AN APPLICATION OF p-ADIC INTEGRATION TO THE DYNAMICS OF A BIRATIONAL TRANSFORMATION PRESERVING A FIBRATION

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ABSTRACT. Let $f\colon X \dashrightarrow X$ be a birational transformation of a projective manifold X whose Kodaira dimension $\kappa(X)$ is non-negative. We show that, if there exist a meromorphic fibration $\pi\colon X \dashrightarrow B$ and a pseudo-automorphism $f_B\colon B \dashrightarrow B$ which preserves an ample line bundle $L \in \mathrm{Pic}(B)$ and such that $f_B \circ \pi = \pi \circ f$, then f_B has finite order. As a corollary we show that, for projective irreducible symplectic manifolds of type $K3^{[n]}$ or generalized Kummer, the first dynamical degree characterizes the birational transformations admitting a Zariski-dense orbit.

1. INTRODUCTION

Let $f\colon X \dashrightarrow X$ be a birational transformation of a complex projective manifold. A natural question when studying the dynamical properties of f is the existence of an equivariant meromorphic fibration, i.e. of a dominant meromorphic map with connected fibres $\pi\colon X \dashrightarrow B$ onto a projective manifold and of a birational transformation $f_B\colon B \dashrightarrow B$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, dashed, "f"] \arrow[d, dashed, "\pi"'] & M \arrow[d, dashed, "\pi"] \\ B \arrow[r, dashed, "f_B"] & B \end{tikzcd}.$$

The transformation f is called *imprimitive* (see [HKZ15]) if there exists a non-trivial f-equivariant fibration (i.e. such that $0 < \dim B < \dim X$), and primitive otherwise; imprimitive birational transformations should intuitively be simpler to study than primitive ones, as their dynamics decomposes into smaller dimensional dynamical systems: the base and the fibres. The goal of the present paper is to study the action on the base induced by an imprimitive transformation. Our main result is the following:

Theorem A. *Let X be a complex projective manifold and let $f\colon X \dashrightarrow X$ be a birational transformation. Suppose that there exist a meromorphic fibration $\pi\colon X \dashrightarrow B$ onto a projective manifold B and a pseudo-automorphism $f_B\colon B \dashrightarrow B$ such that $f_B \circ \pi = \pi \circ f$. Assume that*

(1) *the Kodaira dimension $\kappa(X)$ of X is non-negative;*
(2) *f_B preserves an ample line bundle L.*

Then f_B has finite order.

Remark 1.1. The second assumption of Theorem A is automatically verified if $\pm K_B$ is ample; remark however that, if K_B is ample, then in particular B is of general type, so that the group of birational transformations is finite, which implies the conclusion of the

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Theorem. The condition that $-K_B$ is ample is verified if X is irreducible symplectic and π is a holomorphic fibration.

The following corollary has the advantage of requiring only numerical hypothesis on the action of f_B, instead of having to compute its action on the Picard group.

Corollary B. *Let X be a projective manifold and let $f\colon X \dashrightarrow X$ be a birational transformation. Suppose that there exist a meromorphic fibration $\pi\colon X \dashrightarrow B$ onto a projective manifold B and a pseudo-automorphism $f_B\colon B \dashrightarrow B$ such that $f_B \circ \pi = \pi \circ f$. Assume that*

(1) *the Kodaira dimension $\kappa(X)$ is non-negative;*
(2) $\mathrm{Pic}^0(B) = 0$*;*
(3) *the induced linear maps $(f_B^N)^*\colon H^*(B,\mathbb{C}) \to H^*(B,\mathbb{C})$ have bounded norm as $N \to +\infty$.*

Then f_B has finite order.

Remark 1.2. Remark that the second and third assumptions of Corollary B are automatically satisfied if $\mathrm{Pic}^0(X) = 0$ and the induced linear maps $(f^N)^*\colon H^*(X,\mathbb{C}) \to H^*(X,\mathbb{C})$ have bounded norm.

