est réduit à Cas 2); nous le prouverons sous certaines hypothèses supplémentaires. En fait les seuls exemples connus de surfaces de type général munies de feuilletages holomorphes réguliers minimaux sont les quotients du bidisque et la question de savoir s'il existe d'autres exemples de tels feuilletages reste ouverte depuis l'apparition de [Bru97] il y a une vingtaine d'années. S'il n'existe pas de tels feuilletages alors Cas 7) est automatiquement vide.

Dans Cas 7), nous notons par $\partial_h(U)$ le bord de U dans $\mathbb{D}\times\mathbb{P}^1$. Deux questions se posent naturellement: 1) quand est-ce que U est biholomorphe au bidisque? 2) Lorsque U est biholomorphe au bidisque, est-ce que nous sommes nécessairement dans Cas 2)? Nous donnons quelques réponses partielles. Voir Section 7.4.2 pour la notion de mouvement holomorphe à distorsion bornée utilisée dans l'énoncé suivant.

Théorème 0.3.5

1. *Supposons que nous sommes dans Cas 7) de Théorme 0.3.2. Supposons que U est le graphe d'un mouvement holomorphe à distorsion bornée. Alors U est biholomorphe au bidisque.*
2. *Supposons que nous sommes dans Cas 7) de Théorme 0.3.2 et que U est biholomorphe au bidisque. Supposons que $\partial_h(U)$ est localement connexe de mesure de Lebesgue nulle. Alors nous sommes dans Cas 2).*
3. *Supposons que nous sommes dans Cas 7) de Théorme 0.3.2. Supposons que $\partial_h(U)$ est C^1 à un point. Alors nous sommes dans Cas 2).*

Les deux hypothèses artificielles que nous supposons dans la seconde partie du Théorme ci-dessus sont partiellement inspirées par ce qui se passe en dimension une:

Remarque 0.3.6 La conjecture d'Ahlfors, connue sous le nom de «Ahlfors measure conjecture» en anglais, affirme que l'ensemble limite d'un groupe kleinien classique (au sens usuel, c.-à-d. un sous-groupe discret de $\mathrm{PGL}_2(\mathbf{C})$) est soit $\mathbb{P}^1$ tout entier soit de mesure de Lebesgue nulle. La conjecture a été démontrée pour les groupes de fonction par Bonahon [Bon86] et complètement par Calegari-Gabai [CG06] et Agol [Ago]. En ce qui concerne la connexité locale, Mj [Mj14] a prouvé, pour les groupes kleiniens classiques de type fini, que l'ensemble limite est localement connexe s'il est connexe.

Si nous ne supposons plus que U est simplement connexe et que X est Kählérienne, alors nous avons la classification conditionnelle ci-dessous. Voir Section 1.3 pour la notion de transformation birationnelle loxodromique utilisée dans l'énoncé.

Théorème 0.3.7 *Soit (Y,U,Γ,X) un groupe kleinien birationnel en dimension deux. Supposons que Γ n'est pas virtuellement un groupe cyclique engendré par un élément loxodromique et que Γ ne contient pas d'éléments loxodromiques lorsque X est de classe VII. Alors quitte à faire une conjugaison géométrique et quitte à prendre un sous-groupe d'indice fini de Γ, nous sommes dans l'une des situations dans le tableau suivant:*

Groupes kleiniens birationnels en dimension deux				
	Y	U	Γ	X
1	$B \times \mathbb{P}^1$ *où* B *est une surface de Riemann compacte*	$B \times D_1$ *où* D_1 *est une composante invariante d'un groupe kleinien classique* Γ_1	$\{\mathrm{Id}\} \times \Gamma_1 \subset \mathrm{Aut}(B) \times \mathrm{Aut}(\mathbb{P}^1)$	$B \times (D_1/\Gamma_1)$
2	$\mathbb{P}(\mathscr{E})$ *où* $\mathscr{E}$ *est une extension de* $\mathscr{O}_B$ *par* $\mathscr{O}_B$ *et* B *est une surface de Riemann compacte*	*le complément d'une section de la fibration rationnelle*	*isomorphe à* $\mathbf{Z}^2$	*un fibré principal elliptique*
3	*une surface réglée éclatée*	*un ouvert de Zariski dont l'intersection avec toute fibre de la fibration rationnelle est* $\mathbf{C}^*$	*isomorphe à* $\mathbf{Z}$	*une fibration elliptique dont toute fibre singulière est du type* mI_0

	Y	U	Γ	X
4	$B \times \mathbb{P}^1$ *où* B *est une courbe elliptique*	$B \times D_1$ *où* D_1 *est une composante invariante d'un groupe kleinien classique* Γ_1	$\exists \rho : \Gamma_1 \to \mathrm{Aut}(B), \forall \gamma \in \Gamma, \exists \gamma_1 \in \Gamma_1, \gamma = (\rho(\gamma_1), \gamma_1) \in \mathrm{Aut}(B) \times \mathrm{Aut}(\mathbb{P}^1)$	*fibré en* B *au-dessus de* D_1/Γ_1
5	*une surface géométriquement réglée indécomposable au-dessus d'une courbe elliptique ayant une section d'auto-intersection nulle*	*le complément de la section d'auto-intersection nulle*	*isomorphe à* $\mathbf{Z}^2$	*tore complexe*
6	*une surface géométriquement réglée décomposable au-dessus d'une courbe elliptique*	*le complément de deux sections disjointes*	*isomorphe à* $\mathbf{Z}$	*tore complexe*
7	$\mathbb{P}^2$	$\mathbf{C}^2$	*isomorphe à* $\mathbf{Z}^4$	*tore complexe*
8	$\mathbb{P}^2$	$\mathbf{C} \times \mathbf{C}^*$	*isomorphe à* $\mathbf{Z}^3$	*tore complexe*
9	$\mathbb{P}^2$	$\mathbf{C}^* \times \mathbf{C}^*$	*isomorphe à* $\mathbf{Z}^2$	*tore complexe*
10	$\mathbb{P}^2$	$\mathbf{C}^2$	*un groupe de transformations affines, une extension de* $\mathbf{Z}^2$ *par* $\mathbf{Z}^2$	*surface de Kodaira primaire*
11	$\mathbb{P}^2$	$\mathbf{C}^2 \setminus \{0\}$	*isomorphe à* $\mathbf{Z}$	*surface de Hopf*

	Y	U	Γ	X
12	$\mathbb{P}^2$	$\mathbb{H}\times\mathbf{C}$	*un groupe résoluble de transformations affines*	*surface d'Inoue*
13	$\mathbb{P}^2$	$\mathbb{B}^2$	*un réseau cocompact dans* $\mathrm{PU}(1,2)$	*quotient de la boule*
14	$\mathbb{P}^1\times\mathbb{P}^1$	$\mathbb{D}^1\times\mathbb{D}^1$	*un réseau irréductible cocompact dans* $\mathrm{PSL}_2(\mathbf{R})\times\mathrm{PSL}_2(\mathbf{R})$	*quotient du bidisque*
15	*une surface de Hirzebruch éclatée au-dessus d'au plus deux fibres*	*un ouvert de Zariski dont l'intersection avec toute fibre de la fibration rationnelle est* $\mathbf{C}^*$	*un groupe cyclique engendré par* $(x,y)\mapsto(ax,by)$	*surface de Hopf*
16	*une surface de Hirzebruch éclatée au-dessus d'au plus une fibre*	*un ouvert de Zariski dont l'intersection avec toute fibre de la fibration rationnelle est* $\mathbf{C}^*$	*un groupe cyclique engendré par* $(x,y)\mapsto(x+a,by)$	*surface de Hopf*
17	$\mathbb{P}^1\times\mathbb{P}^1$	$D_1\times D_2$ *où* D_i *est une composante invariante d'un groupe kleinien classique* Γ_i	$\Gamma_1\times\Gamma_2\subset\mathrm{PGL}_2(\mathbf{C})\times\mathrm{PGL}_2(\mathbf{C})$	$(D_1/\Gamma_1)\times(D_2/\Gamma_2)$

	Y	U	Γ	X
18	$\mathbb{P}^1 \times \mathbb{P}^1$	$D_1 \times \mathbf{C}^*$ *où* D_1 *est une composante invariante d'un groupe kleinien classique* Γ_1	*isomorphe à une extension centrale de* Γ_1 *par* $\mathbf{Z}$*, tout élément a la forme* $(x,y) \dashrightarrow (\gamma_1(x), R(x)y)$ *où* $\gamma_1 \in \Gamma_1$ *et* $R \in \mathbf{C}(x)^*$	*fibré principal elliptique au-dessus de* D_1/Γ_1
19	$\mathbb{P}^1 \times \mathbb{P}^1$	$D_1 \times \mathbf{C}$ *où* D_1 *est une composante invariante d'un groupe kleinien classique* Γ_1	*isomorphe à une extension centrale de* Γ_1 *by* $\mathbf{Z}^2$*, tout élément a la forme* $(x,y) \dashrightarrow (\gamma_1(x), y + R(x))$ *où* $\gamma_1 \in \Gamma_1$ *et* $R \in \mathbf{C}(x)$	*fibré principal elliptique au-dessus de* D_1/Γ_1
20	$\mathbb{P}^1 \times \mathbb{P}^1$	$D_1 \times \mathbb{P}^1$ *où* D_1 *est une composante invariante d'un groupe kleinien classique* Γ_1	*isomorphe à* Γ_1*, préserve la projection de* $\mathbb{P}^1 \times \mathbb{P}^1$ *sur le premier facteur*	*surface géométriquement réglée au-dessus de* D_1/Γ_1
21	$\mathbb{P}^1_1 \times \mathbb{P}^1_2$	*le graphe d'un mouvement holomorphe au-dessus de* $\mathbb{D} \subset \mathbb{P}^1_1$ *d'un domaine connexe* $U_0 \subset \mathbb{P}^1_2$	*préserve la projection sur* $\mathbb{P}^1_1$	*surface de type général munie d'un feuilletage régulier minimal*

La surface Y est rationnelle sauf dans les six premiers cas. Le groupe Γ est un groupe d'automorphismes sauf dans les quatres derniers cas. Dans les cas 18), 19), 20), il se peut que Γ n'est conjugué au groupe d'automorphismes d'aucune surface projective.

Remarque 0.3.8 Les groupes kleiniens classiques apparus dans Théorème 0.3.7 sont des groupes de fonction cocompacts. La conjecture de densité de Bers-Sullivan-Thurston a prédit que tout groupe kleinien de type fini est une limite algébrique de groupes géométriquement finis, et a été prouvée par Namazi-Souto [NS12], en tant qu'une généralisation du résultat de Brock-Canary-Minsky's que nous avons mentionné dans Remarque 0.3.3.

Remarque 0.3.9 Nous expliquons maitenant comment Théorème 0.3.2 se déduit de Théorème 0.3.7. Sous l'hypothèse que U est simplement connexe et que X est kählérienne, le groupe Γ isomorphe au groupe de Kähler $\pi_1(X)$. Par Théorème de Hodge l'abélianisation d'un groupe de Kähler est de rang pair, donc n'est pas cyclique. Ainsi l'hypothèse de Théorème 0.3.2 implique l'hypothèse de Théorème 0.3.7. Il suffit donc de remarquer que dans la liste donnée dans Théorème 0.3.7 les seuls cas où U est simplement connexe et X est kählérienne sont les sept cas dans Théorème 0.3.2.

0.4 Structures birationnelles

Nous introduisons dans cette section la notion de structure birationnelle qui a déjà émergé dans les travaux de Dloussky [Dlo16]. C'est une généralisation de structure géométrique d'Ehresmann. L'une des motivations pour cette généralisation est la suivante: la surface de Riemann quotient d'un groupe kleinien classique est munie d'une $(\mathrm{PGL}_2(\mathbf{C}),\mathbb{P}^1)$-structure; si (Y,Γ,U,X) est un groupe kleinien birationnel, alors X est munie d'une $(\mathrm{Bir}(Y),Y)$-structure. Une autre motivation est que les structures birationnelles comprennent toutes les structures géométriques modelées sur les actions algébriques par des groupes algébriques. Nous n'allons pas discuter des structures birationnelles dans le corps de cette thèse et nous renvoyons les lecteurs à [Zhab] et Appendice B Section 2.2 pour plus de détails et pour les preuves des propositions données dans cette section.

Définition 0.4.0.1 *Soit V une variété complexe. Soit Y une variété projective complexe lisse. Une $(\mathrm{Bir}(Y),Y)$-structure sur V est la donnée d'un atlas maximal de cartes locales $\varphi_i : U_i \to Y_i$ tel que*

- *les U_i sont des ouverts de V et forment un recouvrement;*
- *les Y_i sont des variétés projectives lisses birationnelles à Y;*
- *les φ_i sont des biholomorphismes sur image;*
- *les changements de coordonnées $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ sont des difféomorphismes holomorphes qui s'étendent en applications birationnelles de Y_j vers Y_i.*

Définition 0.4.0.2 *Soit V une variété complexe. Soit Y une variété projective complexe lisse. Une $(\mathrm{Bir}(Y),Y)$-structure stricte sur V est la donnée d'un atlas maximal de cartes locales $\varphi_i : U_i \to Y$ tel que*

- *les U_i sont des ouverts de V et forment un recouvrement;*
- *les φ_i sont des biholomorphismes sur image;*
- *les changements de coordonnées $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ sont des difféomorphismes holomorphes qui s'étendent en transformations birationnelles de Y.*

Soit Y' un modéle lisse birationnel de Y. Il découle directement de la défintion qu'une $(\mathrm{Bir}(Y),Y)$-structure sur V la même chose qu'une $(\mathrm{Bir}(Y'),Y')$-structure sur V, et qu'une $(\mathrm{Bir}(Y),Y)$-structure stricte induit une $(\mathrm{Bir}(Y),Y)$-structure. Mais en général une $(\mathrm{Bir}(Y),Y)$-structure n'induit pas une $(\mathrm{Bir}(Y),Y)$-structure stricte. Une variété projective lisse birationnelle à Y admet toujours une $(\mathrm{Bir}(Y),Y)$-structure, mais non nécessairement une $(\mathrm{Bir}(Y),Y)$-structure stricte. Par exemple si Y est une surface K3 alors un éclatement de Y ne possède pas de $(\mathrm{Bir}(Y),Y)$-structures strictes (cf. [Zhab] et Appendice B Section 2.2).

Comme pour les structures géométriques classiques, nous disposons des applications développantes et des représentations d'holonomie pour les structures birationnelles:

Proposition 0.4.1 *Soit Y une variété projective complexe lisse. Soit V une $(\mathrm{Bir}(Y),Y)$-variété. Notons par $\tilde{V}$ le revêtement universel de V et π l'application quotient. Fixons un point de base $v \in V$ et un point $w \in \tilde{V}$ tels que $\pi(w) = v$. Il existe un modèle birationnel lisse Z de Y, a homomorphisme* $\mathrm{Hol} : \pi_1(V,v) \to \mathrm{Bir}(Z)$ *et une application méromorphe $\pi_1(V,v)$-equivariante* $\mathrm{Dev} : \tilde{V} \dashrightarrow Z$ *tels que*

$$\forall f \in \pi_1(V,v), \mathrm{Dev} \circ f = \mathrm{Hol}(f) \circ \mathrm{Dev}.$$

Si $(Z',\mathrm{Hol}',\mathrm{Dev}')$ un autre tel triple, alors il existe une application birationnelle σ de Z vers Z' telle que $\mathrm{Hol}' = \sigma \,\mathrm{Hol}\, \sigma^{-1}$ *et* $\mathrm{Dev}' = \sigma \circ \mathrm{Dev}$. *Nous pouvons choisir $(Z,\mathrm{Hol},\mathrm{Dev})$ de sorte que* Dev *soit holomorphe à w.*

Remark 0.4.2 Une application développante est localement birationnelle, c.-à-d. elle s'écrit sous une expression birationnelle dans certaines cartes locales holomorphes.

Comme les variétés rationnelles ont les groupes de transformations les plus compliqués, il est naturel de poser:

Question 0.4.3 *1. Est-ce qu'une $(\mathrm{Bir}(\mathbb{P}^n),\mathbb{P}^n)$-structure induit toujours une $(\mathrm{Bir}(\mathbb{P}^n),\mathbb{P}^n)$-structure stricte?*

2. *Est-ce que toute variéte rationnelle lisse X de dimension n admet une* $(\mathrm{Bir}(\mathbb{P}^n),\mathbb{P}^n)$*-structure stricte?*

Proposition 0.4.4 *Questions 0.4.3.1 et 0.4.3.2 sont équivalentes.*

Une variété rationnelle lisse X de dimension n est dite *uniformément rationnelle* si tout $x \in X$ admet un voisinage Zariski ouvert qui est isomorphe à un ouvert de Zariski dans l'espace affine $\mathbb{A}^n$. Étant rationnelle, X possède un tel point; il s'agit de savoir si un tel voisinage existe pour tout point, d'où la terminologie “uniformément rationnelle”. Gromov a demandé:

Question 0.4.5 (Gromov [Gro89] page 885, voir aussi [BB14]) *Est-ce que toute variété rationnelle complexe lisse est uniformément rationnelle?*

Il se trouve que la question de Gromov est équivalente à Question 0.4.3 dont la formulation ne semble pas être complètement algébrique à première vue:

Proposition 0.4.6 *Une variété projective rationnelle lisse sur* $\mathbf{C}$ *de dimension n est uniformément rationnelle si et seulement si elle admet une* $(\mathrm{Bir}(\mathbb{P}^n),\mathbb{P}^n)$*-structure stricte.*

Toute variété rationnelle lisse complexe de dimension une ou deux est uniformément rationnelle. Autrement dit pour les structures birationnelles modelées sur $\mathbb{P}^2$, il n'est pas nécessaire de distinguer les structures birationnelles strictes et les non-strictes. La question de Gromov est encore ouverte en dimension ≥ 3 (cf. [BB14]).

0.5 Stratégie et plan

0.5.1 Stratégie principale

Soit (Y,U,Γ,X) un groupe kleinien birationnel en dimension deux. Pour simplicité et pour expliquer la stratégie dans le cas le plus important, nous supposons que Y est une surface rationnelle, que X est kählérienne et que U est simplement connexe. Le groupe fondamental $\pi_1(X)$ peut alors être identifié à Γ. Nous donnons ici une esquisse des étapes de la preuve de Théorème 0.3.2. Les quatres sujets principaux dans la preuve sont:

1. dynamique des transformations birationnelles ou des groupes de transformations birationnelles;
2. groupes fondamentaux des variétés kählériennes compactes c.-à-d. les groupes de Kähler;

3. feuilletages holomorphes sur des surfaces projectives complexes;
4. propriétés analytiques complexes des espaces de Teichmüller.

Nous expliquons brièvement comment ces sujets intéragissent parmi eux. Manin [Man86] a introduit une action fidèle de $\mathrm{Bir}(Y)$ par isométries sur un espace hyperbolique $\mathbb{H}_Y$ de dimension infinie. Ainsi nous obtenons une action de $\Gamma = \pi_1(X)$ sur $\mathbb{H}_Y$. Très fortes contraintes sur les actions isométriques des groupes de Kähler sur les espaces hyperboliques de dimension finie ont été trouvées par Carleson-Toledo [CT89] et elles ont été généralisées par Delzant-Py [DP12] aux espaces hyperboliques de dimension infinie; leurs travaux sont basés sur la théorie très développée des applications harmoniques sur les variétés kählériennes, c.-à-d. la théorie de Hodge non abélienne. Nous comibnons les résultats de Delzant-Py avec la propriété kleinienne de Γ pour montrer que l'action de Γ sur $\mathbb{H}_Y$ admet des points fixes dans $\mathbb{H}_Y$ ou dans son bord. Cantat a montré dans [Can11] que les actions sur Y des groupes de transformations birationnelles, de type fini et avec des points fixes dans $\mathbb{H}_Y \cup \partial\mathbb{H}_Y$, ont des caractéristiques très particulières. En ce qui concerne les groupes kleiniens birationnels, nous déduisons des travaux de Cantat que soit Γ est conjugué à un groupe d'automorphismes, soit à un groupe préservant une fibration rationnelle.

Si Γ est conjugué à un groupe d'automorphismes, alors nous pouvons réduire le problème aux groupes kleiniens complexes projectifs. Comme nous avons déjà dit, les groupes kleiniens complexes projectifs en dimension deux sont complètement classifiés par Kobayashi-Ochiai, Mok-Yeung, Klingler et Cano-Seade.

Considérons le cas où Γ préserve une fibration rationnelle. Nous observons que la fibration rationnelle induit un feuilletage holomorphe régulier sur X. Brunella a obtenu une classification quaisment complète des feuilletages holomorphes réguliers sur des surfaces complexes compactes dans [Bru97]; il reste un cas ouvert concernant les feuilletages réguliers minimaux transversalement hyperboliques. En mettant la classification de Brunella sur place avec les propriétés des groupes de transformations birationnelles préservant une fibration rationnelle, nous examinons quel feuilletage sur X pourrait être réalisé par un groupe kleinien birationnel et comment il serait réalisé. Un résultat technique que nous allons utiliser à plusieurs reprises dans cette partie de la preuve est une description détaillée des centralisateurs des transformations birationnelles préservant une fibration rationnelle. Cette description a été précédemment obtenue par l'auteur [Zhaa] et est donnée dans Appendice A.

Pour les feuilletages minimaux transversalement hyperboliques, notre classification n'est pas complète. Dans ce cas nous construisons une lamination par disques holomorphes dans le bord de U dans Y. En utilisant le λ-lemme de Slodkowski et les propriétés universelles des espaces de Teichmüller, nous relions la dynamique de Γ à la dynamique des groupes modulaires de Teichmüller sur les espaces de Teichmüller et les espaces fibrés de Bers.

0.5.2 Plan

Chapitre 1 présente les préliminaires nécessaires sur les groupes de transformations birationnelles des surfaces projectives. Il s'agit de l'action de $\mathrm{Bir}(\mathbb{P}^2)$ sur un espaces hyperbolique de dimension infinie, la classification des éléments de $\mathrm{Bir}(\mathbb{P}^2)$ selon leurs croissances de degrés, la classification des sous-groupes de type fini de $\mathrm{Bir}(\mathbb{P}^2)$ et la classification des sous-groupes abéliens libres de $\mathrm{Bir}(\mathbb{P}^2)$.

Dans Chapitre 2 nous prouvons qu'il n a pas de groupes kleiniens birationnels sur les variétés projectives de dimension de Kodaira ≥ 0. C'est très lié à un théorème de Nakamura-Ueno que nous présenterons une preuve.

Dans Chapitre 3 nous présentons la classification déjà connue des groupes kleiniens complexes projectives. Nous en donnons une preuve dans le cas où X est kählérienne, sauf la rigidité par déformation des quotients de la boule.

Chapitre 4 concerne les représentations des groupes de Kähler. D'abord nous expliquons comment les techniques d'applications harmoniques permettent de donner de contraintes sur les actions de groupes de Kähler sur les espaces hyperboliques. Puis nous déduisons Théorème de Delzant-Py sur les représentations des groupes de Kähler dans $\mathrm{Bir}(\mathbb{P}^2)$ de leurs travaux sur actions de groupes de Kähler sur les espaces hyperboliques. À la fin nous appliquons Théorème de Delzant-Py aux groupes kleiniens birationnels, et prouvons qu'ils fixent des points dans $\mathbb{H}_Y \cup \partial \mathbb{H}_Y$.

Dans Chaptitre 5 nous étudions les groupes kleiniens birationnels préservant une fibration rationnelle. Soit (Y, Γ, U, X) un groupe kleinien birationnel en dimension deux tel que Γ préserve une fibration rationnelle. Comme nous avons dit dans Section 0.5.1, dans ce cas X est munie d'un feuilletage holomorphe régulier. Nous commençons ce chapitre par une présentation de la classification de Brunella des feuilletages réguliers sur des surfaces complexes compactes. Ensuite nous divisons l'étude en plusieurs sous-cas, selon le type de feuilletage sur X et selon la dynamique de Γ. Toute situation possible sera examinée dans ce chapitre sauf celle des feuilletages minimaux transversalement hyperboliques. Puisqu'il y a beaucoup de sous-cas dans ce chapitre, nous recommendons les lecteurs de lire Chaptire 6 en parallèle.

Chaptitre 6 est un chapitre récapulatif où les résultats des chapitres précédents sont regroupés. Il sert aussi comme un guide pour lire les chapitres précédents, surtout Chapitre 5. Nous recommendons les lecteurs de lire Chaptire 6 en parallèle avec les chapitres précédents.

Dans Chapter 7 nous étudions Cas 7) de Théorème 0.3.2. La première section du chapitre est une introduction aux feuilletages minimaux transversalement hyperboliques sur des surfaces de type général, dans l'état actuell du sujet. Puis nous employons essentiellement un approche de théorie de Teichmüller. Nous construisons un mouvement holomorphe associé à notre groupe kleinien birationnel et en utlisant ce mouvement holomorphe nous prouvons ensuite quelques résultats qui renforcent Cas 7) de Théorème 0.3.2. Nous allons relier Γ aux groupes modulaires de Teichmüller en dimension infinie. Les préliminaires sur les mouvements holomorphes et sur les espaces de Teichmüller seront rappelés.

CHAPTER 1

GROUPS OF BIRATIONAL TRANSFORMATIONS

This chapter is a glossary of notions and theorems on groups of birational transformations of surfaces that we will use in this thesis. We refer the reader to [Can18] and the references therein for most materials presented in this chapter except for the last section. Background on birational geometry of surfaces can be found in [Bea96] and [Bar+04].

1.1 Birational transformations

Let Y be a complex projective manifold. An automorphism of Y is a holomorphic diffeomorphism from Y to itself. The group of automorphisms of Y has a natural structure of complex Lie group and is denoted by $\mathrm{Aut}(Y)$. The connected component of the identity is denoted by $\mathrm{Aut}^0(Y)$. A birational transformation of Y is an isomorphism between two Zariski open and dense subsets of Y which can not be extended to any larger Zariski open subset. Equivalently a birational transformation of Y is a $\mathbf{C}$-algebra automorphism of the function field of Y. The group of birational transformations of Y is denoted by $\mathrm{Bir}(Y)$. It contains $\mathrm{Aut}(Y)$ as a subgroup.

The group of birational transformations of $\mathbb{P}^n$ is called the Cremona group. The Cremona group is the $\mathbf{C}$-algebra automorphism group of the field of rational functions $\mathbf{C}(X_1,\cdots,X_n)$. The elements of the Cremona group have a concrete description in homogeneous coordinates: they are of the form

$$[x_0;\cdots;x_n] \dashrightarrow [P_0;\cdots;P_n]$$

where $P_i, 0\leq i\leq n$ are homogeneous polynomials in $(x_0,\cdots,x_n)$ and they have an inverse map in the same form.

In this text we almost only deal with algebraic surfaces. Unless stated otherwise, the Cremona group always means $\mathrm{Bir}(\mathbb{P}^2)$. Here are two subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ that we will encounter several times in this text.

The toric subgroup In the projective plane $\mathbb{P}^2$, the complement of three coordinates axes is isomorphic to $\mathbf{C}^* \times \mathbf{C}^*$. A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z})$ acts by monomial transformation on $\mathbf{C}^* \times \mathbf{C}^*$: $(x,y) \mapsto (x^a y^b, x^c y^d)$. The group $\mathbf{C}^* \times \mathbf{C}^*$ acts on itself by multiplication. We obtain thus an embedding of $\mathbf{C}^* \times \mathbf{C}^* \rtimes \mathrm{GL}_2(\mathbf{Z})$ into the Cremona group $\mathrm{Bir}(\mathbb{P}^2)$. We call its image *the toric subgroup* of the Cremona group. The toric subgroup depends on a choice of homogeneous coordinates in $\mathbb{P}^2$. Different choices of coordinates yield conjugated subgroups of the Cremona group and the conjugation is by an element of $\mathrm{PGL}_3(\mathbf{C})$.

The Jonquières group. Fix an affine chart of $\mathbb{P}^2$ with coordinates (x,y). *The Jonquières group* is the subgroup of the Cremona group of all transformations of the form

$$(x,y) \mapsto \left(\frac{ax+b}{cx+d}, \frac{A(x)y+B(x)}{C(x)y+D(x)}\right), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{C}), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{C}(x)).$$

In other words, the Jonquières group is the maximal group of birational transformations of $\mathbb{P}^1 \times \mathbb{P}^1$ permuting the fibres of the projection onto the first factor; it is isomorphic to the semidirect product $\mathrm{PGL}_2(\mathbf{C}) \ltimes \mathrm{PGL}_2(\mathbf{C}(x))$. A different choice of the affine chart yields a conjugation by an element of $\mathrm{PGL}_3(\mathbf{C})$. More generally a conjugation by an element of the Cremona group yields a maximal group preserving a pencil of rational curves; conversely any two such groups are conjugated in $\mathrm{Bir}(\mathbb{P}^2)$. The Jonquières group is “infinite-dimensional” since it contains all transformations of the form $(x,y) \mapsto (x, y+P(x))$ where $P(x)$ is a polynomial of arbitrary degree. We denote by Jonq the Jonquières group and by Jonq_0 the subgroup $\mathrm{PGL}_2(\mathbf{C}(x))$.

1.2 An infinite dimensional hyperbolic space

Let Y be a smooth complex projective surface. Denote by $\mathrm{H}^{1,1}(Y,\mathbf{Z})$ the intersection of $\mathrm{H}^{1,1}(Y,\mathbf{R})$ with the image of $\mathrm{H}^2(Y,\mathbf{Z})$ in $\mathrm{H}^2(Y,\mathbf{R})$; it is a free abelian group of rank $\rho(Y)$. The intersection form is a symmetric quadratic form on $\mathrm{H}^{1,1}(Y,\mathbf{Z})$ with signature $(1, \rho(Y)-1)$. An automorphism of Y induces a pull-back linear transformation on $\mathrm{H}^{1,1}(Y,\mathbf{Z})$ which preserves the intersection form.

If $p : Y' \to Y$ is a birational morphism, then the induced pull-back linear map $p^* : \mathrm{H}^{1,1}(Y,\mathbf{Z}) \to \mathrm{H}^{1,1}(Y',\mathbf{Z})$ is injective; $\mathrm{H}^{1,1}(Y',\mathbf{Z})$ is the orthogonal sum of $p^*(\mathrm{H}^{1,1}(Y,\mathbf{Z}))$ with the subspace generated by exceptional curves of p. If $p_1 : Y_1 \to Y$ and $p_2 : Y_2 \to Y$ are two birational morphisms then there are birational morphisms $q_1 : Y_3 \to Y_1$ and $q_2 : Y_3 \to Y_2$ such that $p_1 \circ q_1 =$

$p_2 \circ q_2$.

Thus we can define the direct limit of the groups $\mathrm{H}^{1,1}(Y', \mathbf{Z})$ where $p : Y' \to Y$ runs over all birational morphisms over Y. The limit will be denoted by $\mathrm{PM}(Y)$ and is called the *Picard-Manin* space. It is a free abelian group of infinite rank with a symmetric bilinear form of signature $(1, \infty)$. By construction $\mathrm{H}^{1,1}(Y, \mathbf{Z})$ embeds into $\mathrm{PM}(Y)$.

Consider two birational morphisms $p_1 : Y_1 \to Y$ and $p_2 : Y_2 \to Y$. A point $z_1 \in Y_1$ is said to be equivalent to $z_2 \in Y_2$ if $p_1^{-1} \circ p_2$ is a local isomorphism at z_2 and sends z_2 to z_1. The *bubble space* $\mathscr{B}(Y)$ is the union of all points of surfaces over Y modulo the above equivalence relation. To any point of $z \in \mathscr{B}(Y)$ we associate an element e_z of $\mathrm{PM}(Y)$ as follows. If z is represented by $z_1 \in Y_1$ then we consider the blow-up $Y_1' \to Y_1$ at z_1 and let e_z be the divisor class of the exceptional divisor; it is an element of $\mathrm{H}^{1,1}(Y_1', \mathbf{Z}) \subset \mathrm{PM}(Y)$. We have a direct sum decomposition

$$\mathrm{PM}(Y) = \mathrm{H}^{1,1}(Y, \mathbf{Z}) \oplus \bigoplus_{z \in \mathscr{B}(Y)} \mathbf{Z} e_z.$$

Denote by $\mathscr{Z}(Y)$ the completion of $\mathrm{PM}(Y) \otimes \mathbf{R}$. An element of $\mathscr{Z}(Y)$ can be written as an infinite sum $u + \sum_{z \in \mathscr{B}(Y)} a_z e_z$ where $u \in \mathrm{H}^{1,1}(Y, \mathbf{Z})$ and $\sum_{z \in \mathscr{B}(Y)} a_z^2 < +\infty$. The intersection form extends continuously to $\mathscr{Z}(Y)$. Let $\kappa \in \mathrm{H}^{1,1}(Y, \mathbf{Z})$ be an ample class. The set of vectors v of $\mathscr{Z}(Y)$ such that $v \cdot v = 1$ and $v \cdot \kappa > 0$ is an infinite dimensional hyperbolic space. We denote it by $\mathbb{H}_Y$.

Let $f \in \mathrm{Bir}(Y)$. There is a sequence of blow-ups $Y' \to Y$ such that f lifts to a morphism $f' : Y' \to Y$. The pull-back action of f' induces an isometry $(f')^* : \mathscr{Z}(Y) \to \mathscr{Z}(Y')$. Identifying $\mathscr{Z}(Y)$ with $\mathscr{Z}(Y')$, we obtain an isometry f^* of $\mathscr{Z}(Y)$. Denote by f_* the isometry $(f^{-1})^*$.

Theorem 1.2.1 (Manin [Man86]) *The map $f \mapsto f_*$ is an injective homomorphism from* $\mathrm{Bir}(Y)$ *to the group of isometries of $\mathscr{Z}(Y)$. Each f_* preserves $\mathbb{H}_Y$ and this gives an injective homomorphism* $\mathrm{Bir}(Y) \to \mathrm{Isom}(\mathbb{H}_Y)$.

1.3 Degree growth

Let $f \in \mathrm{Bir}(Y)$. Let $\kappa \in \mathrm{H}^{1,1}(Y, \mathbf{R})$ be an ample class of self-intersection 1. The *degree* of f with respect to κ is $\deg_\kappa(f) = f_*\kappa \cdot \kappa$. The *translation length* of f_* on $\mathbb{H}_Y$ is $L(f_*) = \inf_{x \in \mathbb{H}_Y} d(x, f_*(x))$.

Proposition 1.3.1 *The sequence $(\deg_\kappa(f^n))^{\frac{1}{n}}$ converges to a real number $\lambda(f) \geq 1$, called the dynamical degree of f; its logarithm $\log(\lambda(f))$ is the translation length $L(f_*)$.*

Theorem 1.3.2 ([Giz80], [Can01a], [Can01b], [DF01]) *Elements of* $\mathrm{Bir}(Y)$ *are classified into four types:*

1. *The sequence* $(\deg_\kappa(f^n))^{\frac{1}{n}}$ *is bounded, f is birationally conjugate to an automorphism of a smooth birational model of Y and a positive iterate of f lies in the connected component of identity of the automorphism group of that surface. f_* is an elliptic isometry of $\mathbb{H}_Y$. We call f an* elliptic *element.*
2. *The sequence* $(\deg_\kappa(f^n))^{\frac{1}{n}}$ *grows linearly, f preserves a unique pencil of rational curves and f is not conjugate to an automorphism of any birational model of Y. f_* is a parabolic isometry of $\mathbb{H}_Y$. We call f a* Jonquières twist*.*
3. *The sequence* $(\deg_\kappa(f^n))^{\frac{1}{n}}$ *grows quadratically, f is conjugate to an automorphism of a smooth birational model preserving a unique genus one fibration. f_* is a parabolic isometry of $\mathbb{H}_Y$. We call f a* Halphen twist*.*
4. *The sequence* $(\deg_\kappa(f^n))^{\frac{1}{n}}$ *grows exponentially, i.e. $\lambda(f) > 1$. f_* is a loxodromic isometry of $\mathbb{H}_Y$. We call f a* loxodromic *element.*

Loxodromic elements have very rich dynamics (cf. [Can01a], [DF01], [BD05], [Duj06]) and a general birational transformation is loxodromic in the following sense:

Theorem 1.3.3 (J-Y. Xie [Xie15]) *Let $d \geq 2$ be an integer. Denote by* $\mathrm{Bir}_d(\mathbb{P}^2)$ *the space of birational transformations of degree d. Then for any $\lambda < d$, the subset* $\{f \in \mathrm{Bir}_d(\mathbb{P}^2) | \lambda(f) > \lambda\}$ *is Zariski open and dense in* $\mathrm{Bir}_d(\mathbb{P}^2)$.

1.4 Tits alternative

We say that a group G satisfies *the Tits alternative* if any subgroup of G contains either a solvable subgroup of finite index, or a non-abelian free subgroup; we say that a group G satisfies *the finitely generated Tits alternative* if any finitely generated subgroup of G contains either a solvable subgroup of finite index, or a non-abelian free subgroup. Tits [Tit72] proved that any linear algebraic group over a field of characteristic zero satisfies the Tits alternative and any linear algebraic group over a field of positive characteristic satisfies the finitely generated Tits alternative. Lamy [Lam01a] proved that the group $\mathrm{Aut}(\mathbf{C}^2)$ of polynomial automorphisms of the affine plane $\mathbf{C}^2$ satisfies the Tits alternative. For $\mathrm{Bir}(\mathbb{P}^2)$, the Tits alternative has been proved by Cantat [Can11] for finitely generated subgroups and in general by Urech [Ure]. Furthermore for solvable subgroups, [Can11], [Dés15], [Ure] refined the Tits alternative according to geometric properties of the birational action on $\mathbb{P}^2$. We summarize the classification of subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ in the following theorem. We list the contribution to each case in the classification: Case 1) is due to Weil [Wei55], Case 4) is due to Gizatullin [Giz80], Case 5) is due to Blanc-Cantat [BC16], Case 6) is due to Déserti [Dés15] and Urech [Ure], the bulk of the classification is done by Cantat [Can11] and is finished by Urech [Ure].

Theorem 1.4.1 (Strong Tits alternative for $\mathrm{Bir}(Y)$) *Let Y be a smooth complex projective surface and let Γ be an infinite subgroup of $\mathrm{Bir}(Y)$. Then up to taking a finite index subgroup Γ fits in one of the following mutually exclusive situations:*

1. *There is a projective surface Y' and a birational map $\phi : Y \to Y'$ such that $\Gamma' = \phi\Gamma\phi^{-1}$ is contained in $\mathrm{Aut}^0(Y')$.*
2. *There is a projective surface Y' and a birational map $\phi : Y \to Y'$ such that $\Gamma' = \phi\Gamma\phi^{-1}$ preserves a rational fibration on Y'. All elements of Γ are elliptic but their degrees are not uniformly bounded and the group Γ is not conjugate to a group of automorphisms of any projective surface.*
3. *There is a projective surface Y' and a birational map $\phi : Y \to Y'$ such that $\Gamma' = \phi\Gamma\phi^{-1}$ preserves a rational fibration on Y'. At least one element of Γ is a Jonquières twist, i.e. has linear degree growth.*
4. *There is a projective surface Y' and a birational map $\phi : Y \to Y'$ such that $\Gamma' = \phi\Gamma\phi^{-1}$ acts by automorphisms and preserves a genus one fibration. The group Γ is virtually a free abelian group of rank ≤ 8.*
5. *Γ is virtually a cyclic group generated by a loxodromic element.*
6. *Γ is solvable but not virtually abelian. There is a birational map $\phi : Y \to \mathbf{C}^* \times \mathbf{C}^*$ such that $\Gamma' = \phi\Gamma\phi^{-1}$ is virtually contained in the toric subgroup $\mathrm{Aut}(\mathbf{C}^* \times \mathbf{C}^*) = (\mathbf{C}^* \times \mathbf{C}^*) \rtimes \mathrm{GL}_2(\mathbf{Z})$. The loxodromic elements of Γ form an infinite cyclic group.*
7. *Γ contains a non-abelian free group all of whose non-trivial elements are loxodromic.*
8. *Γ is a torsion group, is not finitely generated, and is not conjugate to a group of automorphisms of any projective surface.*

In the first case Γ is called an *elliptic subgroup* of $\mathrm{Bir}(Y)$; in the third or the fourth case, Γ is called a *parabolic subgroup*; in the seventh case Γ is called *non-elementary*. The last case is impossible for finitely generated subgroups. It is still an open question whether the second case is possible for finitely generated subgroups. There are examples of finitely generated subgroups in all other cases.

We will only need Theorem 1.4.1 for finitely generated subgroups. In other words Case 8) is irrelevant in this text.

1.5 Centralizers

In this section we collect a series of results that we will use several times in this text. They can all be found, with proofs, in [Zhaa] or in Appendix A. Let Y be a smooth rational surface and $r : Y \to \mathbb{P}^1$ be a rational fibration. The subgroup of $\mathrm{Bir}(Y)$ that preserves the fibration r will be identified with the Jonquières group. For an element $f \in \mathrm{Jonq}$ (resp. a subgroup $\Gamma \subset \mathrm{Jonq}$) we denote by f_B (resp. Γ_B) the element (resp. subgroup) of $\mathrm{Aut}(\mathbb{P}^1)$ induced by the action on the base of the fibration. The centralizer in $\mathrm{Bir}(Y)$ of an element $f \in \mathrm{Jonq}$ is denoted by $\mathrm{Cent}(f)$. If f is a Jonquières twist then $\mathrm{Cent}(f)$ is necessarily a subgroup of Jonq and we denote by $\mathrm{Cent}_0(f)$ the normal subgroup of $\mathrm{Cent}(f)$ that preserves fiberwise the fibration r.

Theorem 1.5.1 ([BD15]) *Let $f \in \mathrm{Bir}(\mathbb{P}^2)$ be an elliptic element of infinite order. There exists an affine chart with affine coordinates (x,y) on which f acts by automorphism of the following form:*

1. $(x,y) \mapsto (\alpha x, \beta y)$ *where $\alpha, \beta \in \mathbf{C}^*$ are such that the kernel of the group homomorphism $\mathbf{Z}^2 \to \mathbf{C}^*, (i,j) \mapsto \alpha^i\beta^j$ is generated by $(k,0)$ for some $k \in \mathbf{Z}$;*
2. $(x,y) \mapsto (\alpha x, y+1)$ *where $\alpha \in \mathbf{C}^*$.*

The centralizer $\mathrm{Cent}(f)$ *of f in* $\mathrm{Bir}(Y)$ *is described as follows:*

1. *In the first case*

$$\mathrm{Cent}(f) = \{(x,y) \dashrightarrow (\eta(x), yR(x^k)) | R \in \mathbf{C}(x), \eta \in \mathrm{PGL}_2(\mathbf{C}), \eta(\alpha x) = \alpha\eta(x)\}.$$

   If α is not a root of unity, i.e. if $k = 0$, then R must be constant.

2. *In the second case*

$$\mathrm{Cent}(f) = \{(x,y) \dashrightarrow (\eta(x), y+R(x)) | \eta \in \mathrm{PGL}_2(\mathbf{C}), \eta(\alpha x) = \alpha\eta(x), R \in \mathbf{C}(x), R(\alpha x) = R(x)\}.$$

   If α is not a root of unity then R must be constant and $\eta(x) = \beta x$ for some $\beta \in \mathbf{C}^*$.

Theorem 1.5.2 ([Zhaa], see Appendix A) *Let $f \in$* Jonq *be a Jonquières twist such that f_B has infinite order. Then the exact sequence*

$$\{1\} \to \mathrm{Cent}_0(f) \to \mathrm{Cent}(f) \to \mathrm{Cent}_B(f) \to \{1\}$$

satisfies the following two assertions.

— *If* $\mathrm{Cent}_0(f)$ *is not trivial then it is* $\{(x,y) \mapsto (x,ty), t \in \mathbf{C}^*\}$, $\{(x,y) \mapsto (x,y+t), t \in \mathbf{C}\}$, *or a group with two elements generated by* $(x,y) \mapsto (x,-y)$ *or a Jonquières involution;*

— $\mathrm{Cent}_B(f) \subset \mathrm{PGL}_2(\mathbf{C})$ *is isomorphic to the product of a finite cyclic group with* $\mathbf{Z}$*. The infinite cyclic subgroup generated by* f_B *has finite index in* $\mathrm{Cent}_B(f)$*.*

We will also use the above two theorems in the following form:

Theorem 1.5.3 ([Zhaa], see Appendix A) *Let* $\Gamma \subset \mathrm{Jonq}$*. Suppose that* Γ *is isomorphic to* $\mathbf{Z}^n$ *and that* Γ_B *is infinite. Then there is a subgroup* Γ' *of finite index of* Γ *with generators* $\gamma_1, \cdots, \gamma_n$ *such that one of the following assertions holds up to conjugation in* Jonq*:*

1. *For any i,* γ_i *has the form* $(x,y) \mapsto (a_i x, b_i y)$ *with* $a_i, b_i \in \mathbf{C}^*$*.*
2. *For any i,* γ_i *has the form* $(x,y) \mapsto (x+a_i, b_i y)$ *with* $a_i \in \mathbf{C}$ *and* $b_i \in \mathbf{C}^*$*.*
3. *For any i,* γ_i *has the form* $(x,y) \mapsto (a_i x, y+b_i)$ *with* $a_i \in \mathbf{C}^*$ *and* $b_i \in \mathbf{C}$*.*
4. *For any i,* γ_i *has the form* $(x,y) \mapsto (x+a_i, y+b_i)$ *with* $a_i, b_i \in \mathbf{C}$*.*
5. γ_1 *is a Jonquières twist of the form* $(x,y) \dashrightarrow (\eta(x), yR(x))$ *where* $\eta \in \mathrm{PGL}_2(\mathbf{C})$ *and* $R \in \mathbf{C}(x)^*$*. For any* $i > 1$*,* γ_i *has the form* $(x,y) \mapsto (x, b_i y)$ *with* $b_i \in \mathbf{C}^*$*.*
6. γ_1 *is a Jonquières twist of the form* $(x,y) \dashrightarrow (\eta(x), y+R(x))$ *where* $\eta \in \mathrm{PGL}_2(\mathbf{C})$ *and* $R \in \mathbf{C}(x)$*. For any* $i > 1$*,* γ_i *has the form* $(x,y) \mapsto (x, y+b_i)$ *with* $b_i \in \mathbf{C}$*.*

In general we have:

Theorem 1.5.4 ([Giz80],[Can11],[CD12b],[BD15],[BC16],[Zhaa]) *Let* Γ *be a subgroup of* $\mathrm{Bir}(\mathbb{P}^2)$ *which is isomorphic to* $\mathbf{Z}^2$*. Then* Γ *has a pair of generators* (f,g) *such that one of the following (mutually exclusive) situations happens up to conjugation in* $\mathrm{Bir}(\mathbb{P}^2)$*:*

1. f,g *are elliptic elements and* $\Gamma \subset \mathrm{Aut}(X)$ *for some projective surface X;*
2. f,g *are Halphen twists which preserve the same genus one fibration on a rational surface X, and* $\Gamma \subset \mathrm{Aut}(X)$*;*
3. *one or both of the* f,g *are Jonquières twists, and there exist* $m,n \in \mathbf{N}^*$ *such that the finite index subgroup of* Γ *generated by* f^m *and* g^n *is in an* 1*-dimensional algebraic subgroup over* $\mathbf{C}(x)$ *of* $\mathrm{Jonq}_0 = \mathrm{PGL}_2(\mathbf{C}(x))$*;*
4. f *is a Jonquières twist with an action of infinite order on the base, and g is elliptic. In some affine chart, we can write* f,g *in one of the following forms:*
   — *g is* $(x,y) \mapsto (\alpha x, \beta y)$ *and f is* $(x,y) \dashrightarrow (\eta(x), yR(x^k))$ *where* $\alpha, \beta \in \mathbf{C}^*, \alpha^k = 1, R \in \mathbf{C}(x), \eta \in \mathrm{PGL}_2(\mathbf{C}), \eta(\alpha x) = \alpha\eta(x)$ *and* η *is of infinite order;*
   — *g is* $(x,y) \mapsto (\alpha x, y+1)$ *and f is* $(x,y) \dashrightarrow (\eta(x), y+R(x))$ *where* $\alpha \in \mathbf{C}^*, R \in \mathbf{C}(x), R(\alpha x) = R(x), \eta \in \mathrm{PGL}_2(\mathbf{C}), \eta(\alpha x) = \alpha\eta(x)$ *and* η *is of infinite order.*

CHAPTER 2

NON-NEGATIVE KODAIRA DIMENSION

2.1 Invariant measure

Kodaira dimension. Let Y be a smooth projective variety of dimension n. The canonical bundle of Y is the line bundle K_Y of holomorphic n-forms, i.e. $K_Y = \bigwedge^n \Omega_X$ is the n-th exterior power of the holomorphic cotangent bundle. If for any $m \in \mathbf{N}^*$ the line bundle K_Y^m has no non-trivial sections, then the *Kodaira dimension* of Y is $\kappa(Y) = -\infty$. Otherwise for some m the linear system $|K_Y^m|$ induces a rational map from Y to a projective space of dimension $\dim \mathrm{H}^0(Y, K_Y^m) - 1$; denote the dimension of the image by D_m. Then the *Kodaira dimension* of Y is $\kappa(Y) = \max_{m \in \mathbf{N}^*}\{D_m\}$.

Invariant open subsets. The objective of this section is to prove the following theorem:

Theorem 2.1.1 *Let Y be a smooth projective variety of non-negative Kodaira dimension. Let $f \in \mathrm{Bir}(Y)$. Then any Fatou component of f is non-wandering and recurrent. For any f-invariant open subset $V \subset Y$, the action of f on V is recurrent.*

Here f-invariant means that f and f^{-1} are regular on V and $f(V) = V$; recurrent means that for any open subset $W \subset V$, for almost all $x \in W$ (with respect to Lebesgue measure), there are infinitely many $n \in \mathbf{N}$ such that $f^n(x) \in W$.

If we have a birational Kleinian group (Y, Γ, U, X), then for any $\gamma \in \Gamma$ the open subset U is γ-invariant but not recurrent. Therefore the following theorem is a consequence of Theorem 2.1.1.

Theorem 2.1.2 *There is no birational kleinian group (Y, Γ, U, X) for which Y is a projective variety with non-negative Kodaira dimension.*

Poincaré recurrence theorem says that a measure preserving dynamical system on a space of finite measure is always recurrent. Thus Thereom 2.1.1 is a consequence of the following statement (the particular case $\Gamma = \mathbf{Z}$ is sufficient):

Lemma 2.1.3 *Let Y be a smooth projective variety of non-negative Kodaira dimension. Let Γ be a subgroup of* $\mathrm{Bir}(Y)$. *Then there is a Γ-invariant probability measure on Y absolutely continuous with respect to the Lebesgue measure.*

Remark that a measure which is absolutely continuous with respect to the Lebesgue measure associated with some Riemannian metric does not charge algebraic sets of codimension ≥ 1, so the pull-forward of such a measure by a birational transformation is well-defined and we can talk about Γ-invariant measure. More precisely for ν such a measure, $\gamma \in \Gamma$ a birational transformation and $E \subset Y$ a subset, $\gamma_* \nu(E) = \nu(\gamma^{-1}(E \backslash \mathrm{Ind}(\gamma^{-1})))$.

We prove Lemma 2.1.3 in the sequel of this section. Let us explain firstly the construction of the measure. When Y is an abelian variety or a Calabi-Yau variety, the canonical bundle is trivial and we just use the measure associated with the natural non-vanishing volume form defined up to a multiplicative scalar. In the general case we take a non-zero pluricanonical section $\alpha \in \mathrm{H}^0(Y, mK_Y)$ which in local coordinates writes as $\alpha(z_1, \ldots, z_n)(dz_1 \wedge \cdots \wedge dz_n)^m$ where n is the dimension of Y. Then the differential $2n$-form

$$(\alpha \wedge \overline{\alpha})^{\frac{1}{m}} = i^{n^2} |\alpha(z)|^{\frac{2}{m}} dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z_1} \wedge \cdots \wedge d\overline{z_n} \tag{2.1}$$

is a singular volume form which is > 0 on the complement of the divisor $\{\alpha = 0\}$. The finite measure defined by this singular volume form is of full support and is absolutely continuous with respect to the Lebesgue measure associated with any Riemannian metric on Y. A priori this measure need not to be Γ-invariant. Hence the bulk of the proof is the existence of a $\mathrm{Bir}(Y)$-invariant pluricanonical form. This follows immediately from Nakamura-Ueno's theorem which asserts that the linear representation of $\mathrm{Bir}(Y)$ on the space of pluricanonical forms has finite image. We give an elementary argument in this section which is already sufficient for our purpose and we include in the next section a proof of Nakamura-Ueno's theorem taken from [Uen75] Chapter VI.

For a pluricanonical form $\alpha \in \mathrm{H}^0(Y, mK_Y)$ and a birational transformation $f \in \mathrm{Bir}(Y)$, we can pull back α by f outside the indeterminacy locus of f and then extend it in a unique way by Hartogs' Principle. In this way we obtain a linear representation

$$\rho : \mathrm{Bir}(Y) \to \mathrm{GL}\left(\mathrm{H}^0(Y, mK_Y)\right).$$

The pull-back of an arbitrary measure by a birational transformation is not well defined. However we can pull back the measure associated with the singular volume form (2.1) because it does not charge algebraic subsets of codimension ≥ 1. The pull-back measure is the measure associated with the pull-back pluricanonical form. Lemma 2.1.3 is a consequence of the following statement:

Lemma 2.1.4 *Let Y be a smooth projective variety of non-negative Kodaira dimension. Let Γ be a subgroup of* $\mathrm{Bir}(Y)$. *For any $m \in \mathbf{N}^*$ such that the linear system $|K_Y^m|$ is not empty, there is $\alpha \in \mathrm{H}^0(Y, K_Y^m) \backslash \{0\}$ such that for any $\gamma \in \Gamma$ we have $\gamma^* \alpha = \alpha$.*

Proof There is a norm on the vector space $\mathrm{H}^0(Y, K_Y^m)$:

$$\|\alpha\|^2 = \int_Y (\alpha \wedge \overline{\alpha})^{\frac{1}{m}}.$$

For $f \in \mathrm{Bir}(Y)$, a change of variables gives

$$\|\alpha\|^2 = \int_Y (\alpha \wedge \overline{\alpha})^{\frac{1}{m}} = \int_Y \left(f^* \alpha \wedge \overline{f^* \alpha}\right)^{\frac{1}{m}} = \|f^* \alpha\|^2.$$

This means that the image of the representation $\rho : \mathrm{Bir}(Y) \to \mathrm{GL}(\mathrm{H}^0(Y, K_Y^m))$ is contained in the compact group preserving the norm $\|\cdot\|$. Denote by G the Zariski closure of $\rho(\mathrm{Bir}(Y))$ in $\mathrm{GL}(\mathrm{H}^0(Y, K_Y^m))$. The group G is compact and admits a left and right invariant Haar measure ν. Let β be a non-zero element of $\mathrm{H}^0(Y, K_Y^m)$. Then

$$\alpha = \int_G g(\beta) d\nu(g) \in \mathrm{H}^0(Y, K_Y^m)$$

satisfies the desired properties. □

2.2 Nakamura-Ueno theorem

The existence of a finite measure invariant under the group of birational transformations is a particular instance of the following stronger result:

Theorem 2.2.1 (Nakamura-Ueno) *Let Y be a smooth projective variety of Kodaira dimension ≥ 0. For any $m \in \mathbf{N}^*$, the linear representation $\rho : \mathrm{Bir}(Y) \to \mathrm{GL}\left(\mathrm{H}^0(Y, mK_Y)\right)$ has finite image.*

Remark 2.2.2 The theorem holds also for dominant rational self-maps by [NZ09].

Lemma 2.2.3 *For any $f \in \mathrm{Bir}(Y)$, the eigenvalues of $\rho(f)$ are algebraic integers.*

Proof Let us consider an eigenvector $\alpha \in \mathrm{H}^0(Y, mK_Y)$ such that $f^*\alpha = \lambda\alpha$ for $f \in \mathrm{Bir}(Y)$ and $\lambda \in \mathbf{C}$.

Let us first look at the case $m = 1$. The cohomology group $\mathrm{H}^0(Y, K_Y)$ can be seen as a subspace of $\mathrm{H}^n(Y, \mathbf{C})$. The birational transformation f induces a pull-back action on $\mathrm{H}^n(Y, \mathbf{C})$ too: if Z is the desingularization of the graph of f and if $\pi_1, \pi_2 : Z \to Y$ are projections onto domain and target, then $f^* : \mathrm{H}^n(Y, \mathbf{C}) \to \mathrm{H}^n(Y, \mathbf{C})$ is the composition of π_2^* and the Gysin morphism π_{1*} defined by Poincaré dual. In this way the pull-back action of f on $\mathrm{H}^n(Y, \mathbf{C})$ is the restriction of that on $\mathrm{H}^n(Y, \mathbf{C})$. Since the action f^* on $\mathrm{H}^n(Y, \mathbf{C})$ preserves the non-torsion part of $\mathrm{H}^n(Y, \mathbf{Z})$, the complex number λ, being an eigenvalue, must be an algebraic integer whose degree is bounded by the n-th Betti number of Y.

When $m > 1$, λ is still an algebraic integer: we can reduce to the case of $m = 1$ by using a covering trick as follows. Write α in local coordinates as $a(z_1, \ldots, z_n)(dz_1 \wedge \cdots \wedge dz_n)^m$ and consider the degree m cover W of Y defined in the total space T of the line bundle K_Y by

$$w^m = a(z_1, \ldots, z_n)$$

where w denote the fiber coordinate. The variety W is possibly singular and non-connected. Let μ denote a m-th root of λ and m_μ denote the scaling automorphism of T defined by $(z_1, \ldots, z_n, w) \mapsto (z_1, \ldots, z_n, \mu w)$. A birational transformation $f \in \mathrm{Bir}(Y)$ induces a birational transformation f^* of T. The composition $m_\mu \circ f^*$ is a birational transformation of T preserving W. We desingularize $W \subset T$ by blowing up T, obtaining a finite surjective morphism $q : \bar{W} \to Y$. We have a birational map h of $\bar{W}$ induced by $m_\mu \circ f^*$. By lifting the n-form $w dz_1 \wedge \cdots \wedge dz_n$ from T to $\bar{W}$, we obtain a holomorphic n-form β on $\bar{W}$ such that

$$(\beta)^m = q^*\alpha \quad \text{and} \quad h^*\beta = \mu\beta.$$

The case $m = 1$ implies that μ is an algebraic integer, thus λ too. Furthermore the degree of λ is bounded by the n-th Betti number of $\bar{W}$. □

According to [Uen75], the following lemma is due to Deligne.

Lemma 2.2.4 *The degrees of the eigenvalues (as algebraic integers) are uniformly bounded for all birational transformations $f \in \mathrm{Bir}(Y)$.*

Proof By the previous proof, it suffices to show that the Betti numbers of $\bar{W}$ are uniformly bounded. We see from the fact that $\bar{W}$ only depends on α but not f that the degree of the eigenvalue only depends on the eigenvector but not the endomorphism. We will use the notation

$W_\alpha, \bar{W}_\alpha$ to emphasize that these varieties are constructed using the pluricanonical form α; remark that if we use a multiple of α then the varieties stay isomorphic. The varieties W_α form a family over $\mathbb{P}(\mathrm{H}^0(Y, mK_Y))$ in the following way. Let $(\alpha_0, \alpha_1, \cdots, \alpha_N)$ be a base of $\mathrm{H}^0(Y, mK_Y)$ and let $[u_0; u_1; \cdots; u_N]$ be the corresponding homogeneous coordinates on $\mathbb{P}(\mathrm{H}^0(Y, mK_Y))$. Then the subvariety $\mathfrak{W}$ of $\mathbb{P}(\mathrm{H}^0(Y, mK_Y)) \times Z$ defined by

$$w^m = \sum_{j=0}^{N} u_j \alpha_j$$

gives the construction of W_α in family. Resolving singularities of the fiber over the generic point of $\mathbb{P}(\mathrm{H}^0(Y, mK_Y))$, we obtain a family of $\bar{W}_\alpha$ over a Zariski open set of $\mathbb{P}(\mathrm{H}^0(Y, mK_Y))$. Denote by A_1 this Zariski open set and by B_1 the complement. Consider the family W_α over B_1. By resolving singularities of the fiber over the generic point of B_1, we obtain a Zariski open set A_2 of B_1 over which we obtain a family $\bar{W}_\alpha$. We continue this process with $B_2 = B_1 \backslash A_2$. Finally we get a stratification $\mathbb{P}(\mathrm{H}^0(Y, mK_Y)) = \sqcup A_j$ and a stratified family of $\bar{W}_\alpha$: over each A_j the family is a genuine fibration of algebraic varieties. Then Thom-Mather isotopy theorem (cf. [Mat12] Proposition 11.1) gives the uniform boundedness of Betti numbers of the $\bar{W}_\alpha$, thus also that of degrees of the eigenvalues. □

Lemma 2.2.5 *The eigenvalues are roots of unity.*

Proof First from the equation

$$\int_Y (\alpha \wedge \overline{\alpha})^{\frac{1}{m}} = \int_Y \left(f^*\alpha \wedge \overline{f^*\alpha}\right)^{\frac{1}{m}} = |\lambda|^{\frac{2}{m}} \int_Y (\alpha \wedge \overline{\alpha})^{\frac{1}{m}}$$

we obtain $|\lambda| = 1$. Then we embed Y in some projective space so that Y and f are given by some polynomial formulas. For a field automorphism σ of $\mathbf{C}$, changing the coefficients of these polynomial formulas by using σ, we get a new variety with a birational transformation for which the previous argument gives $|\sigma(\lambda)| = 1$. As λ is an algebraic integer, this implies it is a root of unity. □

Lemma 2.2.6 *For any $f \in \mathrm{Bir}(Y)$, the linear automorphism $\rho(f)$ is semisimple, i.e. diagonalizable.*

Proof Suppose that $\rho(f)$ is not semisimple. By looking at the Jordan normal form we find two linearly independent pluricanonical forms $\alpha_1, \alpha_2 \in \mathrm{H}^0(Y, mK_Y)$ such that

$$f^*\alpha_1 = \lambda \alpha_1 + \alpha_2,\ f^*\alpha_2 = \lambda \alpha_2.$$

Then

$$\int_Y \left((f^l)^*(\alpha_1)\wedge\overline{(f^l)^*(\alpha_1)}\right)^{\frac{1}{m}} = i^{n^2} l^{\frac{2}{m}} \int_Y |\frac{\alpha_1(z)}{l}+\frac{\alpha_2(z)}{\lambda}|^{\frac{2}{m}} dz_1\wedge\cdots\wedge dz_n\wedge d\overline{z_1}\wedge\cdots\wedge d\overline{z_n}.$$

The right-hand side goes to infinity as l goes to infinity. However the left-hand side is always equal to $\int_Y (\alpha_1\wedge\overline{\alpha_1})^{\frac{1}{m}}$, contradiction. □

Proof (of Theorem 2.2.1) Finally we are about to conclude. The fact that f^* is semisimple and has roots of unity as eigenvalues implies that it is of finite order. Furthermore this order is uniformly bounded. So we have proved that the elements in the image of $\rho : \mathrm{Bir}(Y) \to \mathrm{GL}\left(\mathrm{H}^0(Y, mK_Y)\right)$ are of uniformly bounded finite order. Burnside's Theorem says that the image is itself a finite group. □

2.3 Genus one fibrations.

For surfaces with Kodaira dimension 0, the invariant measure is easy to construct and the birational transformation groups of general type surfaces are finite. So the most interesting application of Theorem 2.1.2 in dimension two is for elliptic surfaces of kodaira dimension 1. However for elliptic surfaces we can also use an elementary topological argument. This argument works also for genus one fibrations on rational surfaces and we will need this later.

Lemma 2.3.1 *Let Λ be an infinite group of automorphisms of an elliptic curve E. Then the orbit of a point $v \in E$ is not discrete and its closure is a real subtorus (not necessarily connected).*

Proof Topologically we identify E as the quotient of $\mathbf{R}^2$ by a lattice L. A finite index subgroup $\Lambda' \subset \Lambda$ is an infinite group of translations. Take an arbitrary point $v \in E$. We can suppose that v is the origin. Then the orbit $\Lambda' \cdot v$ is a subgroup of E. Since it is infinite, it is not discrete. Its closure is thus a real subtorus of positive dimension. □

Lemma 2.3.2 *Let $Y \to C$ be a genus one fibration. Let U be an open set of Y which is preserved by an infinite finitely generated subgroup Γ of* $\mathrm{Bir}(Y)$. *Suppose that Γ preserves the genus one fibration $Y \to C$ and that the induced action on C is finite. Then the action of Γ on U is not discrete.*

Proof There is an induced morphism $\phi : \Gamma \to \mathrm{Aut}(C)$. We denote by Γ^0 the kernel of this morphism, its elements preserve fibrewise the genus one fibration. If the image of ϕ is finite, then Γ^0 is infinite and finitely generated. We can always find a fibre F of the genus one fibration such that 1) F is a smooth elliptc curve; 2) F intersects U; 3) Γ^0 acts faithfully on F. Then lemma 2.3.1 shows that the action of Γ on F is nowhere discrete. In particular the action of Γ on U is not discrete. $\square$

Lemma 2.3.2 and Theorem 2.2.1 (recall that Theorem 2.2.1 is easy in dimension two) imply

Corollary 2.3.3 *There is no birational Kleinian groups on an elliptic surface of Kodaira dimension* 1.

Remark that if an automorphism of a rational surface preserves a genus one fibration, then its induced action on the base has finite order (cf. [CF03] Proposition 3.6). Thus by Lemma 2.3.2 we have:

Corollary 2.3.4 *Let* (Y,Γ,U,X) *be a birational kleinian group on a ratioal surface* Y. *Then* Γ *does not contain an infinite parabolic subgroup which preserves a genus one fibration.*

CHAPTER 3

KLEINIAN SUBGROUPS OF $\mathrm{PGL}_3(\mathbf{C})$

3.1 Introduction

In this chapter we study groups with Kleinian property that act by automorphisms on the projective plane $\mathbb{P}^2$. This case has been studied by several people and a complete answer has been obtained:

Theorem 3.1.1 (Kobayashi-Ochiai [KO80], Mok-Yeung [MY93], Klingler [Kli98] [Kli01]) *Let $(\mathbb{P}^2, U, \Gamma, X)$ be an infinite Kleinian group such that $\Gamma \subset \mathrm{PGL}_3(\mathbf{C})$. Then up to taking finite index subgroup and up to conjugation inside $\mathrm{PGL}_3(\mathbf{C})$, we are in one of the following situations:*

1. *$U = \mathbf{C} \times \mathbf{C}, \mathbf{C} \times \mathbf{C}^*$ or $\mathbf{C}^* \times \mathbf{C}^*$ and X is a complex torus; here $\mathbf{C} \times \mathbf{C}, \mathbf{C} \times \mathbf{C}^*, \mathbf{C}^* \times \mathbf{C}^*$ are the standard Zariski open subsets of $\mathbb{P}^2$ and Γ is a lattice in U viewed as a Lie group.*
2. *U is the Euclidean ball $\mathbb{B}^n$ embedded in the standard way in $\mathbb{P}^2$ and X is a ball quotient.*
3. *$U = \mathbf{C}^2 \backslash \{0\}$ where $\mathbf{C}^2$ is embedded in the standard way in $\mathbb{P}^2$ as a Zariski open set and X is a Hopf surface.*
4. *U is the standard Zariski open set $\mathbf{C}^2 \subset \mathbb{P}^2$ and X is a primary Kodaira surface.*
5. *$U = \mathbb{H} \times \mathbf{C}$ embedded in the standard way in the Zariski open subset $\mathbf{C}^2$ and X is an Inoue surface.*
6. *U is biholomorphic to $\mathbb{H} \times \mathbf{C}^*$ and X is an affine-elliptic bundle; here $\mathbb{H} \times \mathbf{C}^*$ is not embedded in a standard way in $\mathbf{C}^2$, see Section 3.3.1 for terminology and description.*

We refer to Section 3.3 for precise descriptions of all the cases. Remark that X is Kähler only in the first two cases.

Whenever we have such a Kleinian group, the quotient surface is equipped with a complex projective structure. Theorem 3.1.1 breaks down to two different problems. The first one is to determine which surfaces admit complex projective structures and the second one is to find all complex projective structures on those candidates. The first problem was initiated by

Kobayashi-Ochiai [KO80] and previous work of Gunning played an important role (see Theorem 3.2.5); it is completed by Klingler in [Kli98]. In [Kli98], the second problem is also investigated and is almost completed, except the particularly difficult case of ball quotients. The fact that the natural complex projective structure on a ball quotient is the only one is proved by Mok-Yeung in [MY93]. Their proof is based on a deep theorem of Mok [Mok87]; in [Kli01] Klingler gave a different algebraic proof. We should also mention the work of Cano-Seade [CS14] where more precise information and an orbifold version of Theorem 3.1.1 are obtained. In higher dimension, Jahnke-Radloff [JR15] classifies all projective manifolds admitting complex projective structures. Admitting Mok-Yeung's theorem on uniqueness for ball quotients, we will give in this chapter a quick proof of Theorem 3.1.1 when X is assumed to be Kähler. We follow mainly the arguments of [JR15].

Though not every complex projective structure arises from Kleinian subgroups of $\mathrm{PGL}_3(\mathbf{C})$, it is easier to study firstly the geometry of surfaces with complex projective structures than to look directly at the dynamics of subgroups of $\mathrm{PGL}_3(\mathbf{C})$. However we will see in other chapters that it is not the case for general birational Kleinian groups: we will need a good understanding on groups of birational transformations.

3.2 Complex affine and projective structures

Here we are interested in two types of geometric structures, complex projective structure and complex affine structure. We will show that the existence of such a structure on X imposes strong restrictions on Chern classes of X. We will omit the factor $2\pi i$ in all formulas of this section.

Atiyah class and Chern classes. Let X be a compact complex manifold of dimension n. Let Ω_X be the cotangent bundle of X. Let E be a holomorphic vector bundle of rank r on X. Let $(U_\alpha, \psi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbf{C}^r)$ be a system of local holomorphic trivializations. The local transition matrices are denoted by $\psi_{\alpha\beta}$. The $d\psi_{\alpha\beta}$s are matrices whose entries are local differential forms. Thus the $\psi_\beta^{-1} \circ (\psi_{\alpha\beta}^{-1} d\psi_{\alpha\beta}) \circ \psi_\beta\}$s are local sections of $\Omega_X \otimes \mathrm{End}(E)$. The *Atiyah class* of E is the element of $\mathrm{H}^1(X, \Omega_X \otimes \mathrm{End}(E))$, denoted by $a(E)$, given by the Čech cocyle $\{U_{\alpha\beta}, \psi_\beta^{-1} \circ (\psi_{\alpha\beta}^{-1} d\psi_{\alpha\beta}) \circ \psi_\beta\}$. The Atiyah class is the obstruction for the holomorphic splitting of the first jet sequence ([Ati57])

$$0 \to \Omega_X \otimes E \to J_1(E) \to E \to 0.$$

A *holomorphic connection* on E is a $\mathbf{C}$-linear map of sheaves $D : E \to \Omega_X \otimes E$ such that

$$D(f \cdot s) = \partial(f) \otimes s + f \cdot D(s)$$

for any local holomorphic function f on X and any local holomorphic section s of E. It is easy to check that E admits a holomorphic connection if and only if its Atiyah class $a(E)$ is trivial (see [Huy05] 4.2).

Let S_k be the $\mathrm{GL}_r(\mathbf{C})$-invariant homogeneous polynomial on $\mathrm{End}(\mathbf{C}^r)$ of degree k such that for any $M \in \mathrm{End}(\mathbf{C}^r)$ we have

$$\det(\mathrm{Id} - M) = 1 + S_1(M) + S_2(M) + \cdots + S_r(M).$$

If we think of $a(E) \in \mathrm{H}^1(X, \Omega_X \otimes \mathrm{End}(E))$ as a Dolbeault cohomology class with value in $\mathrm{End}(E)$, then we can define by using cup-product $S_k(a(E))$ as an element of $\mathrm{H}^k(X, \Omega_X^k) = \mathrm{H}^{k,k}(X, \mathbf{C})$. Atiyah used Chern-Weil theory to obtain:

Theorem 3.2.1 (Atiyah [Ati57]) *If X is a compact Kähler manifold. Then $c_k(E) = S_k(a(E))$ where $c_k(E)$ is the k-th Chern class of E.*

Chen-Ogiue inequality The following theorem, combined with Aubin and Yau's solution of Calabi conjecture, is very usefull for studying existence of complex affine or complex projective stuctures. See our next section for the definition of ball quotient.

Theorem 3.2.2 (Chen-Ogiue [CO75]) *Let (X, κ) be a compact Kähler-Einstein manifold of dimension n. Then*

$$\int_X \left(nc_1^2 - 2(n+1)c_2\right) \wedge \kappa^{n-2} \leq 0.$$

The equality holds if and only if X is $\mathbb{P}^n$, an étale quotient of a torus or a ball quotient.

Complex affine structures. There is a natural holomorphic connection given by the usual differential operator ∂ on the holomorphic tangent and cotangent bundle of the affine space $\mathbf{C}^n$. It is invariant under the group of affine transformations. If X is equipped with a complex affine structure, then ∂ induces a holomorphic connection on the tangent bundle of X. Therefore the Atiyah class of its tangent bundle is trivial. And by Theorem 3.2.1 we have

Proposition 3.2.3 *If X admits a complex affine structure then $a(T_X) = 0$. If moreover X is Kähler, then X is a complex torus.*

Proof By Theorem 3.2.1 all Chern classes of X are trivial. By Yau's solution of Calabi conjecture, X admits a Kähler-Einstein metric because the first Chern class is trivial. The only compact Kähler-Einstein manifolds with trivial Chern classes are complex tori by Theorem 3.2.2. $\square$

Complex projective structures. Now let X be a compact complex manifold of dimension $n \geq 2$ equipped with a complex projective structure and let $(U_\alpha, (z_{\alpha 1}, \cdots, z_{\alpha n}))$ be a system of local coordinates compatible with the complex projective structure. Using Einstein notation for tensors, the Atiyah class of the tangent bundle

$$a(T_X) \in \mathrm{H}^1(X, \Omega_X \otimes \mathrm{End}(T_X)) = \mathrm{H}^1(X, \Omega_X \otimes T_X \otimes \Omega_X)$$

is represented by the Čech cocycle $(U_{\alpha\beta}, \Lambda^k_{\alpha\beta jl} dz_{\alpha j} \otimes \frac{\partial}{\partial z_{\alpha k}} \otimes dz_{\alpha l})$ where

$$\Lambda^k_{\alpha\beta jl} = \sum_b \frac{\partial z_{\alpha k}}{\partial z_{\beta b}} \frac{\partial^2 z_{\beta b}}{\partial z_{\alpha j} \partial z_{\alpha l}}.$$

By hypothesis the changes of coordinates are linear fractional, i.e. of the form

$$z_{\beta j} = \frac{e_0 + \sum_k e_k z_{\alpha k}}{f_0 + \sum_k f_k z_{\alpha k}}.$$

By a calculation (cf. [Gun78] pages 48-50) this implies (in fact equivalent to, but we only need one implication):

$$\Lambda^k_{\alpha\beta jl} = \delta^k_j \sigma_{\alpha\beta l} + \delta^k_l \sigma_{\alpha\beta j} \tag{3.1}$$

where δ is the Kronecker symbol and

$$\sigma_{\alpha\beta j} = \frac{1}{n+1} \frac{\partial \log\det\left((\frac{\partial z_{\beta b}}{\partial z_{\alpha a}})_{b,a} \right)}{\partial z_{\alpha j}}.$$

The local holomorphic functions $\det(\frac{\partial z_{\beta b}}{\partial z_{\alpha a}})_{b,a}$ are the transition functions of the canonical bundle K_X. Thus $(U_{\alpha\beta}, (n+1)\sum_j \sigma_{\alpha\beta j} dz_{\alpha j})$ is the Čech cocycle representing the first Chern class $c_1(K_X)$ which is viewed as an element of $\mathrm{H}^1(X, \Omega_X)$ (cf. [Huy05] 4.2.20).

The identification $\Omega_X \otimes T_X \otimes \Omega_X = \Omega_X \otimes \mathrm{End}(T_X) = \mathrm{End}(\Omega_X) \otimes \Omega_X$ allows us to make two different changes of coefficients from $\mathrm{H}^1(X, \Omega_X)$ to $\mathrm{H}^1(X, \Omega_X \otimes \mathrm{End}(E))$. Hence Equation (3.1) shows

Proposition 3.2.4 *If a compact complex manifold X has a complex projective structure, then*

$$a(T_X) = \frac{1}{n+1}(c_1(K_X)\otimes \mathrm{Id}_{T_X} + \mathrm{Id}_{\Omega_X}\otimes c_1(K_X)) \in \mathrm{H}^1(X, \Omega_X \otimes \mathrm{End}(E))$$

where the notation $\otimes$ at the right hand side means change of coefficients for cohomology.

Theorem 3.2.5 (Gunning [Gun78]) *If X is a compact Kähler manifold of dimension ≥ 2 with a complex projective structure, then its Chern classes satisfy $c_l = \frac{1}{(n+1)^l}\binom{n+1}{l} c_1^l$. In particular for surfaces we have $c_1^2 = 3c_2$.*

Proof We keep in mind the following commutative diagram for comparison between Dolbeault and Čech cohomology with value in a vector bundle V.

$$\begin{tikzcd}
\Omega_X \otimes V \arrow[r] \arrow[d] & C^0(\{U_\alpha\}, \Omega_X \otimes V) \arrow[r] \arrow[d] & C^1(\{U_\alpha\}, \Omega_X \otimes V) \arrow[d] \\
\mathscr{A}^{1,0}(V) \arrow[r] \arrow[d, "\bar\partial"] & C^0(\{U_\alpha\}, \mathscr{A}^{1,0}(V)) \arrow[r, "\delta_1"] \arrow[d, "\bar\partial"] & C^1(\{U_\alpha\}, \mathscr{A}^{1,0}(V)) \\
\mathscr{A}^{1,1}(V) \arrow[r] & C^0(\{U_\alpha\}, \mathscr{A}^{1,1}(V)) &
\end{tikzcd}$$

As the sheaf of C^∞-differential forms is fine (has a partition of unity), the Čech cocycle $(U_{\alpha\beta}, \sum_j \sigma_{\alpha\beta j} dz_{\alpha j})$ representing $\frac{1}{n+1}c_1(K_X)$ is exact in the C^∞-category. There exist C^∞ 1-forms $\Theta_\alpha = \sum_j \theta_{\alpha j} dz_{\alpha j}$ on U_α such that $\sum_j \sigma_{\alpha\beta j}\partial z_{\alpha j} = \Theta_\alpha - \Theta_\beta$. The $(1,1)$-forms $\bar\partial\Theta_\alpha$ glue together to give a global $(1,1)$-form on X. This global $(1,1)$-form represents $\frac{1}{n+1}c_1(K_X)$ viewed as a Dolbeault cohomology class, we will denote it by $\bar\partial\Theta$.

Consider the following local matrix valued $(1,1)$-form on U_α:

$$\Omega_{jk} = \bar\partial\Theta\delta_j^k + \bar\partial\theta_{\alpha j}\wedge dz_{\alpha k}.$$

Though it may not be a global form, it almost represents the Dolbeault cohomology class of $a(T_X)$ (compare with Proposition 3.2.4) and we will use it to compute $S_k(a(T_X))$. We will omit the index α in the sequel. We have

$$\det(\mathrm{Id} - \Omega) = \sum_{l=0}^{n} \left(1 - \bar\partial\Theta\right)^{n-l} (-1)^l P_l$$

where P_l is the sum of all principal minors of size l of Ω. The minor corresponding to indexes $j_1, \cdots, j_l$ is

$$\begin{aligned}&\sum_{\varepsilon} \bar{\partial}\theta_{j_1} \wedge dz_{\varepsilon(j_1)} \wedge \cdots \wedge \bar{\partial}\theta_{j_l} \wedge dz_{\varepsilon(j_l)} \\ =& l! \bar{\partial}\theta_{j_1} \wedge dz_{j_1} \wedge \cdots \wedge \bar{\partial}\theta_{j_l} \wedge dz_{j_l}.\end{aligned}$$

Thus

$$P_l = l! \sum_{(j_1,\cdots,j_l)} \bar{\partial}\theta_{j_1} \wedge dz_{j_1} \wedge \cdots \wedge \bar{\partial}\theta_{j_l} \wedge dz_{j_l} = (\bar{\partial}\Theta)^l$$

and

$$\begin{aligned}\det(\mathrm{Id} - \Omega) &= \sum_{l=0}^{n} \left(1 - \bar{\partial}\Theta\right)^{n-l} (-\bar{\partial}\Theta)^l \\ &= \sum_{l=0}^{n} \binom{n+1}{l} (-\bar{\partial}\Theta)^l.\end{aligned}$$

Remark that this $(1,1)$-form is globally defined on X. By Theorem 3.2.1, the Chern class $c_l(X)$ is represented by $\binom{n+1}{l}(-\bar{\partial}\Theta)^l$ if X is Kähler. Hence the formula in the theorem. $\square$

Remark 3.2.6 If we do not use Atiyah class but identify Ω with the $(1,1)$-part of the curvature of some connection on T_X, then the above computation gives also information when X is not Kähler (see [KO80]). For example for non-Kähler surfaces the conclusion $c_1^2 = 3c_2$ still holds.

3.3 Examples

3.3.1 Dimension two.

Here we list all examples in dimension two, some of them exist in any dimension.

Projective spaces. A projective space has a natural complex projective structure for which the holonomy is trivial and the developping map is the identity.

Complex tori. A complex torus is a quotient of the commutative Lie group $\mathbf{C}^n$ by a lattice, thus is equipped naturally with a complex affine structure from its universal covering. There are many complex projective structures on a complex tori, all of them are affine (cf. [Ben94], [Kli98], the main arguments will be given in Section 3.4.2). Vitter [Vit72] describe all complex affine structures on a complex tori X: they are in bijection with holomorphic connections on the tangent bundle of X with zero torsion and zero curvature. If on an open set U with coordinates $z_1,\dots,z_n$ the connection is given by $d+\omega$ where $\omega=\sum_j A_j dz_j$ is a $\mathrm{End}(T_X)$-valued differential form, then an affine coordinate u on U can be obtained by solving

$$\frac{\partial u}{\partial z_j} = a_j dz_j, \quad da_j = a_j \omega. \tag{3.2}$$

Denote by $dz_1,\cdots,dz_n$ the standard base of holomorphic 1-forms on X induced by the universal cover. A connection with zero torsion and curvature is given by $\omega=\sum_k A_k dz_k$ where the A_k are complex $(n\times n)$-matrices such that for any j,k we have $A_jA_k=A_kA_j$ and the k-th colomn of A_j equals to the j-th column of A_k. We will be only interested in the cases where the holonomy representation gives a Kleinian group. We will see in the next section that for a two dimensional torus the only possibilities correspond to the infinite coverings $\mathbf{C}\times\mathbf{C}$, $\mathbf{C}\times\mathbf{C}^*$ or $\mathbf{C}^*\times\mathbf{C}^*$ embedded in the standard way in $\mathbb{P}^2$.

When $n=1$, Equation (3.2) is easy to solve. The complex affine structures on an elliptic curve are parametrized by $\mathbf{C}$. For z the coordinate on the universal cover $\mathbf{C}$, a developping map of the affine structure corresponding to adz with $a\in\mathbf{C}^*$ is given by $u=\frac{e^{az}}{a^2}$; it is equivalent to the developping map $\frac{e^{az}-1}{a}$. For $a=0$, the corresponding affine structure is just the natural one induced by $\mathbf{C}$. These complex affine structures form a holomorphic family in the sense that

$$\mathbf{C}^2\to\mathbf{C},(a,z)\mapsto\begin{cases}\frac{e^{az}-1}{a} & a\neq 0\\ z \quad a=0\end{cases}$$

is a holomorphic map.

Hopf manifolds. Let $\gamma\in\mathrm{GL}_n(\mathbf{C})$ be a contracting linear automorphism of $\mathbf{C}^n$ such that $\gamma^n(x)\to 0$ for any $x\in\mathbf{C}^n\backslash\{0\}$. Then the quotient of $\mathbf{C}^n\backslash\{0\}$ by the action of γ is a compact manifold with a complex affine structure. Such manifolds are among the so called Hopf manifolds and are not Kähler.

Primary Kodaira surfaces. Consider the subgroup G of $\mathrm{Aff}_2(\mathbf{C})$ generated by the following four elements

$$\begin{aligned} z &\mapsto z+\begin{pmatrix}0\\1\end{pmatrix}, \quad z\mapsto \begin{pmatrix}1&0\\c&1\end{pmatrix}z+\begin{pmatrix}1\\0\end{pmatrix}\\ z &\mapsto z+\begin{pmatrix}0\\a\end{pmatrix}, \quad z\mapsto \begin{pmatrix}1&0\\d&1\end{pmatrix}z+\begin{pmatrix}b\\0\end{pmatrix}. \end{aligned}$$

The quotient $\mathbf{C}^2/G$ is a primary Kodaira surface, i.e. a non Kähler principal bundle of elliptic curves over an elliptic curve; its Kodaira dimension is zero (cf. [Bar+04]).

Inoue surfaces. Let $M\in\mathrm{SL}_3(\mathbf{Z})$ be a matrix with eigenvalues $\alpha,\beta,\bar\beta$ such that $\alpha>1$ and $\beta\neq\bar\beta$. Note that α is irrational and $|\beta|<1$. We choose a real eigenvector (a_1,a_2,a_3) corresponding to α and a complex eigenvector (b_1,b_2,b_3) corresponding to β. Let G_M be the subgroup of $\mathrm{Aff}(\mathbf{C}^2)$ generated by

$$\begin{aligned} g_0 &:(x,y)\mapsto(\alpha x,\beta y)\\ g_i &:(x,y)\mapsto(x+a_i,y+b_i)\quad \text{for } i=1,2,3. \end{aligned}$$

Denote by $\mathbb{H}$ the upper half plane, viewed as an open subset of $\mathbf{C}$. The action of G_M preserves $\mathbb{H}\times\mathbf{C}$; it is free and properly discontinuous. The quotient $\mathbb{H}\times\mathbf{C}/G_M$ is a compact non-Kähler surface without curves called an *Inoue surface of type* S^0. It has a complex affine structure by construction.

Consider the following solvable Lie group which is a subgroup of $\mathrm{Aff}_2(\mathbf{C})$:

$$\mathrm{Sol}^0=\left\{\begin{pmatrix}|\lambda|^{-2}&0&a\\0&\lambda&b\\0&0&1\end{pmatrix},\lambda\in\mathbf{C}^*,a\in\mathbf{R},b\in\mathbf{C}\right\}.$$

The group Sol^0 is a semi-direct product $(\mathbf{C}\times\mathbf{R})\rtimes\mathbf{C}^*$. It acts transitively on $\mathbb{H}\times\mathbf{C}$; the stabilizer of a point is isomorphic to $\mathbb{S}^1$. The group G_M defining the Inoue surface S_M is a lattice in Sol^0; conversely any torsion free lattice of Sol^0 gives an Inoue surface of type S^0.

Let $n \in \mathbf{N}^*$. Consider the group of upper-triangular matrices

$$\Lambda_n = \left\{ \begin{pmatrix} 1 & x & \frac{z}{n} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x,y,z \in \mathbf{Z} \right\}.$$

The center of Λ_n is the infinite cyclic group C_n generated by $\begin{pmatrix} 1 & 0 & \frac{1}{n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The quotient Λ_n/C_n is isomorphic to $\mathbf{Z}^2$. Let $N \in \mathrm{SL}_2(\mathbf{Z})$ be a matrix with eigenvalues $\alpha, \frac{1}{\alpha}$ such that $\alpha > 1$. Let φ be an automorphism of the group of real upper-triangular matrices which preserves Λ_n, acts trivially on C_n and acts on $\Lambda_n/C_n \cong \mathbf{Z}^2$ as N. We form a semi-direct product $\Gamma_N = \Lambda_n \rtimes \mathbf{Z}$ where the $\mathbf{Z}$ factor acts on Λ_n as φ. The group Γ_N acts on the group of real upper-triangular matrices which is identified with $\mathbf{R}^3 = \mathbf{R} \times \mathbf{C}$. Define an action of Γ_N on $\mathbb{H} \times \mathbf{C} = \mathbf{R}^{>0} \times \mathbf{R} \times \mathbf{C}$ with Λ_n acting trivially on $\mathbf{R}^{>0}$ and $1 \in \mathbf{Z}$ acting on $\mathbb{H}$ as $x \mapsto \alpha x$. This action is holomorphic and the quotient $S_N = \mathbb{H} \times \mathbf{C}/\Gamma_N$ is a compact non-Kähler surface called an *Inoue surface of type S^+*.

