ON THE COHOMOLOGICAL ACTION OF AUTOMORPHISMS OF COMPACT KÄHLER THREEFOLDS

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Abstract. Extending well-known results on surfaces, we give bounds on the cohomological action of automorphisms of compact Kähler threefolds. More precisely, if the action is virtually unipotent we prove that the norm of $(f^n)^*$ grows at most as cn^4; in the general case, we give a description of the spectrum of f^*, and bounds on the possible conjugates over $\mathbb{Q}$ of the dynamical degrees $\lambda_1(f), \lambda_2(f)$. Examples on complex tori show the optimality of the results.

An automorphism $f\colon X \to X$ of a compact Kähler manifold induces by pull-back of forms a linear automorphism

$$f^*\colon H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{Z})$$

which preserves the cohomology graduation, the Hodge decomposition and complex conjugation.

Question 1. *What else can one say on f^*? More precisely, can one give constraints on f^* which depend only on the dimension of X (and not on the dimension of $H^*(X)$)?*

This is an interesting question in its own right since the cohomology of a manifold is a powerful tool to describe its geometry; furthermore, the cohomological action of an automorphism is relevant when studying its dynamics: one can deduce its topological entropy from its spectrum (see Theorem 1.3.3), and in the surface case knowing f^* allows to establish the existence of f-equivariant fibrations (see Theorem 2.2.1). It turns out that the restriction of f^* to the even cohomology encodes most of the interesting informations (see Section 1.3), therefore we focus on this part of the action; furthermore, in dimension 3 the action on $H^0(X)$ and on $H^6(X)$ is trivial, and the action on $H^4(X)$ can be deduced from the action on $H^2(X)$ (see Proposition 1.1.1.(3)), so we only describe the latter.

The situation of automorphisms (and, more generally, of birational transformations) of curves and surfaces is well understood (see Section 2). We address here the three-dimensional case.

The first result describes the situation where f^* does not have any eigenvalue of modulus > 1, i.e. the dynamical degrees $\lambda_i(f)$ are equal to 1 (see Definition 1.3.1).

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Theorem A. *Let X be a compact Kähler threefold and let $f\colon X \to X$ be an automorphism such that $\lambda_1(f) = 1$ and whose action on $H^*(X)$ has infinite order. Then the induced linear automorphism $f_2^*\colon H^2(X,\mathbb{C}) \to H^2(X,\mathbb{C})$ is virtually unipotent and has a unique Jordan block of maximal dimension $m = 3$ or 5. In particular, the norm of $(f^n)^*$ grows either as cn^2 or as cn^4 as n goes to infinity.*

For the proof of slightly more general results, see Theorem 3.0.1 and Proposition 3.1.1.

Next we give a description of the spectrum of f^* in terms of the dynamical degrees:

Theorem B. *Let X be a compact Kähler threefold and let $f\colon X \to X$ be an automorphism having dynamical degrees $\lambda_1 = \lambda_1(f)$ and $\lambda_2 = \lambda_2(f)$ (see Definition 1.3.1). Let λ be an eigenvalue of $f_2^*\colon H^2(X,\mathbb{C}) \to H^2(X,\mathbb{C})$; then there exists a positive integer N such that*

$$|\lambda|^{(-2)^N} \in \left\{1, \lambda_1, \lambda_2^{-1}, \lambda_1^{-1}\lambda_2\right\}.$$

If furthermore $\lambda_2 \notin \{\lambda_1^2, \sqrt{\lambda_1}\}$, then λ_1 and λ_2^{-1} are the only eigenvalues of f_2^* having modulus λ_1 or λ_2^{-1}.

For a proof see Proposition 4.3.1 and 4.4.1.