Proof. Since the induced linear maps $(f_B^N)^*\colon H^*(B,\mathbb{C}) \to H^*(B,\mathbb{C})$ have bounded norm as $N \to +\infty$, by **[?]** , up to replacing B by a birational model and f_B by an iterate, we may assume that f_B is an automorphism and that $f_B \in \mathrm{Aut}^0(B)$. In particular, f_B has trivial action on $H^*(B,\mathbb{C})$ and thus, since line bundles on B are uniquely determined by their numerical class, on $\mathrm{Pic}(B)$. Therefore f_B is an automorphism which preserves an ample line bundle, hence by Theorem A it has finite order. $\square$

1.1. **The case of irreducible symplectic manifolds.** Theorem A is particularly interesting in the case where X is an irreducible symplectic manifold. A compact Kähler manifold is said to be *irreducible symplectic* (or hyperkähler) if it is simply connected and the vector space of holomorphic 2-forms is spanned by a nowhere degenerate form. Irreducible symplectic manifolds form, together with Calabi-Yau manifolds and complex tori, one of the three fundamental classes of compact Kähler manifolds with trivial Chern class.

We will say that X is of type $K3^{[n]}$ (resp. generalized Kummer) if it is deformation equivalent to the Hilbert scheme of n points on a $K3$ surface (resp. to a generalized Kummer variety); all known irreducible symplectic manifolds are of type $K3^{[n]}$, generalized Kummer or deformation equivalent to one of two sporadic examples by O'Grady. See [GHJ03] for a complete introduction to irreducible symplectic manifolds.

If $f\colon M \dashrightarrow M$ is a meromorphic transformation of a compact Kähler manifold M, for $p = 0, 1, \dots, \dim(M)$ the p-th *dynamical degree* of f is

$$\lambda_p(f) := \limsup_{N\to+\infty} ||(f^N)_p^*||^{\frac{1}{n}} \geq 1,$$

where $(f^N)_p^*\colon H^{p,p}(M) \to H^{p,p}(M)$ is the linear map induced by f^N and $||\cdot||$ is any norm on the vector space $End(H^{p,p}(M))$. Note that in the case of an automorphism, $(f^N)^* = (f^*)^N$ so that $\lambda_p(f)$ is just the spectral radius (i.e. the maximal modulus of eigenvalues) of f_p^* on $H^{p,p}(M)$; since f_p^* preserves a closed salient cone, we actually have that $\lambda_p(f)$ is a real eigenvalue of f_p^*.

Theorem 1.3 (Hu, Keum and Zhang, [HKZ15])**.** *Let X be a $2n$-dimensional projective irreducible symplectic manifold of type $K3^{[n]}$ or of type generalized Kummer and let $f \in Bir(X)$ be a birational transformation with infinite order; the first dynamical degree*

$\lambda_1(f)$ is equal to 1 if and only if there exist a rational Lagrangian fibration $\pi\colon X \dashrightarrow \mathbb{P}^n$ and $g \in Aut(\mathbb{P}^n) = PGL_{n+1}(\mathbb{C})$ such that $\pi \circ f = g \circ \pi$.

This allows to prove the following corollary:

Corollary C. *Let f be a birational transformation of a projective irreducible symplectic manifold X of type $K3^{[n]}$ or generalized Kummer; then f admits a Zariski-dense orbit if and only if the first dynamical degree $\lambda_1(f)$ is > 1.*

Proof. Assume first that $\lambda_1(f) > 1$; then by [LB, Main Theorem] the very general orbits of f are Zariski-dense.

Assume conversely that $\lambda_1(f) = 1$; by Theorem 1.3 there exists a Lagrangian fibration $\pi\colon X \dashrightarrow \mathbb{P}^n$ such that the induced transformation $g\colon \mathbb{P}^n \to \mathbb{P}^n$ is biregular, and particular preserves the ample line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. Note that, by [Huy99, Proposition 9.1], the induced linear automorphism $f^*\colon H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ has also infinite order. Thus one can apply Theorem A to deduce that g has finite order. In particular all the orbits of f are contained in a finite union of fibres of π, hence they are not Zariski-dense. $\square$

2. Elements of p-adic integration

In this section we give an introduction to p-adic integration; see [CLNS14], [Pop11, Chapter 3] and [Igu00].