The group Γ_N can be identified with a lattice in one of the two following solvable Lie groups which are subgroups of $\mathrm{Aff}_2(\mathbf{C})$ (cf. [Kli98]):

$$\mathrm{Sol}^1 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & d & c \\ 0 & 0 & 1 \end{pmatrix}, a,b,c,d \in \mathbf{R}, d > 0 \right\},\ \mathrm{Sol}^{1'} = \left\{ \begin{pmatrix} 1 & a & b + i\log(d) \\ 0 & d & c \\ 0 & 0 & 1 \end{pmatrix}, a,b,c,d \in \mathbf{R}, d > 0 \right\}.$$

Conversely any torsion free lattice of these two groups gives an Inoue surface of type S^+. Note that a finite unramified cover of an Inoue surface of type S^+ is an Inoue surface of type S^+.

Concretely Γ_N has four generators g_0, g_1, g_2, g_3 which act on $\mathbb{H} \times \mathbf{C}$ as:

$$\begin{aligned} g_0 &: (x,y) \mapsto (\alpha x, y + t) \\ g_i &: (x,y) \mapsto (x + a_i, y + b_i x + c_i) \quad i = 1,2 \\ g_3 &: (x,y) \mapsto (x, y + \frac{b_1 a_2 - b_2 a_1}{n}) \end{aligned}$$

where t is a complex number, (a_1, a_2) (resp. (b_1, b_2)) is a real eigenvector of N corresponding to the eigenvalue α (resp. α^{-1}) and c_1, c_2 are some complex numbers. Thus Inoue surface of type S^+ are also equipped with complex affine structures by construction.

An Inoue surface of type S^- has a double cover which is an Inoue surface of type S^+. It is the quotient of $\mathbb{H}\times\mathbf{C}$ by a group generated by affine transformations g_0, g_1, g_2, g_3 of the form

$$\begin{aligned} g_0 &: (x,y) \mapsto (\alpha x, -y) \\ g_i &: (x,y) \mapsto (x+a_i, y+b_i x+c_i) \quad i=1,2 \\ g_3 &: (x,y) \mapsto (x, y+\frac{b_1a_2-b_2a_1}{n}) \end{aligned}$$

and has a complex affine structure.

Affine-elliptic bundles. Let $S=\mathbb{H}/\pi_1(S)$ be a compact hyperbolic Riemann surface. Let $\bar{\rho}: \pi_1(S)\to \mathrm{PGL}_2(\mathbf{C})$ and $\phi:\mathbb{H}\to\mathbb{P}^1$ be the holonomy and developping map of some complex projective structure on S. By [Gun67], $\bar{\rho}$ always lifts to a representation $\rho:\pi_1(S)\to\mathrm{GL}_2(\mathbf{C})$. Consider $\mathbf{C}^2\backslash\{0\}$ as a $\mathbf{C}^*$-bundle over $\mathbb{P}^1$ and denote it by W. Consider the $\mathbf{C}^*$-bundle ϕ^*W over $\mathbb{H}$ obtained by pulling back W via ϕ. The natural complex affine structure on W as an open set of $\mathbf{C}^2$ induces a complex affine structure on ϕ^*W. The group $\pi_1(S)$ acts on ϕ^*W via its natural action on $\mathbb{H}$ and the representation ρ. This action preserves the affine structure on ϕ^*W. Multiplication in the fibers also preserves this affine structure. Hence the quotient of ϕ^*W by the group generated by $\pi_1(S)$ and a multiplication in the fibers is a compact elliptic surface with a complex affine structure. We call them affine-elliptic bundles as in [Kli98].

Ball quotients. Let $\mathbb{B}^n$ be the unit ball $\{[z_0;\cdots;z_n];|z_0|^2-\sum_{j=1}^n|z_j|^2>0\}\subset\mathbb{P}^n$. The group of biholomorphisms of $\mathbb{B}^n$ is $\mathrm{PU}(1,n)$. If Γ is a torsion-free cocompact lattice of $\mathrm{PU}(1,n)$, then $\mathbb{B}^n/\Gamma$ is a projective manifold of general type equipped with a complex projective structure.

Classification.

Theorem 3.3.1 (Kobayashi-Ochiai [KO80], Klingler [Kli98]) *If a compact complex surface X admits a complex projective structure, then X is biholomorphic to the projective plane, a complex torus, a Hopf surface, a primary Kodaira surface, an Inoue surface, an affine-elliptic bundle or a ball quotient.*

Theorem 3.3.2 (Klingler [Kli98], Mok-Yeung [MY93]) *The examples described in this section exhausts all possible complex projective structures on compact complex surfaces.*

Remark 3.3.3 We can see from the examples described above that the uniqueness of complex projective structure on a given surface does not always hold: some of the examples depend on parameters. See [Kli98] for complete description of deformations of complex projective structures on surfaces.

3.3.2 Projective examples in higher dimension

Kuga fiber varieties. Kuga fiber varieties are total spaces of families of abelian varieties over Shimura varieties. Only those which fiber over a Shimura curve have projective structures. The details of the construction are a little bit lengthy but classical. We give here a rough presentation to show how the projective structure arises and refer to [JR15] for details. Let Γ be a torsion free quaternionic cocompact lattice of $\mathrm{SL}_2(\mathbf{R})$ such that $\mathbb{H}/\Gamma$ is a compact hyperbolic Riemann surface. To such a lattice we can associate a representation $\rho : \Gamma \to \mathrm{GL}_g(\mathbf{R})$ for some $g \in \mathbf{N}$ and a lattice Λ in the additive group $M_{g,2}(\mathbf{R})$ of $(g \times 2)$-matrices that satisfy various arithmetic properties. Consider the group Γ_Λ formed by matrices in $\mathrm{GL}_{g+2}(\mathbf{R})$ of the following block form:

$$\gamma_\lambda = \begin{pmatrix} \rho(\gamma) & \rho(\gamma)\lambda \\ 0_{g\times 2} & \gamma \end{pmatrix} \in \mathrm{GL}_{g+2}(\mathbf{R}), \quad \gamma \in \Gamma, \lambda \in \Lambda.$$

The group Γ_Λ is a semi-direct product $\Lambda \rtimes \Gamma$ which acts on $\mathbf{C}^g \times \mathbb{H} \subset \mathbb{P}^{g+1}$ as follows:

$$\gamma_\lambda(z,\tau) = \left(\frac{\rho(\gamma)\left(z + \lambda \begin{pmatrix} \tau \\ 1 \end{pmatrix}\right)}{c\tau + d}, \frac{a\tau+b}{c\tau+d} \right), \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The arithmetic properties of ρ and Λ guarantees that the quotient $(\mathbf{C}^g \times \mathbb{H})/\Gamma_\Lambda$ is a polarized family of abelian varieties over $\mathbb{H}/\Gamma$. In particular it is a projective manifold with a complex projective structure. Finally remark that these Kuga fiber varieties exist for any $g \geq 2$ but not for $g = 1$.

We mention the following classification in higher dimension:

Theorem 3.3.4 (Jahnke-Radloff [JR15]) *Projective spaces, complex tori, ball-quotients and Kuga fiber varieties over a quaternionic Shimura curve are the only projective manifolds that admit complex projective structures.*

3.4 Proof for X Kähler

3.4.1 Existence of complex projective structures

In this section a *rational curve* on a compact complex manifold is a non-constant holomorphic map $\mathbb{P}^1 \to X$.

Proposition 3.4.1 *If a compact Kähler manifold X of dimension n has a rational curve and a complex projective structure, then $X = \mathbb{P}^n$.*

Proof Let $\iota : \mathbb{P}^1 \to X$ be a rational curve and $h : \tilde{X} \to \mathbb{P}^n$ be the developping map. Then $\iota^* T_X = (h \circ \iota)^* T_{\mathbb{P}^n}$ is an ample vector bundle on $\mathbb{P}^1$. This implies that X is rationally connected (cf. [Deb01] 4.3). A rationally connected manifold is simply connected (cf. [Deb01] 4.3). Thus the holonomy is trivial and the developping map $X \to \mathbb{P}^n$ must be an isomorphism. □

Theorem 3.4.2 *If X is of general type and admits a complex projective structure, then X is a ball quotient.*

Proof For d large enough, the pluricanonical linear system $|dK_X|$ is base point free (cf. [KM98]) and the Iitaka fibration is a birational morphism. The exceptional fibers of the Iitaka fibration contain rational curves by [Deb01] Proposition 1.43. Proposition 3.4.1 then implies that the Iitaka fibration is an isomorphism. In other words the canonical bundle K_X is ample. Thus X admits a Kähler-Einstein metric. Once X is Kähler-Einstein, the equality $c_1^2 = \frac{2(n+1)}{n} c_2$ given by Theorem 3.2.5 implies by Theorem 3.2.2 that X is a ball quotient (in dimension two see Yau's theorem [Yau77]). □

Theorem 3.4.3 (Kobayashi-Ochiai [KO80]) *The only compact Kähler surfaces with complex projective structures are $\mathbb{P}^2$, complex tori and ball quotients.*

Proof Let $X \neq \mathbb{P}^2$ be a compact Kähler surface with a complex projective structure. By Proposition 3.4.1 the Kodaira dimension $\kappa(X)$ is $0, 1$ or 2 because a surface with Kodaira dimension $-\infty$ has many rational curves. If $\kappa(X) = 2$ then X is a ball quotient by Proposition 3.4.2. If $\kappa(X) = 0, 1$, then Theorem 3.2.5 implies $c_1^2 = c_2 = 0$. If moreover $\kappa(X) = 0$ then Enriques-Kodaira's classification of surfaces implies that X is a complex torus.

Now suppose by contradiction that $\kappa(X) = 1$. X is an elliptic surface, there is a fibration $f : X \to C$ over a smooth curve C such that general fibers are smooth elliptic curves. By Noether's equality ([Bar+04] I.5) the Euler characteristic $\chi(\mathscr{O}_X)$ is $\frac{1}{12}(c_1^2 + c_2) = 0$. Therefore by [Bar+04]

V.12.2 and the remark preceding it, the only singular fibers of f are multiples of smooth elliptic curves and the fibration is locally trivial around a general fiber. Now think of C as an orbicurve of which the special points correspond to multiple fibers. It is a hyperbolic orbicurve uniformized by $\mathbb{H}$ (cf. [Bar+04] V.12 and [BN06]). In particular there is a covering $B \to C$ where B is an ordinary curve. Then $X \times_C B$ is an unramified covering of X and the fibration $X \times_C B \to B$ has no multiple fibers. An unramified covering of a manifold with a complex projective structure has an induced complex projective structure. Therefore up to replacing X with $X \times_C B$ we may assume that $f : X \to C$ is a genus one fibration with no multiple nor singular fibers. Up to taking another unramified covering, we can also assume $f_* K_{X/C} = \mathscr{O}_C$ (cf. [Bar+04] III.18). Then the canonical bundle formula ([Bar+04] V.12) gives $K_X = f^* K_C$. The pull-back map $f^* K_C \to \Omega_X$ gives a short exact sequence:

$$0 \longrightarrow f^* K_C = K_X \xrightarrow{\iota} \Omega_X \xrightarrow{\rho} \mathscr{O}_X \longrightarrow 0. \tag{3.3}$$

Using $c_1^2 = c_2 = 0$ we have $h^{2,0} = h^{1,0} + 1$ by Noether's equality ([Bar+04] I.5). This means that Sequence (3.3) is exact at H^0 level and splits holomorphically. Consider the maps α : $\mathrm{H}^1(X, \mathscr{O}_X \otimes \mathscr{O}_X^* \otimes \Omega_X) \to \mathrm{H}^1(X, \mathscr{O}_X \otimes T_X \otimes \Omega_X)$ and $\beta : \mathrm{H}^1(X, \Omega_X \otimes T_X \otimes \Omega_X) \to \mathrm{H}^1(X, \mathscr{O}_X \otimes T_X \otimes \Omega_X)$ induced respectively by $\mathrm{Id} \otimes \rho^* \otimes \mathrm{Id}$ and $\rho \otimes \mathrm{Id} \otimes \mathrm{Id}$. We have $\beta(a(\Omega_X)) = \alpha(a(\mathscr{O}_X)) = 0$. By Proposition 3.2.4 we have $a(\Omega_X) = \frac{1}{n+1}(c_1(K_X) \otimes \mathrm{Id} + \mathrm{Id} \otimes c_1(K_X))$. Thus

$$0 = \beta(a(\Omega_X)) = \beta(\frac{1}{n+1} \mathrm{Id} \otimes c_1(K_X)) = \frac{1}{n+1} \alpha(\mathrm{Id} \otimes c_1(K_X)).$$

However α is injective because Sequence (3.3) splits. Hence $c_1(K_X) = 0$, contradicting the fact $\kappa(X) = 1$. □

3.4.2 Complex tori

To finish the proof of Theorem 3.1.1 for those Kleinian groups with Kähler quotients, it remains to prove that the only ways to realize complex tori and ball quotients are the ones described in Theorem 3.1.1. We said that the fact that the natural complex projective structure on a ball quotient is unique is a difficult result and we simply refer to [MY93] and [Kli01] for the proof. Here we give a proof for complex tori. The proof is based on some arguments of Yves Benoist [Ben94] that apply not only to tori but also to compact manifolds with nilpotent fundamental groups. The contribution of [Kli98] to Theorem 3.1.1 is to apply Benoist's arguments to all possible surfaces. We focus here on complex tori: the proof is simpler and represents the main idea.

Let $(\mathbb{P}^2, U, \Gamma, X)$ be a Kleinian group such that $\Gamma \subset \mathrm{PGL}_3(\mathbf{C})$ and X is a complex torus. There is a developing map $D : \overline{X} = \mathbf{C}^2 \to U$ and a holonomy representation $h : \mathbf{Z}^4 = \pi_1(X) \to \Gamma$. Let N be a maximal connected nilpotent subgroup of $\mathrm{PGL}_3(\mathbf{C})$ that contains the identity component of the complex Zariski closure of $\Gamma \subset \mathrm{PGL}_3(\mathbf{C})$. Then $N \cap \Gamma$ is a finite index subgroup of Γ. Denote by $\overline{N}$ the universal covering of the complex Lie group N and $\pi : \overline{N} \to N$ the covering map.

Lemma 3.4.4 (Benoist [Ben94]) *There is a holomorphic action of $\overline{N}$ on $\overline{X}$ lifting the action of N on $\mathbb{P}^2$, i.e.*

$$D(\phi \cdot x) = \pi(\phi) \cdot D(x), \quad x \in \overline{X}, \quad \phi \in \overline{N}.$$

Proof As N is unipotent, there are normal subgroups $\{N_k\}_{0 \le k \le l}$ of N such that $\{e\} = N_l \subset \cdots \subset N_0 = N$ and N_k/N_{k-1} is one-dimensional. We make a proof by recurrence on k. Suppose that the action of N_{k-1} on $\mathbb{P}^2$ lifts to an action of $\overline{N_{k-1}}$ on $\overline{X}$. Denote by L_k the Lie algebra of N_k. Let a be an element of L_k such that $L_k = L_{k-1} \oplus \mathbf{C}a$. The element a corresponds to a vector field A on $\mathbb{P}^2$ that integrates to $\exp(a) \in \mathrm{PGL}_3(\mathbf{C})$. We pull back A by D, obtaining a vector field $\overline{A}$ on $\overline{X}$. Denote by Φ_t the flow of $\overline{A}$. It suffices to prove that Φ_t is defined everywhere for any time t because then the action of $\overline{N_k}$ on $\overline{X}$ could be defined by

$$(\overline{exp}(ta)f) \cdot w = \Phi_t(f \cdot w) \quad f \in \overline{N_{k-1}}, \quad w \in \overline{X}.$$

Denote by t_w the lifetime of Φ_t at $w \in \overline{X}$, i.e. the supremum of t for which Φ_t is defined at w. We firstly prove that $t_w = t_{\gamma w}$ for any $\gamma \in \pi_1(X)$ such that $h(\gamma) \in N \cap \Gamma$. We define a path p_t in N_k as follows:

$$p_t = \exp(ta) h(\gamma) \exp(ta)^{-1} h(\gamma)^{-1}.$$

Since N_{k-1} is a normal subgroup of N_k, the path p_t is in fact in N_{k-1}. We lift p_t to a path $\overline{p}_t$ on $\overline{N_{k-1}}$ such that $\overline{p}_0 = \mathrm{Id}$. Consider the following path on $\overline{X}$:

$$q_t = \overline{p}_t \gamma \Phi_t(w).$$

It is only defined for $t < t_w$ and is an integral curve of $\overline{A}$ starting from γw because $D(q_t) = \exp(ta) D(\gamma w)$. Hence $t_{\gamma w} \ge t_w$. By considering γ^{-1} we have $t_{\gamma w} = t_w$.

Thus the function $w \mapsto t_w$ comes from a function defined on a compact quotient of $\overline{X}$. It is a standard fact that the lifetime t_w is a lower semicontinuous function in w. By compactness it has a minimum on $\overline{X}$ and this minimum is strictly positive. This implies that the flows Φ_t are in fact defined for all t. The proof is finished. $\square$

As an immediate consequence we have:

Corollary 3.4.5 *The open set U is a union of N-orbits.*

Proof Let $z = D(w) \in U$ with $w \in \tilde{X}$. Then $N \cdot z = D(\tilde{N} \cdot w) \subset U$. $\square$

We need a basic result from Lie theory to describe the orbits of N. Define

$$N(\lambda, d) = \{g \in \mathrm{GL}(\mathbf{C}^d) | (g - \lambda \,\mathrm{Id}) \text{ is strictly upper triangular}\}.$$

Lemma 3.4.6 *Let M be a maximal connected nilpotent subgroup of $\mathrm{GL}_n(\mathbf{C})$. Then there is a unique decomposition $\mathbf{C}^n = \oplus_{1 \le k \le l} E_k$ and $M = M_1 \times \cdots \times M_l$ such that $M_k = M \cap \mathrm{GL}(E_k)$ is conjugate in $\mathrm{GL}(E_k)$ to $N(\lambda_k, \dim E_k)$ for some $\lambda_k \in \mathbf{C}^*$.*

Proof It suffices to prove the corresponding result for Lie algebras. Denote by L and L_k the Lie algebras of M and M_k. By [Var84] Theorem 3.5.8, the L-algebra $\mathbf{C}^n$ decomposes into a direct sum of L-algebras $\oplus_{1 \le k \le l} E_k$ where each E_k corresponds to a weight λ_k. Engel's theorem implies that the representation of M in $\mathrm{GL}(E_k)$ is contained in $N(\lambda_k, \dim E_k)$ up to conjugation. Hence the maximality of M allows us to conclude. $\square$

Now we combine the two lemmas to obtain:

Theorem 3.4.7 (Klingler [Kli98]) *The open set U is one of the Zariski open subsets $\mathbf{C} \times \mathbf{C}$, $\mathbf{C}^* \times \mathbf{C}$ and $\mathbf{C}^{*2}$ of $\mathbf{C}^2 \subset \mathbb{P}^2$ and Γ is a lattice in U viewed as a Lie group.*

Proof There are d orbits of $N(\lambda, d)$ in $\mathbf{C}^d \backslash \{0\}$ and they are of the form

$$\{(z_1, \cdots, z_d) \in \mathbf{C}^d, z_j \neq 0, z_{j-1} = \cdots = z_1 = 0\}$$

where the coordinates $(z_1, \cdots, z_d)$ are written in a base with respect to which $N(\lambda, d)$ is upper-triangular. Recall that N is a maximal connected nilpotent subgroup of $\mathrm{PGL}_3(\mathbf{C})$ such that $N \cap \Gamma$ is of finite index in Γ. Lemma 3.4.6 implies that N has an open orbit in $\mathbb{P}^2$ of the form $\mathbb{P}(\mathbf{C}^{*3}), \mathbb{P}(\mathbf{C}^{*2} \times \mathbf{C})$ or $\mathbb{P}(\mathbf{C}^* \times \mathbf{C}^2)$, i.e. the complement of three lines, two lines or one line.

Such an open orbit must be contained in U by Corollary 3.4.5. As U is a union of N-orbits by Corollary 3.4.5, there are only eight possibilities for U: $\mathbb{P}^2$ minus $0,1,2$ or 3 points, $\mathbf{C}^2\setminus\{0\}$, $\mathbf{C}\times\mathbf{C}$, $\mathbf{C}^*\times\mathbf{C}$ and $\mathbf{C}^{*2}$. The first five open sets cannot be infinite coverings of a complex torus. Therefore U is one of the three desired Zariski open sets. $\square$

CHAPTER 4

REPRESENTATIONS OF KÄHLER GROUPS INTO THE CREMONA GROUP

4.1 Birational Kleinian groups are elementary

The goal of this chapter is to show

Theorem 4.1.1 *Let (Y, U, Γ, X) be a birational Kleinian group in dimension two. If X is not a surface of class VII then Γ is an elementary subgroup of* $\mathrm{Bir}(Y)$.

It is based on the following theorem of Delzant-Py (the terminology *factors through a fibration* is explained below):

Theorem 4.1.2 (Delzant-Py [DP12]) *Let X be a compact Kähler manifold and Y a rational surface. Let $\rho : \pi_1(X) \to \mathrm{Bir}(Y)$ be a non-elementary representation, then one of the following two cases occurs:*

1. *There is a birational map $\varphi : Y \dashrightarrow \mathbf{C}^* \times \mathbf{C}^*$ such that $\varphi\rho\varphi^{-1}$ is a representation into the toric group $\mathbf{C}^* \times \mathbf{C}^* \rtimes \mathrm{GL}_2(\mathbf{Z})$.*
2. *There is a finite index subroup Γ' of Γ corresponding to a finite étale cover X' of X such that the induced representation $\pi_1(X') \to \Gamma' \to \mathrm{Bir}(\mathbb{P}^2)$ factors through a fibration $X' \to \Sigma$ onto a hyperbolic orbicurve Σ.*

Factorization through orbicurves. In this article a *hyperbolic orbicurve* Σ will be a Riemann surface with finitely many marked points with multiplicities, obtained as a quotient of the Poincaré half-plane by the action of a cocompact lattice in $\mathrm{PSL}_2(\mathbf{R})$; marked points are images of the fixed points of the action and multiplicities are orders of the stabilizers. The cocompact lattice is isomorphic to the orbifold fundamental group $\pi_1^{orb}(\Sigma)$ of Σ. A continuous map from a complex manifold X to Σ is holomorphic if it lifts to a holomorphic map from the universal cover $\tilde{X}$ of X to the half-plane, i.e. if there exists a holomorphic map from $\tilde{X}$ to $\mathbb{H}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \mathbb{H} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \Sigma \end{array}$$

A *fibration* of X onto Σ is a holomorphic surjective map $f : X \to \Sigma$ with connected fibres. A fibration $f : X \to \Sigma$ induces a homomorphism $f_* : \pi_1(X) \to \pi_1^{orb}(\Sigma)$. We say that a group representation $\rho : \pi_1(X) \to G$ *factors through the fibration* f if there exists a homomorphism $\hat{\rho} : \pi_1^{orb}(\Sigma) \to G$ such that $\rho = \hat{\rho} \circ f_*$.

We will show in Section 4.5 how to deduce Theorem 4.1.1 from Delzant-Py's work. Since the use of Delzant-Py's work is a crucial step in our study of birational Kleinian groups in dimension two, we will firstly explain where Theorem 4.1.2 comes from. Fundamental groups of compact Kähler manifolds are called *Kähler groups*. The proof of Theorem 4.1.2 can be divided into two parts: the first part (cf. Theorem 4.2.1) is about representations of Kähler groups into isometry groups of hyperbolic spaces and the second part consists of applying the first part to the action of the Cremona group on an infinite dimensional hyperbolic space. The first part relies on methods from non-abelian Hodge theory, i.e. the theory of harmonic maps on compact Kähler manifolds (or on their universal covers). These methods have a long history and are highly involved. We give a sketch of the strategy in Section 4.2. In Section 4.3 we will show in detail how this first part is applied to representations into Cremona groups. Readers who for the moment are not interested in the proof of Delzant-Py's theorem can skip Sections 4.2 and 4.3. In the last section of this chapter we discuss to which extent the proof of Theorem 4.1.1 can be adapted to surfaces of class VII.

4.2 Harmonic maps and factorization

Hyperbolic spaces. We recall here the hyperboloid construction of hyperbolic spaces. Let H be a real Hilbert space. Let $(e_i)_i$ be a Hilbert basis of H; we assume nothing on the cardinality of the base. For $A = \sum a_j e_j \in H$, define $<A,A>= a_0^2 - \sum_{j>0} a_j^2$. Then $<,>$ defines a symmetric bilinear form of signature $(1, \dim(H)-1)$ on H. The hyperboloid $\{A \in E \mid <A,A>=1\}$ has two connected components. We can choose the component with first coordinate $a_0 > 0$ to be a model of the hyperbolic space. The tangent space at a point A can be identified with the orthogonal of $\mathbf{R}A$ in H and the Riemannian metric is given by the opposite of the restriction of $<,>$ to this orthogonal. If the Hilbert space H has finite dimension $m+1$, then the hyperbolic space is of dimension m and will be denoted by $\mathbb{H}^m$. If H is infinite dimensional, then no matter the cardinality of its basis is we will denote by $\mathbb{H}^\infty$ the corresponding hyperbolic space, by abuse of notation. The boundary $\partial\mathbb{H}^m$ of the hyperbolic space can be identified with the set of isotropic lines in H.

We say that an isometric action of a finitely generated group on $\mathbb{H}^m (m \in \mathbf{N} \cup \{\infty\})$ is *elementary* if it fixes a point or a pair of points (i.e. a geodesic) in $\mathbb{H}^m \cup \partial\mathbb{H}^m$, *non-elementary* otherwise. The action is said to be *minimal* if $\mathbb{H}^m$ contains no non-empty closed totally geodesic strict subspace that is invariant.

Carlson-Toledo and Delzant-Py's theorem. We can now state the main theorem of this section the proof of which we will try to explain:

Theorem 4.2.1 (Carlson-Toledo [CT89], Delzant-Py [DP12]) *Let X be a compact Kähler manifold. Let $\rho : \pi_1(X) \to \mathrm{Isom}(\mathbb{H}^m)$ be a non-elementary representation into the isometry group of a possibly infinite dimensional hyperbolic space. Assume that ρ is minimal, i.e. that $\mathbb{H}^m$ contains no non-trivial closed ρ-invariant totally geodesic subspaces. Then one of the following two cases happens:*

1. *ρ factors through a fibration onto a hyperbolic orbicurve;*
2. *$\rho = \Psi \circ \theta$ where $\theta : \pi_1(X) \to \mathrm{PSL}_2(\mathbf{R})$ is a homomorphism with dense image and $\Psi : \mathrm{PSL}_2(\mathbf{R}) \to \mathrm{Isom}(\mathbb{H}^m)$ is a continuous homomorphism. This is only possible when $m = 2$ or ∞ by minimality.*

The theorem is due to Carlson-Toledo [CT89] when m is finite and the image of ρ is contained in a cocompact lattice of $\mathrm{Isom}(\mathbb{H}^m)$. Though the strategy is essentially the same, the generalization to $\mathbb{H}^\infty$ needs still a lot of extra work. Especially it relies on the heavy machinery of Korevaar-Schoen [KS93], [KS97]. We will present the main steps of the proof and say where the infinite dimensional case presents more technical difficulties.

Harmonic maps. Let $f : M \to N$ be a smooth map between two finite dimensional Riemannian manifolds. The *energy density* $e_f(x)$ of f at a point $x \in M$ is by definition $\|df_x\|^2$ where $\|df_x\|$ is the operator norm of df_x with respect to the Riemannian metrics on M and N. For compact M, the total energy of f is $E(f) = \int_M e_f(x)$ and f is a *harmonic map* if it is a critical point of the energy functional E. For N of non-positive sectional curvature, Eells-Sampson [ES64] proved that in each homotopy class of continuous maps from M to N there is a harmonic representative.

Assume still that M is compact. Let $\rho : \pi_1(M) \to \mathrm{Isom}(N)$ be a group morphism into the isometry group of N. A ρ-equivariant map is a smooth map f from the universal covering $\tilde{M}$ of M to N such that $f(\gamma \cdot x) = \rho(\gamma)\dot{f}(x)$ for any $x \in \tilde{M}$ and $\gamma \in \pi_1(M)$. In this situation the energy density of f descends to a function on M, thus is integrable. Therefore the notion of harmonic maps makes sense for such equivariant maps. For N of non-positive sectional curvature, the existence of ρ-equivariant harmonic maps is established by Corlette [Cor88], Donaldson [Don87] and Labourie [Lab91] under some reductive assumption on ρ. We refer to [Amo+96] pages 68-69 for this reductive assumption because we will not need it.

Note that the above results are sufficient for the use of harmonic maps in the proof of Carlson-Toledo's early version of Theorem 4.2.1. However to treat $\mathbb{H}^\infty$, we need a more general notion of harmonic maps due to Korevaar-Schoen [KS93].

Now assume that (N,d) is merely a metric space, not necessarily locally compact. To define harmonic maps from a compact Riemannian manifold M to N, or equivariant harmonic maps from $\tilde{M}$ to N, we only need a new definition of energy density. Let $f : M \to N$ be a map. For $\varepsilon > 0$ and $x \in M$, denote by $S(x,\varepsilon)$ the hypersurface in M of points at distance ε from x. Define

$$e_f(\varepsilon,x) = \int_{S(x,\varepsilon)} \frac{d(f(x),f(y))^2}{\varepsilon^2} \frac{d\nu(y)}{\varepsilon^{d-1}}$$

where ν is the measure on $S(x,\varepsilon)$ induced by the Riemannian metric of M and d is the dimension of M. If the $e_f(\varepsilon,x)$ are regular enough, for example uniformly bounded in ε, then $e_f(\varepsilon,x)dx$ converge weakly, as ε goes to 0, to a measure $e_f(x)dx$ absolutely continous with respect to the Lebesgue measure (cf. [KS93]). The energy density is thus defined to be the L^1 function $e_f(x)$. It coincides with the previous definition when f is a smooth map with values in a finite dimensional Riemannian manifold N. The following is probably the most difficult ingredient in the proof of Delzant-Py's theorem:

Theorem 4.2.2 (Korevaar-Schoen [KS97]) *Let N be a CAT(-1) complete metric space and $\rho : \pi_1(M) \to \mathrm{Isom}(N)$ be an action which fixes no points on the boundary of N, i.e. no equivalence class of geodesic rays. Then there exists a ρ-equivariant harmonic map from $\tilde{M}$ to N.*

Harmonic maps on Kähler manifolds. From now on we assume that the compact Riemannian manifold M has a complex structure and that the Riemannian metric on M is induced by a hermitian metric with fundamental form κ. We also assume that N is a Riemannian manifold.

Let us first assume $\dim(N) < +\infty$. We will denote by f a smooth map $M \to N$ or a smooth equivariant map $\tilde{M} \to N$. Denote by TN the tangent bundle of N and f^*TN the pulled-back bundle on M. We denote by $\mathscr{A}^k(f^*TN)$ the sheaf of complex k-forms with value in the pull-back tangent bundle. We will think of df as a section of $\mathscr{A}^1(f^*TN)$. Denote by ∇ both the Levi-Civita connection on TN and the induced connection on f^TN. We define a connection $d_\nabla : \mathscr{A}^k(f^*TN) \to \mathscr{A}^{k+1}(f^*TN)$ as follows:

$$d_\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \otimes \nabla s.$$

We refer to [Amo+96] for the following formula:

Proposition 4.2.3 *The map f is harmonic if and only if $d_\nabla(\kappa^{n-1} \wedge d^c f) = 0$.*

We say that f is *pluriharmonic* if $d_\nabla d^c f = 0$. This means that f is harmonic restricted to any holomorphic disc in M.

Example 4.2.4 ([ES64]) Holomorphic maps between Kähler manifolds are harmonic.

To state the next result concerning harmonic maps, we need Koszul-Malgrange theorem. If $\delta : \mathscr{A}^k(V) \to \mathscr{A}^{k+1}(V)$ is a connection on a complex vector bundle V over a complex manifold, then its $(0,1)$-part $\delta^{0,1}$ is the composition

$$\mathscr{A}^{0,k}(V) \to \mathscr{A}^k(V) \xrightarrow{\delta} \mathscr{A}^{k+1}(V) \to \mathscr{A}^{0,k+1}(V)$$

where the first arrow is the natural injection and the last arrow is the natural projection.

Theorem 4.2.5 (Koszul-Malgrange [KM58]) *A complex vector bundle on a complex manifold has a structure of holomorphic vector bundle if it has a connection δ with $(\delta^{0,1})^2 = 0$. Then $\delta^{0,1}$ is $\bar\partial$ with respect to that holomorphic structure.*

We say that a Riemannian manifold has non-positive Hermitian sectional curvature if $R(X,Y,\bar X,\bar Y) \leq 0$ where R is the usual curvature tensor extended to the complexified tangent bundle and X, Y are arbitrary vectors in the complexified tangent space. In particular such a Riemannian manifold has non-positive sectional curvature.

Theorem 4.2.6 (Siu [Siu80], Sampson [Sam86]) *Let M be a compact Kähler manifold and N a Riemannian manifold with non-positive Hermitian sectional curvature. Let f be a harmonic map $M \to N$ or an equivariant harmonic map $\tilde M \to N$. Then*

1. *f is pluriharmonic, i.e. $d_\nabla d^c f = 0$;*

2. $(d_{\nabla}^{0,1})^2 = 0$ *so that* $f^*TN \otimes \mathbf{C}$ *is a holomorphic vector bundle by Koszul-Malgrange theorem;*
3. *The complex linear part* $df^{1,0}$ *of* df *is a holomorphic* 1*-form with values in the holomorphic vector bundle* $f^*TN \otimes \mathbf{C}$*.*

This nice theorem makes the transition from Riemannian geometry to complex geometry; it is a kind of Bochner-type formula (cf. [Amo+96] Chapter 6).

When the target N is an infinite dimensional manifold, Korevaar-Schoen's harmonic map is a priori not necessarily a smooth map. Nevertheless when the harmonic map is smooth, it is still characterized by Proposition 4.2.3 and Theorem 4.2.6 still holds: the same proof works (cf. [DP12]).

Hyperbolic targets. Now we come back to the setting of Theorem 4.2.1, i.e. X is compact Kähler, $\mathbb{H}^m$ is a possibly infinite dimensional hyperbolic space and $\rho : \pi_1(X) \to \mathrm{Isom}(\mathbb{H}^m)$ is a non-elementary representation. As ρ is non-elementary, we can apply Korevaar-Schoen's theorem to find a ρ-equivariant harmonic map $f : \tilde{X} \to \mathrm{Isom}(\mathbb{H}^m)$. Delzant-Py verified that for hyperbolic targets the harmonic maps constructed by Korevaar-Schoen are actually smooth. Therefore Theorem 4.2.6 holds for f. Using the fact that $\mathbb{H}^m$ is a symmetric space of rank one, we have

Proposition 4.2.7 (Sampson [Sam86]) *The real rank of the differential* df *is everywhere* ≤ 2*. In other words, the holomorphic* 1*-form* $df^{1,0}$ *with values in the holomorphic vector bundle* $f^*TN \otimes \mathbf{C}$ *is everywhere of complex rank* ≤ 1*.*

The proof is same for finite or infinite dimensional $\mathbb{H}^m$ (cf. [Amo+96], [DP12]). The next step is to prove

Theorem 4.2.8 (Delzant-Py [DP12]) *Under the assumption of Theorem 4.2.1, there is a holomorphic map* g *from* $\tilde{X}$ *to the upper half plane* $\mathbb{H}$ *and a smooth harmonic map* $u : \mathbb{H} \to \mathbb{H}^m$ *such that* $f = u \circ g$*. There exist two group morphisms* $\rho_1 : \pi_1(X) \to \mathrm{PSL}_2(\mathbf{R})$ *and* $\rho_2 : \rho_1(\pi_1(X)) \to \mathrm{Isom}(\mathbb{H}^m)$ *such that* $\rho = \rho_2 \circ \rho_1$ *and* g *is* ρ_1*-equivariant and* u *is* ρ_2*-equivariant.*

The main remaining task for proving Theorem 4.2.8 is to construct the holomorphic map g from the holomorphic form df. The fact that df is of rank ≤ 1 indicates that the target should be a Riemann surface. The holomorphic form df defines a singular holomorphic foliation on $\tilde{X}$. Intuitively the fibers of g are unions of leaves of this foliation. Carlson-Toledo did the construction of g for harmonic maps $f : X \to \mathbb{H}^m$. For equivariant harmonic maps the foliation is more difficult to study because of the non-compactness of $\tilde{X}$. The infinite dimension of $\mathbb{H}^m$ does not bother for the construction of g. Nevertheless when $\mathbb{H}^m$ is of infinite dimension, one needs to prove that the harmonic map u is smooth after the construction of the factorization $f = u \circ g$.

Once we have Theorem 4.2.8, the first case in Theorem 4.2.1 is automatic: if ρ_1 has discrete image, then $g : \tilde{X} \to \mathbb{H}$ descends to $X \to \mathbb{H}/\rho_1(\pi_1(X))$. The second case of Theorem 4.2.1 is another novelty in Delzant-Py's work; one needs to show

Proposition 4.2.9 (Delzant-Py) *Under the assumption of Theorem 4.2.8 and using the same notations, if the image of ρ_1 is not discrete in $\mathrm{PSL}_2(\mathbf{R})$, then it is dense in the euclidean topology. Furthermore ρ_2 extends to $\mathrm{PSL}_2(\mathbf{R})$ and u is equivariant with respect to the extended ρ_2.*

Such wild factorizations really exist for $\mathbb{H}^\infty$, but we will see in next section that they don't occur for applications to the Cremona group.

4.3 Representations into the Cremona group

4.3.1 Factorization

In this section we prove Theorem 4.1.2 from Theorem 4.2.1.

Proof (of Theorem 4.1.2) Consider a non-elementary representation

$$\rho : \pi_1(X) \to \mathrm{Bir}(Y) \hookrightarrow \mathrm{Isom}(\mathbb{H}^\infty_Y).$$

We denote by Γ both the images of ρ in $\mathrm{Bir}(Y)$ and $\mathrm{Isom}(\mathbb{H}^\infty_Y)$; it is a non-elementary subgroup. There exists a unique ρ-invariant subspace $\mathbb{W}$ of $\mathbb{H}^\infty_Y$ on which the action of $\pi_1(X)$ is minimal (see the first paragraph of Section 4.2 for the definition of minimal) and non-elementary. We can apply Theorem 4.2.1 to $\pi_1(X) \to \mathrm{Isom}(\mathbb{W})$.

We firstly show that we are not in the second case of Theorem 4.2.1. Suppose by contradiction that our representation ρ factorizes through a dense representation $\rho_1 : \pi_1(X) \to \mathrm{PSL}_2(\mathbf{R})$. By Theorem 4.2.8 and Proposition 4.2.9, there are a representation $\rho_2 : \mathrm{PSL}_2(\mathbf{R}) \to \mathrm{Isom}(\mathbb{H}^\infty_Y)$ such that $\rho = \rho_2 \circ \rho_1$, a ρ_1-equivariant holomorphic map $g : X \to \mathbb{H}$ and a ρ_2-equivariant smooth map $u : \mathbb{H} \to \mathbb{H}^\infty_Y$. We claim first that there is an element γ of $\pi_1(X)$ such that $\rho_1(\gamma)$ is an infinite order elliptic element of $\mathrm{PSL}_2(\mathbf{R})$. This is because being elliptic in $\mathrm{PSL}_2(\mathbf{R})$ is characterized by the open condition trace$\in (-2, 2)$, and because the elements of finite order are all in the subgroup $\mathrm{SO}_2(\mathbf{R})$. Let q be the fixed point of $\rho_1(\gamma)$ in $\mathbb{H}$. Then $u(q)$ is a fixed point of $\rho(\gamma)$. In particular $\rho(\gamma)$ is an elliptic element. The map u is of rank two at some point because its image is not contained in a geodesic. Because u is ρ_2-equivariant, it is also of rank two at q. Hence the tangent space at q is sent by u to a plane in the tangent space at $q(u)$ which is invariant under $\rho(\gamma)$ and on which $\rho(\gamma)$ acts by irrational rotation. Lemma 4.3.1 in the next subsection says that this is impossible.

Therefore we are in the first case of Theorem 4.2.1, i.e. $\pi_1(X) \to \mathrm{Isom}(\mathbb{W})$ factorizes through a fibration $X \to \Sigma$ onto a hyperbolic orbicurve Σ. The kernel of $\pi_1(X) \to \pi_1^{orb}(\Sigma)$, denoted by H, fixes pointwise $\mathbb{W}$. The image $\rho(H) \subset \mathrm{Bir}(Y)$ is thus an elliptic subgroup and its normalizer, which contains Γ, is non-elementary.

If the normalizer of an infinite elliptic group in $\mathrm{Bir}(\mathbb{P}^2)$ is non-elementary, then the whole normalizer is conjugate to a subgroup of the toric group by a theorem of S. Cantat (see [DP12][Appendix]). Thus, if $\rho(H)$ is infinite there is a birational map $\varphi : Y \dashrightarrow \mathbf{C}^* \times \mathbf{C}^*$ such that $\varphi\rho\varphi^{-1}$ is a representation into the toric group $\mathbf{C}^* \times \mathbf{C}^* \rtimes \mathrm{GL}_2(\mathbf{Z})$, i.e. we are in the first case of Theorem 4.1.2.

Now consider the case where $\rho(H)$ is a finite group. We want to show that we are in the second case of Theorem 4.1.2, i.e. there exists a finite index subgroup of $\pi_1(X)$ corresponding to a finite unramified covering $X' \to X$ such that ρ restricted to $\pi_1(X')$ factorizes through $X' \to X \to \Sigma$. What may cause problem is the finite group $\rho(H)$ which acts trivially on $\mathbb{W}$ but not on the whole space $\mathbb{H}^\infty_Y$. If the representation ρ factorizes through the group $\rho(H) \times \pi_1^{orb}(\Sigma)$, then we can get rid of $\rho(H)$ by taking the preimage of $\{1\} \times \pi_1^{orb}(\Sigma)$ in $\pi_1(X)$. Thus the task is to show that such a factorization can be achieved after replacing $\pi_1(X)$ with a finite index subgroup. Let us look at the conjugation action of Γ on the normal subgroup $\rho(H)$. Since $\rho(H)$ is finite, the kernel of this action is a finite index subgroup Γ_1 of Γ; the elements of Γ_1 commute with $\rho(H)$. We take further a finite index subgroup Γ_2 of Γ_1 such that its image Λ_2 in $\pi_1^{orb}(\Sigma)$ is a surface group thus torsion free. By construction $\rho(H) \cap \Gamma_2$ is a finite abelian subgroup of Γ_2. Consider the following central extension:

$$1 \to \rho(H) \cap \Gamma_2 \to \Gamma_2 \to \Gamma_2/\left(\rho(H) \cap \Gamma_2\right).$$

The extension corresponds to an element of $\mathrm{H}^2(\Gamma_2/\left(\rho(H) \cap \Gamma_2\right), \rho(H) \cap \Gamma_2)$. The group $\Gamma_2/\left(\rho(H) \cap \Gamma_2\right)$ being a quotient of Λ_2, we can pull back this class to $\mathrm{H}^2(\Lambda_2, \rho(H) \cap \Gamma_2)$. As Λ_2 is a surface group, $\mathrm{H}^2(\Lambda_2, \rho(H) \cap \Gamma_2)$ is identified with the second singular cohomology group on some compact Riemann surface and a cohomology class becomes trivial after taking some covering. Therefore we can take a finite index subgroup $\Lambda_3 \subset \Lambda_2$ and its preimage $\Gamma_3 \subset \Gamma_2$ such that the corresponding element in $\mathrm{H}^2(\Lambda_3, \rho(H) \cap \Gamma_3)$ is trivial. Then ρ restricted to the preimage of Γ_3 factorizes through $\rho(H) \times \pi_1^{orb}(\Sigma)$. The proof of Theorem 4.1.2 is finished. □

4.3.2 Discreteness

Recall that the action of $\mathrm{Bir}(Y)$ on $\mathbb{H}_Y^\infty$ comes from the action of $\mathrm{Bir}(Y)$ on a Hilbert space $\mathscr{Z}_Y$ in which the hyperboloid model of $\mathbb{H}_Y^\infty$ is embedded. The Hilbert space $\mathscr{Z}_Y$ is the orthogonal direct sum of $\mathrm{H}^{1,1}(Y, \mathbf{R})$ and another Hilbert space $\mathscr{B}_Y$ with a Hilbert basis indexed by the points on Y and on all of its blow-ups, i.e. the points in the bubble space of Y.

Lemma 4.3.1 *Let $f \in \mathrm{Bir}(Y)$ be an elliptic element. If its action on $\mathscr{Z}_Y$ preserves a two-dimensional linear subspace, then it does not act by irrational rotation on this plane.*

Proof Suppose by contradiction that f acts as an irrational rotation on a plane P. As f is elliptic, a power of f is conjugate to an automorphism isotopic to identity on some surface birational to Y. Up to replacing f with some positive power of such an iterate, we may assume that f is an automorphism of Y isotopic to identity. Therefore the action of f on $\mathrm{H}^{1,1}(Y, \mathbf{R})$ is trivial and the plane P must be contained in $\mathscr{B}_Y$. Let (u, v) be a basis of P for which we have

$$f(u) = \cos(\theta)u + \sin(\theta)v, \quad f(v) = -\sin(\theta)u + \cos(\theta)v$$

where θ is the angle of the irrational rotation. Write u, v in the standard base of $\mathscr{B}_Y$:

$$u = \sum a_x e_x, \quad v = \sum b_x e_x$$

where e_x is the vector indexed by a point x on Y or some of its blow-ups. Since f is an automorphism on Y, $f(e_x)$ is just $e_{f(x)}$ for any point x on Y or its blow-ups. Take a x such that a_x or b_x is not zero. We have

$$a_{f^n(x)} = \cos(n\theta)a_x + \sin(n\theta)b_x, \quad b_{f^n(x)} = -\sin(n\theta)a_x + \cos(n\theta)b_x.$$

For any n the point $f^n(x)$ is different from x because otherwise (a_x, b_x) would be a vector in $\mathbf{R}^2$ fixed by the irrational rotation of angle $n\theta$. But then we have

$$\sum_n \left(a_{f^n(x)}^2 + b_{f^n(x)}^2\right) = \sum_n (a_x^2 + b_x^2) = +\infty.$$

This is a contradiction because the $a_{f^n(x)}, b_{f^n(x)}$ are coefficients of the two vectors u, v. □

4.4 Conjugation

The goal of this section is to prove a preparatory lemma which will be used in the proof of Theorem 4.1.1 and also in other chapters. Roughly speaking the lemma says that a conjugation of birational Kleinian groups is always a geometric conjugation. Throughout this section (Y,Γ,U,X) will be a birational kleinian group.

Remark 4.4.1 Since Γ acts regularly on U, the indeterminacy points and contracted curves of elements of Γ are disjoint from U.

Lemma 4.4.2 *Suppose that there exists a birational map $\phi : Y \dashrightarrow Y'$ to a second surface Y' and a non-empty Zariski open set $Z' \subset Y'$ such that $\Gamma' = \phi\circ\Gamma\circ\phi^{-1}$ is in* $\mathrm{Aut}(Z')$*. Then there exist a third surface Y'', a non-empty Zariski open set $Z'' \subset Y''$ and a birational kleinian group (Y'',Γ'',U'',X'') geometrically conjugate to (Y,Γ,U,X) such that $\Gamma'' \in$* $\mathrm{Aut}(Z'')$.

The Zariski open set Z'' can be obtained by blowing up Z'. If $Y' = Z'$, then $Y'' = Z''$; in this case if moreover $\Gamma' \subset$ $\mathrm{Aut}^0(Y')$*, then a finite index subgroup of Γ'' is in* $\mathrm{Aut}^0(Y'')$.

Proof We first claim that ϕ^{-1} does not contract any curve intersecting Z' onto a point in U. Suppose by contradiction that $u \in U$ is an indeterminacy point of ϕ and a curve C' intersecting Z' is contracted onto u by ϕ^{-1}. Pick a non trivial element $\gamma \in \Gamma$. Denote $\phi\circ\gamma\circ\phi^{-1}$ by γ'. As γ acts freely by diffeomorphism on U, $\gamma(u)$ is a point different from u. And as γ' is an automorphism of Z', the strict transform $\gamma'(C')$ is still a curve which intersects Z'. Thus $\gamma'(C')$ is contracted by ϕ^{-1} onto $\gamma(u)$. This implies that $\{\gamma(u)|\gamma\in\Gamma\}$ is an infinite set of indeterminacy points of ϕ, which is impossible.

Replacing Y with the blow up at the indeterminacy points of ϕ outside U, we can assume from the beginning that ϕ^{-1} does not contract any curve intersecting Z'. We can also assume that no (-1)-curves contracted by ϕ onto a point of Z' lies outside U. This means that the connected chains of (rational) curves contracted by ϕ onto a point of Z' intersect U in their components which are (-1)-curves. In particular if an indeterminacy point of ϕ^{-1} is in Z' then it is the image of a point in U at which ϕ is regular.

Pick $u' \in Z'$ an indeterminacy point of ϕ^{-1}. Let C be the connected chain of curves on Y contracted onto u'. Then $C\cap U \neq \emptyset$ and for $\gamma\in\Gamma$ the strict transform $\gamma(C) = \overline{\gamma(C\backslash\{\text{ind points of }\gamma\})}$ is still a chain of curves intersecting U. As γ' acts by automorphism on Z', $\gamma'(u')$ is still a point in Z' and $\gamma(C)$ is contracted onto it. Therefore Γ' permutes the indeterminacy points of ϕ^{-1} in Z'.

Let $Y_1 \xrightarrow{\phi_1} Y'$ be the blow-up at the indeterminacy points of ϕ^{-1} that are in Z'. Let Z_1 be the preimage $\phi_1^{-1}(Z') \subset Y_1$. Since Γ' permutes the indeterminacy points of ϕ^{-1} in Z', the group $\Gamma_1 = \phi_1^{-1} \circ \Gamma' \circ \phi_1$ acts by automorphisms on Z_1. Some finite index subgroup of Γ_1 is in $\mathrm{Aut}^0(Y_1)$ if we had $Y' = Z'$ and $\Gamma' \subset \mathrm{Aut}(Y')$. Then we continue the process for Y_1, Z_1 and Γ_1. After finitely many times, say m times, we get Y_m, Z_m, Γ_m such that the birational map $\varphi : Y_m \dashrightarrow Y$ induced by ϕ^{-1} has no indeterminacy points in Z_m. Then our hypothesis at the beginning of the second paragraph implies that φ restricted to Z_m is an isomorphism onto image. Hence Γ acts by automorphisms on the Zariski open set $\varphi(Z_m) \subset Y$. The proof is finished. □

4.5 From Delzant-Py's theorem to birational Kleinian groups

This section is a proof of Theorem 4.1.1. Let (Y, Γ, U, X) be a birational kleinian group in dimension two. We want to show that Γ is elementary. We have proved that the Kodaira dimension of Y is necessarily $-\infty$ (cf. 2.1.2). If Y is a non-rational ruled surface, $\mathrm{Bir}(Y)$ preserves the ruling thus has no non-elementary subgroups. Thus we can and will assume that Y is a rational surface.

The open set U is an infinite intermediate Galois covering of X and Γ, being the deck transformation group of the covering, is a quotient of $\pi_1(X)$. Thus we have a representation $\rho : \pi_1(X) \to \mathrm{Bir}(Y)$, composition of the quotient morphism $\pi_1(X) \twoheadrightarrow \Gamma$ and the inclusion $\Gamma \hookrightarrow \mathrm{Bir}(Y)$.

The proof is divided into three independent parts. In the first part we show that Γ is not conjugate to an elementary subgroup of the toric group. In the second part we show that the representation ρ does not factorize through a curve. In the third part we make the additional assumption that the quotient X is a non-Kähler surface which is not of class VII; in this case we essentially only need to treat elliptic surfaces.

If X is Kähler we can apply Theorem 4.1.2 to this representation. There are two cases if ρ is non-elementary: either it factorizes through an orbicurve or it is conjugate to a subgroup of the toric subgroup. Thus the first part and the second part of this section allow us to finish the proof of Theorem 4.1.1 for Kähler X.

4.5.1 Toric subgroup

A valuation on the function field $\mathbf{C}(Y) = \mathbf{C}(x, y)$ is a $\mathbf{Z}$-valued function v on $\mathbf{C}(Y)^*$ such that

1. $v(a) = 0$ for any $a \in \mathbf{C}^*$;
2. $v(PQ) = v(P) + v(Q)$ and $v(P+Q) \geq \min(v(P), v(Q))$ for any $P, Q \in \mathbf{C}(Y)^*$;
3. $v(\mathbf{C}(Y)^*) = \mathbf{Z}$.

The elements of $\mathbf{C}(Y)^*$ with valuations ≥ 0 together with 0 form the valuation ring A_v; those with valuations > 0 form a maximal ideal M_v. If the residue field A_v/M_v has transcendence degree 1 over $\mathbf{C}$, then v is called a *divisorial valuation*. The set of divisorial valuations is in bijection with the set of irreducible hypersurfaces in all birational models of Y (cf. [ZS75]). If v_E is the divisorial valuation associated with an irreducible hypersurface E in some birational model of Y, then for any $P \in \mathbf{C}(Y)^*$ the value $v_E(P)$ is the vanishing order of P along E. We can identify $\mathrm{Bir}(Y)$ with the group of automorphisms of $\mathbf{C}(Y)$. Thus $\mathrm{Bir}(Y)$ acts on the set of valuations by precomposition. It preserves the subset of divisorial valuations.

Lemma 4.5.1 *Suppose that Y is a smooth compactification of $\mathbf{C}^* \times \mathbf{C}^*$. Let $\gamma \in \mathrm{Bir}(Y)$ be a loxodromic map in the toric subgroup. Then any irreducible component of $Y \backslash (\mathbf{C}^* \times \mathbf{C}^*)$ is contracted by a power of γ.*

Proof The complement $Y \backslash (\mathbf{C}^* \times \mathbf{C}^*)$ has only finitely many irreducible components. Let v_E be a divisorial valuation with E outside of $\mathbf{C}^* \times \mathbf{C}^*$. It is sufficient to prove that v_E has an infinite orbit under γ. Let (x, y) be the stardard coordinates on $\mathbf{C}^* \times \mathbf{C}^*$. As E is outside $\mathbf{C}^* \times \mathbf{C}^*$, at least one of x, y has non-zero valuation.

The tranformation γ can be written as $(x, y) \mapsto (\alpha x^a y^b, \beta x^c y^d)$ where $\alpha, \beta \in \mathbf{C}^*$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z})$ is a hyperbolic matrix. We have for any $n \in \mathbf{Z}$

$$\begin{pmatrix} (\gamma^n \cdot v_E)(x) \\ (\gamma^n \cdot v_E)(y) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^n \begin{pmatrix} v_E(x) \\ v_E(y) \end{pmatrix}.$$

Since $\begin{pmatrix} v_E(x) \\ v_E(y) \end{pmatrix}$ is a non-zero integer vector and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a hyperbolic matrix, the orbit of $\begin{pmatrix} v_E(x) \\ v_E(y) \end{pmatrix}$ under $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is infinite. This implies that the orbit of v_E under γ is infinite. $\square$

Proposition 4.5.2 *Suppose that there is a birational map $\phi : Y \dashrightarrow \mathbf{C}^* \times \mathbf{C}^*$ such that $\phi \Gamma \phi^{-1}$ is contained in the toric subgroup. Then Γ contains no loxodromic element.*

Proof By Lemma 4.4.2 we can suppose that Y is a compactification of $\mathbf{C}^* \times \mathbf{C}^*$ and $\Gamma \subset \mathbf{C}^* \times \mathbf{C}^* \rtimes \mathrm{GL}_2(\mathbf{Z})$. Suppose by contradiction that Γ contains a loxodromic element η. Then by Lemma 4.5.1 any irreducible component of $Y \backslash (\mathbf{C}^* \times \mathbf{C}^*)$ is contracted by a power of η. This implies that the open set U on which η acts by biholomorphism is a subset of $\mathbf{C}^* \times \mathbf{C}^*$.

Consider the exponential map π from $\mathbf{C}^2$ to $\mathbf{C}^* \times \mathbf{C}^*$; it is a covering map. The exponential also gives rise to a homomorphism $\rho : \mathrm{Aff}_2(\mathbf{C}) = \mathbf{C}^2 \rtimes \mathrm{GL}_2(\mathbf{Z}) \to \mathbf{C}^* \times \mathbf{C}^* \rtimes \mathrm{GL}_2(\mathbf{Z})$. Then π is ρ-equivariant. Consider a connected component $\bar{U}$ of $\pi^{-1}(U)$ and the subgroup $\bar{\Gamma}$ of $\rho^{-1}(\Gamma)$ that preserves $\bar{U}$. Then $\bar{\Gamma}$ is a complex affine Kleinian group. We can apply Theorem 3.1.1 to $\bar{\Gamma}$. There are five possibilities for X: it is a torus, a Hopf surface, an Inoue surface, a primary Kodaira surface or an affine-elliptic bundle. From the descriptions that we give in Section 3.3, we see that, for the linear part of $\bar{\Gamma}$ to have a hyperbolic matrix, the only possibilities are Inoue surfaces and affine-elliptic bundles. But in these two cases the group cannot be included in $\mathrm{GL}_2(\mathbf{Z})$. For Inoue surfaces we can see this directly from the formulas (see also [Zhab]) and for affine-elliptic bundles it is because the group comes from a cocompact lattice in $\mathrm{PGL}_2(\mathbf{R})$. Hence the hypothesis that Γ contains a loxodromic element is absurd. □

4.5.2 Factorization through curves

Proposition 4.5.3 *Suppose that Γ is a non-elementary subgroup. Then the representation ρ : $\pi_1(X) \twoheadrightarrow \Gamma \hookrightarrow \mathrm{Bir}(Y)$ does not factorize through a hyperbolic orbicurve.*

Proof Suppose by contradiction that ρ is non-elementary and factorizes through a hyperbolic orbicurve Σ. Let F be a general fibre of the fibration $X \to \Sigma$. The image of $\phi : \pi_1(F) \to \pi_1(X)$ is in the kernel of $\pi_1(X) \to \pi_1^{orb}(\Sigma)$, thus in the kernel of ρ. This contradicts Lemma 4.5.4 below. □

Lemma 4.5.4 *If Γ contains a loxodromic element of* $\mathrm{Bir}(Y)$*, then for any compact curve C on X the image of the composition $\pi_1(C) \to \pi_1(X) \twoheadrightarrow \Gamma$ is infinite.*

Proof By Theorem 2.1.2 we can assume that Y is a birationally ruled surface. As all birational transformations of a non rational ruled surface preserve the ruling thus cannot be loxodromic, we can and will further assume that Y is rational.

Suppose by contradiction that C is a compact curve on X such that $\pi_1(C) \to \Gamma$ has finite image. Denote by D the normalization of C. Then there is a finite unramified cover $\bar{D} \to D$ such that the composition $\pi_1(\bar{D}) \to \pi_1(D) \to \pi_1(X) \twoheadrightarrow \Gamma$ is trivial. This implies the existence of a map $\iota : \bar{D} \to U$ lifting $\bar{D} \to D \to C \to X$. Only a finite subgroup of Γ preserves $\iota(\bar{D})$ by Lemma

4.5.5 below. Therefore $\{\gamma(\imath(\bar{D}))\}_{\gamma\in\Gamma}$ form an infinite family of disjoint smooth compact curves in $U \subset Y$. Every element of Γ permutes these curves. Denote by $\alpha_j, j \in \mathbf{N}^*$ the classes of these curves in the Picard group which is isomorphic to the Néron-Severi group because Y is rational. The intersection number $\alpha_i \cdot \alpha_j$ is zero for $i \neq j$ since the corresponding curves are disjoint. As the Néron-Severi group has finite rank, we can suppose that for some $r \in \mathbf{N}^*$, the classes $\alpha_1, \cdots, \alpha_r$ are linearly independent and for any $n > r$ the class α_n is equal to a linear combination of the $\alpha_j, j \leq r$. Among the $\alpha_j, j \leq r$, at most one has zero self-intersection because otherwise there would exist a two-dimensional totally isotropic space which contradicts the Hodge index theorem. When we write α_n as a linear combination of the the $\alpha_j, j \leq r$, if the coefficient before α_j is non-zero then α_j is of zero self-intersection because $\alpha_n \cdot \alpha_j = 0$. This implies that all but finitely many of the α_j are equal to a class β of zero self-intersection. Thus the linear system associated with β has dimension ≥ 1. Since the group Γ permutes the curves $\gamma(\imath(\bar{D}))$, the class β is Γ-invariant. Hence we obtain a pencil of curves invariant under Γ. However a loxodromic birational transformation preserves no pencils of curves. □

Lemma 4.5.5 *An infinite order automorphism of an irreducible compact curve cannot be free and properly discontinuous.*

Proof An automorphism of a singular irreducible curve permutes the singular points and is not free. An automorphism of $\mathbb{P}^1$ has a fixed point. An automorphism of infinite order of a genus one curve without fixed points is an irrational translation; its action is not properly discontinous by Lemma 2.3.1. An automorphism of a general type curve is of finite order. □

4.5.3 Elliptic surfaces

According to Kodaira's classification of surfaces (cf. [Bar+04]), a non-Kähler surface which is not of class VII fits in one of the two following possibilities: either X is a primary or secondary Kodaira surface or X is an elliptic surface with Kodaira dimension 1. The fundamental group of a Kodaira surface is solvable and has no non-elementary representations into $\mathrm{Bir}(Y)$. It remains to consider the elliptic surfaces of Kodaira dimension 1.

Assume that $X \to \Sigma$ is a genus one fibration; we consider the base curve Σ as an orbicurve whose multiple points correspond to multiple fibers. Assume that X is of Kodaira dimension 1. This is equivalent to say that Σ is a hyperbolic orbicurve. We have an exact sequence

$$1 \to H \to \pi_1(X) \xrightarrow{\varphi} \pi_1^{orb}(\Sigma) \to 1$$

where H is the image of the fundamental group of a regular fiber. Suppose by contradiction that the representation $\rho : \pi_1(X) \to \Gamma \hookrightarrow \mathrm{Bir}(Y)$ is non-elementary. By Proposition 4.5.3 $\rho(H)$ is not finite. Hence $\rho(\pi_1(F))$ must be an infinite abelian group. Since Γ is non-elementary, there exists at least one element $a \in \pi_1(X)$ such that $\varphi(a)$ is of infinite order and $\rho(a)$ is a loxodromic birational transformation. Consider $< \varphi(a) >$ the infinite cyclic subgroup of $\pi_1(\Sigma)$ generated by $\varphi(a)$ and denote by G the subgroup $\varphi^{-1}(< \varphi(a) >)$ of $\pi_1(X)$. Then G is an extension of $< \varphi(a) >$ by $\pi_1(F) = \mathbf{Z}^2$; in particular it is solvable. The group $\rho(G)$ is solvable but not virtually cyclic; it contains a loxodromic element. By the strong Tits alternative we infer that $\rho(G)$ is up to conjugation contained in the toric subgroup and that $\rho(\pi_1(F))$ is an infinite elliptic subgroup. The whole group Γ normalizes $\rho(\pi_1(F))$. Thus again by Cantat's theorem (cf. Appendix [DP12]) we infer that Γ is up to conjugation contained in the toric subgroup. This contradicts Proposition 4.5.2.

4.6 Non-Kähler surfaces

One may need to go through Section 4.2 before reading this section.

All known surfaces of class VII have solvable fundamental groups. Non-elementary subgroups of $\mathrm{Bir}(Y)$ are not solvable. Thus an analogue of Theorem 4.1.1 for non-Kähler X does not seem to be very interesting. But this does not mean that such a result is easy to prove. We try to see to which extent the methods used for Theorem 4.1.1 can be carried out for X of class VII.

Let X be an n-dimensional compact hermitian manifold with fundamental form κ. Let N be a complete Riemannian manifold and let $f : X \to N$ be a smooth map. Recall that the map f is harmonic if $d_\nabla(\kappa^{n-1} \wedge d^c f) = 0$ (cf. Proposition 4.2.3) and that the map f is pluriharmonic if $d_\nabla d^c f = 0$. In [JY93] Jost-Yau introduced a different notion of harmonicity: the map f is called *hermitian harmonic* if $\kappa^{n-1} \wedge d_\nabla d^c f = 0$. If X is Kähler, then harmonicity and hermitian harmonicity coincide. The following is an analogue of Eells-Sampson's theorem:

Theorem 4.6.1 (Jost-Yau [JY93]) *Let X be a compact hermtian manifold. Let N be a compact Riemannian manifold with negative sectional curvature. Let $g : X \to N$ be a continuous map not homotopic to a map onto a geodesic. There exists a hermitian harmonic map homotopic to g.*

Though it is not clear how to generalize the notion of hermitian harmonic maps for general metric space targets, we can still use the definition $\kappa^{n-1} \wedge d_\nabla d^c f = 0$ for targets that are infinite dimensional Riemannian manifolds. We would like to have the following:

Hope 4.6.2 *Let X be a compact hermtian surface. Let $\mathbb{H}^m$ be a hyperbolic space of possibly infinite dimension. Let $\rho : \pi_1(X) \to \mathrm{Isom}(\mathbb{H}^m)$ be a non-elementary discrete representation of $\mathrm{Isom}(\mathbb{H}^m)$. There is a ρ-equivariant hermitian harmonic map $\tilde{X} \to N$.*

Theorem 4.6.1 asserts that the hope is true when m is finite and the representation is into a cocompact lattice of $\mathrm{Isom}(\mathbb{H}^m)$.

We have also an analogue of Siu-Sampson's theorem:

Theorem 4.6.3 (Jost-Yau [JY93]) *Let X be a hermtian manifold of dimension n whose fundamental form κ satisfies $dd^c\kappa^{n-2} = 0$. Let N be a Riemannian manifold with non-positive hermtian sectional curvature. Then any hermtian harmonic map from X to N is pluriharmonic.*

Remark that if X is a surface, then the condition on X is vacuous. In [CT97] Carlson-Toledo already used Jost-Yau's work to generalize their theorem (cf. Theorem 4.2.1) to fundamental groups of surfaces of class VII. Once the existence of the desired hermtian harmonic maps is guaranteed, all the other discussions in Section 4.2 work in the same way. In the proof of Theorem 4.1.1, we only use the Kähler hypothesis on X in Theorem 4.2.1. Hence Hope 4.6.2 would imply

Hope 4.6.4 *There is no non-elementary birational Kleinian groups in dimension two.*

Remark 4.6.5 The finite dimensional case of Hope 4.6.2, which is more likely to be true, would imply that there is no non-elementary groups of automorphisms with Kleinian property in dimension two.

As we said, there is no known compact non-Kähler surface of class VII with non-solvable fundamental group. So the discussion in this section could be a too difficult approach towards a vacuous statement.

CHAPTER 5

RULED SURFACES

In this chapter we study birational Kleinian groups preserving a rational fibration. The first section is a glossary of regular holomorphic foliations on compact complex surfaces. Proofs begin from the second section. One may start from the second section and goes back to the first section whenever needed. Since the study in this chapter is divided into many subcases, we recommend the reader to read this chapter in parallel with Chapter 6 which outlines what we do in this chapter.

5.1 Foliated surfaces

Let $\{U_j\}_{j\in J}$ be an open covering of a complex surface X. For each $j \in J$, let v_j be a nowhere vanishing holomorphic vector field defined on U_j. We require that on each $U_i \cap U_j$, there is a nowhere vanishing holomorphic function g_{ij} such that $v_i = g_{ij} v_j$. Two such collections $\{U_j, v_j\}$, $\{U'_k, v'_k\}$ are equivalent if on each $U_j \cap U'_k$, there is a nowhere vanishing holomorphic function h_{jk} such that $v_j = h_{jk} v'_k$. A *regular holomorphic foliation* $\mathscr{F}$ on a complex surface X is an equivalence class of such a collection $\{U_j, v_j\}$. The condition on the local vector fields v_j means their local integral curves can be glued together. A maximal glued integral curve is called a *leaf*. Equivalently a regular holomorphic foliation can also be defined as (the equivalence class of) a collection of nowhere vanishing holomorphic 1-forms ω_j on U_j such that on $U_j \cap U_i$ the two forms ω_j, ω_i differ multiplicatively by a nowhere vanishing holomorphic function. See [Bru15] for background on holomorphic foliations.

Let us describes some examples of regular holomorphic foliations.

Fibrations. Let X be compact complex surface and let $f : X \to B$ be a fibration whose singular fibers are all multiples of smooth curves. The fibration equips X with a regular foliation whose leaves are the underlying manifolds of the fibers. Let mF_b be a multiple fiber lying over a point $b \in B$. Let w be a local coordinate on B which vanishes at b, let h be a local equation of F_b in X, then $f^*(dw)/h^{m-1}$ is a local differential form defining the foliation.

Linear foliations on tori. Let Λ be a lattice in $\mathbf{C}^2$ and $X = \mathbf{C}^2/\Lambda$ a two dimensional complex torus. Let (z,w) be the natural coordinates on $\mathbf{C}^2$. A constant holomorphic differential $adz+bdw$ on $\mathbf{C}^2$ with $a,b \in \mathbf{C}$ and $ab \neq 0$ descends onto X and defines a regular foliation $\mathscr{F}$ on X; this is called a *linear foliation* on the torus X. Choosing such a linear foliation amounts to choose a one dimensional $\mathbf{C}$-linear subspace $W = \ker(adz+bdw)$ of $\mathbf{C}^2$. If $W \cap \Lambda$ is a lattice in W, then the leaves of $\mathscr{F}$ are elliptic curves and the foliation is a fibration. If $W \cap \Lambda$ is non empty but not a lattice, then the leaves of $\mathscr{F}$ are biholomorphic to $\mathbf{C}^*$. If $W \cap \Lambda$ is empty, then the leaves of $\mathscr{F}$ are biholomorphic to $\mathbf{C}$ and are dense in X. When the leaves of $\mathscr{F}$ are not compact, we say the linear foliation $\mathscr{F}$ is *irrational*.

Hopf surfaces A Hopf surface has a finite unramified cover which is a primary Hopf surface and a primary Hopf surface is the quotient of $\mathbf{C}^2 \backslash \{0\}$ by a transformation of the form

$$H_{(\alpha,\beta,\gamma,m)} : (z,w) \mapsto (\alpha z + \gamma w^m, \beta w), \quad \alpha,\beta,\gamma \in \mathbf{C}, m \in \mathbf{N}^*, 0 < |\alpha| \leq |\beta| < 1$$

with $\alpha = \beta^m$ if $\gamma \neq 0$. We call $\mathbf{C}^2/ < H_{(\alpha,\beta,\gamma,m)} >$ a standard primary Hopf surface. See [Kod66] for the above assertion and the following:

1. The standard primary Hopf surface has an elliptic fibration if and only if $\gamma = 0$ and $\alpha^k = \beta^l$ for some $k,l \in \mathbf{N}^*$.
2. If $\gamma = 0$ but $\alpha^k \neq \beta^l$ for any $k,l \in \mathbf{N}^*$, then the two smooth elliptic curves given by $z = 0$ and $w = 0$ are the only curves on the primary Hopf surface.
3. If $\gamma \neq 0$ then the smooth elliptic curve given by $w = 0$ is the only curve on the primary Hopf surface.

Suppose that $\gamma = 0$. The complex lines in $\mathbf{C}^2$ parallel to $z = 0$ and to $w = 0$ form two regular foliations on $\mathbf{C}^2 \backslash \{0\}$ that descend to regular foliations on the Hopf surface. A vector field of the form $az\frac{\partial}{\partial z} + bw\frac{\partial}{\partial w}$ with $a,b \in \mathbf{C}$ and $ab \neq 0$ is invariant under $(z,w) \mapsto (\alpha z + \gamma w^m, \beta w)$, thus descends to the Hopf surface. It gives rise to a regular foliation of which the leaves in $\mathbf{C}^2 \backslash \{0\}$ different from the two axes are given by (e^{az}, ce^{bz}). When $a = b$ this is just the foliation by complex vector lines.

Suppose that $\gamma \neq 0$. The complex lines parallel to $w = 0$ give a regular foliation on the Hopf surface. There exist also regular foliations associated with certain vectors fields on $\mathbf{C}^2 \backslash \{0\}$ of Poincaré-Dulac's form

$$[z(1+aw) + wF(z,w)]\frac{\partial}{\partial z} + bw\frac{\partial}{\partial w}$$

where $a \in \mathbf{C}, b \in \mathbf{C}^*$ and F vanishes at 0 of order 1. When $m = 1$ the normal form $(z,w) \mapsto (\alpha z + \gamma w, \beta w)$ is a linear transformation and the foliation by complex vector lines in $\mathbf{C}^2 \backslash \{0\}$ is one of these foliations.

The foliations described above will be called *obvious foliations on Hopf surfaces*.

Inoue surfaces. See [Ino74] for the precise formulas defining Inoue surfaces. An Inoue surface of type S^0 is the quotient of $\mathbb{H} \times \mathbf{C}$ by a group generated by affine transformations g_0, g_1, g_2, g_3 of the form

$$\begin{aligned} g_0 :& (x,y) \mapsto (\alpha x, \beta y) \\ g_i :& (x,y) \mapsto (x + a_i, y + b_i) \quad \text{for } i = 1,2,3. \end{aligned}$$

The vertical and horizontal foliations on $\mathbb{H} \times \mathbf{C}$ descend to two regular foliations on the Inoue surface of type S^0. An Inoue surface of type S^+ is the quotient of $\mathbb{H} \times \mathbf{C}$ by a group generated by affine transformations g_0, g_1, g_2, g_3 of the form

$$\begin{aligned} g_0 &: (x,y) \mapsto (\alpha x, y + t) \\ g_i &: (x,y) \mapsto (x + a_i, y + b_i x + c_i) \quad i = 1,2 \\ g_3 &: (x,y) \mapsto (x, y + \frac{b_1 a_2 - b_2 a_1}{n}) \end{aligned}$$

An Inoue surface of type S^- is the quotient of $\mathbb{H} \times \mathbf{C}$ by a group generated by affine transformations g_0, g_1, g_2, g_3 of the form

$$\begin{aligned} g_0 &: (x,y) \mapsto (\alpha x, -y) \\ g_i &: (x,y) \mapsto (x + a_i, y + b_i x + c_i) \quad i = 1,2 \\ g_3 &: (x,y) \mapsto (x, y + \frac{b_1 a_2 - b_2 a_1}{n}) \end{aligned}$$

An Inoue surface of type S^- has a double cover which is an Inoue surface of type S^+. The vertical foliation on $\mathbb{H} \times \mathbf{C}$ descends to a regular foliation on an Inoue surface of type S^+ or S^-. The foliations described above will be called *obvious foliations on Inoue surfaces*.

Suspensions. Let M be a Riemann surface. We denote by $\hat{M}$ a Galois covering of M with deck transformation group G. Let N be another Riemann surface. Let $\alpha : G \to \mathrm{Aut}(N)$ be a homomorphism of groups. The group G acts on $\hat{M} \times N$ in the following way: $g \cdot (m,n) = (g \cdot$

$m, \alpha(g) \cdot n)$. Then the quotient $X = (\hat{M} \times N)/G$ fibers onto $M = \hat{M}/G$ via the first projection; the fibers are all isomorphic to N. The foliation by $\{\hat{M} \times \{n\}\}_{n \in N}$ descends to a foliation on X which is everywhere transverse to the fibration. We call such a foliation a *suspension of N over M* with monodromy $\alpha : G \to \mathrm{Aut}(N)$. If M, N are compact, then X is compact; this is the only case we will consider. If moreover $\alpha(G)$ is a finite subgroup of $\mathrm{Aut}(N)$, then the leaves of the foliation are compact and the foliation is in fact a fibration. When $\alpha(G)$ is infinite (since N is compact this is only possible if N is $\mathbb{P}^1$ or an elliptic curve) the leaves are non-compact and we call such a suspension *infinite*.

Remark that if X is a compact complex surface with an infinite suspension foliation, then X is Kähler. This is clear if it is a suspension of $\mathbb{P}^1$ in which case X is ruled. Assume that it is a suspension of an elliptic curve. We use the notations in the previous paragraph. Up to replacing G by a subgroup of finite index we can assume that $\alpha(G) \subset \mathrm{Aut}(N)$ is an abelian group of translations on the elliptic curve N. Thus there is an action of N on X by translations in the fibers of the elliptic bundle $X \to M$. In other words $X \to M$ is a principal elliptic bundle (cf. [Bar+04] V.5.1). Furthermore the suspension process says exactly that the bundle $X \to M$ is defined by a locally constant cocycle. By [Bar+04] V.5.1 and V.5.3 we infer that the second Betti number of X is even. This implies that X is Kähler by [Bar+04] IV.3.1.

Turbulent foliations. Let X be a compact complex surface and $X \to B$ an elliptic fibration with constant functional invariant (i.e. all regular fibers are isomorphic) whose singular fibers are all multiples of smooth elliptic curves. Let $\mathscr{F}$ be a regular foliation on X. If a finite number of fibers of the elliptic fibration are $\mathscr{F}$-invariant and all other fibers are transverse to $\mathscr{F}$, then $\mathscr{F}$ is called a *turbulent foliation*. The underlying elliptic fibration is locally trivial outside the invariant fibers of $\mathscr{F}$; the trivialization is given by the foliation. The invariant fibers are regular or multiples of elliptic curves. Locally around an invariant fiber, let (z, w) be a system of local coordinates such that the fibration is given by $(z, w) \mapsto z^m$ where m is the multiplicity of the fiber. Then the foliation $\mathscr{F}$ is locally defined by a local differential form $dz - A(z)dw$ where A is a holomorphic function which vanishes at 0.

Transversely hyperbolic foliations. Let X be a compact complex surface with a regular holomorphic foliation $\mathscr{F}$. A closed positive current T of bidegree $(1,1)$ is called *$\mathscr{F}$-invariant* if in a neighbourhood of any point we have $T \wedge \omega = 0$ where ω is some holomorphic one-form which defines locally the foliation. The foliation $\mathscr{F}$ is called *transversally hyperbolic* if there exists an $\mathscr{F}$-invariant closed positive current T such that, in any open set of X where the leaves of $\mathscr{F}$

are given by the level sets of a submersion z, there exists a subharmonic function φ which only depends on z and satifies

$$T = \frac{i}{\pi}\partial\bar{\partial}\varphi \quad \text{and} \quad \frac{\partial^2\varphi}{\partial z\partial\bar{z}} = -e^{2\varphi}.$$

Roughly speaking, the current T equips the space of leaves with a hyperbolic metric. For example, if $X \to B$ is a smooth fibration onto a hyperbolic Riemann surface, then the corresponding foliation is transversally hyperbolic.

Let Γ be a torsion free cocompact irreducible lattice in $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$, then the bidisk quotient $\mathbb{D} \times \mathbb{D}/\Gamma$ is a general type surface with two regular transversally hyperbolic holomorphic foliations induced by the product structure of $\mathbb{D} \times \mathbb{D}$ which are called *tautological foliations*. Since Γ is irreducible, its projections in the two factors $\mathrm{PSL}_2(\mathbf{R})$ are dense and the leaves of the tautogical foliations are dense in $\mathbb{D} \times \mathbb{D}/\Gamma$. This is the only known example of transversally hyperbolic regular foliations on general type surfaces which is not a fibration. See Section 7.1 for more details.

An almost complete classification. Brunella classified all regular holomorphic foliations on compact complex surfaces.

Theorem 5.1.1 (M.Brunella [Bru97]) *Let $(X, \mathscr{F})$ be a regularly foliated compact complex surface. Then one of the following situations holds:*

1. *$\mathscr{F}$ comes from a fibration of X onto a curve;*
2. *$\mathscr{F}$ is an irrational linear foliation on a complex torus;*
3. *$\mathscr{F}$ is an obvious foliation on a non-elliptic Hopf surface;*
4. *$\mathscr{F}$ is an obvious foliation on an Inoue surface;*
5. *$\mathscr{F}$ is an infinite suspension of $\mathbb{P}^1$ or an elliptic curve over a compact Riemann surface;*
6. *$\mathscr{F}$ is a turbulent foliation with at least one invariant fiber;*
7. *$\mathscr{F}$ is a transversely hyperbolic foliation with dense leaves whose universal cover is a fibration of disks over a disk.*

Remark 5.1.2 The overlaps in the above list are case 2 with case 5, case 3 with case 6, case 5 with case 6. When a complex torus admits an elliptic fibration, it is straightforward to see that an irrational linear foliation is also a suspension of an elliptic curve. For overlaps between case 3 and case 6, or case 5 and case 6, see Section 5.5.1.

5.2 Invariant rational fibration

In this chapter we suppose that (Y,Γ,U,X) is a birational kleinian group and there is a Γ-invariant rational fibration $r : Y \to B$ over a smooth projective curve B. Note that when B is not a rational curve, any birational transformation group of Y preserves automatically r. We have a group homomorphism $\Gamma \to \mathrm{Aut}(B)$ whose image will be denoted by Γ_B. The ruling r is equivariant with respect to the action of Γ on Y and that of Γ_B on B. By contracting (-1)-curves which are contained in the fibers of r but which are disjoint from U, we can assume, without loss of generality, that if a fiber of r intersects U, then its irreducible components of self-intersection -1 all intersect U.

We recall a basic fact about non relatively minimal ruled surface:

Proposition 5.2.1 *Let F be a singular fiber of a ratinoal fibration. Then F is a tree of rational curves whose components are of self-intersection* ≤ -1.

Proof The singular fiber F is obtained by successive blow-ups from a regular fiber of a relatively minimal rational fibration, thus a tree of rational curves. The second assertion is because original regular fiber is of self-intersection 0 and that a blow-up decreases the self-intersection of the components. □

Lemma 5.2.2 *Let F be a fiber which intersects U. Then no element of Γ has an indeterminacy point on F and no component of F is contracted by an element of Γ.*

Proof Let $\gamma \in \Gamma$ be a non trivial birational transformation. Let us first show that γ does not contract anything on F. If F is a regular fiber, then it has only one irreducible component and this component intersects U by hypothesis. Since the action of Γ on U is regular, F can not be contracted. Now assume that F is a singular fiber. Let $Y \xleftarrow{\varepsilon} Z \xrightarrow{\delta} Y$ be the minimal resolution of indeterminacy of γ, i.e. Z is a smooth projective surface, ε, δ are birational morphisms and $\gamma = \delta \circ \varepsilon^{-1}$. The irreducible components of a singular fiber are of self-intersection ≤ -1 and the self-intersection numbers decrease after blowing up. Thus, by Lemma 5.2.1, the strict transforms of the irreducible components of F in Z are of self-intersection ≤ -2, unless those of some (-1)-components of F. But by the hypothesis of our initial setting the (-1)-components of F all intersect U and can not be contracted by γ. This means δ can not contract any (-1)-curve on the total transform of F in Z, i.e. γ does not contract any component of F.

We remark that the total transform of F by γ is also a fiber of r which intersects U. The above reasoning, applied to γ^{-1}, says that γ^{-1} does not contract anything onto F, i.e. γ does not have any indeterminacy point on F. □

Corollary 5.2.3 *The group Γ acts by holomorphic diffeomorphisms on $r^{-1}(r(U))$. In particular if $r(U) = B$ then $\Gamma \subset \mathrm{Aut}(Y)$.*

The following proposition shows how holomorphic foliations come in our study.

Proposition 5.2.4 *Let (Y,Γ,U,X) be a birational kleinian group. Suppose $\Gamma \subset \mathrm{Bir}(Y)$ preserves a fibration $r : Y \to B$. If a fiber F of the invariant fibration intersects U, then $F \cap U$ contains no singular point of F. Therefore the fibration descends to a regular holomorphic foliation on X.*

Proof Suppose by contradiction that F is a singular fiber and $p \in F \cap U$ is a singular point of F. If $\gamma \in \Gamma$, then $\gamma(p)$ is a singular point of the singular fiber $\gamma(F)$ since Γ preserve the fibration and acts by holomorphic diffeomorphisms on U. But a fibration has only finitely many singular fibers and each singular fiber has only finitely many singular points, so the infinite group Γ could not act freely on U. □

5.3 Finite action on the base

In this section we will consider the case where Γ_B is a finite group. We will prove:

Theorem 5.3.1 *Let (Y,Γ,U,X) be a birational kleinian group on a surface Y which is ruled over a curve B. Assume that Γ preserves the ruling, and Γ induces a finite action on the base curve B. Then up to geometric conjugation and up to taking finite index subgroup, we are in one of the following situations:*

1. *$Y = B \times \mathbb{P}^1$, $\Gamma \subset \{\mathrm{Id}\} \times \mathrm{PGL}_2(\mathbf{C})$ and $U = B \times D$ where $D \subset \mathbb{P}^1$ is an invariant component of the domain of discontinuity of Γ viewed as a classical Kleinian group.*
2. *Y is $\mathbb{P}(\mathscr{E})$ where $\mathscr{E}$ is an extension of $\mathscr{O}_B$ by $\mathscr{O}_B$. The extension determines a section s of the ruling. We have $U = Y - s$. The subgroup of $\mathrm{Aut}_B(Y)$ fixing s is isomorphic to $\mathbf{C}$ in which Γ is a lattice. The group Γ is isomorphic to $\mathbf{Z}^2$ and the surface X is a principal elliptic fiber bundle.*
3. *Y is obtained by blowing up a decomposable ruled surface, U is a Zariski open set of Y whose intersection with each fiber is biholomorphic to $\mathbf{C}^*$. The group Γ is isomorphic to $\mathbf{Z}$ and is generated by an automorphism which acts by multiplication in each fiber. The quotient surface X is an elliptic fibration over B with isomorphic regular fibers and whose only singular fibers are multiples of smooth elliptic curves.*

There are examples for each of the three cases.

In the first subsection we will study how Γ acts on each fiber of the ruling. In the second subsection we present some results on automorphisms of ruled surfaces due to Maruyama that we will need. In the third subsection we finish the proof of Theorem 5.3.1.

We start the proof of Theorem 5.3.1 by making the following reduction: *replacing* Γ *with a finite index subgroup, we can and will assume that* $\Gamma_B = \{\mathrm{Id}\}$ *and that every component of singular fibers of r is invariant by* Γ.

5.3.1 Fiberwise study

Lemma 5.3.2 *We have $r(U) = B$, $\Gamma \subset \mathrm{Aut}(Y)$ and the foliation $\mathscr{F}$ on X is a fibration.*

Proof The set $V = r(U)$ is open because a holomorphic map is an open map. Since $\Gamma_B = \{\mathrm{Id}\}$ the fibration r induces a commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\pi} & X \\ \big\downarrow{\scriptstyle r} & & \big\downarrow{\scriptstyle f} \\ V & \longrightarrow & V \end{array}$$

As X is compact, V should be compact too. This implies that V is the whole curve B. By Corollary 5.2.3, we have $\Gamma \in \mathrm{Aut}(Y)$. The leaves of the foliation on X are the fibers of the map $X \to V$. They are compact, so the foliation is a fibration. □

Proposition 5.3.3 *We have:*

- *If $F \cong \mathbb{P}^1$ is a general fiber of $r : Y \to B$, then $F \cap U$, as a subset of $\mathbb{P}^1$, is a union of connected components of the discontinuity of a classical Kleinian group.*
- *If moreover $r : Y \to B$ has at least one singular fiber, then for any singular or non-singular fiber F, the intersection $F \cap U$ is biholomorphic to $\mathbf{C}^*$ and Γ is isomorphic to $\mathbf{Z}$.*

Proof For any fiber F_b (possibly singular) over a point $b \in B$, there is a homomorphism $\Gamma \to \mathrm{Aut}(F_b)$ with image Γ_b and the action of Γ_b on F_b preserves $U \cap F_b$. The group Γ_b is isomorphic to Γ because the action of Γ on U is free. We have a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\pi} & X \\ \big\downarrow{\scriptstyle r} & & \big\downarrow{\scriptstyle f} \\ B & \longrightarrow & B \end{array} \tag{5.1}$$

□

where f is a surjective proper holomorphic map. By [Bar+04] III.11 the sheaf $f_*\mathscr{O}_X$ is locally free and the non-singular fibers of f have the same number of connected components. For $b \in B$ a general point, F_b is isomorphic to the projective line $\mathbb{P}^1$. The quotient of $U \cap F_b$ by the action of Γ_b is a general fiber of f, that is, a disjoint union of smooth compact curves. This means that Γ_b is a classical Kleinian group and $U \cap F_b$ is a union of connected components of the discontinuity set of Γ_b.

Suppose that Y has a singular fiber F_b. Every irreducible component of F_b contains at least one singular point of F_b; by our hypothesis the singular points of F_b are fixed by Γ_b. If an irreducible component of F_b intersects U then it contains one or two singular points of F_b because otherwise it would be pointwise fixed by Γ_b. In particular Γ_b restricted to each component of F_b is isomorphic to a solvable subgroup of $\mathrm{PGL}_2(\mathbf{C})$. So if there is at least one singular fiber, then the group Γ is solvable. A solvable torsion free classical Kleinian group is isomorphic to $\mathbf{Z}$ or $\mathbf{Z}^2$ and its set of discontinuity is biholomorphic to $\mathbf{C}$ or $\mathbf{C}^*$ (cf. [Har78]). This implies that the intersection of U with any fiber is biholomorphic to $\mathbf{C}$ or $\mathbf{C}^*$. In particular $f: X \to B$ has connected fibers. Thus for the singular fiber F_b the intersection $F_b \cap U$ is connected and is contained in one component C_b of F_b. The component C_b is necessarily a (-1)-curve because by our hypothesis every (-1)-curve in the fiber F_b intersects U.