Finally, we describe the (moduli of) Galois conjugates of $\lambda_1(f)$ over $\mathbb{Q}$:

Theorem C. *Let X be a compact Kähler threefold and let $f\colon X \to X$ be an automorphism having dynamical degrees $\lambda_1 = \lambda_1(f)$ and $\lambda_2 = \lambda_2(f)$. Then λ_1 is an algebraic integer, all of whose conjugates over $\mathbb{Q}$ have modulus belonging to the following set:*

$$\left\{\lambda_1, \lambda_2^{-1}, \lambda_1^{-1}\lambda_2, \sqrt{\lambda_1^{-1}}, \sqrt{\lambda_2}, \sqrt{\lambda_1\lambda_2^{-1}}\right\}.$$

See Proposition 4.6.7 and 4.7.2 for a proof and for a more detailed description of all possible subcases.

In Section 1 we introduce the problem and the tools which will be used in the proofs, namely the generalized Hodge index theorem, an application of Poincaré's duality and some elements of the theory of algebraic groups; in Section 2 we present the known results in dimension two. In the rest of the paper we treat the case of dimension three: in Section 3 we give a proof of Theorem A and describe examples on complex tori which show the optimality of the result; similarly, in Section 5 and Section 6 we prove Theorem B and C respectively, and describe further examples on tori which show the optimality of the claims; finally, in Section 7 we address the problem to determine whether f^* can be neither (virtually) unipotent nor semisimple (see Proposition 5.0.1).

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1. Introduction and main tools

Throughout this Section, we denote by $f\colon X \to X$ an automorphism of a compact Kähler manifold X of complex dimension d and by

$$f^*\colon H^*(X,\mathbb{R}) \to H^*(X,\mathbb{R})$$

the induced linear automorphism on cohomology. We still denote by $f^*\colon H^*(X,\mathbb{C}) \to H^*(X,\mathbb{C})$ the complexification of f^*, and by f^*_k (resp. $f^*_{p,q}$) the restriction of f^* to $H^k(X,\mathbb{R})$ (resp. to $H^{p,q}(X)$).

1.1. **First constraints.** The induced linear automorphism f^* preserves some additional structure on the cohomology space $H^*(X,\mathbb{R})$:

Proposition 1.1.1.
(1) *$f^*(H^i(X,\mathbb{R})) = H^i(X,\mathbb{R})$ for all $i = 0,\dots,2d$ and $f^*(H^{p,q}(X)) = H^{p,q}(X)$ for $p,q = 0,\dots d$;*
(2) *$f^*(u \wedge v) = f^*(u) \wedge f^*(v)$ for all $u, v \in H^*(X,\mathbb{R})$;*
(3) *the Poincaré's duality $H^i(X,\mathbb{R}) \cong H^{2d-i}(X,\mathbb{R})^\vee$ induces an identification*
$$f^*_i = ((f^*_{2d-i})^{-1})^\vee;$$
(4) *f^* is defined over $\mathbb{Z}$; in other words, the coefficients of f^* with respect to an integral basis of $H^*(X,\mathbb{R})$ are integers;*
(5) *f^* preserves the convex salient cones $\mathcal{K}_p \subset H^{p,p}(M,\mathbb{R})$ generated by the classes of positive currents (see [7]).*

Remark that properties (1) – (4) are algebraic, while property (5) is not.

1.2. **The unipotent and semi-simple parts.** Let V be a finite dimensional real vector space and let $g\colon V \to V$ be a linear endomorphism. It is well-known that there exists a unique decomposition
$$g = g_u \circ g_s = g_s \circ g_u,$$
where g_u is unipotent (i.e. $(g_u - \mathrm{id}_V)^{\dim V} = 0$) and g_s is semisimple (i.e. diagonalizable over $\mathbb{C}$). This is a special case of the following more general statement:

Theorem 1.2.1 (Jordan decomposition)**.** *Let V be a finite dimensional and let $G \subset \mathrm{GL}(V)$ be a commutative algebraic group. Then, denoting by $G_u \subset G$ (respectively by $G_s \subset G$) the subset of unipotent (respectively semisimple) elements of G, G_u and G_s are closed subgroups of G and the product morphism induces an isomorphism of real algebraic groups*
$$G \cong G_u \times G_s.$$