2.1. **p-adic and local fields.** We remind that, for a prime number p, the p-adic norm on $\mathbb{Q}$ is defined as

$$\left|p^n \cdot \frac{a}{b}\right| = p^{-n} \qquad p \nmid a, \quad p \nmid b.$$

We denote $\mathbb{Q}_p$ the metric completion of $(\mathbb{Q}, |\cdot|_p)$; every element of $\mathbb{Q}_p$ can be uniquely written as a Laurent series

$$a = \sum_{n=n_0}^{+\infty} a_n p^n \qquad a_i \in \{0, 1, \ldots, p-1\}.$$

Denote by $\mathbb{Z}_p$ the closed unit ball in $\mathbb{Q}_p$; it is an integrally closed local subring of $\mathbb{Q}_p$ with maximal ideal $p\mathbb{Z}_p$ and residue field $\mathbb{F}_p$; its field of fractions is $\mathbb{Q}_p$, and it is a compact, closed and open subset of $\mathbb{Q}_p$.

A *p-adic field* is a finite extension K of $\mathbb{Q}_p$ for some prime p; on K there exists a unique absolute value $|\cdot|_K$ extending $|\cdot|_p$. We denote by $\mathcal{O}_K$ the closed unit ball in K.

A *local field* is a field K with a valuation $|\cdot|\colon K \to \mathbb{R}_{\geq 0}$ such that K with the induced topology is locally compact.

Theorem 2.1. *A local field of characteristic 0 is isomorphic either to $\mathbb{R}$ or $\mathbb{C}$ endowed with the usual absolute values (archimedean case) or to a finite extension of $\mathbb{Q}_p$ for some prime number p endowed with the unique extension of $|\cdot|_p$ (non-archimedean case).*

2.2. **Measure on K.** On a locally compact topological group G there exists a measure μ, unique up to scalar multiplication, called the *Haar measure of G* such that:

- any continuous function $f\colon G \to \mathbb{C}$ with compact support is μ-integrable;
- μ is G-invariant to the left.

Other important properties of the Haar measure are as follows: every Borel subset of G is measurable; $\mu(A) > 0$ for every nonempty open subset of G.

We consider $G = (\mathbb{Q}_p, +)$, and take on it the Haar measure μ normalized so that

$$\mu(\mathbb{Z}_p) = 1.$$

Example 2.2. It is easy to show that for every $m \geq 0$ one has $\mu(p^m\mathbb{Z}_p) = p^{-m}$.

More generally, on a p-adic field K we consider the Haar measure μ such that

$$\mu(\mathcal{O}_K) = 1.$$

2.3. **Integration on K-analytic manifolds.** Let K be a p-adic field with norm $|\cdot|$. For any open subset $U \subset K^n$, a function $f: U \to K$ is said to be K-analytic if locally around each point it is given by a convergent power series. Similarly, we call $f = (f_1, \dots, f_m): U \to K^m$ a K-analytic map if all the f_i are analytic.

As in the real and complex context, we define a K-analytic manifold of dimension n as a Hausdorff topological space locally modelled on open subsets of K^n and with K-analytic change of charts.

Example 2.3.

(1) Every open subset $U \subset K^n$ is a K-analytic manifold of dimension n; in particular, the set $\mathcal{O}_K^n \subset K^n$ is a K-analytic manifold.
(2) The projective space $\mathbb{P}^n_K$ over K is a K-analytic manifold.
(3) Every smooth algebraic variety over K is a K-analytic manifold; in order to see this one needs a K-analytic version of the implicit function theorem (see [CLNS14, §1.6.4]).