Suppose by contradiction that Γ is isomorphic to $\mathbf{Z}^2$. Then the fibration $f: X \to B$ is an elliptic fibration of which general fibers are all isomorphic, i.e. it is isotrivial. Let Δ be a small disk centred at $b \in B$ and let (x,y) be the local coordinates around the singular fiber F_b where r is given by x and F_b is defined by $x = 0$. The isotriviality of f implies that, locally over $\Delta \backslash \{0\}$, up to holomorphic (not necessarily algebraic) conjugation the action of Γ is generated by two transformations $\gamma_j : (x,y) \mapsto (x, y + a_j), j = 1,2$ where $\mathbf{Z}a_1 + \mathbf{Z}a_2$ is a lattice in $\mathbf{C}$. In other words there is a sequence of blow-downs $r^{-1}(\Delta) \to \Delta \times \mathbb{P}^1$ which sends F_b to $\mathbb{P}^1$ such that the action of Γ on $r^{-1}(\Delta)$ is the lift of the action of $\mathbf{Z}^2$ on $\Delta \times \mathbb{P}^1$ generated by $\gamma_j : (x,y) \mapsto (x, y + a_j), j = 1,2$. To obtain $r^{-1}(\Delta)$ from $\Delta \times \mathbb{P}^1$ we blow up the common fixed (infinitely near) points of $\gamma_j, j = 1,2$. However in this case the action of $\gamma_j, j = 1,2$ on the exceptional divisors are always trivial. Thus we have $r^{-1}(\Delta) = \Delta \times \mathbb{P}^1$ which contradicts that F_b is a singular fiber.

5.3.2 Automorphism groups of geometrically ruled surfaces

Before continuing our study of birational Kleinian groups preserving a rational fibration, we collect some preliminaries on automorphism groups of geometrically ruled surfaces. They are studied by Maruyama in [Mar71]; we will present some of his main results. Our notations are those of [Har77] but are different from those of [Mar71]. Let Y be a geometrically ruled surface. The geometrically ruled surface Y over B is isomorphic to the projective bundle $\mathbb{P}(\mathscr{E})$ where $\mathscr{E}$ is a rank two vector bundle (not unique) over B such that $\mathrm{H}^0(B,\mathscr{E}) \neq \{0\}$ and $\mathrm{H}^0(B,\mathscr{E}\otimes\mathscr{L}) = \{0\}$ for every line bundle $\mathscr{L}$ with $\deg(\mathscr{L}) < 0$. The opposite of the degree of $\bigwedge^2 \mathscr{E}$, denoted by e, is an invariant of Y; the number $-e$ is the minimal self-intersection number of a section of $Y \to B$. A section with self-intersection number $-e$ is called a minimal section; the sub line bundle of $\mathscr{E}$ which corresponds to the minimal section is called a maximal line bundle. The geometrically ruled surface Y is called decomposable if there exists a decomposition $\mathscr{E} = \mathscr{L}_1 \oplus \mathscr{L}_2$ where $\mathscr{L}_1, \mathscr{L}_2$ are line bundles, otherwise Y is called indecomposable. There exist two sections which do not intersect each other if and only if Y is decomposable.

The group of automorphisms of Y which preserve fiberwise the ruling is denoted by $\mathrm{Aut}_B(Y)$. Theorem 2 of [Mar71] says:

Theorem 5.3.4 (Maruyama [Mar71])

1. *If $e < 0$, then $\mathrm{Aut}_B(Y)$ is a finite group.*
2. *If $e \geqslant 0$ and if Y is indecomposable, then $\mathrm{Aut}_B(Y) \cong \mathbf{C}^k$ where $k = \dim \mathrm{H}^0(B, (\det\mathscr{E})^{-1} \otimes \mathscr{L}^2)$ and $\mathscr{L} \subset \mathscr{E}$ is the unique maximal line bundle.*
3. *If $Y \cong \mathbb{P}^1 \times B$, then $\mathrm{Aut}_B(Y) \cong \mathrm{PGL}_2(\mathbf{C})$.*
4. *If $Y \cong \mathbb{P}(\mathscr{E})$ with $\mathscr{E} = \mathscr{L} \oplus (\mathscr{L} \otimes \mathscr{N})$, $\mathscr{N}^2 = \mathscr{O}_B$ and $\mathscr{N} \neq \mathscr{O}_B$, then $\mathrm{Aut}_B(Y)$ is an extension of $\mathbf{C}^*$ by $\mathbf{Z}/2\mathbf{Z}$.*
5. *If Y is decomposable and does not fit in the previous two cases, then $\mathrm{Aut}_B(Y)$ is isomorphic to the following subgroup of $\mathrm{GL}_{k+1}(\mathbf{C})$:*

$$H_k = \left\{ \begin{pmatrix} \alpha & t_1 & \cdots & \cdots & t_k \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & & \cdots & 0 & 1 \end{pmatrix} \in \mathrm{GL}_{k+1}(\mathbf{C}) \right\}$$

where $k = \dim \mathrm{H}^0(B, (\det\mathscr{E})^{-1} \otimes \mathscr{L}^2)$ and $\mathscr{L}$ is a maximal line bundle.

In the second and last case, the action of an element of H_k on the fiber F_b over $b \in B$ is basically (with some abusive identifications) $x \mapsto \alpha x + t_1 l_1(b) + \cdots + t_k l_k(b)$ where $l_1, \cdots, l_k$ form a base of $\mathrm{H}^0(B, (\det\mathscr{E})^{-1} \otimes \mathscr{L}^2)$.

Corollary 5.3.5 *If $\mathrm{Aut}_B(Y)$ is infinite and not solvable, then $Y = B \times \mathbb{P}^1$.*

5.3.3 Classification

We turn back to our problem without assuming that Y is geometrically ruled. We still assume that $\Gamma_B = \{\mathrm{Id}\}$ and that the components of every singular fiber of r is fixed by Γ. By Proposition 5.3.3 we know that if Y is not geometrically ruled then the intersection of U with a general fiber is biholomorphic to $\mathbf{C}^*$.

Non-abelian case

Proposition 5.3.6 *If Γ is not solvable, then $Y = \mathbb{P}^1 \times B$ and $U = W \times B$ where $W \subset \mathbb{P}^1$ is a connected component of the set of discontinuity of a non-elementary classical Kleinian group. The quotient surface X is the product of B with a hyperbolic curve.*

Proof In particular Γ is not cyclic and Proposition 5.3.3 implies that Y is geometrically ruled. By Corollary 5.3.5 $Y = \mathbb{P}^1 \times B$. In this case $\mathrm{Aut}_B(Y) = \mathrm{PGL}_2\,\mathbf{C}$ and the intersection of U with each fiber is the set of discontinuity of some classical Kleinian group. This classical kleinian group does not depend on the fiber. Since U is connected, the intersection of U with each fiber is the same component of the discontinuity set. $\square$

Translation in the fibers

As Γ is isomorphic to a classical Kleinian group by Proposition 5.3.3, if it is solvable then it is isomorphic to $\mathbf{Z}^2$ or $\mathbf{Z}$, depending on whether the intersection of U with a fiber is $\mathbf{C}$ or $\mathbf{C}^*$.

We first investigate the case where $\Gamma = \mathbf{Z}^2$; in this case Γ acts on each fiber by translations (translation means parabolic automorphism of $\mathbb{P}^1$). In case 2 (resp. case 5) of Maruyama's theorem, the action is given, in local coordinate z of the fiber, by

$$z \mapsto z + t_1 e_1 + \cdots + t_k e_k \text{ (resp. } z \mapsto \alpha z + t_1 e_1 + \cdots + t_k e_k) \tag{5.2}$$

where $e_1, \cdots, e_k$ extend to a global base of $\mathrm{H}^0(B, (\det \mathscr{E})^{-1} \otimes \mathscr{L}^2)$. Since a line bundle has a nowhere vanishing section if and only if it is trivial, an automorphism can act by non-trivial translation in every fiber only if the line bundle $(\det \mathscr{E})^{-1} \otimes \mathscr{L}^2$ is trivial. Writing $0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{L}' \to 0$, we have $(\det \mathscr{E})^{-1} \otimes \mathscr{L}^2 = \mathscr{L} \otimes (\mathscr{L}')^{-1}$. Therefore the triviality of $(\det \mathscr{E})^{-1} \otimes \mathscr{L}^2$ implies that $\mathscr{L}^{-1} \otimes \mathscr{E}$ is an extension of two trivial line bundles. Thus we obtain:

Proposition 5.3.7 *Suppose that the intersection of U with a fiber is* $\mathbf{C}$. *There is a rank two vector bundle $\mathscr{E}$ on B such that*

1. *Y is isomorphic to $\mathbb{P}(\mathscr{E})$;*
2. *We can write $\mathscr{E}$ as an extension $0 \to \mathscr{O}_B \to \mathscr{E} \to \mathscr{O}_B \to 0$ such that $U = Y - s(B)$ where $s : B \to Y$ is the section determined by this extension, i.e. $s \mapsto s(b)$ with $s(b) = \mathbb{P}(\mathscr{O}_b) \in \mathbb{P}(\mathscr{E}_b)$.*

As $(\det \mathscr{E})^{-1} \otimes \mathscr{L}^2 = \mathscr{O}_B$, the subgroup of $\mathrm{Aut}_B(Y)$ fixing $s(B)$ is isomorphic to $\mathbf{C}$ by Theorem 5.3.4, in which Γ is a lattice. The surface X is an elliptic fiber bundle over B with fiber isomorphic to $\mathbf{C}/\Gamma$.

Remark 5.3.8 The action of $\mathbf{C}$ on U descends to an action of $\mathbf{C}/\Gamma$ on X; this implies that X is a principal elliptic bundle by Lemma V.5.1 of [Bar+04]. A principal elliptic bundle is topologically a product of the circle S^1 with an S^1-bundle (Proposition V.5.2 of [Bar+04]). We claim that in our case the corresponding S^1-bundle is a product. The reason is that the rank two vector bundle $\mathscr{E}$ is topologically trivial because an extension of a trivial bundle by a trivial bundle always splits in the topological category. If $B = \mathbb{P}^1$ and X is Kähler, then the only possibility is $X = (\mathbf{C}/\Gamma) \times \mathbb{P}^1$; if $B = \mathbb{P}^1$ and X is Kähler, then X is a complex torus or a hyperelliptic surface (see [Bar+04] Chapter V).

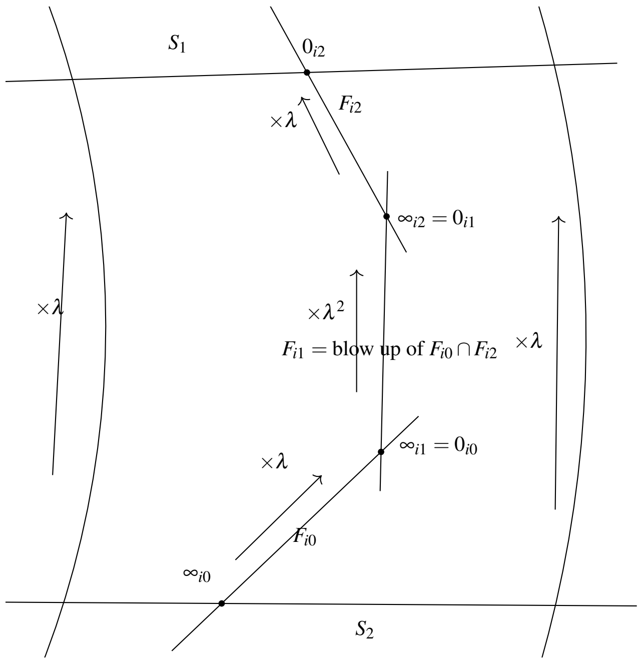
Multiplication in the fibers

We now investigate the case where $\Gamma = \mathbf{Z}$. We first show that Y can be non geometrically ruled (cf. Proposition 5.3.3).

Given a decomposable geometrically ruled surface $Y_0 = \mathbb{P}(\mathscr{E})$ over B, after tensoring $\mathscr{E}$ with a line bundle, we can suppose that it decomposes as $\mathscr{E} = \mathscr{O}_B \oplus \mathscr{L}$. The decomposition determines two sections S_1, S_2 on Y_0 without intersection. Pick a finite family of fibers $F_1, \cdots, F_n$ over $b_1, \cdots, b_n \in B$. Choose a positive integer $m \leqslant n$. Denote by $p_1, \cdots, p_m$ the intersection points of $F_1, \cdots, F_m$ with S_1 and $p_{m+1}, \cdots, p_n$ the intersection points of $F_{m+1}, \cdots, F_n$ with S_2. Construct a birationally ruled surface Y over B as follows: Y is obtained from Y_0 by successive blow-ups at $p_1, \cdots, p_n$ and some infinitely near points. For $i \leq m$ (resp. $i > m$) the successive blow-ups on the fiber over p_i are executed either at the intersection point of two irreducible components of the fiber or at the intersection point of the fiber with (the strict transform of) S_1 (resp. S_2). We denote by $\pi : Y \to Y_0$ the contraction map. The singular fibers of $Y \xrightarrow{\phi} B$ are chains of smooth rational curves, i.e. the singular fiber corresponding to F_i is

$$\tilde{F}_i = \sum_{k=0}^{l(i)} d_{i,k} F_{i,k};$$

here $F_{i,0}$ is the strict transform of F_i and the $F_{i,k}$ are all smooth rational curves among which the only non vanishing intersection pairings are $F_{i,k} \cdot F_{i,k+1} = 1$. We have $d_{i,0} = 1$ and $d_{i,1}, \cdots, d_{i,l(i)} \in \mathbf{N}^+$. Each $F_{i,k}$ has two distinguished points $\{0_{i,k}, \infty_{i,k}\}$ which are the two intersection points of $F_{i,k}$ with other components of the fiber or with (strict transforms of) the two sections S_1, S_2. Let's say that the one which points to S_1 is $0_{i,k}$.



By Theorem 5.3.4, the subgroup of $\mathrm{Aut}_B(Y_0)$ fixing S_1, S_2 is isomorphic to $\mathbf{C}^*$; it acts by multiplication in the fibers. The $\mathbf{C}^*$-action lifts to an action on Y by automorphisms because the blow-ups which we did to obtain Y are all at the fixed points of the $\mathbf{C}^*$-action. An element $\lambda \in \mathbf{C}^*$ acts on a component $F_{i,k}$ of the singular fiber $\tilde{F}_i$ by multiplication by $\lambda^{d_{i,k}}$ with fixed points $\{0_{i,k}, \infty_{i,k}\}$.

Now for each $\tilde{F}_i$, choose a component $F_{i,k_0(i)}$ and consider the open set

$$U = Y \backslash \left(S_1 \bigcup S_2 \bigcup (\cup_i \cup_{k \neq k_0(i)} F_{i,k}) \right).$$

We remark that U is Zariski open in Y. The ruling ϕ restricted to U is a surjective holomorphic map onto B with fibers biholomorphic to $\mathbf{C}^*$. The quotient of U by $\lambda \in \mathbf{C}^*, |\lambda| \neq 1$ is a compact complex surface X which admits an elliptic fibration over B. If a fiber of $X \to B$ comes from a non-singular fiber of $\phi : Y \to B$, i.e. the fiber over a point that is not one of the $p_1, \cdots, p_n$, then it is isomorphic to the elliptic curve $\mathbf{C}^* / < \lambda >$. If a fiber of $X \to B$ comes from F_i, then it is the quotient of $\mathbf{C}^* = F_{i,k_0(i)} \backslash \{0_{i,k_0(i)}, \infty_{i,k_0(i)}\}$ by $< \lambda^{d_{i,k_0(i)}} >$. Hence the only singular fibers of the elliptic fibration $X \to B$ are multiples of smooth elliptic curves, the multiplicities being the $d_{i,k_0(i)}$.

Actually the above construction exhausts all possibilities.

Proposition 5.3.9 *If the intersection of U with a fiber is $\mathbf{C}^*$, i.e. if $\Gamma = \mathbf{Z}$ then X is obtained from the above process. In particular the surface X is an elliptic fibration over B whose singular fibers are all multiples of smooth elliptic curves.*

Proof Recall that by Lemma 5.3.2 $\Gamma \subset \mathrm{Aut}(Y)$. We have also assumed that Γ fixes each irreducible component of a fiber. Therefore the action of Γ descends to an action by automorphisms, after blowing down some exceptional curves in a fiber. Continuing the blow down untill a geometrically ruled surface, we infer that the action of Γ on Y comes from a $\mathbf{C}^*$-action on a geometrically ruled surface Y_0; and Y is obtained from Y_0 by blowing-up fixed points (or infinitely near fixed points) of the $\mathbf{C}^*$-action. This is exactly how the above process works. □

We denote by X_0 the quotient of $Y_0 \backslash S_1 \cup S_2$ by λ. It is a principal elliptic fiber bundle over B because the $\mathbf{C}^*$-action descends to X_0 (Proposition V.5.2 of [Bar+04]). The surface X is obtained from X_0 by logarithmic transformations which replace the fibers over $b_1, \cdots, b_n \in B$ with $d_{1,k_0(1)}E_1, \cdots, d_{n,k_0(n)}E_n$ where $E_i = \mathbf{C}^* / < \lambda^{d_{i,k_0(i)}} >$.

5.4 Elliptic base

We start with a lemma that will be used in this section as well as in other sections of this chapter.

Lemma 5.4.1 *Suppose that Γ_B is an infinite group. Then B is an elliptic curve or $\mathbb{P}^1$. Up to geometric conjugation of birational Kleinian groups, Y is geometrically ruled, $r(U) = B$ and $\Gamma \subset \mathrm{Aut}(Y)$.*

Proof The open set $r(U)$ is invariant under the infinite group Γ_B. Suppose by contradiction that $r(U) \neq B$. Then according to Lemma 2.3.1, it is a connected component of the complement of a finite union of real subtori, thus a band. The action of Γ_B on the band $r(U)$ is by translations along the direction of the band. Such an action on a band is never cocompact, i.e. the union of the images under Γ of a compact subset of the band never covers the band. This contradicts the cocompactness of the action of Γ on U.

By Corollary 5.2.3, the fact $r(U) = B$ implies that $\Gamma \subset \mathrm{Aut}(Y)$. Since Γ_B is infinite and B is an elliptic curve, every point of B has an infinite Γ_B-orbit. If there existed a singular fiber of

r, then its translations by elements of Γ would give rise to an infinite number of singular fibers. Therefore Y is geometrically ruled. $\square$

In this section we prove:

Theorem 5.4.2 *Let (Y,Γ,U,X) be a birational kleinian group on a surface Y which is ruled over an elliptic curve B. Then up to geometric conjugation and up to taking finite index subgroup, we are in one of the following situations:*

1. *$Y = \mathbb{P}^1 \times B$, $U = D \times B$ where $D \subset \mathbb{P}^1$ is a connected component of the discontinuity domain of a classical Kleinian group. X is a fiber bundle over B or a suspension of B.*
2. *$Y = R_0$, $U = Y - s$ and U has a structure of algebraic group which is isomorphic to* $\mathrm{Aut}^0(Y)$. *As a complex analytic Lie group U is isomorphic to $\mathbf{C}^* \times \mathbf{C}^*$ and the group Γ is isomorphic to a lattice in U. The quotient X is a torus.*
3. *Y is decomposable with $e = 0$, $U = Y - (s_1 \cup s_2)$ and U has a structure of algebraic group which is isomorphic to* $\mathrm{Aut}^0(Y)$. *The quotient X is a torus.*
4. *Y is obtained by blowing up a decomposable ruled surface, U is a Zariski open set of Y whose intersection with each fiber is isomorphic to $\mathbf{C}^*$. The group Γ is generated by an automorphism which acts by multiplication in each fiber. The quotient X is an elliptic fibration over B whose only singular fibers are multiples of smooth elliptic curves.*

Remark 5.4.3 In every case of the above theorem, the open set U is not simply connected. Except in the first case, U is a Zariski open set.

Let (Y,Γ,U,X) be a birational Kleinian group satisfying the hypothesis of the theorem. The ruling $r : Y \to B$ is identified with the Albanese morphism and is preserved by each element of Γ. We have an exact sequence

$$0 \to \Gamma_r \to \Gamma \to \Gamma_B \to 0$$

where Γ_r preserves fiberwise the fibration and $\Gamma_B \subset \mathrm{Aut}(Y)$.

Proposition 5.4.4 *If B is an elliptic curve and Γ_B is infinite, then X is a complex torus with a linear irrational foliation or a suspension of B. In the two cases we have $r(U) = B$.*

Proof We apply Theorem 5.1.1 to investigate the possibilities for X. The orbit of a point on B under Γ_B is not discrete and its closure contains a real subtorus (cf. Lemma 2.3.1). This implies that no leaf of the foliation on X is a submanifold of X. Thus the foliation is not a fibration, is not a turbulent foliation because a turbulent foliation has at least one invariant elliptic curve, is not an obvious foliation on a Hopf surface because the smooth elliptic curves on the Hopf surface are invariant. Here the base curve B is elliptic so the foliation is transversely Euclidean; this rules out transversely hyperbolic foliations on surfaces of general type (cf. [Bru97] p592 first paragraph) and obvious foliations on Inoue surfaces because (cf. [Zhab]). The foliation on X cannot be a suspension of $\mathbb{P}^1$ because otherwise U would be covered by $\mathbb{P}^1$s and the only $\mathbb{P}^1$ in Y are the fibers of the rational fibration so there is no $\mathbb{P}^1$ transverse to the foliation. There are only two possibilities left: either X is a complex torus with an irrational linear foliation or X is an infinite suspension of a hyperbolic curve over an elliptic curve (note that an infinite suspension of an elliptic curve over an elliptic curve is an irrational linear foliation on a torus).□

Remark 5.4.5 Automorphisms of geometrically ruled surfaces are faily well understood and we do not really need the full strength of Brunella's work to study those groups with Kleinian property, as we shall see shortly.

We use the notations of section 5.3.2. We recall that $\mathrm{Aut}_B(Y)$ is the subgroup of $\mathrm{Aut}(Y)$ which preserves each fiber; it is described by Theorem 5.3.4. It is well known that, up to isomorphisms, there are only two indecomposable geometrically ruled surfaces over an elliptic curve B: R_0 with $e=0$ and R_1 with $e=1$ (see [Har77][Chapter 5]). Now we describe the whole automorphism group of the geometrically ruled surface Y:

Theorem 5.4.6 (Maruyama [Mar71]) *Let Y be a geometrically ruled surface over an elliptic curve B.*

1. *If $Y=\mathbb{P}^1\times B$, then $\mathrm{Aut}(Y)=\mathrm{PGL}_2(\mathbf{C})\times\mathrm{Aut}(B)$.*
2. *If Y is decomposable and $e\neq 0$, then we have an exact sequence of algebraic groups*

$$0\to\mathrm{Aut}_B(Y)\to\mathrm{Aut}(Y)\to G\to 0$$

where G is a finite subgroup of $\mathrm{Aut}(B)$.

3. *If Y is decomposable with $e=0$ and if $Y\neq\mathbb{P}^1\times B$, then we have an exact sequence of algebraic groups*

$$0\to\mathrm{Aut}_B(Y)\to\mathrm{Aut}(Y)\to G\to 0$$

where $\mathrm{Aut}_B(Y)$ is isomorphic to $\mathbf{C}^*$ or an extension of $\mathbf{C}^*$ by $\mathbf{Z}/2\mathbf{Z}$, and G is a subgroup of $\mathrm{Aut}(B)$ containing $\mathrm{Aut}^0(B)$. If s_1, s_2 are the minimal sections, then $Y-(s_1\cup s_2)$ is a principal $\mathbf{C}^*$-bundle over B which has a structure of commutative algebraic group.

The algebraic group $Y - (s_1 \cup s_2)$ *is isomorphic to the connected component of identity* $\mathrm{Aut}^0(Y)$.

4. *If* $Y = R_0$, *then we have an exact sequence of algebraic groups*

$$0 \to \mathbf{C} \to \mathrm{Aut}(Y) \to \mathrm{Aut}(B) \to 0.$$

If s *is the minimal section, then* $Y - s$ *is a principal* $\mathbf{C}$*-bundle over* B *which has a structure of commutative algebraic group. The algebraic group* $Y - s$ *is isomorphic to the connected component of identity* $\mathrm{Aut}^0(Y)$.

5. *If* $Y = R_1$, *then we have*

$$0 \to \mathrm{Aut}_B(Y) \to \mathrm{Aut}(Y) \to \mathrm{Aut}(B) \to 0$$

where $\mathrm{Aut}_B(Y)$ *is isomorphic to* $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

Remark 5.4.7 In the case of $Y = R_0$, the group law on $Y - s$ is explicitly computed in terms of the Weierstrass function in [LMP09]. As a complex analytic Lie group, it is isomorphic to $\mathbf{C}^* \times \mathbf{C}^*$ but the isomorphism is not algebraic. The variety $Y - s$ is the space of linear connections on B and the holomorphic isomorphism $Y - s \to \mathbf{C}^* \times \mathbf{C}^*$ is the Riemann-Hilbert mapping (cf. [LMP09]).

Remark 5.4.8 We describe here more in detail the geometry of R_1. The sections of $R_1 \to B$ which have self-intersection 1 are in bijection with line bundles of degree one on B; thus they are parametrized by B (cf. [Mar71]). There exist two sections of self-intersection 1 passing through a general point of R_1; through some points only one section of self-intersection 1 passes. This assertion is summarized in a morphism from $B \times B$ to R_0 which is a ramified double cover: a point $(b, \beta) \in B \times B$ is sent to the intersection point of the fiber over b with the section of self-intersection 1 parametrized by β. The involution associated with this ramified double cover is $(b, \beta) \mapsto (b, -b - \beta)$ (cf. [Dia]). Let $a, a' \in B$ such that $a = 2a'$. The automorphism of $B \times B$ defined by $(b, \beta) \mapsto (b + a, \beta - a')$ commutes with the involution, thus descends to an automorphism of R_1 whose action on the base B is $b \mapsto b + a$. Theorem 5.4.6 says that every automorphism of R_1 comes from this construction.

Proof (of Theorem 5.4.2) We already treated the case when Γ_B is finite in theorem 5.3.1. This is covered by case 1), 2), 4) of Theorem 5.4.2.

From now on we assume that Γ_B is infinite. We apply Theorem 5.4.6 case by case. From the infiniteness of $\mathrm{Aut}_B(Y)$ we infer that Y is not decomposable with $e \neq 0$. The surface Y is not R_1 because $\mathrm{Aut}(R_1)$ is compact and an infinite subgroup of a compact group cannot act discontinuously.

Assume that Y is decomposable with $e = 0$. The two minimal sections s_1 and s_2 are disjoint from U because they are invariant under $\mathrm{Aut}(Y)$. The universal cover of the complex analytic Lie group $Y - (s_1 \cup s_2)$ is necessarily $\mathbf{C}^2$ because it is commutative. Hence we obtain immediately that $U = Y - (s_1 \cup s_2)$ and X is a complex torus. Assume that $Y = R_0$. The same reasoning tells us that $U = Y - s$ and X is a torus.

Assume that $Y = \mathbb{P}^1 \times B$. By Proposition 5.4.4 the foliation on X induced by $Y \to B$ is a linear foliation on a torus or a suspension of B over a hyperbolic curve. If X is a torus then the intersection of U with any fiber of $Y \to B$ is biholomorphic to $\mathbf{C}$ or $\mathbf{C}^*$; the conclusion follows immediately in this case. Assume that X is a suspension over a hyperbolic curve. The fibration $Y \to \mathbb{P}^1$ induces a second regular foliation on X which is transverse to the first one; this second foliation must be the elliptic fibration subjacent to the suspension. The conclusion follows. □

5.5 Turbulent foliations, Hopf surfaces, suspensions of $\mathbb{P}^1$

From now on we consider the case where $B = \mathbb{P}^1$ and Γ_B is infinite. By Proposition 6.2.1 the rational fibration $Y \to B$ induces a regular foliation $\mathscr{F}$ on X. We apply Theorem 5.1.1 to the induced foliation on X and study it case by case. We will treat all cases in this chapter except Inoue surfaces and the case where X is of general type and $\mathscr{F}$ is a minimal transversely hyperbolic foliation. Minimal transversely hyperbolic foliations will be studied in the next chapter. The case of Inoue surfaces is treated in [Zhab] and Appendix B where results stronger than what we need are proved; a corollary of the main result is:

Theorem 5.5.1 ([Zhab], Appendix B) *Let S be an Inoue surface. Then up to geometric conjugation there is only one birational Kleinian group (Y, Γ, U, X) such that $X = S$. It corresponds to the standard construction of the Inoue surface by taking the quotient of $\mathbb{H} \times \mathbf{C}$ by a group of affine transformations.*

We need here only the case where Γ preserves a rational fibration. As the proof is still a bit lengthy, we refer simply the reader to [Zhab] and Appendix B.

5.5.1 Some examples

The Hirzebruch surface $\mathbf{F}_n$ is the projectivization of the rank two vector bundle $\mathscr{O}\oplus\mathscr{O}(n)$ on $\mathbb{P}^1$. The subbundles $\mathscr{O}$ and $\mathscr{O}(n)$ determine two sections. The complement of these two sections, denoted by W_n, is isomorphic to the principal $\mathbf{C}^*$-bundle over $\mathbb{P}^1$ associated with the line bundle $\mathscr{O}(n)$. The quasi-projective variety W_0 is a product $\mathbf{C}^*\times\mathbb{P}^1$. The quasi-projective variety W_1 is isomorphic to $\mathbf{C}^2\backslash\{0\}$. For $n\geq 1$, W_1 is an unramified cover of degree n of W_n; the cover is given in each fiber by $w\mapsto w^n$.

The automorphism group of $\mathbf{F}_n$ is connected and fits in the following exact sequence (cf. [Mar71]):

$$1\to H_{n+1}\to \mathrm{Aut}(\mathbf{F}_n)\to\mathrm{Aut}(\mathbb{P}^1)\to 1$$

where H_{n+1} is the group defined in Theorem 5.3.4. Using coordinates we have

$$\mathrm{Aut}(\mathbf{F}_n)=\left\{(x,y)\dashrightarrow\left(\frac{ax+b}{cx+d},\frac{y+t_0+t_1x+\cdots+t_nx^n}{(cx+d)^n}\right)\mid\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{GL}_2(\mathbf{C}),t_0,\cdots,t_n\in\mathbf{C}\right\}.$$

The action around the fiber at infinity is obtained by the change of coordinates $(x,y)\mapsto(\frac{1}{x},\frac{y}{x^n})$. In particular for the automorphism $(x,y)\mapsto(ax,by)$ with $a,b\in\mathbf{C}^*$, the action on the fiber at infinity is multiplication by $\frac{b}{a^n}$; for the automorphism $(x,y)\mapsto(x+c,by)$ with $a,b\in\mathbf{C}^*$, the action on the fiber at infinity is multiplication by b.

Example 5.5.2 Take $a,b\in\mathbf{C}^*$ such that $0<|b|<1$ and $|b|<|a|^n$. The automorphism $f_{n,a,b}$: $(x,y)\mapsto(ax,by)$ of $\mathbf{F}_n$ preserves W_n and acts freely properly discontinuously and cocompactly on W_n. The action of $f_{1,a,b}$ on W_1 is nothing else but the linear contraction $(z,w)\mapsto(bz,\frac{b}{a^n}w)$ on $\mathbf{C}^2\backslash\{0\}$. Let X denote the quotient surface $=W_n/<f_{n,a,b}>$. If a is a root of unity then the foliation on X induced by the rational fibration is a genus one fibration.

Assume that a is not a root of unity. When $n=0$, the quotient is a ruled surface over the elliptic curve $\mathbf{C}^*/<b>$; the foliation induced by the rational fibration is a suspension of $\mathbb{P}^1$. When $n>0$ the surface X is a Hopf surface with two elliptic curves isomorphic to $\mathbf{C}^*/<b>$ and $\mathbf{C}^*/<\frac{b}{a^n}>$; the foliation on X comes from the foliation by straight lines on the universal cover $\mathbf{C}^2\backslash\{0\}$ (one of the obvious foliations). No matter what n is, a and b are multiplicatively dependent if and only if X is equipped with an elliptic fibration; in this case the foliation on X is a turbulent foliation with respect to that elliptic fibration.

We can modify the above examples by adapting the construction in Section 5.3.3. We blow up $\mathbf{F}_n$ at the fibers over $x=0$ or $x=\infty$ to obtain a new surface Y; the points we blow up are intersection points of irreducible components with other irreducible components or with the strict transforms of the two sections of $\mathbf{F}^n$. The points we blowed up are fixed points of the $f_{n,a,b}$. Thus $f_{n,a,b}$ acts also by automorphism on Y; its action on an irreducible component of a fiber is by multiplication. Let $U \subset Y$ be a $\mathbf{C}^*$-fibration over $\mathbb{P}^1$ which is the complement in Y of the union of strict transforms of the two sections with some components of the singular fibers, as in Section 5.3.3. Then for suitable choice of a, b, the action of $f_{n,a,b}$ on U is free, properly discontinuous and cocompact. The quotient is either a Hopf surface or a ruled surface over an elliptic curve. If the quotient is a ruled surface then U is foliated by rational curves of self-intersection 0; this forces U to be a product. Hence the quotient is a ruled surface if and only if the birational Kleinian group is conjugate to the action of $f_{0,a,b}$ on $\mathbf{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

Example 5.5.3 Take $0 < |b| < 1$ and $c \in \mathbf{C}^*$. The automorphism $g_{n,b,c} : (x,y) \mapsto (x+c, by)$ of $\mathbf{F}_n$ preserves W_n; its action on the fiber at infinity is multiplication by b. If $n = 0$ then the quotient of W_0 by $g_{0,b,c}$ is a ruled surface over an elliptic curve equipped with a suspension foliation. The action of $g_{1,b,c}$ on W_1 is nothing else but the contraction $(z,w) \mapsto (bz + bcw, bw)$ on $\mathbf{C}^2 \backslash \{0\}$. For $n \geq 1$ the quotient of W_n by $g_{n,b,c}$ is a Hopf surface with only one elliptic curve. It has un unramified cover which is a primary Hopf surface corresponding to the normal form $(z,w) \mapsto (bz + \gamma w, bw)$ where γ is some number depending on n, b, c. Remark that this construction does not give all primary Hopf surfaces with only one elliptic curve, but only those with linear normal form. The foliation induced by the rational fibration corresponds to the foliation by lines in $\mathbf{C}^2 \backslash \{0\}$.

As in Example 5.5.2 we can modify the above construction by blowing up the invariant fiber.

Example 5.5.4 Consider two automorphisms $f_i : (x,y) \mapsto (x + a_i, y + b_i), i = 1,2$ of $\mathbb{P}^1 \times \mathbb{P}^1$. Suppose that $\mathbf{Z}b_1 + \mathbf{Z}b_2$ is a lattice in $\mathbf{C}$. Then f_1 and f_2 preserve $U = \mathbb{P}^1 \times \mathbf{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$; the data $(\mathbb{P}^1 \times \mathbb{P}^1, < f_1, f_2 >, U)$ give a birational Kleinian group. The quotient $X = U / < f_1, f_2 >$ is a ruled surface over the elliptic curve $\mathbf{C}/(\mathbf{Z}b_1 + \mathbf{Z}b_2)$ and the foliation on X induced by the rational fibration is a suspension. If moreover $\mathbf{Z}a_1 + \mathbf{Z}a_2$ is a lattice in $\mathbf{C}$ such that the elliptic curve $\mathbf{C}/(\mathbf{Z}a_1 + \mathbf{Z}a_2)$ is isogeneous to $\mathbf{C}/(\mathbf{Z}b_1 + \mathbf{Z}b_2)$, then X is equipped with an elliptic fibration so that the foliation is turbulent. The turbulent foliation has one compact leaf.

Consider two automorphisms g_1, g_2 of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by $g_1 : (x,y) \mapsto (x, y + b_1)$ and $g_2 : (x,y) \mapsto (ax, y + b_2)$. Suppose that $\mathbf{Z}b_1 + \mathbf{Z}b_2$ is a lattice in $\mathbf{C}$. Then g_1 and g_2 preserve $U = \mathbb{P}^1 \times \mathbf{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$; the data $(\mathbb{P}^1 \times \mathbb{P}^1, < g_1, g_2 >, U)$ give a birational Kleinian group. The quotient $X = U / < g_1, g_2 >$ is a ruled surface over the elliptic curve $\mathbf{C}/(\mathbf{Z}b_1 + \mathbf{Z}b_2)$ and the foliation on X induced by the rational fibration is a suspension. If moreover $< a >$ is a lattice in $\mathbf{C}^*$ such that the elliptic curve $\mathbf{C}^*/ < a >$ is isogeneous to $\mathbf{C}/(\mathbf{Z}b_1+\mathbf{Z}b_2)$, then X is equipped with an elliptic fibration so that the foliation is turbulent. The turbulent foliation has two compact leaves.

Remark 5.5.5 There are no analogues of Example 5.5.4 for $\mathbf{F}_n$ with $n \geq 1$. For $n \geq 1$ the action of $(x,y) \mapsto (x+a, y+b)$ on the fiber at infinity is trivial and $(x,y) \mapsto (ax, y+b)$ acts as multiplication by $\frac{1}{a^n}$ on the fiber at infinity.

We can see in another way that there are no such analogues when $n \geq 1$. Let U_n be the complement of the minimal section in $\mathbf{F}_n$; it is a $\mathbf{C}$-bundle over $\mathbb{P}^1$. If the quotient of U_n by an automorphism group isomorphic to $\mathbf{Z}^2$ is a compact surface then the compact surface would be Kähler because a compact complex surface with even second Betti number is Kähler (cf. Theorem IV.3.1 [Bar+04]). However U_n is not the universal cover of any compact Kähler surface because if the universal cover of a compact Kähler surface is non-compact and quasi-projective then it is $\mathbf{C}^2$ or $\mathbf{C} \times \mathbb{P}^1$ (cf. [Cla]).

5.5.2 Some characterizations

Proposition 5.5.6 *If $r(U) = B$ then $\mathscr{F}$ is a turbulent foliation, an obvious foliation on a Hopf surface or a suspension of $\mathbb{P}^1$.*

Proof We assume that $r(U) = B$.

Assume that X is of general type and that $\mathscr{F}$ is a minimal transversely hyperbolic foliation. In this case there is a fibration $f : \tilde{X} \to \mathbb{D}$ from the universal cover $\tilde{X}$ to the disk $\mathbb{D}$ with fibers all biholomorphic to $\mathbb{D}$; furthermore there is a representation $\rho : \pi_1(X) \to \mathrm{PSL}_2(\mathbf{R})$ with dense orbits with respect to which f is equivariant (cf. Theorem 7.1.1). The leaves of $\mathscr{F}$ come from the fibers of f. In our situation $\mathscr{F}$ comes also from the fibration $U \to r(U)$. We obtain thus a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathbb{D} & \longrightarrow & r(U) \end{array}$$

where all arrows are equivariant. The map $\mathbb{D} \to r(U)$ is surjective and equivariant under the homomorphism $\rho(\pi_1(X)) \to \Gamma_B$. In particular $r(U)$ is a hyperbolic Riemann surface, thus a proper subset of B.

Assume that $\mathscr{F}$ is a linear foliation on a torus. A linear foliation on $\mathbf{C}^2$ is induced by the projection of $\mathbf{C}^2$ onto some subvector-space of dimension one. The induced action of $\pi_1(X)$ on that subspace, which is isomorphic to $\mathbf{C}$, is by translations; the image of $\pi_1(X)$ is a free abelian group of rank $2,3$ or 4, depending on whether the leaves of $\mathscr{F}$ are elliptic curves, $\mathbf{C}^*$ or $\mathbf{C}$. As in the previous case, we obtain a surjective holomorphic map $\mathbf{C} \to r(U)$ that is equivariant with respect to the homomorphism $\pi_1(X) \to \Gamma_B$. This forces $r(U)$ to be $\mathbf{C}$ or $\mathbf{C}^*$.

$\mathscr{F}$ is not a fibration by Lemma 5.7.1 that we prove later. Inoue surfaces are excluded by Proposition 5.5.1. By Theorem 5.1.1 the only possibilities for which $r(U) = B$ are those listed in the proposition. □

Proposition 5.5.7 *If $\mathscr{F}$ is a turbulent foliation then up to a geometric conjugation preserving the rational fibration and up to replacing Γ with a finite index subgroup we are in one of the cases described in Examples 5.5.2 and 5.5.4.*

Proof Let $f : X \to C$ be the elliptic fibration subjacent to the turbulent foliation. At least one fiber of f is a leaf of $\mathscr{F}$; let E be such a fiber. Denote by π the covering map from U to X. A connected component of $\pi^{-1}(E)$ is a connected component of the intersection of U with some fiber of r; it must be biholomorphic to $\mathbf{C}$ or $\mathbf{C}^*$. Recall that we have an exact sequence $\pi_1(E') \to \pi_1(X) \to \pi_1^{orb}(C) \to 1$ where E' is a general fiber of f. If E'' is a multiple fiber of f then the image of $\pi_1(E'')$ in $\pi_1(X)$ includes the image of $\pi_1(E')$ as a subgroup of finite index. Let $G \subset \Gamma$ be the image of the composition $\pi_1(E') \to \pi_1(X) \to \Gamma$. Then G is a normal subgroup of Γ; it is the image of $\pi_1(E)$ in Γ if E is non-multiple and is a finite index subgroup of it if E is multiple. Let F be a fiber of r such that $\pi(U \cap F) = E$. The action of G preserves $U \cap F$ and $(U \cap F)/G = E$. Furthermore for any fiber E' of f, for any connected component Ω of $\pi^{-1}(E')$, G preserves Ω and Ω/G is an elliptic curve which is E' if E' is non multiple. Thus U/G is an elliptic fibration over a possibly non-compact base. G is cyclic if $U \cap F = \mathbf{C}^*$ and is isomorphic to $\mathbf{Z}^2$ if $U \cap F = \mathbf{C}$.

Denote by G_B the image of G in Γ_B. We claim that G_B is infinite. Take a general fiber E' of f which is transverse to the turbulent foliation $\mathscr{F}$. Let $\Omega \subset U$ be a connected component of $\pi^{-1}(E')$. Ω is transverse to the rational fibration r and r restricted to Ω is a covering map. Therefore $r(\Omega) \subset r(U)$ is $\mathbf{C}^*$ or $\mathbf{C}$. The claim follows because $\Omega/G = E'$ and $r|_\Omega$ is equivariant under the homomorphism $G \to G_B$.

We claim that an element of infinite order of Γ is contained in G. Let $\gamma \in \Gamma$ be of infinite order. $\gamma(U \cap F)$ is also a connected component of $\pi^{-1}(E)$ and $\gamma(U \cap F)/G$ is also E because G is a normal subgroup. This implies in particular that G preserves the fiber $\gamma(F)$. As G_B is infinite, there are at most two fibers preserved by G. By replacing γ with γ^2 we have $\gamma(F) = F$. The action of $< \gamma, G >$ on $U \cap F$ is discrete if and only if γ is already in G because $(U \cap F)/G = E$ is already compact and γ is of infinite order.

Therefore G is a normal abelian subgroup of Γ of finite index. Up to raplacing Γ with a finite index subgroup of G, we can and will suppose that $G = \Gamma$ and that Γ, Γ_B are free abelian groups of rank one or two. Let us say that the fiber F is over the point at infinity ∞ of $B = \mathbb{P}^1$. Denote by $R \subset B$ the set of fixed points of G_B. It contains ∞ and has at most two elements; we let the other element be 0 if it exists. $r(U)\backslash R$ is $\mathbf{C}$ or $\mathbf{C}^*$; Γ_B acts respectively by translations or multiplications. Any point in $r(U)\backslash R$ has an infinite Γ_B-orbit. Thus by Corollary 5.2.3 r has no singular fibers over $r(U)\backslash R$. The fibers over points of R correspond to fibers of $f : X \to C$ that are tangent to $\mathscr{F}$. Denote by S the subset of C over which the fibers are tangent to $\mathscr{F}$; it has the same cardinality as R. The foliation $\mathscr{F}$ restricted to $f^{-1}(C\backslash S)$ is everywhere transverse to the elliptic fibration f, thus is a suspension. We have

$$\pi^{-1}\left(f^{-1}(C\backslash S)\right) = r^{-1}(r(U)\backslash R) \cap U$$

and we denote this open set by M.

Let Ω be the preimage in M of a fiber E' of f over $C\backslash S$. It is biholomorphic to $\mathbf{C}$ if Γ is of rank two and to $\mathbf{C}^*$ if Γ is of rank one. It is transverse to the rational fibration r. It is invariant under Γ; the quotient is E'. The projection $\Omega \to r(U)\backslash R$ induced by r is equivariant with respect to $\Gamma \to \Gamma_B$ and is an unramified covering. This implies that $(r(U)\backslash R)/\Gamma_B$ is an elliptic curve isogeneous to E' and we are in one of the following situations:

1. Γ and Γ_B are cyclic, $r(U)\backslash R$ is $\mathbf{C}^*$;
2. Γ and Γ_B have rank two, $r(U)\backslash R$ is $\mathbf{C}$;
3. Γ is of rank two, Γ_B is cyclic, $r(U)\backslash R$ is $\mathbf{C}^*$.

Let us consider firstly the first two cases. In these two cases the covering map $\Omega \to r(U)\backslash R$ is finite. For any $x \in r(U)\backslash R$, $r^{-1}(x) \cap \Omega$ is a finite set. Thus a leaf of $\mathscr{F}$ transverse to f intersects a fixed fiber only a finite number of times. This implies that for any $x \in r(U)\backslash R$, $r^{-1}(x) \cap U$ is biholomorphic to a finite unramified cover of $C\backslash S$, thus a Riemann surface of finite type. This is only possible if $C = \mathbb{P}^1$ because $r^{-1}(x) \cap U$ is an open set of $\mathbb{P}^1$. In the first case we obtain that $R = \{0, \infty\}$, $r(U) = B$ and $r^{-1}(b) \cap U$ is biholomorphic to $\mathbf{C}^*$. In the second case we obtain that $R = \{\infty\}$, $r(U) = B$ and $r^{-1}(b) \cap U$ is biholomorphic to $\mathbf{C}$. The fact $r(U) = B$ allows us to suppose $\Gamma \subset \mathrm{Aut}(Y)$ by Corollary 5.2.3.

Recall that Y has no singular fibers over $r(U)\backslash R$. In the first case we can conclude from the discussions in Example 5.5.2 that we are in Example 5.5.2. We need to show that if we are in the second case then we are in Example 5.5.4. In this case U is a $\mathbf{C}$ bundle over $\mathbb{P}^1$ and is necessarily a Zariski open set. Then Remark 5.5.5 allows us to conclude.

Finally let us consider the third case. In this case $r^{-1}(\infty)\cap U$ is isomorphic to $\mathbf{C}$ and Γ acts by translations on them. In this case an infinite cyclic subgroup H of Γ preserves fiber by fiber the rational fibration. The projection $\Omega \to r(U)\backslash R$ is an infinite cyclic covering, i.e. the exponential map from $\mathbf{C}$ to $\mathbf{C}^*$. Using again the fact the foliation on M is a suspension of E' over $C\backslash S$, we deduce that for any $b\in r(U)\backslash R$, $r^{-1}(b)\cap U$ is an infinite cyclic cover of $C\backslash S$. Moreover the quotient of $r^{-1}(b)\cap U$ by H is a finite cover of $C\backslash S$, thus a Riemann surface of finite type. Therefore H can be viewed as a cyclic (not neccessarily cocompact) classical Kleinian group in the fiber $r^{-1}(b)$. Then we have two possibilities because S has cardinality 1 or 2: 1) $r^{-1}(b)\cap U$ is $\mathbf{C}$, $C=\mathbb{P}^1$ and $C\backslash S$ is $\mathbf{C}^*$; 2) $r^{-1}(b)\cap U$ is the complement in $\mathbf{C}^*$ of one or two orbits under a multiplication map and C is an elliptic curve.

Suppose by contradiction that C is an elliptic curve. Then for any $b\in r(U)\backslash R=\mathbf{C}^*$, $r^{-1}(b)\cap U$ is biholomorphic to the complement in $\mathbf{C}^*$ of one or two orbits under a multiplication map because S has cardinality 1 or 2. In this case $\Gamma=\mathbf{Z}^2$ is generated by an element γ_1 whose action over $r(U)\backslash R=\mathbf{C}^*$ is by multiplications in the fibers and another element γ_2 whose image in Γ_B is a multiplication. By Theorem 1.5.3 (the part we use here is due to [CD12b]), γ_1,γ_2 can be written respectively as $(x,y)\mapsto(x,ay)$ and $(x,y)\mapsto(bx,R(x)y)$ with $a,b\in\mathbf{C}^*$ and $R\in\mathbf{C}(x)^*$. Recall that we have an Γ-invariant fiber F over $x=\infty$ such that $F\cap U=\mathbf{C}$ and Γ acts by translations on $F\cap U$. This gives a contradiction because γ_1 acts on any component of F by multiplication.

Therefore we obtain that $C=\mathbb{P}^1$, $C\backslash S=\mathbf{C}^*$ and $r^{-1}(b)\cap U=\mathbf{C}$. As S has cardinality two, we have $R=\{0,\infty\}$ and $r(U)=B$. Again $r(U)=B$ implies $\Gamma\subset \mathrm{Aut}(Y)$ by Corollary 5.2.3. Thus we infer that U is the complement in Y of a section, thus a $\mathbf{C}$-bundle over $\mathbb{P}^1$. This implies again by Remark 5.5.5 that Y is $\mathbb{P}^1\times\mathbb{P}^1$. Hence we are in Example 5.5.4. $\square$

Proposition 5.5.8 *Suppose that $\mathscr{F}$ is an obvious foliation on a non-elliptic Hopf surface. Then up to a geometric conjugation preserving the rational fibration and up to replacing Γ with a finite index subgroup we are in one of the following situations:*

1. *X has two elliptic curves and we are in Example 5.5.2.*
2. *X has one elliptic curve and we are in Example 5.5.3.*
3. *U is the standard open set $\mathbf{C}^2\backslash\{0\}$ of $Y=\mathbb{P}^1\times\mathbb{P}^1$ and Γ is generated by the normal form $(x,y)\mapsto(\alpha x+\gamma y^m,\beta y)$.*

Proof Since the fundamental group of Hopf surface is virtually cyclic, the open set U is a finite quotient of $\mathbf{C}^2\backslash\{0\}$. If $\mathscr{F}$ comes from the foliation by parallel lines on $\mathbf{C}^2\backslash\{0\}$, then the intersection of U with a fiber of r is $\mathbf{C}$ except for one which is $\mathbf{C}^*$. Such an open set is necessarily $\mathbf{C}^2 \setminus \{0\}$. As the normal form is already birational and preserves a rational fibration, our situation is necessarily conjugate to it.

Assume now that $\mathscr{F}$ is not a foliation by parallel lines. The space of leaves of any infinite cover of X is $\mathbb{P}^1$. Thus $r(U) = \mathbb{P}^1$. By Corollary 5.2.3 this implies $\Gamma \subset \mathrm{Aut}(Y)$. The open set is necessarily one of the Zariski open sets appeared in Examples 5.5.2 and 5.5.3 because every leaf is biholomorphic to $\mathbf{C}^*$. Hence we are in Example 5.5.2 if Γ_B is generated by a multiplication and in Example 5.5.3 if Γ_B is generated by a translation. $\square$

Proposition 5.5.9 *Suppose that $r(U) = B$ and that $\mathscr{F}$ is a suspension foliation. Then it is a suspension of $\mathbb{P}^1$ and we have, up to a geometric conjugation preserving the rational fibration and up to replacing Γ with a finite index subgroup, that $Y = \mathbb{P}^1 \times \mathbb{P}^1$, $\Gamma \subset \mathrm{PGL}_2(\mathbf{C}) \times \mathrm{PGL}_2(\mathbf{C})$. The projection of Γ onto the second factor is a classical Kleinian group; we have $U = \mathbb{P}^1 \times D$ where $D \subset \mathbb{P}^1$ is a connected component of the domain of discontinuity of that classical Kleinian group.*

Proof As the leaves in U are parametrized by $r(U) = \mathbb{P}^1$, the suspension must be over $\mathbb{P}^1$. Therefore any unramified cover of X is covered by smooth rational curves. In particular U is covered by rational curves of self-intersection 0 which are transverse to the rational fibration. This implies that we have up to geometric conjugation $U \subset \mathbb{P}^1 \times \mathbb{P}^1$ and the the rational curves in U are fibers of the other rational fibration. The conclusion follows. $\square$

5.6 Suspensions

We continue our study under the hypothesis $B = \mathbb{P}^1$ and Γ_B is infinite. We denote the covering map $U \to X$ by π. Denote by Γ_r the subgroup of Γ that preserves the fibration $r : Y \to B$ fiber by fiber. We have an exact sequence

$$1 \to \Gamma_r \to \Gamma \to \Gamma_B \to 1.$$

By Corollary 5.2.3 Γ acts by holomorphic diffeomorphisms on $r^{-1}(r(U))$. In this section we consider the case where the induced foliation $\mathscr{F}$ on X is an infinite suspension of N over M. Either N is an elliptic curve or it is $\mathbb{P}^1$. We have a fibration $f : X \to C$ of which all fibers are isomorphic to N; the foliation $\mathscr{F}$ is transverse to this fibration f. The foliation induced by f lifts to a second foliation $\mathscr{G}$ on U, transverse to the rational fibration r. By the transversality, we infer that, restricted to every leaf of $\mathscr{G}$, r is an unramified covering map onto $r(U)$. The covering map π restricted to a leaf of $\mathscr{G}$ is a covering map onto a fiber of f, which is isomorphic to N. From this we deduce immediately:

Lemma 5.6.1 *If N is $\mathbb{P}^1$ then $r(U) = B$ and the situation is described by Proposition 5.5.9. If N is an elliptic curve then $r(U)$ is isomorphic to $\mathbf{C}$ or $\mathbf{C}^*$.*

From now on we assume that N is an elliptic curve. Recall the construction of suspension. There exist an infinite covering $\bar{M}$ of M with deck transformation group G and a representation $\alpha : G \to \mathrm{Aut}(N)$ such that X is the quotient of $\bar{M} \times N$ by the action of G defined by $g \cdot (m,n) = (g \cdot m, \alpha(g) \cdot n)$. Up to replacing $\bar{M}$ with a quotient, we can assume that α is injective. Up to replacing G with a subgroup of finite index, we can assume that it is an abelian group, i.e. $\alpha(G)$ is a group of translations on N. The fundamental group of $\bar{M} \times N$ is $\pi_1(\bar{M}) \times \pi_1(N)$ and the fundamental group of X fits in the exact sequence

$$1 \to \pi_1(\bar{M}) \times \pi_1(N) \to \pi_1(X) \to G \to 1.$$

Denote by $\tilde{M}$ the universal cover of M. The universal cover of X is $\tilde{M} \times N$. Any element of $\pi_1(X)$ acts on $\tilde{M} \times \mathbf{C}$ in a diagonal way; the action on the $\mathbf{C}$ factor is a translation. Thus $\pi_1(N)$ is in the center of $\pi_1(X)$. Our birational Kleinian group Γ is a quotient of $\pi_1(X)$. Denote respectively by Γ_1 and Γ_2 the images of $\pi_1(N)$ and $\pi_1(\bar{M})$ in Γ; they are normal subgroups of Γ. Up to replacing Γ by a subgroup of finite index, we can and will assume that Γ_1 is a free abelian group of rank one or two. The center of Γ includes Γ_1. Remark that we do not have necessarily $\Gamma_1 \cap \Gamma_2 = \{1\}$ because we are taking quotients. Denote by Γ_3 the quotient $\Gamma/(\Gamma_1\Gamma_2)$; it is isomorphic to a quotient of G. Denote by q the epimorphism $\Gamma \to \Gamma_3$.

Lemma 5.6.2

1. *Γ_1 preserves each leaf of the foliation $\mathscr{G}$. The image Γ_{1B} of Γ_1 in Γ_B is an infinite group of translations if $r(U) = \mathbf{C}$ or an infinite group of multiplications if $r(U) = \mathbf{C}^*$.*
2. *r has no singular fibers over $r(U)$ and $r^{-1}(r(U)) = r(U) \times \mathbb{P}^1$.*
3. *Let Ω be a connected component of the intersection of U with a fiber of the rational fibration r and let Γ_Ω be the subgroup of Γ that preserves Ω. Then $\Gamma_\Omega = \Gamma_2 \subset \Gamma_r$.*

Proof The foliation $\mathscr{G}$ on U is the pull-back of the elliptic fibration on X. The first assertion is a consequence of the following two observations that we made earlier: 1) r restricted to a leaf of $\mathscr{G}$ is a covering map onto $r(U)$ and that $r(U)$ is $\mathbf{C}^*$ or $\mathbf{C}$; 2) π restricted to a leaf of $\mathscr{G}$ is a covering map onto a fiber of the elliptic fibration f.

By Corollary 5.2.3 Γ acts by holomorphic diffeomorphisms on $r^{-1}(r(U))$. If there was a singular fiber then there would be an infinite number of them by the first assertion. Hence the second assertion.

Let us prove now the third assertion. The connected component Ω is a leaf of $\pi^*\mathscr{F}$ and $\pi(\Omega)$ is a leaf of $\mathscr{F}$; $\pi(\Omega) = \Omega/\Gamma_\Omega$ is biholomorphic to $\bar{M}$. Thinking of a fundamental group as a deck transformation group, it follows from the definition of Γ_2 that $\Gamma_\Omega = \Gamma_2$. As a consequence Γ_Ω is a normal subgroup of Γ and does not depend on Ω. This implies that $\Gamma_\Omega \subset \Gamma_r$. $\square$

Lemma 5.6.3 *Γ_B is not virtually cyclic.*

Proof Suppose by contradiction that Γ_B is virtually cyclic. We know already that Γ_{1B} is infinite. Thus Γ_B is virtually Γ_{1B}. By the first assertion in the above lemma, we know that the action of Γ_B on $r(U)$ is discrete. Therefore $r(U)/\Gamma_B$ is an orbifold Riemann surface and we have a surjective holomorphic map from X to $r(U)/\Gamma_B$ whose fibers are leaves of $\mathscr{F}$. This contradicts that $\mathscr{F}$ is an infinite suspension foliation. $\square$

Lemma 5.6.4 *Up to geometric conjugation of birational Kleinian groups realized by elementary transformations, we have $Y = \mathbb{P}^1 \times \mathbb{P}^1$ (we let r to be the projection onto the first factor) and Γ_1 is a subgroup of one of the following four groups: $\{(x,y) \mapsto (ax,by) | a,b \in \mathbf{C}^*\}, \{(x,y) \mapsto (ax,y+b) | a \in \mathbf{C}^*, b \in \mathbf{C}\}, \{(x,y) \mapsto (x+a,by) | a \in \mathbf{C}, b \in \mathbf{C}^*\}$ or $\{(x,y) \mapsto (x+a,y+b) | a,b \in \mathbf{C}\}$. In the first two cases $r(U) = \mathbf{C}^*$; in the last two cases $r(U) = \mathbf{C}$.*

Proof Recall that Γ_1 is in the center of Γ. The action on B of the centralizer of a Jonquières twist is virtually cyclic by Theorem 1.5.2. Thus by the previous lemma we conclude that Γ_1 contains no Jonquières twists and is an elliptic free abelian group. Then there exist a birational map $\phi : Y \to \mathbb{P}^1 \times \mathbb{P}^1$, composition of elementary transformations, such that $\phi\Gamma_1\phi^{-1}$ is included in one of the four groups in the statement (see Theorem 1.5.3). It remains to show that the conjugation ϕ can be chosen to be a geometric conjugation of birational Kleinian group; the reason is as follows. Up to post-composition with an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ we observe that the elementary transformations composing ϕ can be made over any point in B. As $r(U)$ is a proper subset of B, we can do elementary transformations on a fiber over a point outside $r(U)$. Such an elementary transformation has no effects on U or on the dynamics of Γ on U; in other words elementary transformations outside U realize a geometric conjugation of Γ. $\square$

Proposition 5.6.5 *Γ is a subgroup of* $\mathrm{PGL}_2(\mathbf{C}) \times \mathrm{PGL}_2(\mathbf{C})$ *acting on $Y = \mathbb{P}^1 \times \mathbb{P}^1$. The open subset $U \subset Y$ is $\Omega_1 \times \Omega_2$ where Ω_2 is $\mathbf{C}$ or $\mathbf{C}^*$ and $\Omega_1 \subset \mathbb{P}^1$ is biholomorphic to $\bar{M}$.*

Proof We see from the previous lemma that Γ_1 acts separately on the x and y, the action on the x coordinate being Γ_{1B}. Assume first that its action on the y coordinate is finite. Since Γ_1 is a central abelian subgroup of Γ, we can replace Γ with a subgroup of finite index to assume that the action of Γ_1 on the y coordinate is trivial. Let L be a leaf of $\mathscr{G}$. Recall that L is biholomorphic to $\mathbf{C}$ or $\mathbf{C}^*$, is transverse to the rational fibration $(x,y) \mapsto x$, is preserved by Γ_1 and L/Γ_1 is isomorphic to N, being a fiber of $X \to C$. This forces L to be $r(U) \times \{y\}$ for some $y \in \mathbb{P}^1$. Hence the open set U is as described in the statement. We obtain also that the foliation $\mathscr{G}$ is induced by the projection of $\mathbb{P}^1 \times \mathbb{P}^1 = Y$ onto the second factor. Therefore Γ preserves this second rational fibration as well. This implies that Γ is a subgroup of $\mathrm{PGL}_2(\mathbf{C}) \times \mathrm{PGL}_2(\mathbf{C})$ (cf.).

Now assume that the action of Γ_1 on the y coordinate has transformations of infinite order. Then Γ_1 contains elements with infinite actions on both x and y coordinates. Recall that Γ_1 is in the center of Γ. Thus by Theorem 1.5.1, Γ is a subgroup of $\mathrm{PGL}_2(\mathbf{C}) \times \mathrm{PGL}_2(\mathbf{C})$. Therefore the projection of $\mathbb{P}^1 \times \mathbb{P}^1 = Y$ onto the second factor induces a foliation $\mathscr{F}'$ on X. This foliation is transverse to $\mathscr{F}$ and is not the elliptic fibration f because the sets $\{y = \text{constant}\}$ are not invariant under Γ_1. Remark that X has Kodaira dimension ≥ 0 because it is a genus one bundle over M which is a compact Riemann surface with infinite fundamental group. Proposition 4.3 of [Bru97] tells us that a regular foliation on a genus one bundle of non-negative Kodaira dimension is either the genus one fibration or a (not necessarily infinite) suspension or a turbulent foliation. We said that $\mathscr{F}'$ is not the fibration; it is not a turbulent foliation either by the characterizations given in Proposition 5.5.7. Therefore $\mathscr{F}'$ is another (possibly finite) suspension foliation. The genus one fibration, $\mathscr{F}$ and $\mathscr{F}'$ are transverse to each other. Thus these three foliations trivialize the projectivized tangent bundle of X. Consequently X has trivial first Chern class, and is a complex torus or a K3 surface. As there are no regular foliations on a K3 surface (cf. [Bru97] 5.Proposition 7), X is a complex torus. As a consequence, $\mathscr{F}$ is a irrational linear foliation on X and its leaves are biholomorphic to $\mathbf{C}$ or $\mathbf{C}^*$. This implies that the intersection of U with a fiber of r is $\mathbf{C}$ or $\mathbf{C}^*$. Together with the fact $\Gamma \subset \mathrm{PGL}_2(\mathbf{C}) \times \mathrm{PGL}_2(\mathbf{C})$, this implies that U is either $\mathbf{C} \times \mathbf{C}$, or $\mathbf{C} \times \mathbf{C}^*$, or $\mathbf{C}^* \times \mathbf{C}$, or $\mathbf{C}^* \times \mathbf{C}^*$. Thus in this case the proposition also holds. $\square$

5.7 Fibrations

5.7.1 Classical Kleinian groups in base and fibers

We continue our study under the hypothesis $B = \mathbb{P}^1$ and Γ_B is infinite. In this section we consider the case where the induced foliation $\mathscr{F}$ on X is a fibration $f : X \to C$. We denote the covering map $U \to X$ by π. Denote by Γ_r the subgroup of Γ that preserves the fibration $r : Y \to B$ fiber by fiber. We have an exact sequence

$$1 \to \Gamma_r \to \Gamma \to \Gamma_B \to 1.$$

By Corollary 5.2.3 Γ acts by holomorphic diffeomorphisms on $r^{-1}(r(U))$. For a point $x \in r(U)$ we denote by Γ_x the subgroup of Γ that preserves the fiber $r^{-1}(x)$. The group Γ_x acts by automorphims on $r^{-1}(x)$. The group Γ_r is a normal subgroup of Γ_x for any x.

Lemma 5.7.1 *1. For any $x \in r(U)$, the intersection of U with $r^{-1}(x)$ has finitely many connected components.*

2. *If Ω is a connected component of $U \cap r^{-1}(x)$, then the subgroup Γ_Ω of Γ preserving Ω is a finite index subgroup of Γ_x.*
3. *For any $x \in r(U)$, the fiber $r^{-1}(x)$ is non-singular, i.e. $r^{-1}(x) = \mathbb{P}^1$.*
4. *$r(U)$ is a proper subset of $B = \mathbb{P}^1$.*

Proof Let Ω be a connected component of $U \cap r^{-1}(x)$ for some point $x \in r(U)$. It is a leaf of the foliation $\pi^*\mathscr{F}$ on U, i.e. $\pi(\Omega)$ is a fiber of $f : X \to C$. We assume that it is not a multiple fiber. Denote by Γ_Ω the subgroup of Γ that preserves Ω; it is a subgroup of Γ_x. The quotient of Ω by Γ_Ω is the fiber $\pi(\Omega)$. The group Γ_Ω is the image of the composition $\pi_1(\pi(\Omega)) \to \pi_1(X) \to \Gamma$. The fibration $f : X \to C$ induces an exact sequence

$$\pi_1(\pi(\Omega)) \to \pi_1(X) \to \pi_1^{orb}(C) \to 1.$$

Therefore Γ_Ω is a normal subgroup of Γ and it does not depend on the leaf Ω as long as $\pi(\Omega)$ is a general fiber of $f : X \to C$; in particular Γ_Ω preserves fiberwise the rational fibration, i.e. $\Gamma_\Omega \subset \Gamma_r$. For a leaf Ω' such that $\pi(\Omega')$ is a multiple fiber, Γ_Ω is a subgroup of finite index of $\Gamma_{\Omega'}$.

Suppose that $r^{-1}(x)$ is a singular fiber of the rational fibration. In this case the group Γ_Ω preserves the set of singular points of $r^{-1}(x)$ and is solvable. Thus Ω is biholomorphic to $\mathbf{C}$

or $\mathbf{C}^*$ and the intersection of U with an irreducible component of $r^{-1}(x)$ is either empty or connected. Thus the first assertion of our lemma is true if $r^{-1}(x)$ is a singular fiber. Let us assume now that $r^{-1}(x)$ is not singular, i.e. is isomorphic to $\mathbb{P}^1$. Then we can think of Γ_Ω as a classical Kleinian group with an invariant component Ω. The intersection of U with $r^{-1}(x)$ is a union of connected components of the discontinuity domain of the Kleinian group Γ_Ω; by the above discussion all of them are invariant, or invariant under a finite index subgroup of Γ_Ω if $\pi(\Omega)$ is a multiple fiber. Ahlfors's finiteness theorem [Ahl64] (see also [Sul85]) says that $(U\cap r^{-1}(x))/\Gamma_\Omega$ is a finite union of compact Riemann surfaces. This implies that $r^{-1}(x)\cap U$ has only finitely many connected components.

In the first paragraph of the proof we infered $\Gamma_\Omega \subset \Gamma_r$; we also have an inclusion $\Gamma_r \subset \Gamma_x$. The above finiteness of connected components implies that Γ_Ω has finite index in Γ_x. From this and the infiniteness of Γ_B we deduce that the orbit of any point $x \in r(U)$ has an infinite orbit under Γ_B. As Γ acts by holomorphic diffeomorphisms on $r^{-1}(r(U))$, we conclude that r has no singular fibers over $r(U)$.

Any element of Γ_B has a fixed point in $B=\mathbb{P}^1$. If $r(U)=B$ then a fixed point x of some element of infinite order of Γ_B would give rise to a Γ_x such that Γ_r has infinite index in Γ_x, contradicting the second point of the lemma. □

Up to making elemantary transformations on a fiber over a point outside $r(U)$ we can and will assume that $Y=\mathbb{P}^1\times\mathbb{P}^1$ and that r is the projection onto the first factor. We will denote by x and y the first and the second coordinate of $\mathbb{P}^1\times\mathbb{P}^1$. Γ is a subgroup of the Jonquières group $\mathrm{PGL}_2(\mathbf{C}(x))\rtimes \mathrm{PGL}_2(\mathbf{C})$ and Γ_r is a subgroup of $\mathrm{PGL}_2(\mathbf{C}(x))$. An element of Γ can be written as

$$(x,y)\mapsto\left(\frac{ax+b}{cx+d},\frac{A(x)y+B(x)}{C(x)y+D(x)}\right),\quad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{PGL}_2(\mathbf{C}),\quad \begin{pmatrix} A & B\\ C & D\end{pmatrix}\in \mathrm{PGL}_2(\mathbf{C}(x)).$$

For a point $x\in B$, we denote by $\mathbb{P}^1_x$ the fiber of r over x, and by U_x the intersection $U\cap\mathbb{P}^1_x$. For any $x\in B$ and any connected component Ω of $\mathbb{P}^1_x\cap U$, the group Γ_Ω is a subgroup of finite index of Γ_r and Γ_r is a subgroup of finite index of Γ_x by Lemma 5.7.1. Γ_Ω is finitely generated because it is a quotient of the fundamental group of a compact Riemann surface. Thus Γ_r and Γ_x are also finitely generated groups. As a finitely generated subgroup of $\mathrm{PGL}_2(\mathbf{C})$, Γ_B has a torsion free subgroup of finite index by Selberg's lemma (cf. [Sel60], [Alp87]). Up to replacing Γ with the preimage of such a subgroup of Γ_B, we can and will assume that Γ_B *is torsion free*. This implies that *for any $x\in B$, we have $\Gamma_x=\Gamma_r$.*

As a group of birational transformations Γ_r is determined by its actions in all fibers of r. We can identify all the $\mathbb{P}^1_x, x \in B$ to a fixed $\mathbb{P}^1$ via the projection of $\mathbb{P}^1 \times \mathbb{P}^1$ onto the second factor. In this way we think of Γ_r as a family of representations $\{\rho_x : \Gamma_r \to \mathrm{Aut}(\mathbb{P}^1_x) = \mathrm{PGL}_2(\mathbf{C})\}_x$ parametrized by $x \in B \backslash I$ where $I \subset B$ is the set of points over which the fibers of r contain indetermincay points of birational transformations in Γ_r. The set I is finite because Γ_r is finitely generated. Thus Γ_r is an algebraic family of representations over the quasi-projective curve $B \backslash I$. Remark that $I \cap r(U) = \emptyset$. We will denote by $G_x \subset \mathrm{PGL}_2(\mathbf{C})$ the image of ρ_x. It is a classical Kleinian group and U_x is a finite union of some of its connected components.

Lemma 5.7.2 *Γ_B is a classical Kleinian group and $r(U)$ is an invariant component of its domain of discontinuity.*

Proof The rational fibration r induces a surjective holomorphic map r' from the quotient U/Γ_r to $r(U)$ with compact fibers. The group of deck transformations of the covering $U/\Gamma_r \to X$ is isomorphic to $\Gamma_B = \Gamma/\Gamma_r$. The action of Γ_B on U/Γ_r permutes the fibers of r'. Since the fibers of r' are compact, this action is discrete on the base which is $r(U)$. We have an induced holomorphic map $X = (U/\Gamma_r)/\Gamma_B \to r(U)/\Gamma_B$. Therefore $r(U)/\Gamma_B$ is a compact Riemann surface. □

Denote by C' the quotient compact Riemann surface $r(U)/\Gamma_B$ and by f' the morphism $X \to D$. As each fiber of f' is a union of finitely many leaves of $\mathscr{F}$, there is a morphism $g : C \to C'$ such that $f' = g \circ f$. Since C' is not $\mathbb{P}^1$, C is not $\mathbb{P}^1$ either. Now think of C as an orbicurve with multiple points corresponding to multiple fibers of f. As the underlying compact Riemann surface is not $\mathbb{P}^1$ it is uniformizable as an orbicurve; there is a finite orbifold covering $\bar{C}$ of C which is an orbicurve with no multiple points, i.e. a compact Riemann surface. Using the base change $\bar{C} \to C$, we obtain a commutative diagram

$$\begin{array}{ccc} \bar{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \bar{C} & \longrightarrow & C \end{array}$$

where $\bar{X} \to \bar{C}$ is a fibration without multiple fibers and $\bar{X} \to X$ is un unramified covering (cf. [Bar+04] III.9). Thus replacing Γ with a subgroup of finite index we can and will assume that $f : X \to C$ *has no multiple fibers*. Thus for any $x \in r(U)$, for any connected component Ω of $\mathbb{P}^1_x \cap U$, the group Γ_Ω is identified with the image of $\pi_1(F)$ in Γ where F is any fiber of f. From this we deduce that for any $x \in r(U)$, the finite group Γ_r/Γ_Ω acts freely on the set of connected components of $\mathbb{P}^1_x \cap U$. Note that the fibers of f are the Ω/Γ_Ω and the fibers of f' are the $(\mathbb{P}^1_x \cap U)/\Gamma_r$. This implies that

Lemma 5.7.3 *The morphism $C \to C'$ is an unramified covering. The number of connected components of $\mathbb{P}^1_x \cap U$ does not depend on $x \in r(U)$.*

Remark 5.7.4 We have a fibration $f : X \to C$ without singular nor multiple fibers. If C is elliptic then the fibration is necessarily locally trivial (cf. [Bar+04] III.15.4). If the fibers of f are elliptic or rational then it is also locally trivial (cf. [Bar+04] V). Hence the only case where f is not locally trivial is the case of Kodaira fibrations, where the base and the fibers all have genus ≥ 2.

5.7.2 Representation varieties

We refer to [HP04] for a detailed introduction to $\mathrm{PGL}_2(\mathbf{C})$-character varieties; here we give a brief account for what we need. Let G be a finitely generated abstract group. The set of all representations of G into $\mathrm{PGL}_2(\mathbf{C})$ is denoted by $R(G)$. Given a presentation $G = \langle g_1, \cdots, g_s | h_i, i \in I \rangle$, we have an embedding:

$$\begin{cases} R(G) \to \mathrm{PGL}_2(\mathbf{C}) \times \cdots \times \mathrm{PGL}_2(\mathbf{C}) \\ \rho \to (\rho(g_1), \cdots, \rho(g_s)) \end{cases}.$$

$\mathrm{PGL}_2(\mathbf{C})$ is a complex algebraic group, the product $\mathrm{PGL}_2(\mathbf{C}) \times \cdots \times \mathrm{PGL}_2(\mathbf{C})$ is a complex affine algebraic variety. The image of $R(G)$ in $\mathrm{PGL}_2(\mathbf{C}) \times \cdots \times \mathrm{PGL}_2(\mathbf{C})$ is determined by the ideal generated by the algebraic functions corresponding to the relations $\{h_i\}_{i \in I}$. Therefore $R(G)$ has naturally a structure of complex affine algebraic variety (which in fact does not depend on the choice of the presentation), we call it the $\mathrm{PGL}_2(\mathbf{C})$-*representation variety of* G. The action by conjugation of $\mathrm{PGL}_2(\mathbf{C})$ on $R(G)$ is algebraic with respect to this structure of algebraic variety. The topological quotient $R(G)/\mathrm{PGL}_2(\mathbf{C})$ is usually not Hausdorff, so it is more convenient to consider the quotient $\chi(G) = R(G)//\mathrm{PGL}_2(\mathbf{C})$ in the sense of geometric invariant theory. Since $\mathrm{PGL}_2(\mathbf{C})$ is reductive, $\chi(G)$ is a complex affine variety. Denote by τ the projection morphism $R(G) \to \chi(G)$. We call $\chi(G)$ the $\mathrm{PGL}_2(\mathbf{C})$-*character variety of* G.

A representation $\rho : G \to \mathrm{PGL}_2(\mathbf{C})$ is called irreducible if its action on $\mathbb{P}^1$ does not fix any point. If two representations $\rho, \rho' \in R(G)$ are conjugate, then they are mapped by τ to the same point in $\chi(G)$. The converse is not true in general, but if ρ and ρ' are irreducible and $\tau(\rho) = \tau(\rho')$ then ρ and ρ' are conjugate.

We have a homomorphism $G \to \mathrm{Aut}(G)$ where each element of G acts on G by conjugation; the automorsphism in the image $\mathrm{Inn}(G)$ are called *inner automorphisms*. $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ and the quotient $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the *outer automorphism group* of G and is denoted by $\mathrm{Out}(G)$. An element $\eta \in \mathrm{Aut}(G)$ induces an automorphism of $R(G)$ defined by $\rho \mapsto \rho \circ \eta$; we have thus an action of $\mathrm{Aut}(G)$ on $R(G)$ by automorphisms of algebraic variety. Via $\tau : R(G) \to \chi(G)$ it induces an action of $\mathrm{Out}(G)$ on $\chi(G)$ by automorphisms.

5.7.3 Isotrivial fibrations

Let us return to our group Γ. Recall that Γ_r is the data of an algebraic family of representations $\{\rho_x : \Gamma_r \to \mathrm{Aut}(\mathbb{P}^1_x) = \mathrm{PGL}_2(\mathbf{C})\}_x$ parametrized by the quasiprojective curve $B\backslash I$. In other words Γ_r gives a morphism of algebraic varieties

$$\Phi : B\backslash I \to R(\Gamma_r).$$

From the short exact sequence $1 \to \Gamma_r \to \Gamma \to \Gamma_B \to 1$ we deduce a homomorhism $\Gamma_B \to \mathrm{Out}(\Gamma_r)$, thus an action of Γ_B on $\chi(\Gamma_r)$. Let us describe concretely this action.

For $\gamma \in \Gamma$, denote its image in Γ_B by γ_B. For $x, x' \in B$ such that $\gamma_B(x) = x'$ and such that γ is regular on $r^{-1}(x)$, denote by $\gamma_{xx'}$ the induced map $\mathbb{P}^1_x \to \mathbb{P}^1_{x'}$. Identifying $\mathbb{P}^1_x \to$ with $\mathbb{P}^1_{x'}$ via the projection of $\mathbb{P}^1 \times \mathbb{P}^1$ onto the second factor, we think of the morphism $\gamma_{xx'}$ as an element of $\mathrm{PGL}_2(\mathbf{C})$.

Pick two points $x, x' \in B\backslash I$ such that $\gamma_B(x) = x'$ for some $\gamma \in \Gamma$ and such that γ is regular on $\mathbb{P}^1_x$. We have $\gamma_{xx'}(\rho_x(\Gamma_r))\gamma_{xx'}^{-1} = \rho_{x'}(\Gamma_r)$ but not necessarily $\gamma_{xx'} \circ \rho_x(\alpha) \circ \gamma_{xx'}^{-1} = \rho_{x'}(\alpha)$ for every $\alpha \in \Gamma_r$. In other words γ induces an automorphism $\eta_\gamma \in \mathrm{Aut}(\Gamma_r)$ such that $\rho_x \circ \eta_\gamma = \rho_{x'}$ whenever $x' = \gamma_B(x)$. The image of η_γ in $\mathrm{Out}(\Gamma_r)$ depends only on γ_B. In summary we have

Lemma 5.7.5 *We have an action of Γ_B on $\chi(\Gamma_r)$ by automorphisms such that the morphism $\tau \circ \Phi : B\backslash I \to \chi(\Gamma_r)$ satisfies that for any $\beta \in \Gamma_B$ and $x \in r(U)$ we have $\beta \cdot (\tau \circ \Phi(x)) = \tau \circ \Phi(\beta(x))$.*

Corollary 5.7.6 *The image $\tau \circ \Phi(B\backslash I)$ is either a point or an irreducible algebraic curve in the affine variety $\chi(\Gamma_r)$ which is invariant under the action of Γ_B. If it is a curve then the action of Γ_B on it is faithful.*

Proof Lemma 5.7.5 implies that $\tau \circ \Phi(r(U))$ is a set invariant under Γ_B. It is a Zariski dense subset of $\tau \circ \Phi(B\backslash I)$. Hence the invariance. Recall that we assumed that Γ_B is torsion free. Let β be a non-trivial element of Γ_B. There is a point $x \in r(U)$ which has an infinite β-orbit. If β acts trivially on $\tau \circ \Phi(r(U))$ then by Lemma 5.7.5 the whole β-orbit of x is sent by $\tau \circ \Phi$ to a single point. As $\tau \circ \Phi$ is an algebraic morphism from a curve to another algebraic variety, this is possible only if $\tau \circ \Phi$ is a constant map. $\square$

Lemma 5.7.7 *$\tau \circ \Phi$ is a constant map.*

Proof Assume by contradiction that the morphism $\tau \circ \Phi : B \backslash I \to \chi(\Gamma_r)$ is not constant. Then Γ_B acts faithfully on the irreducible algebraic curve $\tau \circ \Phi(B \backslash I)$. The curve $\tau \circ \Phi(B \backslash I)$ is not projective because $\chi(\Gamma_r)$ is an affine variety. By Lemma 5.7.8 below we infer that the classical Kleinian group Γ_B is solvable. Therefore $C' = r(U)/\Gamma_B$ is necessarily an elliptic curve. Thus by Lemma 5.7.3 C is also an elliptic curve. Consequently the fibration $f : X \to C$ is locally trivial.

Consider the exact sequence $\pi_1(F) \to \pi_1(X) \to \pi_1(C) \to 1$ where F is any fiber of f. If the fibers of f have genus one, then after a finite base change we get a finite unramified cover X' of X which is a principal elliptic bundle and the image of the fundamental group of any fiber is a central subgroup of $\pi_1(X')$ (cf. [Bar+04] V.5). If the fibers of f have genus ≥ 2 then after a finite base change we get a finite unramified cover X' of X which is simply a product (cf. [Bar+04] V.6). Thus the local triviality of f implies that the homomorphism $\pi_1(C) \to \mathrm{Out}(\mathrm{Im}(\pi_1(F) \to \pi_1(X)))$ has finite image. Recall that the image of $\pi_1(F)$ in Γ is Γ_Ω and it is a subgroup of finite index in Γ_r. Hence the homomorphism $\Gamma_B \to \mathrm{Out}(\Gamma_r)$ has also finite image. This contradicts Corollary 5.7.6 that Γ_B acts faithfully on $\tau \circ \Phi(B \backslash I)$. □

Lemma 5.7.8 *The automorphism group of an irreducible quasi-projective curve is infinite and non-solvable if and only if the curve is $\mathbb{P}^1$.*

Proof Let M be an irreducible quasi-projective curve and $\bar{M}$ be its normalization which is a smooth quasi-projective curve. Let $\hat{M}$ be a non-singular compactification of $\bar{M}$; it is a smooth projective curve. Any automorphism of M lifts to $\bar{M}$ and extends to $\hat{M}$. Hence $\mathrm{Aut}(M)$ is a subgroup of the automorphism group of a smooth projective curve. The automorphism group of a hyperbolic curve is finite. The automorphism group of a curve of genus one is an extension of a finite cyclic group by the elliptic curve itself, thus solvable. The automorphism group of $\mathbb{P}^1$ preserving any finite subset is either finite or solvable. Hence if $\mathrm{Aut}(M)$ is infinite and non-solvable then $M = \bar{M} = \hat{M}$. □

Proposition 5.7.9 *1. The fibration $f : X \to C$ is locally trivial.*

2. There exists $\phi \in \mathrm{PGL}_2(\mathbf{C}(x))$ without indeterminacy points over $r(U)$ such that $\phi\Gamma_r\phi^{-1}$ is included in the subgroup $\{\mathrm{Id}\} \times \mathrm{PGL}_2(\mathbf{C})$ of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$.

3. After conjugating by ϕ, $U = r(U) \times D$ where $D \subset \mathbb{P}^1$ is an invariant connected component of the domain of discontinuity of Γ_r viewed as a classical Kleinian group via the second projection.

Proof If the fibers of f are rational curves then it is automatically locally trivial. In this case Γ_r is a trivial group and for any $x \in r(U)$, $\mathbb{P}^1_x \cap U = \mathbb{P}^1_x$.

Assume that the fibers of f have genus ≥ 2. Then for any $x \in r(U)$ the classical Kleinian group $\alpha_x(\Gamma_r)$ is non-elementary. Thus from the previous lemma we deduce that the representations $\alpha_x, x \in r(U)$ are conjugate to each other. The conjugation depends algebraically in x because α_x is an algebraic family. The second assertion of the proposition is rephrasement of this. We can thus assume that $\Gamma_r \subset \{\mathrm{Id}\} \times \mathrm{PGL}_2(\mathbf{C})$. Therefore for any $x \in r(U)$, $\mathbb{P}^1_x \cap U$ is a union of finitely many connected components of a fixed classical Kleinian group. By connectedness of U we infer the third assertion. Also we obtain that $(\mathbb{P}^1_x \cap U)/\Gamma_r$ does not depend on $x \in r(U)$, which implies that f is locally trivial.

From now on we assume that the fibers of f have genus one. As f has no singular nor multiple fibers it is locally trivial. Since the only infinite coverings of an elliptic curve are $\mathbf{C}$ and $\mathbf{C}^*$, we infer that we are in one of the following situations:

1. For any $x \in r(U)$, $\mathbb{P}^1_x \cap U$ is biholomorphic to $\mathbf{C}$; Γ_r is isomorphic to $\mathbf{Z}^2$.
2. For any $x \in r(U)$, $\mathbb{P}^1_x \cap U$ is biholomorphic to $\mathbf{C}^*$; Γ_r is isomorphic to $\mathbf{Z}$.

In both cases we deduce immediately from the local triviality that $\alpha_x, x \in r(U)$ are conjugate to each other. As previously the conjugation depends algebraically on x and we obtain the second assertion. The last assertion follows immediately. □

Proposition 5.7.10 *Suppose that the fibers of f have genus ≥ 2. Then up to geometric conjugation and up to finite index subgroup the birational Kleinian group (Y, Γ, U, X) is such that $Y = \mathbb{P}^1 \times \mathbb{P}^1$, $\Gamma = \Gamma_B \times \Gamma_r$ and $U = D_1 \times D_2$ where D_1, D_2 are respectively connected components of the classical Kleinian groups Γ_B, Γ_r.*

Proof Let F be a fiber of f. Since F has genus ≥ 2, after replacing X with a finite unramified cover we can and will assume that X is a product $F \times C$ and f is the projection onto C (cf. [Bar+04] V.6). In particular $\pi_1(X)$ splits into a product $\pi_1(F) \times \pi_1(C)$. Recall that the image of $\pi_1(F)$ in Γ is exactly Γ_r. Therefore the image of $\pi_1(C)$ in Γ maps isomorphically onto Γ_B so that Γ is actually isomorphic to the product $\Gamma_B \times \Gamma_r$. By Proposition 5.7.9 we can assume that Γ_r a subgroup of $\{\mathrm{Id}\} \times \mathrm{PGL}_2(\mathbf{C})$. We need to prove that $\Gamma_B \subset \mathrm{PGL}_2(\mathbf{C}) \times \{\mathrm{Id}\}$.

Let γ be an element of Γ_B viewed as a subgroup of Γ. As γ commutes with Γ_r, for any $x \in r(U)$, the restriction of γ to $\mathbb{P}^1_x$ realizes a conjugation between α_x and $\alpha_{\gamma_B(x)}$ which are the same representation after identifying $\mathbb{P}^1_x$ with $\mathbb{P}^1_{\gamma_B(x)}$. In other words for any $x \in r(U)$ the birational transformation γ realizes a self-conjugation of the classical Kleinian group Γ_r. However a non-elementary classical Kleinian group has no non-trivial self-conjugations (the limit set has more than three points and is fixed by a self-conjugation). The conclusion follows. □

Proposition 5.7.11 *Suppose that the fibers of f have genus* 1. *Then up to geometric conjugation and up to finite index subgroup the birational Kleinian group (Y,Γ,U,X) is in one of the two following situations*

1. $Y=\mathbb{P}^1\times\mathbb{P}^1$, $U=D_1\times\mathbf{C}^*$ *and D_1 is a connected component of the classical Kleinian group Γ_B. Γ is a subgroup of* $J^+ := \{(x,y) \dashrightarrow (\eta(x),R(x)y) \mid \eta\in \mathrm{PGL}_2(\mathbf{C}), R\in\mathbf{C}(x)^*\} = \mathbf{C}(x)^*\rtimes \mathrm{PGL}_2(\mathbf{C})$. $\Gamma_r\subset\{\mathrm{Id}\}\times\mathrm{PGL}_2(\mathbf{C})$ *is a cyclic central subgroup of* Γ.
2. $Y=\mathbb{P}^1\times\mathbb{P}^1$, $U=D_1\times\mathbf{C}$ *and D_1 is a connected component of the classical Kleinian group Γ_B. Γ is a subgroup of* $J^* := \{(x,y) \dashrightarrow (\eta(x),y+R(x)) \mid \eta\in \mathrm{PGL}_2(\mathbf{C}), R\in\mathbf{C}(x)\} = \mathbf{C}(x)\rtimes \mathrm{PGL}_2(\mathbf{C})$. $\Gamma_r\subset\{\mathrm{Id}\}\times\mathrm{PGL}_2(\mathbf{C})$ *is a cyclic central subgroup of Γ isomorphic to* $\mathbf{Z}^2$.