Differential forms are defined in the usual way via charts: on a chart with coordinates $x_1, \dots, x_n$, a differential form of degree k can be written as

$$\alpha = \sum_{|I|=k} f_I(x_1, \dots, x_n) dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

with $f_I: U \to K$ functions on U; if the f_I are K-analytic we say that the form is analytic. Now take a maximal degree analytic differential form ω; let $\phi: U \to K^n$ be a local chart, defining local coordinates $x_1, \dots, x_n$. In these coordinates we can write

$$\phi_* \omega = f(x_1, \dots, x_n) dx_1 \wedge \dots \wedge dx_n.$$

Then one can define a Borel measure $|\omega|$ on U as follows: for any open subset $A \subset U$, we set

$$|\omega|(A) = \int_{\phi(A)} |f(x)|_K \, d\mu,$$

where μ is the usual normalized Haar measure on $\phi(U) \subset K^n$.
Similarly, let ω be a maximal degree pluri-form, i.e. a section of the analytic sheaf $(\Omega^n_X)^{\otimes m}$ for some $m > 0$; let $\phi: U \to K^n$ be a local chart, defining local coordinates $x_1, \dots, x_n$. In these coordinates we can write

$$\phi_* \omega = f(x_1, \dots, x_n)(dx_1 \wedge \dots \wedge dx_n)^{\otimes m}.$$

Then one can define a Borel measure $\sqrt[m]{|\omega|}$ on U as follows: for any open subset $A \subset U$, we set

$$\sqrt[m]{|\omega|}(A) = \int_{\phi(A)} \sqrt[m]{|f(x)|_K} d\mu,$$

where μ is the usual normalized Haar measure on $\phi(U) \subset K^n$.

Now let ω be a global section of Ω^n_X (resp. $(\Omega^n_X)^{\otimes m}$). To define a Borel measure $|\omega|$ (resp. $\sqrt[m]{|\omega|}$) on the whole manifold X, one uses partitions of unity exactly as in the real case. The only thing to check is that $|\omega|$ (resp. $\sqrt[m]{|\omega|}$) transforms precisely like differential forms when changing coordinates, which is a consequence of the following K-analytic version of the change of variables formula.

Theorem 2.4 (Change of variables formula)**.** *Let U be an open subset of K^n and let $\phi\colon U \to K^n$ be an injective K-analytic map whose Jacobian J_ϕ is invertible on U. Then for every measurable positive (resp. integrable) function $f\colon \phi(U) \to \mathbb{R}$*

$$\int_{\phi(U)} f(y)d\mu(y) = \int_U f(\phi(x)) \left|\det J_\phi(x)\right|_K d\mu(x).$$

3. Proof of Theorem A

In this section we give the proof of Theorem A. The strategy of the proof goes as follows:

(1) A multiple mL of L induces an embedding $B \hookrightarrow \mathbb{P}^N$; since f_B preserves L, it is the restriction of a linear automorphism $g \in \mathrm{PGL}_{N+1}(\mathbb{C})$.
(2) Find an f-invariant volume form ω on X (for X irreducible symplectic, $\omega = (\sigma \wedge \bar\sigma)^n$, where $2n = \dim X$ and σ is a symplectic form).
(3) The push-forward of μ by π defines a f_B-invariant measure vol on B not charging positive codimensional subvarieties; using this it is not hard to put $g \in \mathrm{PGL}_{N+1}(\mathbb{C})$ in diagonal form with only complex numbers of modulus 1 on the diagonal.
(4) Define the field of coefficients k: roughly speaking, a finitely generated (but not necessarily finite) extension of $\mathbb{Q}$ over which X, B, f, the volume form and all the relevant maps are defined.
(5) Apply a key lemma: if one of the coefficients α of g weren't a root of unity, there would exist an embedding $k \hookrightarrow K$ into a local field K such that $|\rho(\alpha)| \neq 1$. Then the same measure-theoretic argument as in point (3) leads to a contradiction.
A similar idea appears in the proof of Tits alternative for linear groups, see [Tit72].