Proof A finite unramified cover of X is a principal elliptic bundle (cf. [Bar+04] V.5); up to replacing X with this cover we have that Γ_r is central in Γ. We need to show that Γ is included in J^+ or J^*; other assertions follow from Proposition 5.7.9. This is because Γ_r is central in Γ; the centralizer of Γ_r is respectively J^+ or J^* by Thereom 1.5.1. □

Let us see two examples where the group Γ is not conjugate to a subgroup of $\mathrm{PGL}_2(\mathbf{C})\times\mathrm{PGL}_2(\mathbf{C})$.

Example 5.7.12 Consider the subgroup Γ of $\mathrm{Aff}_2(\mathbf{C})$ generated by the following four elements

$$(x,y)\mapsto(x,y+1),\quad (x,y)\mapsto(x,y+a)$$
$$(x,y)\mapsto(x+1,y+bx),\quad (x,y)\mapsto(x+c,y+dx)$$

where $a,c\in\mathbf{C}^*$ and $b,d\in\mathbf{C}$ are such that $bd\in\mathbf{C}^*$. The quotient $\mathbf{C}^2/\Gamma$ is a primary Kodaira surface.

Example 5.7.13 Consider the group Γ generated by the following two elements

$$\begin{aligned}&(x,y)\mapsto(x,ay)\\&(x,y)\mapsto(bx,cx^ky),\quad a,b,c\in\mathbf{C}^*,k\in\mathbf{Z}^*\ |a|,|b|\neq 1.\end{aligned}$$

By lifting the deck transformations to $\mathbf{C}^2$ we see that the quotient $\mathbf{C}^*\times\mathbf{C}^*/\Gamma$ is a primary Kodaira surface. Remark that Γ is included in the toric subgroup and that Γ is not an elliptic subgroup because $(x,y)\mapsto(bx,cx^ky)$ has linear degree growth.

We reformulate Proposition 5.7.9 in the case of rational fibrations; we cannot say much more about this situation.

Proposition 5.7.14 *Suppose that the fibers of f are rational. Then up to geometric conjugation the birational Kleinian group (Y,Γ,U,X) is such that $Y=\mathbb{P}^1\times\mathbb{P}^1$, $U=D_1\times\mathbb{P}^1$, Γ is isomorphic to Γ_B and D_1 is a connected component of the classical Kleinian group Γ_B.*

Example 5.7.15 Consider the group Γ generated by

$$(x,y)\mapsto(bx,cx^k y),\quad b,c\in\mathbf{C}^*,k\in\mathbf{Z}^*\ |b|\neq 1.$$

The quotient $\mathbf{C}^*\times\mathbb{P}^1/\Gamma$ is a geometrically ruled surface over an elliptic curve; it is a decomposable ruled surface because of the two disjoint sections $\{y=0\},\{y=\infty\}$. This is our only example of birational Kleinian group for which Γ is non-elliptic and X is Kähler.

5.8 Complex tori

Proposition 5.8.1 *Assume that $B=\mathbb{P}^1$, Γ_B is infinite and X is a complex torus. Then up to geometric conjugation the birational Kleinian group (Y,Γ,U,X) satisfies that*

— *$Y=\mathbb{P}^1\times\mathbb{P}^1$. U is one of the three Zariski open sets: $\mathbf{C}^2$, $\mathbf{C}\times\mathbf{C}^*$ or $\mathbf{C}^*\times\mathbf{C}^*$.*

— *Γ can be identified with a lattice in U with its natural structure of algebraic group, i.e. the elements of Γ have respectively the form $(x,y)\mapsto(x+a,y+b)$, $(x,y)\mapsto(x+a,by)$ or $(x,y)\mapsto(ax,by)$.*

Proof The fundamental group of a complex torus is isomorphic to $\mathbf{Z}^4$. After replacing Γ with a subgroup of finite index we can and will assume that Γ and Γ_B are free abelian groups. Remark that the only regular foliations on complex tori are turbulent foliations and linear foliations and we know by Proposition 5.5.7 that turbulent foliations on tori do not appear in our study.

Assume that the $\mathscr{F}$ is a linear foliation, not necessarily irrational. A linear foliation on $\mathbf{C}^2$ is induced by the projection of $\mathbf{C}^2$ onto some subvector-space of dimension one. The induced action of $\pi_1(X)$ on that subspace, which is isomorphic to $\mathbf{C}$, is by translations; the image of $\pi_1(X)$ is a free abelian group of rank $2,3$ or 4, depending on whether the leaves of $\mathscr{F}$ are elliptic curves, $\mathbf{C}^*$ or $\mathbf{C}$. As in the previous case, we obtain a surjective holomorphic map $\mathbf{C}\to r(U)$ that is equivariant with respect to the homomorphism $\pi_1(X)\to\Gamma_B$. This forces $r(U)$ to be $\mathbf{C}$ or $\mathbf{C}^*$ and that Γ_B is respectively a discrete group of translations or multiplications. In particular every point of $r(U)$ has an infinite Γ_B-orbit. Thus r does not have singular fibers over $r(U)$ because by

Corollary 5.2.3 Γ acts by diffeomorphisms on $r^{-1}(r(U))$. Consequently U is a $\mathbf{C}$ or $\mathbf{C}^*$ bundle over $\mathbf{C}$ or $\mathbf{C}^*$. This implies that U is biholomorphic to $\mathbf{C}^2$, $\mathbf{C}\times\mathbf{C}^*$ or $\mathbf{C}^*\times\mathbf{C}^*$, though we do not know yet if U is the corresponding standard Zariski open set; in other words we know that $r^{-1}(r(U))\backslash U$ is one or two holomorphic sections of r over $r(U)$ but we do not know if these sections are algebraic. However we do know now that Γ is a free abelian group of rank ≥ 2.

We assume firstly that Γ_B is free of rank ≥ 2. In this case the descriptions of Γ and U given in the statement follows from Theorem 1.5.3; we only need to observe that the change of coordinates that we need to write the transformations in the adequate forms is a geometric conjugation of birational Kleinian groups. This is because a change of coordinates is a sequence of elementary transformations. If we let the elementary transformations to be done outside U, then the conjugation is geometric. We make also a remark on the hypothesis that the rank of Γ_B is ≥ 2: it is always satisfied when $\mathscr{F}$ is irrational because the action of Γ_B on $r(U)$ is not discrete.

Consider now the case where Γ_B is cyclic. In this case $r(U)$ is necessarily $\mathbf{C}^*$, $\mathscr{F}$ is an elliptic fibration and Γ_r, the subgroup of Γ preserving fiberwise the rational fibration, is a free abelian group of rank 1 or 2. If Γ_r is of rank 1 then $r^{-1}(x)\cap U=\mathbf{C}^*$ for any $x\in r(U)$ and Γ_r acts by multiplications in the fibers; if Γ_r is of rank 2 then $r^{-1}(x)\cap U=\mathbf{C}$ for any $x\in r(U)$ and Γ_r acts by translations in the fibers.

Firstly look at the case where Γ_r is of rank 1. By Theorem 1.5.3 up to conjugation Γ_r is generated by an element $\gamma_1:(x,y)\mapsto(x,ay)$ and Γ is generated by γ_1 and $\gamma_2:(x,y)\mapsto(bx,R(x)y)$ with $R\in\mathbf{C}(x)^*$; the conjugation being a sequence of elementary transformations can be done outside U so it is a conjugation of birational Kleinian groups. Therefore U is the standard Zariski open set $\mathbf{C}^*\times\mathbf{C}^*$. As γ_2 acts in a regular way over $r(U)=\mathbf{C}^*$, the rational function R has no zeros nor poles over $\mathbf{C}^*$. Thus $R(x)=x^k$ for some $k\in\mathbf{Z}$. If $k\neq 0$ then the quotient $\mathbf{C}^*\times\mathbf{C}^*$ would be a primary Kodaira surface (cf. Example 5.7.13). Hence the conclusion.

Finally look at the case where Γ_r is of rank 2. By Theorem 1.5.3 up to conjugation Γ_r is generated by two elements $\gamma_j:(x,y)\mapsto(x,y+a_j)$, $j=1,2$ and Γ is generated by γ_1,γ_2 and $\gamma_3:(x,y)\mapsto(bx,y+R(x))$ with $R\in\mathbf{C}(x)$; the conjugation being a sequence of elementary transformations can be done outside U so it is a geometric conjugation of birational Kleinian groups. Therefore U is the standard Zariski open set $\mathbf{C}^*\times\mathbf{C}$. As γ_2 acts in a regular way over $r(U)=\mathbf{C}^*$, the rational function R has no poles over $\mathbf{C}^*$. Thus $R(x)=\frac{Q(x)}{x^k}$ for some $k\in\mathbf{Z}$ and

$Q \in \mathbf{C}[x]$. The iterates of γ_3 can be written as

$$\gamma_3^n : (x,y) \mapsto (b^n x, y + S(x)) \quad \text{where} \quad S(x) = \frac{1}{x^k} \sum_{j=0}^{n-1} \frac{Q(bx)}{b^k}.$$

Thus γ_3 has bounded degree growth and is an elliptic element. By conjugating Γ with elements of the form $(x,y) \mapsto (x, y + \frac{P(x)}{x^k})$ with $P \in \mathbf{C}[x]$ we do not change γ_1, γ_2 and can put γ_3 in the form $(x,y) \mapsto (bx,y)$. Hence the conclusion. $\square$

CHAPTER 6

SYNTHESIS

In this chapter we explain how what we have proved so far can be assembled into Theorem 0.3.2 and Theorem 0.3.7. It suffices to prove Theorem 0.3.7 by Remark 0.3.9.

Now we begin the proof of Theorem 0.3.7. Let (Y,Γ,U,X) be a birational Kleinian group in dimension two. By Theorem 2.1.2 Y is a birational to a ruled surface.

6.1 Non-rational ruled surfaces

Suppose that Y is not rational. Then there is a rational fibration $r : Y \to B$ onto a curve B of genus ≥ 1. The fibration r is induced by the Albanese morphism of Y. Thus every birational transformation of Y preserves the fibration r. Therefore we are in the setting of Chapter 5 and Γ induces an action $\Gamma_B \subset \mathrm{Aut}(B)$ on the base. If Γ_B is a finite group then all possibilities are classified by Theorem 5.3.1; this gives the first three cases in Theorem 0.3.7. If Γ_B is infinite then B is an elliptic curve and all possibilities are classified by Theorem 5.4.2; this gives cases 1), 4), 5) and 6) in Theorem 0.3.7.

6.2 Rational surfaces

Now we assume that Y is a rational surface. We apply Theorem 1.4.1 to Γ. There are eight cases in Theorem 1.4.1, let us look at them case by case. In the first subsection we deal with the cases where a pencil of rational curves is preserved by Γ. We deal with the remaining cases in the second subsection.

6.2.1 Rational fibrations

Assume that Γ preserves a pencil of rational curves. This hypothesis includes Cases 2), 3) and partially case 1) of Theorem 1.4.1.

Lemma 6.2.1 *Suppose $\Gamma \subset \mathrm{Bir}(Y)$ preserves a pencil of rational curves on Y. Then the birational Kleinian group is birationally conjugate to (Y', Γ', U', X) such that the pencil of rational curves becomes a regular rational fibration of Y' onto $\mathbb{P}^1$ and Γ' preserves this fibration.*

Proof Since the action of Γ on U is free, the set of base points of the pencil does not intersect U. We can blow up the base points to get the fibration. $\square$

Therefore we can and will assume that Γ preserves a rational fibration $r : Y \to B$ where $B = \mathbb{P}^1$ because Y is rational. Hence we are in the setting of Chapter 5. As always we denote by $\Gamma_B \subset \mathrm{Aut}(B)$ the action of Γ on B. By Proposition 6.2.1 the rational fibration r induces on X a regular holomorphic foliation $\mathscr{F}$.

If Γ_B is a finite group then all possibilities are classified by Theorem 5.3.1; this gives the first three cases in Theorem 0.3.7. In this case the foliation $\mathscr{F}$ is a fibration.

Assume now that Γ_B is infinite. We distinguish two cases, the first one where $r(U) = B$ and the second one where $r(U)$ is a proper subset of B.

The case $r(U) = B$

If $r(U) = B$ then by Proposition 5.5.6 the foliation $\mathscr{F}$ is a turbulent foliation, an obvious foliation on a Hopf surface or a suspension of $\mathbb{P}^1$.

Assume that $\mathscr{F}$ is a turbulent foliation. Then by Proposition 5.5.7, up to taking a subgroup of finite index of Γ, we are in Examples 5.5.2 and 5.5.4. In the situation of these two examples the quotient X is a Hopf surface or a geometrically ruled surface over an elliptic curve. If X is a Hopf surface then this gives Case 15) of Theorem 0.3.7. If X is a geometrically ruled surface over an elliptic curve, then the situations described in Examples 5.5.2 and 5.5.4 are particular cases of Case 20) in Theorem 0.3.7.

Assume that $\mathscr{F}$ is an obvious foliation on a Hopf surface. Then Proposition 5.5.8 classifies all possibilities. There are three cases in Proposition 5.5.8. The last case does not satisfy $r(U) = B$. The first two cases satisfy $r(U) = B$ and they correspond to Cases 15) and 16) of Theorem 0.3.7.

Assume that $\mathscr{F}$ is a suspension of $\mathbb{P}^1$. Then Proposition 5.5.9 says that we are in Case 20) of Theorem 0.3.7.

The case $r(U) \neq B$

Assume now that $r(U) \neq B$.

If $\mathscr{F}$ is an obvious foliation on an Inoue surface then Theorem 5.5.1 says that we are in Case 12) of Theorem 0.3.7.

If $\mathscr{F}$ is an obvious foliation on a Hopf surface then Proposition 5.5.8 classifies all possibilities; only the last case of Proposition 5.5.8 satisfies $r(U) \neq B$ and it corresponds to Case 11) in Theorem 0.3.7.

By Propositions 5.5.8 and 5.5.9 the fact that $r(U) \neq B$ implies that $\mathscr{F}$ is not a turbulent foliation, nor a suspension of $\mathbb{P}^1$.

If $\mathscr{F}$ is a suspension of an elliptic curve, then Proposition 5.6.5 says that the situation is a particular case of Case 18) or 19) of Theorem 0.3.7. Here Γ is a group of automorphisms. However note that in general Γ is not necessarily a group of automorphisms in Cases 18) and 19) of Theorem 0.3.7.

Assume that $\mathscr{F}$ is a fibration. If the genus of a fiber is ≥ 2 then Proposition 5.7.10 says that we are in Case 17) of Theorem 0.3.7. If the fibers are elliptic curves then Proposition 5.7.11 says that we are in Cases 18) and 19) of Theorem 0.3.7. If the fibers are $\mathbb{P}^1$ then Proposition 5.7.14 says that we are in Case 20) of Theorem 0.3.7.

If $\mathscr{F}$ is an irrational linear foliation on a complex torus, then Proposition 5.8.1 says that we are in Cases 7), 8) or 9) of Theorem 0.3.7.

According to Theorem 5.1.1 the only remaining possibility for $\mathscr{F}$ is a minimal transversely hyperbolic regular foliation on a surface of general type. This case is studied in Chapter 7 and the description given in Theorems 0.3.2 and 0.3.7 will follow from Proposition 7.2.8.

6.2.2 Other cases in strong Tits alternative

We dealt with the case where Γ preserves a rational fibration in the previous subsection and now we consider other possibilities for Γ, i.e. Cases 1), 4), 5), 6) and 7) of Theorem 1.4.1. Note that we do not need to consdier Case 8) of Theorem 1.4.1 because being a quotient of the fundamental group of a compact manifold, Γ is finitely generated.

Genus one fibrations. Let us consider the fourth case of Theorem 1.4.1. There is a ratioal surface Y' and a birational map $\phi : Y \to Y'$ such that $\Gamma' = \phi\Gamma\phi^{-1}$ acts by automorphisms and preserves a genus one fibration. By Lemma 4.4.2 we can choose Y' and ϕ so that the conjugation ϕ is a geometric conjugation of birational Kleinian groups. Then Corollary 2.3.4 says that this case is impossible.

Elliptic groups. Let us consider the first case of Theorem 1.4.1. In this case Γ is an elliptic subgroup, i.e. there is a projective surface Y' and a birational map $\phi : Y \to Y'$ such that $\Gamma' = \phi\Gamma\phi^{-1}$ is a group of automorphisms of Y' and a finite index subgroup of Γ' is in $\mathrm{Aut}^0(Y')$. By Lemma 4.4.2 we can choose Y' and ϕ so that the conjugation ϕ is a geometric conjugation of birational Kleinian groups. Thus we can and will assume that $\Gamma \subset \mathrm{Aut}^0(Y)$.

Assume that Y is not $\mathbb{P}^2$. Then there is a birational morphism $\varphi : Y \to Y'$ to a Hirzebruch surface Y' and we have $\varphi \circ \mathrm{Aut}^0(Y) \circ \varphi^{-1} \subset \mathrm{Aut}^0(Y')$. Since the Aut^0 of a Hirzebruch surface preserves a rational fibration, Γ preserves a rational fibration on Y. Hence we reduce the situation to the case where a rational fibration is preserved and where the classification is already done.

In the remaining case $Y = \mathbb{P}^2$ and $\Gamma \subset \mathrm{PGL}_3(\mathbf{C}) = \mathrm{Aut}(\mathbb{P}^2)$. This is exactly the case of complex projective Kleinian groups. The complete classification is given by Theorem 3.1.1; this gives Cases 7)–13) in Theorem 0.3.7.

Toric subgroup. Proposition 4.5.2 says that if Γ is conjugate in $\mathrm{Bir}(Y)$ to a subgroup of the toric subgroup then it contains no loxodromic elements. Thus Case 6) of Theorem 1.4.1 is impossible.

Non-elementary subgroups. Theorem 4.1.1 says that Γ is not a non-elementary subgroup of $\mathrm{Bir}(Y)$ under the hypothesis that X is not of class VII, thus rules out Case 7) of Theorem 1.4.1.

Virtually cyclic subgroups. The author is not able to handle Case 5) of Theorem 1.4.1 and we put it as a hypothesis in Theorem 0.3.7.

CHAPTER 7

MINIMAL TRANSVERSELY HYPERBOLIC FOLIATIONS

In this chapter we deal with a birational Kleinian group (Y,Γ,U,X) such that Γ preserves a rational fibration $r : Y \to B$ with $B = \mathbb{P}^1$ which induces a minimal transversely hyperbolic regular foliation on X. The first section is a state-of-the-art introduction to these foliations; it is completely independent and some of the results are not used in the sequel. From the second section we work with the birational Kleinian group (Y,Γ,U,X). We will show that a part of the boundary of U in Y is the graph of a holomorphic motion, i.e. is laminated by holomorphic disks. Everything in the rest of this chapter is based upon this structure of holomorphic motion. There will be two types of results: 1) under some additional hypothesis U is biholomorphic to the bidisk (see Theorems 7.4.10, 7.4.23); 2) under the hypothesis that U is a bidisk, we try to rigidify the dynamics of Γ (see Theorems 7.4.21, 7.4.23). Preliminaries on holomorphic motions and on Teichmüller theory will be recalled.

7.1 Corlette-Simpson and Brunella-McQuillan

We introduced minimal transversely hyperbolic regular foliations on general type surfaces in Section 5.1. We explained that examples of such foliations are given by quotients of $\mathbb{D}\times\mathbb{D}$ by torsion free irreducible cocompact lattices in $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$ and that no other examples are known. The two natural foliations on a quotient of the bidisk will be called *tautological foliations*. We will call *a Brunella exotic foliation* a minimal transversely hyperbolic regular foliation on a general type surface which is not a tautological foliation on a bidisk quotient. In this section we collect some known properties of these foliations. Two important results are Theorem 7.1.11 which says that Brunella exotic foliations are similar to bidisk quotients in some sense and Theorem 7.1.13 which says that they are drastically different in some other sense.

Let X be a smooth complex projective surface of general type equipped with a transversely hyperbolic regular foliation $\mathscr{F}$ which is not a foliation induced by a fibration. Denote by $\tilde{X}$ the universal cover of X. The pioneer work of Brunella asserts:

Theorem 7.1.1 (Brunella [Bru97]) *The canonical bundle of X is ample. The foliation $\mathscr{F}$ is minimal, i.e. every leaf of $\mathscr{F}$ is dense in X. There is a homomorphism $\rho : \pi_1(X) \to \mathrm{PSL}_2(\mathbf{R})$ and a ρ-equivariant holomorphic map $s : \tilde{X} \to \mathbb{D}$ such that*

— *every fiber of s is biholomorphic to $\mathbb{D}$;*

— *$\mathscr{F}$ is induced by s.*

and

Theorem 7.1.2 (Brunella [Bru97]) *If X is a bidisk quotient, then $\mathscr{F}$ is one of the two tautological foliations. We have always $c_1(X)^2 \geq 2c_2(X)$. If the equality holds then X is a bidisk quotient.*

Proposition 7.1.3 *X and $\tilde{X}$ are Kobayashi hyperbolic manifolds.*

Proof A complex manifold is Kobayashi hyperbolic if and only if its universal cover is (cf. [Kob98]). Thus it suffices to prove that X is Kobayashi hyperbolic. Since X is compact, Brody Lemma (cf. [Bro78]) says that it is Kobayashi hyperbolic if and only if there are no non-constant holomorphic map from $\mathbf{C}$ to X. Suppose by contradiction that $\xi : \mathbf{C} \to X$ is a non-constant holomorphic map. Then it lifts to a non-constant holomorphic map $\tilde{\xi} : \mathbf{C} \to \tilde{X}$. The composition $s \circ \tilde{\xi}$ has values in $\mathbb{D}$, thus is constant. Therefore the image of $s \circ \tilde{\xi}$ is contained in a fiber of s. But as a fiber of s is also a disk, $s \circ \tilde{\xi}$ must be constant, contradiction. □

The fact that every leaf of $\mathscr{F}$ is dense implies that the $\rho(\pi_1(X))$-orbit of every point in $\mathbb{D}$ is dense in $\mathbb{D}$. Thus we have:

Proposition 7.1.4 *The image of ρ does not factorize through a hyperbolic orbicurve and is dense in $\mathrm{PSL}_2(\mathbf{R})$ with respect to the usual topology.*

Proof We apply Theorem 4.2.1 to ρ; either it factorizes through a hyperbolic orbicurve or it has dense image in $\mathrm{PSL}_2(\mathbf{R})$. Thus it suffices to prove that ρ does not factorize through a hyperbolic orbicurve. Suppose by contradiction that there is a fibration $f : X \to \Sigma$ and a homomorphism $\eta : \pi_1^{orb}(\Sigma) \to \mathrm{PSL}_2(\mathbf{R})$ such that $\rho = \eta \circ f_*$. Let F be a general fiber of f. It is a compact Riemann surface of genus ≥ 2 because X is of general type; its universal cover is $\mathbb{D}$. Let $g : \mathbb{D} \to \tilde{X}$ be a map that lifts the embedding $F \to X$. By our hypothesis the composition $\pi_1(F) \to \pi_1(X) \xrightarrow{\rho} \mathrm{PSL}_2(\mathbf{R})$ is trivial. Thus $g(\mathbb{D})$ is included in a fiber of $r : \tilde{X} \to \mathbb{D}$. This implies that $g(\mathbb{D})$ is a fiber of r and F is a leaf of $\mathscr{F}$, contradiction. □

A little lemma that we will use several times is

Lemma 7.1.5 *A dense torsion free subgroup of* $\mathrm{PSL}_2(\mathbf{R})$ *contains elements of infinite order.*

Proof This follows from the fact that being a non trivial elliptic element is an open condition in $\mathrm{PSL}_2(\mathbf{R})$. Indeed if an element is represented by a matrix in $\mathrm{SL}_2(\mathbf{R})$ then the element is elliptic if and only the trace of the matrix has absolute value < 2. $\square$

Known results. The results that we will present in the rest of this section will not be directly used in the sequel. A reader who is mainly interested in the results of Section 0.3 can skip directly to the next section.

We firstly mention a theorem describing the group of biholomorphisms of $\tilde{X}$.

Theorem 7.1.6 (Nadel [Nad90], Frankel [Fra95]) *If Y is a smooth complex projective variety with ample canonical bundle and if the universal cover $\tilde{Y}$ is not a product of a bounded hermitian symmetric domain with some other manifold, then $\pi_1(Y)$ has finite index in the group of self-biholomorphisms of $\tilde{Y}$.*

Non-abelian Hodge theory gives very strong constraints on representations of Kähler groups into $\mathrm{PGL}_2(\mathbf{C})$. We introduce first some terminology. Let $n \geq 2$ and Λ be a torsion free irreducible lattice in a product of n copies of $\mathrm{PSL}_2(\mathbf{R})$. Then the quotient of a product of n disks by Λ is a quasi-projective variety Z of dimension n by Baily-Borel's theorem [BB66]; we will call Z a *polydisk Shimura variety*. The fundamental group of Z is identified with Γ and has n natural faithful representations into $\mathrm{PSL}_2(\mathbf{R})$ with dense images induced by projections of $\mathrm{PSL}_2(\mathbf{R}) \times \cdots \times \mathrm{PSL}_2(\mathbf{R})$ onto its factors. These n representations $\tau_i, 1 \leq i \leq n$ of $\pi_1(Z)$ are called *tautological representations*. There are n regular codimension 1 foliations on Z induced by the product structure of the polydisk; we call them *tautological foliations* and denote them by $\mathscr{G}_i, 1 \leq i \leq n$. The i-th tautological representation is the holonomy of the i-th tautological foliation.

Theorem 7.1.7 (Corlette-Simpson [CS08]) *Let Y be a smooth complex projective variety. Let $\varphi : \pi_1(Y) \to \mathrm{PGL}_2(\mathbf{C})$ be a representation with Zariski dense image. Then up to replacing Y with a finite unramified cover there are two possibilities:*

1. *There is a fibration $f : Y \to C$ onto a compact hyperbolic Riemann surface C and a representation $\eta : \pi_1(C) \to \mathrm{PGL}_2(\mathbf{C})$ such that $\varphi = \eta \circ f_*$.*
2. *There is a morphism $f : Y \to Z$ to a polydisk Shimura variety Z and a tautological representation $\tau_i : \pi_1(Z) \to \mathrm{PSL}_2(\mathbf{R})$ such that $\varphi = \tau_i \circ f_*$.*

Remark 7.1.8 The two cases in the above theorem are not mutually exclusive. In the second case the polydisk Shimura variety is not necessarily projective; a priori neither Z nor f are uniquely determined by φ.

Remark 7.1.9 The theorem still holds when Y is a smooth quasiprojective variety by [CS08], [LPT16]. The projective case we use here was known before [CS08]; a proof can be obtained by combining [Cor88], [Sim92], [Sim91], [GS92].

Remark 7.1.10 $\mathbb{P}^1(\mathbf{C})$ is the boundary of the three dimensional real hyperbolic space $\mathbb{H}^3$ and the action of $\mathrm{PGL}_2(\mathbf{C})$ on $\mathbb{P}^1(\mathbf{C})$ is induced by the isometric action of $\mathrm{PGL}_2(\mathbf{C})$ on $\mathbb{H}^3$. We can compare Theorem 7.1.7 with Theorem 4.2.1. Note that the representation is minimal in the sense of hyperbolic geometry if and only if it is Zariski dense with respect to the real algebraic structure of $\mathrm{PGL}_2(\mathbf{C})$; a complex Zariski dense subgroup of $\mathrm{PGL}_2(\mathbf{C})$ may not be minimal, for example $\mathrm{PSL}_2(\mathbf{R})$.

As $\mathrm{PSL}_2(\mathbf{R}) \subset \mathrm{PGL}_2(\mathbf{C})$ we will consider $\rho : \pi_1(X) \to \mathrm{PSL}_2(\mathbf{R})$ also as a representation into $\mathrm{PGL}_2(\mathbf{C})$. Since its image is dense in $\mathrm{PSL}_2(\mathbf{R})$ with respect to the usual topology, it is dense in $\mathrm{PGL}_2(\mathbf{C})$ with respect to the Zariski topology of the complex variety $\mathrm{PGL}_2(\mathbf{C})$. Hence we can apply Corlette-Simpson's theorem to our representation ρ. Since ρ does not factorize through a curve by Proposition 7.1.4, it factorizes through a polydisk Shimura variety Z with its i-th tautological representation τ_i. It is only natural to observe that:

Theorem 7.1.11 (Touzet [Tou16]) *Up to replacing X with a finite unramified cover, there is a morphism $f : X \to Z$ such that $\rho = \tau_i \circ f_*$ and the foliation $\mathscr{F}$ is induced by the corresponding tautological foliation on Z, i.e. $\mathscr{F} = f^*\mathscr{G}_i$.*

Proof The above discussion gives us the existence of a morphism $g : X \to Z$ such that $\rho = \tau_i \circ g_*$. This morphism may not be unique and it may not satisfy $\mathscr{F} = g^*\mathscr{G}_i$; we will show that there is at least one with desired properties. The morphism lifts to a holomorphic map $(g_1, \cdots, g_n) : \tilde{X} \to \mathbb{D} \times \cdots \times \mathbb{D}$ which is equivariant; here the g_i are holomorphic functions on $\tilde{X}$ with values in $\mathbb{D}$. Recall that s is the fibration $\tilde{X} \to \mathbb{D}$. We define a new holomorphic map $\tilde{f} = (g_1, \cdots, g_{i-1}, s, g_{i+1}, \cdots, g_n) : \tilde{X} \to \mathbb{D} \times \cdots \times \mathbb{D}$. This map $\tilde{f}$ is also equivariant because $\rho = \tau_i \circ g_*$. Thus it descends to a morphism $f : X \to Z$. By construction it satisfies $\mathscr{F} = f^*\mathscr{G}_i$.□

Remark 7.1.12 By Margulis superrigidity there are no homomorphisms with infinite image from an irreducible lattice in $(\mathrm{PSL}_2(\mathbf{R}))^k$ to an irreducible lattice in $(\mathrm{PSL}_2(\mathbf{R}))^l$ if $l, k \geq 2$ and $l \neq k$. Therefore when X is a bidisk quotient, the polydisk Shimura variety Z must be X itself. This recovers the fact that bidisk quotients and Brunella exotic foliations do not mix.

The tangent bundle $T_{\mathscr{F}}$ of the foliation $\mathscr{F}$ is the sub-line bundle of the tangent bundle of X defined by the directions tangent to the foliation $\mathscr{F}$. The canonical bundle $K_{\mathscr{F}}$ of $\mathscr{F}$ is the dual of $T_{\mathscr{F}}$. The canonical bundle of a singular foliation can be easily defined in the same way and there is a theory of minimal model program for foliations on surfaces established by Brunella and McQuillan. We refer to [Bru15] for the foliated MMP theory. Here we just mention that bidisk quotients play a special role in the theory: compact bidisk quotients or compactification of quasiprojective bidisk quotients are the only singular foliations on projective surfaces that violate the abundance principle. We only state the result for our regular foliation $\mathscr{F}$ on X:

Theorem 7.1.13 (Brunella-McQuillan [Bru03], [McQ08]) — *If X is a bidisk quotient, then $K_{\mathscr{F}}$ is nef and not numerically trivial. The Kodaira dimension of $K_{\mathscr{F}}$ is $-\infty$, i.e. for any $m \in \mathbf{N}^*$, $K_{\mathscr{F}}^{\otimes m}$ has no sections.*
— *If X is a Brunella exotic foliation, then $K_{\mathscr{F}}$ is nef and big, i.e. $K_{\mathscr{F}}$ is nef, has Kodaira dimension 2 and $K_{\mathscr{F}} \cdot K_{\mathscr{F}} > 0$.*

plurisubharmonic variation of leafwise Poincaré metrics. Let V be a small open subset of X where $\mathscr{F}$ is generated by a holomorphic vector field v. This vector field induces a local trivialization of $T_{\mathscr{F}}$ thus $K_{\mathscr{F}}$ over V. For $p \in V$ let L_p be the leaf through p and denote by $\|v(p)\|_{poin}$ the norm of the vector $v(p)$ with respect to the Poincaré metric on L_p. One of the main ingredient of Thereom 7.1.13 is the following result.

Theorem 7.1.14 ([Bru03]) *The function $h(p) = \log \|v(p)\|_{poin}$ is continuous and plurisubharmonic.*

Thus the function h is the local weight of a singular hermitian metric on $K_{\mathscr{F}}$ (cf. [Dem92] for singular hermitian metrics), in the local trivialization induced by v. The curvature of the metric can be locally written as $\Omega = \frac{-1}{2\pi i}\partial\overline{\partial}h$. The function h is pluriharmonic if and only if Ω is a closed positive current. The Chern class of $K_{\mathscr{F}}$ is the de Rham cohomology class of Ω. A reformulation of Theroem 7.1.14 is

Theorem 7.1.15 ([Bru03]) *The Poincaré metrics on the leaves of $\mathscr{F}$ induce a singular Hermitian metric on $K_{\mathscr{F}}$; the curvature of this current is a closed positive current.*

Note that the existence of a singular Hermitian metric whose curvature is a closed positive current is just a consequence of the fact that $K_{\mathscr{F}}$ is nef (cf. [Dem92]); the feature of the above theorem is to construct such a metric in a natural way from the geometry of the foliation.

7.2 Holomorphic motions

7.2.1 Holomorphic motions and λ-Lemma

We refer to [AM01] and [Dou95] for more details on holomorphic motions.

A real valued function defined on an interval $[a,b]$ is *absolutely continuous* if there is a Lebesgue integrable function g on $[a,b]$ such that $f(x) = f(a) + \int_a^x g(t)dt$ for all $x \in [a,b]$. A continuous map $f : A \to \mathbf{C}$ defined on a domain $A \subset \mathbf{C}$ is *absolutely continuous on lines (ACL)* if for each closed rectangle $\{x+iy | a \leq x \leq b, c \leq y \leq d\} \subset A$, the function $x \mapsto f(x+iy)$ is absolutely continuous on $[a,b]$ for almost all $y \in [a,b]$ and $y \mapsto f(x+iy)$ is absolutely continuous on $[c,d]$ for almost all $x \in [a,b]$. As an absolutely function is almost everywhere differentiable, an ACL map has partial derivatives almost everywhere. Let $\kappa \in [1,+\infty)$. A sense-preserving continuous map $f : A \to \mathbf{C}$ which is homeomorphic onto the image is κ-*quasiconformal* if 1) f is ACL; 2) $|\partial f(z)| + |\overline{\partial} f(z)| \leq \kappa \Big(|\partial f(z)| - |\overline{\partial} f(z)| \Big)$ almost everywhere. If $f : A \to \mathbf{C}$ is quasiconformal then its *complex dilatation*, or its *Beltrami differential*, is the Borel measurable function $\mu_f : z \mapsto \frac{\overline{\partial} f(z)}{\partial f(z)}$. If f is κ-quasiconformal then $|\mu_f(z)| \leq \frac{\kappa-1}{\kappa+1} < 1$. The *dilatation* of f is the infimum of κ for which f is κ-quasiconformal. A 0-quasiconformal map is a holomorphic map. We can define quasiconformal maps for domains in $\mathbb{P}^1$.

Now let A be an arbitrary subset of $\mathbb{P}^1$. An injective map $A \to \mathbb{P}^1$ is said to be *quasiconformal* if it is the restriction of a quasi-conformal map.

A *holomorphic motion* of a subset $A \subset \mathbb{P}^1$ over a complex manifold T is a map $\Psi : A \times T \to \mathbb{P}^1$ such that

1. for any $a \in A$, the map $z \mapsto \Psi(a,z)$ is holomorphic;
2. for any $z \in T$, the map $a \mapsto \Psi(a,z) = \Psi_z(a)$ is an injection;
3. Ψ_0 is the identity map.

The base manifold of a holomorphic motion will always be $\mathbb{D}$ unless stated otherwise. A holomorphic motion Ψ is called *constant* if Ψ does not depend on z. Two holomorphic motions Ψ_1, Ψ_2 of A are *equivalent (resp. algebraically equivalent)* if there is a holomorphic map $f : \mathbb{D} \to \mathrm{PGL}_2(\mathbf{C})$ (resp. which extends to a rational function on $\mathbf{C}$) such that $\Psi_2(a,z) = \Psi_1(f(z)(a), z)$. A holomorphic motion is trivial (resp. algebraically trivial) if it is *equivalent* (resp. algebraically equivalent) to a constant holomorphic motion.

Theorem 7.2.1 (λ-Lemma [MnSS83], [Lyu83]) *If $\Psi : A \times \mathbb{D} \to \mathbb{P}^1$ is a holomorphic motion of $A \in \mathbb{P}^1$, then Ψ extends to a holomorphic motion $\overline{\Psi} : \bar{A} \times \mathbb{D} \to \mathbb{P}^1$ of the closure $\bar{A}$ and $\overline{\Psi}$ is necessarily continuous.*

Remark 7.2.2 The continuity implies that the extension is unique.

Two naturally raised questions are whether Ψ can be extended over a neighbourhood of $0 \in \mathbb{D}$ to a set larger than $\bar{A}$, and how regular the functions Ψ_z are. Some partial answers are obtained in [ST86] and [BR86]. Using different methods, Slodkowski proved the following optimal result:

Theorem 7.2.3 (Slodkowski's Extended λ-Lemma [Slo91]) *If $\Psi : A \times \mathbb{D} \to \mathbb{P}^1$ is a holomorphic motion of $A \subset \mathbb{P}^1$, then it has an extension $\tilde{\Psi} : \mathbb{P}^1 \times \mathbb{D} \to \mathbb{P}^1$ such that*

1. *$\tilde{\Psi}$ is a holomorphic motion of $\mathbb{P}^1$;*
2. *For any $z \in \mathbb{D}$, $\tilde{\Psi}_z$ is a quasiconformal self-homeomorphism of $\mathbb{P}^1$ of dilatation not exceeding $\frac{1+|z|}{1-|z|}$*
3. *$\tilde{\Psi}$ is continuous.*

When we have a holomorphic motion Ψ of A, we denote by A_z the set $\Psi_z(A) \subset \mathbb{P}^1$; A_z is homeomorphic to A for any $z \in \mathbb{D}$. The *graph* of a holomorphic motion Ψ of A is $\{(z,w) \in \mathbb{D} \times \mathbb{P}^1 | \exists a \in A, w = \Psi(z,a)\}$. A holomorphic motion is not uniquely determined by its graph; for example $\mathbb{D} \times \mathbb{P}^1$ is the graph of every holomorphic motion of $\mathbb{P}^1$. A subset of the graph of Ψ of the form $\{(z,w) \in \mathbb{D} \times \mathbb{P}^1 | w = \Psi(z,a)\}$ where a is fixed is called a *leaf* of the holomorphic motion.

7.2.2 Construction of a holomorphic motion

From now on in this chapter we assume that (Y, Γ, U, X) is a birational Kleinian group in dimension two such that Γ preserves a rational fibration $r : Y \to B$ with $B = \mathbb{P}^1$ which induces a minimal transversely hyperbolic regular foliation on X. As in Chapter 5 we denote by Γ_B the image of Γ in $\mathrm{Aut}(B)$ and we denote by γ_B the image of an element $\gamma \in \Gamma$ in Γ_B. By Selberg's lemma, up to replacing Γ with a subgroup of finite index we will assume that Γ_B is torsion free. Up to contracting (-1)-curves outside U we assume that every (-1)-curve contained in some fiber of r intersects U.

When we speak about disks (resp. circles) in $\mathbb{P}^1$ we mean round disks (resp. circles) that are images of the unit disk (resp. circle) $\mathbb{D}$ under $\mathrm{Aut}(\mathbb{P}^1)$; topological disks in $\mathbb{P}^1$ will just be called simply connected domains or domains biholomorphic to $\mathbb{D}$, and topological circles will be called Jordan curves. The image of an injective holomorphic map from $\mathbb{D}$ to a complex manifold will be called a *holomorphic disk*.

The terminology and notations from Section 7.1 will be used for our X. We recall the following properties. There is a fibration $s : \tilde{X} \to \mathbb{D}$ from the universal cover $\tilde{X}$ to the disk $\mathbb{D}$ with fibers all biholomorphic to $\mathbb{D}$ and a representation $\rho : \pi_1(X) \to \mathrm{PSL}_2(\mathbf{R})$ with dense orbits with respect to which s is equivariant (cf. Theorem 7.1.1). The leaves of $\mathscr{F}$ come from the fibers of s.

Proposition 7.2.4 *Up to conjugation in* $\mathrm{Aut}(B)$*, the projection* $r(U)$ *is the unit disk in* $\mathbb{P}^1$ *and* Γ_B *is a dense subgroup of* $\mathrm{PSL}_2(\mathbf{R})$*. The action of* Γ_B *on* $r(U)$ *is just the action of* $\rho(\pi_1(X))$ *on* $\mathbb{D}$*. The fibers of* $r : U \to r(U)$ *are connected.*

Proof In our situation $\mathscr{F}$ comes from the map $r : U \to r(U)$. We obtain thus a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathbb{D} & \xrightarrow[q]{} & r(U) \end{array}$$

where all arrows are equivariant. In particular $q : \mathbb{D} \to r(U)$ is equivariant under a homomorphism $\eta : \rho(\pi_1(X)) \to \Gamma_B \subset \mathrm{PGL}_2(\mathbf{C})$. Recall that we have assumed that Γ_B is torsion free by Selberg's lemma. We know that $\rho(\pi_1(X))$ is dense in $\mathrm{PSL}_2(\mathbf{R})$ (cf. Proposition 7.1.4), thus contains elliptic elements of infinite order by Lemma 7.1.5. Let $g \in \rho(\pi_1(X))$ be an elliptic element of infinite order and let $x \in \mathbb{D}$ be a point which is not the center of g. The closure of the g-orbit of x is a circle in $\mathbb{D}$. The image of this circle in $r(U)$ is a curve contained in the closure of the $\eta(g)$-orbit of $q(x)$. This implies that $\eta(g)$ is an elliptic element of infinite order of $\mathrm{PGL}_2(\mathbf{C}) = \mathrm{Aut}(B)$. Thus being $\eta(g)$-invariant $r(U)$ is a disk and q sends concentric circles associated with g to concentric circles associated with $\eta(g)$. If we normalize so that the center of g and $\eta(g)$ are origins and that $r(U)$ is the unit disk, then we see that q is the identity map. This implies that the fiber of $r : U \to r(U)$ over $b \in B$ is connected, being the image of the fiber of s over $q^{-1}(b)$. □

Every Γ_B-orbit in $r(U)$ being infinite and dense in $r(U)$, we have:

Corollary 7.2.5 $r : Y \to B$ *has no singular fibers over* $r(U)$*.*

Therefore the open set $r^{-1}(r(U))$ is $\mathbb{D} \times \mathbb{P}^1$. We can perform elementary transformations outside U to realize a geometric conjugation $Y \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ of birational Kleinian groups. From now on in this chapter we will assume that $Y = \mathbb{P}^1 \times \mathbb{P}^1$, $r(U) = \mathbb{D}$.

For any Γ-invariant metric on U, the *injectivity radius* is the supremum of R such that balls of radii R inject into $X = U/\Gamma$; it is a positive number. We have seen in Corollary 7.1.3 that X

and U are Kobayashi hyperbolic. Thus the Kobayashi hyperbolic metric is a Γ-invariant metric on U.

We denote by ∂U the boundary of U in $Y = \mathbb{P}^1 \times \mathbb{P}^1$. The boundary of U in $\mathbb{D} \times \mathbb{P}^1$ will be called the horizontal boundary and be denoted by $\partial_h U$. Note that $r(\partial_h U) = \mathbb{D}$ and it is only a subset of the total boundary ∂U in $Y = \mathbb{P}^1 \times \mathbb{P}^1$. A holomorphic disk in $\mathbb{P}^1 \times \mathbb{P}^1$ is called *a horizontal holomorphic disk over a simply connected domain* $D \subset B$ if it is a holomorphic section of r over D.

Let K be a subset of U, a point p in ∂U is said to be *a K-accessible point* if every neighborhood of p intersects $\gamma(K)$ for infinitely many γ in Γ. We say that a subset $V \subset U$ is a *weak fundamental domain* if 1) Γ-translates of V cover U; 2) the projection $V \to U/\Gamma$ is proper. The following two lemmas are taken from [Kul78].

Lemma 7.2.6 *There exists a compact connected weak fundamental domain.*

Proof We denote the covering map $U \to X$ by π. For any $x \in X$ and $x' \in U$ such that $\pi(x') = x$, pick a compact connected neighborhood $U_{x'} \subset U$ of x' such that $U_x = \pi(U_{x'})$ is a compact neighborhood of x. Since X is compact, there exist finitely many points $x_i, 1 \leq i \leq k$ in X such that the associated U_{x_i} cover X. Then the union of $U_{x_i'}$ is a compact subset of U which projects surjectively onto X. We can make this union connected by translating the $U_{x_i'}$ by elements of Γ. □

Lemma 7.2.7 *Every point in ∂U is a K-accessible point for any compact connected weak fundamental domain K.*

Proof Let p be a point in ∂U and V a compact connected neighborhood of p in the closure $\bar{U}$. Assume by contradiction that p is not K-accessible. Then $V - \partial U$ is covered by finitely many translates of K, so it is compact. But this implies that ∂U is open in V, which is absurd. □

Proposition 7.2.8 *The horizontal boundary $\partial_h U \subset \mathbb{D} \times \mathbb{P}^1$ is laminated by horizontal holomorphic disks over $\mathbb{D} = r(U)$. In other words, $\partial_h U$ is the graph of some holomorphic motion.*

Proof The holomorphic leaves in the lamination will be constructed as limits of holomorphic disks in U.

Let p be a point in $\partial_h U$. We want to construct a holomorphic disk in $\partial_h U$ containing p. Let K be a weak fundamental domain in U. We cover K by finitely many small round bidisks $(D_1^j \times D_2^j)_{j \in J}$ such that

1. The D_1^js are disks in $r(U) = \mathbb{D}$ with hyperbolic radii all equal to δ.
2. The D_2^js are disks in $\mathbb{P}^1$.
3. For any $j \in J$, $D_1^j \times D_2^j \subset U$.
4. For $j \in J$ denote by E_1^j the disk having the same center as D_1^j but with hyperbolic radius 4δ. For any j the radius of $E_1^j \times D_2^j$ with respect to the Kobayashi hyperbolic metric of U is less than half of the injectivity radius.

A consequence of the fourth point is:

— For any j_1, j_2, there is at most one element $\gamma \in \Gamma$ such that $E_1^{j_1} \times D_2^{j_1}$ intersects $\gamma(E_1^{j_2} \times D_2^{j_2})$. In particular for any j, $E_1^j \times D_2^j$ injects into X.

By Lemma 7.2.7 p is K-accessible, there is a sequence $(\gamma_n)_n$ of distinct elements of Γ such that for every neighborhood V of p, V intersects $\gamma_n(K)$ for large enough n. As K is covered by the finitely many bidisks $D_1^j \times D_2^j, j \in J$, we infer that there exists a j such that V intersects $\gamma_n(D_1^j \times D_2^j)$ for infinitely many n. Note that the $\gamma_n(D_1^j \times D_2^j)$ are disjoint.

Projecting everything to $r(U) = \mathbb{D}$, we infer that $r(p) \in \mathbb{D}$ is a D_1^j-accessible point for the action of Γ_B. Denote by $D_{r(p)}$ the disk of hyperbolic radius 2δ centered at $r(p)$. For n large enough, the center of the disk $\gamma_{nB}(D_1^j)$ is contained in $D_{r(p)}$. As the E_1^j have radii 4δ, $\gamma_{nB}(E_1^j)$ contains $D_{r(p)}$ for large enough n. After throwing a finite number of terms in the sequence $(\gamma_n)_n$, we will assume that every $\gamma_{nB}(E_1^j)$ contains $D_{r(p)}$. Now consider the sequence of regions $L_n = \gamma_n(E_1^j \times D_2^j) \cap (D_{r(p)} \times \mathbb{P}^1)$; they are holomorphic bidisks because $\gamma_{nB}(E_1^j)$ contains $D_{r(p)}$. Let $\beta_n : D_{r(p)} \times \mathbb{D} \to L_n, (z,w) \mapsto (z, b_n(z,w))$ be a sequence of biholomorphic parametrizations of the L_n.

Lemma 7.2.9 *There is a subsequence of $(\beta_n)_n$ which converges locally uniformly to a holomorphic map β such that*

1. *the image of β is a horizontal disk D_p over $D_{r(p)}$;*
2. *D_p is contained in $\partial_h U \cap (D_{r(p)} \times \mathbb{P}^1)$;*
3. *D_p contains p;*
4. *D_p does not depend on the chosen subsequence.*

Proof Choose three disjoint horizontal disks R_1, R_2, R_3 in L_0. There is a holomorphic diffeomorphism ϕ of $D_{r(p)} \times \mathbb{P}^1$ preserving fiberwise $r : D_{r(p)} \times \mathbb{P}^1 \to D_{r(p)}$ such that $\phi(R_i), i = 1,2,3$ are $D_{r(p)} \times \{0\}, D_{r(p)} \times \{1\}$ and $D_{r(p)} \times \{\infty\}$. Up to conjugating everything by ϕ, we can and will assume that R_1, R_2, R_3 are $D_{r(p)} \times \{0\}, D_{r(p)} \times \{1\}$ and $D_{r(p)} \times \{\infty\}$. Thus $(D_{r(p)} \times \mathbb{P}^1) \backslash L_0$ is a subset of $D_{r(p)} \times (\mathbf{C}^* \setminus \{1\})$ which is the product of two hyperbolic Riemann surfaces; for any $n \geq 1$, L_n is contained in $D_{r(p)} \times (\mathbf{C}^* \setminus \{1\})$. We apply Montel's theorem to the parametrization mappings β_n and obtain a subsequence of $(\beta_n)_n$ which converges uniformly on compact sets to a limit holomorphic mapping $\beta : D_{r(p)} \times \mathbb{D} \to (U \cup \partial_h U), (z,w) \mapsto (z, b(z,w))$. For z fixed, the function $w \mapsto b(z,w)$ must be constant because it is a limit function of a sequence of holomorphic maps $\mathbb{D} \to \mathbf{C}^* \setminus \{1\}$ with disjoint images. Therefore the image of β is a horizontal disk D_p over D. Since every neighbourhood of p intersects L_n, D_p contains p. Suppose by contradiction that a point $q \in D_p$ is in U. Then q is contained in a translate of the weak fundamental domain K. Thus there exist $\gamma' \in \Gamma$ and $k \in J$ such that $\gamma'(D_1^k \times D_2^k)$ is an open neighbourhood of q. Therefore $\gamma'(D_1^k \times D_2^k)$ intersects L_n, thus $\gamma_n(E_1^j \times D_2^j)$ for n large enough. However by our initial hypothesis there is at most one n such that $\gamma_n(E_1^j \times D_2^j)$ intersects $\gamma'(D_1^k \times D_2^k)$, contradiction. Hence $D_p \subset \partial_h(U)$.

Suppose by contradiction that using the above process we can construct two distinct limit horizontal disks D_s, D_t over $D_{r(p)}$ with non-empty intersection which correspond to two subsequences $s(n), t(n)$. For large enough n we choose a horizontal holomorphic disk $D_{s(n)}$ over S in $L_{s(n)}$ (resp. $D_{t(n)}$ in $L_{t(n)}$). The holomorphic disk $D_{s(n)}$ (resp. $D_{t(n)}$) is the graph of a holomorphic function $f_{s(n)}$ (resp. $f_{t(n)}$) on S. The sequence of holomorphic functions $f_{s(n)}$ (resp. $f_{t(n)}$) converges locally uniformly to a holomorphic function f_s (resp. f_t). The intersection $D_s \cap D_t$ is a discrete set in D_s and in D_t. We chose a small relatively compact disk $S \subset D_{r(p)}$ which contains the projection of a point in $D_s \cap D_t$. Thus $f_s - f_t$ has a zero in S. For n large enough, $f_{s(n)} - f_{t(n)}$ is close to $f_s - f_t$ on $\bar{S}$; by Rouché's theorem $f_{s(n)} - f_{t(n)}$ has also a zero in S. This contradicts the fact that the regions L_n are disjoint. □

Remark that, if for each n we choose a horizontal disk M_n over $D_{r(p)}$ contained in L_n, then the corresponding subsequence of $(M_n)_n$ converges also to D_p. To construct D_p we have chosen a j such that p is $D_1^j \times D_2^j$-accessible. The following lemma shows that D_p does not depend on the choice of j (by taking $p = q$ in the following lemma), and not even on p.

Lemma 7.2.10 *Let $q \in \partial_h(U)$ and let $k \in J$ such that q is $D_1^k \times D_2^k$-accessible. Let D_q be the limit horizontal disk constructed by the above process. Then either $D_p \cap D_q = \emptyset$ or they intersect in a relative open subset, i.e. D_p and D_q glue together to a Riemann surface.*

Proof We can assume that $r(q)$ is at hyperbolic distance $< 4\delta$ from $r(p)$; otherwise D_p and D_q are trivially disjoint because their projections in $r(U) = \mathbb{D}$ are disjoint. Let $(\gamma_n^j)_n$ (resp. $(\gamma_n^k)_n$) be a sequence of elements in Γ such that $L_n = \gamma_n^j((E_1^j \times D_2^j) \cap (D_{r(p)} \times \mathbb{P}^1))$ (resp. $M_n = \gamma_n^k((D_1^k \times D_2^k) \cap (D_{r(q)} \times \mathbb{P}^1)))$ converges to a limit horizontal disk D_p^j (resp. D_q^k). If the $\gamma_n^j(E_1^j \times D_2^j)$ and the $\gamma_n^k(E_1^k \times D_2^k)$ are all disjoint, then the parts over $D_{r(p)} \cap D_{r(q)}$ of the limit disks coincide by Rouché's theorem as at the end of the proof of the previous lemma. Suppose that infinitely many of them intersect. We said that there is at most one $h \in \Gamma$ such that $E_1^j \times D_2^j$ intersects $h(E_1^k \times D_2^k)$. Therefore up to taking subsequences of $(\gamma_n^j)_n$ and $(\gamma_n^k)_n$, we can assume that for any n, $\gamma_n^k = \gamma_n^j \circ h$. Then $(L_n \cap K_n)_n$ is a sequence of open domains that converges to both an open subset of D_p^j and an open subset of D_q^k. Thus D_p^j glues with D_q^k by analytic continuation.□

In summary, once we fixed K and the finitely many round bidisks $(D_1^j \times D_2^j)_{j\in J}$ covering K, we have a process to construct locally for each $p \in \partial_h(U)$ a unique horizontal disk D_p such that

1. $D_p \subset \partial_h(U)$.
2. $p \in D_p$.
3. $r(D_p) \subset \mathbb{D}$ is a disk of hyperbolic radius 2δ.
4. for $p \neq q$, if $D_p \cap D_q \neq \emptyset$ then $D_p \cup D_q$ is a connected Riemann surface.

To finish the proof it remains to show that the projection in $r(U) = \mathbb{D}$ of a maximal connected horizontal Riemann surface glued from the D_ps is the whole $\mathbb{D}$. Suppose by contradiction that L is such a maximal Riemann surface with $r(L)$ contained strictly in $\mathbb{D}$. Then there exits a point $p \in L$ such that $r(p)$ is at distance $< \delta$ from the boundary of $r(L)$ in $\mathbb{D}$. By glueing L with D_p, we get a contradiction to the maximality of L because $r(D_p)$ has radius 2δ. □

7.2.3 Action on the holomorphic motion

We fix some notations that we will use throughout this chapter. Denote by ψ the holomorphic motion we constructed in the previous subsection; its graph is $\partial_h(U)$. For $x \in \mathbb{D} = r(U)$, we will denote by U_x the intersection $U \cap \mathbb{P}_x^1$. Denote by A_x the boundary of U_x in $\mathbb{P}_x^1$ and denote A_0 by A. Then we have

Proposition 7.2.11 *$\partial_h(U)$ is the graph of ψ which is a holomorphic motion of A over $\mathbb{D}$. The intersection of $\partial_h(U)$ with $\mathbb{P}_x^1$ is A_x.*

The extended λ-Lemma (cf. Theorem 7.2.3) allows us to extend ϕ to the whole $\mathbb{P}^1$. The extension is not unique but when there is no ambiguity we will always use Ψ to denote an extension of ψ. As a consequence of the extended λ-Lemma we have

Proposition 7.2.12 *U is the graph of a holomorphic motion.*

Another consequence is

Proposition 7.2.13 *A horizontal holomorphic disk over $\mathbb{D}$ contained in $\partial_h(U)$ is necessarily a leaf of ψ and ψ is the unique holomorphic motion whose graph is $\partial_h(U)$.*

Proof Let $\{(x,h(x))|x \in \mathbb{D}\}$ be a horizontal holomorphic disk over $\mathbb{D}$. Let Ψ be a holomorphic motion of U_0 with graph U. Consider two holomorphic motions of $U_0 \cup \{h(0)\}$. The first is defined by $(x,w) \mapsto \Psi(x,w)$ if $x \in U_0$ and $(x,w) \mapsto h(x)$ if $w = h(0)$. The second is defined by $(x,w) \mapsto \Psi(x,w)$ if $x \in U_0$ and $(x,w) \mapsto \psi(h(0))$ if $w = h(0)$. Since $h(0)$ is in the closure of U_0, these two holomorphic motions coincide by the continuity assertion in λ-Lemma (cf. Theorem 7.2.1). $\square$

Corollary 7.2.14 *Γ permutes the leaves of the holomorphic motion ψ. This induces a homomorphism $\zeta : \Gamma \to \mathrm{Qc}(A)$ into the group of quasiconformal homeomorphisms of A.*

Proof By Corollary 5.2.3 Γ acts by holomorphic diffeomorphisms on $\mathbb{D} \times \mathbb{P}^1$. The image of a leaf in $\partial_h(U)$ by an element of Γ is still a leaf in $\partial_h(U)$ by Proposition 7.2.13. Let $\gamma \in \Gamma$. The homeomorphism $\zeta(\gamma) : A \to A$ is the composition $\psi^{-1}_{\gamma_B(0)} \circ \gamma|_{A_0}$. It is quasiconformal by λ-Lemma. $\square$

7.3 Teichmüller Spaces and Bers fiber spaces

Before we continue to investigate the birational kleinian group Γ and the transversely hyperbolic foliation $\mathscr{F}$, we introduce some preliminaries on Teichmüller theory in this section. All assertions without proofs and explicitly mentioned references can be found in [Leh87].

7.3.1 Teichmüller spaces

The following fundamental result is at the heart of the preliminaries that we are going to introduce in this section.

Theorem 7.3.1 (Ahlfors-Bers [AB60]) *Let μ be a bounded measurable function in $A\mathbb{P}^1$ such that $\|\mu\|_\infty < 1$. Then there exists a unique quasiconformal map $f_\mu : \mathbb{P}^1 \to \mathbb{P}^1$ which fixes $0, 1, \infty$ and whose Beltrami differential agrees with μ almost everywhere. Such a quasiconformal map depends holomorphically in μ, i.e. for ν another bounded measurable function, for any $a \in \mathbb{P}^1$, $f_{\mu+z\nu}(a)$ is holomorphic in z for z in a neighbourhood of 0.*

Let G be a Fuchsian group, i.e. an arbitrary discrete subgroup of $\mathrm{PSL}_2(\mathbf{R})$, acting on the upper half plane $\mathbb{H}$, hence also on the lower half plane $\mathbb{L}$. The space of Beltrami differentials for G is

$$M(G) = \{\mu \in L^\infty(\mathbb{H}) \mid \|\mu\|_\infty < 1, \forall g \in G, (\mu \circ g)\frac{\overline{\partial} g}{\partial g} = \mu\}.$$

For $\mu \in M(G)$ denote by w_μ the unique quasiconformal self-map of $\mathbb{P}^1$ that fixes $0, 1, \infty$, is holomorphic in $\mathbb{L}$ and whose Beltrami differential equals to μ almost everywhere in $\mathbb{H}$. For $\mu \in M(G)$ denote by w^μ the unique quasiconformal self-map of $\mathbb{H}$ that fixes $0, 1, \infty \in \partial\mathbb{H}$ and whose Beltrami differential equals to μ almost everywhere in $\mathbb{H}$ (a quasiconformal self-map of $\mathbb{H}$ always extends to the boundary circle).

Proposition 7.3.2 (see [Leh87] III.1.3) *Let $\mu, \nu \in M(G)$. The following are equivalent*

1. *w_μ, w_ν agree on $\mathbb{L}$.*
2. *w^μ, w^ν agree on the circle $\mathbf{R} \cup \{\infty\}$.*

Two elements $\mu, \nu \in M(G)$ are called *equivalent* if they satisfy the conditions in the above theorem. The equivalence class of μ is denoted by $[\mu]$. The set of equivalence classes is the *Teichmüller space of* G, denoted by $T(G)$. Thus the domains $w_\mu(\mathbb{H})$ and $w_\mu(\mathbb{L})$ depend only on $[\mu] \in T(G)$. If $\mu \in M(G)$ and $g \in G$ then $w_\mu \circ g \circ (w_\mu)^{-1}$ agrees with an element of $\mathrm{PGL}_2(\mathbf{C}) = \mathrm{Aut}(\mathbb{P}^1)$ in $\mathbb{L}$ and $w^\mu \circ g \circ (w^\mu)^{-1}$ agrees with an element $h(g,\mu)$ of $\mathrm{PGL}_2(\mathbf{C}) = \mathrm{Aut}(\mathbb{P}^1)$ on $\mathbf{R} \cup \{\infty\}$. More precisely, if we set $g_\mu = w_\mu \circ (w^\mu)^{-1} \circ h(g,\mu) \circ w^\mu \circ w_\mu^{-1}$ in the closure of $w_\mu(\mathbb{H})$ and $g_\mu = w_\mu \circ g \circ (w_\mu)^{-1}$ in $w_\mu(\mathbb{L})$, then $g_\mu \in \mathrm{PGL}_2(\mathbf{C})$. We denote by G_μ the group formed by the g_μ; it is a quasifuchsian group which depends only on $[\mu] \in T(G)$.

If f is a holomorphic function in a domain with nowhere vanishing derivative, then its *Schwarzian derivative* is the holomorphic function

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

The Schwarzian derivative of a function is zero if and only if the function is the restriction of a transformation in $\mathrm{PGL}_2(\mathbf{C}) = \mathrm{Aut}(\mathbb{P}^1)$. The space of quadratic differentials $Q(G)$ is the space of all holomorphic functions S on $\mathbb{L}$ such that $S = (S \circ g)(g')^2$ for all $g \in G$. We have a map from $M(G)$ to $Q(G)$ that sends μ to $S_{w_\mu|_{\mathbb{L}}}$.

Theorem 7.3.3 (Bers embedding) *The map $M(G) \to Q(G), \mu \mapsto S_{w_\mu|_{\mathbb{L}}}$ is holomorphic and induces an embedding $T(G) \hookrightarrow Q(G)$.*

Thus the Teichmüller space $T(G)$ is equipped with a structure of complex (Banach) manifold. When $G = \{\mathrm{Id}\}$ is the trivial group, we denote $T(G)$ by $T(1)$ and call it the *universal Teichmüller space*. For $G_1 \subset G_2$ we have natural inclusions $Q(G_2) \subset Q(G_1)$ and $T(G_2) \subset T(G_1)$. Thinking of the $T(G)$s as open subsets of the $Q(G)$s, and thinking of the $Q(G)$s as sub-vectorspaces of $Q(1)$, we have $T(G) = Q(G) \cap T(1)$. Hence the terminology "universal Teichmüller space". If C is a hyperbolic Riemann surface then it is the quotient of $\mathbb{H}$ by a fuchsian group G; the Teichmüller space $T(C)$ of C is defined to be $T(G)$.

7.3.2 Bers fiber spaces

The map $T(G) \times \mathbb{P}^1 \to \mathbb{P}^1, ([\mu], y) \mapsto w_\mu(y)$ is a holomorphic motion over $T(G)$ by the holomorphic dependence of solutions of Beltrami differential equations. The Bers fiber space $F(G)$ is a complex manifold that fibers over $T(G)$ with fibers all biholomorphic to $\mathbb{D}$; it is defined to be the following open subset of $T(G) \times \mathbb{P}^1$:

$$F(G) = \{([\mu], y) \in T(G) \times \mathbb{P}^1 | y \in w_\mu(\mathbb{H})\}.$$

It is the graph of a holomorphic motion of $\mathbb{H}$ over $T(G)$ and its boundary in $T(G) \times \mathbb{P}^1$ is the graph of a holomorphic motion of $\mathbf{R} \cup \{\infty\}$. The domain $F(G)$ itself is not a bounded domain in $T(G) \times \mathbb{P}^1$, but it is biholomorphic to one:

Theorem 7.3.4 ([Ber73]) *There is an embedding $F(G) \to T(G) \times \mathbb{P}^1$ which preserves fiberwise the fibration $F(G) \to T(G)$ such that the image is contained in $T(G) \times \mathbb{D}$.*

From now on *we will assume that the fuchsian group G is torsion free*, i.e. $\mathbb{H}/G$ is a Riemann surface. The group G acts on $F(G)$ by $g(([\mu], y)) = ([\mu], g_\mu(y))$. The action is discontinuous and by biholomorphisms. The quotient $V(G) = F(G)/G$ is a complex (Banach) manifold which fibers over $T(G)$ with fibers all quasiconformal to $\mathbb{H}/G$. We call $V(G)$ the *Teichmüller curve*, or the *universal family*.

It turns out that $F(G)$ is itself a Teichmüller space:

Theorem 7.3.5 ([Ber73]) *Let $\dot{G}$ be a torsion free fuchsian group such that $\mathbb{H}/\dot{G}$ is biholomorphic to the complement of a point in $\mathbb{H}/G$. Then $F(G)$ is biholomorphic to $T(\dot{G})$.*

Denote by QC the group of quasiconformal self-maps of $\mathbb{H}$ and by QC_0 the subgroup of those quasiconformal self-maps that fix all points on the boundary circle $\mathbf{R} \cup \{\infty\}$. Denote by $N(G)$ and $N_c(G)$ the normalizers of G in respectively QC and $\mathrm{PSL}_2(\mathbf{R})$. Define $\mathrm{mod}(G)$ to be the group $N(G)/(N(G) \cap QC_0)$. The quotient map $N(G) \to N(G)/(N(G) \cap QC_0)$ is injective restricted to G so that G may be considered as a subgroup of $\mathrm{mod}(G)$. It is actually a normal subgroup and we define $\mathrm{Mod}(G)$ to be the quotient group $\mathrm{mod}(G)/G$.

Let $\omega \in QC$. For $\mu \in M(G)$ consider the Beltrami differential of the map $w^\mu \circ \omega^{-1}$ that we denote by $\omega_*(\mu)$. It is a Beltrami differential for $\omega G \omega^{-1}$ and its class $[\omega_*(\mu)] \in T(\omega G \omega^{-1})$ only depends on $[\mu] \in T(G)$ and on the image of ω in QC/QC_0. This defines a biholomorphism from $T(G)$ to $T(\omega G \omega^{-1})$. It can be extended to a biholomorphism $F(G) \to F(\omega G \omega^{-1})$, $([\mu], y) \mapsto ([\omega_*(\mu)], \hat{y})$ where $\hat{y} = w_{\omega_*(\mu)} \circ \omega \circ (w_\mu)^{-1}(y)$. These assertions are proved in [Ber73] and lead to

Theorem 7.3.6 ([Ber73]) *The group* $\mathrm{Mod}(G)$ *acts by biholomorphisms on* $T(G)$ *and* $V(G)$ *and the group* $\mathrm{mod}(G)$ *acts by biholomorphisms on* $F(G)$ *so that the maps* $F(G) \to V(G)$ *and* $V(G) \to T(G)$ *are equivariant. The action of* $\mathrm{mod}(G)$ *on* $F(G)$ *and the action of* $\mathrm{Mod}(G)$ *on* $V(G)$ *are faithful while the action of* $\mathrm{Mod}(G)$ *on* $T(G)$ *is not always faithful.*

We will call $\mathrm{Mod}(G)$ the *Teichmüller modular group* and $\mathrm{mod}(G)$ the *extended Teichmüller modular group*.

Remark 7.3.7 We said that the boundary of $F(G)$ in $T(G) \times \mathbb{P}^1$ is a graph of a holomorphic motion of the circle over $T(G)$. The action of $\mathrm{Mod}(G)$ on $F(G)$ extends to this boundary; $\mathrm{Mod}(G)$ permutes the leaves of the holomorphic motion in the boundary and induces an action on the circle which is the space of leaves. This action on the circle is nothing else but the action of QC/QC_0 on the boundary circle of $\mathbb{H}$ as we can see from the formula $([\mu], y) \mapsto ([\omega_*(\mu)], \hat{y})$ where $\hat{y} = w_{\omega_*(\mu)} \circ \omega \circ (w_\mu)^{-1}(y)$.

The group of self-biholomorphisms of $T(G)$ has been studied in [Roy71], [EK74], [EG96], [Lak97], [Mar03], and that of $F(G), V(G)$ in [EF85], [She06], [HS06]. We will need essentially only [EF85]. For completeness we gather all known results in the following statement:

Theorem 7.3.8
1. *If G is torsion free and $\mathbb{H}/G$ is not a Riemann surface of genus g with n punctures such that $2g+n \leq 4$, then the group of self-biholomorphisms of $T(G)$ is* $\mathrm{Mod}(G)$.
2. *If G is torsion free then the group of self-biholomorphisms of $V(G)$ preserving the fibration $V(G) \to T(G)$ is* $\mathrm{Mod}(G)$.
3. *If G is torsion free and $\mathbb{H}/G$ is not a once punctured disk then the group of self-biholomorphisms of $V(G)$ is* $\mathrm{Mod}(G)$.

4. *If G is torsion free and $\mathbb{H}/G$ is not a twice punctured affine plane or a once punctured elliptic curve then the group of self-biholomorphisms of $F(G)$ preserving the fibration $F(G) \to T(G)$ is* $\mathrm{mod}(G)$.

7.3.3 Universal properties

A holomorphic family of Riemann surfaces with fiber model S is the data of two complex (Banach) manifolds C and E, a connected Riemann surface S and a holomorphic submersion $f : E \to C$ locally trivial in the topological category with fibers all homeomorphic to S. We define an *admissible holomorphic family of Riemann surfaces* of fiber model S as a holomorphic family of Riemann surfaces (E,C,S,f) satisfying the following additional conditions:

1. C is simply connected;
2. for any $x \in C$, there is an open neighbourhood $W \subset C$ of x such that there exists a homeomorphic trivialization $t : W \times S \to f^{-1}(W)$ such that $w \mapsto t(w,s)$ is a holomorphic map from W to E for any fixed $s \in S$.

When the context is clear, we just say that E is an admissible holomorhic family. A *morphism* of admissible holomorphic families from (E_1,C_1,S_1,f_1) to (E_2,C_2,S_2,f_2) is a pair of holomorphic maps $h : C_1 \to C_2$ and $\hat{h} : E_1 \to E_2$ such that for any $x \in C$, $\hat{h}$ restrited to the fiber $f_1^{-1}(x)$ is a bijective map onto $f_2^{-1}(h(x))$; in particular it satifies $f_2 \circ \hat{h} = h \circ f_1$. A *morphism* $(h,\hat{h})$ of admissible holomorphic families is an *isomorphism* if $h, \hat{h}$ are biholomorphic. See also [Nag88] p360 for the following theorem:

Theorem 7.3.9 (Earle-Fowler [EF85] Theorem 2, Paragraphs 7.7 and 7.8) *Let G be a torsion free Fuchsian group and S be the Riemann surface $\mathbb{H}/G$. The natural fibration $V(G) \to T(G)$ is an admissible family of Riemann surfaces with fiber model S such that:*

1. *Given any admissible holomorphic family of Riemann surfaces (E,C,S,f), there is a morphism of admissible holomorphic families $(h,\hat{h})$ from E to $V(G)$.*
2. *The morphism above $(h,\hat{h})$ is unique up to an automorphism of $V(G)$. In other words if $(h_1,\hat{h}_1)$ and $(h_2,\hat{h}_2)$ are two morphisms from E to $V(G)$ then there is an isomorphism $(g,\hat{g})$ of admissible holomorphic families of $V(G)$ to itself such that $h_2 = g \circ h_1$ and $\hat{h}_2 = \hat{g} \circ \hat{h}_1$.*

Notations.

— We denote by $\mathrm{Aut}(T(G))$ the group of self biholomorphisms of $T(G)$.

— We denote by $\mathrm{Aut}(V(G))$ (resp. $\mathrm{Aut}(F(G))$) the group of automorphisms of $V(G)$ (resp. $F(G)$) as an admissible holomorphic family of Riemann surfaces over $T(G)$ with fiber model $\mathbb{H}/G$ (resp. $\mathbb{H}$).

Theorem 7.3.8 tells us that $\mathrm{Aut}(V(G))$ coincides with the Teichmüller modular group and that $\mathrm{Aut}(T(G))$ (resp. $\mathrm{Aut}(F(G))$) coincides with the Teichmüller modular group (resp. the extended Teichmüller modular group) unless in some exceptional cases.

7.4 From birational Kleinian groups to Teichmüller modular groups

Let us come back to birational Kleinian groups.

7.4.1 Morphisms into Teichmüller spaces

By Slodkowski's extended λ-Lemma, we know that U is the graph of a holomorphic motion of U_0 (cf. Proposition 7.2.12). As a consequence $(U,\mathbb{D},r,U_0)$ is an admissible holomorphic family of Riemann surfaces with fiber model U_0. In the sequel we think of U as an admissible family. Since Γ preserves the fibration r, it is a group of automorphisms of admissible family.

Theorem 7.4.1 *There is a morphism of admissible families $\hat{\varphi} : U \to V(U_0)$ which is equivariant under a homomorphism $\Gamma \to \mathrm{Aut}(V(U_0))$. More precisely there are holomorphic maps $\varphi : \mathbb{D} \to T(U_0)$ and $\hat{\varphi} : U \to V(U_0)$, a homomorphism $\hat{\theta} : \Gamma \to \mathrm{Aut}(V(U_0)) = \mathrm{Mod}(U_0)$ and a homomorphism $\theta : \Gamma \to \mathrm{Aut}(T(U_0))$ which factorizes through $\Gamma \to \Gamma_B$ such that*

— *$\hat{\varphi}$ restricted to U_x for any $x \in \mathbb{D}$ is a bijection onto the fiber of $V(U_0) \to T(U_0)$ over $\varphi(x)$;*

— *$\varphi \circ \gamma_B = \theta(\gamma) \circ \varphi$ and $\hat{\varphi} \circ \gamma = \hat{\theta}(\gamma) \circ \hat{\varphi}$ for any $\gamma \in \Gamma$.*

If φ is not a constant map then φ is a holomorphic embedding; the image is geodesic with respect to the Teichmüller metric on $T(U_0)$.

Proof By Theorem 7.3.9 there is a morphism of admissible families $(\varphi, \hat{\varphi})$ from U to $V(U_0)$, unique up to automorphisms of $V(U_0)$. For any $\gamma \in \Gamma$, $(\varphi \circ \gamma_B, \hat{\varphi} \circ \gamma)$ is also a morphism of admissible families. By uniqueness there is an automorphism $(\theta(\gamma), \hat{\theta}(\gamma)) \in \mathrm{Aut}(V(U_0))$ such that $\varphi \circ \gamma_B = \theta(\gamma) \circ \varphi$ and $\hat{\varphi} \circ \gamma = \hat{\theta}(\gamma) \circ \hat{\varphi}$. Thus the equivariance.

Let us show that if φ is not constant then it is an embedding. Since every Γ_B-orbit is dense in $\mathbb{D}$ and φ is equivariant, if the differential of φ does not vanish at one point then it vanishes nowhere. Therefore it suffices to prove that φ is injective if not constant. Suppose that there are $x, y \in \mathbb{D}$ such that $\varphi(a) \neq \varphi(b)$. By Lemma 7.1.5 we can find an elliptic element $g \in \Gamma_B$ of infinite order. Up to conjugating g by other elements in Γ_B, we can assume that x, y are not the center of g and are on two distinct concentric circles with center the center of g. As φ is equivariant, for any $n \in \mathbf{Z}$, $g^n(x)$ and $g^n(y)$ have the same image under φ. The closures of the g-orbits of x and y are two concentric circles and φ takes the same values on these two circles. The quotient of the annulus domain between these two circles by identifying them is an elliptic curve. Thus φ induces a holomorphic function on that elliptic curve. This implies that φ is constant.

The Teichmüller metric on $T(U_0)$ is a Finsler metric, i.e. it is determined by a continuous function defined on the tangent bundle of $T(U_0)$ which associates to a tangent vector its norm (cf. [EE67]); it is preserved by the Teichmüller modular group. Consider the norm function N of the pull-back Teichmüller metric on the unit tangent bundle of $\mathbb{D}$ with respect to the Poincaré metric. Note that $\mathrm{PSL}_2(\mathbf{R})$ acts transitively on the unit tangent bundle. By equivariance and the density of Γ_B in $\mathrm{PSL}_2(\mathbf{R})$, N is a continuous function which is constant on a dense subset of the unit tangent bundle of $\mathbb{D}$. Thus N is a constant function, i.e. the image of φ is complex geodesic. $\square$

Recall that the universal cover $\tilde{X}$ has a fibration $s : \tilde{X} \to \mathbb{D}$ equivariant under a representation $\rho : \pi_1(X) \to \mathrm{PSL}_2(\mathbf{R})$. The fibration $r : U \to \mathbb{D}$ is induced by s and $\rho(\pi_1(X)) = \Gamma_B$ (cf. Proposition 7.2.4). Since U is an admissible family, $(\tilde{X}, \mathbb{D}, s, \mathbb{D})$ is also an admissible family (cf. [EF85] Theorem 3). Then Theorem 7.3.9 implies

Lemma 7.4.2 *$\tilde{X}$ is biholomorphic to the graph of a holomorphic motion of a simply connected domain over $\mathbb{D}$.*

Along the same way we have

Theorem 7.4.3 *There is a morphism of admissible families $\tilde{X} \to V(1) = F(1)$ which is equivariant under a homomorphism $\pi_1(X) \to \mathrm{Aut}(V(1)) = \mathrm{Mod}(1) = QC/QC_0$. The classifying map $\mathbb{D} \to T(1)$ is either constant or a holomorphic embedding with geodesic image.*

Corollary 7.4.4 *$\tilde{X}$ is biholomorphic to a bounded domain in $\mathbf{C}^2$.*

Proof Recall that $V(1) = F(1)$ is biholomorphic to a bounded domain in $T(1) \times \mathbf{C}$ (cf. Theorem 7.3.4). If $\tilde{X}$ is a bidisk there is nothing to prove. Otherwise the above theorem tells us that it is just the part of $V(1)$ over a holomorphic disk in $T(1)$. $\square$

Variation of Poincaré metrics. Usually complex analytic data associated with a domain vary in a real analytic way when the domain moves in a holomorphic motion (cf. [EM00]). In particular we have

Theorem 7.4.5 (Rodin [Rod86]) *Let Ψ be a holomorphic motion of a simply connected domain Ω over $\mathbb{D}$. Let $a \in \Omega$. There is a neighbourhood W of $0 \in \mathbb{D}$ such that*

1. *$a \in \Omega_x = \Psi_x(\Omega)$ for $x \in W$.*
2. *Let $R_x : \mathbb{D} \to \Omega_x$ be the biholomorphism normalized so that $R_x(0) = a$ and $R'_x(0) > 0$. Then $R(x,y) = R_x(y)$ is real analytic for $x \in W$ and $y \in \mathbb{D}$.*

With Lemma 7.4.2 the above theorem implies that leafwise Poincaré metrics of the foliation $\mathscr{F}$ vary in a real analytic way. Recall that in general we know that $K_{\mathscr{F}}$ is nef and big, and by Theorem 7.1.15 that leafwise Poincaré metrics give rise to a continuous singular hermitian metric on $K_{\mathscr{F}}$. Here we obtain something better:

Proposition 7.4.6 *Leafwise Poincaré metrics give a smooth Hermitian metric and $K_{\mathscr{F}}$ is Hermitian semipositive.*

7.4.2 Quasisymmetric groups

A *quasisymmetric* self-homeomorphism of the circle is a homeomorphism which extends to a quasiconformal self-homeomorphism of the closed disk. The group of quasisymmetric self-homeomorphisms of the circle is exactly $QC/QC_0 = \mathrm{Mod}(1)$. A direct definition of quasisymmetric hemeomorphism is as follows. Let f be a self-homeomorphism of the circle $\mathbf{R} \cup \{\infty\} \subset \mathbb{P}^1$ such that $f(\infty) = \infty$. Then f is κ-quasisymmetric if and only if there exists $\kappa \in \mathbf{R}$ such that $\frac{1}{\kappa} \leq \frac{f(x+t)-f(x)}{f(x)-f(x-t)} \leq \kappa$ for all $x \in \mathbf{R}$ and all $t > 0$. For a given κ there is a κ' such that any quasisymmetric self-homeomorphism of the circle which extends to a κ-quasiconformal self-homeomorphism of $\mathbb{D}$ is κ'-quasisymmetric.

Let Ψ be a holomorphic motion over $\mathbb{D}$. Recall that λ-lemma asserts that for any $z \in \mathbb{D}$, Ψ_z is a quasiconformal map with dilatation $\leq \frac{1+|z|}{1-|z|}$. We say that the holomorphic motion Ψ has *bounded distortion* if there is $\kappa \in \mathbf{R}$ such that Ψ_z is κ-quasiconformal for any $z \in \mathbb{D}$.

Theorem 7.4.7 (Hinkkanen [Hin85],[Hin90], Markovic [Mar06]) *A subgroup of* $\mathrm{Mod}(1)$ *such that every element is κ-quasisymmetric for some $\kappa \in \mathbf{R}$ is conjugate to a subgroup of* $\mathrm{PSL}_2(\mathbf{R})$ *by a κ'-quasisymmetric self-homeomorphism where κ' depends only on κ.*

Remark 7.4.8 If the subgroup extends to a group of quasiconformal self-homeomorphisms of $\mathbb{D}$, i.e. if there is a section of $QC \to QC/QC_0$ over this subgroup, then the theorem is proved much earlier by Tukia [Tuk80] and Sullivan [Sul81].

Lemma 7.4.9 *Suppose that $\tilde{X}$ is not the bidisk. Then the image of the homomorphism $\pi_1(X) \to$ $\mathrm{Mod}(1)$ in Theorem 7.4.3 is not conjugate in $\mathrm{Mod}(1)$ to a subgroup of $\mathrm{PSL}_2(\mathbf{R})$.*

Proof Suppose the contrary. Up to post-composing the classifying map $\tilde{X} \to V(1)$ by the element of $\mathrm{Mod}(1)$ which realizes the conjugation, we can assume that the homomorphism $\alpha : \pi_1(X) \to \mathrm{Mod}(1)$ has image in $\mathrm{PSL}_2(\mathbf{R})$. By Theorem 7.4.3 we have the following equivariant commutative diagram

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & V(1) \\ \downarrow & & \downarrow \\ \mathbb{D} & \longrightarrow & T(1) \end{array}$$

where horizontal arrows are embeddings. Thus the representation $\rho : \pi_1(X) \to \mathrm{PSL}_2(\mathbf{R})$ associated with the foliation $\mathscr{F}$ is the composition of α with some homomorphism $\alpha(\pi_1(X)) \to$ $\mathrm{PSL}_2(\mathbf{R})$. As ρ does not factorize through an orbicurve by Proposition 7.1.4, α does not factorize through an orbicurve either. Thus by Theorem 4.2.1 the image of α is a dense subgroup of $\mathrm{PSL}_2(\mathbf{R})$. Let $(f_n)_n$ be a sequence of elements of $\pi_1(X)$ such that $\alpha(f_n)$ approaches to Id in $\mathrm{PSL}_2(\mathbf{R})$. We want to show that as elements of $\mathrm{Aut}(\tilde{X})$ the sequance $(f_n)_n$ converges to Id too; this will contradict the discreteness of $\pi_1(X) \subset \mathrm{Aut}(\tilde{X})$.

We denote $\alpha(f_n)$ by ω_n if we think of it as an element of $\mathrm{PSL}_2(\mathbf{R})$ and by g_n if we think of it as an element of $\mathrm{Mod}(1)$. In other words ω_n acts on $\mathbb{H} \cup \partial\mathbb{H}$ and g_n acts on $T(1)$ and $V(1)$. We need to show that g_n converges to the identity map. We recall how g_n acts on $T(1)$ and $V(1)$ (see the paragraph before Theorem 7.3.6). If $[\mu] \in T(1)$ is represented by a Beltrami differential μ then $g_n([\mu])$ is represented by the Beltrami differential of $\omega_{n*}(\mu) = w^{\mu} \circ \omega_n^{-1}$. As ω_n is holomorphic, the formula for changes of variables (cf. [Leh87] I.4.2) gives

$$\omega_{n*}(\mu)(z) = \mu(z) \left(\frac{\omega_n'(z)}{|\omega_n'(z)|} \right)^2 .$$

Thus $\omega_{n*}(\mu)$ converges to μ as ω_n goes to Id. The action of g_n on $V(1) = F(1)$ is described as follows. If $([\mu], y) \in V(1)$ then its image under g_n is $([\omega_{n*}(\mu)], \hat{y})$ where

$$\hat{y} = w_{\omega_{n*}(\mu)} \circ \omega_n \circ w_{\mu}^{-1}(y).$$

By Theorem 7.3.1, when $\omega_{n*}(\mu)$ is close to μ, $w_{\omega_{n*}(\mu)}$ is close to w_μ. Thus $\hat{y}$ converges to y as ω_n goes to Id. Hence g_n converges pointwise to Id. □

Theorem 7.4.10 *If U is the graph of a holomorphic motion of bounded distorsion then $\tilde{X}$ is biholomorphic to the bidisk.*

Proof If U is the graph of some holomorphic motion of bounded distorsion then the same is true for $\tilde{X}$. Suppose that $\tilde{X}$ is not the bidisk and is the graph of a holomorphic motion Ψ of bounded distorsion. We think of $\tilde{X}$ as embedded in $V(1)$. Denote by $\tilde{X}_x$ the fiber of $s : \tilde{X} \to \mathbb{D}$ over $x \in \mathbb{D}$. Since the action of Mod(1) on $T(1)$ is transitive we can assume that $\tilde{X}_0$ is $\mathbb{H}$. An element of $\pi_1(X)$ acts on $\tilde{X}$ by biholomorphisms and acts on the circle $\partial\mathbb{H}$ by quasisymmetric homeomorphisms via the embedding $\alpha : \pi_1(X) \to \mathrm{Mod}(1)$. The relation between these two actions is as follows. Let $f \in \pi_1(X)$. The fiber $\tilde{X}_0$ is sent by f to another fiber $\tilde{X}_x$ where $x = \rho(f)(0)$, then it can be sent back to itself by following the holomorphic motion Ψ; in this way we obtain a quasiconformal self-homeomorphism of $\tilde{X}_0 = \mathbb{H}$ associated with f. The extension of this quasiconformal map to the boundary circle is nothing else but the quasisymmetric homeomorhism $\alpha(f)$ (cf. Theorem 7.3.6 and Remark 7.3.7); in particular it does not depend on the holomorphic motion Ψ. Thus we see that if Ψ has bounded distorsion then every element of $\alpha(\pi_1(X))$ is κ-quasisymmetric for some fixed κ. By Theorem 7.4.7 $\alpha(\pi_1(X))$ is conjugate to a subgroup of $\mathrm{PSL}_2(\mathbf{R})$ in Mod(1), contradiction to Lemma 7.4.9. □

Remark 7.4.11 Since U is determined by $\partial_h(U)$, the condition that U is not the graph of any holomorphic motion of bounded distorsion is a condition on $\partial_h(U)$. But it is weaker than saying that the unique holomorphic motion of which $\partial_h(U)$ is the graph has bounded distorsion. For example the complement of the closure of $F(1) = V(1)$ in $T(1) \times \mathbb{P}^1$ is biholomorphic to $T(1) \times \mathbb{H}$ but its boundary is the same as the boundary of $F(1)$ and does not have bounded distorsion.

On the other hand even if $\partial_h(U)$ is the graph of a constant holomorphic motion, U may be the graph of a holomorphic with unbounded distorsion; see [McM07] for such an example.

Actually what we proved is a theorem of independant interest for Brunella exotic foliations:

Theorem 7.4.12 *The universal cover of a Brunella exotic foliation is not the graph of a holomorphic motion of bounded distorsion.*

Remark 7.4.13 We hope that our approach (cf. Question 7.5.6) shed some light on the study of Brunella exotic foliations, though Theorem 7.4.12 is still far from answering the question whether Brunella exotic foliations exist. It seems that Theorem 7.4.12 is so far the only result on Brunella exotic foliations after the work of Brunella himself [Bru97], [Bru03].

7.4.3 Carathéodory's theorem

In this subsection we assume that *U is simply connected and $\partial_h(U)$ is locally connected*. In other words we assume that for any $x \in \mathbb{D}$, U_x is a simply connected domain in $\mathbb{P}^1_x$ and A_x is locally connected. We recall a classical theorem of Carathéodory:

Theorem 7.4.14 (Carathéodory [Pom92] Chapter 2) *Let $W \subset \mathbb{P}^1$ be a simply connected open set. Let $f : \mathbb{D} \to W$ be a biholomorphism. Then the following assertions are equivalent*

1. *f extends continuously to the closures $\overline{\mathbb{D}} \to \overline{W}$.*
2. *∂W is locally connected.*
3. *$\mathbb{P}^1 \backslash W$ is locally connected.*

The extension is a bijection if and only if it is a homeomorphism if and only if ∂W is a Jordan curve.

Recall that by Theorem 7.4.1 there is a morphism of admissible families given by $\varphi : \mathbb{D} \to T(1), \hat{\varphi} : U \to V(1)$ and $\hat{\theta} : \Gamma \to \mathrm{Aut}(V(1))$. More precisely there are holomorphic maps $\varphi : \mathbb{D} \to T(U_0)$ and $\theta : \Gamma \to \mathrm{Aut}(T(U_0)), \hat{\varphi} : U \to V(U_0)$. For convenience we introduce the following notations:

1. If U is not biholomorphic to the bidisk, then we denote $\hat{\varphi}(U)$ by U'. The boundary of $V(1) = F(1)$ in $T(1) \times \mathbb{P}^1$ is the graph of a holomorphic motion of the circle over $T(1)$, we denote by $\partial_h(U')$ the part of that boundary over $\varphi(U)$. We identify $\mathbb{D}$ with $\varphi(\mathbb{D})$. Then $\partial_h(U')$ is the graph of a holomorphic motion of a quasicircle A' over $\mathbb{D}$. Since $\mathrm{Mod}(1)$ acts transitively on $T(1)$, up to postcomposing $\hat{\varphi}$ by an element of $\mathrm{Aut}(V(1))$ we can and will assume that $U'_0 = \mathbb{D}$ and A'_0 is the unit circle.
2. If U is biholomorphic to the bidisk, $U' = \mathbb{D} \times \mathbb{D}$, $\partial U' = \mathbb{D} \times \mathbb{S}^1$ and $\hat{\varphi}$ is a fiber preserving biholomorphism $U \to U'$.

In other words the admissible family $U' \to \mathbb{D}$ is isomorphic to $U \to \mathbb{D}$, but is embedded in $\mathbb{D} \times \mathbb{P}^1$ in a different way so that the boundary is the graph of a holomorphic motion of a Jordan curve.

Lemma 7.4.15 *$\hat{\varphi}^{-1}$ extends to a continuous map $F : \partial_h(U') \to \partial_h(U)$ which sends leaves to leaves.*

Proof Consider the restriction of $\hat{\varphi}^{-1}$ to the fiber U'_0. It extends to a continuous map $f_0 : A'_0 = (S)^1 \to A_0$ by Carathéodory's theorem. Let Ψ, Ψ' be holomorphic motions of respectively $A_0 \cap$

U_0 and $A'_0 \cap U_0$ such that 1) $U \cap \partial_h(U)$ and $U' \cap \partial_h(U')$ are graphs of respectively Ψ and Ψ'; 2) $\hat{\varphi}$ sends the leaves of Ψ in U to leaves of Ψ' in U', i.e. Ψ coincides in U with Ψ' via φ. Define a map $F : \partial_h(U') \cap U' \to \partial_h(U) \cap U$ by

$$F\left((x, \Psi'_x(y))\right) = (x, \Psi_x(f_0(y))).$$

The map F restricted to U' is nothing else but $\hat{\varphi}^{-1}$. It is continuous because topologically it is just the map $\mathbb{D} \times \mathbb{D} \cap (S)^1 \to \mathbb{D} \times \mathbb{D} \cap A$ that sends (x,y) to (x,y) if $y \in \mathbb{D}$ and to $(x, f_0(y))$ if $y \in (S)^1$. It sends leaves to leaves by construction. $\square$

Recall that Γ acts on $\partial_h(U)$ (cf. Corollary 7.2.14) and also on $\partial_h(U')$ (cf. Remark 7.3.7) via $\hat{\theta} : \Gamma \to \mathrm{Mod}(1)$; both actions send leaves to leaves. Since a continuous map from a Hausdorff space to a Hausdorff space extends in at most one way to its closure and F is Γ-equivariant in U', we have

Lemma 7.4.16 *F is Γ-equivariant.*

Lemma 7.4.17 *If U is biholomorphic to the bidisk then F is injective, i.e. A is a Jordan curve*

Proof Since F sends the leaves in $\partial_h(U')$ to leaves in $\partial_h(U)$, it induces a continuous map $g : A' = \mathbb{S}^1 \to A$. Γ is isomorphic to an irreducible lattice in $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$. The map g is Γ-equivariant, the action of Γ on A' being induced by the projection of Γ onto the second factor $\mathrm{PSL}_2(\mathbf{R})$. Suppose by contradiction that $g(a) = g(b)$ for two distincts points $a, b \in A'$. Let $c, d \in A'$ such that $g(c) \neq g(d)$. Then there are neighbourhoods W_c, W_d respectively of c and d such that $g(W_c) \cap g(W_d) = \emptyset$. As $\mathrm{PSL}_2(\mathbf{R})$ acts 2-transitively on $\mathbb{S}^1$ and the projection of Γ in $\mathrm{PSL}_2(\mathbf{R})$ is dense, there is $\gamma \in \Gamma$ such that $\gamma \cdot a \in W_c$ and $\gamma \cdot b \in W_d$. By equivariance we have $g(\gamma \cdot a) = g(\gamma \cdot b)$, contradiction. $\square$

Lemma 7.4.18 *If A is locally connected and U is biholomorphic to the bidisk then $V = \mathbb{D} \times \mathbb{P}^1 \backslash (U \cap \partial_h(U))$ is biholomorphic to the bidisk.*

Proof By the previous lemma $\partial_h(U)$ is the graph of a holomorphic motion of a Jordan curve. Thus $V = \mathbb{D} \times \mathbb{P}^1 \backslash (U \cap \partial_h(U))$ is the graph of a holomorphic motion of a simply connected domain, i.e. is an admissible family over $\mathbb{D}$ of fiber model $\mathbb{D}$. Suppose by contradiction that V is not biholomorphic to the bidisk. As in Theorem 7.4.1 we have an embedding of admissible family $\hat{\phi} : V \to V(1)$, equivariant under a homomorphism $\hat{\varsigma} : \Gamma \to \mathrm{Mod}(1)$. If we think of $\hat{\varsigma}(\Gamma)$ as a quasisymmetric group of the circle, then its action is just conjugate to the action of Γ on the Jordan curve A. Since U is biholomorphic to the bidisk, this action is conjugate to the action of a dense subgroup of $\mathrm{PSL}_2(\mathbf{R})$ on $\mathbb{S}^1$. As in Lemma 7.4.9, we obtain a sequence of birational transformations in Γ which approaches the identity map on V. Thus the sequence approaches the identity on U as well, contradiction. $\square$

Lemma 7.4.19 *If A is locally connected and has zero Lebesgue measure in $\mathbb{P}^1$ and if U is biholomorphic to the bidisk, then ψ is a trivial holomorphic motion.*

Proof By the previous lemma we know that U and V are biholomorphic to the bidisk under the hypothesis. We can extend to ψ to a holomorphic motion Ψ of $\mathbb{P}^1$ such that for any $x \in \mathbb{D}$, Ψ_x is holomorphic in $\mathbb{P}^1 \backslash A$. Since A has zero Lebesgue measure, the Beltrami differential of Ψ_x vanishes almost everywhere for any x. By Theorem 7.3.1 Ψ_x is holomorphic everywhere and is in fact a Möbius transformation. Thus ψ is a trivial holomorphic motion. $\square$

7.4.4 Trivial holomorphic motions

Proposition 7.4.20 *Suppose that the holomorphic motion ψ of which $\partial_h(U)$ is the graph is trivial. Then up to geometric conjugation in the Jonquières group, the birational Kleinian group (Y, Γ, U, X) satisfies*

— $Y = \mathbb{P}^1 \times \mathbb{P}^1$ *and* $U = \mathbb{D} \times \mathbb{D}$;

— Γ *is an irreducible cocompact lattice in* $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R}) \subset \mathrm{PGL}_2(\mathbf{C}) \times \mathrm{PGL}_2(\mathbf{C}) = \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$.

Proof Under the hypothesis U is biholomorphic to a product. Thus $\tilde{X}$ is the bidisk. Then $\rho : \pi_1(X) \to \mathrm{PSL}_2(\mathbf{R})$ is just the projection of an irreducible lattice in $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$ onto one factor. In particular ρ is injective and $U = \tilde{X}$. Let $H : \mathbb{D} \times \mathbb{P}^1 \to \mathbb{D} \times \mathbb{P}^1, (x,y) \mapsto (x, h(x)(y))$ be the biholomorphism that sends $\partial_h(U)$ to $\mathbb{D} \times A \subset \mathbb{D} \times \mathbb{P}^1$ where h is a holomorphic map from $\mathbb{D}$ to $\mathrm{PGL}_2(\mathbf{C})$. Then $H\Gamma H^{-1}$ preserves both the projections $\mathbb{D} \times \mathbb{P}^1 \to \mathbb{D}$ and $\mathbb{D} \times \mathbb{P}^1 \to \mathbb{P}^1$, i.e. it is a subgroup of $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PGL}_2(\mathbf{C})$. By Margulis superrigidity the projection of $H\Gamma H^{-1}$ in the $\mathrm{PGL}_2(\mathbf{C})$ factor is in $\mathrm{PSL}_2(\mathbf{R})$ up to conjugation. This means that A is a round disk in $\mathbb{P}^1$.

To finish the proof we need to prove that the conjugation H is algebraic, i.e. h is the restriction to $\mathbb{D}$ of a rational map on $\mathbb{P}^1$.

The fact that Γ is isomorphic to a cocompact irreducible lattice in $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$ implies the following two observations:

1. Γ has subgroups isomorphic to $\mathbf{Z}^2$ because $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$ has real rank two.

2. Elements of $\rho(\Gamma) \subset \mathrm{PSL}_2(\mathbf{R})$, or equivalently elements of Γ_B, are either elliptic or loxodromic because every element of a cocompact lattice is semisimple.

Thus by Theorem 1.5.3 there are two birational transformations $\gamma_1, \gamma_2 \in \Gamma$ which generate a subgroup isomorphic to $\mathbf{Z}^2$ and can be written, up to conjugation in Jonq, as $\gamma_i : (x,y) \mapsto (a_i x, b_i y), i = 1,2$ or $\gamma_i : (x,y) \mapsto (a_i x, b_i + y), i = 1,2$. Here the latter case is impossible because the holomorphic conjugation H, in $\mathbb{D} \times \mathbb{P}^1$, is supposed to make $\gamma_i, i = 1,2$ also multiplicative in the y coordinate. Hence we can write them as $\gamma_i : (x,y) \mapsto (a_i x, b_i y), i = 1,2$.

The two horizontal disks $S_1 = \{y = 0, x \in \mathbb{D}\}, S_2 = \{y = \infty, x \in \mathbb{D}\}$ are invariant under $\gamma_i, i = 1,2$. We want to show that they are leaves in $\partial_h(U)$. Consider the actions of $\gamma_i, i = 1,2$ on the x coordinate, i.e. the $\gamma_{iB}, i = 1,2$. The subgroup of $\mathbf{C}^*$ generated by a_1, a_2 is isomorphic to $\mathbf{Z}^2$, thus dense. Therefore there is a sequence of birational transformations $(\delta_n)_n$ in the subgroup generated by γ_1, γ_2 such that δ_{nB} tends to the identity. Up to extracting a subsequence from $(\delta_n)_n$, $\delta_n|_U$ converges to a holomorphic map from U to $U \cap \partial U$. By discontinuity of Γ, the limit map has values in ∂U; as δ_{nB} tends to the identity, the image of the limit map is a horizontal disk in $\partial_h(U)$. Since each δ_n has the form $(x,y) \mapsto (c_n x, d_n y)$, the limit horizontal disk is either S_1 or S_2. By considering the sequence $(\delta_n^{-1})_n$, we infer that S_1 and S_2 are both leaves in $\partial_h(U)$.

Thus the holomorphic conjugation map H has the form $(x,y) \mapsto (x, h(x)y)$ where h is a holomorphic function $\mathbb{D} \to \mathbf{C}^*$. To conclude that h is algebraic, it suffices to exhibit another algebraic leaf in $\partial_h(U)$. We can conjugate the γ_1, γ_2 in Γ to obtain $\gamma_3, \gamma_4 \in \Gamma$ such that $< \gamma_3, \gamma_4 >$ is isomorphic to $\mathbf{Z}^2$ but γ_3, γ_4 do not commute with γ_1, γ_2. We apply the above discussion to γ_3, γ_4 and obtain other algebraic leaves in $\partial_h(U)$, actually infinitely many. The proof is finished. □

As a corollary of Lemma 7.4.19 and Proposition 7.4.20, we obtain

Theorem 7.4.21 *Suppose that there is a point $x \in \mathbb{D}$ such that the intersection of ∂U with the fiber of r over x is locally connected and has zero Lebesgue measure in $\mathbb{P}^1$. Suppose that U is biholomorphic to the bidisk. Then up to geometric conjugation in the Jonquières group, the birational Kleinian group (Y, Γ, U, X) satisfies*

— *$Y = \mathbb{P}^1 \times \mathbb{P}^1$ and $U = \mathbb{D} \times \mathbb{D}$;*

— *Γ is an irreducible cocompact lattice in* $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R}) \subset \mathrm{PGL}_2(\mathbf{C}) \times \mathrm{PGL}_2(\mathbf{C}) = \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$.

Circles If we do not assume that U is biholomorphic to the bidisk, then we can prove the same conclusion under a strong hypothesis on A.

Lemma 7.4.22 *If A_x is a round circle for some $x \in \mathbb{D}$ then the holomorphic motion ψ is trivial.*

Proof Suppose that A_x is a circle. Then for any $\gamma \in \Gamma$, $A_{\gamma_B(x)} = \gamma(A_x)$ is also a circle. Thus for z in a dense subset of $\mathbb{D}$, A_z is a circle. Let a,b,c,d be three distinct points in A. Consider the function $CR(a,b,c,d)$ which associates to $z \in \mathbb{D}$ the cross ratio of $(\psi_z(a), \psi_z(b), \psi_z(c), \psi_z(d))$. It is a holomorphic in z and takes value in $\mathbf{R}$ for z contained in a dense subset of $\mathbb{D}$. Thus $CR(a,b,c,d)$ is constant with value in $\mathbf{R}$. If we fix a,b,c and let d vary, then we see that for any z and any d, $\psi_z(d)$ lies on the circle that passes through $\psi_z(a), \psi_z(b), \psi_z(c)$. This implies that the circle that passes through $\psi_z(a), \psi_z(b), \psi_z(c)$ is exactly A_z for any $z \in \mathbb{D}$. Let $h(z) \in \mathrm{PGL}_2(\mathbf{C})$ be the Möbius transformation that sends $\psi_z(a), \psi_z(b), \psi_z(c)$ respectively to $0, 1, \infty$; it depends holomorphically on z. Then the biholomorphism $(x,y) \mapsto (x, h(x)(y))$ of $\mathbb{D} \times \mathbb{P}^1$ sends $\partial_h(U)$ to $\mathbb{D} \times \partial\mathbb{H}$. $\square$

As a corollary of Proposition 7.4.20 and Lemma 7.4.22, we have

Theorem 7.4.23 *Suppose that there is a point $x \in \mathbb{D}$ such that the intersection of ∂U with the fiber of r over x is a round circle. Then up to geometric conjugation in the Jonquières group, the birational Kleinian group (Y, Γ, U, X) satisfies*

— $Y = \mathbb{P}^1 \times \mathbb{P}^1$ *and* $U = \mathbb{D} \times \mathbb{D}$*;*

— Γ *is an irreducible cocompact lattice in* $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R}) \subset \mathrm{PGL}_2(\mathbf{C}) \times \mathrm{PGL}_2(\mathbf{C}) = \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$.

7.5 Smoothness of the boundary

We say that $\partial_h(U)$ is C^1 at a point $p \in \partial_h(U)$ if there are a neighbourhood W of p in Y such that $\partial_h(U) \cap W$ is a C^1 real hypersurface of W. The goal of this section is to prove:

Theorem 7.5.1 *Suppose that $\partial_h(U)$ is C^1 at a point. Then up to geometric conjugation in the Jonquières group, the birational Kleinian group (Y, Γ, U, X) satisfies*

— $Y = \mathbb{P}^1 \times \mathbb{P}^1$ *and* $U = \mathbb{D} \times \mathbb{D}$*;*

— Γ *is an irreducible cocompact lattice in* $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R}) \subset \mathrm{PGL}_2(\mathbf{C}) \times \mathrm{PGL}_2(\mathbf{C}) = \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$.

Remark 7.5.2 We did not assume that U is simply connected in Theorem 7.5.1. Thus Theorem 7.5.1 is stronger than what we stated in Theorem 0.3.5.

Under the hypothesis of Theorem 7.5.1, we can find an open subset $W \subset \mathbb{D} \times \mathbb{P}^1$ such that

1. $W = W_1 \times W_2$ where W_1 is a disk in $\mathbb{D}$ of hyperbolic radius 3δ for some $\delta > 0$ and W_2 is a disk in $\mathbb{P}^1$;
2. $\partial_h(U) \cap W$ is a connected C^1 real hypersurface of W;
3. for any $x \in W_1$, $(\partial_h(U) \cap W) \cap \mathbb{P}^1_x = A_x \cap W$ is a connected C^1 open arc, i.e. the image of a C^1 embedding from $(0,1)$ to $\mathbb{P}^1_x$.

Let $p \in W$ be a point such that $r(p)$ is the center of the disk W_1. Let $K \subset U$ be a compact connected weak fundamental domain (cf. Lemma 7.2.6). As in the proof of Proposition 7.2.8, we cover K by finitely many small round bidisks $(D_1^j \times D_2^j)_{j \in J}$ such that

1. The D_1^js are disks in $r(U) = \mathbb{D}$ with hyperbolic radii all equal to δ and the D_2^js are disks in $\mathbb{P}^1$.
2. For any $j \in J$, $M_j = D_1^j \times D_2^j \subset U$.

Since Γ_B is dense in $\mathrm{PSL}_2(\mathbf{R})$, we can find a transformation $\gamma \in \Gamma$ such that γ_B is an elliptic element with center $x \in D_1^j$. We denote by $\gamma_x \in \mathrm{Aut}(\mathbb{P}^1_x) = \mathrm{PGL}_2(\mathbf{C})$ the restriction of γ to $\mathbb{P}^1_x$.

By Lemma 7.2.7 the point p is K-accessible. Thus there is at least one $j \in J$ such that p is M_j-accessible. Let $j \in J$ and $(\gamma_n)_n$ be a sequence of elements of Γ such that for any neighbourhood Ω of p there exists $N(\Omega) \in \mathbf{N}$ such that $\gamma_n(M_j) \cap \Omega \neq \emptyset$ for any $n > N(\Omega)$. Since $r(W)$ has radius 3δ and D_1^j has radius δ, we can and will assume that for any n, $\gamma_{nB}(D_1^j) \subset r(W)$. We take a Cartan KAK decomposition in $\mathrm{PSL}_2(\mathbf{R})$: $\gamma_{nB} = F_n G_n H_n$ where $F_n, H_n \in \mathrm{PSO}(2)$ and G_n is represented by a diagonal matrix in $\mathrm{SL}_2(\mathbf{R})$. By extracting a subsequence we can and will assume that F_n, H_n converge in $F_n, H_n \in \mathrm{PSO}(2)$. Since $\gamma_{nB}(D_1^j) \subset W_1$ for any n, we can and will also assume that G_n converges to a loxodromic element. In other words we can and will assume that γ_{nB} converges to $\gamma_{Blim} \in \mathrm{PSL}_2(\mathbf{R})$.

Consider the restriction of γ_n to $\mathbb{P}^1_x$. It is an isomorphism from $\mathbb{P}^1_x$ to $\mathbb{P}^1_{\gamma_B(x)}$; we denote it by γ_{nx}. By identifying $\mathbb{P}^1_x$ with $\mathbb{P}^1_{\gamma_B(x)}$, we think of γ_{nx} as an element of $\mathrm{PGL}_2(\mathbf{C})$. We take a Cartan KAK decomposition: $\gamma_{nx} = f_{nx} g_{nx} h_{nx}$ where $f_{nx}, h_{nx} \in \mathrm{PU}(2)$ and g_{nx} is represented by a diagonal matrix in $\mathrm{SL}_2(\mathbf{R})$. Up to extracting a subsequence, we can and will assume that f_{nx}, h_{nx} converge respectively to f_{xlim} and g_{xlim} in $\mathrm{PU}(2)$. By discontinuity of Γ, we can and will assume that g_{nx} converges to a map $g_{xlim} : \mathbb{P}^1 \to \mathbb{P}^1$ that fixes one point and maps all other points to a point. In other words we can and will assume that γ_{nx} converges to a map $\gamma_{xlim} : \mathbb{P}^1_x \to \mathbb{P}^1_y$ such that

1. $y = \gamma_{Blim}(x) \in r(W)$;
2. except one point $q_x \in \mathbb{P}^1_x$, γ_{xlim} maps all points of $\mathbb{P}^1_x$ to a point in $A_y \cap W = (\partial_h(U) \cap \mathbb{P}^1_y) \cap W$;

3. for any compact subset $C \subset \mathbb{P}^1_x \backslash \{q_x\}$, there exists $N \in \mathbf{N}$ such that for any $n > N$, $\gamma_{nx}(C) \in W$.

Lemma 7.5.3 *A_x is a Jordan curve containg q_x.*

Proof For n large enough, $y = \gamma_{nB}(x) \in W_1$. Consider $A_y \cap W = A_y \cap W_2$; it is a connected C^1 arc. The image of $A_y \cap W_2$ by γ_{nx}^{-1} is the part of A_x in the open disk $\gamma_{nx}^{-1}(W_2)$; it is also a C^1 arc. By our hypothesis the union of the $\gamma_{nx}^{-1}(W_2)$s is $\mathbb{P}^1_x \backslash \{q_x\}$. Thus $A_x \cap (\mathbb{P}^1_x \backslash \{q_x\})$ is a connected $\mathbf{C}^1$ curve. Furthermore q_x is its cluster point and is the only one. Thus $A_x = \big(A_x \cap (\mathbb{P}^1_x \backslash \{q_x\})\big) \cup \{q_x\}$ is either a Jordan curve or there is a homeomorphism $h : [0,1] \to A_x$ such that $h(0) = q_x$. The latter is not possible because for any neighbourhood Ω of q_x, $A_x \backslash \Omega$ is diffeomorphic to an open interval in W. $\square$

Lemma 7.5.4 *If γ_x is a loxodromic element of* $\mathrm{PGL}_2(\mathbf{C})$*, then A_x is a circle.*

Proof Suppose that γ_x is a loxodromic element. We choose adequate coordinate on $\mathbb{P}^1_x$ so that γ_x is $z \mapsto \lambda z$ for some $\lambda \in \mathbf{C}^*$. At least one of its two fixed points is not q_x; let us say that $0 \neq q_x$.

The boundary A_x is a closed subset of $\mathbb{P}^1_x$ invariant under γ_x. Under positive or negative iterates of γ_x, every point of $\mathbb{P}^1_x \backslash \{0, \infty\}$ converges to 0 or ∞. Thus $0, \infty \in A_x$. Since $\gamma_{nx}(0) \in W$ for n large enough, locally around 0 the boundary A_x is a C^1 arc. As the tangent direction to A_x at 0 should be preserved by γ_x, we infer that $\lambda \in \mathbf{R}^*$. By γ_x-invariance, to prove that A_x is circle, it suffices to prove that A_x is a circular arc locally around 0. If A_x is the pure imaginary axis then it is a circle. Assume that A_x is not the pure imaginary axis and parametrize it loccaly around 0 as $u + \eta(u)i$, where $u \in \mathbf{R}$ and η is a real C^1 function. The γ_x-invariance gives $\eta(\lambda u) = \lambda(\eta(u))$ for any u in a neighbourhood of 0 in $\mathbf{R}$. By differentiating we obtain $\eta'(u) = \eta'(\lambda u)$ and thus $\eta'(u) = \eta'(0)$ is a constant function. Hence A_x is a circle. $\square$

By Theorem 7.4.23 we have immediately

Corollary 7.5.5 *If γ_x is a loxodromic element of* $\mathrm{PGL}_2(\mathbf{C})$*, then the conclusion of Theorem 7.5.1 holds.*

Note that γ_x is not elliptic because otherwise its action on $\mathbb{P}^1_x$ would not be discontinuous. To finish the proof of Theorem 7.5.1 it suffices to prove that γ_x is not parabolic.

Now we assume by contradiction that γ_x is parabolic. Since A_x is a Jordan curve by Lemma 7.5.3, U_x is a simply connected domain. Let $\varphi : \mathbb{H} \to U_x$ be a biholomorphism. By Carathéodory Theorem (cf. Theorem 7.4.14) it extends to a homeomorphism $\mathbb{H} \cup \partial\mathbb{H} \to U_x \cup A_x$. Thus as γ_x

fixes only one point in A_x, $\gamma_x' = \varphi^{-1} \circ \gamma_x \circ \varphi$ has only one fixed point in $\partial\mathbb{H}$ and is a parabolic element of $\mathrm{PSL}_2(\mathbf{R})$. Without loss of generality, we can assume that γ_x' is the map $z \mapsto z+1$. For $u \in \mathbf{R}$ let L_u be the horizontal segment in $\mathbb{H}$ connecting ui and $1+ui = \gamma_x'(ui)$. Since a holomorphic map can only decrease the Kobayashi hyperbolic distance, the length of $\varphi(L_u)$ with respect to the Kobayashi distance of U is smaller than the length of L_u with respect to the Poincaré distance which goes to 0 when u goes to $+infty$. However for any u the curve $\varphi(L_u)$ descends to a loop in X which represents the same non trivial class in $\pi_1(X)$. This contradicts the compactness of X. Thus γ_x is not parabolic and the proof of Theorem 7.5.1 is finished.

QUESTIONS

Cocompact Fatou components

We have no classification for birational Kleinian groups (Y,Γ,U,X) in dimension two where Γ is virtually cyclic. If (Y,Γ,U,X) is a birational Kleinian group such that Γ is generated by a birational transformation γ, then U is contained in an invariant Fatou component of γ. More generally one may ask for a classification of invariant Fatou components of birational transformations of surfaces on which the action is cocompact. First of all do such Fatou components exist?

The only virtually cyclic birational Kleinian group that we are able to treat is the case where X is an Inoue surface (cf. [Zhab] and Appendix B); in this case we combined birational dynamics with the very particular geometry of foliations on Inoue surfaces.

Lifting classical Kleinian groups

In Cases 18), 19), 20) of Theorem 0.3.7, can we give a more precise classification of $\Gamma \subset \mathrm{Jonq}$. More generally what can we say about subgroups of $\mathrm{PGL}_2(\mathbf{C}(x)) \rtimes \mathrm{PGL}_2(\mathbf{C})$ that project isomorphically onto classical Kleinian groups in $\mathrm{PGL}_2(\mathbf{C})$?

Irreducible cocompact lattices in $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$

Let Γ be an irreducible cocompact lattice in $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$ and $\rho : \Gamma \to \mathrm{Bir}(\mathbb{P}^2)$ be an embedding. Is $\rho(\Gamma)$ necessarily birationally conjugate to a subgroup of $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R}) \subset \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$? If the unswer is yes, then in Case 7) of Theorem 0.3.7 we are automatically in Case 2) under the additional hypothesis that U is biholomorphic to the bidisk.

Embeddings into $\mathrm{Bir}(\mathbb{P}^2)$ of irreducible lattices in semisimple Lie groups of rank ≥ 2 have been studied in [Dés06a], [Can11], [CX18], [CC19]. If a lattice has property (T), or congruence subgroup property, or property FW, then the classification is complete. However irreducible cocompact lattices in $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$ are still not known to have any of these three properties.

Brunella exotic foliations and Teichmüller modular groups

We extract from our approach in Chapter 7 the following problem (see Theorem 7.4.1).

Let $T(G)$ be a Teichmüller space where G is a torsion free fuchsian group. Let $\pi : V(G) \to T(G)$ be the universal curve. The automorphism group of $V(G)$ that preserves the fibration π is $\mathrm{Mod}(G)$. Let D be a holomorphic disk in $T(G)$ totally geodesic with respect to the Teichmüller metric and let $E \to D$ be the restriction of the universal curve. Let Γ_E be the subgroup of $\mathrm{Mod}(G)$ that preserves E. We denote by Γ_D its image in $\mathrm{PSL}_2(\mathbf{R}) = \mathrm{Aut}(D)$.

Assume that $\mathbb{H}/G$ is a compact Riemann surface. Then Γ_D is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{R})$ related to the so called *Veech group*. Veech proved that E/Γ_E is never compact (cf. [Vee89]). If moreover Γ_D is a lattice in $\mathrm{PSL}_2(\mathbf{R})$, then it is a non-uniform lattice and E/G is a fibration over the quasi-projective curve D/Γ_D, called a *Teichmüller curve*. The study of Teichmüller curves is currently very active because of its relations to billard dynamics and algebraic geometry; variations of Hodge structures play an important role in the study of Teichmüller curves (cf. [Möl]).

Assume that $\mathbb{H}/G$ is not a Riemann surface of finite type. Then $T(G)$ and $V(G)$ are infinite dimensional complex Banach manifolds. It is possible for Γ_D to be a dense subgroup of $\mathrm{PSL}_2(\mathbf{R})$ (cf. [PSV11]).

Question 7.5.6 *For infinite dimensional $T(G)$, is it possible that Γ_D is dense in* $\mathrm{PSL}_2(\mathbf{R})$ *and*

1. *E/Γ_E is a quasi-projective surface?*
2. *E/Γ_E is compact?*

If yes what is the classification?

Suppose that Γ is a subgroup of finite index of Γ_E such that $X = E/\Gamma_E$ is a smooth quasi-projective surface. Then the fibration $E \to D$ induces a regular transversally hyperbolic foliation $\mathscr{F}$ on X; if X is compact then it is a Brunella exotic foliation. As in Section 7.1, by using Theorem 7.1.7 from non-abelian Hodge theory, we know that up to taking a finite étale cover there is a morphism $f : X \to Z$ to a polydisk Shimura variety Z such that $\mathscr{F}$ is the pullback of one of the tautological foliations on Z (cf. [Tou16]). The most interesting case would be where f is an embedding. Question 7.5.6 is thus also a question about certain subvarieties of polydisk Shimura varieties.

APPENDIX A

CENTRALIZERS OF ELEMENTS OF INFINITE ORDER IN PLANE CREMONA GROUPS

A.1 Introduction

Let $\mathscr{K}$ be an algebraically closed field. The *plane Cremona group* $\mathrm{Cr}_2(\mathbf{K})$ is the group of birational transformations of the projective plane $\mathbb{P}^2_{\mathscr{K}}$. It is isomorphic to the group of $\mathscr{K}$-algebra automorphisms of $\mathscr{K}(X_1,X_2)$, the function field of $\mathbb{P}^2_{\mathscr{K}}$. Using a system of homogeneous coordinates $[x_0;x_1;x_2]$, a birational transformation $f \in \mathrm{Cr}_2(\mathbf{K})$ can be written as

$$[x_0 : x_1 : x_2] \dashrightarrow [f_0(x_0,x_1,x_2) : f_1(x_0,x_1,x_2) : f_2(x_0,x_1,x_2)]$$

where f_0, f_1, f_2 are homogeneous polynomials of the same degree without common factor. This degree does not depend on the system of homogeneous coordinates. We call it the *degree* of f and denote it by $deg(f)$. Geometrically it is the degree of the pull-back by f of a general projective line. Birational transformations of degree 1 are homographies and form $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{K}}) = \mathrm{PGL}_3(\mathbf{K})$, the group of automorphisms of the projective plane.

Four types of elements. Following the work of M.H. Gizatullin, S. Cantat, J. Diller and C. Favre, we can classify an element $f \in \mathrm{Cr}_2(\mathbf{K})$ into exactly one of the four following types according to the growth of the sequence $(deg(f^n))_{n\in\mathbf{N}}$ (The standard reference [DF01] is written for $\mathscr{K} = \mathbf{C}$ but it is known that the same proof works over an algebraically closed field $\mathscr{K}$ of characteristic different from 2 and 3. The only problem with characteristics 2 and 3 is that the important ingredient [Giz80] does not deal with quasi-elliptic fibrations. This minor issue has been clarified in [CD12a] and [CGL] so that the following classification holds for arbitrary characteristic.):

1. The sequence $(deg(f^n))_{n\in\mathbf{N}}$ is bounded, f is birationally conjugate to an automorphism of a rational surface X and a positive iterate of f lies in the connected component of the identity of the automorphism group $\mathrm{Aut}(X)$. We call f an *elliptic* element.
2. The sequence $(deg(f^n))_{n\in\mathbf{N}}$ grows linearly, f preserves a unique pencil of rational curves and f is not conjugate to an automorphism of any rational surface. We call f a *Jonquières twist*.
3. The sequence $(deg(f^n))_{n\in\mathbf{N}}$ grows quadratically, f is conjugate to an automorphism of a rational surface preserving a unique elliptic fibration. We call f a *Halphen twist*.
4. The sequence $(deg(f^n))_{n\in\mathbf{N}}$ grows exponentially and f is called *loxodromic*.

The Jonquières group Fix an affine chart of $\mathbb{P}^2$ with coordinates (x,y). *The Jonquières group* Jonq is the subgroup of the Cremona group of all transformations of the form

$$(x,y) \dashrightarrow \left(\frac{ax+b}{cx+d}, \frac{A(x)y+B(x)}{C(x)y+D(x)}\right), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{K}), \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{PGL}_2(\mathscr{K}(x)).$$

In other words, Jonq is the group of all birational transformations of $\mathbb{P}^1 \times \mathbb{P}^1$ permuting the fibres of the projection onto the first factor; it is isomorphic to the semi-direct product $\mathrm{PGL}_2(\mathbf{K}) \ltimes \mathrm{PGL}_2(\mathscr{K}(x))$. A different choice of the affine chart yields a conjugation by an element of $\mathrm{PGL}_3(\mathbf{K})$. More generally a conjugation by an element of the Cremona group yields a group preserving a pencil of rational curves; conversely any two such groups are conjugate in $\mathrm{Cr}_2(\mathbf{K})$. Elements of Jonq are either elliptic or Jonquières twists. We denote by $\mathrm{Jonq}_0(\mathbf{K})$ the normal subgroup of Jonq that preserves fibrewise the rational fibration, i.e. the subgroup of those transformations of the form $(x,y) \dashrightarrow \left(x, \frac{A(x)y+B(x)}{C(x)y+D(x)}\right)$; it is isomorphic to $\mathrm{PGL}_2(\mathscr{K}(x))$. A Jonquières twist of the de Jonquières group will be called a *base-wandering Jonquières twist* if its action on the base of the rational fibration is of infinite order.

If $\mathscr{K} = \overline{\mathbf{F}_p}$ is the algebraic closure of a finite field, then $\mathscr{K}, \mathscr{K}^*$ and $\mathrm{PGL}_2(\mathbf{K})$ are all torsion groups. Thus, if $\mathscr{K} = \overline{\mathbf{F}_p}$ then base-wandering Jonquières twists do not exist. Whenever $\mathrm{char}(\mathbf{K}) = 0$, or $\mathrm{char}(\mathbf{K}) = p > 0$ and $\mathscr{K} \neq \overline{\mathbf{F}_p}$, there exist base-wandering Jonquières twists.

The group of automorphisms of a Hirzebruch surface will be systematically considered as a subgroup of the Jonquières group in the following way:

$$\mathrm{Aut}(\mathbf{F}_n) = \left\{ (x,y) \dashrightarrow \left(\frac{ax+b}{cx+d}, \frac{y+t_0+t_1x+\cdots+t_nx^n}{(cx+d)^n}\right) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathscr{K}), t_0,\cdots,t_n \in \mathscr{K} \right\}.$$

Main results.

Theorem A.1.1 *Let $f \in \mathrm{Cr}_2(\mathbf{K})$ be an element of infinite order. If the centralizer of f is not virtually abelian, then f is an elliptic element and a power of f is conjugate to an automorphism of $\mathbb{A}^2$ of the form $(x,y) \mapsto (x, y+1)$ or $(x,y) \mapsto (x, \beta y)$ with $\beta \in \mathscr{K}^*$.*

Theorem A.1.2 *Let Γ be a subgroup of $\mathrm{Cr}_2(\mathbf{K})$ which is isomorphic to $\mathbf{Z}^2$. Then Γ has a pair of generators (f,g) such that one of the following (mutually exclusive) situations happens up to conjugation in $\mathrm{Cr}_2(\mathbf{K})$:*

1. *f,g are elliptic elements and $\Gamma \subset \mathrm{Aut}(X)$ where X is a rational surface;*
2. *f,g are Halphen twists which preserve the same elliptic fibration on a rational surface X, and $\Gamma \subset \mathrm{Aut}(X)$;*
3. *one or both of the f,g are Jonquières twists, and there exist $m,n \in \mathbf{N}^*$ such that the finite index subgroup of Γ generated by f^m and g^n is in an 1-dimensional torus over $\mathscr{K}(x)$ in $\mathrm{Jonq}_0(\mathbf{K}) = \mathrm{PGL}_2(\mathscr{K}(x))$;*
4. *f is a base-wandering Jonquières twist and g is elliptic. In some affine chart, we can write f,g in one of the following forms:*
   - *g is $(x,y) \mapsto (\alpha x, \beta y)$ and f is $(x,y) \dashrightarrow (\eta(x), yR(x^k))$ where $\alpha, \beta \in \mathscr{K}^*, \alpha^k = 1, R \in \mathscr{K}(x), \eta \in \mathrm{PGL}_2(\mathbf{K}), \eta(\alpha x) = \alpha \eta(x)$ and η is of infinite order;*
   - *(only when $\mathrm{char}(\mathbf{K}) = 0$) g is $(x,y) \mapsto (\alpha x, y+1)$ and f is $(x,y) \dashrightarrow (\eta(x), y+R(x))$ where $\alpha \in \mathscr{K}^*, R \in \mathscr{K}(x), R(\alpha x) = R(x), \eta \in \mathrm{PGL}_2(\mathbf{K}), \eta(\alpha x) = \alpha \eta(x)$ and η is of infinite order.*

Remark A.1.3 When K is the algebraic closure of a finite field, the above list can be shortened since there is no elliptic elements of infinite order nor base-wandering Jonquières twists.

Remark A.1.4 From Theorem A.1.2 it is easy to see that (we will give a proof), when Γ is isomorphic to $\mathbf{Z}^2$, the degree function $deg : \Gamma \to \mathbf{N}$ is governed by the word length function with respect to some generators in the following sense. In the first case of the above theorem it is bounded. In the second case it is up to a bounded term a positive definite quadratic form over $\mathbf{Z}^2$. In the third case, if f is elliptic then deg is up to a bounded term $f^i \circ g^j \mapsto c|j|$ for some $c \in \mathbf{Q}_+$; otherwise we can choose two generators f_0, g_0 of $\Gamma \cap \mathrm{Jonq}_0(\mathbf{K})$ such that deg restricted to $\Gamma \cap \mathrm{Jonq}_0(\mathbf{K})$ is up to a bounded term $f_0^i \circ g_0^j \mapsto c_1|i| + c_2|j|$ for some $c_1, c_2 \in \mathbf{Q}_+$. In the fourth case the degree function is up to a bounded term $f^i \circ g^j \mapsto c|i|$ for some $c \in \mathbf{Q}_+$. Note that if f and g are two Jonquières twists of Jonq that do not necessarily commute, then the degree of $f^i \circ g^j$ is always dominated by $deg(f)|i| + deg(g)|j|$ (see Lemma 5.7 [BC16]).

A direct corollary of Theorem A.1.2 is:

Corollary A.1.5 *Let $G \subset \mathrm{Cr}_2(\mathbf{K})$ be a subgroup isomorphic to $\mathbf{Z}^2$. If G is not an elliptic subgroup then there exists a non-trivial element of G which preserves each member of a pencil of rational or elliptic curves.*

Theorem A.1.2 is based on several known results. The main new feature is the fourth case. We reformulate this special case as a corollary (see Theorem B.1.1 for a more precise reformulation):

Corollary A.1.6 *Let $G \subset \mathrm{Jonq}$ be a subgroup isomorphic to $\mathbf{Z}^2$. Suppose that the action of G on the base of the rational fibration is faithful. Then G is an elliptic subgroup.*

A *maximal abelian subgroup* is an abelian subgroup which is not strictly contained in any other abelian subgroup. Over the field of complex numbers, finite abelian subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ have been classified in [Bla07] and Déserti [Dés06b] has a rough classification of maximal abelian subgroups. We will use Theorem A.1.2 to classify maximal abelian subgroups of $\mathrm{Cr}_2(\mathbf{K})$ which contain at least one element of infinite order, see Theorem A.4.1.

Previously known results. Let us begin with the group of polynomial automorphism of the affine plane $\mathrm{Aut}(\mathbb{A}^2)$. It can be seen as a subgroup of $\mathrm{Cr}_2(\mathbf{K})$. It is the amalgamated product of the group of affine automorphisms with the so called *elementary group*

$$\mathrm{El}(\mathscr{K}) = \{(x,y) \mapsto (\alpha x+\beta, \gamma y+P(x)) | \alpha,\beta,\gamma \in \mathscr{K}, \alpha\beta \neq 0, P \in \mathscr{K}[x]\}.$$

Let $\mathscr{K}$ be the field of complex numbers. S. Friedland and J. Milnor showed in [FM89] that an element of $\mathrm{Aut}(\mathbf{C}^2)$ is either conjugate to an element of $\mathrm{El}(\mathscr{K})$ or to a gengeralized Hénon map, i.e. a composition $f_1 \circ \cdots \circ f_n$ where the f_i are Hénon maps of the form $(x,y) \mapsto (y, P_i(y) - \delta_i x)$ with $\delta_i \in \mathbf{C}^*$, $P_i \in \mathbf{C}[y]$, $deg(P_i) \geq 2$. S. Lamy and C. Bisi showed in [Lam01b] and [Bis04] that the centralizer in $\mathrm{Aut}(\mathbf{C}^2)$ of a generalized Hénon map is finite by cyclic, and that of an element of $\mathrm{El}(\mathbf{C})$ is uncountable (see also [Bis08] for partial extensions to higher dimension). Note that, when viewed as elements of $\mathrm{Bir}(\mathbb{P}^2)$, a generalized Hénon map is loxodromic and an element of $\mathrm{El}(\mathbf{C})$ is elliptic.

As regards the Cremona group, centralizers of loxodromic elements are known to be finite by cyclic (S. Cantat [Can11], J. Blanc-S. Cantat [BC16]). Centralizers of Halphen twists are virtually abelian of rank at most 8 (M.K. Gizatullin [Giz80], S. Cantat [Can11]). When $\mathscr{K}$ is the field of complex numbers, centralizers of elliptic elements of infinite order are completely described by J. Blanc-J. Déserti in [BD15] and centralizers of Jonquières twists in $\mathrm{Jonq}_0(\mathbf{K})$ are completely described by D. Cerveau-J. Déserti in [CD12b]. Centralizers of base-wandering Jonquières twists are also studied in [CD12b] but they were not fully understood, for example the results in loc. cit. are not sufficient for classifying pairs of Jonquières twists generating a copy of $\mathbf{Z}^2$. Thus, in order to obtain a classification of embeddings of $\mathbf{Z}^2$ in $\mathrm{Cr}_2(\mathbf{K})$, we need a detailed study of centralizers of base-wandering Jonquières twists, which is the main task of this article. Regarding the elements of finite order and their centralizers in $\mathrm{Cr}_2(\mathbf{K})$, the problem is of a rather different flavour and we refer the readers to [Bla07], [DI09], [Ser10], [Ure] and the references therein.

Remark A.1.7 There is a topology on $\mathrm{Cr}_2(\mathbf{K})$, called Zariski toplogy, which is introduced by M. Demazure and J-P. Serre in [Dem70] and [Ser10]. Note that the Zariski topology does not make $\mathrm{Cr}_2(\mathbf{K})$ an infinite dimensional algebraic group (cf. [BF13]). With respect to the Zariski topology, the centralizer of any element of $\mathrm{Cr}_2(\mathbf{K})$ is closed (J-P. Serre [Ser10]). When K is a local field, J. Blanc and J-P. Furter construct in [BF13] an Euclidean topology on $\mathrm{Cr}_2(\mathbf{K})$ which when restricted to $\mathrm{PGL}_3(\mathbf{K})$ coincides with the Euclidean topology of $\mathrm{PGL}_3(\mathbf{K})$; centralizers are also closed with respect to the Euclidean toplogy. In particular the intersection of the centralizer of an element in $\mathrm{Cr}_2(\mathbf{K})$ with an algebraic subgroup G of $\mathrm{Cr}_2(\mathbf{K})$ is a closed subgroup of G, with respect to the Zariski topology of G (and with respect to the Euclidean topology when the later is present).

Comparison with other results. S.Smale asked in the '60s if, in the group of diffeomorphisms of a compact manifold, the centralizer of a generic diffeomorphism consists only of its iterates. There has been a lot of work on this question, see for example [**BCW09**] for an affirmative answer in the C^1 case. Similar phenomenons also appear in the group of germs of 1-dimensional holomorphic diffeomorphisms at $0 \in \mathbf{C}$ ([É81]). See the introduction of [CD12b] for more references in this direction. With regard to $\mathrm{Cr}_2(\mathbf{K})$, it is known that loxodromic elements form a Zariski dense subset of $\mathrm{Cr}_2(\mathbf{K})$ (cf. [Xie15], [BD05]) and that their centralizers coincide with the cyclic group formed by their iterates up to finite index (cf. [BC16]). Centralizers of general Jonquières twists are also finite by cyclic (Remark A.3.5).

One may compare our classification of $\mathbf{Z}^2$ in $\mathrm{Cr}_2(\mathbf{K})$ to the following two theorems where the situations are more rigid. The first can be seen as a continuous counterpart and is proved by F. Enriques [Enr93] and M. Demazure [Dem70], the second can be seen as a torsion counterpart and is proved by A. Beauville [Bea07]:

1. *If $\mathscr{K}^{*r}$ embeds as an algebraic subgroup into* $\mathrm{Cr}_2(\mathbf{K})$*, then $r \leq 2$; if $r = 2$ then the embedding is conjugate to an embedding into the group of diagonal matrices Δ in* $\mathrm{PGL}_3(\mathscr{K})$.
2. *If $p \geq 5$ is a prime number different from the characteristic of $\mathscr{K}$ and if $(\mathbf{Z}/p\mathbf{Z})^r$ embeds into* $\mathrm{Cr}_2(\mathbf{K})$*, then $r \leq 2$; if $r = 2$ then the embedding is conjugate to an embedding into the group of diagonal matrices Δ in* $\mathrm{PGL}_3(\mathscr{K})$.

The classification of $\mathbf{Z}^2$ in $\mathrm{Cr}_2(\mathbf{K})$ is a very natural special case of the study of finitely generated subgroups of $\mathrm{Cr}_2(\mathbf{K})$; and information on centralizers can be useful for studying homomorphisms from other groups into $\mathrm{Cr}_2(\mathbf{K})$, see for example [Dés06a]. We refer the reader to the surveys [Fav10],[Can18] for representations of finitely generated groups into $\mathrm{Cr}_2(\mathbf{K})$ and [CX18] for general results in higher dimension.

A.2 Elements which are not base-wandering Jonquières twists

This section contains a quick review of some scattered results about centralizers from [Can11],[BD15],[CD12b Some of the proofs are reproduced, because the original proofs were written over $\mathbf{C}$ on the one hand, and because we will need some by-products of the proofs on the other hand.

A.2.1 Loxodromic elements

Theorem A.2.1 ([BC16] Corollary 4.7) *Let $f \in \mathrm{Cr}_2(\mathbf{K})$ be a loxodromic element. The infinite cyclic group generated by f is a finite index subgroup of the centralizer of f in $\mathrm{Cr}_2(\mathbf{K})$.*

Proof We provide a proof which is simpler than [BC16]. The Cremona group $\mathrm{Cr}_2(\mathbf{K})$ acts faithfully by isometries on an infinite dimensional hyperbolic space $\mathbb{H}$ and the action of a loxodromic element is loxodromic in the sense of hyperbolic geometry (see [Can11], [Can18]). In particular there is an f-invariant geodesic $Ax(f)$ on which f acts by translation and the translation length is $\log(\lim_{n\to\infty} deg(f^n)^{1/n})$. The centralizer $\mathrm{Cent}(f)$ preserves $Ax(f)$ and by considering translation lengths we get a morphism $\phi : \mathrm{Cent}(f) \to \mathbf{R}$. We claim that the image of ϕ is discrete thus cyclic. Let us see first how the conclusion follows from the claim. Let $x \in \mathbb{H}$ be a point which corresponds to an ample class and let y be an arbitrary point on $Ax(f)$. Since the kernel $\mathrm{Ker}(\phi)$ fixes $Ax(f)$ pointwise, for any element g of $\mathrm{Ker}(\phi)$ the distance $d(x, g(x))$ is bounded by $2d(x,y)$. This implies that $\mathrm{Ker}(\phi)$ is a subgroup of $\mathrm{Cr}_2(\mathbf{K})$ of bounded degree. If $\mathrm{Ker}(\phi)$ were infinite then its Zariski closure G in $\mathrm{Cr}_2(\mathbf{K})$ would be an algebraic subgroup of strictly positive dimension contained, after conjugation, in the automorphism group of a rational surface. As

$\mathrm{Cent}(f)$ is Zariski closed, the elements of G commute with f. The orbits of a one-parameter subgroup of G would form an f-invariant pencil of curves. This contradicts the fact that f is loxodromic. Consequently $\mathrm{Ker}(\phi)$ is finite and hence $\mathrm{Cent}(f)$ is finite by cyclic.

Now let us prove the claim that the image of ϕ is discrete. This follows directly from a spectral gap property for translation lengths of loxodromic elements proved in [BC16]. We give here an easier direct proof found with S. Cantat. Suppose by contradiction that there is a sequence $(g_n)_n$ of distinct elements of $\mathrm{Cent}(f)$ whose translation lengths on $Ax(f)$ tend to 0 when n goes to infinity. Without loss of generality, we can suppose the existence of a point y on $Ax(f)$ and a real number $\varepsilon > 0$ such that $\forall n, d(y, g_n(y)) < \varepsilon$. Let $x \in \mathbb{H}$ be an element which corresponds to an ample class. Then it follows that

$$\forall n, d(x, g_n(x)) \leq d(x,y) + d(y, g_n(y)) + d(g_n(y), g_n(x)) < 2d(x,y) + \varepsilon =: d,$$

i.e. the sequence $(g_n)_n$ is of bounded degree d. Elements of degree less than d of the Cremona group form a quasi-projective variety $\mathrm{Cr}_2^d(\mathscr{K})$. JunYi Xie proved in [Xie15] that for any $0 < \lambda < \log(d)$, the loxodromic elements of $\mathrm{Cr}_2^d(\mathscr{K})$ whose translation lengths are greater than λ form a Zariski open dense subset of $\mathrm{Cr}_2^d(\mathscr{K})$. Thus the g_n give rise to a strictly ascending chain of Zariski open subsets of $\mathrm{Cr}_2^d(\mathscr{K})$, contradicting the noetherian property of Zariski topology. This finishes the proof. Note that [Xie15] is also used to prove the spectral gap property in [BC16]. □

A.2.2 Halphen twists

We only recall here the final arguments of the proofs.

Theorem A.2.2 ([Giz80] and [Can11] Proposition 4.7) *Let $f \in \mathrm{Cr}_2(\mathbf{K})$ be a Halphen twist. The centralizer $\mathrm{Cent}(f)$ of f in $\mathrm{Cr}_2(\mathbf{K})$ contains a finite index abelian subgroup of rank less than or equal to* 8.

Proof Being a Halphen twist, the birational transformation f is up to conjugation an automorphism of a rational surface and preserves a relatively minimal elliptic fibration. This f-invariant fibration is unique. As a consequence $\mathrm{Cent}(f)$ acts by automorphisms preserving this fibration. It is proved in [Giz80] (see [CGL] for a clarification in characteristics 2 and 3) that the automorphism group of a rational minimal elliptic surface has a finite index abelian subgroup of rank less than 8. □

A.2.3 Elliptic elements of infinite order

In this section we reproduce a part of [BD15]; we follow the original proofs (for $\mathrm{char}(\mathbf{K}) = 0$) in loc. cit. and some extra details are added in case $\mathrm{char}(\mathbf{K}) > 0$.

We omit the proof of the following key proposition which is based on a G-Mori-program for rational surfaces due to J. Manin [Man67] and V. Iskovskih [Isk79].

Proposition A.2.3 ([BD15] Proposition 2.1) *Let S be a smooth rational surface over $\mathscr{K}$. Let $f \in \mathrm{Aut}(S)$ be an automorphism of infinite order whose action on $\mathrm{Pic}(S)$ is of finite order. Then there exists a birational morphism $S \to X$ where X is a Hirzebruch surface $\mathbf{F}_n$ ($n \neq 1$) or the projective plane $\mathbb{P}^2$, which conjugates f to an automorphism of X.*

Proposition A.2.4 ([BD15] Proposition 2.3) *Let $f \in \mathrm{Cr}_2(\mathbf{K})$ be an elliptic element of infinite order. Then f is conjugate to an automorphism of $\mathbb{P}^2$. Furthermore there exists an affine chart with affine coordinates (x,y) on which f acts by automorphism of the following form:*

1. $(x,y) \mapsto (\alpha x, \beta y)$ *where $\alpha, \beta \in \mathscr{K}^*$ are such that the kernel of the group homomorphism $\mathbf{Z}^2 \to \mathscr{K}^*, (i,j) \mapsto \alpha^i\beta^j$ is generated by $(k,0)$ for some $k \in \mathbf{Z}$;*
2. $(x,y) \mapsto (\alpha x, y+1)$ *where $\alpha \in \mathscr{K}^*$ and α is of infinite order if* $\mathrm{char}(\mathbf{K}) > 0$.

Remark A.2.5 If $\mathscr{K} = \overline{\mathbf{F}_p}$ then every elliptic element is of finite order.

As a byproduct of the proof of Proposition A.2.4, we will get the following:

Proposition A.2.6 *Let f be an automorphism of a Hirzebruch surface which preserves the rational fibration fibre by fibre (we do not assume that f is of infinite order). Then there exists an affine chart on which f acts as an automorphism of the following form:*

1. $(x,y) \mapsto (x, \beta y)$ *where $\beta \in \mathscr{K}^*$;*
2. $(x,y) \mapsto (x, y+1)$.

Here x is the coordinate on the base of the rational fibration.

Proof (of Proposition A.2.4) Proposition A.2.3 says that f is conjugate to an automorphism of $\mathbb{P}^2$ or of a Hirzebruch surface.

Let's consider first the case when $f \in \mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbf{K})$. By putting the corresponding matrix in Jordan normal form, we can find an affine chart on which f is, up to conjugation, of one of the following form: 1) $(x,y) \mapsto (\alpha x, \beta y)$; 2) $(x,y) \mapsto (\alpha x, y+1)$; 3) $(x,y) \mapsto (x+y, y+1)$. If $\mathrm{char}(\mathbf{K}) > 0$ then f can not be of the third form since it would be of finite order; if $\mathrm{char}(\mathbf{K}) = 0$

then in the third case f is conjugate by $[x:y:z] \dashrightarrow [xz-\frac{1}{2}y(y-z):yz:z^2]$ to $(x,y)\mapsto(x,y+1)$. We now show that in the first case α,β can be chosen to verify the conditon in the proposition. Let $\phi:(x,y)\mapsto(\alpha x,\beta y)$ be a diagonal automorphism, we denote by $\Delta(\phi)$ the kernel of the group morphism $\mathbf{Z}^2\to\mathscr{K}^*,(i,j)\mapsto\alpha^i\beta^j$. For $M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{GL}2Z$, we denote by $M(\phi)$ the diagonal automorphism $(x,y)\mapsto(\alpha^a\beta^b x,\alpha^c\beta^d y)$, i.e. the conjugate of ϕ by the monomial map $(x,y)\dashrightarrow(x^ay^b,x^cy^d)$. We have the relation $\Delta(M(\phi))=(M^\intercal)^{-1}(\Delta(\phi))$. This implies that up to conjugation by a monomial map we can suppose that our elliptic element f satisfies $\Delta(f)=<(k_1,0),(0,k_1k_2)>$ where $k_1,k_2\in\mathbf{Z}$. Since f is of infinite order, k_1k_2 must be 0.

If $f\in\mathrm{Aut}(\mathbf{F}_0)=\mathrm{Aut}(\mathbb{P}^1\times\mathbb{P}^1)$, then we reduce to the case of $\mathbb{P}^2$ by blowing up a fixed point and contracting the strict transforms of the two rulings passing through the point. If $f\in\mathrm{Aut}(\mathbf{F}_n)$ for $n\geq 2$ and if f has a fixed point which is not on the exceptional section, then we can reduce to $\mathbf{F}_{n-1}$ by making an elementary transformation at the fixed point.

Suppose now that $f\in\mathrm{Aut}(\mathbf{F}_n),n\geq 2$ and its fixed points are all on the exceptional section. By removing the exceptional section and an invariant fibre of the rational fibration, we get an open subset isomorphic to $\mathbb{A}^2$ on which f can be written as: $(x,y)\mapsto(\alpha x,\beta y+Q(x))$ or $(x,y)\mapsto(x+1,\beta y+Q(x))$ where $\alpha,\beta\in\mathscr{K}^*$ and Q is a polynomial of degree $\leq n$.

In the first case, the fact that there is no extra fixed point on the fibre $x=0$ implies $\beta=1$ and $Q(0)\neq 0$. The action on the fibre at infinity can be obtained by a change of variables $(x',y')=(1/x,y/x^n)$, so the fact that there is no extra fixed point on it implies $\beta=\alpha^n$ and $\deg(Q)=n$. This forces α to be a primitive r-th root of unity for some $r\in\mathbf{N}$. Conjugating f by $(x,y)\mapsto(x,y+\gamma x^d)$, we replace $Q(x)$ with $Q(x)+\gamma(\alpha^d-1)x^d$. This allows us to eliminate the term x^d of Q unless $\alpha^d=1$. So we can assume that f is of the form $(x,y)\mapsto(\alpha x,y+\tilde{Q}(x^r))$ where $\alpha^r=1$ and $\tilde{Q}\in\mathscr{K}[x]$. Then f is conjugate to $(x,y)\mapsto(\alpha x,y+1)$ by $(x,y)\dashrightarrow(x,y/\tilde{Q}(x^r))$. Remark that this case does not happen in positive characteristic because an automorphism of this form would be of finite order. Note that in this paragraph we did not use the fact that f is of infinite order, so that Proposition A.2.6 is proved.

Suppose now we are in the second case. There is no extra fixed point if and only if $\beta=1$ and $\deg(Q)=n$. If $\mathrm{char}(\mathbf{K})>0$ and if $\beta=1$, then f would be of finite order. Therefore we can assume $\mathrm{char}(\mathbf{K})=0$. In that case, we can decrease the degree of Q by conjugating f by a well chosen birational transformation of the form $(x,y)\dashrightarrow(x,y+\gamma x^{n+1})$ with $\gamma\in\mathscr{K}^*$. By induction we get $(x,y)\mapsto(x+1,y)$ at last. □

Once we have the above normal forms, explicit calculations can be done:

Theorem A.2.7 ([BD15] Lemmas 2.7 and 2.8) *Let $f \in \mathrm{Cr}_2(\mathbf{K})$ be an elliptic element of infinite order.*

1. *If f is of the form $(x,y) \mapsto (\alpha x, \beta y)$ where $\alpha, \beta \in \mathscr{K}^*$ are such that the kernel of the group homomorphism $\mathbf{Z}^2 \to \mathscr{K}^*, (i,j) \mapsto \alpha^i\beta^j$ is generated by $(k,0)$ for some $k \in \mathbf{Z}$, then the centralizer of f in $\mathrm{Cr}_2(\mathbf{K})$ is*

$$\mathrm{Cent}(f) = \{(x,y) \dashrightarrow (\eta(x), yR(x^k)) | R \in \mathscr{K}(x), \eta \in \mathrm{PGL}_2(\mathbf{K}), \eta(\alpha x) = \alpha\eta(x)\}.$$

2. *If $\mathrm{char}(\mathbf{K}) = 0$ and if f is of the form $(x,y) \mapsto (\alpha x, y+1)$, then the centralizer of f in $\mathrm{Cr}_2(\mathbf{K})$ is*

$$\mathrm{Cent}(f) = \{(x,y) \dashrightarrow (\eta(x), y+R(x)) | \eta \in \mathrm{PGL}_2(\mathbf{K}), \eta(\alpha x) = \alpha\eta(x), R \in \mathscr{K}(x), R(\alpha x) = R(x)\}.$$

If α is not a root of unity then R must be constant and $\eta(x) = \beta x$ for some $\beta \in \mathscr{K}^*$.

3. *If $\mathrm{char}(\mathbf{K}) = p > 0$ and if f is of the form $(x,y) \mapsto (\alpha x, y+1)$ (where α must be of infinite order), then the centralizer of f in $\mathrm{Cr}_2(\mathbf{K})$ is*

$$\mathrm{Cent}(f) = \{(x,y) \dashrightarrow (R(y)x, y+t) | t \in \mathscr{K}, R(y) = S(y)S(y-1)\cdots S(y-p+1), S \in \mathscr{K}(y)\}.$$

Remark A.2.8

$$\{\eta \in \mathrm{PGL}_2(\mathbf{K}) | \eta(\alpha x) = \alpha\eta(x)\} = \begin{cases} \mathrm{PGL}_2(\mathbf{K}) & \text{if} \quad \alpha = 1 \\ \{x \mapsto \gamma x^{\pm 1} | \gamma \in \mathscr{K}^*\} & \text{if} \quad \alpha = -1 \\ \{x \mapsto \gamma x | \gamma \in \mathscr{K}^*\} & \text{if} \quad \alpha \neq \pm 1 \end{cases}$$

Proof *First case.* We treat first the case where f is of the form $(x,y) \mapsto (\alpha x, \beta y)$. Let $(x,y) \dashrightarrow (\frac{P_1(x,y)}{Q_1(x,y)}, \frac{P_2(x,y)}{Q_2(x,y)})$ be an element of $\mathrm{Cent}(f)$; here $P_1, P_2, Q_1, Q_2 \in \mathscr{K}[x,y]$. The commutation relation gives us

$$\frac{P_1(\alpha x, \beta y)}{Q_1(\alpha x, \beta y)} = \frac{\alpha P_1(x,y)}{Q_1(x,y)}, \quad \frac{P_2(\alpha x, \beta y)}{Q_2(\alpha x, \beta y)} = \frac{\beta P_2(x,y)}{Q_2(x,y)}$$

which imply that P_1, P_2, Q_1, Q_2 are eigenvectors of the $\mathscr{K}$-linear automorphism $\mathscr{K}[x,y] \to \mathscr{K}[x,y], g(x,y) \mapsto g(\alpha x, \beta y)$. Therefore each one of the P_1, P_2, Q_1, Q_2 is a product of a monomial in x, y with a polynomial in $\mathscr{K}[x^k]$. Then we must have $\frac{P_1(x,y)}{Q_1(x,y)} = xR_1(x^k)$ and $\frac{P_2(x,y)}{Q_2(x,y)} = yR_2(x^k)$ for some $R_1, R_2 \in \mathscr{K}(x)$. The first factor $\frac{P_1(x,y)}{Q_1(x,y)}$ only depends on x, so for f to be birational it must be an element of $\mathrm{PGL}_2(\mathbf{K})$. The conclusion in this case follows.

Second case. We now treat the case where $\mathrm{char}(\mathbf{K}) = 0$ and where f is of the form $(x,y) \mapsto (\alpha x, y+1)$. Let $(x,y) \dashrightarrow (\frac{P_1(x,y)}{Q_1(x,y)}, \frac{P_2(x,y)}{Q_2(x,y)})$ be an element of $\mathrm{Cent}(f)$. We have

$$\frac{P_1(\alpha x, y+1)}{Q_1(\alpha x, y+1)} = \frac{\alpha P_1(x,y)}{Q_1(x,y)} \quad \frac{P_2(\alpha x, y+1)}{Q_2(\alpha x, y+1)} = \frac{P_2(x,y)}{Q_2(x,y)} + 1. \tag{A.1}$$

The first equation implies that P_1, Q_1 are eigenvectors of the $\mathscr{K}$-linear automorphism $\mathscr{K}[x,y] \to \mathscr{K}[x,y], g(x,y) \mapsto g(\alpha x, y+1)$. We view an element of $\mathscr{K}[x,y]$ as a polynomial in x with coefficients in $\mathscr{K}[y]$. Since the only eigenvector of the $\mathscr{K}$-linear automorphism $\mathscr{K}[y] \to \mathscr{K}[y], g(y) \mapsto g(y+1)$ is 1 (this is not true if $\mathrm{char}(\mathbf{K}) > 0$), we deduce that P_1, Q_1 depend only on x. Thus, $\frac{P_1(x,y)}{Q_1(x,y)}$ is an element η of $\mathrm{PGL}_2(\mathbf{K})$.

We derive $\psi = \frac{P_2}{Q_2}$ and get

$$\frac{\partial \psi}{\partial y}(\alpha x, y+1) = \frac{\partial \psi}{\partial y}(x,y), \quad \frac{\partial \psi}{\partial x}(\alpha x, y+1) = \alpha^{-1}\frac{\partial \psi}{\partial x}(x,y).$$

As before, this means that $\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}$ only depend on x (not true if $\mathrm{char}(\mathbf{K}) > 0$). Hence, we can write ψ as $ay + B(x)$ with $a \in \mathscr{K}^*$ and $B \in \mathscr{K}(x)$. Then equation (A.1) implies $B(\alpha x) = B(x) + 1 - a$, which implies further $x\frac{\partial B}{\partial x}(x)$ is invariant under $x \mapsto \alpha x$. If α is of infinite order, then $\frac{\partial B}{\partial x}(x) = \frac{c}{x}$ for some constant $c \in \mathscr{K}$. This is only possible if $c = 0$. So B is constant and $a = 1$ in this case. If α is a primitive k-th root of unity, then $(\eta(x), ay + B(x))$ commutes with $f^k : (x,y) \mapsto (x, y+k)$. This yields $a = 1$ and $B(\alpha x) = B(x)$.

Third case. We finally treat the case where $\mathrm{char}(\mathbf{K}) = p > 0$ and where f is of the form $(x,y) \mapsto (\alpha x, y+1)$ with α of infinite order. Let $g \in \mathrm{Cent}(f)$. Then g commutes with $f^p : (x,y) \mapsto (\alpha^p x, y)$ which is in the form of case 1 (the roles of x,y are exchanged). Thus, we know that g writes as $(A(y)x, \eta(y))$ where $\eta \in \mathrm{PGL}_2(\mathbf{K})$ and $A \in \mathscr{K}(x)$. Then $f \circ g = g \circ f$ implies that η is $y \mapsto y + R$ for some $R \in \mathscr{K}$ and that $A(y+1) = A(y)$. The last equation implies $A(y) = S(y)S(y-1)\cdots S(y-p+1)$ for some $S \in \mathscr{K}(x)$. $\square$

For later use, we determine when an element of the centralizers appeared in Theorem B.2.3 is elliptic. Though we will use some of the materials of Section A.3.1 in the proofs, we find it more natural to state these facts here.

Lemma A.2.9 *Let $f : (x,y) \dashrightarrow (\eta(x), yR(x)), \eta \in \mathrm{PGL}_2(\mathbf{K}), R \in \mathscr{K}(x)$ be an elliptic element. Then*

1. *either $R \in \mathscr{K}$,*

2. *or* $R(x) = \frac{rS(x)}{S(\eta(x))}$ *with* $r \in \mathscr{K}^*$ *and* $S \in \mathscr{K}(x) \backslash \mathscr{K}$.

Proof If η is the identity, then we see easily, by looking at the degree growth, that f is elliptic if and only if R is constant.

From now on assume that η is not the identity. We claim that f is conjugate by an element of $\mathrm{Jonq}_0(\mathbf{K})$ to an automorphism of a Hirzebruch surface. By Corollary A.3.8, this does not hold if and only if η is of finite order d and f^d is a Jonquières involution (see Corollary A.3.9 for the terminology). However if η is of finite order d then f^d is of the form $(x,y) \dashrightarrow (\eta(x), y\tilde{R}(x))$ with $\tilde{R}(x) = R(x)\cdots R(\eta^{d-1}(x))$, which is never a Jonquières involution. This proves the claim.

By Theorem A.3.6, the conjugation which turns f into an automorphism of a Hirzebruch surface is a sequence of elementary transformations. After conjugation it preserves the two strict transforms of the two sections $\{y=0\}$ and $\{y=\infty\}$. Therefore there exists $g \in \mathrm{Jonq}_0(\mathbf{K})$ of the form $(x,y) \dashrightarrow (x, yS(x)), S \in \mathscr{K}(x)$ such that $g \circ f \circ g^{-1}$ is $(x,y) \dashrightarrow (\eta(x), ry)$ with $r \in \mathscr{K}^*$. Hence f is $(x,y) \dashrightarrow (\eta(x), y\frac{rS(x)}{S(\eta(x))}$. □

Remark A.2.10 In the above lemma S may not be unique. If η has finite order and $T \in \mathscr{K}(x)$ is such that $T(x) = T(\eta(x))$, then $\frac{S(x)}{S(\eta(x))} = \frac{T(x)S(x)}{T(\eta(x))S(\eta(x))}$.

Lemma A.2.11 *Let* $f : (x,y) \dashrightarrow (\eta(x), y+R(x)), \eta \in \mathrm{PGL}_2(\mathbf{K}), R \in \mathscr{K}(x)$ *be an elliptic element. Then*

1. *either* η *has finite order,*
2. *or for a coordinate* x' *such that* η *is* $x' \mapsto x'+1$ *or* $x' \mapsto \nu x'$ *with* $\nu \in \mathscr{K}^*$*,* R *is a polynomial in* x'.

Proof It is clear that, if η has finite order then the degree of f^n is bounded for all $n \in \mathbf{Z}$. Assume that η has infinite order, then for some coordinate x', η writes as $\eta'(x') = x' \mapsto \nu x' + u$ with $\nu, u \in \mathscr{K}$. In coordinates (x', y), write the transformation f as $(x,y) \dashrightarrow (\eta'(x'), y + R'(x'))$ where $R'(x') = \frac{P(x')}{Q(x')}$ with $P, Q \in \mathscr{K}[x']$. For $n \in \mathbf{N}^*$, the iterate f^n is

$$(x,y) \dashrightarrow \left(\eta'(x'), y + \frac{P(x')}{Q(x')} + \cdots + \frac{P(\eta'^{n-1}(x'))}{Q(\eta'^{n-1}(x'))}\right).$$

If $Q \notin \mathscr{K}$, then the number of different factors of the polynomials $Q(x'), \cdots, Q(\eta'^{n-1}(x'))$ would go to infinity when n tends to infinity, which would imply that the degrees of the f^n are not bounded. Therefore for f to be elliptic, R' must be a polynomial. □

A.2.4 Jonquières twists with trivial action on the base

We follow [CD12b] in this section.

Lemma A.2.12 *Let* $f \in \mathrm{Jonq}$ *be a Jonquières twist. Let* $\mathrm{Cent}(f)$ *be the centralizer of* f *in* $\mathrm{Cr}_2(\mathbf{K})$. *Then* $\mathrm{Cent}(f) \subset \mathrm{Jonq}$.

Proof The rational fibration preserved by a Jonquières twist f is unique, thus is also preserved by $\mathrm{Cent}(f)$. □

Let us consider centralizers of Jonquières twists in $\mathrm{Jonq}_0(\mathbf{K}) = \mathrm{PGL}_2(\mathscr{K}(x))$ which is a linear algebraic group over the function field $\mathscr{K}(x)$. Let $f \in \mathrm{Jonq}_0(\mathbf{K})$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_2(\mathscr{K}(x))$ be a matrix representing f where $A,B,C,D \in \mathscr{K}[x]$. We introduce the function $\Delta := \frac{\mathrm{Tr}^2}{\det}$ which is well defined in PGL and is invariant by conjugation. This invariant Δ indicates the degree growth:

Lemma A.2.13 ([CD12b] Theorem 3.3 [Xie15] Proposition 6.6) *The rational function* $\Delta(f)$ *is constant if and only if* f *is an elliptic element.*

Proof Let t_1, t_2 be the two eigenvalues of the matrix M which are elements of the algebraic closure of $\mathscr{K}(x)$. The invariant $\Delta(f)$ equals to $t_1/t_2 + t_2/t_1 + 2$. Since $\mathscr{K}$ is algebraically closed, $\Delta(f) \in \mathscr{K}$ if and only if $t_1/t_2 \in \mathscr{K}$. If $t_1 = t_2$, then by conjugating M to a triangular matrix we can write f in the form $(x,y) \dashrightarrow (x, y + a(x))$ with $a \in \mathscr{K}(x)$ and it follows that f is an elliptic element.

Suppose now that $t_1 \neq t_2$. Let $\zeta : C \to \mathbb{P}^1$ be the curve corresponding to the finite field extension $\mathscr{K}(x) \hookrightarrow \mathscr{K}(x)(t_1)$, here ζ is the identity map on $\mathbb{P}^1$ if $t_1, t_2 \in \mathscr{K}(x)$. The birational transformation f induces a birational transformation f_C on $C \times \mathbb{P}^1$ by base change. The induced map f_C is of the form $(x, (t_1/t_2)y)$ where t_1/t_2 is viewed as a function on C. The degree growth of f_C which is the same as f is linear if and only if t_1/t_2 is not a constant, i.e. if and only if $\Delta(f)$ is not a constant. □

From now on we suppose that f is a Jonquières twist so that $\Delta(f) \notin \mathscr{K}$. We still denote by t_1, t_2 the two eigenvalues of M as in the above proof, we know that $t_1 \neq t_2$.

We first study the centralizer $\mathrm{Cent}_0(f)$ of f in $\mathrm{Jonq}_0(\mathbf{K}) = \mathrm{PGL}_2(\mathscr{K}(x))$. Let L be the finite extension of $\mathscr{K}(x)$ over which M is diagonalisable; it is $\mathscr{K}(x)$ itself or a quadratic extension of $\mathscr{K}(x)$, depending on whether or not t_1, t_2 are in $\mathscr{K}(x)$. The centralizer $\mathrm{Cent}_0^L(f)$ of f in $\mathrm{PGL}_2(L)$ is isomorphic to the multiplicative group L^*. So $\mathrm{Cent}_0(f)$, being contained in $\mathrm{Cent}_0^L(f)$

and containing all the iterates of f, must be a 1-dimensional torus over $\mathscr{K}(x)$. It is split if $L=\mathscr{K}(x)$, i.e. if $t_1,t_2\in\mathscr{K}(x)$.

If $L=\mathscr{K}(x)$, then up to conjugation f can be written as $(x,y)\dashrightarrow(x,b(x)y)$ with $b\in\mathscr{K}(x)^*$ and $\mathrm{Cent}_0(f)=\{(x,y)\dashrightarrow(x,\gamma(x)y)|\gamma\in\mathscr{K}(x)^*\}$.

If L is a quadratic extension of $\mathscr{K}(x)$ and if $\mathrm{char}(\mathbf{K})\neq 2$, we can put f in a simpler form and write $\mathrm{Cent}_0(f)$ explicitly as follows. We may assume that the matrix $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ has entry $C=1$, after conjugation by $\begin{pmatrix}C & 0\\ 0 & 1\end{pmatrix}$. Once we have $C=1$, a conjugation by $\begin{pmatrix}2 & D-A\\ 0 & 2\end{pmatrix}$ allows us to put M in the form $\begin{pmatrix}A & B\\ 1 & A\end{pmatrix}$ with $A,B\in\mathscr{K}[x]$. Therefore $\mathrm{Cent}_0(f)$ is $\{Id,(x,y)\dashrightarrow(x,\frac{C(x)y+B(x)}{y+C(x)})|C\in\mathscr{K}(x)\}$ as the ($\mathscr{K}(x)$-points of the) later algebraic group is easily seen to commute with f. Note that B is not a square in $\mathscr{K}(x)$ because M is not diagonalisable over $\mathscr{K}(x)$ and that the transformation $f:(x,y)\dashrightarrow(x,\frac{A(x)y+B(x)}{y+A(x)})$ fixes pointwise the hyperelliptic curve defined by $y^2=B(x)$.

Now we look at the whole centralizer of f. For $\eta\in\mathrm{PGL}_2(\mathbf{K})$ and $f\in\mathrm{Jonq}_0(\mathbf{K})$ represented by a matrix $\begin{pmatrix}A(x) & B(x)\\ C(x) & D(x)\end{pmatrix}$, we denote by f_η the element of $\mathrm{Jonq}_0(\mathbf{K})$ represented by $\begin{pmatrix}A(\eta(x)) & B(\eta(x))\\ C(\eta(x)) & D(\eta(x))\end{pmatrix}$. Let $f\in\mathrm{Jonq}_0(\mathbf{K})$ be a Jonquières twist and $g:(x,y)\dashrightarrow(\eta(x),\frac{a(x)y+b(x)}{c(x)y+d(x)})$ be an element of Jonq. Writing down the commutation equation , we see that g commutes with f if and only if f is conjugate to f_η in $\mathrm{PGL}_2(\mathscr{K}(x))$ by the transformation represented by $\begin{pmatrix}a(x) & b(x)\\ c(x) & d(x)\end{pmatrix}$. We have thus $\Delta(f)(x)=\Delta(f_\eta)(x)=\Delta(f)(\eta(x))$. Recall that $\Delta(f)\in\mathscr{K}(x)$ is not in $\mathscr{K}$. As a consequence the group

$$\{\eta\in\mathrm{PGL}_2(\mathbf{K}),\Delta(f)(x)=\Delta(f)(\eta(x))\}$$

is a finite subgroup of $\mathrm{PGL}_2(\mathbf{K})$. We then obtain:

Theorem A.2.14 ([CD12b]) *Let* $f\in\mathrm{Jonq}_0(\mathbf{K})$ *be a Jonquières twist preserving the rational fibration fibre by fibre. Let* $\mathrm{Cent}(f)$ *be the centralizer of* f *in* $\mathrm{Cr}_2(\mathbf{K})$*. Then* $\mathrm{Cent}(f)\subset\mathrm{Jonq}$ *and* $\mathrm{Cent}_0(f)=\mathrm{Cent}(f)\cap\mathrm{Jonq}_0(\mathbf{K})$ *is a finite index normal subgroup of* $\mathrm{Cent}(f)$*. The group* $\mathrm{Cent}_0(f)$ *has a structure of a* 1*-dimensional torus over* $\mathscr{K}(x)$*. In particular* $\mathrm{Cent}(f)$ *is virtually abelian.*

Remark A.2.15 In [CD12b], the authors give explicit description of the quotient $\mathrm{Cent}(f)/\mathrm{Cent}_0(f)$ when $\mathrm{char}(\mathbf{K}) = 0$.

Finite action on the base. If $f \in \mathrm{Jonq}$ is a Jonquières twist which has a finite action on the base, then $f^k \in \mathrm{Jonq}_0(\mathbf{K})$ for some $k \in \mathbf{N}$. As $\mathrm{Cent}(f) \subset \mathrm{Cent}(f^k)$, we can use Theorem A.2.14 to describe $\mathrm{Cent}(f)$:

Corollary A.2.16 *If $f \in \mathrm{Jonq}$ is a Jonquières twist which has a finite action on the base, then $\mathrm{Cent}(f)$ is virtually contained in a 1-dimensional torus over $\mathscr{K}(x)$. In particular $\mathrm{Cent}(f)$ is virtually abelian.*

We are contented with this coarse description of $\mathrm{Cent}(f)$ because this causes only a finite index problem as regards the embeddings of $\mathbf{Z}^2$ to $\mathrm{Cr}_2(\mathbf{K})$. We give an example to show how we expect $\mathrm{Cent}(f)$ to look like:

Example A.2.17 If f is $(x,y) \dashrightarrow (a(x), R(x)y)$ where $R \in \mathscr{K}(x)$ and $a \in \mathrm{PGL}_2(\mathbf{K})$ is of order $k < +\infty$. Then all maps of the form $(x,y) \dashrightarrow (x, S(x)S(a(x))\cdots S(a^{k-1}(x))y)$ with $S \in \mathscr{K}(x)$ commute with f.

A.3 Base-wandering Jonquières twists

We introduce some notations. For a Hirzebruch surface X, let us denote by π the projection of X onto $\mathbb{P}^1$, i.e. the rational fibration. When $X = \mathbb{P}^1 \times \mathbb{P}^1$, π is the projection onto the first factor. For $x \in \mathbb{P}^1$, we denote by F_x the fibre $\pi^{-1}(x)$. If f is a birational transformation of a Hirzebruch surface X which preserves the rational fibration, we denote by $\overline{f} \in \mathrm{PGL}_2(\mathbf{K})$ the induced action of f on the base $\mathbb{P}^1$ and we will consider f as an element of Jonq.

Assume now that f is a Jonquières twist such that $\overline{f} \in \mathrm{PGL}_2(\mathbf{K})$ if of infinite order, we will call it a *base-wandering Jonquières twist*. We have an exact sequence:

$$\{1\} \to \mathrm{Cent}_0(f) \to \mathrm{Cent}(f) \to \mathrm{Cent}_b(f) \to \{1\} \tag{A.2}$$

where $\mathrm{Cent}_0(f) = \mathrm{Cent}(f) \cap \mathrm{Jonq}_0(\mathbf{K})$ and $\mathrm{Cent}_b(f) \subset \mathrm{Cent}(\overline{f}) \subset \mathrm{PGL}_2(\mathbf{K})$. The action $\overline{f}$ on the base is conjugate to $x \mapsto \alpha x$ with $\alpha \in \mathscr{K}^*$ of infinite order or to $x \mapsto x+1$. The later case is only possible if $\mathrm{char}(\mathbf{K}) = 0$. Thus $\mathrm{Cent}_b(f)$ is a subgroup of $\{x \mapsto \gamma x, \gamma \in \mathscr{K}^*\}$ or of $\{x \mapsto x+\gamma, \gamma \in \mathscr{K}\}$. In both cases $\mathrm{Cent}_b(f)$ is abelian. We first remark:

Lemma A.3.1 *All elements of $\mathrm{Cent}_0(f)$ are elliptic.*

Proof By Theorem A.2.14, a Jonquières twist in $\mathrm{Jonq}_0(\mathbf{K})$ can not have a base-wandering Jonquières twist in its centralizer. □

The rest of the article will essentially be occupied by the proof of the following theorem:

Theorem A.3.2 *Let $f \in \mathrm{Jonq}$ be a base-wandering Jonquières twist. The exact sequence*

$$\{1\} \to \mathrm{Cent}_0(f) \to \mathrm{Cent}(f) \to \mathrm{Cent}_b(f) \to \{1\}$$

satisfies

— $\mathrm{Cent}_0(f) = \mathrm{Cent}(f) \bigcap \mathrm{Jonq}_0(\mathbf{K})$*, if not trivial, is* $\{(x,y) \mapsto (x,ty), t \in \mathscr{K}^*\}$*,* $\{(x,y) \mapsto (x,y+t), t \in \mathscr{K}\}$*,* $\langle (x,y) \mapsto (x,-y) \rangle$ *or* $\langle$*a Jonquières involution*$\rangle$*;*

— $\mathrm{Cent}_b(f) \subset \mathrm{PGL}_2(\mathbf{K})$ *is isomorphic to the product of a finite cyclic group with* $\mathbf{Z}$*. The infinite cyclic subgroup generated by* $\overline{f}$ *has finite index in* $\mathrm{Cent}_b(f)$*.*

Proof (of Theorem B.1.1) The theorem is a consequence of Proposition A.3.15, Corollary A.3.19 and Proposition A.3.31. □

Corollary A.3.3 *The centralizer of a base-wandering Jonquières twist is virtually abelian.*

Proof This results directly from the fact that $\mathrm{Cent}_b(f)$ is virtually the cyclic group generated by $\overline{f}$. □

Remark A.3.4 Theorem B.1.1 is optimal in the sense that $\mathrm{Cent}_b(f)$ can be $\mathbf{Z}$ (Remark A.3.5) or a product of $\mathbf{Z}$ with a non trivial finite cyclic group (Example A.3.30) and $\mathrm{Cent}_0(f)$ can be trivial, isomorphic to $\mathscr{K}$, $\mathscr{K}^*$ or $\mathbf{Z}/2\mathbf{Z}$ (Section A.3.2).

Remark A.3.5 A general base-wandering Jonquières twist can not be written as $(\eta(x), yR(x^k))$ or $(\eta(x), y+R(x))$. So the centralizer of a general Jonquières twist f differs from the infinite cyclic group $\langle f \rangle$ only by some finite groups. For example, for a generic choice of $\alpha, \beta \in \mathscr{K}^*$, the centralizer of $f_{\alpha,\beta} : (x,y) \dashrightarrow (\alpha x, \frac{\beta y+x}{y+1})$ is $\langle f_{\alpha\beta} \rangle$, this is showed by J. Déserti in [Dés08].

A.3.1 Algebraically stable maps

If f is a birational transformation of a smooth algebraic surface X over $\mathscr{K}$, we denote by $\mathrm{Ind}(f)$ the set of indeterminacy points of f. We say that f is *algebraically stable* if there is no curve V on X such that the strict transform $f^k(V) \subset \mathrm{Ind}(f)$ for some integer $k \geq 0$. There always exists a birational morphism $\hat{X} \to X$ which lifts f to an algebraically stable birational transformation of $\hat{X}$ ([DF01] Theorem 0.1). The following theorem says that for $f \in \mathrm{Jonq}$, we can get a more precise algebraically stable model:

Theorem A.3.6 *Let f be a birational transformation of a ruled surface X that preserves the rational fibration. Then there is a rational ruled surface $\hat{X}$ and a birational map $\varphi : X \dashrightarrow \hat{X}$ such that*

- *the only singular fibres of $\hat{X}$ are of the form $D_0 + D_1$ where D_0, D_1 are (-1)-curves, i.e. $\hat{X}$ is a conic bundle;*
- *$f_{\hat{X}} = \varphi \circ f \circ \varphi^{-1}$ is an algebraically stable birational transformation of $\hat{X}$ and it preserves the rational fibration of $\hat{X}$ which is induced by that of X;*
- *$f_{\hat{X}}$ sends singular fibres isomorphically to singular fibres and all indeterminacy points of $f_{\hat{X}}$ and its iterates are located on regular fibres.*
- *φ is a sequence of elementary transformations and blow-ups.*

Let $z \in X$ be an indeterminacy point of f. Let $X \xleftarrow{u} Y \xrightarrow{v} X$ be a minimal resolution of the indeterminacy point z, i.e. u, v are birational maps which are regular around the fibre over $\pi(z)$, u^{-1} is a series of n blow-ups at z or at its infinitely near points and n is minimal among possible integers.

Lemma A.3.7 *The total transform by u^{-1} in Y of $F_{\pi(z)}$, the fibre containing z, is a chain of $(n+1)$ rational curves $C_0 + C_1 + \cdots + C_n$: C_0 is the strict transform of $F_{\pi(z)}$, $C_0^2 = C_n^2 = -1$, $C_i^2 = -2$ for $0 < i < n$ and $C_i \cdot C_{i+1} = 1$ for $0 \leq i < n$.*

Proof Let us write $u : Y \to X$ as $Y = Y_n \xrightarrow{u_n} Y_{n-1} \cdots \xrightarrow{u_2} Y_1 \xrightarrow{u_1} Y_0 = X$ where each u_i is a single contraction of a (-1)-curve and C_i is (the strict transform) of the contracted (-1)-curve. By an abuse of notation, we will use C_i to denote all strict transforms of the (-1)-curve contracted by u_i. The connectedness of the fibres and the preservation of the fibration imply that for each i, the map $f \circ u_1 \circ \cdots \circ u_i$ has at most one indeterminacy point on a fibre. To prove the lemma, it suffices to show that the indeterminacy point of $f \circ u_1 \circ \cdots \circ u_i$ which by construction lies in C_i is not the intersection point of C_i with C_{i-1}.