3.1. Invariant volume form on X.

Remark that, given a holomorphic n-form Ω (n being the dimension of X), the pull-back $f^*\Omega$ is defined outside the indeterminacy locus of f; the latter being of codimension ≥ 2, by Hartogs principle we can extend $f^*\Omega$ to an n-form on the whole X. This action determines a linear automorphism

$$f^*\colon H^0(X, K_X) \to H^0(X, K_X).$$

Similarly, for all $m \geq 0$ one can define linear automorphisms

$$f_m^*\colon H^0(X, mK_X) \to H^0(X, mK_X).$$

Since X has non-negative Kodaira dimension, there exists $m > 0$ such that mK_X has a section. The complex vector space $H^0(X, mK_X)$ has finite dimension, thus there exists an eigenvector $\Omega \in H^0(X, mK_X) \setminus \{0\}$:

$$f^*\Omega = \xi\Omega.$$

The section Ω can be written in local holomorphic coordinates $x_1, \dots, x_n$ as

$$\Omega = f(x)(dx_1 \wedge \ldots \wedge dx_n)^{\otimes m}$$

for some (local) holomorphic function f. Thus locally

$$\Omega \wedge \overline{\Omega} = |f(x)|^2 (dx_1 \wedge \ldots \wedge dx_n)^{\otimes m} \wedge (d\bar x_1 \wedge \ldots \wedge d\bar x_n)^{\otimes m}.$$

It can be checked that the local form

$$\omega = \frac{(-1)^{n(n-1)/2}}{i^n} \sqrt[m]{|f(x)|^2} dx_1 \wedge \ldots \wedge dx_n \wedge d\bar x_1 \wedge \ldots \wedge d\bar x_n$$

is a volume form; since it is canonically associated to Ω, such local expressions glue together to define a volume form on X

$$\omega = \frac{(-1)^{n(n-1)/2}}{i^n} \sqrt[m]{\Omega \wedge \overline{\Omega}}.$$

Since the measure μ of integration by ω doesn't charge positive codimensional analytic subvarieties and since f is birational, f preserves the (finite) total measure:

$$\int_X f^*(\omega) = \int_X \omega;$$

this implies that $|\xi| = 1$, and in particular that the volume form ω is f-invariant.
The push-forward by π induces a measure vol on B: for all Borel set $A \subset B$, we set

$$\mathrm{vol}(A) := \int_{\pi^{-1}(A)} \omega.$$

The measure vol is f_B-invariant.

3.2. **A first reduction of g.** In a given system of homogeneous coordinates on $\mathbb{P}^N$, an automorphism $g \in \mathrm{Aut}(\mathbb{P}^N) = \mathrm{PGL}_{N+1}(\mathbb{C})$ is represented by a matrix M acting linearly on such coordinates; M is well-defined up to scalar multiplication. We will say that g is semi-simple if M is; in this case there exist homogeneous coordinates $Y_0, \ldots, Y_n$ such that the action of g on these coordinates can be written

$$g([Y_0 : \ldots : Y_N]) = \begin{bmatrix} 1 & & & \\ & \alpha_1 & & \\ & & \ddots & \\ & & & \alpha_N \end{bmatrix} \underline{Y} = [Y_0 : \alpha_1 Y_1 : \ldots : \alpha_N Y_N].$$

By an abuse of terminology, we will call the α_i the eigenvalues of g; they are not well-defined, but the property that they are all of modulus 1 is.

Lemma 3.1. *The automorphism g is semi-simple and its eigenvalues have all modulus* 1.

Proof. Let us prove first that g is semi-simple. If this were not the case, the Jordan form of g (which is well-defined up to scalar multiplication) would have a non-trivial Jordan block, say of dimension $k \geq 2$. It turns out that the computations are clearer if we consider the lower triangular Jordan form. In some good homogeneous coordinates $Y_0, \ldots Y_n$ of $\mathbb{P}^N$, after rescaling the coefficients of g we can write

$$g(\underline{Y}) = \begin{bmatrix} 1 & & & \mathbf{0} & & & \\ 1 & 1 & & & & & \\ & \ddots & \ddots & & & \mathbf{0} & \\ \mathbf{0} & & 1 & 1 & & & \\ & & & & \alpha_k & & \mathbf{0} \\ & \mathbf{0} & & & & \ddots & \\ & & & & \bigstar & & \alpha_N \end{bmatrix} \underline{Y}.$$