Suppose by contradiction that C_{i+1} is obtained by blowing up the intersection point of C_i with C_{i-1}. Then for $j > i$, the auto-intersection of C_i on X_j is less than or equal to -2. Let us write $v : Y \to X$ as $Y = Y_n \xrightarrow{v_n} Y_{n-1} \cdots \xrightarrow{v_2} Y_1 \xrightarrow{v_1} Y_0 = X$ where each v_i is a single contraction of a (-1)-curve. Since C_i is contracted by v, there must exist an integer k such that $v_{k+1} \circ \cdots \circ v_n(C_i)$ is the (-1)-curve on Y_k contracted by v_k. This is possible only if the $C_j, j > i$ are all contracted by $v_k \circ \cdots \circ v_n$. But by the minimality of the integer n, C_n can not be contracted by v. □

Proof (of Theorem A.3.6) Our proof is inspired by the proof of Theorem 0.1 of [DF01]. Let $p_1, \cdots, p_k \in X$ be the indeterminacy points of f. By Lemma A.3.7, for $1 \leq i \leq k$ the minimal resolution of f at p_i writes as

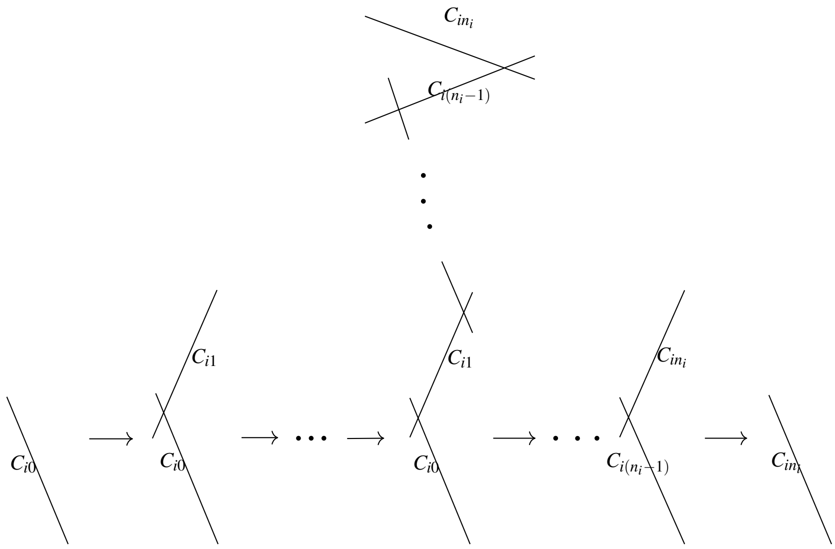
$$X = X_{i0} \xleftarrow{u_{i1}} X_{i1} \xleftarrow{u_{i2}} \cdots \xleftarrow{u_{in_i}} X_{in_i} = Y_{in_i} \xrightarrow{v_{in_i}} \cdots \xrightarrow{v_{i2}} Y_{i1} \xrightarrow{v_{i1}} Y_{i0} = X$$

where $u_{i1}, \cdots, u_{in_i}, v_{i1}, \cdots, v_{in_i}$ are single contractions of (-1)-curves and X_{in_i} has one singular fibre which is a chain of rational curves $C_{i0} + \cdots + C_{in_i}$. Let us write the global minimal resolution of indeterminacy of f by keeping in mind the rational fibration:

$$\begin{array}{ccccccccccccc}
X = X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \cdots & \xrightarrow{f_{2n-2}} & X_{2n-1} & \xrightarrow{f_{2n-1}} & X_{2n} = X \\
\downarrow{\pi} & & \downarrow{\pi} & & & & \downarrow{\pi} & & & & \downarrow{\pi} & & \downarrow{\pi} \\
\mathbb{P}^1 & \xrightarrow{\overline{f_0}} & \mathbb{P}^1 & \xrightarrow{\overline{f_1}} & \cdots & \xrightarrow{\overline{f_{n-1}}} & \mathbb{P}^1 & \xrightarrow{\overline{f_n}} & \cdots & \xrightarrow{\overline{f_{2n-2}}} & \mathbb{P}^1 & \xrightarrow{\overline{f_{2n-1}}} & \mathbb{P}^1
\end{array}$$

where $n = n_1 + \cdots + n_k$ and

- $f_0, \cdots, f_{n-1}$ are blow-ups which correspond to the inverses of $u_{11}, \cdots, u_{1n_1}, \cdots, u_{k1}, \cdots, u_{kn_k}$;
- $f_n, \cdots, f_{2n-1}$ are blow-downs which correspond to $v_{11}, \cdots, v_{1n_1}, \cdots, v_{k1}, \cdots, v_{kn_k}$;
- X_n has k singular fibres which are chains of rational curves $C_{i0} + \cdots + C_{in_i}, 1 \leq i \leq k$;
- the abusive notation π is self-explaining and we will also denote by C_{il} its strict transforms (if it remains a curve) on the surfaces X_j. On $X_0 = X_{2n}$, it is possible that $C_{i'0} = C_{in_i}$ for $1 \leq i, i' \leq k$.



For any $j \in \mathbf{N}$, we let $X_j = X_{j \mod 2n}$ and $f_j = f_{j \mod 2n}$. If f_j blows up a point $r_j \in X_j$, then we denote by V_{j+1} the exceptional curve on X_{j+1}. If f_j contracts a curve $W_j \subset X_j$ then we denote by s_{j+1} the point $f_j(W_j) \in X_{j+1}$. For each V_j (resp. W_j), there is an i such that V_j (resp. W_j) is among $C_{i0}, \cdots, C_{in_i}$. Suppose that f is not algebraically stable on H. Then there exist integers $1 \leq M < N$ such that f_M contracts W_M and

$$f_{N-1} \circ \cdots \circ f_M(W_M) = r_N \in \mathrm{Ind}(f_N).$$

We can assume that $n \leq N \leq 2n-1$ and the length $(N-M)$ is minimal. Observe first that the minimality of the length implies for all $M \leq j < N-1$, the point $t_{j+1} := f_j \circ \cdots \circ f_M(W_M) = f_j \circ \cdots \circ f_{M+1}(s_{M+1})$ is neither an indeterminacy point nor a point on a curve contracted by f_{j+1}. Secondly we assert that for all $M \leq j < N-1$, t_{j+1} is not on the singular fibres of X_{j+1}. Indeed if some t_{j+1} was on a singular fibre of X_{j+1}, then the sequence of points $t_{j+1}, t_{j+2}, \cdots$ would meet a contracted curve before meeting the first indeterminacy point r_N (look at the picture), which contradicts our first observation. The second observation further implies that for $M \leq j < N-1$ such that $j + 2n < N-1$, t_{j+1}, t_{j+2n+1} are not on the same fibre of $X_{j+1} = X_{j+2n+1}$ because otherwise there would exist $j < j' < j+2n+1$ such that $j' = M \mod 2n$ and $t_{j'}$ would be on the singular fibre containing W_M.

Since f_{N-1} maps isomorphically the fibre of X_{N-1} containing t_{N-1} (which is regular by the above observation) to the fibre of X_N containing r_N, the fibre containing r_N is just one rational curve. As f_N is a blow-up, the fibre of X_{N+1} containing V_{N+1} is the union of two (-1)-curves, let us say, C_{k0} and $C_{k1} = V_{N+1}$. Then the fibre of X_N containing r_N is just C_{k0}. Similarly the singular fibre of X_M containing W_M is $C_{mn_m} + C_{m(n_m-1)}$ for some $1 \leq m \leq k$.

First case. Suppose that $m = k$ and $n_k = 1$. Let $a \in \mathbf{N}$ be the minimal integer such that $M + 2an > N$. Then for $N < j \leq M + 2an$, the surface X_j has a singular fibre $C_{k0} + C_{k1}$ and the maps $f_N, \cdots, f_{M+2an-1}$ are all regular on $C_{k0} + C_{k1}$. Now we blow-up $t_{M+1}, \cdots, t_{N-1}, r_N$. For $j_1 = j_2 \mod 2n$, we showed that t_{j_1}, t_{j_2} are not on the same fibre of $X_{j_1} = X_{j_2}$. This means that these blow-ups only give rise to singular fibres which are unions of two (-1)-curves. We denote by $\hat{X}_j$ the modified surfaces, and $\hat{f}_j$ the induced maps. Then every $\hat{X}_j$ has singular fibres of the form $C_{k0} + C_{k1}$ and every $\hat{f}_j$ is regular around these singular fibres. Let $\hat{f} = \hat{f}_{2n-1} \circ \cdots \circ \hat{f}_0$. The number of indeterminacy points of $\hat{f}$ (it was k for f) has decreased by one. Note that $\hat{f}$ exchanges the two components C_{k0} and C_{k1}. This fact will be used in the proof of Corollary A.3.8.

Second case. Suppose that $m = k$ and $n_k > 1$ or simply $m \neq k$. We blow-up r_N and contract the strict transform of the initial fibre containing r_N which is C_{k0}, obtaining a new surface $\hat{X}_N$ whose corresponding fibre is now the single rational curve C_{k1}. We perform elementary transformations at $t_{N-1}, \cdots, t_{M+1}$, i.e. we blow-up X_j at t_j and contract the strict transform of the initial fibre, replacing X_j with $\hat{X}_j$. This process has no ambiguity: if $j_1 = j_2 \mod 2n$, we showed that t_{j_1}, t_{j_2} are not on the same fibre of $X_{j_1} = X_{j_2}$, so the corresponding elementary transformations do not interfere with each other. Let us denote by $\hat{f}_M, \cdots, \hat{f}_N$ the maps induced by $f_M, \cdots, f_N$.

We now analyse the effects of $\hat{f}_M, \cdots, \hat{f}_N$. First look at f_N, it lifts to a regular isomorphism after blowing up r_N. Thus $\hat{f}_N$ is the blow-up at the point e_N of $\hat{X}_N$ to which C_{k0} is contracted. After this step, the map going from X_{N-1} to $\hat{X}_N$ induced by f_{N-1} is as following: it contracts the fibre containing t_{N-1} to e_N and blows up t_{N-1}. Then we make elementary transformations at $t_{N-1}, \cdots, t_{M+1}$ in turn. The maps $\hat{f}_{N-1}, \cdots, \hat{f}_{M+1}$ are all regular on the modified fibres, thus they are still single blow-ups or single blow-downs. The behaviour of $\hat{f}_M$ differs from the previous ones: it does not contract $C_{m(n_m-1)}$ any more, but contracts C_{mn_m}.

The hypothesis $m \neq k$ (or $m = k$, $n_k > 1$) forbids $C_{k0} \subset X_{N+1}$ to go back into the fibre of $X_{M+2na} = X_M$ containing W_M without being contracted. More precisely this implies the existence of $N' > N$ such that

— $X_{N+1}, \cdots, X_{N'}$ all contain C_{k0} and C_{k1};
— $f_{N+1}, \cdots, f_{N'-1}$ are regular on C_{k0} and $f_{N'}$ contracts C_{k0};
— if $a \in \mathbf{N}$ is the minimal integer such that $M + 2na > N$, then $N' < M + 2na$.

On the surfaces $X_{N+1},\cdots,X_{N'}$, C_{k0} is always a (-1)-curve, we contract all these C_{k0} and obtain new surfaces $\hat{X}_{N+1},\cdots,\hat{X}_{N'}$. The second and the third property listed above mean that the new induced maps $\hat{f}_N,\cdots,\hat{f}_{N'}$ are all single blow-ups, single blow-downs or simply isomorphisms.

In summary we get a commutative diagram:

$$\begin{array}{ccccccccccccc}
\hat{X}_0 & \xrightarrow{\hat{f}_0} & \hat{X}_1 & \xrightarrow{\hat{f}_1} & \cdots & \xrightarrow{\hat{f}_{n-1}} & \hat{X}_n & \xrightarrow{\hat{f}_n} & \cdots & \xrightarrow{\hat{f}_{2n-2}} & \hat{X}_{2n-1} & \xrightarrow{\hat{f}_{2n-1}} & \hat{X}_{2n}=\hat{X}_0 \\
\downarrow & & \downarrow & & & & \downarrow & & & & \downarrow & & \downarrow \\
X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \cdots & \xrightarrow{f_{2n-2}} & X_{2n-1} & \xrightarrow{f_{2n-1}} & X_{2n}=X_0
\end{array}$$

where the vertical arrows are composition of elementary transformations and blow-ups. Let us remark that:

— the first vertical arrow $\hat{X}_0 \dashrightarrow X_0$ is a composition of elementary transformations.

— the blow-ups or the contractions of the $\hat{f}_j$ only concern the k singular fibres and the exceptional curves are always among $C_{10},\cdots,C_{1n_1},\cdots,C_{k0},\cdots,C_{kn_k}$;

— there is no more C_{k0}. We then do a renumbering: $C_{k1},\cdots,C_{kn_k}$ become $C_{k0},\cdots,C_{k(n_k-1)}$.

Let $\hat{f}=\hat{f}_{2n-1}\circ\cdots\circ\hat{f}_0$. We repeat the above process. Either we are in the first case and k decreases, or we are in the second case and the total number of $C_{10},\cdots,C_{1n_1},\cdots,C_{k0},\cdots,C_{kn_k}$ decreases. As a consequence, after a finite number of times, either we get an algebraically stable map $\hat{f}$, or we will get rid of all the $C_{10},\cdots,C_{1n_1},\cdots,C_{k0},\cdots,C_{kn_k}$. In the later case $\hat{f}$ is a regular automorphism, thus automatically algebraically stable. □

Theorem A.3.6 also gives a geometric complement to the study of elements of finite order of Jonq in [Bla11] Section 3. In particular the proof of Theorem A.3.6 implies the following corollary (which is already known, see for example [Bla11]), one special case of which will be used in the next section:

Corollary A.3.8 *Let $f\in$ Jonq be an elliptic element. If f is not conjugate to an automorphism of a Hirzebruch surface, then it is a conjugate to an automorphism of a conic bundle and the order of f is $2k$ for some $k\in\mathbf{N}^*$. Moreover f^k is in $\mathrm{Jonq}_0(\mathbf{K})$ and exchanges the two components of some singular fibres of the conic bundle.*

Proof We see by Theorem A.2.4 that an elliptic element of infinite order is always conjugate to an automorphism of Hirzebruch surface. Hence our hypothesis implies immediately that f is of finite order. We can assume that f is an algebraically stable map on a conic bundle X which satisfies the conditions of Theorem A.3.6. We claim that f is an automorphism of X. Suppose by contradiction that p is an indeterminacy point of f. It must lie on a regular fibre F of X. The fact that f is of finite order and the algebraic stability of f imply that f^{-1} has an indeterminacy point on F different from p. But then f can not be of finite order, contradiction.

Since by hypothesis X is not a Hirzebruch surface, it must have some singular fibres. By the proof of Theorem A.3.6 (see the *First case* in the proof), for each singular fibre there exists an iterate of f which exchanges the two components of that fibre. Since there are finitely many singular fibres, we can find an integer $k>0$ such that f^k is in $\mathrm{Jonq}_0(\mathbf{K})$ and exchanges the two components of at least one singular fibre. If we consider f^k as an element of $\mathrm{PGL}_2(\mathscr{K}(x))$, it is not diagonalizable over $\mathscr{K}(x)$. As we have seen in Section A.2.4, the map f^k, being non diagonalizable, fixes pointwise a hyperelliptic curve whose projection onto $\mathbb{P}^1$ is induced by the rational fibration. The map f^{2k} does not exchange the components of the singular fibres, so it is conjugate to an automorphism of a Hirzebruch surface and is diagonalizable over $\mathscr{K}(x)$. A diagonalizable map does not fix any hyperelliptic curve like this unless the map is trivial. Hence $f^{2k}=\mathrm{Id}$. □

See [Bla11] Section 3, especially Proposition 3.3 and Lemma 3.9, for more information on such elliptic elements of finite order; see also [DI09]. We will use a special case of the above corollary:

Corollary A.3.9 *Let $f\in\mathrm{Jonq}_0(\mathbf{K})$ be an elliptic element which is not conjugate to an automorphism of a Hirzebruch surface. Then f is of order 2 and is conjugate to an automorphism of a conic bundle on which it fixes pointwise a hyperelliptic curve whose projection onto the base $\mathbb{P}^1$ is a ramified double cover. In some affine chart f writes as $(x,y)\dashrightarrow(x,\frac{a(x)}{y})$ with $a\in\mathscr{K}[x]$. The hyperelliptic curve is given by the equation $y^2=a(x)$.*

Such involutions are well known and are called *Jonquières involutions*, see [BB00].

Remark A.3.10 An element of the form $(x,y)\dashrightarrow(\eta(x),yR(x))$ or $(x,y)\dashrightarrow(\eta(x),y+R(x))$ with $\eta\in\mathrm{PGL}_2(\mathbf{K})$ and $R\in\mathscr{K}(x)$ is never a Jonquières twist. Thus by Theorem B.2.3, a Jonquières twist never commutes with an elliptic element of infinite order.

We will need an abelian elliptic group version of Theorem A.3.6:

Corollary A.3.11 *Let $G\subset\mathrm{Jonq}$ be a finitely generated abelian elliptic subgroup without Jonquières involutions. We can conjugate G to a group of automorphisms of a Hirzebruch surface. The conjugation is a sequence of elementary transformations.*

Proof Let $f_1,\cdots,f_d\in G$ be a finite set of generators of G. We apply Theorem A.3.6 to f_1, then to f_2, etc. Remark that by the proof of Theorem A.3.6, the elementary transformations of the conjugation are made at the indeterminacy points of the f_i. However G is an abelian group, so that if p is an indeterminacy point of f_i and g is another element of G, then either g fixes p or p is an indeterminacy point of g too. Therefore after applying Theorem A.3.6 to f_{i+1}, the previous ones $f_1,\cdots,f_i$ remain automorphisms. □

A.3.2 The group $\mathrm{Cent}_0(f)$

Let f be a base-wandering Jonquières twist. In [CD12b], it is proved by explicit calculations, in the case where $\mathscr{K}=\mathbf{C}$, that $\mathrm{Cent}_0(f)$ is isomorphic to $\mathbf{C}^*$, $\mathbf{C}^*\rtimes\mathbf{Z}/2\mathbf{Z}$, $\mathbf{C}$ or a finite group (this is not optimal). Their arguments do not work directly when $\mathrm{char}(\mathbf{K})>0$. With a more precise description of elements of $\mathrm{Jonq}_0(\mathbf{K})$, we simplify their arguments and improve their results.

Let $g\in\mathrm{Cent}_0(f)$ be non trivial. Then either g is conjugate to an automorphism of a Hirzebruch surface or g is a Jonquières involution as in Corollary A.3.9. In the first case, by proposition A.2.6, we can write g as $(x,y)\mapsto(x,\beta y)$ or $(x,y)\mapsto(x,y+1)$.

Lemma A.3.12 *Suppose that there exists a non trivial $g\in\mathrm{Cent}_0(f)$ that writes as $(x,y)\mapsto(x,\beta y)$ with $\beta\in\mathscr{K}^*$. Either f is of the form $(a(x),R(x)y^{-1})$ and $\mathrm{Cent}_0(f)$ is an order two group generated by the involution $(x,y)\mapsto(x,-y)$, or f is of the form $(a(x),R(x)y)$ and $\mathrm{Cent}_0(f)$ is $\{(x,y)\mapsto(x,\gamma y),\gamma\in\mathscr{K}^*\}$.*

Proof The map g preserves $\{y=0\}$ and $\{y=\infty\}$ and these two curves are the only g-invariant sections. Thus f permutes these two sections and is necessarily of the form $(x,y)\dashrightarrow(a(x),R(x)y^{\pm1})$ where $R\in\mathscr{K}(x)$ and $a\in\mathrm{PGL}_2(\mathbf{K})$ is of infinite order. If f is $(a(x),R(x)y^{-1})$, then $\beta=-1$. For the discussion which follows, it is not harmfull to replace f by f^2 so that we can assume f is $(a(x),R(x)y)$.

The only f-invariant sections are $\{y=0\}$ and $\{y=\infty\}$. Indeed an invariant section s satisfies

$$s(a^n(x))=R(x)\cdots R(a^{n-1}(x))s(x)\quad\forall n\in\mathbf{N}.$$

If s was not $\{y=0\}$ nor $\{y=\infty\}$, then the two sides of the equations are rational fractions and by comparing the degrees (of numerators and denominators) we get a contradiction because R is not constant. Thus, an element of $\mathrm{Cent}_0(f)$ permutes the two f-invariant sections and is of the form $(x,A(x)y)$ or $(x,\frac{A(x)}{y})$ with $A\in\mathscr{K}(x)$. In the first case the commutation relation implies $A(a(x))=A(x)$ which further implies that A is a constant. In the second case the commutation relation gives $A(a(x))^{-1}R(x)^2A(x)=1$ which further implies that $(a(x),R(x)^2y)$ is conjugate by

$(x, A(x)y)$ to an elliptic element $(a(x), y)$. This is not possible because the map $f' : (x,y) \dashrightarrow (a(x), R(x)^2 y)$ is a Jonquières twist. Indeed the iterates f^n, f'^n are respectively

$$(a^n(x), R(x)\cdots R(a^{n-1}(x))y) \quad \text{and} \quad (a^n(x), (R(x)\cdots R(a^{n-1}(x)))^2 y)$$

and they have the same degree growth.

Reciprocally all elements of the form $(x,y) \mapsto (x, \beta y)$ with $\beta \in \mathscr{K}^*$ commute with $f : (x,y) \dashrightarrow (a(x), R(x)y)$ and we have already observed that $(x,y) \mapsto (x,-y)$ is the only non trivial element of $\mathrm{Jonq}_0(\mathbf{K})$ which commutes with $(a(x), R(x)y^{-1})$. $\square$

Lemma A.3.13 *Suppose that there exists a non trivial $g \in \mathrm{Cent}_0(f)$ that writes as $(x,y) \mapsto (x, y+1)$. Then f is of the form $(a(x), y+S(x))$ with $S \in \mathscr{K}(x)$ and $\mathrm{Cent}_0(f)$ is $\{(x, y+\gamma), \gamma \in \mathscr{K}\}$.*

Proof The section $\{y = \infty\}$ is the only g-invariant section. Thus f preserves this section and is of the form $(x,y) \dashrightarrow (a(x), R(x)y + S(x))$ where $R, S \in \mathscr{K}(x)$ and $a \in \mathrm{PGL}_2(\mathbf{K})$ is of infinite order. Writing down the relation $f \circ g = g \circ f$, we see that $R = 1$. Thus f is $(a(x), y+S(x))$ where S belongs to $\mathscr{K}(x)$ but not to $\mathscr{K}[x]$ since f is a Jonquières twist. The only f-invariant section is $\{y = \infty\}$. Indeed an invariant section s satisfies

$$s(a^n(x)) = s(x) + S(x) + \cdots + S(a^{n-1}(x)) \quad \forall n \in \mathbf{N}.$$

If s was not $\{y = \infty\}$, then the two sides of the equations are rational fractions. The degree of the right-hand side grows linearly in n while the degree of the left-hand side does not depend on n, contradiction. Thus, an element of $\mathrm{Cent}_0(f)$ fixes $\{y = \infty\}$ and is of the form $(x, A(x)y + B(x))$ with $A, B \in \mathscr{K}(x)$. Writing down the commutation relation, we get

$$A(x)y + B(x) + S(x) = A(a(x))y + A(a(x))S(x) + B(a(x)).$$

The fact that a is of infinite order implies that A is a constant. Then the equation is reduced to

$$B(x) + (1-A)S(x) - B(a(x)) = 0.$$

If $A \neq 1$, then $f : (x,y) \dashrightarrow (a(x), y+S(x))$ would be conjugate by $(x, y + \frac{B(x)}{1-A})$ to the elliptic elment $(a(x), y)$. Therefore $A = 1$ and B is a constant. Reciprocally we see that all elements of the form $(x,y) \mapsto (x, y+\beta)$ with $\beta \in \mathscr{K}$ commute with $f : (x,y) \dashrightarrow (a(x), y+S(x))$. $\square$

Lemma A.3.14 *Assume that no non-trivial element of* $\mathrm{Cent}_0(f)$ *is conjugate to an automorphism of a Hirzebruch surface and that* $\mathrm{Cent}_0(f)$ *has a non-trivial element* g*. Then* g *is a Jonquières involution and is the only non-trivial element of* $\mathrm{Cent}_0(f)$*.*

Proof By Lemma A.3.1, g is an elliptic element. By Corollary A.3.9, g acts on a conic bundle X and fixes pointwise a hyperelliptic curve C. The map f induces an action on C, equivariant with respect to the ramified double cover. The action of f on C is infinite, this is possible only if the action of f on the base is up to conjugation $x \mapsto \alpha x$ and if C is a rational curve whose projection on the base $\mathbb{P}^1$ is ramified over $x = 0, x = \infty$. Then the only singular fibres of X are over $x = 0, x = \infty$. If f had an indeterminacy point on these two fibres, then it would be a fixed point of g because g commutes with f. But the only fixed point of g on a singular fibre is the intersection point of the two components, which can not be an indeterminacy point by Lemma A.3.7. Therefore the Jonquières twist f must have an indeterminacy point over a point whose orbit in the base is infinite. This implies that the indeterminacy points of all the iterates of f form an infinite set. As g commutes with all the iterates of f, it fixes an infinite number of these indeterminacy points. Thus, the hyperelliptic curve C associated to g is the Zariski closure of these indeterminacy points and is uniquely determined by f. However C determines g too by Corollary A.3.9 (see [Bla11] for more general results). Therefore g is uniquely determined by f and is the only non trivial element of $\mathrm{Cent}_0(f)$. □

Putting together the three previous lemmas, we obtain the following improvement of [CD12b]:

Proposition A.3.15 *Let* f *be a base-wandering Jonquières twist. If* $\mathrm{Cent}_0(f)$ *is not trivial, then it is* $\{(x,y) \mapsto (x,ty), t \in \mathscr{K}^*\}$, $\{(x,y) \mapsto (x,y+t), t \in \mathscr{K}\}$, $\langle (x,y) \mapsto (x,-y) \rangle$ *or* $\langle$*a Jonquières involution*$\rangle$.

A.3.3 Persistent indeterminacy points

general facts

Let f be a birational transformation of a surface X. An indeterminacy point $x \in X$ of f will be called *persistent* if 1) for every $i > 0$, f^{-i} is regular at x; and 2) there are infinitely many curves contracted onto x by the iterates f^{-n}, $n \in \mathbf{N}$. This notion of persistence and the following idea appeared first in a non published version of [Can11], and it is also applied to some particular examples in [Dés08].

Proposition A.3.16 *Let* f *be an algebraically stable birational transformation of a surface* X*. Suppose that there exists at least one persistent indeterminacy point with an infinite backward* *orbit. Let n denote the number of such indeterminacy points. Then the centralizer* $\mathrm{Cent}(f)$ *of f admits a morphism* $\varphi : \mathrm{Cent}(f) \to \mathscr{S}_n$ *to the symmetric group of order n satisfying the following property: for any* $g \in \mathrm{Ker}(\varphi)$*, there exists* $l \in \mathbf{Z}$ *such that* $g \circ f^l$ *preserves fibre by fibre a pencil of rational curves.*

Proof The algebraic stability of f will be used throughout the proof, we will not recall it each time. Denote by $p_1, \cdots, p_n$ the persistent indeterminacy points of f. Let g be a birational transformation of X which commute with f. Fix an index $1 \le n_0 \le n$. Since $\{f^{-i}(p_{n_0}), i > 0\}$ is infinite, there exists $k_0 > 0$ such that g is regular at $f^{-k}(p_{n_0})$ for all $k \ge k_0$. For infinitely many $j > 0$, f^{-j} contracts a curve onto p_1, denote these curves by $C^j_{n_0}$. There exists $k_1 > 0$ such that g does not contract $C^k_{n_0}$ for all $k \ge k_1$. We deduce, from the above observations and the fact that f and g commute, that for $k \ge k_0$ the point $g(f^{-k}(p_{n_0}))$ is an indeterminacy point of some f^m with $0 < m \le k_0 + k_1$. Then there exists $0 \le m_0 < m$ such that

— for $0 \le i \le m_0$, f^i is regular at $g(f^{-k}(p_{n_0}))$;

— $f^{m_0}(g(f^{-k}(p_{n_0}))) = g(f^{m_0-k}(p_{n_0}))$ is an indeterminacy point of f.

By looking at $g(f^{-k}(p_{n_0}))$ and $C^{k'}_{n_0}$ for infinitely many k, k', we see that the above indeterminacy point does not depend on k and is persistent with an infinite backward orbit. So it is $p_{\sigma_g(n_0)}$ for some $1 \le \sigma_g(n_0) \le n$. This gives us a well defined map $\sigma_g : \{1, \cdots, n\} \to \{1, \cdots, n\}$.

Now let g, h be two elements of $\mathrm{Cent}(f)$. Then by considering a sufficiently large k for which g is regular at $f^{-k}(p_{n_0})$ and h is regular at $g(f^{-k}(p_{n_0}))$, we see that $\sigma_h \circ \sigma_g = \sigma_{h \circ g}$. By taking $h = g^{-1}$ we see that σ_g is bijective. We have then a group homomorphism φ from $\mathrm{Cent}(f)$ to the symmetric group $\mathscr{S}_n$ which sends g to σ_g.

Assume that n_0 is a fixed point of σ_g, this holds in particular when $g \in \mathrm{Ker}(\varphi)$. We keep the previous notations. Since $g(f^{-k}(p_{n_0}))$ is an indeterminacy point of f^m whose forward orbit meets p_{n_0}, for an appropriate choice of $l \le k$ we have

$$g \circ f^l(f^{-k}(p_{n_0})) = f^{-k}(p_{n_0})$$

for all $k \ge k_0$. This implies further

$$g \circ f^l(C^{k'}_{n_0}) = C^{k'}_{n_0}$$

for all sufficiently large k'. We conclude by Lemma A.3.17 below. $\square$

The proof of the following lemma in [Can10] is written over $\mathbf{C}$ for rational self-maps. It is observed in [Xie15] that the same proof works in all characteristics for birational transformations.

Lemma A.3.17 *A birational transformation of a smooth algebraic surface which preserves infinitely many curves preserves each member of a pencil of curves.*

persistent indeterminacy points for Jonquières twists

We examine the notion of persistence in the Jonquières group and give a complement to Theorem A.3.6:

Proposition A.3.18 *Let f be a Jonquières twist acting algebraically stably on a conic bundle X as in the statement of Theorem A.3.6. Then an indeterminacy point p of f is persistent if and only if the orbit of $\pi(p) \in \mathbb{P}^1$ under $\overline{f}$ is infinite. And in that case, every $f^{-i}, i \in \mathbf{N}^*$ contracts a curve onto p.*

Proof If $\pi(p)$ has a finite orbit then p certainly can not be persistent. Let us assume that the orbit of $\pi(p)$ is infinite. Then $\overline{f}$ is conjugate to $x \mapsto \alpha X$ with $\alpha \in K^*$ of infinite order or to $x \mapsto x+1$ (only when $\mathrm{char}(\mathbf{K}) = 0$). By the algebraic stability of f, f^{-i} is regular at p for all $i > 0$ and all the points $f^{-i}(p), i > 0$ are on distinct fibres. Denote by x_0, x_1 the points $\pi(p), \overline{f}(\pi(p))$. By Theorem A.3.6, we know that the fibres F_{x_0}, F_{x_1} are not singular. Thus f is regular on $F_{x_0} \backslash \{p\}$ and contracts it onto a point $q \in F_{x_1}$; f^{-1} is regular on $F_{x_1} \backslash \{q\}$ and contracts it onto p. Now pick a point x_n in the forward orbit of x_0 by $\overline{f}$ and consider the fibre F_{x_n}. The fibre F_{x_n} cannot be contracted onto q by $f^{-(n-1)}$ because of the algebraic stability of f. As a consequence it is contracted by f^{-n} onto p. □

Corollary A.3.19 *Let f be a Jonquières twist acting algebraically stably on a conic bundle X as in the statement of Theorem A.3.6. Suppose that the base action $\overline{f} \in \mathrm{PGL}_2(\mathbf{K})$ is of infinite order and there is an indeterminacy point of f located on a fibre $F_x \subset X$ such that $\overline{f}(x) \neq x$.*

1. *If $\overline{f}$ is of the form $x \mapsto x+1$ then $\mathrm{Cent}_b(f)$ is isomorphic to $\mathbf{Z}$;*
2. *if $\overline{f}$ is of the form $x \mapsto \alpha x$ then $\mathrm{Cent}_b(f)$ is isomorphic to the product of $\mathbf{Z}$ with a finite cyclic group.*

Note that the first case does not occur when $\mathrm{char}(\mathbf{K}) \neq 0$.

Proof Proposition A.3.18 shows that the birational transformation f satisfies the hypothesis of Proposition A.3.16. Let n denote the number of persistent indeterminacy points of f with infinite backward orbits. Let $g \in \mathrm{Cent}(f)$. Proposition A.3.16 says that $g^{n!} \circ f^l$ preserves every member of a pencil of rational curves for some $l \in \mathbf{Z}$. The proof of Proposition A.3.16 shows that certain members of this pencil of rational curves are fibres of the initial rational fibration on X, so this pencil of rational curves is the initial rational fibration. This means $\overline{g}^{n!} \circ \overline{f}^l = \mathrm{Id} \in \mathrm{PGL}_2(\mathbf{K})$.

When $\mathrm{char}(\mathbf{K}) = 0$ and $\overline{f}$ is $x \mapsto x+1$, its centralizer in $\mathrm{PGL}_2(\mathbf{K})$ is isomorphic to the additive group $\mathscr{K}$ and this group is torsion free. Thus, $\mathrm{Cent}_b(f)$ is contained in an infinite cyclic group in which $< \overline{f} >$ is of index $\leq n!$. The conclusion follows in this case.

When $\overline{f}$ is $x \mapsto \alpha x$ with α of infinite order, its centralizer in $\mathrm{PGL}_2(\mathbf{K})$ is isomorphic to the multiplicative group $\mathscr{K}^*$. The difference is that, in this case it is possible that $\overline{g}$ is of finite order $\leq n!$. Thus, we may have an additional finite cyclic factor of $\mathrm{Cent}_b(f)$. □

A.3.4 Local analysis around a fibre

Now we need to study the case where there is no persistent indeterminacy points. In this section we will work in the following setting:

- Let f be a base-wandering Jonquières twist. We can suppose that $\overline{f}$ is $x \mapsto \alpha x$ or $x \mapsto x+1$.
- Up to taking an algebraically stable model as in Theorem A.3.6, we can suppose that f is a birational transformation of a conic bundle X which satisfies the properties in Theorem A.3.6.
- We assume that the only indeterminacy points of f are on the fibres F_0, F_∞.

Without loss of generality, let us suppose that f has an indeterminacy point p on the fibre F_∞. By algebraic stability f^{-1} has an indeterminacy point $q \neq p$ on F_∞. If $x \in \mathbb{P}^1$ is not 0 nor ∞, then the orbit of x under $\overline{f}$ is infinite and the fibre F_x is regular. As f has an indeterminacy point on F_∞, the fibre F_∞ is also regular. Assume that F_0 is singular, then it is the union of two (-1)-curves and f exchanges the two components. Since the aim of this section is to prove that $\mathrm{Cent}_b(f)$ is finite by cyclic, it is not harmful to replace f with f^2 so that the two components of F_0 are no more exchanged and we can assume that F_0 is regular. Thus, we can suppose that

- the surface X is a Hirzebruch surface.

If $\overline{f}$ is $x \mapsto \alpha x$, then $\mathrm{Cent}_b(f)$ is contained in $\{(x \mapsto \gamma x), \gamma \in \mathscr{K}^*\}$ and all elements of $\mathrm{Cent}_b(f)$ fix 0 and ∞. Similarly if $\overline{f}$ is $x \mapsto x+1$ then all elements of $\mathrm{Cent}_b(f)$ fix ∞. Thus F_0 or F_∞ is $\mathrm{Cent}(f)$-invariant (under total transforms), we will study the (semi-)local behaviour of the elements in $\mathrm{Cent}(f)$ around such an invariant fibre.

An infinite chain

We blow up X at p,q the indeterminacy points of f,f^{-1}, obtaining a new surface X_1. The fibre of X_1 over 0 is a chain of three rational curves $C_{-1}+C_0+C_1$ where C_1 (resp. C_{-1}) is the exceptional curve corresponding to p (resp. q) and C_0 is the strict transform of $F_\infty \subset X$. Now f induces a birational transformation f_1 of X_1. As in Lemma A.3.7, we know that f_1 (resp. f_1^{-1}) has an indeterminacy point p_2 (resp. q_2) on C_1 (resp. C_{-1}) which is disjoint from C_0. We then blow up p_2,q_2 and repeat the process. We have:

— for every $n \in \mathbf{N}$, a surface X_n on which f induces a birational transformation f_n;

— the fibre of X_n over 0 is a chain of rational curves $C_{-n},\cdots,C_0,\cdots,C_n$;

— f_n (resp. f^{-n}) has an indeterminacy point p_{n+1} (resp. q_{n+1}) on C_n (resp. C_{-n}) disjoint from C_{n-1} (resp. $C_{-(n-1)}$).

Let g be a birational transformation of X which commutes with f. We already observed that F_∞ is an invariant fibre of g. If g is regular on F_∞, then the commutativity implies that g preserves the set $\{p,q\}$. Suppose that g is not regular on F_∞. Then g (resp. g^{-1}) has an indeterminacy point p' (resp. q') on F_∞. Replacing g by g^{-1} or f by f^{-1}, we can suppose that $p' \neq q$. Then for every point $x \in F_\infty$ such that $x \neq p,p'$, we have that $g(q)=g(f(x))=f(g(x))$ is a point, thus equals q. This further implies $q=q'$. Then we apply the same argument to g,f^{-1}, obtaining $p=p'$. In summary, g is either regular on F_∞ and preserves $\{p,q\}$, or the set of indeterminacy points of g,g^{-1} on F_∞ is exactly $\{p,q\}$.

We lift g to a birational transformation on X_n. By repeating the above arguments, we deduce that for all $n \in \mathbf{N}$ the two indeterminacy points of f_n,f_n^{-1} on the fibre $F_\infty \subset X_n$ coincide with that of g_n,g_n^{-1} if the later exist. This means that for a C_i given, and for sufficiently large n, the rational curve C_i is a component of the fibre of X_n and g_n maps it to another component C_j of the fibre. In other words g acts on the infinite chain of rational curves $\sum_{n\in\mathbf{Z}} C_n$. The dual graph of this infinite chain of rational curves is a chain of vertices indexed by $\mathbf{Z}$. The action of f on the dual graph is just a non trivial translation. The isomorphism group of the dual graph is isomorphic to $\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$. Those isomorphisms which commute with a non trivial translation coincide with the subgroup of translations $\mathbf{Z}$. The above considerations can be summarized as follows:

Lemma A.3.20 *There is a group homomorphism $\Phi : \mathrm{Cent}(f) \to \mathbf{Z}$ such that $g(C_n) = C_{\Phi(g)+n}$ for $g \in \mathrm{Cent}(f)$. An element $g \in \mathrm{Cent}(f)$ is in the kernel of Φ if and only if $g(C_n) = C_n$ for every $n \in \mathbf{Z}$. In other words an element g of the kernel of Φ is regular on the fibre F_∞ and fixes the indeterminacy points of f,f^{-1} on this fibre.*

Lemma A.3.21 *Let g be an element of* $\mathrm{Cent}(f)$*. Let $x \in \mathbb{P}^1$ be a point not fixed by $\overline{f}$. Then g can not have any indeterminacy points on the fibre F_x over x.*

Proof By our hypothesis f is regular on all fibres F_{x_n} where $\{x_n, n \in \mathbf{Z}\}$ denote the orbit of x under $\overline{f}$. If g had an indeterminacy point p on F_x, then $f(p), f^2(p), \cdots$ would give us an infinite number of indeterminacy points of g. $\square$

Corollary A.3.22 *Suppose that $\overline{f}$ is conjugate to $x \mapsto x+1$ (in particular* $\mathrm{char}(\mathbf{K}) = 0$*). Let $g \in \mathrm{Cent}(f)$ be in the kernel of $\Phi : \mathrm{Cent}(f) \to \mathbf{Z}$. Then g is an automorphism of X. Furthermore g preserves the rational fibration fibre by fibre.*

Proof Lemma A.3.20 says that g does not have any indeterminacy point on the fibre F_∞. Lemma A.3.21 says that g does not have any indeterminacy point elsewhere neither. Thus, g is an automorphism. Since $\overline{g}$ commutes with $\overline{f} : x \mapsto x+1$, $\overline{g}$ is $x \mapsto x+v$ for some $v \in \mathscr{K}$. Suppose by contradiction that $v \neq 0$. Then g is an elliptic element of infinite order and $f \in \mathrm{Cent}(g)$. We can apply Theorem B.2.3 to g, f and put them in normal form. As f is a Jonquières twist, the rational fibration preserved simultaneously by f and g is unique and it must be the rational fibration appeared in the normal form. Hence, Theorem B.2.3 forbids $\overline{f}, \overline{g}$ to be both non-trivial and of the form $x \mapsto x + sth$. $\square$

When $\overline{f}$ is of the form $x \mapsto \alpha x$, there are two special fibres F_0, F_∞ and the above easy argument does not work.

Formal considerations along a fibre

In the rest of this section we will assume that $\overline{f}$ is $x \mapsto \alpha x$. There are two invariant fibers F_∞ and F_0 in this case. We assume that f has an indeterminacy point q on F_0.

The idea of what we do in the sequel is as follows. Let us look at the case where $\mathscr{K} = \mathbf{C}$. The indeterminacy point $q \in F_0$ of f^{-1} is a fixed point of f, at which the differential of f has two eigenvalues 0 and α; the fibre directon is superattracting and in the transverse direction f is just $x \mapsto \alpha x$. Therefore there is a local invariant manifold at q for f, which is a local holomorphic section of the rational fibration. Likewise, there is a local invariant manifold at $p \in F_0$, the indeterminacy point of f. These two local holomorphic sections allow us to conjugate locally holomorphically f to $(\alpha x, a(x)y)$ where a is a germ of holomorphic function. The structure of Jonquières maps is nice enough to allow us to apply this geometric idea over any field in an elementary way. We need just to work with formal series instead of polynomials.

From now on we fix $f:(x,y)\dashrightarrow(\alpha x,\frac{A(x)y+B(x)}{C(x)y+D(x)})$ where $\alpha\in\mathscr{K}^*$ is of infinite order and $A,B,C,D\in\mathscr{K}[x]$. Without loss of generality, we suppose that 1) the point $(0,0)$ (resp. $(0,\infty)$) is an indeterminacy point of f (resp. f^{-1}); 2) one of the A,B,C,D is not a multiple of x. This implies

$$B(0)=C(0)=D(0)=0,\ A(0)\neq 0. \tag{A.3}$$

We will consider A,B,C,D as elements of the ring of formal series $\mathscr{K}[\![x]\!]$. We will also view f as an element of the formal Jonquières group $\mathrm{PGL}_2(\mathscr{K}((x)))\rtimes\mathscr{K}^*$ whose elements are formal expressions of the form $(\mu x,\frac{a(x)y+b(x)}{c(x)y+d(x)})$ where $\mu\in\mathscr{K}^*$ and a,b,c,d belong to $\mathscr{K}((x))$, the fraction field of $\mathscr{K}[\![x]\!]$.

Normal form. We want to conjugate f to a formal expression of the form $(\alpha x,\beta(x)y),\beta\in\mathscr{K}((x))$ by some formal expression $(x,\frac{E(x)y+F(x)}{G(x)y+H(x)})$ with $E,F,G,H\in\mathscr{K}[\![x]\!]$. This amounts to say that we are looking for $E,F,G,H\in\mathscr{K}[\![x]\!]$ such that $EF-GH\neq 0$ and

$$\begin{pmatrix}E(\alpha x) & F(\alpha x)\\ G(\alpha x) & H(\alpha x)\end{pmatrix}^{-1}\begin{pmatrix}A(x) & B(x)\\ C(x) & D(x)\end{pmatrix}\begin{pmatrix}E(x) & F(x)\\ G(x) & H(x)\end{pmatrix}$$

is a diagonal matrix. By writing out the explicit expressions of the up-right entry and the down-left entry of this matrix product, we obtain two equations to solve:

$$F(x)H(\alpha x)A(x)+H(x)H(\alpha x)B(x)-F(x)F(\alpha x)C(x)-H(x)F(\alpha x)D(x)=0 \tag{A.4}$$

$$-E(x)G(\alpha x)A(x)-G(x)G(\alpha x)B(x)+E(x)E(\alpha x)C(x)+G(x)E(\alpha x)D(x)=0 \tag{A.5}$$

We will use minuscules to denote the coefficients of the formal series, e.g. $E(x)=\sum_{i\in\mathbf{N}}e_ix^i$. Let us first look at the constant terms of equations (A.4), (A.5), they give

$$-e_0g_0a_0-g_0^2b_0+e_0^2c_0+e_0g_0d_0=0=f_0h_0a_0+h_0^2b_0-f_0f_0c_0-f_0h_0d_0.$$

Since $b_0=c_0=d_0=0$ and $a_0\neq 0$ (see Equation (A.3)), we must have $e_0g_0=f_0h_0=0$. We can choose $f_0=g_0=0$ and $e_0=h_0=1$, this guarantees in particular that our solution will satisfy $EH-FG\neq 0$.

Remark that the equations (A.4) and (A.5) involve respectively only E,G and F,H, and they have exactly the same form. So it suffices to show the existence of E,G which satisfy equation (A.4). The constant term is done, let us look at the x term. This leads to a linear equation in e_1,g_1 with coefficients involving a_0,b_0,c_0,d_0,e_0,g_0 and α. Therefore there exists at least one solution for e_1, g_1. Then we turn to the next term and get a linear equation in e_2, g_2, and so on. Hence, we can find E,F,G,H which satisfy the desired properties. To sum up, we have:

Lemma A.3.23 *There exists $E,F,G,H \in \mathscr{K}[\![x]\!]$ such that:*

— $E(0) = H(0) = 1$ *and* $F(0) = G(0) = 0$*, in particular* $\begin{pmatrix} E & F \\ G & H \end{pmatrix} \in \mathrm{PGL}_2(\mathscr{K}((x)))$*;*

— $(x, \frac{E(x)y+F(x)}{G(x)y+H(x)})$ *conjugates f to $(\alpha x, \beta(x)y)$ for some $\beta \in \mathscr{K}((x))$;*

Projective line over $\mathscr{K}((x))$. We call an element of $\mathbb{P}^1(\mathscr{K}((x))) = \mathscr{K}((x)) \bigcup \{\infty\}$ a formal section. We say a formal section $\theta(x)$ passes through the origin if $\theta(0) = 0$. An element $u = (\mu x, \frac{a(x)y+b(x)}{c(x)y+d(x)})$ of the formal Jonquières group $\mathrm{PGL}_2(\mathscr{K}((x))) \rtimes \mathscr{K}^*$ acts on $\mathbb{P}^1(\mathscr{K}((x)))$ in the following way:

$$\theta(x) \mapsto u \cdot \theta(x) = \begin{cases} \infty & \text{if } c(\mu^{-1}x)\theta(\mu^{-1}x) + d(\mu^{-1}x) = 0 \\ \frac{a(\mu^{-1}x)\theta(\mu^{-1}x)+b(\mu^{-1}x)}{c(\mu^{-1}x)\theta(\mu^{-1}x)+d(\mu^{-1}x)} & \text{otherwise} \end{cases},$$

$$\infty \mapsto \begin{cases} \infty & \text{if } c = 0 \\ \frac{a(\mu^{-1}x)}{c(\mu^{-1}x)} & \text{if } c \neq 0 \end{cases}.$$

Geometrically this is saying that a formal section of the rational fibration is sent to another by a formal Jonquières transformation. Remark that this action on $\mathbb{P}^1_{\mathscr{K}((x))}$ is not an automorphism of $\mathscr{K}((x))$-algebraic variety. In scheme theoretic language, we have a commutative diagram:

$$\begin{tikzcd} \mathbb{P}^1_{\mathscr{K}((x))} \arrow[r, "\theta \mapsto u \cdot \theta"] \arrow[d] & \mathbb{P}^1_{\mathscr{K}((x))} \arrow[d] \\ \mathrm{Spec}(\mathscr{K}((x))) \arrow[r, "\mu x \leftarrow x"] & \mathrm{Spec}(\mathscr{K}((x))). \end{tikzcd}$$

Thus, we have a group homomorphism from $\mathrm{PGL}_2(\mathscr{K}((x))) \rtimes \mathscr{K}^*$ to the group of such twisted automorphisms of $\mathbb{P}^1_{\mathscr{K}((x))}$.

Now let $g \in \mathrm{Cent}(f)$ be an element in the kernel of Φ. Recall (see Lemma A.3.20) that g is regular on the fibre F_0 and fixes $(0,0),(0,\infty)$. We showed that f is conjugate by $\begin{pmatrix} E & F \\ G & H \end{pmatrix}$ to a formal expression $\hat{f}$ of the form $(\alpha x, \beta(x)y)$. We conjugate g by $\begin{pmatrix} E & F \\ G & H \end{pmatrix}$ too to get a formal expression $\hat{g}$. Then $\hat{g}$ commutes with $\hat{f}$.

Recall that, by Lemma A.3.23, we get $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ when we evaluate the formal expression $\begin{pmatrix}E & F\\ G & H\end{pmatrix}$ at $x=0$. Together with the fact that $g \in \mathrm{Ker}(\Phi)$, this implies that we get $y \mapsto \delta_0 y$ for some $\delta_0 \in \mathscr{K}^*$ when we evaluate $\hat{g}$ at $x=0$.

Let us consider the actions of $\hat{f}, \hat{g}$ on $\mathbb{P}^1_{\mathscr{K}((x))}$ as described above. Since $\hat{f}$ is in diagonal form, it fixes the points 0 and ∞ of $\mathbb{P}^1_{\mathscr{K}((x))}$.

Lemma A.3.24 *If $\theta \in \mathbb{P}^1_{\mathscr{K}((x))}$ satisfies $\theta(0)=0$ and $\hat{f}\cdot\theta(x)=\theta(x)$, then $\theta=0$.*

Proof The equation $\hat{f}\cdot\theta(x)=\theta(x)$ writes as $\beta(\alpha^{-1}x)\theta(\alpha^{-1}x)=\theta(x)$, i.e. $\theta(\alpha x)^{-1}\beta(x)\theta(x)=1$. Suppose by contradiction that θ is not 0. Then we can write $\theta(x)$ as $x^r\tilde{\theta}(x)$ where $r>0$ and $\tilde{\theta}(0)\neq 0$. Hence we have $\tilde{\theta}(\alpha x)^{-1}\beta(x)\tilde{\theta}(x)=\alpha^r$. This implies that $\hat{f}$ is conjugate by $(x,\tilde{\theta}(x)y)$ to $(\alpha x, \alpha^r y)$. Since $\tilde{\theta}(0)\neq 0$ and $\begin{pmatrix}E(0) & F(0)\\ G(0) & H(0)\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, this implies that the initial Jonquières twist f is regular on the fibre F_0, contradiction. □

Since $\hat{g}$ is $y\mapsto\delta_0 y$ at $x=0$, it sends the formal section $0\in\mathbb{P}^1(\mathscr{K}((x)))$ to another former section passing through the origin. The fact that $\hat{f}$ and $\hat{g}$ commute and the fact that 0 is the only fixed formal section of $\hat{f}$ which passes through the origin imply that $\hat{g}$ fixes $0\in\mathbb{P}^1_{\mathscr{K}((x))}$. Likewise $\hat{g}$ fixes ∞ too. Therefore $\hat{g}$ writes as $(\gamma x, \delta(x)y)$ where $\gamma\in\mathscr{K}^*$ and $\delta\in\mathscr{K}((x))$ satisfies $\delta(0)=\delta_0\neq 0$.

Normal forms for a pair. Let us assume for the moment that γ is not a root of unity; we are going to prove that this is impossible. We want to, under this hypothesis, conjugate $\hat{g}=(\gamma x,\delta(x)y)$ to $(\gamma x,\delta(0)y)$ by $h=(x,\xi(x)y)$ for some $\xi\in\mathscr{K}[\![x]\!]$. Remark that the conjugate of $\hat{f}$ by h will still be in diagonal form.

We write $\delta=\frac{\omega}{\sigma}$ where $\omega,\sigma\in\mathscr{K}[\![x]\!]$ satisfies $\omega(0)\neq 0,\sigma(0)\neq 0$ and $\frac{\omega(0)}{\sigma(0)}=\delta(0)$. We will write ξ as $\sum_{i\in\mathbf{N}}\xi_i x^i$, and likewise for σ,ω.

After conjugation by $h=(x,\xi(x)y)$, $\hat{g}$ becomes

$$\tilde{g}=h\circ\hat{g}\circ h^{-1}=(\gamma x,\frac{\xi(\gamma x)}{\xi(x)}\frac{\omega(x)}{\sigma(x)}y).$$

Therefore the equation we want to solve is

$$\xi(\gamma x)\omega(x)=\frac{\omega_0}{\sigma_0}\xi(x)\sigma(x). \tag{A.6}$$

The constant terms of the two sides are automatically equal, let us just choose $\xi_0 = 1$. Comparing the other terms, we obtain

$$\begin{aligned}
&\xi_0\omega_1 + \gamma\xi_1\omega_0 = \frac{\omega_0}{\sigma_0}(\xi_0\sigma_1 + \xi_1\sigma_0)\\
&\xi_0\omega_2 + \gamma\xi_1\omega_1 + \gamma^2\xi_2\omega_0 = \frac{\omega_0}{\sigma_0}(\xi_0\sigma_2 + \xi_1\sigma_1 + \xi_2\sigma_0)\\
&\cdots
\end{aligned}$$

which are equivalent to

$$\begin{aligned}
&(\gamma - 1)\omega_0\xi_1 = \frac{\omega_0}{\sigma_0}\xi_0\sigma_1 - \xi_0\omega_1\\
&(\gamma^2 - 1)\omega_0\xi_2 = \frac{\omega_0}{\sigma_0}(\xi_0\sigma_2 + \xi_1\sigma_1) - \xi_0\omega_2 - \gamma\xi_1\omega_1\\
&\cdots.
\end{aligned}$$

For the i-th term, we have a linear equation whose coefficient before ξ_i is $(\gamma^i - 1)\omega_0$. Since $\omega \neq 0$ and we have supposed that γ is not a root of unity, The above equations always have solutions. In summary, we have the following intermediate lemma (we will get from this lemma a contradiction so its hypothesis is in fact absurd):

Lemma A.3.25 *Suppose that $g \in \mathrm{Ker}(\Phi)$ and the action of g on the base is of infinite order. Then we can conjugate f and g, simultaneously by an element in $\mathrm{PGL}_2(\mathscr{K}((x)))$ whose evaluation at $x = 0$ is $\mathrm{Id} : y \mapsto y$, to*

$$\tilde{g} = (\gamma x, \delta y),\ \tilde{f} = (\alpha x, \beta(x) y)$$

where $\alpha, \gamma, \delta \in \mathscr{K}^*$, $\beta \in \mathscr{K}((x))^*$ and α, γ are of infinite order.

Writing down the equation $\tilde{f} \circ \tilde{g} = \tilde{g} \circ \tilde{f}$, we get $\delta\beta(x) = \delta\beta(\gamma x)$. As $\delta \neq 0$, we get $\beta(x) = \beta(\gamma x)$. We write $\beta = \frac{\beta^{num}}{\beta^{den}}$ with $\beta^{num}, \beta^{den} \in \mathscr{K}[\![x]\!]$ such that at least one of the $\beta_0^{num}, \beta_0^{den}$ is not 0. The equation becomes

$$\beta^{num}(x)\beta^{den}(\gamma x) = \beta^{den}(x)\beta^{num}(\gamma x).$$

By comparing the coeffcients of two sides, we get

$$\forall k \in \mathbf{N},\ \sum_{i+j=k} \beta_i^{num}\beta_j^{den}\gamma^j = \sum_{i+j=k} \beta_i^{den}\beta_j^{num}\gamma^j.$$

Then by induction on k we get from these equations:

1. either $\beta^{num} = 0$ (when $\beta_0^{num} = 0$), this is impossible;
2. or $\beta^{den} = 0$ (when $\beta_0^{den} = 0$), this is again impossible;
3. or $\beta^{num} = \kappa\beta^{den}$ for some $\kappa \in \mathscr{K}^*$ (when $\beta_0^{num}\beta_0^{den} \neq 0$). Then$\tilde{f} = (\alpha x, \kappa y)$, this contradicts the fact that the original birational transformation f has an indeterminacy point on the fibre F_0 because to get $\tilde{f}$ we only did conjugations whose evaluation at $x = 0$ are the identity $y \mapsto y$.

Thus, we get

Proposition A.3.26 *Suppose that* $g \in \mathrm{Ker}(\Phi)$. *Then* $\overline{g}$ *is of finite order and* g *is an elliptic element of* $\mathrm{Cr}_2(\mathbf{K})$.

Proof We have already showed that $\overline{g}$ can not be of infinite order. Then an iterate g^k is in $\mathrm{Jonq}_0(\mathbf{K})$ and $f \in \mathrm{Cent}(g^k)$. By Theorem A.2.14, an element which commutes with a Jonquières twist in $\mathrm{Jonq}_0(\mathbf{K})$ can not have an infinite action on the base. As $\overline{f}$ is of infinite order, g^k must be elliptic. So g must be elliptic. □

Another fibre

The base action $\overline{f} \in \mathrm{PGL}_2(\mathbf{K})$ is $x \mapsto \alpha x$, it has two fixed points 0 and ∞. Recall that we are always under the hypothesis that the indeterminacy points of f are on the fibres F_0, F_∞. We have done analysis around the fibre F_0 on which f has an indeterminacy point. We will denote by Φ_0 the homomorphism Φ we considered before. In case f has also an indeterminacy point on F_∞, we denote the corresponding homomorphism by Φ_∞. We are going to reduce the proof to a situation where the following lemma applies.

Lemma A.3.27 *The image of* $\mathrm{Aut}(X)\bigcap\mathrm{Ker}(\Phi_0) \subset \mathrm{Cent}(f)$ *in* $\mathrm{Cent}_b(f) \subset \mathrm{PGL}_2(\mathbf{K})$ *is a finite cyclic group.*

Proof We recall first that the automorphism group of a Hirzebruch surface is an algebraic group (see [Mar71]). An element of $\mathrm{Cent}(f)$ which is regular everywhere on H must be in $\mathrm{Ker}(\Phi_0)$. Thus, $\mathrm{Aut}(X)\bigcap\mathrm{Ker}(\Phi_0) = \mathrm{Aut}(X)\bigcap\mathrm{Cent}(f)$ is an algebraic subgroup of $\mathrm{Aut}(H)$. An automorphism of a Hirzebruch surface always preserves the rational fibration and there is a morphism of algebraic groups from $\mathrm{Aut}(X)$ to $\mathrm{PGL}_2(\mathbf{K})$ (see [Mar71]). The image of $\mathrm{Aut}(X)\bigcap\mathrm{Ker}(\Phi_0) \subset \mathrm{Cent}(f)$ in $\mathrm{Cent}_b(f) \subset \mathrm{PGL}_2(\mathbf{K})$ is an algebraic subgroup Λ of $\mathrm{PGL}_2(\mathbf{K})$.

By Proposition A.3.26, the elements of Λ are all multiplication by roots of unity. If Λ was infinite then it would equal to its Zariski closure in $\mathrm{PGL}_2(\mathbf{K})$ and would be isomorphic to the multiplicative group $\mathscr{K}^*$. But the existence of a base-wandering Jonquières twist means that $\mathscr{K}^*$ contains elements of infinite order, for example α. This contradicts the fact that $\Lambda = \mathscr{K}^*$ is of torsion. The conclusion follows. $\square$

We first look at the case where we have two homomorphisms Φ_0, Φ_∞ to use:

Proposition A.3.28 *If f has an indeterminacy point on F_∞, then* $\mathrm{Ker}(\Phi_0) = \mathrm{Ker}(\Phi_\infty)$ *is a subgroup of* $\mathrm{Aut}(X)$. *The image of* $\mathrm{Ker}(\Phi_0)$ *in* $\mathrm{Cent}_b(f) \subset \mathrm{PGL}_2(\mathbf{K})$ *is a finite cyclic group.*

Proof Let g be an element of $\mathrm{Ker}(\Phi_0)$. By Proposition A.3.26 g is an elliptic element of $\mathrm{Cr}_2(\mathbf{K})$. If $\Phi_\infty(g)$ were not trivial, then g would act by a non trivial translation on the corresponding infinite chain of rational curves and could not be conjugate to any automorphism. This means g must belong to $\mathrm{Ker}(\Phi_\infty)$ and consequently g must be an automorphism of H. The second part of the statement follows from Lemma A.3.27. $\square$

When f is regular on F_∞, we may need to do a little more, but we get more precise information as well:

Proposition A.3.29 *If f has no indeterminacy points on F_∞, then* $\mathrm{Ker}(\Phi_0)$ *is a finite cyclic group whose elements are automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ of the form $(x,y) \mapsto (\gamma x, y)$ with γ a root of unity.*

Proof Assume that f is regular on F_∞. Let $g \in \mathrm{Ker}(\Phi_0)$ be a non trivial element, it is regular on F_0. By Lemma A.3.21, an indeterminacy point of g can only be located on F_∞.

Suppose that g has an indeterminacy point p on F_∞. Then g^{-1} also has an indeterminacy point q on F_∞. If $p \neq q$, then g would act by translation on the corresponding infinite chain of rational curves. This means that g would never be conjugate to an automorphism of some surface and contradicts Proposition A.3.26 which asserts that g is elliptic. Thus, we have $p = q$. The facts that f commutes with g and that f is regular on F_∞ imply $f(p) = p$. We blow up the Hirzebruch surface X at p to get a new surface X' and induced actions f', g'. The induced action f' is still regular on the fibre F'_∞ and preserves both of the two irreducible components. If g' has an indeterminacy point on F'_∞, then as before it coincides with the indeterminacy point of g'^{-1} and must be fixed by f'. Then we can keep blowing up indeterminacy points of maps induced from g, or contracting g-invariant (-1)-curves in the fibre, without loosing the regularity of the map induced by f. As g is elliptic, we will get at last a surface $\hat{X}$ with induced actions $\hat{f}, \hat{g}$

which are all regular on the fibre over ∞. We can suppose that $\hat{X}$ is minimal among the surfaces with this property. In particular $\hat{g}$ is an automorphism of $\hat{X}$. Moreover, the proof of Theorem A.3.6 shows that $\hat{X}$ is a conic bundle and the only possible singular fibre is $\hat{F}_\infty$. We claim that $\hat{F}_\infty$ is in fact regular. Suppose by contradiction that $\hat{F}_\infty$ is singular. Then it is a chain of two (-1)-curves and $\hat{g}$ exchanges the two components. However the conic bundle $\hat{X}$ is obtained from a Hirzebruch surface by a single blow-up, it has a unique section of negative self-intersection which passes through only one of the two components of the singular fibre. As a consequence, the automorphism $\hat{g}$ can not exchange the two components, contradiction. Thus, replacing X by $\hat{X}$, we can suppose from the beginning that g is an automorphism of the Hirzebruch surface X.

Suppose by contradiction that g preserves only finitely many sections of the rational fibration. Since f commutes with g, we can assume, after perhaps replacing f by some of its iterates, that f and g preserve simultaneously a section of the rational fibration. Removing this section and the fibre F_0 from H, we get an open set isomorphic to $\mathbb{A}^2$ restricted to which f writes as $(x',y') \mapsto (\alpha^{-1}x', A(x')y' + B(x'))$ where $A, B \in \mathscr{K}(x')$. The rational function A must be a constant because f acts as an automorphism on this affine open set. Likewise the rational function B must be a polynomial. But then $(deg(f^n))_{n\in\mathbf{N}}$ would be a bounded sequence. This contradicts the fact that f is a Jonquières twist.

Hence, if $g \in \mathrm{Ker}(\Phi)$ is non-trivial then it preserves necessarily infinitely many sections. This forces g to preserve each member of a pencil of rational curves on X whose general members are sections (see Lemma A.3.17). This is only possible if $X = \mathbb{P}^1 \times \mathbb{P}^1$ and g acts as $(x,y) \mapsto (\gamma x, y)$ with $\gamma \in \mathscr{K}^*$; here the projection of $\mathbb{P}^1 \times \mathbb{P}^1$ onto the first factor is the original rational fibration we were looking at. This allows us to conclude by Lemma A.3.27. $\square$

Example A.3.30 Let μ be a k-th root of unity, the pair $f : (x,y) \mapsto (\alpha x, \frac{(1+x^k)y+x^k}{(2+x^k)y+1+x^k}), g : (x,y) \mapsto (\mu x, y)$ satisfy the conditions in Proposition A.3.29.

Now let f be a base-wandering Jonquières twist which satisfies the hypothesis made at the beginning of Section A.3.4; in particular f is regular outside $F_0 \bigcap F_\infty$ and $\overline{f}$ is $x \mapsto \alpha x$ or $x \mapsto x+1$. The image $\Phi_\infty(\mathrm{Cent}(f))$ is an infinite cyclic subgroup of $\mathbf{Z}$ and is isomorphic to $\mathbf{Z}$, it is generated by $\Phi_\infty(g)$ for some $g \in \mathrm{Cent}(f)$. Then for any $h \in \mathrm{Cent}(f)$, there exists $k \in \mathbf{Z}$ such that $g^{-k} \circ h \in \mathrm{Ker}(\Phi_\infty)$. Thus, $\overline{g}^{-k} \circ \overline{h}$ belongs to the image of $\mathrm{Ker}(\Phi_\infty)$ in $\mathrm{Cent}_b(f)$. By Corollary A.3.22, Proposition A.3.28 and Proposition A.3.29, the image of $\mathrm{Ker}(\Phi_\infty)$ in $\mathrm{Cent}_b(f)$ is at worst finite cyclic. Note that $\mathrm{Cent}_b(f)$ is always abelian. Therefore we obtain the last piece of information to prove Theorem B.1.1:

Proposition A.3.31 *Let f be a base-wandering Jonquières twist which satisfies the hypothesis made at the beginning of Section A.3.4. Let g be an element of* $\mathrm{Cent}(f)$ *such that $\Phi_0(g)$ generates the image of Φ. Then* $\mathrm{Cent}_b(f)$ *is the product of a finite cyclic group with the infinite cyclic group generated by $\overline{g}$.*

A.4 Proofs of the main results

Proof (of Theorem A.1.1) Centralizers of loxodromic elements are virtually cyclic by Theorem A.2.1 of Blanc-Cantat. It is proved in [Giz80],[Can11] that centralizers of Halphen twists are virtually abelian (see Theorem A.2.2). Centralizers of Jonquières twists whose actions on the base are of finite order are contained in tori over the function field $\mathscr{K}(x)$, thus are abelian ([CD12b] see Theorem A.2.14). Our Theorem A.3.3 says that centralizers of base-wandering Jonquières twists are virtually abelian. Centralizers of infinite order elliptic elements (due to [BD15]) are described in Theorem B.2.3, from which we see directly that the only infinite order elliptic elements which admit non virtually abelian centralizers are those given here. □

Proof (of Theorem A.1.2) The proof is a direct combination of Theorems A.2.1, A.2.2, B.2.3, A.2.14 and B.1.1. □

Proof (of Remark A.1.4) In the first case Γ is an elliptic subgroup, so the degree function is bounded.

In the second case, the two Halphen twists f and g are automorphisms of a rational surface X preserving an elliptic fibration $X \to \mathbb{P}^1$. The elliptic fibration is induced by the linear system corresponding to mK_X for some $m \in \mathbf{N}^*$. For $n \in \mathbf{N}$, the actions of f^n and g^n on $Pic(X)$ are respectively

$$D \mapsto D - mn(D\cdot K_X)\Delta_i + \left(-\frac{m^2}{2}(D\cdot K_X)\cdot(n\Delta_i)^2 + m(D\cdot(n\Delta_i))\right)K_X, \quad i = 1,2$$

where $(\cdot)$ denotes the intersection form and $\Delta_i \in Pic(X)$ satisfies $\Delta_i \cdot K_X = 0$ (cf. [Giz80], [BD15]). Therefore the action of $f^i \circ g^j$ on $Pic(X)$ is

$$\begin{aligned} &D \mapsto D - mi(D\cdot K_X)\Delta_1 - mj(D\cdot K_X)\Delta_2 + \lambda_{ij}K_X \quad \text{where} \\ &\lambda_{ij} = -\frac{m^2}{2}(D\cdot K_X)\cdot\left(i^2\Delta_1^2 + j^2\Delta_2^2\right) + mD\cdot(i\Delta_1 + j\Delta_2) - ijm^2(D\cdot K_X)(\Delta_1\cdot\Delta_2). \end{aligned}$$

Let Λ be an ample class on X. Then the degree of $f^i \circ g^j$ is up to a bounded term (cf. [BD15] Section 5)

$$\Lambda \cdot (f^i \circ g^j)^* \Lambda = \Lambda^2 - \frac{m^2}{2}(\Lambda \cdot K_X)^2 \left(i^2\Delta_1^2 + j^2\Delta_2^2\right) - ijm^2(\Lambda \cdot K_X)^2(\Delta_1 \cdot \Delta_2).$$

Note that Δ_1^2 and Δ_2^2 are negative.

Let us consider the third case. Firstly assume that $\Gamma \cap \mathrm{Jonq}_0(\mathbf{K})$ is contained in a split torus over $\mathscr{K}(x)$. Then up to conjugation we can find two generators $f_0 : (x,y) \dashrightarrow (x, \frac{P(x)}{Q(x)}y), g_0 : (x,y) \dashrightarrow (x, \frac{R(x)}{S(x)}y)$ of $\Gamma \cap \mathrm{Jonq}_0(\mathbf{K})$ such that $P,Q,R,S \in \mathscr{K}[x]$ do not have common factors. If f_0 is elliptic, then $Q = 1$ and $P \in \mathscr{K}$, so the degree of $f_0^i g_0^j$ is $|j|(deg(R) + deg(S)) + 1$. If f_0, g_0 are both Jonquières twists, then the degree of $f_0^i g_0^j$ is $|i|(deg(P) + deg(Q)) + |j|(deg(R) + deg(S)) + 1$. Now assume that $\Gamma \cap \mathrm{Jonq}_0(\mathbf{K})$ is contained in a non-split torus over $\mathscr{K}(x)$. The torus becomes split over a quadratic extension L of $\mathscr{K}(x)$. The field L is the function field of a double cover of $\mathbb{P}^1$, it has also a notion of degree. On $\mathscr{K}(x)$, the L-degree function is a multiple of the $\mathscr{K}(x)$-degree function. Therefore the arguments in the split case still work.

In the fourth case the description of the degree function follows directly from the explicit expressions. □

Theorem A.4.1 *Let $G \subset \mathrm{Cr}_2(\mathbf{K})$ be a maximal abelian subgroup which has at least one element of infinite order. Then up to conjugation one of the following possibilities holds:*

1. *G is $\{(x,y) \mapsto (\alpha x, \beta y) | \alpha, \beta \in \mathscr{K}^*\}$, $\{(x,y) \mapsto (\alpha x, y+v) | \alpha \in \mathscr{K}^*, v \in \mathscr{K}\}$ or $\{(x,y) \mapsto (x+u, y+v) | u, v \in \mathscr{K}\}$;*
2. *G is the product of $\{(x,y) \mapsto (x, \beta y) | \beta \in \mathscr{K}^*\}$ with an infinite torsion group G_1. Each element of G_1 is of the form*

$$(x,y) \dashrightarrow \left(\eta(x), y\frac{S(x)}{S(\eta(x))}\right) \text{ with } \eta \in \mathrm{PGL}_2(\mathbf{K}), S \in \mathscr{K}(x)$$

   and the morphism from G_1 to $\mathrm{PGL}_2(\mathbf{K})$ embeds G_1 as a subgroup of the group of roots of unity of $\mathscr{K}$ or a subgroup of the additive group $\mathscr{K}$. All elements of G are elliptic but G is not conjugate to a group of automorphisms of any rational surface.
3. *G has a finite index subgroup contained in $\mathrm{Jonq}_0(\mathbf{K}) = \mathrm{PGL}_2(\mathscr{K}(x))$.*
4. *A finite index subgroup G' of G is a cyclic group generated by a base-wandering Jonquières twist.*

5. *A finite index subgroup G' of G is isomorphic to $\mathscr{K}^* \times \mathbf{Z}$ (resp. $\mathscr{K} \times \mathbf{Z}$) where the first factor is $\{(x,y) \mapsto (x, \beta y) | \beta \in \mathscr{K}^*\}$ (resp. $\{(x,y) \mapsto (x, y+v) | v \in \mathscr{K}\}$) and the second factor is generated by a base-wandering Jonquières twist, as in the fourth case of Theorem A.1.2;*
6. *A finite index subgroup G' of G is isomorphic to $\mathbf{Z}^s$ with $s \leq 8$ and G' preserves fibrewise an elliptic fibration;*
7. *A finite index subgroup G' of G is a cyclic group generated by a loxodromic element.*

The existence of a type two maximal abelian group is less obvious than the others. We give here two examples.

Example A.4.2 Let $q \in \mathbf{N}^*$. Let $(\xi_n)_n$ be a sequence of elements of $\mathscr{K}^*$ such that ξ_n is a primitive q^n-th root of unity and $\xi_n^q = \xi_{n-1}$. Let $(R_n)_n$ be a sequence of non-constant rational fractions. For $i \in \mathbf{N}$, put

$$f_{i+1} : (x,y) \dashrightarrow (\xi_{i+1}x, yS_{i+1}(x)) \text{ with } S_{i+1}(x) = \frac{R_i(x^{q^i})}{R_i(\xi_1 x^{q^i})} \frac{R_{i-1}(x^{q^{i-1}})}{R_{i-1}(\xi_2 x^{q^{i-1}})} \cdots \frac{R_1(x)}{R_1(\xi_i x)}.$$

We have $f_{i+1}^q = f_i$ for all $i \in \mathbf{N}^*$ so that the group G_1 generated by all the f_i is an infinite torsion abelian group. Let $T_i(x) = R_i(x^{q^i}) \cdots R_1(x^q)$. The conjugation by $(x,y) \dashrightarrow (x, yT_i(x))$ sends the group generated by $f_1, \cdots, f_i$ into the cyclic elliptic group $\{(x,y) \mapsto (\xi_i^j x, y) | j = 0, 1, \cdots, q^i - 1\}$. However the degree of f_i goes to infinity when i tends to infinity, which implies that G_1 can not be conjugate to an automorphism group. The product of G_1 with $\{(x,y) \mapsto (x, \beta y) | \beta \in \mathscr{K}^*\}$ is a maximal abelian subgroup of $\mathrm{Cr}_2(\mathbf{K})$; the maximality follows directly from Theorem B.2.3.

Example A.4.3 We can give an additive version of Example A.4.2. Suppose that $\mathrm{char}(\mathbf{K}) = p > 0$. Let $(t_n)_n$ be a sequence of elements of $\mathscr{K}$ linearly independant over $\mathbf{F}_p$. Let $R \in \mathscr{K}(x)$ be a non-constant rational fraction. For $i \in \mathbf{N}$, put

$$f_{i+1} : (x,y) \dashrightarrow (x + t_{i+1}, yS_{i+1}(x)) \text{ with } S_{i+1}(x) = \frac{\prod_{(a_1,\cdots,a_i) \in \mathbf{F}_p^i} R(x - \sum_{k=1}^i a_k t_k)}{\prod_{(a_1,\cdots,a_i) \in \mathbf{F}_p^i} R(x + t_{i+1} - \sum_{k=1}^i a_k t_k)}.$$

Let G_1 be the group generated by all the f_i. The product of G_1 with $\{(x,y) \mapsto (x, \beta y) | \beta \in \mathscr{K}^*\}$ is a maximal abelian subgroup of $\mathrm{Cr}_2(\mathbf{K})$.

Proof (of Theorem A.4.1) Let G be a maximal abelian subgroup of $\mathrm{Cr}_2(\mathbf{K})$. Note that if f is a non-trivial element of G, then G is the maximal abelian subgroup of $\mathrm{Cent}(f)$.

If G contains a loxodromic element f, then G is included in $\mathrm{Cent}(f)$ and is virtually the cyclic group generated by f by Theorem A.2.1; this corresponds to the last case of the above statement. If G contains a Halphen twist, then by Theorem A.2.2 it is virtually a free abelian group of rank ≤ 8 which preserves fibrewise an elliptic fibration; this corresponds to the sixth case.

Assume that G contains a base-wandering Jonquières twist f. Theorem B.1.1 says that $\mathrm{Cent}(f)$ is virtually isomorphic to $\mathscr{K}^* \times \mathbf{Z}$, $\mathscr{K} \times \mathbf{Z}$ or $\mathbf{Z}$. Thus the same is true for G. This correponds to the fourth and the fifth case.

Assume that G contains a non-base-wandering Jonquières twist f. Theorem A.2.14 says that $\mathrm{Cent}(f)$ is virtually isomorphic to an abelian subgroup of $\mathrm{PGL}_2(\mathscr{K}(x))$, so the same is true for G. This is the third case.

In the rest of the proof we assume that G *contains only elliptic elements*. Note that G is not necessarily an elliptic subgroup because it may not be finitely generated.

Assume that $\mathrm{char}(\mathbf{K}) = 0$ and G contains an element $f : (x,y) \mapsto (\alpha x, y+1)$ with $\alpha \in \mathscr{K}^*$. By Theorem B.2.3 we have

$$\mathrm{Cent}(f) = \{(x,y) \dashrightarrow (\eta(x), y+R(x)) | \eta \in \mathrm{PGL}_2(\mathbf{K}), \eta(\alpha x) = \alpha\eta(x), R \in \mathscr{K}(x), R(\alpha x) = R(x).\}$$

If α has infinite order, then $G = \mathrm{Cent}(f) = \{(x,y) \dashrightarrow (\gamma x, y+v) | \gamma \in \mathscr{K}^*, v \in \mathscr{K}\}$ and we are in the first case. Assume at first that G has an element g with an infinite action on the base of the rational fibration $(x,y) \mapsto x$. If the action of g on the base is conjugate to $x \mapsto \beta x$ with $\beta \in \mathscr{K}^*$, then up to conjugation in Jonq we can suppose that g is just our initial element $f : (x,y) \dashrightarrow (\alpha x, y+1)$ (see Proposition A.2.4), so that G is isomorphic to $\mathscr{K}^* \times \mathscr{K}$. If the the action of g on the base is conjugate to $x \mapsto x+1$, then by choosing an appropriate coordinate x, the two elements f and g are respectively $(x,y) \mapsto (x+1, y+R(x))$ and $(x,y) \mapsto (x, y+1)$ where R is a polynomial by Lemma A.2.11. We can conjugate g and f, simultaneously by $(x,y) \dashrightarrow (x, y+S(x))$ for some $S \in \mathscr{K}[x]$, to $(x,y) \mapsto (x+1, y)$ and $(x,y) \mapsto (x, y+1)$. Then we have

$$G = \mathrm{Cent}(f) \bigcap \mathrm{Cent}(g) = \{(x,y) \mapsto (x+u, y+v) | u, v \in \mathscr{K}\}.$$

We are still under the hypothesis that $\mathrm{char}(\mathbf{K}) = 0$ and G contains an element $f : (x,y) \mapsto (\alpha x, y+1)$ with $\alpha \in \mathscr{K}^*$. Assume now that no element of G has an infinite action on the base of the rational fibration $(x,y) \mapsto x$. Then the description of $\mathrm{Cent}(f)$ implies that G is a subgroup of

$$\{(x,y) \mapsto (\delta x, y+R(x)) | \delta \in \mathscr{K}^*, R \in \mathscr{K}(x)\}.$$

Consider the projection $\pi : G \to \mathrm{PGL}_2(\mathbf{K})$ which records the action on the base. Denote by G_0 the kernel of π and by G_b the image of π. We identify G_b as a subgroup of the multiplicative group of roots of unity of $\mathscr{K}$. We want to prove that G_b is finite so that G is virtually contained in $\mathrm{Jonq}_0(\mathbf{K}) = \mathrm{PGL}_2(\mathscr{K}(x))$. Assume that G_b is an infinite subgroup of the group of roots of unity. We first claim that G_0 is isomorphic to $\mathscr{K}$. Let $h : (x,y) \mapsto (x, y + R(x)), R \in \mathscr{K}(x)$ be an element of G_0 and $g : (x,y) \mapsto (\beta x, y + S(x)), S \in \mathscr{K}(x)$ be an element of G. The commutation relation $f \circ g = g \circ f$ implies $R(x) = R(\beta x)$. Here β can be any element of the infinite group G_b. This implies that R is constant, which proves the claim. Let H_γ be a finite subgroup of G_b, it is a cyclic group generated by $x \mapsto \gamma x$ for some $\gamma \in \mathscr{K}^*$. Let $g : (x,y) \mapsto (\gamma x, y + R(x))$ be an element of G such that $\pi(g)$ is $x \mapsto \gamma x$. By Lemma A.2.11 R is a polynomial. We can conjugate g by an element of the form $(x,y) \mapsto (x, y + P(x)), P \in \mathscr{K}[x]$ to $(x,y) \mapsto (\gamma x, y)$ and the polynomial P is unique up to addition by a constant. In fact, the conjugation by $(x,y) \mapsto (x, y + P(x))$ sends the subgroup $\pi^{-1}(H_\gamma)$ of G into $\{(x,y) \mapsto (\delta x, y + t), t \in \mathscr{K}\}$ because any element h of $\pi^{-1}(H_\gamma)$ is equal to $g^n \circ g_0$ for some $n \in \mathbf{Z}$ and $g_0 \in G_0$. The unicity of P implies that, if we take a finite subgroup H_ν which contains strictly H_γ, then the conjugation by $(x,y) \mapsto (x, y + P(x))$ still sends the subgroup $\pi^{-1}(H_\nu)$ into $\{(x,y) \mapsto (\delta x, y+t), t \in \mathscr{K}\}$. This further implies that the conjugation by $(x,y) \mapsto (x, y + P(x))$ sends the whole group G into $\{(x,y) \mapsto (\delta x, y+t), t \in \mathscr{K}\}$. Then by the maximality of G, it is isomorphic to $\mathscr{K}^* \times \mathscr{K}$ and we are in the first case of the statement. Note that we have made the hypothesis that G_b is torsion, so here $\mathscr{K}$ must be the algebraic closure of a finite field.

Assume that G contains an element $f : (x,y) \mapsto (\alpha x, \beta y)$ where $\alpha, \beta \in \mathscr{K}^*$ and β has infinite order. If α also has infinite order, then Theorem B.2.3 implies immediately that $G = \mathrm{Cent}(f)$ is isomorphic to $\mathscr{K}^* \times \mathscr{K}^*$ and we are in the first case. Assume that α has finite order but G contains an element $f_1 : (x,y) \dashrightarrow (\alpha_1 x, yR(x))$ where $R \in \mathscr{K}(x)$ and $\alpha_1 \in \mathscr{K}^*$ has infinite order. By Corollary A.3.11 the two elements f and f_1 are simultaneously conjugate to $(x,y) \mapsto (\alpha x, \beta y)$ and $(x,y) \mapsto (\alpha_1 x, ry)$ with $r \in \mathscr{K}^*$. Thus, Theorem B.2.3, when applied respectively to f and f_1, shows that $G = \mathrm{Cent}(f) \bigcap \mathrm{Cent}(f_1)$ is isomorphic to the diagonal group $\mathscr{K}^* \times \mathscr{K}^*$. Hence we are in the first case.

According to the classification of normal forms of elliptic elements of infinite order (see Proposition A.2.4), the only remaining cases are the two following: 1) G contains an element $f : (x,y) \mapsto (\alpha x, \beta y)$ where $\alpha \in \mathscr{K}^*$ has finite order and $\beta \in \mathscr{K}^*$ has infinite order but G contains no elements $(x,y) \dashrightarrow (\alpha_1 x, yR(x))$ with α_1 of infinite order; 2) $\mathrm{char}(\mathbf{K}) = p > 0$ and G contains an element $f : (x,y) \mapsto (x+1, \beta y)$ with $\beta \in \mathscr{K}^*$ of infinite order. In both cases $\mathrm{Cent}(f)$ is a subgroup of the Jonquières group by Theorem B.2.3. Denote by π the projection of G into

$\mathrm{PGL}_2(\mathbf{K})$. If $\pi(G)$ is finite then we are in the third case of Theorem A.4.1. So we assume that $\pi(G)$ is infinite. Then $\pi(G)$ is isomorphic to an infinite subgroup of the group of roots of unity or an infinite subgroup of $\mathscr{K}$, and it is an infinite torsion abelian group. We want to show that we are in the second case of Theorem A.4.1. By Lemma A.2.9, each element of G is of the form $(x,y) \dashrightarrow (\eta(x), y\frac{rS(x)}{S(\eta(x))})$ with $\eta \in \mathrm{PGL}_2(\mathbf{K}), r \in \mathscr{K}^*, S \in \mathscr{K}(x)$. If $(x,y) \dashrightarrow (\eta(x), y\frac{rS(x)}{S(\eta(x))})$ is an element of G, then $(x,y) \dashrightarrow (\eta(x), y\frac{S(x)}{S(\eta(x))})$ is also an element of G because it commutes with every other element. However the later has the same order in G as η in $\mathrm{PGL}_2(\mathbf{K})$. This means that G has a subgroup isomorphic to $\pi(G)$, so that G is isomorphic to the product of this subgroup with the kernel of π. To finish the proof, it suffices to show that the kernel of π is $\{(x,y) \mapsto (x, \beta y) | \beta \in \mathscr{K}^*\}$. This is because $(x,y) \mapsto (x, \beta y)$ are the only possible elliptic elements by Lemma A.2.9. $\square$

APPENDIX B

UNIQUENESS OF BIRATIONAL STRUCTURES ON INOUE SURFACES

B.1 Introduction

Inoue surfaces are compact non-Kähler complex surfaces discovered by Inoue in [Ino74]. They are of class VII in Enriques-Kodaira's classification of compact complex surfaces, and are the only compact complex sufaces with Betti numbers $b_1 = 1, b_2 = 0$ (cf. [Tel94]). There are three types of Inoue surfaces: S^0, S^+ and S^-. Their universal covers are isomorphic to $\mathbb{H} \times \mathbf{C}$ where $\mathbb{H}$ is the upper half plane and the deck transformations can be written as restrictions of complex affine transformations of $\mathbf{C}^2$. Therefore the Inoue surfaces are equipped with a natural complex affine structure. Klingler proves in [Kli98] that the natural complex affine structure is the unique complex projective structure carried by Inoue surfaces. In this article we prove the following:

Theorem B.1.1 *If an Inoue surface is equipped with a* $(\mathrm{Bir}(X), X)$*-structure for some complex projective surface X, then X is a rational surface and the* $(\mathrm{Bir}(X), X)$*-structure is induced by the natural* $(\mathrm{Aff}_2(\mathbf{C}), \mathbf{C}^2)$*-structure.*

Remark B.1.2 Roughly speaking, a birational structure is an atlas of local charts with birational changes of coordinates. The precise definition and basic properties will be given in Section B.2.2. It is a generalization of the classical (G, X)-structure; if we think of a geometric structure as a way to patch coordinates, then it is the most general algebraic geometric structure (the changes of coordinates are rational).

In a recent article [KS], Kwon and Sullivan introduced some generalized notions of geometric structures for which they allow a family of Lie groups $(G_i)_i$ acting on the same space X. The group of birational transformations of a surface, though not a classical Lie group itself, is generated by Lie groups acting by holomorphic diffeomorphisms on different birational models of X. So, the geometric structure of [KS] share interesting similarities with $(\mathrm{Bir}(X), X)$-structures.

Note that the group of birational transformations of a variety of dimension ≥ 3 may not be generated by its connected algebraic subgroups (cf. [BY]). Kwon and Sullivan proved in [KS] that every prime orientable three manifold admits such a generalized geometric structure. Their result is somewhat analogous to Dloussky's conjecture mentionned in Remark B.1.4 below.

Remark B.1.3 Compared to the four-pages-long proof in [Kli98] of the uniqueness of complex projective structure, our proof is more involved because the group $\mathrm{Bir}(\mathbb{P}^2)$ of birational transformations of $\mathbb{P}^2$ is much larger than $\mathrm{PGL}_3(\mathbf{C})$. Also the fact that in our case the developing map is a priori not holomorphic but only meromorphic will be the cause of some technical complications.

Remark B.1.4 Complex projective structures on compact complex surfaces are classified by Klingler in [Kli98]. There exist compact non-Kähler complex surfaces which have $(\mathrm{Bir}(\mathbb{P}^2), \mathbb{P}^2)$-structures but no complex projective structures, for example some Hopf surfaces; and Dloussky conjectured in [Dlo16] that every surface of class VII admits a $(\mathrm{Bir}(\mathbb{P}^2), \mathbb{P}^2)$-structure.

Remark B.1.5 If Y is a complex projective surface, we say that a subgroup Γ of $\mathrm{Bir}(Y)$ has the *Kleinian property* if the following three conditions are satisfied: 1) the group $\Gamma \subset \mathrm{Bir}(Y)$ acts by holomorphic diffeomorphisms on a Euclidean open set $U \subset Y$, i.e. an open set for the Euclidean topology but not necessarily for the Zariski topology; 2) the action of Γ on U is free and properly discontinuous; 3) the quotient U/Γ is compact. Once we have a birational Kleinian group, the quotient surface is equipped naturally with a birational structure. Thus, we can view Theorem B.1.1 as a result about Fatou components of (groups of) birational transformations. For a systematic study of birational Kleinian groups we refer to the forthcoming article [**Zhaoklein**].

Plan and strategy. Section B.2 concerns two subjects of independent interest. The notion of birational structure appeared already in the work of Dloussky [Dlo16] but several subtleties, that do not appear in classical geometric structures, were not addressed in that paper. In Section B.2.2 we give two different definitions of $(\mathrm{Bir}(\mathbb{P}^n), \mathbb{P}^n)$-structure. For $n = 2$ they are the same but for $n \geq 3$ whether they are the same is equivalent to an open question of Gromov. In Section B.2.3 we study Ahlfors-Nevanlinna currents attached to entire curves (see the work of Brunella and McQuillan in [McQ98], [**Bru99**]): for nice (uniform) families of entire curves, we show how to construct families of Ahlfors-Nevalinna currents, with a fixed cohomology class; this may be useful to people interested in holomorphic foliations or Kobayashi hyperbolicity.

After these preliminaries we prove Theorem B.1.1. The construction of Inoue surfaces of type S^0 (resp. $S^{\pm}$) will be recalled in Section B.3 (resp. B.4). For simplicity, let us focus, here, on

Inoue surfaces of type S^0. Rather different tools are used, depending on the size of the image of the holonomy representation. When the holonomy is injective, the two principal ingredients are the classification of solvable subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ due to Cantat ([Can11]), Déserti ([Dés15]) and Urech ([Ure]), and the classification of subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ ismorphic to $\mathbf{Z}^2$, obtained in [Zhaa]. With these results, we can reduce the structure group from $\mathrm{Bir}(\mathbb{P}^2)$ to $\mathrm{PGL}_3(\mathbf{C})$, and then apply Klingler's previous work [Kli98]. When the holonomy representation is not injective, we can suppose that its image is cyclic. Then, the strategy is geometric: an Inoue surface is foliated by compact real submanifolds of dimension three that are themselves foliated by entire curves, i.e. Riemann surfaces isomorphic to $\mathbf{C}$. Via the developing map, we obtain families of Levi-flat hypersurfaces foliated by entire curves, in some projective surfaces. The proof is then based on the following three tools that are described in Section B.2: 1) our deformation lemma for Ahlfors-Nevanlinna currents; 2) the relation between these currents and the transverse invariant measures of Plante ([Pla75]) and Sullivan ([Sul76]); 3) properties of the pull-back action of a birational transformation on currents (as in [DF01] and [Can01a]).

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B.2 Preliminaries

B.2.1 Groups of birational transformations

Let X be a smooth complex projective surface. We denote by $\mathrm{Bir}(X)$ the group of birational transformations of X. An element f of $\mathrm{Bir}(X)$ has a pull back action f^* on $\mathrm{H}^{1,1}(X,\mathbf{R})$ (cf. [DF01]). Note that in general $(f^*)^n \neq (f^n)^*$. Fix an ample class $H \in \mathrm{H}^{1,1}(X,\mathbf{R})$, the *$H$-degree* of f is the intersection number $f^*H \cdot H$.

The *plane Cremona group* $\mathrm{Bir}(\mathbb{P}^2)$ is the group of birational transformations of the projective plane $\mathbb{P}^2_{\mathbf{C}}$. It is isomorphic to the group of $\mathbf{C}$-algebra automorphisms of $\mathbf{C}(X_1,X_2)$, the function field of $\mathbb{P}^2_{\mathbf{C}}$. Using a system of homogeneous coordinates $[x_0;x_1;x_2]$, a birational transformation $f \in \mathrm{Bir}(\mathbb{P}^2)$ can be written as

$$[x_0 : x_1 : x_2] \dashrightarrow [f_0(x_0,x_1,x_2) : f_1(x_0,x_1,x_2) : f_2(x_0,x_1,x_2)]$$

where f_0, f_1, f_2 are homogeneous polynomials of the same degree without common factor. This degree does not depend on the system of homogeneous coordinates and is the degree of f with respect to the class of a projective line. Birational transformations of degree 1 are homographies and form $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbf{C})$, the group of automorphisms of the projective plane. See [Can18] for more about the Cremona group.

Algebraically stable maps. If f is a birational transformation of a smooth projective surface X, we denote by $\mathrm{Ind}(f)$ the set of indeterminacy points of f. We say that f is *algebraically stable* if there are no curves V on X such that $f^k(V) \subset \mathrm{Ind}(f)$ for some integer $k \geq 0$. There always exists a birational morphism $\hat{X} \to X$ which lifts f to an algebraically stable birational transformation of $\hat{X}$ ([DF01] Theorem 0.1). An algebraically stable map f satisfies $(f^*)^n = (f^n)^*$ (cf. [DF01]).

Four types of elements. Fix a Euclidean norm $\|\cdot\|$ on $\mathrm{H}^{1,1}(X, \mathbf{R})$. The two sequences $(\|(f^n)^*\|)_n$ and $((f^n)^* H \cdot H)_n$ have the same asymptotic growth. Elements of $\mathrm{Bir}(X)$ are classified into four types (cf. [DF01]):

1. The sequence $(\|(f^n)^*\|)_{n\in\mathbf{N}}$ is bounded, f is birationally conjugate to an automorphism of a smooth birational model of X and a positive iterate of f lies in the connected component of identity of the automorphism group of that surface. We call f an *elliptic* element.
2. The sequence $(\|(f^n)^*\|)_{n\in\mathbf{N}}$ grows linearly, f preserves a unique pencil of rational curves and f is not conjugate to an automorphism of any birational model of X. We call f a *Jonquières twist*.
3. The sequence $(\|(f^n)^*\|)_{n\in\mathbf{N}}$ grows quadratically, f is conjugate to an automorphism of a smooth birational model preserving a unique genus one fibration. We call f a *Halphen twist*.
4. The sequence $(\|(f^n)^*\|)_{n\in\mathbf{N}}$ grows exponentially and f is called *loxodromic*. The limit $\lambda(f) = \lim_{n\to+\infty}(\|(f^n)^*\|)^{\frac{1}{n}}$ exists and we call it the *dynamical degree* of f. If f is an algebraically stable map on X, then there is a nef cohomology class $v_f^+ \in \mathrm{H}^{1,1}(X, \mathbf{R})$, unique up to multiplication by a constant, such that $f^* v_f^+ = \lambda(f) v_f^+$. If moreover v_f^+ has zero self-intersection, then f is conjugate to an automorphism.

Loxodromic automorphisms. We refer the reader to [Can14] for details of the materials presented in this paragraph. Let X be a smooth projective surface and f be an automorphism of X which is loxodromic. The dynamical degree $\lambda(f)$ is a simple eigenvalue for the pullback action

f^* on $\mathrm{H}^{1,1}(X,\mathbf{R})$ and it is the unique eigenvalue of modulus larger than 1. Let $v_f^+ \in \mathrm{H}^{1,1}(X,\mathbf{R})$ be a non-zero eigenvector associated with $\lambda(f)$; we have $f^*v_f^+ = \lambda(f)v_f^+$. By considering f^{-1}, we can also find a non-zero eigenvector v_f^- such that $f^*v_f^- = \frac{1}{\lambda(f)}v_f^-$. The two cohomology classes v_f^+, v_f^- are nef and of self-intersection 0, they are uniquely determined up to scalar multiplication. They are irrational in the sense that the two lines $\mathbf{R}v_f^+$ and $\mathbf{R}v_f^-$ contain no non-zero elements of $\mathrm{H}^{1,1}(X,\mathbf{R}) \cap \mathrm{H}^2(X,\mathbf{Z})$. We will need the following theorem of Cantat which has been generalized to higher dimension by Dinh and Sibony:

Theorem B.2.1 (Cantat [Can01a], [Mon12], [DS05], [DS10]) *There is a unique closed positive current T_f^+ (resp. T_f^-) whose cohomology class is v_f^+ (resp. v_f^-). It satisfies $f^*T_f^+ = \lambda(f)T_f^+$ (resp. $f^*T_f^- = \frac{1}{\lambda(f)}T_f^-$).*

The Jonquières group. Fix an affine chart of $\mathbb{P}^2$ with coordinates (x,y). *The Jonquières group* Jonq is the subgroup of the Cremona group of all transformations of the form

$$(x,y) \mapsto \left(\frac{ax+b}{cx+d}, \frac{A(x)y+B(x)}{C(x)y+D(x)}\right), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{C}), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{C}(x)).$$

In other words, the Jonquières group is the maximal group of birational transformations of $\mathbb{P}^1 \times \mathbb{P}^1$ permuting the fibers of the projection onto the first factor; it is isomorphic to the semidirect product $\mathrm{PGL}_2(\mathbf{C}) \ltimes \mathrm{PGL}_2(\mathbf{C}(x))$. A different choice of the affine chart yields a conjugation by an element of $\mathrm{PGL}_3(\mathbf{C})$. More generally a conjugation by an element of the Cremona group yields a maximal group preserving a pencil of rational curves; conversely any two such groups are conjugate in $\mathrm{Bir}(\mathbb{P}^2)$.

Elements of the Jonquières group are either elliptic or Jonquières twists. We will need the following results:

Theorem B.2.2 ([Zhaa]) *Let G be a subgroup of* Jonq *which is isomorphic to $\mathbf{Z}^2$. Then G has a pair of generators (f,g) such that one of the following (mutually exclusive) situations happens:*

1. *f,g are elliptic elements and $G \subset \mathrm{Aut}(X)$ where X is a rational surface;*
2. *f is a Jonquières twist, and a finite index subgroup of G preserves each fiber of the f-invariant fibration;*
3. *f is a Jonquières twist with action of infinite order on the base of the rational fibration and g is an elliptic element whose action on the base is of finite order. In some affine chart, we can write f,g in one of the following forms:*

— *g is $(x,y) \mapsto (\alpha x, \beta y)$ and f is $(x,y) \dashrightarrow (\eta(x), yR(x^k))$ where $\alpha, \beta \in \mathbf{C}^*, \alpha^k = 1, R \in \mathbf{C}(x), \eta \in \mathrm{PGL}_2(\mathbf{C}), \eta(\alpha x) = \alpha\eta(x)$ and η is of infinite order;*

— *g is $(x,y) \mapsto (\alpha x, y+1)$ and f is $(x,y) \dashrightarrow (\eta(x), y+R(x))$ where $\alpha \in \mathbf{C}^*, R \in \mathbf{C}(x), R(\alpha x) = R(x), \eta \in \mathrm{PGL}_2(\mathbf{C}), \eta(\alpha x) = \alpha\eta(x)$ and η is of infinite order.*

Theorem B.2.3 ([BD15] Lemmata 2.7 and 2.8) *Let $f \in \mathrm{Bir}(\mathbb{P}^2)$ be an elliptic element of infinite order.*

1. *If f is of the form $(x,y) \mapsto (x, \nu y)$ where $\nu \in \mathbf{C}^*$ has infinite order, then the centralizer of f in $\mathrm{Bir}(\mathbb{P}^2)$ is*

$$\{(x,y) \dashrightarrow (\eta(x), yR(x)) | \eta \in \mathrm{PGL}_2(\mathbf{C}), R \in \mathbf{C}(x)\}.$$

2. *If f is of the form $(x,y) \mapsto (x, y+v)$ with $v \in \mathbf{C}^*$, then the centralizer of f in $\mathrm{Bir}(\mathbb{P}^2)$ is*

$$\{(x,y) \dashrightarrow (\eta(x), y+R(x)) | \eta \in \mathrm{PGL}_2(\mathbf{C}), R \in \mathbf{C}(x)\}.$$

Tits alternative and solvable subgroups. Déserti and Urech refined for finitely generated solvable subgroups, the strong Tits alternative proved by Cantat in [Can11]; we state the solvable version:

Theorem B.2.4 (Cantat, Déserti, Urech [Can11], [Dés15], [Ure]) *Let $G \subset \mathrm{Bir}(X)$ be a solvable subgroup. Exactly one of the following cases holds up to conjugation.*

1. *G is a subgroup of automorphisms of a birational model Y and a finite index subgroup of G is in $\mathrm{Aut}(Y)^0$ the connected component of identity of $\mathrm{Aut}(Y)$; the elements of G are all elliptic and G is called an elliptic subgroup.*
2. *G preserves a rational fibration and has at least one Jonquières twist.*
3. *G is a virtually abelian group whose elements are Halphen twists; there is a birational model Y on which the action of G is by automorphisms and preserves an elliptic fibration.*
4. *X is a rational surface and G is contained in the group generated by $\{(\alpha x, \beta y) | \alpha, \beta \in \mathbf{C}^*\}$ and one loxodromic monomial transformation $(x^p y^q, x^r y^s)$ where $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z})$ is a hyperbolic matrix.*
5. *X is an abelian surface and G is contained in the group generated by translations and one loxodromic transformation.*

B.2.2 Geometric structures

Let us first recall the classical notion of (G,X)-structures in the sense of Ehresmann (cf. [Ehr36], see also [Thu97]):

Definition B.2.5 *Let X be a connected real analytic manifold and let G be a Lie group which acts real analytically faithfully on X. Let V be a real analytic manifold. A (G,X)-structure on V is a maximal atlas of local charts $\phi_i : U_i \to X$ such that*

— *the U_i are open sets of V and form a covering;*

— *the ϕ_i are diffeomorphisms onto their images;*

— *the changes of coordinates $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ are restrictions of elements of G.*

A (G,X)-manifold is a manifold which is equipped with a (G,X)-structure.

The group of birational transformations of an algebraic variety is not a Lie group in the classical sense, see [BF13] for its topology. Its action on the variety is not a classical set-theoretic group action either. We give here two non-equivalent definitions of birational structure. The first one is more flexible and is the notion of birational structure that we use in this article.

Definition B.2.6 *Let V be a complex manifold. Let X be a smooth complex projective variety. A $(\mathrm{Bir}(X),X)$-structure on V is a maximal atlas of local charts $\varphi_i : U_i \to X_i$ such that*

— *the U_i are open sets of V and form a covering;*

— *the X_i are smooth projective varieties birational to X;*

— *the φ_i are biholomorphic onto their images;*

— *the changes of coordinates $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ are holomorphic diffeomorphisms which extend to birational maps from X_j to X_i.*

Definition B.2.7 *Let V be a complex manifold. Let X be a smooth complex projective variety. A strict $(\mathrm{Bir}(X),X)$-structure on V is a maximal atlas of local charts $\varphi_i : U_i \to X$ such that*

— *the U_i are open sets of V and form a covering;*

— *the φ_i are biholomorphic onto their images;*

— *the changes of coordinates $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ are holomorphic diffeomorphisms which extend to birational transformations of X.*

Remark B.2.8 Let X' be a smooth birational model of X. It follows directly from the definition that a $(\mathrm{Bir}(X),X)$-structure on V is the same thing as a $(\mathrm{Bir}(X'),X')$-structure on V, and that a strict $(\mathrm{Bir}(X),X)$-structure induces a $(\mathrm{Bir}(X),X)$-structure. But in general it is not true that a strict $(\mathrm{Bir}(X), X)$-structure on V gives rise to a strict $(\mathrm{Bir}(X'), X')$-structure on V, see Example B.2.15.

Holonomy and developing map. For a classical (G,X)-manifold V, there exist a group homomorphism $\mathrm{Hol} : \pi_1(V) \to G$ and a local diffeomorphism Dev from $\tilde{M}$, the universal cover of V, to X such that

$$\forall \gamma \in \pi_1(V), \mathrm{Dev} \circ \gamma = \mathrm{Hol}(\gamma) \circ \mathrm{Dev}.$$

The map Dev is called *the developing map* and Hol is called the *holonomy representation*. A (G,X)-structure is uniquely determined by its holonomy and its developing map, up to composition by an element of G.

The same proof as in the classical case shows:

Proposition B.2.9 *Let X be a smooth complex projective variety. Let V be a $(\mathrm{Bir}(X), X)$-manifold. Denote by $\tilde{V}$ the universal cover of V and π the quotient map. Fix a base point $v \in V$ and choose a point $w \in \tilde{V}$ such that $\pi(w) = v$. There exist a smooth birational model Y of X, a homomorphism* $\mathrm{Hol} : \pi_1(V, v) \to \mathrm{Bir}(Y)$ *and a $\pi_1(V, v)$-equivariant meromorphic map* $\mathrm{Dev} : \tilde{V} \dashrightarrow Y$ *such that*

$$\forall f \in \pi_1(V, v), \mathrm{Dev} \circ f = \mathrm{Hol}(f) \circ \mathrm{Dev}.$$

If $(Y', \mathrm{Hol}', \mathrm{Dev}')$ is another such triple, then there exists a birational map σ from Y to Y' such that $\mathrm{Hol}' = \sigma \mathrm{Hol} \sigma^{-1}$ *and* $\mathrm{Dev}' = \sigma \circ \mathrm{Dev}$. *We can choose* $(Y, \mathrm{Hol}, \mathrm{Dev})$ *so that* Dev *is holomorphic at w.*

Proof Let $c : [0,1] \to \tilde{V}$ be a smooth path from $w = c(0)$ to a point $z = c(1)$. The image $c([0,1])$ can be covered by local charts of birational structure $(U_0, \varphi_0 : U_0 \to X_0), \cdots, (U_k, \varphi_k : U_k \to X_k)$ which are pulled-back from local charts on V, such that $U_i \cap U_j$ is connected and is non-empty if $j = i+1$. We denote by g_i the map $\varphi_{i-1} \circ \varphi_i^{-1} \in \mathrm{Bir}(X_i, X_{i-1})$ which is the unique map such that $g_i \circ \varphi_i$ and φ_{i-1} agree on $U_i \cap U_j$; the uniqueness is because of the fact that two birational maps which coincide on a non empty Euclidean open set must be the same. We define $\mathrm{Dev}(z)$ as

$$\mathrm{Dev}(z) = g_1 g_2 \cdots g_k \varphi_k(z).$$

To be rigorous, this expression does not associate a value to any point z: the g_i are birational so we get only a meromorphic expression. Let us see that Dev is a well-defined meromorphic map from $\tilde{V}$ to X_0; it has milder properties than an arbitrary meromorphic map because locally analytically it behaves as a birational map. The unicity of the g_i guarantees that, once c is fixed, Dev does not depend on $U_1, \cdots, U_k$, but only on the initial chart U_0 at the base point w. Choose another path c' from w to z. Since $\tilde{V}$ is simply connected, there exists a homotopy $H : [0,1] \times [0,1] \to \tilde{V}$ between c and c'. We can cover $c([0,1] \times [0,1])$ by local charts of birational structure. The uniqueness of the transition maps then shows that Dev depends only on the homotopy class of c. Around the point w, the map Dev coincides with a coordinate chart, thus is holomorphic.

Let $f \in \pi_1(V, v)$ be a deck transformation. Let $z = f(w)$ in the above construction of Dev. We can suppose that $U_k = f(U_0)$ and $\varphi_k = \varphi_0 \circ f^{-1}$. Then $\mathrm{Dev}(f(w)) = g_1 g_2 \cdots g_k \varphi_0 \circ f^{-1}$. Put $\mathrm{Hol}(f) = g_1 g_2 \cdots g_k$. It belongs to $\mathrm{Bir}(X_0)$. We have $\mathrm{Dev} \circ f = \mathrm{Hol}(f) \circ \mathrm{Dev}$ in a neighbourhood of w. Thus $\mathrm{Dev} \circ f = \mathrm{Hol}(f) \circ \mathrm{Dev}$ by analytic continuation.

Let $(Y', \mathrm{Hol}', \mathrm{Dev}')$ be another such triple. Since the set of points of $\tilde{V}$ where a developing map is not defined or is not locally biholomorphic is locally closed of codimension at least one, there exists an open set U of $\tilde{V}$ restricted to which both Dev and Dev' are biholomorphic. Then $\mathrm{Dev}|_U$ and $\mathrm{Dev}'|_U$ are both local birational charts. They have to be compatible, i.e. $\mathrm{Dev}'|_U \circ (\mathrm{Dev}|_U)^{-1}$ extends to a birational map σ from Y to Y'. By analytic continuation we see that σ satisfies $\mathrm{Hol}' = \sigma \,\mathrm{Hol}\, \sigma^{-1}$ and $\mathrm{Dev}' = \sigma \circ \mathrm{Dev}$. □

Remark B.2.10 A developing map is locally birational; this means that locally it has a birational expression when written in some complex analytic coordinates. Thus a developing map has no ramification. In particular a ramified covering map is never a developing map.

If V is a $(\mathrm{Bir}(X), X)$-manifold, then any finite unramified cover V' of V is equipped with an induced $(\mathrm{Bir}(X), X)$-structure. If $(Y, \mathrm{Hol}, \mathrm{Dev})$ is a holonomy-developing-map triple for V, then the compositions $\pi_1(V') \to \pi_1(V) \xrightarrow{\mathrm{Hol}} \mathrm{Bir}(Y)$ and $V' \to V \xrightarrow{\mathrm{Dev}} Y$ form a pair of holonomy and developing map for V'.

Proposition B.2.11 *Let V be a complex manifold with two $(\mathrm{Bir}(X), X)$-structures. Let $(X_1, \mathrm{Hol}_1, \mathrm{Dev}_1)$ and $(X_2, \mathrm{Hol}_2, \mathrm{Dev}_2)$ be pairs of holonomy and developing map associated with these two $(\mathrm{Bir}(X), X)$-structures. The two $(\mathrm{Bir}(X), X)$-structures are the same if and only if there exists $\sigma \in \mathrm{Bir}(X_1, X_2)$ such that $\mathrm{Hol}_2 = \sigma \,\mathrm{Hol}_1\, \sigma^{-1}$ and $\mathrm{Dev}_2 = \sigma \circ \mathrm{Dev}_1$.*

Proof We need to prove the “if” part. Let z be a point of the universal cover $\tilde{V}$. Without loss of generality, using the "only if" part (Proposition B.2.9), we can suppose that Dev_1 and Dev_2 are both locally biholomorphic at z. Thus on a neighbourhood of z, the restrictions of Dev_1 and Dev_2 give local charts for their corresponding birational structures. The hypothesis implies that these charts are compatible, i.e. contained in a same maximal atlas. The conclusion follows. □

Remark B.2.12 Propositions B.2.9 and B.2.11 hold for strict $(\mathrm{Bir}(X), X)$-structures too. Note that for a strict $(\mathrm{Bir}(X), X)$-structure, the target of the developing map is X itself.

The next proposition says that we could alternatively define a birational structure using holonomy and developing map.

Proposition B.2.13 *Let X be a smooth projective variety. Let V be a compact complex manifold and $\tilde{V}$ its universal cover. Let $D : \tilde{V} \dashrightarrow X$ be a meromorphic map that satisfies the following: for every point $w \in \tilde{V}$, there is a Euclidean neighbourhood W of w and a holomorphic diffeomorphism φ from W to a Euclidean open set of a birational model X_w of X depending on w such that $D|_W \circ \varphi^{-1}$ is the restriction of a birational map. Let $H : \pi_1(V) \to \mathrm{Bir}(X)$ be a homomorphism of groups such that for every $\gamma \in \pi_1(V)$ we have $H(\gamma) \circ D = D \circ \gamma$. Then V has a $(\mathrm{Bir}(X), X)$-structure for which (H, D) is a holonomy/developing pair. If $X_w = X_0$ are the same for all w then we have a strict $(\mathrm{Bir}(X_0), X_0)$-structure.*

Proof Let v be a point of V. Choose a point $w \in \tilde{V}$ which projects onto v, and a sufficiently small neighbourhood W of w which maps bijectively to a neighbourhood U of v. By hypothesis U is biholomorphic to an open subset of a birational model X_w. The hypothesis on the local birational property of D and the equivariance of H imply that different choices of w give the same $(\mathrm{Bir}(X), X)$-structure on U. Thus V is equipped with a $(\mathrm{Bir}(X), X)$-structure. We leave the reader to verify that (H, D) is indeed a corresponding pair of holonomy and developing map. □

Corollary B.2.14 *Let $f : X_1 \dashrightarrow X_2$ be a birational map between two smooth projective varieties X_1, X_2. Let $(X_1, \mathrm{Hol}, \mathrm{Dev})$ be a holonomy/developing triple associated with a $(\mathrm{Bir}(X_1), X_1)$-structure on a compact complex manifold V. Then $(X_2, f\,\mathrm{Hol}\, f^{-1}, f \circ \mathrm{Dev})$ is a holonomy/developing triple associated with the same $(\mathrm{Bir}(X_1), X_1)$-structure.*

Proof The pair $(f\,\mathrm{Hol}\, f^{-1}, f \circ \mathrm{Dev})$ satisfies the conditions of Proposition B.2.13, thus defines a $(\mathrm{Bir}(X_2), X_2)$-structure. This $(\mathrm{Bir}(X_2), X_2)$-structure coincides with the original $(\mathrm{Bir}(X_1), X_1)$-structure by Proposition B.2.11. □

Strict $(\mathrm{Bir}(\mathbb{P}^n), \mathbb{P}^n)$-structures. In general a $(\mathrm{Bir}(X), X)$-structure does not induce a strict $(\mathrm{Bir}(X), X)$-structure. A smooth projective variety birational to X always admits a $(\mathrm{Bir}(X), X)$-structure, but not necessarily a strict $(\mathrm{Bir}(X), X)$-structure as the following example shows:

Example B.2.15 Let X be a projective K3 surface. Let Z be the blow-up of X at some point. Suppose by contradiction that Z admits a strict $(\mathrm{Bir}(X), X)$-structure. Consider the developing map Dev : $Z \dashrightarrow X$. Being locally birational, it induces an injection of the function field of X into that of Z. This means that Dev is a rational dominant map. As a developing map Dev has no ramification and a K3 surface is simply connected, the degree of Dev must be one, i.e. it is birational. Then we infer that Dev is the blow-down of the exceptional curve. However by Proposition B.2.9 we could choose Dev so that Dev is locally biholomorphic around a point on the execptional curve, contradiction.

The reasoning in the previous example shows:

Lemma B.2.16 *Let X be a simply connected smooth variety. For a $(\mathrm{Bir}(X), X)$-structure on X, any developing map is birational. Hence the natural $(\mathrm{Bir}(X), X)$-structure on X is the unique one. If X has a strict $(\mathrm{Bir}(Y), Y)$-structure for some Y birational to X then it is the unique strict $(\mathrm{Bir}(Y), Y)$-structure on X.*

As rational varieties have the most complicated birational transformation groups, it is natural to ask

Question B.2.17
1. *Does a $(\mathrm{Bir}(\mathbb{P}^n), \mathbb{P}^n)$-structure always induce a strict $(\mathrm{Bir}(\mathbb{P}^n), \mathbb{P}^n)$-structure?*
2. *Does every smooth rational variety X of dimension n admit a strict $(\mathrm{Bir}(\mathbb{P}^n), \mathbb{P}^n)$-structure?*

Proposition B.2.18 *Questions B.2.17.1 and B.2.17.2 are equivalent.*

Proof 1) implies 2) because every smooth rational variety admits trivially a non-strict $(\mathrm{Bir}(\mathbb{P}^n), \mathbb{P}^n)$-structure. Suppose that 2) is true. Let V be a complex manifold with a $(\mathrm{Bir}(\mathbb{P}^n), \mathbb{P}^n)$-structure. Let $(U_i, \phi_i : U_i \to X_i)$ be an atlas for the $(\mathrm{Bir}(\mathbb{P}^n), \mathbb{P}^n)$-structure. By hypothesis each X_i has a strict $(\mathrm{Bir}(\mathbb{P}^n), \mathbb{P}^n)$-structure. Via ϕ_i this equips U_i with a strict $(\mathrm{Bir}(\mathbb{P}^n), \mathbb{P}^n)$-structure. Cover U_i by charts $\{U_{ij}\}_j$ of strict $(\mathrm{Bir}(\mathbb{P}^n), \mathbb{P}^n)$-structure induced by the one on X_i; the U_{ij} are identified via ϕ with charts of $(\mathrm{Bir}(\mathbb{P}^n), \mathbb{P}^n)$-structure on X_i. By Lemma B.2.16 the changes from U_{ij} to $U_{i'j'}$ are birational. Therefore the strict $(\mathrm{Bir}(\mathbb{P}^n), \mathbb{P}^n)$-structures on U_i patch together to give a strict $(\mathrm{Bir}(\mathbb{P}^n), \mathbb{P}^n)$-structure on V for which the U_{ij} form an atlas. □

A smooth rational variety X of dimension n is called *uniformly rational* if any $x \in X$ has a Zariski neighbourhood which is isomorphic to a Zariski open set in the affine space $\mathbb{A}^n$. Being rational X has to have such a point; the issue is whether it holds for all points, hence the terminology "uniformly rational". Gromov asked:

Question B.2.19 (Gromov [Gro89] page 885, see also [BB14]) *Is every smooth rational complex variety uniformly rational?*

It turns out that Gromov's question is equivalent to Question B.2.17 of which the formulation seems not quite algebraic at first glance:

Proposition B.2.20 *A rational variety of dimension n is uniformly rational if and only if it admits a strict* $(\mathrm{Bir}(\mathbb{P}^n), \mathbb{P}^n)$*-structure.*

Proof Suppose that X is a uniformly rational variety of dimension n. For any $x \in X$, let U_x be a Zariski neighbourhood isomorphic to an open set of $\mathbb{A}^n$. Then the open sets U_x give an atlas of strict $(\mathrm{Bir}(\mathbb{P}^n), \mathbb{P}^n)$-structure.

Now suppose that X admits a strict $(\mathrm{Bir}(\mathbb{P}^n), \mathbb{P}^n)$-structure. Let $x \in X$. Take a developing map $\mathrm{Dev} : X \dashrightarrow \mathbb{P}^n$ which is holomorphic at x (cf. Proposition B.2.9). By Lemma B.2.16 Dev is birational. Thus Dev realizes an isomorphism from some Zariski neighbourhood of x to a Zariski open set of $\mathbb{A}^n$. $\square$

Question B.2.19 is easy in dimension 1 or 2, and is still open in dimension ≥ 3 (cf. [BB14]). The one dimensional case is trivial because $\mathbb{P}^1$ is the only smooth rational curve. For completeness we include a proof for the two dimensional case:

Proposition B.2.21 *Let X be a smooth rational surface. Then X is uniformly rational and admits a unique strict* $(\mathrm{Bir}(\mathbb{P}^2), \mathbb{P}^2)$*-structure.*

Proof Once we prove that X is uniformly rational, we obtain the existence of strict $(\mathrm{Bir}(\mathbb{P}^2), \mathbb{P}^2)$-structure by Proposition B.2.20 and the uniqueness by Lemma B.2.16.

A Hirzebruch surface is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Cover the base and fiber $\mathbb{P}^1$ respectively by two pieces of $\mathbb{A}^1$, we see that every Hirzebruch surface can be otbtained by patching four pieces of $\mathbb{A}^2$. Every rational surface different from $\mathbb{P}^2$ can be obtained from a Hirzebruch surface by blow-ups. The blow-up of $\mathbb{A}^2$ at one point is the union of two Zariski open sets isomorphic to $\mathbb{A}^2$. Hence the uniform rationality. $\square$

So it does no harm if we do not distinguish $(\mathrm{Bir}(\mathbb{P}^2), \mathbb{P}^2)$-structure from strict $(\mathrm{Bir}(\mathbb{P}^2), \mathbb{P}^2)$-structure. More precisely we have:

Corollary B.2.22 *Let V be a compact complex surface equipped with a* $(\mathrm{Bir}(\mathbb{P}^2), \mathbb{P}^2)$*-structure. Then for any rational surface X, there exists a unique strict* $(\mathrm{Bir}(X), X)$*-structure on V such* *that,* $(\mathrm{Hol}, \mathrm{Dev})$ *is a holonomy/developing pair associated with the strict* $(\mathrm{Bir}(X), X)$*-structure if and only if* $(X, \mathrm{Hol}, \mathrm{Dev})$ *is a holonomy/developing triple associated with the* $(\mathrm{Bir}(\mathbb{P}^2), \mathbb{P}^2)$*-structure.*

Proof By Propositions B.2.18 and B.2.21 V has a strict $(\mathrm{Bir}(\mathbb{P}^2), \mathbb{P}^2)$-structure. Let $(U_i, \phi_i : U_i \hookrightarrow \mathbb{P}^2)$ be an atlas. The strict $(\mathrm{Bir}(X), X)$-structure on V is constructed as follows. Firstly $\mathrm{Aut}(\mathbb{P}^2)$ is transitive so any point of $\mathbb{P}^2$ admits a Zariski neighbourhood which is isomorphic to a Zariski open set of X. By considering the intersection of the Euclidean open set $\phi_i(U_i)$ with these Zariski neighbourhoods, we can further subdivide the atlas (U_i) into an atlas (U_{ij}) such that there are embeddings $\varphi_{ij} : U_{ij} \hookrightarrow X$ such that $\phi_i \circ \varphi_{ij}^{-1}$ extend to birational maps from X to $\mathbb{P}^2$. The atlas $(U_{ij}, \varphi_{ij} : U_{ij} \to X)$ defines the desired strict $(\mathrm{Bir}(X), X)$-structure. $\square$

Local structure of the developing map. Though the developing map Dev is in general not holomorphic, it is by construction locally birational. Thus, at least locally, algebro-geometric reasonings could be applied. In dimension two, the indeterminacy set of Dev is a discrete set of points. We can speak about contracted curves, they are complex analytic subsets of pure dimension 1. A contracted curve has locally a finite number of components. An irreducible contracted curve is a minimal closed connected 1-dimensional analytic subset contracted by Dev.

B.2.3 Entire curves and Ahlfors-Nevanlinna currents

In this section we give a treatment of families of Ahlfors-Nevanlinna currents which are, we believe, of independent interest. Let X be a smooth projective surface. An *entire curve* on X is a non-constant holomorphic map $\xi : \mathbf{C} \to X$. An entire curve ξ is called *transcendental* if its image is not contained in an algebraic curve of X. We can associate to a transcendental entire curve ξ a (a priori non unique) closed positive current, called *Ahlfors-Nevanlinna current*. We need to prove a variant of the construction for a family of entire curves. We recall first the process for a single entire curve (see [McQ98], [**Bru99**] for Ahlfors-Nevanlinna currents and [Dem97] for the functions we use below).

For a differential form $\eta \in \mathscr{A}^2(X)$ and for $r > 0$ we put

$$T_{\xi,r}(\eta) = \int_0^r \frac{ds}{s} \int_{\mathbb{D}_s} \xi^* \eta$$

where $\mathbb{D}_s \subset \mathbf{C}$ is the disk of radius s. We fix a Kähler form $\omega \in \mathscr{A}^{1,1}(X)$. Consider the positive currents defined by

$$\Phi_r(\eta) = \frac{T_{\xi,r}(\eta)}{T_{\xi,r}(\omega)}, \quad \forall \eta \in \mathscr{A}^2(X).$$

The family $\{\Phi_r\}_{r>0}$ is bounded, so we can find a sequence of radii $(r_n)_{n\in\mathbf{N}}$ such that $r_n \to +\infty$ and Φ_{r_n} converges weakly to a positive current Φ. For the limit Φ to be a closed current, we need a smart choice of the sequence (r_n). Let us denote by $A(r)$ the area of $\xi(\mathbb{D}_r)$ and $L(r)$ the length of $\xi(\partial\mathbb{D}_r)$ with respect to the Riemannian metric induced by ω. Then $T_{\xi,r}(\omega)$ may be written as

$$T_{\xi,r}(\omega) = \int_0^r A(s)\frac{ds}{s}.$$

We have

$$\operatorname{limsup}_{r\to\infty} \frac{T_{\xi,r}(\omega)}{\log r} = \infty$$

since ξ is transcendental (cf. [Dem97]). We define

$$S_{\xi,r}(\omega) = \int_0^r L(s)\frac{ds}{s}.$$

For $\beta \in \mathscr{A}^1(X)$, Stokes' theorem and the compactness of X imply the inequality

$$|T_{\xi,r}(d\beta)| \leq \int_0^r \frac{ds}{s}\int_{\partial\mathbb{D}_s} |\xi^*\beta| \leq constant \cdot S_{\xi,r}(\omega),$$

where the constant on the right side depends on β but not on r. Therefore to obtain a closed limit current Φ, we need a sequence of radii $(r_n)_n$ such that

$$\frac{S_{\xi,r_n}(\omega)}{T_{\xi,r_n}(\omega)} \to 0, \quad \text{when } n \to \infty.$$

The existence of such a sequence of radii is guaranteed by the following lemma (see [**Bru99**]):

Lemma B.2.23 (Ahlfors [Ahl35]) *Let $R > 0$, $\varepsilon > 0$ be two positive real numbers. Denote by $B(\xi,\varepsilon)$ the set $\{r > R | S_{\xi,r}(\omega) > \varepsilon T_{\xi,r}(\omega)\}$. Then*

$$\int_{B(\xi,\varepsilon)} \frac{dr}{r\log r} < \infty.$$

In particular $\operatorname{liminf}_{r\to\infty} \frac{S_{\xi,r}(\omega)}{T_{\xi,r}(\omega)} = 0$.

Note that the measure of (R,∞) with respect to $\frac{dr}{r\log r}$ is infinite, so the above lemma implies that we can choose an appropriate sequence of radii simultaneously for a finite number of entire curves:

Lemma B.2.24 *Let $\xi_1, \cdots, \xi_k$ be k transcendental entire curves on X. There exists a sequence of radii $(r_n)_{n\in\mathbf{N}}$ such that for each $i \in \{1, \cdots, k\}$, the sequence $\left(\frac{T_{\xi_i, r_n}(\cdot)}{T_{\xi_i, r_n}(\omega)}\right)_{n\in\mathbf{N}}$ converges weakly to a closed positive current.*

A closed positive current Φ constructed by the above limit process is called an *Ahlfors-Nevanlinna current* associated with the entire curve ξ, it depends on the choice of a sequence of radii $(r_n)_n$.

A cohomology class is called *nef* if its intersections with all curves are non negative. We refer the reader to [McQ98], **[Bru99]** for the following:

Lemma B.2.25 *Let $[\Phi] \in \mathrm{H}^{1,1}(X, \mathbf{R})$ be the cohomology class of an Ahlfors-Nevanlinna current associated with a transcendental entire curve. Then $[\Phi]$ is nef. In particular $[\Phi]^2 \geq 0$.*

We will need to consider some families of entire curves. To treat the Ahlfors-Nevanlinna currents simultaneously in family, we need some control on the variation of entire curves. The following very restricted notion will be sufficient for our proof.

Definition B.2.26 *A family of entire curves parametrized by a real manifold B is a differentiable map $B \times \mathbf{C} \to X, (b,z) \mapsto \xi_b(z)$ such that ξ_b is an entire curve for every $b \in B$. A family of entire curves $(\xi_b)_{b\in B}$ is called uniform if the following condition is satisfied: $\forall b_0 \in B, \forall \delta > 0$, there exists a neighborhood U of b_0 such that*

$$\forall b \in U, \forall z \in \mathbf{C}, \big| |\xi_b'(z)| - |\xi_{b_0}'(z)| \big| < \delta |\xi_{b_0}'(z)|$$

where the absolute values are measured with respect to a fixed Kähler metric on X. In other words a family of entire curves is uniform if nearby pull-backed metrics are close in proportion.

There exist non-trivial uniform families of transcendental entire curves on complex projective surfaces, for example there exist families of Levi-flat hypersurfaces foliated by entire curves (see Remark 1.6 of [Der05]). Our interest in this notion is explained by the following lemma:

Lemma B.2.27 *Let $(\xi_b)_{b\in B}$ be a uniform family of transcendental entire curves on X. Let A be a compact C^∞-path connected subset of B. Then there exists a sequence of radii $(r_n)_{n\in\mathbf{N}}$ using which an Ahlfors-Nevanlinna current associated with ξ_a can be constructed for all $a \in A$. After fixing such a sequence $(r_n)_{n\in\mathbf{N}}$, the Ahlfors-Nevanlinna currents associated with the ξ_a all have the same cohomology class.*

Proof We prove first that there exists a common choice of the sequence of radii. Let us fix a sufficiently small real number $\delta > 0$. By the definition of uniform family and by compactness of A, we can find a finite number of points $a_1, \cdots, a_k$ in A and their neighbourhoods $U_1, \cdots, U_k$ in B such that

— $\forall i \in \{1, \cdots, k\}, \forall a \in U_i, \forall z \in \mathbf{C}, \big||\xi_a'(z)| - |\xi_{a_i}'(z)|\big| < \delta |\xi_{a_i}'(z)|$;
— $A \subset \cup U_i$.

By lemma B.2.24, we can take a sequence of radii $(r_n)_{n \in \mathbf{N}}$ that works for all the $\xi_{a_i}, 1 \le i \le k$. We denote by λ the Lebesgue measure on $\mathbf{C}$. Let $a \in U_i$. We have

$$T_{\xi_a, r}(\omega) = \int_0^r A(s) \frac{ds}{s} = \int_0^r \int_{\mathbb{D}_r} |\xi_a'(z)|^2 d\lambda(z) \frac{ds}{s}$$

and

$$\begin{aligned} |T_{\xi_a, r}(\omega) - T_{\xi_{a_i}, r}(\omega)| &\le \int_0^r \int_{\mathbb{D}_r} ||\xi_a'(z)|^2 - |\xi_{a_i}'(z)|^2| d\lambda(z) \frac{ds}{s} \\ &\le (2\delta + \delta^2) \int_0^r \int_{\mathbb{D}_r} |\xi_{a_i}'(z)|^2 d\lambda(z) \frac{ds}{s} \\ &= (2\delta + \delta^2) T_{\xi_{a_i}, r}(\omega) \end{aligned}$$

Similarly we have

$$|S_{\xi_a, r}(\omega) - S_{\xi_{a_i}, r}(\omega)| \le \delta S_{\xi_{a_i}, r}(\omega).$$

Consequently

$$\frac{S_{\xi_a, r}(\omega)}{T_{\xi_a, r}(\omega)} \le \frac{1+\delta}{1 - 2\delta - \delta^2} \frac{S_{\xi_{a_i}, r}(\omega)}{T_{\xi_{a_i}, r}(\omega)}.$$

In particular we have

$$\lim_{n \to \infty} \frac{S_{\xi_a, r_n}(\omega)}{T_{\xi_a, r_n}(\omega)} = \lim_{n \to \infty} \frac{S_{\xi_{a_i}, r_n}(\omega)}{T_{\xi_{a_i}, r_n}(\omega)} = 0$$

so that the sequence $(r_n)_n$ can be used to construct Ahlfors-Nevanlinna currents for all $a \in A$. Hence we can talk about *the* Ahlfors-Nevanlinna currents Φ_a associated with the ξ_a and this fixed sequence $(r_n)_n$.

Let a, b be two points in A. We now prove that the Ahlfors-Nevanlinna currents Φ_a, Φ_b are cohomologous. It is sufficient to treat the case where $a = a_i$ and $b \in U_i$. Take a C^∞-path $c : [0,1] \to U_i$ such that $c(0) = a, c(1) = b$. We denote by F the induced map $[0,1] \times \mathbf{C} \to$

$X, F(s,z) = \xi_{c(s)}(z)$. Let $\eta \in A^2(X)$. Applying Stokes' theorem, we have

$$T_{\xi_a,r}(\eta) - T_{\xi_b,r}(\eta) = \int_0^r \int_{[0,1]\times\mathbb{D}_s} F^*(d\eta)\frac{ds}{s} - \int_0^r \int_{[0,1]\times\partial\mathbb{D}_s} F^*(\eta)\frac{ds}{s}. \tag{B.1}$$

We denote by Θ_r the current of dimension 3 defined by $\Theta_r(\beta) = \int_0^r \int_{[0,1]\times\mathbb{D}_s} F^*(\beta)\frac{ds}{s}$ for $\beta \in A^3(X)$, and by Ξ_r the current of dimension 2 defined by $\Xi_r(\beta) = \int_0^r \int_{[0,1]\times\partial\mathbb{D}_s} F^*(\beta)\frac{ds}{s}$. We have

$$\frac{T_{\xi_a,r}(\eta)}{T_{\xi_a,r}(\omega)} - \frac{T_{\xi_b,r}(\eta)}{T_{\xi_b,r}(\omega)} = \frac{T_{\xi_a,r}(\eta) - T_{\xi_b,r}(\eta)}{T_{\xi_b,r}(\omega)} + \frac{T_{\xi_b,r}(\omega) - T_{\xi_a,r}(\omega)}{T_{\xi_a,r}(\omega)T_{\xi_b,r}(\omega)} T_{\xi_b,r}(\eta)$$

which with Equation (B.1) implies

$$\frac{T_{\xi_a,r}(\eta)}{T_{\xi_a,r}(\omega)} - \frac{T_{\xi_b,r}(\eta)}{T_{\xi_b,r}(\omega)} = \frac{1}{T_{\xi_b,r}(\omega)}\left(d\Theta_r(\eta) - \Xi_r(\eta)\right) + \frac{T_{\xi_b,r}(\omega) - T_{\xi_a,r}(\omega)}{T_{\xi_a,r}(\omega)} \frac{T_{\xi_b,r}(\eta)}{T_{\xi_b,r}(\omega)}. \tag{B.2}$$

We want to show that along the sequence of radii $(r_n)_{n\in\mathbf{N}}$, the right side of Equation (B.2) converges weakly to an exact current. We first estimate Ξ_r. By compactness of X, we have

$$|\Xi_r(\eta)| = \left| \int_0^r \int_{[0,1]\times\partial\mathbb{D}_s} F^*(\eta)\frac{ds}{s} \right| \leq M(\eta) \int_0^r \int_{[0,1]\times\partial\mathbb{D}_s} |F^*(\omega)|\frac{ds}{s}$$

where $M(\eta)$ is a constant that depends on η but not on r. By Fubini's theorem, we deduce from the above inequality that

$$|\Xi_r(\eta)| \leq M(\eta) \int_0^r \int_0^1 L(c(u),s)du\frac{ds}{s}$$

where $L(c(u),s)$ is the length of $\xi_{c(u)}(\partial\mathbb{D}_s)$ with respect to the Kähler metric defined by ω. Using the fact that the path c lies in U_i, we have further

$$|\Xi_r(\eta)| \leq M(\eta) \int_0^r \int_0^1 (1+\delta)L(a,s)du\frac{ds}{s} = M(\eta)(1+\delta)S_{\xi_a,r}(\omega).$$

This implies that the sequence of currents $(\Xi_{r_n}/T_{\xi_b,r_n}(\omega))_{n\in\mathbf{N}}$ converges weakly to 0.

Now we estimate the last term of Equation (B.2). By Stokes' formula, we have

$$T_{\xi_b,r}(\omega) - T_{\xi_a,r}(\omega) = \int_0^r \int_{[0,1]\times\mathbb{D}_s} F^*(d\omega)\frac{ds}{s} - \int_0^r \int_{[0,1]\times\partial\mathbb{D}_s} F^*(\omega)\frac{ds}{s}. \tag{B.3}$$

Since the form ω is Kähler, we have $d\omega = 0$ and the first term of the right side of (B.3) vanishes.

The second term at the right side of (B.3) is dominated by $S_{\xi_a,r}(\omega)$. It follows immediately that the last term of (B.2) converges weakly to zero along the sequence of radii $(r_n)_n$.

Finally we estimate Θ_r. Note that, since the other terms in Equation (B.2) all converge weakly along the sequence $(r_n)_n$, the sequence $(d\Theta_{r_n}/T_{\xi_b,r_n}(\omega))_n$ converges weakly too. However this does not imply that $(\Theta_{r_n}/T_{\xi_b,r_n}(\omega))_n$ converges weakly. Again using Fubini's theorem and compactness of X, we have

$$|\Theta_r(\beta)| \leq N(\beta) T_{\xi_b,r}(\omega)$$

where $N(\beta)$ is a constant which depends on $|\beta|$ and on δ but not on r. Thus the $\Theta_{r_n}/T_{\xi_b,r_n}(\omega)$ form a bounded family and there exists a subsequence $(r_{n_j})_j$ of $(r_n)_n$ such that $\Theta_{r_{n_j}}/T_{\xi_b,r_{n_j}}(\omega)$ converges weakly to a current Θ. Hence, the weak limit of $(d\Theta_{r_n}/T_{\xi_b,r_n}(\omega))_n$ is exact because

$$\lim_{n\to\infty} \frac{d\Theta_{r_n}(\eta)}{T_{\xi_b,r_n}(\omega)} = \lim_{j\to\infty} d\left(\frac{\Theta_{r_{n_j}}}{T_{\xi_b,r_{n_j}}(\omega)}\right)(\eta) = d\Theta(\eta).$$

The conlusion follows. □

Note that to construct Ahlfors-Nevanlinna currents it is not necessary for ω to be Kähler: a Hermitian metric would be sufficient. However the property $d\omega = 0$ is used in the last part of the above proof.

B.2.4 Transverse invariant measures

All the materials in this section can be found in [Ghy99] and [FS08]. Let M be a compact Hausdorff topological space. A structure of *lamination by Riemann surfaces* on M is an atlas $\mathscr{L}$ of charts $h_i : U_i \to \mathbb{D} \times B_i$ where $\mathbb{D}$ is the unit disk in $\mathbf{C}$, the B_i are topological spaces, the h_i are homeomorphisms and the U_i are open sets of M which cover M; the changes of coordinates $h_{ij} = h_j \circ h_i^{-1}$ are of the form $(f_{ij}(z,b), g_{ij}(b))$ where the f_{ij} are holomorphic in z and the g_{ij} are continuous. A connected component of $V_c = \{(z,b)|b = c\}$ in a chart U_i is called a *plaque*. A minimal connected subset of M which contains all plaques that it intersects is called *a leaf*. A lamination by Riemann surfaces $(M,\mathscr{L})$ is *transversally smooth* if the B_i are real manifolds and if the g_{ij} are smooth maps. A *transverse invariant measure* μ on $(M,\mathscr{L})$ is a family of locally finite positive measures μ_i on the topological spaces B_i such that if $B \subset B_i$ is a measurable set contained in the domain of definition of g_{ij}, then $\mu_i(B) = \mu_j(g_{ij}(B))$.

From now on we make the hypothesis that $(M,\mathscr{L})$ is a lamination by Riemann surfaces contained in a Kähler surface X. This hypothesis is just for convenience of the presentation and everything we will say makes sense without assuming that there is an ambient surface X. Examples to keep in mind are Levi-flat hypersurfaces and saturated sets of holomorphic foliations. We say that a continuous $(1,0)$-form β on X *defines the lamination* $(M,\mathscr{L})$ if $\beta \wedge [V_c] = 0$ for every plaque V_c, where $[V_c]$ is the current of integration on the plaque V_c. A closed positive current Θ of bidimension $(1,1)$ on X is *directed by* $(M,\mathscr{L})$ if it is supported on M and if $\Theta \wedge \beta = 0$ for all β defining $(M,\mathscr{L})$. Our purpose of introducing the above notions is the following theorem:

Theorem B.2.28 (Sullivan [Sul76]) *Let $(M,\mathscr{L})$ be a transversally smooth lamination by Riemann surfaces contained in a Kähler surface X. A transverse invariant measure on $(M,\mathscr{L})$ is the same thing as a closed positive current directed by $(M,\mathscr{L})$ via the following correspondence: in a chart $h_i : U_i \to \mathbb{D} \times B_i$, a closed positive directed current T may be written as*

$$T = \int_{B_i} [V_b] d\mu(b)$$

where μ is a transverse invariant measure and the $[V_b]$ are integrations on plaques.

We will apply Sullivan's theorem to Ahlfors-Nevanlinna currents associated with entire curves tangent to the lamination, thanks to the following construction studied by Plante:

Theorem B.2.29 (Plante [Pla75], see also [Ghy99], [FS08]) *Let $(M,\mathscr{L})$ be a lamination by Riemann surfaces contained in a Kähler surface X. Let $f : \mathbf{C} \to X$ be a transcendental entire curve contained in a leaf of the lamination and let Φ_f be an Ahlfors-Nevanlinna current associated with f. Then Φ_f is directed by $(M,\mathscr{L})$.*

B.3 Inoue surfaces of type S^0.

B.3.1 Description

Let $M \in \mathrm{SL}_3(\mathbf{Z})$ be a matrix with eigenvalues $\alpha, \beta, \bar{\beta}$ such that $\alpha > 1$ and $\beta \neq \bar{\beta}$. Note that α is irrational and $|\beta| < 1$. We choose a real eigenvector (a_1, a_2, a_3) corresponding to α and a complex eigenvector (b_1, b_2, b_3) corresponding to β. Let G_M be the subgroup of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$

generated by

$$\begin{aligned} g_0 &: (x,y) \mapsto (\alpha x, \beta y) \\ g_i &: (x,y) \mapsto (x+a_i, y+b_i) \quad \text{for } i = 1,2,3. \end{aligned}$$

Denote by $\mathbb{H}$ the upper half plane, viewed as an open subset of $\mathbb{P}^1 = \mathbf{C} \cup \{\infty\}$. The action of G_M preserves $\mathbb{H} \times \mathbf{C}$; it is free and properly discontinuous. The quotient $S_M = \mathbb{H} \times \mathbf{C}/G_M$ is a compact non-Kähler surface without curves called an *Inoue surface of type* S^0 ([Ino74]). Note that we should have included the choices of (a_1, a_2, a_3) and (b_1, b_2, b_3) in the notation of S_M. By construction it has an $(\mathrm{Aff}_2(\mathbf{C}), \mathbf{C}^2)$-structure where by $\mathrm{Aff}_2(\mathbf{C})$ we denote the affine transformation group of $\mathbf{C}^2$. In particular an Inoue surface of type S^0 has a natural $(\mathrm{Bir}(\mathbb{P}^2), \mathbb{P}^2)$-structure.

Consider the following solvable Lie group which is a subgroup of $\mathrm{Aff}_2(\mathbf{C})$:

$$\mathrm{Sol}^0 = \left\{ \begin{pmatrix} |\lambda|^{-2} & 0 & a \\ 0 & \lambda & b \\ 0 & 0 & 1 \end{pmatrix}, \lambda \in \mathbf{C}^*, a \in \mathbf{R}, b \in \mathbf{C} \right\}.$$

The group Sol^0 is a semi-direct product $(\mathbf{C} \times \mathbf{R}) \rtimes \mathbf{C}^*$. It acts transitively on $\mathbb{H} \times \mathbf{C}$; the stabilizer of a point is isomorphic to $\mathbb{S}^1$. The group G_M defining the Inoue surface S_M is a lattice in Sol^0; conversely any torsion free lattice of Sol^0 gives an Inoue surface of type S^0. The three elements g_1, g_2, g_3 generate a free abelian group of rank three; denote it by A_M. The group G_M is a semi-direct product $A_M \rtimes \mathbf{Z}$ where the $\mathbf{Z}$ factor is generated by g_0. We have $g_0 g_i g_0^{-1} = g_1^{m_{i1}} g_2^{m_{i2}} g_3^{m_{i3}}$ where the m_{ij} are the entries of the matrix M. Note that a finite unramified cover of an Inoue surface of type S^0 is an Inoue surface of type S^0.

The following lemma says that G_M has few normal subgroups. In particular the commutator $[G_M, G_M]$ is a finite index subgroup of A_M.

Lemma B.3.1 *If K is a non-trivial normal subgroup of G_M, then either K is of finite index in A_M or K is of finite index in G_M.*

Proof The conjugation action of g_0 on A_M is just the action of $M \in \mathrm{SL}_3(\mathbf{Z})$ on $\mathbf{Z}^3$. For all $v \in \mathbf{Z}^3 \setminus \{0\}$, the iterates $M^n v$ generate a finite index subgroup of $\mathbf{Z}^3$. Thus, if $K \cap A_0$ is non trivial, then $K \cap A_0$ is a free $\mathbf{Z}$-module of rank 3 and is of finite index in A_M. To conclude, we need only remark that, by the semi-direct product structure, the intersection of a normal subgroup of G_M with A_M cannot be trivial. □

Lemma B.3.2 *Let $\sigma : G_M \to \mathrm{PGL}_2(\mathbf{C})$ be an injective morphism. Then for some affine coordinate $\mathbb{P}^1 = \{x \in \mathbf{C}\} \cup \{\infty\}$, the images $\sigma(g_i), i = 0, \cdots, 3$, viewed as homographies of $\mathbb{P}^1$, may be written as*

$$\begin{aligned} \sigma(g_i) &: x \mapsto x + u_i, \quad i = 1, 2, 3 \\ \sigma(g_0) &: x \mapsto \nu x \end{aligned}$$

for some $\nu, u_i \in \mathbf{C}^*$.

Proof As $\sigma(g_i), i = 1, 2, 3$ commute with each other, we have two possibilities for them: we can find an affine coordinate x such that they are either $x \mapsto x + u_i$ with $u_i \neq 0$ or $x \mapsto \alpha_i x$ with α_i of infinite order.

Suppose by contradiction that the $\sigma(g_i)(x) = \alpha_i x$. Since A_M is normal, $\sigma(g_0)$ preserves the set of fixed points of $\sigma(A_M)$, which is $\{0, \infty\}$. Hence $\sigma(g_0)(x) = \gamma x^{\pm 1}$. But then the action of $\sigma(g_0)$ on $\sigma(A_M)$ has finite order, a contradiction.

Hence the $\sigma(g_i)$ are $x \mapsto x + u_i$. The invariance of the fixed point ∞ implies that $\sigma(g_0)$ is $x \mapsto \nu x + \delta$ where ν satisfies that $\nu u_i = m_{i1} u_1 + m_{i2} u_2 + m_{i3} u_3$ and δ is arbitrary. Then the change of coordinates $x \mapsto x - \frac{\delta}{\nu}$ allows us to write the $\sigma(g_i)$ as in the statement of the lemma. $\square$

Lemma B.3.3 *The only (possibly singular) holomorphic foliations on S_M are the two obvious ones coming from the horizontal foliation and the vertical foliation of $\mathbb{H} \times \mathbf{C}$.*

Proof This is already observed by Brunella in [Bru97] without proof details. Here we give a proof for completeness. See [Bru15] for the terminology we use concerning holomorphic foliations. Suppose by contradiction that $\mathscr{F}$ is a non-necessarily saturated holomorphic foliation on S_M different from the two obvious ones. Since S_M has no curves ([Ino74]), the singularities of $\mathscr{F}$ are necessarily isolated. We compare $\mathscr{F}$ with one of the two obvious foliations: the tangency locus is empty because otherwise it would be a curve on S_M. Since the tangency locus contains the singularities of $\mathscr{F}$, we deduce that $\mathscr{F}$ is a regular holomorphic foliation, transverse to the two obvious foliations.

We denote by T the tangent bundle of S_M, by T^* its dual and by K the canonical bundle of S_M. We denote by F_0 the normal bundle of one obvious foliation; the normal bundle of the other obvious foliation is then $-K - F_0$ (here we use notations of [Ino74]). Let F be the normal bundle of $\mathscr{F}$. The foliation $\mathscr{F}$ corresponds to a non-zero global section of $T^* \otimes F$. It is proved in [Ino74] (see the first two sentences of sections 6 and 8) that the only line bundles F on S_M such that $T^* \otimes F$ has non-zero sections are $F = F_0$ or $F = -F_0 - K$. In other words, $\mathscr{F}$ and one of the obvious foliations share the same normal bundle. As $\mathscr{F}$ is everywhere transverse to this foliation, the two sections of $T^* \otimes F$ corresponding to the two foliations trivialize the sheaf $T^* \otimes F$, i.e. T^* is isomorphic to $(-F) \oplus (-F)$. However $T^* = (-F_0) \oplus (F_0 + K)$ and F is either F_0 or $-F_0 - K$. This leads to a contradiction as K is not trivial. □

The surface $\overline{S_M} = \mathbb{H} \times \mathbf{C}/A_M$ is an infinite cyclic cover of S_M. As a real manifold, $\overline{S_M}$ admits a fibration $\rho : \overline{S_M} \to \mathbf{R}_+^*$ where $\mathbf{R}_+^* = \{t\sqrt{-1}, t \in \mathbf{R}_+^*\}$ is the vertical axis of the component $\mathbb{H}$ of $\mathbb{H} \times \mathbf{C}$. The fibers of ρ, denoted by F_t, are quotients of $\{x + t\sqrt{-1}, x \in \mathbf{R}\} \times \mathbf{C}$ by A_M; they are real tori of dimension 3. The F_t are Levi-flat hypersurfaces in $\overline{S_M}$ and they are foliated by entire curves coming from the vertical complex lines in $\mathbb{H} \times \mathbf{C}$.

Lemma B.3.4 *Up to multiples, there is only one transverse invariant measure on the Levi flat hypersurface F_t.*

Proof Recall that (a_1, a_2, a_3) is an eigenvector associated with the irrational eigenvalue α of $M \in \mathrm{SL}_3(\mathbf{Z})$. A transverse invariant measure on F_t is induced by a measure on $\mathbf{R} = \{x + t\sqrt{-1}, x \in \mathbf{R}\}$ which is invariant under the group of translations generated by $x \mapsto x + a_i, i = 1,2,3$. This latter group is a dense subgroup of $\mathbf{R}$ so the transverse invariant measure must be a multiple of the Lebesgue measure. □

Lemma B.3.5 *The two obvious foliations on S_M are not transversely Euclidean.*

Proof The two dimensional Euclidean isometry group is the semi-direct product $\mathbf{R}^2 \rtimes \mathrm{SO}_2(\mathbf{R})$, where $\mathbf{R}^2$ is the group of translations and $\mathrm{SO}_2(\mathbf{R})$ is the group of rotations. Suppose by contradiction that one obvious foliation is transversely Euclidean. Then to this transverse Euclidean structure are associated a holonomy representation $\rho : G_M \to \mathbf{R}^2 \rtimes \mathrm{SO}_2(\mathbf{R})$ and a continuous ρ-equivariant developing map $D : T \to \mathbf{R}^2$, where the space of leaves T is $\mathbb{H}$ or $\mathbf{C}$ depending on which of the two obvious foliations we are looking at. We prove first that ρ is injective by contradiction. Suppose that the kernel K of ρ is not trivial, then it is a finite index subgroup of A_M by Lemma B.3.1. As A_M is a group of translations on T which is isomorphic to $\mathbf{Z}^3$, the closure of any A_M-orbit contains at least one real line (for $T = \mathbb{H}$ a subgroup of A_M isomorphic to $\mathbf{Z}^2$ would be sufficient as the a_i are real). The same holds for K-orbits. Then by the ρ-equivariance and the continuity of the developing map, D is constant on each of these real lines. This contradicts the fact that the developing map is locally homeomorphic.

We know now that ρ is injective. As A_M is abelian, we must have $\rho(A_M) \subset \mathbf{R}^2$. The conjugation action of g_0 on $\mathbf{R}^3 = A_M \otimes \mathbf{R}$ and that of $\rho(g_0)$ on $\mathbf{R}^2$ are linear maps. We think of

g_0 and $\rho(g_0)$ as linear maps via their conjugation actions. The group morphism ρ induces a linear map $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ which is equivariant under the actions of g_0 and $\rho(g_0)$, i.e. we have $\pi \circ g_0 = \rho(g_0) \circ \pi$. This is not possible because $\rho(g_0)$ is a rotation while g_0 corresponds to the matrix M whose eigenvalues are $\alpha, \beta, \overline{\beta}$ with $\alpha > 1$ and $|\beta| < 1$. □

B.3.2 Proof of Theorem B.1.1 for Inoue surfaces of type S^0

Let S_M be an Inoue surface of type S^0. We fix a $(\mathrm{Bir}(X), X)$-structure on S_M where X is some projective surface. We want to prove that X is rational and the structure is just the obvious affine structure. Let $(Y, \mathrm{Dev}, \mathrm{Hol})$ be a corresponding holonomy/developing triple as in Proposition B.2.9. We will denote by π the covering map from $\mathbb{H} \times \mathbf{C}$ to S_M.

Lemma B.3.1 says that there are only three possibilities for the holonomy representation. It is easy to rule out the first possibility: if the holonomy had finite image then the developing map would induce a meromorphic locally birational map from a finite unramified cover of S_M to Y, contradicting the fact that S_M has algebraic dimension zero. The second possibility is that the kernel K of the holonomy is a finite index subgroup of A_M. Then $K \rtimes \mathbf{Z}$ has finite index in G_M; in this case by considering the corresponding finite unramified cover of S_M and the induced birational structure, we can suppose that $K = A_M$. We will prove in a first step that this case is not possible either. Then we examine the last possibility where the holonomy representation is injective.

The holonomy is not cyclic

The proof of the following proposition will occupy the rest of this section.

Proposition B.3.6 *The image of the holonomy representation is not cyclic.*

We want to prove it by contradiction. *We can and will assume in the sequel that the kernel of* Hol *is exactly A_M.* Thus the developing map $\mathrm{Dev} : \mathbb{H} \times \mathbf{C} \dashrightarrow Y$ factorizes through $\overline{\mathrm{Dev}} : \overline{S_M} \dashrightarrow Y$. We will call the latter map the developing map too.

Lemma B.3.7 *The developing map $\overline{\mathrm{Dev}}$ has only a finite number of irreducible contracted curves.*

Proof Consider the (real) fibration $\rho : \overline{S_M} \to \mathbf{R}^{*+}$. The fibers F_t are compact and $\overline{dev}$ is locally birational, so each fiber intersects only a finite number of irreducible contracted curves. Thus it is sufficient to prove that every irreducible contracted curve intersects all the fibers. In other words, let $C \subset \overline{S_M}$ be an irreducible contracted curve, then we want to prove that $\rho(C) = \mathbf{R}^{*+}$. Since ρ is proper and C is closed, the image $\rho(C)$ is closed in $\mathbf{R}^{*+}$. It is then sufficient to prove that $\rho(C)$ is open. For this purpose it is more convenient to look at the universal covering $\mathbb{H} \times \mathbf{C}$. Let $\tilde{C} \subset \mathbb{H} \times \mathbf{C}$ be a component of the inverse image of C. Then the projection of $\tilde{C}$ onto $\mathbb{H}$ is not a point because C cannot be contained in a leaf of the foliation. Thus the projection is open since a holomorphic map is open. Therefore $\rho(C)$, identified as the projection of $\tilde{C}$ onto the vertical axis of $\mathbb{H}$, is also open. □

Let $C \subset \overline{S_M}$ be an irreducible contracted curve and $q \in Y$ be the point onto which C is contracted. Take a point $c \in C$ which is not an indeterminacy point of $\overline{\mathrm{Dev}}$. Take a chart of birational structure $U \subset \overline{S_M}$ at c so that the restriction $\overline{\mathrm{Dev}}|_U$ is analytically equivalent to a birational map. By Zariski's decomposition of birational maps, we can blow up Y at q and its infinitely near points to obtain a surface Y' such that the map $U \dashrightarrow Y'$ induced by $\overline{\mathrm{Dev}}|_U$ does not contract $C \cap U$. By analytic continuation, the map $\overline{S_M} \dashrightarrow Y'$ induced by $\overline{\mathrm{Dev}}$ does not contract C. As $\overline{\mathrm{Dev}}$ has only finitely many irreducible contracted curves by Lemma B.3.7, by repeating the above process we can find a rational surface Y^*, obtained by blowing up Y a finite number of times, such that the induced map $\overline{S_M} \dashrightarrow Y^*$ has no contracted curves. By replacing Y with Y^* *we will suppose from now on that* Dev *and* $\overline{\mathrm{Dev}}$ *have no contracted curves.*

However $\overline{\mathrm{Dev}}$ may still have indeterminacy points. We denote by $I \subset \overline{S_M}$ the indeterminacy set of $\overline{\mathrm{Dev}}$, it is a discrete set. The map $\overline{\mathrm{Dev}} : \overline{S_M} \dashrightarrow Y$ is locally biholomorphic outside I. We will call $\overline{\mathrm{Dev}}(\overline{S_M} \backslash I)$ the *image* of $\overline{S_M}$ (it is also the image of Dev).

The deck transformation group of the covering $\overline{S_M} \to S_M$ is isomorphic to the cyclic group G_M/A_M. Denote by g its generator induced by $g_0 \in G_M$. Denote by f the birational transformation $\mathrm{Hol}(g_0)$ of Y. We have $f \circ \overline{\mathrm{Dev}} = \overline{\mathrm{Dev}} \circ g$. By blowing up Y at some of the indeterminacy points of f (and their infinitely near points), we can and will assume that f is algebraically stable (see [DF01]). The only effect of doing so is to add some extra points into the indeteminacy set I of $\overline{\mathrm{Dev}}$.

Lemma B.3.8 *The contracted curves of $f^n, n \in \mathbf{Z}$ are disjoint from the image of* $\overline{\mathrm{Dev}}$.

Proof Suppose by contradiction that a curve $C \subset Y$ contracted by f^n intersects the image of $\overline{\mathrm{Dev}}$. Since $\overline{\mathrm{Dev}}$ is locally biholomorphic where it is defined, the inverse image $\overline{\mathrm{Dev}}^{-1}(C)$ is a curve on $\overline{S_M}$. Using the relation $f^n \circ \overline{\mathrm{Dev}} = \overline{\mathrm{Dev}} \circ g^n$, we see that $g^n(\overline{\mathrm{Dev}}^{-1}(C))$ is a curve on $\overline{S_M}$ contracted by $\overline{\mathrm{Dev}}$. This is a contradiction as we are already in the case where there are no contracted curves. □

Lemma B.3.9 *The birational transformation $f = \mathrm{Hol}(g_0)$ is loxodromic.*

Proof Suppose by contradiction that f is not loxodromic. We first claim that f preserves a pencil of curves. By definition a Jonquières twist or a Halphen twist preserves a pencil of curves. Thus we assume that f is elliptic. An elliptic element comes from a holomorphic vector field on Y. An elliptic element of infinite order exists only if Y is a rational surface, a ruled surface, an elliptic surface or birational to a surface of Kodaira dimension zero covered by an abelian surface.

If Y is birational to a surface covered by an abelian surface, then f preserves a transversely Euclidean foliation coming from a linear foliation on the abelian surface. This foliation can be pulled-back by $\overline{\mathrm{Dev}}$ to a foliation on $\overline{S_M}$ invariant under g. This further induces a holomorphic foliation on S_M which by Lemma B.3.3 coincides with one of the two obvious foliations on S_M. However neither of these foliations is transversely Euclidean by Lemma B.3.5, contradiction.

If Y is rational, then f preserves a pencil of rational curves by Proposition 2.3 of [BD15]. If Y is an elliptic surface of Kodaira dimension one, then f preserves the elliptic fibration of Y. If Y is a non-rational ruled surface, then f preserves the rational fibration. The claim follows.

Now we know that f preserves a pencil of curves. This pencil gives rise to a possibly singular holomorphic foliation S_M which by Lemma B.3.3 coincides with one of the two obvious foliations on S_M. By abuse of notation, we use the same letter $\mathscr{F}$ to denote this foliation on S_M and the one on $\overline{S_M}$. The fact that $\mathscr{F}$ is induced by a pencil of curves on Y implies that the images of the leaves of $\mathscr{F}$ by $\overline{\mathrm{Dev}}$ are contained in algebraic curves. The actions of A_M on the spaces of leaves of both of the two foliations are non-discrete, thus the leaves of the two foliations in $\overline{S_M}$ are not closed. Hence the image of a leaf of $\mathscr{F}$ by $\overline{\mathrm{Dev}}$ cannot be contained in an algebraic curve. $\square$

In the sequel we fix a Kähler metric on Y and we endow $\overline{S_M}\backslash I$ with the Kähler metric pulled back from Y by $\overline{\mathrm{Dev}}$. Before we consider Ahlfors-Nevanlinna currents, a few words need to be said about the Kähler metric. In Section B.2.3, for constructing Ahlfors-Nevanlinna currents the Kähler surface needs to be compact so that the difference between any Riemannian metric and the Kähler metric is everywhere bounded by a constant. In our situation here, though $\overline{S_M}\backslash I$ is not compact, we will be able to use freely all the results of Section B.2.3 because of the following three observations: 1) the Kähler metric on $\overline{S_M}\backslash I$ is pulled back from the compact surface Y; 2) in a small neighborhood U of a point $e \in I$, the map $\overline{\mathrm{Dev}}|_{U\backslash\{e\}}$ factorizes through a compact surface (Zariski's factorization theorem for birational maps); 3) the entire curves with which we will deal lie in a compact subset of $\overline{S_M}$. Roughly speaking, these three observations allow us to think of the part of $\overline{S_M}$ on which we will work as an open set of a compact Kähler surface.

The set of points in $\overline{S_M}$ that are mapped by $\overline{\mathrm{Dev}}$ to indeterminacy points of f is discrete and countable. The indeterminacy set I of $\overline{\mathrm{Dev}}$ is also discrete and countable. Therefore we can find two fibers F_a, F_b of $\rho : \overline{S_M} \to \mathbf{R}^{*+}$ such that:

— $F_a \cup F_b$ is disjoint from I and $\overline{\mathrm{Dev}}(F_a \cup F_b)$ is disjoint from the indeteminacy points of f;

— $g(F_a) = F_b$.

We will view the covering map $\mathbb{H} \times \mathbf{C} \to \overline{S_M}$ and the developing map $\mathrm{Dev} : \mathbb{H} \times \mathbf{C} \dashrightarrow Y$ (where it is defined) as families of entire curves. By choosing an appropriate path in $\mathbb{H}$ from a point of vertical coordinate a to a point of vertical coordinate b, we can extract from the above family a family of entire curves $(\xi_t)_{t\in[a,b]}$ on $\overline{S_M}$ parametrized by the interval $[a,b]$ such that

— $\forall t \in [a,b]$, the image of $\xi_t : \mathbf{C} \to \overline{S_M}$ is disjoint from the indeterminacy set I of $\overline{\mathrm{Dev}}$;

— ξ_t parametrizes a leaf of F_t; in particular ξ_a (resp. ξ_b) parametrizes a leaf of F_a (resp. F_b).

We can push the family $(\xi_t)_t$ forward by $\overline{\mathrm{Dev}}$ to obtain a family of entire curves $(\overline{\mathrm{Dev}} \circ \xi_t)_t$ on Y. As the covering map and the developing map (where it is defined) are locally biholomorphic, the derivative $\xi_t'(z)$ is non-zero for all $t \in [a,b]$ and for all $z \in \mathbf{C}$. We claim that the families $(\xi_t)_t$ and $(\overline{\mathrm{Dev}} \circ \xi_t)_t$ are uniform in the sense of Definition B.2.26. This is clear if $\overline{\mathrm{Dev}}$ has no indeterminacy points because in that case the entire curves $\overline{\mathrm{Dev}} \circ \xi_t$ factorize through the compact sets F_t. Since $\overline{\mathrm{Dev}}$ is locally birational, the same reasonning works after blowing up the indeterminacy points contained in the $F_t, t \in [a,b]$.

By Lemma B.2.27, we can construct a family of Ahlfors-Nevanlinna currents $(\Phi_t)_t$ associated with the uniform family of entire curves $(\xi_t)_t$, after fixing an appropriate sequence of radii once and for all. We construct corresponding Ahlfors-Nevanlinna currents associated with the $\overline{\mathrm{Dev}} \circ \xi_t$: they are the push-forward $\overline{\mathrm{Dev}}_*\Phi_t$. Lemma B.2.27 tells us that the cohomology classes $[\overline{\mathrm{Dev}}_*\Phi_t] \in \mathrm{H}^{1,1}(Y, \mathbf{R})$ are all the same. We also know that they are nef (see Lemma B.2.25).

As (the images of) the entire curves $\overline{\mathrm{Dev}} \circ \xi_a$ and $\overline{\mathrm{Dev}} \circ \xi_b$ are disjoint from the contracted curves and the indeterminacy set of f by Lemma B.3.8, we can push forward the Ahlfors-Nevanlinna current $\overline{\mathrm{Dev}}_*\Phi_a$ by f without any ambiguity. We want to compare the pushed forward current $f_*(\overline{\mathrm{Dev}}_*\Phi_a)$ with $\overline{\mathrm{Dev}}_*\Phi_b$. We have

$$f_*(\overline{\mathrm{Dev}}_*\Phi_a) = \overline{\mathrm{Dev}}_*(g_*\Phi_a).$$

Thus we just need to compare $g_*\Phi_a$ and Φ_b. By Plante's Theorem B.2.29, the closed positive currents Φ_a, Φ_b are respectively directed by the laminations F_a, F_b. As g sends F_a to F_b preserving their lamination structures, the push forward $g_*\Phi_a$ is a closed positive current directed by

F_b. By Sullivan's Theorem B.2.28, the two currents $g_*\Phi_a$ and Φ_b correspond to two transverse invariant measures on F_b. However by Lemma B.3.4, there exists only one transverse invariant measure on F_b up to multiples. Thus, we have

$$\lambda g_*\Phi_a = \Phi_b \quad \text{for some } \lambda \in \mathbf{R}^{*+}.$$

It follows that

$$\lambda f_*(\overline{\mathrm{Dev}}_*\Phi_a) = \overline{\mathrm{Dev}}_*\Phi_b.$$

By the equality of cohomology classes $[\overline{\mathrm{Dev}}_*\Phi_a] = [\overline{\mathrm{Dev}}_*\Phi_b]$, we get

$$\lambda f_*[\overline{\mathrm{Dev}}_*\Phi_a] = [\overline{\mathrm{Dev}}_*\Phi_a]. \tag{B.4}$$

As (the images of) the entire curves $\overline{\mathrm{Dev}} \circ \xi_a$ and $\overline{\mathrm{Dev}} \circ \xi_b$ are disjoint from the contracted curves and the indeterminacy set of f by Lemma B.3.8, we get also

$$f^*[\overline{\mathrm{Dev}}_*\Phi_a] = \lambda [\overline{\mathrm{Dev}}_*\Phi_a]. \tag{B.5}$$

Lemma B.3.10 *The dynamical degree of f is equal to λ or λ^{-1}.*

Proof Denote the dynamical degree of f by $\lambda(f)$. There exists a unique nef cohomology class v_f^+ such that $f^*v_f^+ = \lambda(f)v_f^+$. By Proposition 1.11 of [DF01], we have the following equality for intersection numbers:

$$(f^*v_f^+, [\overline{\mathrm{Dev}}_*\Phi_a]) = (v_f^+, f_*[\overline{\mathrm{Dev}}_*\Phi_a]). \tag{B.6}$$

If $[\overline{\mathrm{Dev}}_*\Phi_a]$ and v_f^+ are proportional, then $\lambda(f) = \lambda$ by Equation (B.5). Assume that they are not proportional; this implies that their intersection is strictly positive because they are both nef. Then Equations (B.4) and (B.6) force the equality between $\lambda(f)$ and $1/\lambda$. □

Replacing f with f^{-1} if necessary, we can and will assume that the dynamical degree of f is λ.

Lemma B.3.11 *We can assume that f acts by automorphism on Y, without loosing any other property that we need.*

Proof We first prove that all the irreducible curves contracted by the iterates f^n are of strictly negative self-intersection. Let E be an irreducible curve contracted by f^n. By Lemma B.3.8, the curve E is disjoint from $\overline{\mathrm{Dev}}(F_a)$ which is the support of $\overline{\mathrm{Dev}}_*\Phi_a$. Therefore the intersection number $[\overline{\mathrm{Dev}}_*\Phi_a]\cdot E$ is zero. As $[\overline{\mathrm{Dev}}_*\Phi_a]$ is nef, we have $[\overline{\mathrm{Dev}}_*\Phi_a]^2 \geq 0$. It follows from Hodge index theorem that $E^2 \leq 0$, with equality if and only if $[\overline{\mathrm{Dev}}_*\Phi_a]^2 = 0$ and E is proportional to $[\overline{\mathrm{Dev}}_*\Phi_a]$. Since $[\overline{\mathrm{Dev}}_*\Phi_a]$ is an eigenvector associated with λ, the equality $[\overline{\mathrm{Dev}}_*\Phi_a]^2 = 0$ would imply that the algebraically stable map f is an automorphism (see Section B.2.1). But then $[\overline{\mathrm{Dev}}_*\Phi_a]$ would be irrational and could not be proportional to E.

We write the Zariski factorization of f as $Y \leftarrow \hat{Y} \rightarrow Y$. Let $E_1,\cdots,E_m$ be the irreducible curves contracted by f. Denote by $\hat{E}_1,\cdots,\hat{E}_m$ their strict transforms in $\hat{Y}$. Among the $\hat{E}_i$, there exists at least one (-1)-curve, let us say, $\hat{E}_1$. Since $\hat{Y}$ is obtained from Y by blow-ups, we have $\hat{E}_1^2 \leq E_1^2$. We have showed that $E_1^2 < 0$. It follows that E_1 is already a (-1)-curve on Y. Now we contract it to obtain a new surface Y_1. We need to verify that all the hypothesis still hold on Y_1. The contraction may give rise to new curves contracted by f, but the new contracted curves on Y_1 come from the curves on Y contracted by f^2. So they are still disjoint from the image of $\overline{\mathrm{Dev}}$ and are of strictly negative self-intersection on Y_1. Hence we can continue the process. This process terminates because the Picard number drops down by one after each step. At last we get a surface on which f contracts no curves, i.e. f acts by automorphism. □

Once we know that f is a loxodromic automorphism, Theorem B.2.1 implies that $\overline{\mathrm{Dev}}_*\Phi_a$ is the unique closed positive current with cohomology class $[\overline{\mathrm{Dev}}_*\Phi_a]$. However the cohomology class of $\overline{\mathrm{Dev}}_*\Phi_b$ is also $[\overline{\mathrm{Dev}}_*\Phi_a]$. This leads to a contradiction because $\overline{\mathrm{Dev}}_*\Phi_a$ and $\overline{\mathrm{Dev}}_*\Phi_b$ are two different currents. Indeed their supports are respectively $\overline{\mathrm{Dev}}(F_a), \overline{\mathrm{Dev}}(F_b)$ and $\overline{\mathrm{Dev}}(F_a) \neq \overline{\mathrm{Dev}}(F_b)$ because otherwise $\overline{\mathrm{Dev}}$ would induce a map from S_M to Y. The proof of Proposition B.3.6 is finished.

Remark B.3.12 The very existence of immersed Levi-flat hypersurfaces as $\overline{\mathrm{Dev}}(F_a)$ imposes strong restrictions on the geometry of Y. For example there are no such immersed Levi-flat hypersurfaces in $\mathbb{P}^2$ (see [Der05]). However there exist families of Levi-flat hypersurfaces on other surfaces and we are not able to conclude directly by the existence of an "immersion" of $\overline{S_M}$ into Y. This is why the geometry of the cyclic covering $\overline{S_M} \to S_M$ plays a crucial role in our proof.

Injective holonomy

Since we have proved that the image of the holonomy is not cyclic, its kernel must be finite by Lemma B.3.1. Thus changing S_M into a finite cover *we can and will assume in the sequel that* Hol *is injective.* We will identify G_M with its image $\mathrm{Hol}(G_M) \in \mathrm{Bir}(Y)$.

Lemma B.3.13 *The group G_M is an elliptic subgroup of* $\mathrm{Bir}(Y)$.

Proof We apply Theorem B.2.4 to the solvable group $G_M \subset \mathrm{Bir}(Y)$. Up to conjugating the holonomy representation there are five possibilities in Theorem B.2.4 and we need to rule out the last four ones.

In case 5) Y is an abelian surface, the group G_M is generated by translations and a loxodromic automorphism. The stable and the unstable foliations of the loxodromic automorphism (see Example 1.1 of [CF03]), which are both linear foliations on Y, are preserved by G_M. Thus they can be pulled back to two holomorphic foliations on S_M. Induced by linear foliations on Y, these two pulled back foliations are transversely Euclidean. But they must coincide with the obvious foliations on S_M by Lemma B.3.3; this contradicts Lemma B.3.5.

In case 4) Y is rational and G_M is in $\mathrm{Bir}(\mathbb{P}^2)$. In this case A_M is contained in $\{(\alpha x, \beta y) | \alpha, \beta \in \mathbf{C}^*\}$ and g_0 is a monomial map $(x^p y^q, x^r y^s)$ such that the matrix $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z})$ is hyperbolic, i.e. has one eigenvalue > 1 and one < 1. The conjugation action of g_0 on A_M is given by $(\alpha, \beta) \mapsto (\alpha^p \beta^q, \alpha^r \beta^s)$. The exponential map semi-conjugates the action of $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ on $\mathbf{C}^2$ to the action of g_0 on $\mathbf{C}^* \times \mathbf{C}^*$. We think of A_M as a $\mathbf{Z}$-module of rank 3 with an irreducible action of g_0. Its preimage $\overline{A_M}$ in $\mathbf{C}^2$ by the exponential map is a $\mathbf{Z}$-module of rank 5 invariant under B. The kernel of the exponential map, generated by $(0, 2\pi i)$ and $(2\pi i, 0)$ is invariant under B and the action is irreducible. Hence either $\overline{A_M}$ is an indecomposable module of rank 5 or there is an indecomposable submodule of rank 3 which is isomorphic to A_M. However an indecomposable module, subgroup of $\mathbf{C}^2$, has to be of even rank because B is a hyperbolic matrix. We obtain thus a contradiction. Hence case 4) is not possible.

Case 3) of Theorem B.2.4 is impossible because G_M is not virtually abelian.

It suffices to show that case 2) of Theorem B.2.4 is not possible for G_M. Suppose the contrary. The rational fibration preserved by G_M can be pulled-back to a holomorphic foliation on $\mathbb{H} \times \mathbf{C}$. By the equivariance of D, this equips S_M with a holomorphic foliation. This foliation must coincide with one of the two obvious foliations on S_M by Lemma B.3.3. Acting on $\mathbb{H} \times \mathbf{C}$, the elements $g_0, \cdots, g_3$ permute the leaves of the foliation. The action of G_M on the spaces of leaves of the two foliations on $\mathbb{H} \times \mathbf{C}$, i.e. its actions on the $\mathbf{C}$-factor and on the $\mathbb{H}$-factor are both non-discrete. This means that, on the $\mathrm{Bir}(Y)$ side, the action of G_M on the base of the rational fibration is non discrete. As the automorphism group of a curve of general type is finite, the base is either $\mathbb{P}^1$ or an elliptic curve. But the base cannot be an elliptic curve neither because otherwise the fibration would be transversely Euclidean. Thus the base of the rational fibration is $\mathbb{P}^1$ and Y is a rational surface.

We have a morphism $\sigma : G_M \to \mathrm{PGL}_2(\mathbf{C})$ that records the action of G_M on the base of the rational fibration. Since this base action is non-discrete, Lemma B.3.1 implies that σ is injective. So g_1, g_2, g_3 have infinite actions on the base. Suppose by contradiction that one of the g_i, say g_1, is a Jonquières twist. Theorem B.2.2 case 3 says that for any h that commutes with g_1, the actions of h and g_1 on the base generate a virtually cyclic group. But we said that the actions of g_1, g_2 and g_3 on the base generate a group isomorphic to $\mathbf{Z}^3$, contradiction. Hence g_1, g_2, g_3 are all elliptic elements of $\mathrm{Bir}(Y)$ and A_M is an elliptic subgroup of $\mathrm{Bir}(Y)$. Up to replacing A_M by a finite index free abelian subgroup, we can assume that A_M is contained in $\mathrm{Aut}^0(Z)$, the connected component of the automorphism group of a rational surface Z. The group $\mathrm{Aut}^0(Z)$ is an algebraic group; we denote by $\overline{A_M}$ the Zariski closure of A_M in $\mathrm{Aut}^0(Z)$. Since A_M is infinite, $\overline{A_M}$ is an algebraic group of dimension ≥ 1.

We want to prove that no element of G_M is a Jonquières twist. For this purpose we apply an argument used by S. Cantat in the appendix of [**DP12**]. Any element of G_M normalizes A_M, thus normalizes $\overline{A_M}$. We have two possibilities for the action of the abelian algebraic group $\overline{A_M}$ on the rational surface Z, either it has a Zariski open orbit, or its orbits form a pencil of curves.

Assume that the orbits of $\overline{A_M}$ form a pencil of curves. This pencil of curves must differ from the original rational fibration preserved by A_M because the actions of g_1, g_2, g_3 on the base of the rational fibration are infinite. Every element of G_M normalizes $\overline{A_M}$, it preserves this pencil of curves. Recall that every element of G_M preserves also the rational fibration, thus preserves simultaneously two pencils of curves. This implies that g_0 is an elliptic element (cf. [DF01]). Therefore the group G_M contains no Jonquières twists.