Take the affine chart $\{Y_0 \neq 0\} \cong \mathbb{C}^N$ with the induced affine coordinates $y_i = Y_i / Y_0$. In these coordinates we can write

$$g(y_1, \ldots, y_N) = (y_1 + 1, y_2 + y_1, \ldots)$$

and thus

$$g^N(y_1, \ldots, y_N) = (y_1 + N, \ldots).$$

Let

$$A = \{(y_1, \dots, y_N) \in \mathbb{C}^N \mid 0 \le \mathrm{Re}(y_1) < 1\};$$

then we have

$$\mathbb{C}^N = \coprod_{N \in \mathbb{Z}} g^N(A);$$

therefore

$$B \cap \mathbb{C}^N = \coprod_{N \in \mathbb{Z}} B \cap g^N(A) = \coprod_{N \in \mathbb{Z}} f_B^N(A \cap B)$$

and

$$\mathrm{vol}(B) = \mathrm{vol}(B \cap \mathbb{C}^N) = \sum_{N \in \mathbb{Z}} \mathrm{vol}(f_B^N(A \cap B)) = \sum_{N \in \mathbb{Z}} \mathrm{vol}(A \cap B) = 0 \text{ or } +\infty,$$

which is a contradiction with the finiteness of vol. This shows that g is diagonalizable.

Next we show that, up to rescaling, in good homogeneous coordinates one can write

$$g(\underline{Y}) = \begin{bmatrix} 1 & & & \\ & \alpha_1 & & \\ & & \ddots & \\ & & & \alpha_N \end{bmatrix} \underline{Y} = [Y_0 : \alpha_1 Y_1 : \ldots : \alpha_N Y_N]$$

with $|\alpha_i| = 1$. Suppose by contradiction that $|\alpha_1| \neq 1$ (for example $|\alpha_1| > 1$), and define

$$A = \{(y_1, \dots, y_N) \in \mathbb{C}^N \mid 1 \le |y_1| < |\alpha_1|\} \subset \mathbb{C}^N.$$

The same argument as above leads to a contradiction. □

3.3. **The field of coefficients.** A key idea of the proof will be to define the "smallest" extension k of $\mathbb{Q}$ over which X, B and all the relevant applications are defined, and to embed k in a local field in such a way as to obtain a contradiction.

Let us fix a cover of X by affine charts $U_1, \dots, U_m$ trivializing the canonical bundle. Each of these U_i is isomorphic to the zero locus of some polynomials $p_{i,1}, \dots, p_{i,n_i}$ in an affine space $\mathbb{C}^{N_i}$; fix some rational functions $g_{i,j} \colon \mathbb{C}^{N_i} \dashrightarrow \mathbb{C}^{N_j}$ giving the changes of coordinates from U_i to U_j. Denote $\phi_{i,j} \colon U_i \cap U_j \to \mathbb{C}^*$ the change of charts for the canonical bundle; such functions are algebraic, therefore they are given by some rational functions $h_{i,j}$ on $\mathbb{C}^{N_i}$ (or, equivalently, $\mathbb{C}^{N_j}$). Let $f \colon \mathbb{C}^{N_i} \dashrightarrow \mathbb{C}^{N_j}$ (resp. $\Omega_i \colon \mathbb{C}^{N_i} \dashrightarrow \mathbb{C}$,) be some rational functions defining f (resp. Ω). Finally, fix homogeneous coordinates on $\mathbb{P}^N$ diagonalizing g (see paragraph 3.2), and let $q_1, \dots, q_M$ be homogeneous polynomials defining $B \subset \mathbb{P}^N$; fix as well some rational maps $\pi_i \colon \mathbb{C}^{N_i} \dashrightarrow \mathbb{C}^{N+1}$ defining π_{U_i} upon passing to quotient.

We define the *field of coefficients* $k = k_\Omega$ as the extension of $\mathbb{Q}$ generated by all the coefficients appearing in the $p_{i,k}, f_{i,j}, g_{i,j}, h_{i,j}, \Omega_i, \pi_i$ and by $\alpha_1, \dots, \alpha_n$; this is a finitely generated (but not necessarily finite) extension of $\mathbb{Q}$ over which X is defined.