Now assume that $\overline{A_M}$ has a Zariski open orbit O. We have three possibilities for O; it is a principal homogeneous space isomorphic to $\mathbf{C}^2$, $\mathbf{C} \times \mathbf{C}^*$ or $\mathbf{C}^* \times \mathbf{C}^*$. Since an element of G_M normalizes $\overline{A_M}$, it acts on O by automorphism of principal homogeneous spaces. If $O = \mathbf{C}^2$ then every element of G_M would be affine, thus elliptic. If $O = \mathbf{C} \times \mathbf{C}^*$ then an element of G_M would be of the form $(ax+b, \alpha y)$ with $a, \alpha \in \mathbf{C}^*, b \in \mathbf{C}$, which is again elliptic. If $O = \mathbf{C}^* \times \mathbf{C}^*$ then every element of G_M would be contained in the group generated by $\{(\alpha x, \beta y) | \alpha, \beta \in \mathbf{C}^*\}$ and $\{(x^p y^q, x^r y^s), \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z})\}$. In this case an element of G_M is either elliptic or loxodromic, but it cannot be loxodromic because we work already under the hypothesis that G_M preserves a rational fibration. Thus we have proved that every element of G_M is elliptic. This implies that G_M is an elliptic subgroup by Theorem B.2.4. $\square$

We proved that G_M is an elliptic subgroup of $\mathrm{Bir}(Y)$. Up to taking a finite index subgroup,

G_M is contained in $\mathrm{Aut}^0(Z)$, the component of identity of the automorphism group of a projective surface Z birational to Y. The Aut^0 of a projective variety is an algebraic group. By Chevalley structure theorem it is an extension of an abelian variety by a linear algebraic group. As $\mathrm{Aut}^0(Z)$ contains the non-abelian infinite group G_M, its linear part is not trivial. Hence Z is ruled by Theorem 14.1 of [Uen75] (see also [Bru15] Chapter 6.3). If Z were a non-rational ruled surface, then G_M would preserve the ruling and the ruling would be pulled back by Dev to one of the two obvious foliations on S_M. Using the fact that G_M acts non-discretely on the space of leaves and the fact that the two obvious foliations are not transversely Euclidean, we obtain a contradiction as in the proof of Lemma B.3.13.

Therefore Z is a rational surface. Since G_M is solvable, it comes from a group of automorphisms of a Hirzebruch surface. *We can and will assume in the sequel that $Z = \mathbb{F}_n$ is a Hirzebruch surface and that $G_M \subset$* $\mathrm{Aut}^0(Z)$ (cf. Corollary B.2.14). Note that from now on we take Z as the target space of the developing map.

Since $\mathrm{Aut}(\mathbb{F}_n)$ preserves the rational fibration on $\mathbb{F}_n$, we have a group homomorphism $\sigma : G_M \to \mathrm{PGL}_2(\mathbf{C})$ which encodes the action of G_M on the base $\mathbb{P}^1$ of the rational fibration. As G_M is solvable, we can assume, maybe after replacing G_M with a subgroup of index two, that $\sigma(G_M) \subset \mathrm{PGL}_2(\mathbf{C})$ fixes at least one point in $\mathbb{P}^1$. Let us decompose $\mathbb{P}^1$ as $\mathbf{C} \cup \{\infty\}$ where ∞ is one of the fixed point of $\sigma(G_M)$. As in the proof of Lemma B.3.13, the rational fibration induces a foliation on S_M which must coincide with one of the two obvious foliations on S_M; and we deduce from this that $\sigma(A_M)$ is not discrete. By Lemma B.3.1, this implies that σ is injective (up to taking a finite index subgroup). By Lemma B.3.2, we can write $\sigma(g_1), \sigma(g_2), \sigma(g_3)$ as $x \mapsto x + u_i$ for some $u_i \neq 0$, and $\sigma(g_0)$ as $x \mapsto \nu x$ for some $\nu \in \mathbf{C}^*$ of infinite order.

Lemma B.3.14 *The developing map* $\mathrm{Dev} : \mathbb{H} \times \mathbf{C} \dashrightarrow \mathbb{F}_n$ *is everywhere defined and is locally biholomorphic.*

Proof First we claim that Dev contracts no curves. Suppose by contradiction that Dev contracts a curve $C \subset \mathbb{H} \times \mathbf{C}$. Let $\gamma \in G_M$ be a non-trivial element. From the relation $\mathrm{Dev} \circ \gamma = \gamma \circ \mathrm{Dev}$ and the fact that γ acts by automorphism on $\mathbb{F}_n$, we deduce that $\gamma(C) \subset \mathbb{H} \times \mathbf{C}$ is also a contracted curve of Dev. Since locally there is only a finite number of contracted curves, the union $\bigcup_{\gamma \in G_M} \gamma(C)$ is a G_M-invariant set locally closed in $\mathbb{H} \times \mathbf{C}$. Therefore the image of C in the quotient S_M is locally closed, i.e. it is a curve on S_M. This contradicts the fact that S_M has no curves.

Suppose by contradiction that $p \in \mathbb{H} \times \mathbf{C}$ is an indeterminacy point of Dev. Take a local chart of birational structure U at p. We factorize $\mathrm{Dev}|_U : U \dashrightarrow \mathbb{F}_n$ as $U \xleftarrow{\pi_1} V \xrightarrow{\pi_2} \mathbb{F}_n$ where π_1

is a composition of (inverses of) blow-ups at p and its infinitely near points, and π_2 is an open embedding because Dev contracts no curves. Note that here we have holomorphic foliations on U and V, pulled back from the rational fibration on $\mathbb{F}_n$. As the foliation on U is regular, the exceptional curve of π_1 is an invariant curve of the foliation on V. This implies that its image by π_2 into $\mathbb{F}_n$ must be contained in a fiber of the rational fibration. However on $\mathbb{F}_n$ there are no (-1)-curves contained in the fibers of the rational fibration, this contradicts the fact that π_2 is an open embedding. $\square$

The above lemma tells us that the birational structure on S_M is in fact a $(\mathrm{Aut}(\mathbb{F}_n), \mathbb{F}_n)$-structure in the classical sense.

Lemma B.3.15 *Any G_M-invariant curve is disjoint from the image of* Dev.

Proof Let C be a curve which intersects the image of Dev, then $\mathrm{Dev}^{-1}(C)$ is a curve on $\mathbb{H} \times \mathbf{C}$. Note that elements of G_M are regular on the intersection of C with the image of Dev. So if C were G_M-invariant, then $\pi(\mathrm{Dev}^{-1}(C))$ would be a curve on S_M. $\square$

First case: Assume that the Hirzebruch surface $\mathbb{F}_n$ is not $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. $n \geq 1$. The fiber over $\infty \in \mathbb{P}^1$ and the exceptional section of $\mathbb{F}_n$ are G_M-invariant curves. Lemma B.3.15 implies that the image of the developing map is contained in the complement of these two invariant curves which is isomorphic to $\mathbf{C}^2$. The automorphisms g_1, g_2, g_3 are of the form $(x,y) \mapsto (x + u_i, \beta_i y + R_i(x))$ where $\beta_i \in \mathbf{C}^*$ and R_i is a polynomial of degree $\leq n$; the automorphism g_0 is $(x,y) \mapsto (\nu x, \beta_0 y + R_0(x))$ where $\beta_0 \in \mathbf{C}^*$ and R_0 is a polynomial of degree $\leq n$.

Assume first that $\beta_1, \beta_2, \beta_3$ are not all equal to 1, for example $\beta_1 \neq 1$. In this case g_1 has a fixed point e on the fiber over ∞ which is not on the exceptional section of $\mathbb{F}_n$. By commutativity, the point e is fixed by A_M; then by the fact that g_0 normalizes A_M, the whole group G_M fixes e. We can blow-up e and contract the strict transform of the initial fiber to get $\mathbb{F}_{n-1}$. The group G_M remains a group of automorphisms of $\mathbb{F}_{n-1}$. Moreover the image of the developing map is not affected by this elementary transformation. Therefore the initial birational structure reduces to a $(\mathrm{Aut}(\mathbb{F}_{n-1}), \mathbb{F}_{n-1})$-structure. We continue this process and reduce the birational structure to a $(\mathrm{Aut}(\mathbb{F}_1), \mathbb{F}_1)$-structure. The Hirzebruch surface $\mathbb{F}_1$ is the blow-up of $\mathbb{P}^2$ at one point; the exceptional divisor is the exceptional section and is disjoint from the image of Dev. Therefore we finally get a $(\mathrm{PGL}_3(\mathbf{C}), \mathbb{P}^2)$-structure.

Assume that $\beta_1 = \beta_2 = \beta_3 = 1$. We can conjugate g_1 inside $\mathrm{Bir}(\mathbb{P}^2)$, by elements of the form $(x,y) \dashrightarrow (x, y + \delta x^d)$, to decrease the degree of R_1 until g_1 becomes $(x,y) \mapsto (x + u_1, y)$;

note that this only modifies the fiber at ∞ so that the conjugation does not affect Dev. After these conjugations, g_2, g_3 become $(x,y) \dashrightarrow (x+u_i, y+\tilde{R}_i(x))$, $i = 2,3$ and g_0 becomes $(x,y) \dashrightarrow (\nu x, \beta_0 y + \tilde{R}_0(x))$ where the $\tilde{R}_i$ are polynomials for $i = 0,2,3$. The commutation relations between g_1 and g_2, g_3 reads:

$$\tilde{R}_i(x) = \tilde{R}_i(x+u_1), \quad i = 2,3;$$

this implies immediately that $\tilde{R}_2$ and $\tilde{R}_3$ are constants. Therefore we have conjugated A_M to a subgroup of $\mathrm{PGL}_3(\mathbf{C})$. Now the transformation $g_0 \circ g_1 \circ g_0^{-1}$ is

$$(x,y) \dashrightarrow (x+\nu u_1, y+\tilde{R}_0(\nu^{-1}x+u_1) - \tilde{R}_0(\nu^{-1}x))$$

For $g_0 \circ g_1 \circ g_0^{-1}$ to be in A_M, the polynomial function $\tilde{R}_0(\nu^{-1}x+u_1) - \tilde{R}_0(\nu^{-1}x)$ needs to be a constant. This implies that the degree of $\tilde{R}_0$ is at most 1, i.e. g_0 is also in $\mathrm{PGL}_3(\mathbf{C})$. We get again a $(\mathrm{PGL}_3(\mathbf{C}), \mathbb{P}^2)$-structure.

Second case: the Hirzebruch surface is $\mathbb{P}^1 \times \mathbb{P}^1$. Considering a finite unramified cover of S_M, we can assume that G_M is included in the identity component of the automorphism group which is $\mathrm{PGL}_2(\mathbf{C}) \times \mathrm{PGL}_2(\mathbf{C})$. Replacing G_M with a index two subgroup if necessary, we have two injective homomorphisms σ_1, σ_2 from the solvable group G_M to $\mathrm{PGL}_2(\mathbf{C})$. The image $\sigma_1(G_M)$ (resp. $\sigma_2(G_M)$) fixes at least one point in the first (resp. second) factor $\mathbb{P}^1$. Removing the two corresponding G_M-invariant curves from $\mathbb{P}^1 \times \mathbb{P}^1$, we get a Zariski open set which is isomorphic to $\mathbf{C}^2$ and in which the image of the developing map is contained. This means that the birational structure is reduced to a complex affine structure.

Bruno Klingler proved in [Kli98] that the only $(\mathrm{PGL}_3(\mathbf{C}), \mathbb{P}^2)$-structure on S_M is the natural one, this finishes the proof of Theorem B.1.1 for Inoue surfaces of type S^0.

B.4 Inoue surfaces of type $S^{\pm}$

B.4.1 Description

Let $n \in \mathbf{N}^*$. Consider the group of upper-triangular matrices

$$\Lambda_n = \left\{ \begin{pmatrix} 1 & x & \frac{z}{n} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x,y,z \in \mathbf{Z} \right\}.$$

The center of Λ_n is the infinite cyclic group C_n generated by $\begin{pmatrix} 1 & 0 & \frac{1}{n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The quotient Λ_n/C_n is isomorphic to $\mathbf{Z}^2$. Let $N \in \mathrm{SL}_2(\mathbf{Z})$ be a matrix with eigenvalues $\alpha, \frac{1}{\alpha}$ such that $\alpha > 1$. Let φ be an automorphism of the group of real upper-triangular matrices which preserves Λ_n, acts trivially on C_n and acts on $\Lambda_n/C_n \cong \mathbf{Z}^2$ as N. We form a semi-direct product $\Gamma_N = \Lambda_n \rtimes \mathbf{Z}$ where the $\mathbf{Z}$ factor acts on Λ_n as φ. The group Γ_N acts on the group of real upper-triangular matrices which is identified with $\mathbf{R}^3 = \mathbf{R} \times \mathbf{C}$. Define an action of Γ_N on $\mathbb{H} \times \mathbf{C} = \mathbf{R}^{>0} \times \mathbf{R} \times \mathbf{C}$ with Λ_n acting trivially on $\mathbf{R}^{>0}$ and $1 \in \mathbf{Z}$ acting on $\mathbb{H}$ as $x \mapsto \alpha x$. This action is holomorphic and the quotient $S_N = \mathbb{H} \times \mathbf{C}/\Gamma_N$ is a compact non-Kähler surface called an *Inoue surface of type* S^+ ([Ino74]). Note that the Inoue surface depends on n, φ, and φ depends on N; we denote it by S_N because N is the most significant parameter.

The group Γ_N can be identified with a lattice in one of the two following solvable Lie groups which are subgroups of $\mathrm{Aff}_2(\mathbf{C})$ (cf. [Kli98]):

$$\mathrm{Sol}^1 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & d & c \\ 0 & 0 & 1 \end{pmatrix}, a,b,c,d \in \mathbf{R}, d > 0 \right\}, \ \mathrm{Sol}^{1'} = \left\{ \begin{pmatrix} 1 & a & b + i\log(d) \\ 0 & d & c \\ 0 & 0 & 1 \end{pmatrix}, a,b,c,d \in \mathbf{R}, d > 0 \right\}.$$

Conversely any torsion free lattice of these two groups gives an Inoue surface of type S^+. Note that a finite unramified cover of an Inoue surface of type S^+ is an Inoue surface of type S^+.

Concretely Γ_N has four generators g_0, g_1, g_2, g_3 which act on $\mathbb{H} \times \mathbf{C}$ as:

$$\begin{aligned} g_0 &: (x,y) \mapsto (\alpha x, y + t) \\ g_i &: (x,y) \mapsto (x + a_i, y + b_i x + c_i) \quad i = 1,2 \\ g_3 &: (x,y) \mapsto (x, y + \frac{b_1 a_2 - b_2 a_1}{n}) \end{aligned}$$

where t is a complex number, (a_1, a_2) (resp. (b_1, b_2)) is a real eigenvector of N corresponding to the eigenvalue α (resp. α^{-1}) and c_1, c_2 are some complex numbers (see [Ino74] for the explicit expressions of c_1, c_2). The center C_n of Λ_n is also the center of Γ_N, it is generated by g_3. The normal subgroup Λ_n is generated by g_1, g_2, g_3. We have

$$\begin{aligned} g_1^{-1} g_2^{-1} g_1 g_2 &= g_3^n \\ g_0 g_i g_0^{-1} &= g_1^{n_{i1}} g_2^{n_{i2}} g_3^{m_i}, \quad i = 1,2 \end{aligned}$$

where n_{ij} are entries of the matrix N and m_1, m_2 are two integers depending on c_1, c_2.

The Inoue surface S_N has an obvious $(\mathrm{Aff}_2(\mathbf{C}), \mathbf{C}^2)$-structure. The surface $\overline{S_N} = \mathbb{H} \times \mathbf{C}/\Lambda_n$ is an infinite cyclic covering space of S_N. As a real manifold, $\overline{S_N}$ admits a fibration $\rho : \overline{S_N} \to \mathbf{R}^{*+}$ where the $\mathbf{R}^*_+ = \{t\sqrt{-1}, t \in \mathbf{R}^*_+\}$ is the vertical axis of $\mathbb{H} \subset \mathbb{H} \times \mathbf{C}$. The fibers, denoted by E_t, are quotients of $\{x + t\sqrt{-1}, x \in \mathbf{R}^{*+}\} \times \mathbf{C}$ by Λ_n; they are compact real nilmanifolds of dimension 3. The E_t are Levi-flat hypersurfaces in $\overline{S_N}$ and they are foliated by entire curves coming from the vertical complex lines in $\mathbb{H} \times \mathbf{C}$.

The analogues of Lemmata B.3.1, B.3.2, B.3.3, B.3.4 and B.3.5 still hold (we omit the details when the proof is exactly the same).

Lemma B.4.1 *If K is a non-trivial normal subgroup of Γ_N, then K has finite index in C_n, Λ_n or Γ_N.*

Proof The conjugation action of g_0 on Λ_n/C_n is just the action of $N \in \mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{Z}^2$; it has no eigenvectors in $\mathbf{Z}^2 \setminus \{0\}$. Thus, if K contains an element of Λ_n which is not in C_n, then it contains Λ_n. To conclude, we need only remark that, by the semi-direct product structure, the intersection of a normal subgroup of Γ_N with Λ_n cannot be trivial. □

Lemma B.4.2 *Let $\sigma : \Gamma_N \to \mathrm{PGL}_2(\mathbf{C})$ be a morphism whose kernel is C_n. Then for some affine coordinate $\mathbb{P}^1 = \{x \in \mathbf{C}\} \cup \{\infty\}$, the images $\sigma(g_i), i = 0, \cdots, 2$, viewed as homographies of $\mathbb{P}^1$, may be written as*

$$\begin{aligned} \sigma(g_i) &: x \mapsto x + u_i, \quad i = 1, 2 \\ \sigma(g_0) &: x \mapsto \nu x \end{aligned}$$

for some $\nu, u_i \in \mathbf{C}^*$.

Lemma B.4.3 *The only (possibly singular) holomorphic foliation on S_N is the obvious one coming from the vertical foliation of $\mathbb{H} \times \mathbf{C}$.*

Proof Let $\mathscr{F}$ be a foliation on S_N. As in the proof of Lemma B.3.3, we infer that $\mathscr{F}$ is saturated and non-singular. We denote by T the tangent bundle of S_M, by T^* its dual and by K the canonical bundle of S_M. We denote by F_0 the normal bundle of the obvious foliation (here we use notations of [Ino74]) and by F the normal bundle of $\mathscr{F}$. The foliation $\mathscr{F}$ corresponds to a non-zero global section of $T^* \otimes F$. It is proved in [Ino74] that $T^* \otimes F$ has non-zero sections if and only if $F = F_0$. In other words, $\mathscr{F}$ and the obvious foliation share the same normal bundle. It is also proved in [Ino74] that the space of global sections of $T^* \otimes F_0$ is one dimensional. Thus, $\mathscr{F}$ must coincide with the obvious foliation. □

Lemma B.4.4 *Up to multiples, there is only one transverse invariant measure on E_t.*

Lemma B.4.5 *The obvious foliation on S_N is not transversely Euclidean.*

Inoue surfaces of type S^- are defined similarly: instead of choosing N in $\mathrm{SL}_2(\mathbf{Z})$, we take a matrix in $\mathrm{GL}_2(\mathbf{Z})$ with determinant (-1). Every Inoue surface of type S^- has a double unramified cover which is an Inoue surface of type S^+. Thus, for our purpose it is sufficient to consider only the Inoue surfaces of type S^+.

B.4.2 Proof of Theorem B.1.1 for Inoue surfaces of type S^+

Many details of the proof will be very similar to the case of Inoue surfaces of type S^0; we will make them brief.

Equip S_N with a $(\mathrm{Bir}(X),X)$-structure and let $(Y,\mathrm{Hol},\mathrm{Dev})$ be a holonomy/developing triple.

Run again the previous proof

Lemma B.4.1 says that there are only four possibilities for the holonomy representation. It is easy to rule out the first possibility: if the holonomy had finite image then the developing map would induce a meromorphic locally birational map from a finite unramified cover of S_N to Y, contradicting the fact that S_N has algebraic dimension zero.

If the kernel K of the holonomy has finite index in Λ_n, then $K \rtimes \mathbf{Z}$ has finite index in Γ_N; in this case by considering the corresponding finite unramified cover of S_N and the induced birational structure, we can suppose that $K = \Lambda_n$. The image of the holonomy is then cyclic. This is not possible: Lemmata B.4.3, B.4.4 and B.4.5 ensure that the proof of Subsection B.3.2 works exactly in the same way for S_N.

We now rule out the case where the kernel K of Hol has finite index in C_n; we will examine the situation of injective holonomy in the next subsection. After taking a finite unramified cover of S_N, we can and will assume that $K = C_n$. Thus, we have an embedding of $\Omega_N = \Gamma_N/C_n \cong \mathbf{Z}^2 \rtimes \mathbf{Z}$ into $\mathrm{Bir}(Y)$. The situation is almost the same as in the case of Inoue surface of type S^0; there we had $\mathbf{Z}^3 \rtimes \mathbf{Z}$, here we have $\mathbf{Z}^2 \rtimes \mathbf{Z}$. We can almost copy the proof of Section B.3.2; we give here a sketch.

Firstly we prove as in Lemma B.3.13 that Ω_N is an elliptic subgroup of $\mathrm{Bir}(Y)$. The only difference in the proof is the fourth case of Theorem B.2.4. In case 4), Ω_N is contained in the group generated by $\{(\alpha x,\beta y)|\alpha,\beta \in \mathbf{C}^*\}$ and one monomial transformation (x^py^q,x^ry^s)

where $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z})$. In this case Ω_N preserves two holomorphic foliations defined by $\iota_1 xdy + \nu_1 ydx$ and $\iota_2 xdy + \nu_2 ydx$ where (ι_i, ν_i) $i = 1,2$ are two eigenvectors of $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$. These two Γ_N-invariant foliations induce two foliations on S_N; this is impossible because there exists only one holomorphic foliation on an Inoue surface of type S^+ by Lemma B.4.3.

Once we know that Ω_N is an elliptic subgroup, we prove as in Section B.3.2 that the $(\mathrm{Bir}(X), X)$-structure is reduced to a $(\mathrm{Aut}(\mathbb{F}_n), \mathbb{F}_n)$-structure, and then to a $(\mathrm{PGL}_3(\mathbf{C}), \mathbb{P}^2)$-structure; the arguments here and there are exactly the same. However the only $(\mathrm{PGL}_3(\mathbf{C}), \mathbb{P}^2)$-structure on S_N is the obvious one by [Kli98] and its holonomy is injective, a contradiction to the hypothesis that the kernel of the holonomy is C_n. Thus, we have proved:

Lemma B.4.6 *The kernel of the holonomy representation* Hol *is trivial.*

Injective holonomy

After Lemma B.4.6 we know that the holonomy representation is injective. From now on we consider Γ_N as a subgroup of $\mathrm{Bir}(Y)$. We apply Theorem B.2.4 to Γ_N. Case 5) is not possible because the stable and the unstable foliations of a loxodromic automorphism on an abelian surface would induce two transversely Euclidean foliations on S_N. Case 4) is not possible because the derived length of $\mathbf{C}^* \times \mathbf{C}^* \rtimes \langle \text{a monomial map} \rangle$ is 2 and that of Γ_N is 3 (here we can also use the foliation argument). Case 3) is impossible because Γ_N is not virtually abelian. The following lemma says that case 2) is not possible either.

Lemma B.4.7 *If the group Γ_N preserves a rational fibration, then it contains no Jonquières twists and Y is rational.*

Proof The rational fibration preserved by Γ_N induces a holomorphic foliation on S_N which coincides with the natural one. The action of g_3, even on $\mathbb{H} \times \mathbf{C}$, does not permute the leaves, so its action on the base of the rational fibration must be trivial. As regards the action of $\Gamma_N \backslash C_n$ on the base, it is non-discrete by considering the action on the space of leaves. Together with the fact that the foliation is not transversely Euclidean, this implies that the base of the rational fibration is necessarily $\mathbb{P}^1$. Thus Y is a rational surface.

Using again the non-discreteness of the base action, we have an embedding $\sigma : \Omega_N = \Gamma_n / C_n \to \mathrm{PGL}_2(\mathbf{C})$. By Lemma B.4.2, we infer that $\sigma(g_0), \sigma(g_1), \sigma(g_2)$ are respectively $x \mapsto \gamma x, x \mapsto x + u_1, x \mapsto x + u_2$ where $\gamma, u_1, u_2 \in \mathbf{C}^*$ are such that $\gamma u_i = n_{i1} u_1 + n_{i2} u_2$ for $i = 1,2$. The sequel of the proof is purely about the group of birational transformations.

Every element of Γ_N commutes with g_3; for $i = 0,1,2$, the group generated by g_i, g_3 is isomorphic to $\mathbf{Z}^2$. By Theorem B.2.2, g_3 must be an elliptic element. Up to conjugation, g_3 is $(x,y) \mapsto (x, y+v_3)$ or $(x,y) \mapsto (x, \nu y)$. Let us first suppose that g_3 is $(x,y) \mapsto (x, \nu y)$ for some $\nu \in \mathbf{C}^*$ of infinite order. By Theorem B.2.3 g_0, g_1, g_2 are respectively $(\gamma x, R_0(x)y)$ and $(x+u_i, R_i(x)y)$, $i = 1,2$ where $R_0, R_1, R_2 \in \mathbf{C}(x)$. The relation $g_1^{-1} g_2^{-1} g_1 g_2 = g_3^n$ reads

$$R_2(x)R_1(x+u_2)R_2(x+u_1)^{-1}R_1(x)^{-1} = \nu^n.$$

For $i = 1,2$ write R_i as $\frac{P_i}{Q_i}$ with $P_i, Q_i \in \mathbf{C}[x]$. Then the above equation becomes

$$\frac{P_2(x)P_1(x+u_2)Q_2(x+u_1)Q_1(x)}{P_2(x+u_1)P_1(x)Q_2(x)Q_1(x+u_2)} = \nu^n.$$

On the left-hand side, the numerator and the denominator have the same degree and the same dominant coefficient. This implies $\nu^n = 1$, which is absurd because ν has infinite order. Thus, g_3 is not of the form $(x,y) \mapsto (x, \nu y)$.

Hence g_3 is of the form $(x,y) \mapsto (x, y+v_3)$. By Theorem B.2.3 g_0, g_1, g_2 can be respectively written as $(\gamma x, y+R_0(x))$ and $(x+u_i, y+R_i(x))$, $i = 1,2$ where $R_0, R_1, R_2 \in \mathbf{C}(x)$.

We will exploit the relation $g_1^{-1} g_2^{-1} g_1 g_2 = g_3^n$ to show that R_1, R_2 must be polynomials. Note that g_3 is elliptic and acts trivially on the base; roughly speaking g_1, g_2 almost commute. Before we continue the proof we recall first some notions. An indeterminacy point x of f will be called persistent if for every $i > 0$, f^{-i} is regular at x and the backward orbit of x is infinite, and if there are infinitely many curves contracted onto x by the iterates $f^{-k}, k \in \mathbf{N}$. A conic bundle is a rational fibration where the only singular fibers are unions of two (-1)-curves. It is proved in [Zhaa] that g_1, being an element of Jonq, acts by algebraically stable transformation on a conic bundle X; moreover the only singular fiber of X lies over the Γ_N-invariant fiber $x = \infty$.

Suppose by contradiction that R_1 is not a polynomial; this implies that g_1 is a Jonquières twist. Some poles of R_1 in $\mathbf{C}$ correspond to persisitent indeterminacy points of g_1 on X (see [Zhaa] for details). Let $e \in X$ be a persistent indeterminacy point of g_1. Since $\{g_1^{-i}(e), i > 0\}$ is infinite, g_2 and g_3 are regular at $g_1^{-k}(e)$ for k large enough. For infinitely many $j > 0$, g_1^{-j} contracts a regular fiber of the conic bundle onto e, denote it by C_j. For k large enough g_2 and g_3 do not contract C_k. Keeping these two observations in mind, from the relation $g_1^k \circ g_2 \circ g_1^{-j} = g_2 \circ g_3^{nk} \circ g_1^{k-j}$ we deduce that $g_2 \circ g_1^{-j}(e)$ is an indeterminacy point of g_1^k for suitable j,k (recall that g_3 does not permute the fibers of the conic bundle). This means that, under the iteration of g_1, the forward orbit of $g_2 \circ g_1^{-j}(e)$ will meet a persistent indeterminacy point e' of g_1. The correspondance $e \mapsto e'$ does not depend on j,k. Thus, up to raplacing g_2 by an iterate g_2^m, we have $e = e'$. Then for some $l \in \mathbf{Z}$, $g_1^l \circ g_2^m(g_1^{-j}(e))$ will be an indeterminacy point of g_1^j, i.e. we have $g_1^l \circ g_2^m(g_1^{-j}(e)) = g_1^{-j}(e)$. Similarly, we have $g_1^l \circ g_2^m(C_k) = C_k$ for k large enough. This means that $g_1^l \circ g_2^m$ preserves the rational fibration fiber by fiber. In particular $lu_1 + mu_2 = 0$, which is impossible because u_1, u_2 generate a non-discrete subgroup of $\mathbf{C}$.

Now we know that R_1, R_2 are polynomials. Consequently g_1, g_2 are elliptic. Let us finish the proof by showing that R_0 is a polynomial too. The element $g_0 g_i g_0^{-1}$ reads

$$(x,y) \dashrightarrow (x + \gamma u_1, y - R_0(\gamma^{-1}x) + R_1(\gamma^{-1}x) + R_0(\gamma^{-1}x + u_1)).$$

The relation $g_0 g_i g_0^{-1} = g_1^{n_{i1}} g_2^{n_{i2}} g_3^{m_i}$ implies that the rational fraction $-R_0(\gamma^{-1}x) + R_1(\gamma^{-1}x) + R_0(\gamma^{-1}x + u_1)$ is a polynomial. This is only possible if R_0 is a polynomial. $\square$

From the above discussions we know that Γ_N is an elliptic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$. The proofs of Lemma B.3.14 and Lemma B.3.15 work exactly in the same way and we reduce the birational structure on S_N to a $(\mathrm{Aut}(\mathbb{F}_k), \mathbb{F}_k)$-structure for $k \neq 1$ or to a $(\mathrm{PGL}_3(\mathbf{C}), \mathbb{P}^2)$-structure as in Section B.3.2. If it is reduced to a $(\mathrm{PGL}_3(\mathbf{C}), \mathbb{P}^2)$-structure then the result of B. Klingler [Kli98] finishes the proof. It cannot be reduced to a $(\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1), \mathbb{P}^1 \times \mathbb{P}^1)$-structure because a finite unramified cover of S_N would have two holomorphic foliations.

Assume that the birational structure is reduced to a $(\mathrm{Aut}(\mathbb{F}_k), \mathbb{F}_k)$-structure for $k \geq 2$. Then Γ_N preserves a rational fibration. Denote by σ the induced homomorphism from Γ_N to $\mathrm{PGL}_2(\mathbf{C})$. Using the same reasoning we have done in the proof of the previous lemma, we can write g_0, g_1, g_2, g_3 as:

$$\begin{aligned} g_0 &: (x,y) \mapsto (\gamma x, y + R_0(x)); \\ g_i &: (x,y) \mapsto (x + u_i, y + R_i(x)),\ i = 1,2; \\ g_3 &: (x,y) \mapsto (x, y + v_3) \end{aligned}$$

where $u_1, u_2, v_3, \gamma \in \mathbf{C}^*$ and R_1, R_2, R_3 are polynomials. Moreover we have

$$\gamma \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{B.7}$$

where $\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}$ is the matrix N. The relation $g_1^{-1}g_2^{-1}g_1g_2 = g_3^n$ reads

$$R_2(x) + R_1(x + u_2) - R_2(x + u_1) - R_1(x) = nv_3. \tag{B.8}$$

For the left side of Equation (B.8) to be a constant, the degrees of R_1, R_2 must be the same. Denote by l their degree. For $i = 1, 2$, the element $g_0 g_i g_0^{-1}$ may be written as

$$(x, y) \mapsto (x + \gamma u_i, y - R_0(\gamma^{-1}x) + R_1(\gamma^{-1}x) + R_0(\gamma^{-1}x + u_i)). \tag{B.9}$$

The relation $g_0 g_i g_0^{-1} = g_1^{n_{i1}} g_2^{n_{i2}} g_3^{m_i}$ implies that the polynomial $-R_0(\gamma^{-1}x) + R_1(\gamma^{-1}x) + R_0(\gamma^{-1}x + u_i)$ has degree l. This is possible only if the degree of R_0 is less than or equal to $(l+1)$. For $i = 1, 2, 3$ and $0 \le j \le l+1$, we denote by r_{ij} the coefficient of x^j in $R_i(x)$.

Suppose by contradiction that $l > 1$. By looking at the dominant coefficients in the equations $g_1^{-1}g_2^{-1}g_1g_2 = g_3^n$ and $g_0 g_i g_0^{-1} = g_1^{n_{i1}} g_2^{n_{i2}} g_3^{m_i}, i = 1, 2$, we obtain

$$r_{1l} l u_2 - r_{2l} l u_1 = 0 \tag{B.10}$$

$$\gamma^{-l} r_{il} + \gamma^{-l}(l+1) u_i r_{0(l+1)} = n_{i1} r_{1l} + n_{i2} r_{2l} \quad i = 1, 2. \tag{B.11}$$

In terms of matrices, Equation (B.11) reads

$$(N - \gamma^{-l}\,\mathrm{Id}) \begin{pmatrix} r_{1l} \\ r_{2l} \end{pmatrix} = \gamma^{-l}(l+1) u_i r_{0(l+1)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

which by Equation (B.7) and Equation (B.10) is equivalent to

$$(\gamma - \gamma^{-l}) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = C \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{B.12}$$

for some non-zero constant C. This is not possible because $u_1 \neq u_2$. Therefore $l \le 1$ and g_1, g_2 are affine transformations. The relation $g_1^{-1}g_2^{-1}g_1g_2 = g_3^n$ now reads

$$r_{11} l u_2 - r_{21} l u_1 = nv_3. \tag{B.13}$$

Equation (B.13) implies that $l \neq 0$, i.e. $l = 1$. Then R_0 is a polynomial of degree at most 2. If R_0 is of degree 2, then we can conjugate $g_0 : (x, y) \mapsto (\gamma x, y + R_0(x))$ by $(x, y) \mapsto (x, y + \delta x^2)$ for an appropriate $\delta \in \mathbf{C}^*$ to decrease the degree of R_0. Moreover the conjugation by $(x, y) \mapsto$

$(x, y+\delta x^2)$ keeps g_1, g_2, g_3 affine transformations. Thus we reduce the birational structure to a complex affine structure. Using again [Kli98], we achieve the proof of Theorem B.1.1 for Inoue surfaces of type $S^{\pm}$.

BIBLIOGRAPHY

[AB60] Lars Ahlfors and Lipman Bers. « Riemann's mapping theorem for variable metrics ». In: *Ann. of Math. (2)* 72 (1960), pp. 385–404. ISSN: 0003-486X. DOI: 10.2307/1970141. URL: https://doi.org/10.2307/1970141.

[Abi75] William Abikoff. « On boundaries of Teichmüller spaces and on Kleinian groups. III ». In: *Acta Math.* 134 (1975), pp. 211–237. ISSN: 0001-5962. DOI: 10.1007/BF02392102. URL: https://doi.org/10.1007/BF02392102.

[Ago] Ian Agol. « Tameness of hyperbolic 3-manifolds ». In: *arxiv.org/abs/math/0405568* ().

[Ahl35] Lars Ahlfors. « Zur Theorie der Überlagerungsflächen ». In: *Acta Math.* 65.1 (1935), pp. 157–194. ISSN: 0001-5962. DOI: 10.1007/BF02420945. URL: https://doi.org/10.1007/BF02420945.

[Ahl64] Lars V. Ahlfors. « Finitely generated Kleinian groups ». In: *Amer. J. Math.* 86 (1964), pp. 413–429. ISSN: 0002-9327. DOI: 10.2307/2373173. URL: https://doi.org/10.2307/2373173.

[Alp87] Roger C. Alperin. « An elementary account of Selberg's lemma ». In: *Enseign. Math. (2)* 33.3-4 (1987), pp. 269–273. ISSN: 0013-8584.

[AM01] Kari Astala and Gaven J. Martin. « Holomorphic motions ». In: *Papers on analysis*. Vol. 83. Rep. Univ. Jyväskylä Dep. Math. Stat. Univ. Jyväskylä, 2001, pp. 27–40.

[Amo+96] Jaume Amorós et al. *Fundamental groups of compact Kähler manifolds*. Vol. 44. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1996, pp. xii+140. ISBN: 0-8218-0498-7. DOI: 10.1090/surv/044. URL: http://dx.doi.org/10.1090/surv/044.

[Ati57] Michael F. Atiyah. « Complex analytic connections in fibre bundles. » English. In: *Trans. Am. Math. Soc.* 85 (1957), pp. 181–207. ISSN: 0002-9947; 1088-6850/e.

[Bar+04] Wolf P. Barth et al. *Compact complex surfaces*. Second. Vol. 4. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Springer-Verlag, Berlin, 2004, pp. xii+436. ISBN: 3-540-00832-2. DOI: 10.1007/978-3-642-57739-0. URL: http://dx.doi.org/10.1007/978-3-642-57739-0.

[BB00] Lionel Bayle and Arnaud Beauville. « Birational involutions of $\mathbf{P}^2$ ». In: *Asian J. Math.* 4.1 (2000). Kodaira's issue, pp. 11–17. ISSN: 1093-6106. DOI: 10.4310/AJM.2000.v4.n1.a2. URL: https://doi.org/10.4310/AJM.2000.v4.n1.a2.

[BB14] Fedor Bogomolov and Christian Böhning. « On uniformly rational varieties. » English. In: *Topology, geometry, integrable systems and mathematical physics. Novikov's seminar: 2012–2014. Selected papers of the seminar, Moscow, Russia, 2012–2014. Dedicated to S. P. Novikov on the occasion of his 75th birthday*. Providence, RI: American Mathematical Society (AMS), 2014, pp. 33–48. ISBN: 978-1-4704-1871-7/hbk.

[BB66] Walter L. Baily Jr. and Armand Borel. « Compactification of arithmetic quotients of bounded symmetric domains ». In: *Ann. of Math. (2)* 84 (1966), pp. 442–528. ISSN: 0003-486X. DOI: 10.2307/1970457. URL: https://doi.org/10.2307/1970457.

[BC16] Jérémy Blanc and Serge Cantat. « Dynamical degrees of birational transformations of projective surfaces. » English. In: *J. Am. Math. Soc.* 29.2 (2016), pp. 415–471. ISSN: 0894-0347; 1088-6834/e.

[BCM12] Jeffrey F. Brock, Richard D. Canary, and Yair N. Minsky. « The classification of Kleinian surface groups, II: The ending lamination conjecture ». In: *Ann. of Math. (2)* 176.1 (2012), pp. 1–149. ISSN: 0003-486X. DOI: 10.4007/annals.2012.176.1.1. URL: https://doi.org/10.4007/annals.2012.176.1.1.

[BD05] Eric Bedford and Jeffrey Diller. « Energy and invariant measures for birational surface maps ». In: *Duke Math. J.* 128.2 (2005), pp. 331–368. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-04-12824-6. URL: https://doi.org/10.1215/S0012-7094-04-12824-6.

[BD15] Jérémy Blanc and Julie Déserti. « Degree growth of birational maps of the plane ». In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 14.2 (2015), pp. 507–533. ISSN: 0391-173X.

[Bea07] Arnaud Beauville. « p-elementary subgroups of the Cremona group ». In: *J. Algebra* 314.2 (2007), pp. 553–564. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2005.07.040. URL: https://doi.org/10.1016/j.jalgebra.2005.07.040.

[Bea96] Arnaud Beauville. *Complex algebraic surfaces*. Second. Vol. 34. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1996, pp. x+132. ISBN: 0-521-49510-5; 0-521-49842-2. DOI: 10.1017/CBO9780511623936. URL: http://dx.doi.org/10.1017/CBO9780511623936.

[Ben94] Yves Benoist. « Nilvariétés projectives. » French. In: *Comment. Math. Helv.* 69.3 (1994), pp. 447–473. ISSN: 0010-2571; 1420-8946/e.

[Ber60] Lipman Bers. « Simultaneous uniformization ». In: *Bull. Amer. Math. Soc.* 66 (1960), pp. 94–97. ISSN: 0002-9904. DOI: 10.1090/S0002-9904-1960-10413-2. URL: https://doi.org/10.1090/S0002-9904-1960-10413-2.

[Ber70] Lipman Bers. « On boundaries of Teichmüller spaces and on Kleinian groups. I ». In: *Ann. of Math. (2)* 91 (1970), pp. 570–600. ISSN: 0003-486X. DOI: 10.2307/1970638. URL: https://doi.org/10.2307/1970638.

[Ber73] Lipman Bers. « Fiber spaces over Teichmüller spaces ». In: *Acta. Math.* 130 (1973), pp. 89–126. ISSN: 0001-5962. DOI: 10.1007/BF02392263. URL: https://doi.org/10.1007/BF02392263.

[BF13] Jérémy Blanc and Jean-Philippe Furter. « Topologies and structures of the Cremona groups ». In: *Ann. of Math. (2)* 178.3 (2013), pp. 1173–1198. ISSN: 0003-486X. DOI: 10.4007/annals.2013.178.3.8. URL: https://doi.org/10.4007/annals.2013.178.3.8.

[Bis04] Cinzia Bisi. « On commuting polynomial automorphisms of $\mathbb{C}^2$ ». In: *Publ. Mat.* 48.1 (2004), pp. 227–239. ISSN: 0214-1493. DOI: 10.5565/PUBLMAT_48104_10. URL: https://doi.org/10.5565/PUBLMAT_48104_10.

[Bis08] Cinzia Bisi. « On commuting polynomial automorphisms of $\mathbb{C}^k$, $k \geq 3$ ». In: *Math. Z.* 258.4 (2008), pp. 875–891. ISSN: 0025-5874. DOI: 10.1007/s00209-007-0203-7. URL: https://doi.org/10.1007/s00209-007-0203-7.

[Bla07] Jérémy Blanc. « Finite abelian subgroups of the Cremona group of the plane ». In: *C. R. Math. Acad. Sci. Paris* 344.1 (2007), pp. 21–26. ISSN: 1631-073X. DOI: 10.1016/j.crma.2006.11.026. URL: https://doi.org/10.1016/j.crma.2006.11.026.

[Bla11] Jérémy Blanc. « Elements and cyclic subgroups of finite order of the Cremona group ». In: *Comment. Math. Helv.* 86.2 (2011), pp. 469–497. ISSN: 0010-2571. DOI: 10.4171/CMH/231. URL: https://doi.org/10.4171/CMH/231.

[BN06] Kai Behrend and Behrang Noohi. « Uniformization of Deligne-Mumford curves. » English. In: *J. Reine Angew. Math.* 599 (2006), pp. 111–153. ISSN: 0075-4102; 1435-5345/e.

[Bon86] Francis Bonahon. « Bouts des variétés hyperboliques de dimension 3 ». In: *Ann. of Math. (2)* 124.1 (1986), pp. 71–158. ISSN: 0003-486X. DOI: 10.2307/1971388. URL: https://doi.org/10.2307/1971388.

[BR86] Lipman Bers and H. L. Royden. « Holomorphic families of injections ». In: *Acta Math.* 157.3-4 (1986), pp. 259–286. ISSN: 0001-5962. DOI: 10.1007/BF02392595. URL: https://doi.org/10.1007/BF02392595.

[Bro78] Robert Brody. « Compact manifolds and hyperbolicity ». In: *Trans. Amer. Math. Soc.* 235 (1978), pp. 213–219. ISSN: 0002-9947. DOI: 10.2307/1998216. URL: https://doi.org/10.2307/1998216.

[Bru03] Marco Brunella. « Subharmonic variation of the leafwise Poincaré metric ». In: *Invent. Math.* 152.1 (2003), pp. 119–148. ISSN: 0020-9910. DOI: 10.1007/s00222-002-0269-0. URL: https://doi.org/10.1007/s00222-002-0269-0.

[Bru15] Marco Brunella. *Birational geometry of foliations*. Vol. 1. IMPA Monographs. Springer, Cham, 2015, pp. xiv+130. ISBN: 978-3-319-14309-5; 978-3-319-14310-1. DOI: 10.1007/978-3-319-14310-1. URL: https://doi.org/10.1007/978-3-319-14310-1.

[Bru97] Marco Brunella. « Feuilletages holomorphes sur les surfaces complexes compactes ». In: *Ann. Sci. École Norm. Sup. (4)* 30.5 (1997), pp. 569–594. ISSN: 0012-9593. DOI: 10.1016/S0012-9593(97)89932-6. URL: http://dx.doi.org/10.1016/S0012-9593(97)89932-6.

[BY] Jérémy Blanc and Egor Yasinsky. « Quotients of groups of birational transformations of cubic del Pezzo fibrations ». In: *preprint, https://arxiv.org/abs/1907.04696* (). URL: https://arxiv.org/abs/1907.04696.

[Can01a] Serge Cantat. « Dynamique des automorphismes des surfaces $K3$ ». In: *Acta Math.* 187.1 (2001), pp. 1–57. ISSN: 0001-5962. URL: https://doi.org/10.1007/BF02392831.

[Can01b] Serge Cantat. « Sur la dynamique du groupe d'automorphismes des surfaces $K3$ ». In: *Transform. Groups* 6.3 (2001), pp. 201–214. ISSN: 1083-4362. DOI: 10.1007/BF01263089. URL: https://doi.org/10.1007/BF01263089.

[Can10] Serge Cantat. « Invariant hypersurfaces in holomorphic dynamics ». In: *Math. Res. Lett.* 17.5 (2010), pp. 833–841. ISSN: 1073-2780. DOI: 10.4310/MRL.2010.v17.n5.a3. URL: https://doi.org/10.4310/MRL.2010.v17.n5.a3.

[Can11] Serge Cantat. « Sur les groupes de transformations birationnelles des surfaces ». In: *Ann. of Math. (2)* 174.1 (2011), pp. 299–340. ISSN: 0003-486X. DOI: 10.4007/annals.2011.174.1.8. URL: http://dx.doi.org/10.4007/annals.2011.174.1.8.

[Can14] Serge Cantat. « Morphisms between Cremona groups, and characterization of rational varieties ». In: *Compos. Math.* 150.7 (2014), pp. 1107–1124. ISSN: 0010-437X. DOI: 10.1112/S0010437X13007835. URL: http://dx.doi.org/10.1112/S0010437X13007835.

[Can18] Serge Cantat. « The Cremona group ». In: *Algebraic geometry: Salt Lake City 2015*. Vol. 97. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2018, pp. 101–142.

[CC19] Serge Cantat and Yves de Cornulier. « Commensurating actions of birational groups and groups of pseudo-automorphisms ». In: *J. Éc. polytech. Math.* 6 (2019), pp. 767–809. ISSN: 2429-7100. DOI: 10.5802/jep.106. URL: https://doi.org/10.5802/jep.106.

[CCE15] Fréderic Campana, Benoît Claudon, and Philippe Eyssidieux. « Représentations linéaires des groupes kählériens: factorisations et conjecture de Shafarevich linéaire ». In: *Compos. Math.* 151.2 (2015), pp. 351–376. ISSN: 0010-437X. DOI: 10.1112/S0010437X14007751. URL: https://doi.org/10.1112/S0010437X14007751.

[CD12a] Serge Cantat and Igor Dolgachev. « Rational surfaces with a large group of automorphisms ». In: *J. Amer. Math. Soc.* 25.3 (2012), pp. 863–905. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-2012-00732-2. URL: https://doi.org/10.1090/S0894-0347-2012-00732-2.

[CD12b] Dominique Cerveau and Julie Déserti. « Centralisateurs dans le groupe de Jonquières ». In: *Michigan Math. J.* 61.4 (2012), pp. 763–783. ISSN: 0026-2285. DOI:

10.1307/mmj/1353098512. URL: https://doi.org/10.1307/mmj/1353098512.

[CF03] Serge Cantat and Charles Favre. « Symétries birationnelles des surfaces feuilletées ». In: *J. Reine Angew. Math.* 561 (2003), pp. 199–235. ISSN: 0075-4102. DOI: 10.1515/crll.2003.066. URL: https://doi.org/10.1515/crll.2003.066.

[CG06] Danny Calegari and David Gabai. « Shrinkwrapping and the taming of hyperbolic 3-manifolds ». In: *J. Amer. Math. Soc.* 19.2 (2006), pp. 385–446. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-05-00513-8. URL: https://doi.org/10.1090/S0894-0347-05-00513-8.

[CGL] Serge Cantat, Vincent Guirardel, and Anne Lonjou. « Generators of proper normal subgroups of the Cremona group ». In: *preprint* ().

[CHK13] Benoît Claudon, Andreas Höring, and János Kollár. « Algebraic varieties with quasi-projective universal cover ». In: *J. Reine Angew. Math.* 679 (2013), pp. 207–221. ISSN: 0075-4102.

[Cla] Benoît Claudon. « Non algebraicity of universal covers of Kähler surfaces ». In: *https://arxiv.org/abs/1001.2379* (). arXiv: 1001.2379.

[CNS13] Angel Cano, Juan Pablo Navarrete, and José Seade. *Complex Kleinian groups*. Vol. 303. Progress in Mathematics. Birkhäuser/Springer Basel AG, Basel, 2013, pp. xx+271. ISBN: 978-3-0348-0480-6; 978-3-0348-0481-3. DOI: 10.1007/978-3-0348-0481-3. URL: https://doi.org/10.1007/978-3-0348-0481-3.

[CO75] Bang-Yen Chen and Koichi Ogiue. « Some characterizations of complex space forms in terms of Chern classes. » English. In: *Q. J. Math., Oxf. II. Ser.* 26 (1975), pp. 459–464. ISSN: 0033-5606; 1464-3847/e.

[Cor88] Kevin Corlette. « Flat G-bundles with canonical metrics ». In: *J. Differential Geom.* 28.3 (1988), pp. 361–382. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214442469.

[CS08] Kevin Corlette and Carlos Simpson. « On the classification of rank-two representations of quasiprojective fundamental groups ». In: *Compos. Math.* 144.5 (2008), pp. 1271–1331. ISSN: 0010-437X. DOI: 10.1112/S0010437X08003618. URL: https://doi.org/10.1112/S0010437X08003618.

[CS14] Angel Cano and José Seade. « On discrete groups of automorphisms of $\mathbb{P}^2_{\mathbb{C}}$ ». In: *Geom. Dedicata* 168 (2014), pp. 9–60. ISSN: 0046-5755. DOI: 10.1007/s10711-012-9816-z. URL: https://doi.org/10.1007/s10711-012-9816-z.

[CT89] James A. Carlson and Domingo Toledo. « Harmonic mappings of Kähler manifolds to locally symmetric spaces ». In: *Inst. Hautes Études Sci. Publ. Math.* (1989), pp. 173–201. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1989__69__173_0.

[CT97] James A. Carlson and Domingo Toledo. « On fundamental groups of class VII surfaces ». In: *Bull. London Math. Soc.* 29.1 (1997), pp. 98–102. ISSN: 0024-6093. DOI: 10.1112/S0024609396001774. URL: https://doi.org/10.1112/S0024609396001774.

[CX18] Serge Cantat and Junyi Xie. « Algebraic actions of discrete groups: the p-adic method ». In: *Acta Math.* 220.2 (2018), pp. 239–295. ISSN: 0001-5962. DOI: 10.4310/ACTA.2018.v220.n2.a2. URL: https://doi.org/10.4310/ACTA.2018.v220.n2.a2.

[Deb01] Olivier Debarre. *Higher-dimensional algebraic geometry.* English. New York, NY: Springer, 2001, pp. xiii + 233. ISBN: 0-387-95227-6/hbk.

[Dem70] Michel Demazure. « Sous-groupes algébriques de rang maximum du groupe de Cremona ». In: *Ann. Sci. École Norm. Sup. (4)* 3 (1970), pp. 507–588. ISSN: 0012-9593. URL: http://www.numdam.org/item?id=ASENS_1970_4_3_4_507_0.

[Dem92] Jean-Pierre Demailly. « Singular Hermitian metrics on positive line bundles ». In: *Complex algebraic varieties (Bayreuth, 1990)*. Vol. 1507. Lecture Notes in Math. Springer, Berlin, 1992, pp. 87–104. DOI: 10.1007/BFb0094512. URL: https://doi.org/10.1007/BFb0094512.

[Dem97] Jean-Pierre Demailly. « Variétés hyperboliques et équations différentielles algébriques ». In: *Gaz. Math.* 73 (1997), pp. 3–23. ISSN: 0224-8999.

[Der05] Bertrand Deroin. « Hypersurfaces Levi-plates immergées dans les surfaces complexes de courbure positive ». In: *Ann. Sci. École Norm. Sup. (4)* 38.1 (2005), pp. 57–75. ISSN: 0012-9593. DOI: 10.1016/j.ansens.2004.10.004. URL: https://doi.org/10.1016/j.ansens.2004.10.004.

[Dés06a] Julie Déserti. « Groupe de Cremona et dynamique complexe: une approche de la conjecture de Zimmer ». In: *Int. Math. Res. Not.* (2006), Art. ID 71701, 27. ISSN: 1073-7928. DOI: `10.1155/IMRN/2006/71701`. URL: `https://doi.org/10.1155/IMRN/2006/71701`.

[Dés06b] Julie Déserti. « Sur les automorphismes du groupe de Cremona ». In: *Compos. Math.* 142.6 (2006), pp. 1459–1478. ISSN: 0010-437X. DOI: `10.1112/S0010437X06002478`. URL: `https://doi.org/10.1112/S0010437X06002478`.

[Dés08] Julie Déserti. « Expériences sur certaines transformations birationnelles quadratiques ». In: *Nonlinearity* 21.6 (2008), pp. 1367–1383. ISSN: 0951-7715. DOI: `10.1088/0951-7715/21/6/013`. URL: `https://doi.org/10.1088/0951-7715/21/6/013`.

[Dés15] Julie Déserti. « On solvable subgroups of the Cremona group ». In: *Illinois J. Math.* 59.2 (2015), pp. 345–358. ISSN: 0019-2082. URL: `http://projecteuclid.org/euclid.ijm/1462450705`.

[DF01] Jeffrey Diller and Charles Favre. « Dynamics of bimeromorphic maps of surfaces ». In: *Amer. J. Math.* 123.6 (2001), pp. 1135–1169. ISSN: 0002-9327. URL: `http://muse.jhu.edu/journals/american_journal_of_mathematics/v123/123.6diller.pdf`.

[DI09] Igor V. Dolgachev and Vasilii A. Iskovskikh. « Finite subgroups of the plane Cremona group ». In: *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I.* Vol. 269. Progr. Math. Birkhäuser Boston, Inc., Boston, MA, 2009, pp. 443–548. DOI: `10.1007/978-0-8176-4745-2_11`. URL: `https://doi.org/10.1007/978-0-8176-4745-2_11`.

[Dia] Adjaratou Arame Diaw. « Geometry of stable ruled surface over an elliptic curve ». In: *preprint, https://arxiv.org/abs/1903.00373* (). arXiv: `1903.00373`.

[Dlo16] Georges Dloussky. « Special birational structures on non-Kählerian complex surfaces ». In: *J. Math. Pures Appl. (9)* 106.1 (2016), pp. 76–122. ISSN: 0021-7824. DOI: `10.1016/j.matpur.2016.02.002`. URL: `https://doi.org/10.1016/j.matpur.2016.02.002`.

[Don87] Simon K. Donaldson. « Twisted harmonic maps and the self-duality equations. » English. In: *Proc. Lond. Math. Soc. (3)* 55 (1987), pp. 127–131. ISSN: 0024-6115; 1460-244X/e.

[Dou95] Adrien Douady. « Prolongement de mouvements holomorphes (d'après Słodkowski et autres) ». In: 227. Séminaire Bourbaki, Vol. 1993/94. 1995, Exp. No. 775, 3, 7–20.

[DP12] Thomas Delzant and Pierre Py. « Kähler groups, real hyperbolic spaces and the Cremona group ». In: *Compos. Math.* 148.1 (2012), pp. 153–184. ISSN: 0010-437X. DOI: 10.1112/S0010437X11007068. URL: http://dx.doi.org/10.1112/S0010437X11007068.

[DS05] Tien-Cuong Dinh and Nessim Sibony. « Green currents for holomorphic automorphisms of compact Kähler manifolds ». In: *J. Amer. Math. Soc.* 18.2 (2005), pp. 291–312. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-04-00474-6. URL: https://doi.org/10.1090/S0894-0347-04-00474-6.

[DS10] Tien-Cuong Dinh and Nessim Sibony. « Super-potentials for currents on compact Kähler manifolds and dynamics of automorphisms ». In: *J. Algebraic Geom.* 19.3 (2010), pp. 473–529. ISSN: 1056-3911. DOI: 10.1090/S1056-3911-10-00549-7. URL: https://doi.org/10.1090/S1056-3911-10-00549-7.

[Duj06] Romain Dujardin. « Laminar currents and birational dynamics ». In: *Duke Math. J.* 131.2 (2006), pp. 219–247. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-06-13122-8. URL: https://doi.org/10.1215/S0012-7094-06-13122-8.

[É81] Jean Écalle. *Les fonctions résurgentes. Tome II.* Vol. 6. Publications Mathématiques d'Orsay 81 [Mathematical Publications of Orsay 81]. Les fonctions résurgentes appliquées à l'itération. [Resurgent functions applied to iteration]. Université de Paris-Sud, Département de Mathématique, Orsay, 1981, pp. 248–531.

[EE67] Clifford J. Earle and James Eells Jr. « On the differential geometry of Teichmüller spaces ». In: *J. Analyse Math.* 19 (1967), pp. 35–52. ISSN: 0021-7670. DOI: 10.1007/BF02788708. URL: https://doi.org/10.1007/BF02788708.

[EF85] Clifford J. Earle and Robert S. Fowler. « Holomorphic families of open Riemann surfaces ». In: *Math. Ann.* 270.2 (1985), pp. 249–273. ISSN: 0025-5831. DOI: 10.1007/BF01456185. URL: https://doi.org/10.1007/BF01456185.

[EG96] Clifford J. Earle and Frederick P. Gardiner. « Geometric isomorphisms between infinite-dimensional Teichmüller spaces ». In: *Trans. Amer. Math. Soc.* 348.3 (1996), pp. 1163–1190. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-96-01490-0. URL: https://doi.org/10.1090/S0002-9947-96-01490-0.

[Ehr36] Charles Ehresmann. « Sur les espaces localement homogènes ». In: *Enseign. Math.* 35 (1936), pp. 317–333.

[EK74] Clifford J. Earle and Irwin Kra. « On isometries between Teichmüller spaces ». In: *Duke Math. J.* 41 (1974), pp. 583–591. ISSN: 0012-7094. URL: http://projecteuclid.org/euclid.dmj/1077310579.

[EM00] Clifford J. Earle and Sudeb Mitra. « Variation of moduli under holomorphic motions ». In: *In the tradition of Ahlfors and Bers (Stony Brook, NY, 1998)*. Vol. 256. Contemp. Math. Amer. Math. Soc., Providence, RI, 2000, pp. 39–67. DOI: 10.1090/conm/256/03996. URL: https://doi.org/10.1090/conm/256/03996.

[Enr93] Federigo Enriques. « Sui gruppi continui di trasformazioni cremoniane nel piano ». In: *Rom. Acc. L. Rend.* (5) II_1 (1893), pp. 468–473.

[ES64] James Eells Jr. and Joseph H. Sampson. « Harmonic mappings of Riemannian manifolds ». In: *Amer. J. Math.* 86 (1964), pp. 109–160. ISSN: 0002-9327. DOI: 10.2307/2373037. URL: https://doi.org/10.2307/2373037.

[Eys04] Philippe Eyssidieux. « Sur la convexité holomorphe des revêtements linéaires réductifs d'une variété projective algébrique complexe ». In: *Invent. Math.* 156.3 (2004), pp. 503–564. ISSN: 0020-9910. DOI: 10.1007/s00222-003-0345-0. URL: https://doi.org/10.1007/s00222-003-0345-0.

[Eys+12] Philippe Eyssidieux et al. « Linear Shafarevich conjecture ». In: *Ann. of Math. (2)* 176.3 (2012), pp. 1545–1581. ISSN: 0003-486X. DOI: 10.4007/annals.2012.176.3.4. URL: https://doi.org/10.4007/annals.2012.176.3.4.

[Fav10] Charles Favre. « Le groupe de Cremona et ses sous-groupes de type fini ». In: *Astérisque* Volume 2008/2009.332 (2010). Séminaire Bourbaki. Exposés 997–1011, Exp. No. 998, vii, 11–43. ISSN: 0303-1179.

[FM89] Shmuel Friedland and John Milnor. « Dynamical properties of plane polynomial automorphisms ». In: *Ergodic Theory Dynam. Systems* 9.1 (1989), pp. 67–99. ISSN: 0143-3857. DOI: 10.1017/S014338570000482X. URL: https://doi.org/10.1017/S014338570000482X.

[Fra89] Sidney Frankel. « Complex geometry of convex domains that cover varieties ». In: *Acta Math.* 163.1-2 (1989), pp. 109–149. ISSN: 0001-5962. DOI: 10.1007/BF02392734. URL: https://doi.org/10.1007/BF02392734.

[Fra95] Sidney Frankel. « Locally symmetric and rigid factors for complex manifolds via harmonic maps ». In: *Ann. of Math. (2)* 141.2 (1995), pp. 285–300. ISSN: 0003-486X. DOI: 10.2307/2118521. URL: https://doi.org/10.2307/2118521.

[FS08] John Erik Fornæss and Nessim Sibony. « Riemann surface laminations with singularities ». In: *J. Geom. Anal.* 18.2 (2008), pp. 400–442. ISSN: 1050-6926. DOI: 10.1007/s12220-008-9018-y. URL: https://doi.org/10.1007/s12220-008-9018-y.

[Ghy99] Étienne Ghys. « Laminations par surfaces de Riemann ». In: *Dynamique et géométrie complexes (Lyon, 1997)*. Vol. 8. Panor. Synthèses. Soc. Math. France, Paris, 1999, pp. ix, xi, 49–95.

[Giz80] M. H. Gizatullin. « Rational G-surfaces ». In: *Izv. Akad. Nauk SSSR Ser. Mat.* 44.1 (1980), pp. 110–144, 239. ISSN: 0373-2436.

[Gri71] Phillip A. Griffiths. « Complex-analytic properties of certain Zariski open sets on algebraic varieties ». In: *Ann. of Math. (2)* 94 (1971), pp. 21–51. ISSN: 0003-486X. DOI: 10.2307/1970733. URL: https://doi.org/10.2307/1970733.

[Gro89] Mikhail Gromov. « Oka's principle for holomorphic sections of elliptic bundles. » English. In: *J. Am. Math. Soc.* 2.4 (1989), pp. 851–897. ISSN: 0894-0347; 1088-6834/e.

[GS92] Mikhail Gromov and Richard Schoen. « Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one ». In: *Inst. Hautes Études Sci. Publ. Math.* 76 (1992), pp. 165–246. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1992__76__165_0.

[Gun67] Robert C. Gunning. « Special coordinate coverings of Riemann surfaces ». In: *Math. Ann.* 170 (1967), pp. 67–86. ISSN: 0025-5831. DOI: 10.1007/BF01362287. URL: https://doi.org/10.1007/BF01362287.

[Gun78] Robert C. Gunning. *On uniformization of complex manifolds: The role of connections.* English. Mathematical Notes. Princeton, New Jersey: Princeton University Press. Tokyo: University of Tokyo Press. II, 141 p. $ 9.50 (1978). 1978.

[Har77] Robin Hartshorne. *Algebraic geometry.* Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9.

[Har78] William James Harvey. « Kleinian groups (a survey) ». In: *Séminaire Bourbaki, 29e année (1976/77)*. Vol. 677. Lecture Notes in Math. Springer, Berlin, 1978, Exp. No. 491, pp. 30–45.

[Hin85] Aimo Hinkkanen. « Uniformly quasisymmetric groups ». In: *Proc. London Math. Soc. (3)* 51.2 (1985), pp. 318–338. ISSN: 0024-6115. DOI: 10.1112/plms/s3-51.2.318. URL: https://doi.org/10.1112/plms/s3-51.2.318.

[Hin90] Aimo Hinkkanen. « The structure of certain quasisymmetric groups ». In: *Mem. Amer. Math. Soc.* 83.422 (1990), pp. iv+87. ISSN: 0065-9266. DOI: 10.1090/memo/0422. URL: https://doi.org/10.1090/memo/0422.

[HP04] Michael Heusener and Joan Porti. « The variety of characters in $\mathrm{PSL}_2(\mathbb{C})$ ». In: *Bol. Soc. Mat. Mexicana (3)* 10.Special Issue (2004), pp. 221–237. ISSN: 1405-213X.

[HS06] Yun Hu and Yuliang Shen. « Isomorphisms of fiber spaces over Teichmüller spaces ». In: *Sci. China Ser. A* 49.7 (2006), pp. 877–892. ISSN: 1006-9283. DOI: 10.1007/s11425-006-2001-9. URL: https://doi.org/10.1007/s11425-006-2001-9.

[Huy05] Daniel Huybrechts. *Complex geometry. An introduction.* English. Berlin: Springer, 2005, pp. xii + 309. ISBN: 3-540-21290-6/pbk.

[Ino74] Masahisa Inoue. « On surfaces of Class VII_0 ». In: *Invent. Math.* 24 (1974), pp. 269–310. ISSN: 0020-9910. DOI: 10.1007/BF01425563. URL: https://doi.org/10.1007/BF01425563.

[Isk79] Vasilii A. Iskovskih. « Minimal models of rational surfaces over arbitrary fields ». In: *Izv. Akad. Nauk SSSR Ser. Mat.* 43.1 (1979), pp. 19–43, 237. ISSN: 0373-2436.

[JR15] Priska Jahnke and Ivo Radloff. « Projective uniformization, extremal Chern classes and quaternionic Shimura curves ». In: *Math. Ann.* 363.3-4 (2015), pp. 753–776. ISSN: 0025-5831. DOI: 10.1007/s00208-015-1185-0. URL: https://doi.org/10.1007/s00208-015-1185-0.

[JY93] Jürgen Jost and Shing-Tung Yau. « A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry ». In: *Acta Math.* 170.2 (1993), pp. 221–254. ISSN: 0001-5962. DOI: 10.1007/BF02392786. URL: https://doi.org/10.1007/BF02392786.

[Kli01] Bruno Klingler. « Un théorème de rigidité non-métrique pour les variétés localement symétriques hermitiennes ». In: *Comment. Math. Helv.* 76.2 (2001), pp. 200–217. ISSN: 0010-2571. DOI: 10.1007/s00014-001-8320-0. URL: https://doi.org/10.1007/s00014-001-8320-0.

[Kli98] Bruno Klingler. « Structures affines et projectives sur les surfaces complexes ». In: *Ann. Inst. Fourier (Grenoble)* 48.2 (1998), pp. 441–477. ISSN: 0373-0956. URL: http://www.numdam.org/item?id=AIF_1998__48_2_441_0.

[KM58] Jean-Louis Koszul and Bernard Malgrange. « Sur certaines structures fibrées complexes. » French. In: *Arch. Math.* 9 (1958), pp. 102–109. ISSN: 0003-889X; 1420-8938/e.

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties. With the collaboration of C. H. Clemens and A. Corti.* English. Vol. 134. Cambridge: Cambridge University Press, 1998, pp. viii + 254. ISBN: 0-521-63277-3/hbk.

[KO80] Shoshichi Kobayashi and Takushiro Ochiai. « Holomorphic projective structures on compact complex surfaces ». In: *Math. Ann.* 249.1 (1980), pp. 75–94. ISSN: 0025-5831. DOI: 10.1007/BF01387081. URL: https://doi.org/10.1007/BF01387081.

[Kob98] Shoshichi Kobayashi. *Hyperbolic complex spaces*. Vol. 318. Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin, 1998, pp. xiv+471. ISBN: 3-540-63534-3. DOI: 10.1007/978-3-662-03582-5. URL: https://doi.org/10.1007/978-3-662-03582-5.

[Kod66] Kunihiko Kodaira. « On the structure of compact complex analytic surfaces. II ». In: *Amer. J. Math.* 88 (1966), pp. 682–721. ISSN: 0002-9327. DOI: 10.2307/2373150. URL: https://doi.org/10.2307/2373150.

[KP12] János Kollár and John Pardon. « Algebraic varieties with semialgebraic universal cover ». In: *J. Topol.* 5.1 (2012), pp. 199–212. ISSN: 1753-8416. DOI: 10.1112/jtopol/jts001. URL: https://doi.org/10.1112/jtopol/jts001.

[KR98] Ludmil Katzarkov and Mahan Ramachandran. « On the universal coverings of algebraic surfaces ». In: *Ann. Sci. École Norm. Sup. (4)* 31.4 (1998), pp. 525–535. ISSN: 0012-9593. DOI: 10.1016/S0012-9593(98)80105-5. URL: https://doi.org/10.1016/S0012-9593(98)80105-5.

[KS] Alice Kwon and Dennis Sullivan. « Geometry on all prime Three Manifolds ». In: *preprint, https://arxiv.org/abs/1906.10820* (). URL: `https://arxiv.org/abs/1906.10820`.

[KS93] Nicholas J. Korevaar and Richard M. Schoen. « Sobolev spaces and harmonic maps for metric space targets. » English. In: *Commun. Anal. Geom.* 1.4 (1993), pp. 561–659. ISSN: 1019-8385; 1944-9992/e.

[KS97] Nicholas J. Korevaar and Richard M. Schoen. « Global existence theorems for harmonic maps to non-locally compact spaces ». In: *Comm. Anal. Geom.* 5.2 (1997), pp. 333–387. ISSN: 1019-8385. DOI: `10.4310/CAG.1997.v5.n2.a4`. URL: `https://doi.org/10.4310/CAG.1997.v5.n2.a4`.

[Kul78] Ravindra S. Kulkarni. « Groups with domains of discontinuity ». In: *Math. Ann.* 237.3 (1978), pp. 253–272. ISSN: 0025-5831. DOI: `10.1007/BF01420180`. URL: `https://doi.org/10.1007/BF01420180`.

[Lab91] François Labourie. « Existence d'applications harmoniques tordues à valeurs dans les variétés à courbure négative. » French. In: *Proc. Am. Math. Soc.* 111.3 (1991), pp. 877–882. ISSN: 0002-9939; 1088-6826/e.

[Lak97] Nikola Lakic. « An isometry theorem for quadratic differentials on Riemann surfaces of finite genus ». In: *Trans. Amer. Math. Soc.* 349.7 (1997), pp. 2951–2967. ISSN: 0002-9947. DOI: `10.1090/S0002-9947-97-01771-6`. URL: `https://doi.org/10.1090/S0002-9947-97-01771-6`.

[Lam01a] Stéphane Lamy. « L'alternative de Tits pour Aut$[\mathbb{C}^2]$ ». In: *J. Algebra* 239.2 (2001), pp. 413–437. ISSN: 0021-8693. DOI: `10.1006/jabr.2000.8701`. URL: `https://doi.org/10.1006/jabr.2000.8701`.

[Lam01b] Stéphane Lamy. « L'alternative de Tits pour Aut$[\mathbb{C}^2]$ ». In: *J. Algebra* 239.2 (2001), pp. 413–437. ISSN: 0021-8693. DOI: `10.1006/jabr.2000.8701`. URL: `https://doi.org/10.1006/jabr.2000.8701`.

[Leh87] Olli Lehto. *Univalent functions and Teichmüller spaces*. Vol. 109. Graduate Texts in Mathematics. Springer-Verlag, New York, 1987, pp. xii+257. ISBN: 0-387-96310-3. DOI: `10.1007/978-1-4613-8652-0`. URL: `https://doi.org/10.1007/978-1-4613-8652-0`.

[LMP09] Frank Loray and David Marín Pérez. « Projective structures and projective bundles over compact Riemann surfaces ». In: *Astérisque* 323 (2009), pp. 223–252. ISSN: 0303-1179.

[LPT16] Frank Loray, Jorge Vitório Pereira, and Frédéric Touzet. « Representations of quasi-projective groups, flat connections and transversely projective foliations ». In: *J. Éc. polytech. Math.* 3 (2016), pp. 263–308. ISSN: 2429-7100. DOI: `10.5802/jep.34`. URL: `https://doi.org/10.5802/jep.34`.

[Lyu83] Mikhail Yu. Lyubich. « Some typical properties of the dynamics of rational mappings ». In: *Uspekhi Mat. Nauk* 38.5(233) (1983), pp. 197–198. ISSN: 0042-1316.

[Man67] Juri I. Manin. « Rational surfaces over perfect fields. II ». In: *Mat. Sb. (N.S.)* 72 (114) (1967), pp. 161–192.

[Man86] Yuri I. Manin. *Cubic forms*. Second. Vol. 4. North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam, 1986, pp. x+326. ISBN: 0-444-87823-8.

[Mar03] Vladimir Markovic. « Biholomorphic maps between Teichmüller spaces ». In: *Duke Math. J.* 120.2 (2003), pp. 405–431. ISSN: 0012-7094. DOI: `10.1215/S0012-7094-03-12028-1`. URL: `https://doi.org/10.1215/S0012-7094-03-12028-1`.

[Mar06] Vladimir Markovic. « Quasisymmetric groups ». In: *J. Amer. Math. Soc.* 19.3 (2006), pp. 673–715. ISSN: 0894-0347. DOI: `10.1090/S0894-0347-06-00518-2`. URL: `https://doi.org/10.1090/S0894-0347-06-00518-2`.

[Mar71] Masaki Maruyama. « On automorphism groups of ruled surfaces ». In: *J. Math. Kyoto Univ.* 11 (1971), pp. 89–112. ISSN: 0023-608X. DOI: `10.1215/kjm/1250523688`. URL: `http://dx.doi.org/10.1215/kjm/1250523688`.

[Mas70] Bernard Maskit. « On boundaries of Teichmüller spaces and on Kleinian groups. II ». In: *Ann. of Math. (2)* 91 (1970), pp. 607–639. ISSN: 0003-486X. DOI: `10.2307/1970640`. URL: `https://doi.org/10.2307/1970640`.

[Mas88] Bernard Maskit. *Kleinian groups*. Vol. 287. Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin, 1988, pp. xiv+326. ISBN: 3-540-17746-9.

[Mat12] John Mather. « Notes on topological stability. » English. In: *Bull. Am. Math. Soc., New Ser.* 49.4 (2012), pp. 475–506. ISSN: 0273-0979; 1088-9485/e.

[McM07] Curtis T. McMullen. « Foliations of Hilbert modular surfaces ». In: *Amer. J. Math.* 129.1 (2007), pp. 183–215. ISSN: 0002-9327. DOI: 10.1353/ajm.2007.0002. URL: https://doi.org/10.1353/ajm.2007.0002.

[McQ08] Michael McQuillan. « Canonical models of foliations ». In: *Pure Appl. Math. Q.* 4.3, Special Issue: In honor of Fedor Bogomolov. Part 2 (2008), pp. 877–1012. ISSN: 1558-8599. DOI: 10.4310/PAMQ.2008.v4.n3.a9. URL: https://doi.org/10.4310/PAMQ.2008.v4.n3.a9.

[McQ98] Michael McQuillan. « Diophantine approximations and foliations ». In: *Inst. Hautes Études Sci. Publ. Math.* 87 (1998), pp. 121–174. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1998__87__121_0.

[Mj14] Mahan Mj. « Cannon-Thurston maps for surface groups ». In: *Ann. of Math. (2)* 179.1 (2014), pp. 1–80. ISSN: 0003-486X. DOI: 10.4007/annals.2014.179.1.1. URL: https://doi.org/10.4007/annals.2014.179.1.1.

[MnSS83] Ricardo Mañé, Paulo Sad, and Dennis Sullivan. « On the dynamics of rational maps ». In: *Ann. Sci. École Norm. Sup. (4)* 16.2 (1983), pp. 193–217. ISSN: 0012-9593. URL: http://www.numdam.org/item?id=ASENS_1983_4_16_2_193_0.

[Mok87] Ngaiming Mok. « Uniqueness theorems of Hermitian metrics of seminegative curvature on quotients of bounded symmetric domains. » English. In: *Ann. Math. (2)* 125 (1987), pp. 105–152. ISSN: 0003-486X; 1939-8980/e.

[Möl] Martin Möller. « Geometry of Teichmüller curves ». In: *Procedings of International Congress of mathematicians -2018 Rio de Janeiro*. Vol. 2, pp. 2011–2028.

[Mon12] Arnaud Moncet. « Thèse de doctorat de mathématiques ». In: *Université de Rennes 1* (2012).

[MY93] Ngaiming Mok and Sai-Kee Yeung. « Geometric realizations of uniformization of conjugates of Hermitian locally symmetric manifolds ». In: *Complex analysis and geometry*. Univ. Ser. Math. Plenum, New York, 1993, pp. 253–270.

[Nad90] Alan Michael Nadel. « Semisimplicity of the group of biholomorphisms of the universal covering of a compact complex manifold with ample canonical bundle ». In: *Ann. of Math. (2)* 132.1 (1990), pp. 193–211. ISSN: 0003-486X. DOI: 10.2307/1971521. URL: https://doi.org/10.2307/1971521.

[Nag88] Subhashis Nag. *The complex analytic theory of Teichmüller spaces*. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1988, pp. xiv+427. ISBN: 0-471-62773-9.

[NS12] Hossein Namazi and Juan Souto. « Non-realizability and ending laminations: proof of the density conjecture ». In: *Acta Math.* 209.2 (2012), pp. 323–395. ISSN: 0001-5962. DOI: 10.1007/s11511-012-0088-0. URL: https://doi.org/10.1007/s11511-012-0088-0.

[NZ09] Noboru Nakayama and De-Qi Zhang. « Building blocks of étale endomorphisms of complex projective manifolds. » English. In: *Proc. Lond. Math. Soc. (3)* 99.3 (2009), pp. 725–756. ISSN: 0024-6115; 1460-244X/e.

[Pla75] Joseph Plante. « Foliations with measure preserving holonomy ». In: *Ann. of Math. (2)* 102.2 (1975), pp. 327–361. ISSN: 0003-486X. DOI: 10.2307/1971034. URL: https://doi.org/10.2307/1971034.

[Pom92] Christian Pommerenke. *Boundary behaviour of conformal maps*. Vol. 299. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1992, pp. x+300. ISBN: 3-540-54751-7. DOI: 10.1007/978-3-662-02770-7. URL: https://doi.org/10.1007/978-3-662-02770-7.

[PSV11] Piotr Przytycki, Gabriela Schmithüsen, and Ferrán Valdez. « Veech groups of Loch Ness monsters ». In: *Ann. Inst. Fourier (Grenoble)* 61.2 (2011), pp. 673–687. ISSN: 0373-0956. DOI: 10.5802/aif.2625. URL: https://doi.org/10.5802/aif.2625.

[Rod86] Burton Rodin. « Behavior of the Riemann mapping function under complex analytic deformations of the domain ». In: *Complex Variables Theory Appl.* 5.2-4 (1986), pp. 189–195. ISSN: 0278-1077. DOI: 10.1080/17476938608814139. URL: https://doi.org/10.1080/17476938608814139.

[Ros79] Jean-Pierre Rosay. « Sur une caractérisation de la boule parmi les domaines de $\mathbf{C}^n$ par son groupe d'automorphismes ». In: *Ann. Inst. Fourier (Grenoble)* 29.4 (1979), pp. ix, 91–97. ISSN: 0373-0956. URL: http://www.numdam.org/item?id=AIF_1979__29_4_91_0.

[Roy71] Halsey L. Royden. « Automorphisms and isometries of Teichmüller space ». In: *Advances in the Theory of Riemann Surfaces (Stony Brook, 1969)*. Ann. of Math. Studies, No. 66. Princeton Univ. Press, 1971, pp. 369–383.

[Sam86] Joseph H. Sampson. « Applications of harmonic maps to Kähler geometry ». In: *Complex differential geometry and nonlinear differential equations (Brunswick, Maine, 1984)*. Vol. 49. Contemp. Math. Amer. Math. Soc., Providence, RI, 1986, pp. 125–134. DOI: 10.1090/conm/049/833809. URL: https://doi.org/10.1090/conm/049/833809.

[Sel60] Atle Selberg. « On discontinuous groups in higher-dimensional symmetric spaces ». In: *Contributions to function theory (internat. Colloq. Function Theory, Bombay, 1960)*. Tata Institute of Fundamental Research, Bombay, 1960, pp. 147–164.

[Ser05] Caroline Series. « A crash course on Kleinian groups ». In: *Rend. Istit. Mat. Univ. Trieste* 37.1-2 (2005), 1–38 (2006). ISSN: 0049-4704.

[Ser10] Jean-Pierre Serre. « Le groupe de Cremona et ses sous-groupes finis ». In: *Astérisque* Volume 2008/2009.332 (2010). Séminaire Bourbaki. Exposés 997–1011, Exp. No. 1000, vii, 75–100. ISSN: 0303-1179.

[She06] Yuliang Shen. « Biholomorphic isomorphisms between Teichmüller curves ». In: *J. Anal. Math.* 100 (2006), pp. 191–209. ISSN: 0021-7670. DOI: 10.1007/BF02916760. URL: https://doi.org/10.1007/BF02916760.

[Sim91] Carlos T. Simpson. « The ubiquity of variations of Hodge structure ». In: *Complex geometry and Lie theory (Sundance, UT, 1989)*. Vol. 53. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1991, pp. 329–348. DOI: 10.1090/pspum/053/1141208. URL: https://doi.org/10.1090/pspum/053/1141208.

[Sim92] Carlos T. Simpson. « Higgs bundles and local systems ». In: *Inst. Hautes Études Sci. Publ. Math.* 75 (1992), pp. 5–95. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1992__75__5_0.

[Siu80] Yum Tong Siu. « The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds ». In: *Ann. of Math. (2)* 112.1 (1980), pp. 73–111. ISSN: 0003-486X. DOI: 10.2307/1971321. URL: http://dx.doi.org/10.2307/1971321.

[Slo91] Zbigniew Slodkowski. « Holomorphic motions and polynomial hulls ». In: *Proc. Amer. Math. Soc.* 111.2 (1991), pp. 347–355. ISSN: 0002-9939. DOI: 10.2307/2048323. URL: https://doi.org/10.2307/2048323.

[ST86] Dennis P. Sullivan and William P. Thurston. « Extending holomorphic motions ». In: *Acta Math.* 157.3-4 (1986), pp. 243–257. ISSN: 0001-5962. DOI: 10.1007/BF02392594. URL: https://doi.org/10.1007/BF02392594.

[Sul76] Dennis Sullivan. « Cycles for the dynamical study of foliated manifolds and complex manifolds ». In: *Invent. Math.* 36 (1976), pp. 225–255. ISSN: 0020-9910. DOI: 10.1007/BF01390011. URL: https://doi.org/10.1007/BF01390011.

[Sul81] Dennis Sullivan. « On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions ». In: *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978).* Vol. 97. Ann. of Math. Stud. Princeton Univ. Press, Princeton, N.J., 1981, pp. 465–496.

[Sul85] Dennis Sullivan. « Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains ». In: *Ann. of Math. (2)* 122.3 (1985), pp. 401–418. ISSN: 0003-486X. DOI: 10.2307/1971308. URL: https://doi.org/10.2307/1971308.

[Tel94] Andrei D. Teleman. « Projectively flat surfaces and Bogomolov's theorem on class VII_0 surfaces ». In: *Internat. J. Math.* 5.2 (1994), pp. 253–264. ISSN: 0129-167X. DOI: 10.1142/S0129167X94000152. URL: https://doi.org/10.1142/S0129167X94000152.

[Thu97] William P. Thurston. *Three-dimensional geometry and topology. Vol. 1*. Vol. 35. Princeton Mathematical Series. Edited by Silvio Levy. Princeton University Press, Princeton, NJ, 1997, pp. x+311. ISBN: 0-691-08304-5.

[Tit72] Jacques Tits. « Free subgroups in linear groups ». In: *J. Algebra* 20 (1972), pp. 250–270. ISSN: 0021-8693. DOI: 10.1016/0021-8693(72)90058-0. URL: https://doi.org/10.1016/0021-8693(72)90058-0.

[Tou16] Frédéric Touzet. « On the structure of codimension 1 foliations with pseudoeffective conormal bundle ». In: *Foliation theory in algebraic geometry*. Simons Symp. Springer, Cham, 2016, pp. 157–216.

[Tuk80] Pekka Tukia. « On two-dimensional quasiconformal groups ». In: *Ann. Acad. Sci. Fenn. Ser. A I Math.* 5.1 (1980), pp. 73–78. ISSN: 0066-1953. DOI: 10.5186/aasfm.1980.0530. URL: https://doi.org/10.5186/aasfm.1980.0530.

[Uen75] Kenji Ueno. *Classification theory of algebraic varieties and compact complex spaces. Notes written in collaboration with P. Cherenack.* English. Vol. 439. Springer, Cham, 1975.

[Ure] Christian Urech. « Subgroups of elliptic elements of the Cremona group ». In: *J. Reine Angew. Math., to appear* ().

[Var84] V. S. Varadarajan. *Lie groups, Lie algebras, and their representations. Reprint of the 1974 edition.* English. Reprint of the 1974 edition. Vol. 102. New York, NY: Springer, 1984, pp. xiii + 430. ISBN: 0-387-90969-9/hbk.

[Vee89] W. A. Veech. « Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards ». In: *Invent. Math.* 97.3 (1989), pp. 553–583. ISSN: 0020-9910. DOI: 10.1007/BF01388890. URL: https://doi.org/10.1007/BF01388890.

[Vit72] Al Vitter. « Affine structures on compact complex manifolds. » English. In: *Invent. Math.* 17 (1972), pp. 231–244. ISSN: 0020-9910; 1432-1297/e.

[Wei55] André Weil. « On algebraic groups of transformations ». In: *Amer. J. Math.* 77 (1955), pp. 355–391. ISSN: 0002-9327. DOI: 10.2307/2372535. URL: https://doi.org/10.2307/2372535.

[Won77] Bun Wong. « Characterization of the unit ball in $\mathbf{C}^n$ by its automorphism group ». In: *Invent. Math.* 41.3 (1977), pp. 253–257. ISSN: 0020-9910. DOI: 10.1007/BF01403050. URL: https://doi.org/10.1007/BF01403050.

[Xie15] Junyi Xie. « Periodic points of birational transformations on projective surfaces ». In: *Duke Math. J.* 164.5 (2015), pp. 903–932. ISSN: 0012-7094. DOI: 10.1215/00127094-2877402. URL: https://doi.org/10.1215/00127094-2877402.

[Yau77] Shing-Tung Yau. « Calabi's conjecture and some new results in algebraic geometry. » English. In: *Proc. Natl. Acad. Sci. USA* 74 (1977), pp. 1798–1799. ISSN: 0027-8424; 1091-6490/e.

[Zhaa] ShengYuan Zhao. « Centralizers of elements of infinite order in plane Cremona groups ». In: *preprint, https://arxiv.org/abs/1907.11203* ().

[Zhab] ShengYuan Zhao. « Uniqueness of birational structures on Inoue surfaces ». In: *preprint, https://arxiv.org/abs/1909.13266* (). arXiv: 1909.13266.

[ZS75] Oscar Zariski and Pierre Samuel. *Commutative algebra. Vol. II*. Reprint of the 1960 edition, Graduate Texts in Mathematics, Vol. 29. Springer-Verlag, New York-Heidelberg, 1975, pp. x+414.

UNIVERSITE BRETAGNE LOIRE / MATHSTIC

UNIVERSITÉ DE RENNES 1

Titre : Groupes kleiniens birationnels).............

Mot clés : transformations birationnelles des surfaces, groupe kleinien, uniformisation, groupe de Kähler, feuilletages, espace de Teichmüller

Résumé : Dans ce mémoire de thèse je considère une généralisation des groupes kleiniens en géométrie algébrique complexe. Le probléme peut aussi être vu comme l'uniformisation des variétés projectives complexes sous une hypothèse algébrico-géométrique sur l'action du groupe de revêtement. Soient Y une variété projective complexe lisse et $U \subset Y$ un ouvert en topologie usuelle. Soit $\Gamma \subset \mathrm{Bir}(Y)$ un groupe infini de transformations birationnelles. Nous imposons les conditions suivantes sur Γ : 1) les points d'indétermination de Γ sont disjoints de U et Γ préserve U, c.-à-d. tout élément de Γ induit un difféomorphisme holomorphe de U ; 2) l'action de Γ sur U est libre, proprement discontinue et cocompacte. Nous appelons un groupe kleinien birationnel a donnée de (Y, Γ, U, X). Dans ce mémoire je donne une classification des groupes kleiniens birationnels en dimension deux. Il s'agit d'une intéraction entre les transformations birationnelles des surfaces, les groupes de Kähler, les feuilletages holomorphes sur des surfaces complexes, et les espaces de Teichmüller.

Title: Birational Kleinian groups

Keywords: birational transformations of surfaces, Kleinian groups, uniformization, Kähler groups, foliations, Teichmüller spaces

Abstract: In this thesis I study a generalisation of Kleinian groups in the setting of complex algebraic geometry. The problem can also be seen as uniformization of projective varieties under an algebro-geometric hypothesis on the group of deck transformations. Let Y be a smooth complex projective variety and $U \subset Y$ an open subset in the usual topology. Let $\Gamma \subset \mathrm{Bir}(Y)$ be an infinite group of birational transformations. We impose the following conditions on Γ: 1) the indeterminacy points of Γ are disjoint from U and Γ preserves U, i.e. any element of Γ induces a holomorphic diffeomorphism of U; 2) the action of Γ on U is free, properly discontinuous and cocompact. A birational Kleinian group is by definition the data of (Y, Γ, U, X). In this thesis I give a classification of birational Kleinian groups in dimension two. It implements an interaction between birational transformations of surfaces, Kähler groups, holomorphic foliations on complex surfaces, and Teichmüller spaces.