Let $\rho \colon k \hookrightarrow K$ be an embedding of k into a local field K; since $\mathbb{R}$ is naturally embedded in $\mathbb{C}$, we may and will assume that K is either $\mathbb{C}$ or a p-adic field. We can now apply a base change in the sense of algebraic geometry to recover a smooth projective scheme over K and all the relevant functions.

Here are the details of the construction: the polynomials $p_{i,k}^\rho = \rho(p_{i,k})$ define affine varieties X_i^ρ of K^{N_i}; the rational functions $g_{i,j}^\rho$ allow to glue the X_i^ρ-s into a complex algebraic variety X^ρ. This variety is actually smooth since smoothness is a local condition which is algebraic in the coefficients of the $p_{i,k}$. Furthermore, by applying ρ to all the relevant rational functions, we can recover a birational transformation $f^\rho \colon X^\rho \dashrightarrow X^\rho$

and a canonical section $\Omega^\rho \in H^0(X^\rho, K_{X^\rho})$. Remark that we can suppose that X^ρ is projective: indeed, $X \subset \mathbb{P}^N(\mathbb{C})$ is the zero locus of some homogeneous polynomials $P_1, \ldots, P_k \in \mathbb{C}[Y_0, \ldots, Y_N]$, and, up to adding the affine open subsets $X_i = X \cap \{Y_i \neq 0\}$ to the above constructions, it is easy to see that $X^\rho \subset \mathbb{P}^N(K)$ is the zero locus of $P_1^\rho, \ldots, P_k^\rho$. Furthermore, applying ρ to the equations of π defines a meromorphic fibration $\pi^\rho \colon X^\rho \dashrightarrow \mathbb{P}^n_K$, and, denoting $g^\rho \colon \mathbb{P}^n_K \to \mathbb{P}^n_K$ the automorphism given by $g^\rho[Y_0 : \ldots : Y_n] = [Y_0 : \alpha_1^\rho Y_1 : \ldots : \alpha_n^\rho Y_n]$, we have $\pi^\rho \circ f^\rho = g^\rho \circ \pi^\rho$:

$$\begin{tikzcd} X^\rho \arrow[r, dashed, "f^\rho"] \arrow[d, dashed, "\pi^\rho"] & X^\rho \arrow[d, dashed, "\pi^\rho"] \\ \mathbb{P}^n_K \arrow[r, "g^\rho"] & \mathbb{P}^n_K \end{tikzcd}$$

We will denote by μ^ρ the measure on X^ρ associated to Ω^ρ: this has been denoted by $\sqrt[m]{|\Omega^\rho|}$ in section 2.3 in the non-archimedean case, while if $K = \mathbb{C}$ it is defined as the measure of integration of

$$\omega^\rho = \frac{(-1)^{n(n-1)/2}}{i^n} \sqrt[m]{\Omega^\rho \wedge \overline{\Omega^\rho}}.$$

In both cases, μ^ρ doesn't charge positive codimensional analytic subvarieties.

Remark 3.2. At this stage we can already prove that the α_i are algebraic numbers all of whose conjugates over $\mathbb{Q}$ have modulus 1. Indeed suppose that this is not the case, say for α_1; by a standard argument in Galois theory (see for example [Lan02]), one can find an embedding $\rho \colon k \hookrightarrow \mathbb{C}$ such that $|\rho(\alpha_1)| \neq 1$. Now, g^ρ preserves the measure vol^ρ induced on $\mathbb{P}^n$ by ω^ρ

$$\mathrm{vol}^\rho(A) := \int_{(\pi^\rho)^{-1}(A)} \omega^\rho,$$

and Lemma 3.1 leads to a contradiction.

If we somehow knew that the α_i are algebraic integers, we could conclude by a lemma of Kronecker's (see [Kro57]) that they are roots of unity. However, this is in general not true for algebraic numbers: for example,

$$\alpha = \frac{3 + 4i}{5}$$

has only $\bar{\alpha}$ as a conjugate over $\mathbb{Q}$, and they both have modulus 1, but they are not roots of unity. In order to exclude this case we will have to use the p-adic argument.

3.4. **Key lemma and conclusion.** In his original proof of the Tits alternative for linear groups [Tit72], Tits proved and used (much like we do in this context) the following simple but crucial lemma:

Lemma 3.3 (Key lemma)**.** *Let k be a finitely generated extension of $\mathbb{Q}$ and let $\alpha \in k$ be an element which is not a root of unity. Then there exist a local field K (with norm $|\cdot|$) and an embedding $\rho \colon k \hookrightarrow K$ such that $|\rho(\alpha)| > 1$.*

We can now show that the number ξ such that $f^*\Omega = \xi\Omega$ is actually a root of unity; this follows from the classical result that f induces a linear map on cohomology preserving the integral structure (see for example [NU73]) and from the fact that all the conjugates of ξ over $\mathbb{Q}$ also have modulus 1 (the method for the proof being the same as the one explained in Remark 3.2), but to the best of my knowledge the present proof using Lemma 3.3 is original.

Lemma 3.4. *Some iterate f^N of f preserves Ω.*

Proof. We need to show that $|\xi|$ is a root of unity. Suppose by contradiction that this is not the case, and define the field of coefficients $k = k_\Omega$; since $f^*\Omega = \xi\Omega$, we have $\xi \in k$. Applying Lemma 3.3, one finds an embedding $\rho\colon k \hookrightarrow K$ into a local field K such that $|\rho(\xi)| \neq 1$.
The measure μ^ρ on X^ρ doesn't charge positive codimensional analytic subvarieties and $f^\rho\colon X^\rho \dashrightarrow X^\rho$ is a birational map, therefore the (finite) total measure is preserved by f^ρ:

$$(f^\rho)^*\mu^\rho(X^\rho) = \mu^\rho(X^\rho).$$

On the other hand $(f^\rho)^*\mu^\rho$ is the measure associated to $(f^\rho)^*\Omega^\rho = \xi^\rho\Omega^\rho$, hence $(f^\rho)^*\mu^\rho = |\xi^\rho| \cdot \mu^\rho$ and $|\xi^\rho| \neq 1$, a contradiction. □

Proof of Theorem A. Suppose by contradiction that one of the eigenvalues, say α_1, is not a root of unity.

We replace f by an iterate f^N preserving Ω, and define the field of coefficients k. Now, since $f^*\Omega = \Omega$, we have $(f^\rho)^*(\Omega^\rho) = \Omega^\rho$, and in particular f_B^ρ preserves the measure vol^ρ on B^ρ induced by the push-forward of μ_ρ:

$$\mathrm{vol}^\rho(A) := \mu^\rho\left((\pi^\rho)^{-1}(A)\right).$$

The measure vol^ρ is non-trivial, finite, and doesn't charge positive codimensional analytic subvarieties of B^ρ, thus we can conclude just as in the proof of Lemma 3.1.

Namely, identify $\{Y_0 \neq 0\} \subset \mathbb{P}^N_K$ with K^N and denote $A := \{(y_1 : \ldots : y_N) \in K^N \,|\, 1 \leq |y_1| < |\alpha_1|\} \subset K^N$ if $|\rho(\alpha_1)| > 1$ (respectively $A := \{(y_1 : \ldots : y_N) \in K^N \,|\, |\alpha_1^\rho| \leq |y_1| < 1\} \subset K^N$ if $|\rho(\alpha_1)| < 1$); then, since vol^ρ doesn't charge positive codimension analytic subsets, we have

$$\mu^\rho(X^\rho) = \mathrm{vol}^\rho(B^\rho) = \mathrm{vol}^\rho(B^\rho \cap K^n) =$$
$$\sum_{N\in\mathbb{Z}} \mathrm{vol}^\rho((g^\rho)^N(A \cap B^\rho)) = \sum_{N\in\mathbb{Z}} \mathrm{vol}^\rho(A \cap B^\rho) = 0 \text{ or } +\infty,$$

a contradiction. □

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