Let us go back to the context of automorphisms of compact Kähler manifold. Let $f\colon X \to X$ be an automorphism of a compact Kähler manifold, $V = H^*(X,\mathbb{R})$, $f^*\colon V \to V$ and
$$G = \overline{\bigcup_{n\in\mathbb{Z}} (f^n)^*}^{Zar} \subset \mathrm{GL}(V);$$
here $\overline{A}^{Zar}$ denotes the Zariski-closure of a set $A \subset \mathrm{GL}(V) \cong \mathbb{R}^{(\dim V)^2}$, i.e. the smallest Zariski-closed subset of $\mathrm{GL}(V)$ containing A. Then, since $\langle g\rangle$ is commutative, so is G, and by Theorem 1.2.1 we have an isomorphism of real algebraic groups
$$G \cong G_u \times G_s;$$
this means in particular that, writing the Jordan decomposition $f^* = f^*_u \circ f^*_s$, we have $f^*_u, f^*_s \in G$, i.e. if f^* satisfies some algebraic constraint, then so do f^*_u and f^*_s. We have therefore:

Lemma 1.2.2.
(1) *$f^*_u(H^i(X,\mathbb{R})) = H^i(X,\mathbb{R})$ for all $i = 0,\dots,2d$ and $f^*_u(H^{p,q}(X)) = H^{p,q}(X)$ for $p,q = 0,\dots d$;*
(2) *$f^*_u(v \wedge w) = f^*_u(v) \wedge f^*_u(w)$ for all $v, w \in H^*(X,\mathbb{R})$;*

(3) *the Poincaré's duality $H^i(X,\mathbb{R}) \cong H^{2d-i}(X,\mathbb{R})^\vee$ induces an identification*

$$(f_u^*)_i = ((f_u^*)_{2d-i}^{-1})^\vee;$$

(4) *f_u^* is defined over $\mathbb{Z}$; in other words, the coefficients of f^* with respect to an integral basis of $H^*(X,\mathbb{R})$ are integers.*

The same properties are still true if one replaces f_u^* by f_s^*.

Remark however that, since preserving a cone is not an algebraic property, f_u^* and f_s^* may not preserve the positive cones $\mathcal{K}_p$.

1.3. **Dynamical degrees.** In this paragraph only, we allow $f\colon M \dashrightarrow M$ to be a dominant meromorphic self-map of a compact Kähler manifold M.

Definition 1.3.1. *The p-th dynamical degree of f is defined as*

$$\lambda_p(f) = \limsup_{n\to+\infty} \|(f^n)^*_{p,p}\|^{\frac{1}{n}},$$

where $\|\cdot\|$ is any matrix norm on the space $\mathcal{L}(H^{p,p}(X,\mathbb{R}))$ of linear maps of $H^{p,p}(X,\mathbb{R})$ into itself.

In the meromorphic case the pull-backs $f^*_{p,p}$ are defined in the sense of currents (see [7]).
One can prove that

$$\lambda_p(f) = \lim_{n\to+\infty} \left(\int_M (f^n)^*\omega^p \wedge \omega^{d-p} \right)^{\frac{1}{n}} \tag{1.1}$$

for any Kähler form ω; see [10], [5] for details.

In the case of holomorphic maps, we have $(f^n)^* = (f^*)^n$, so that $\lambda_p(f)$ is the spectral radius (i.e. the maximal modulus of eigenvalues) of the linear map $f^*_{p,p}$; since f^* also preserves the positive cone $\mathcal{K}_p \subset H^{p,p}(M,\mathbb{R})$, a theorem of Birkhoff [2] implies that $\lambda_p(f)$ is a positive real eigenvalue of f^*_p. In particular, $\lambda_p(f)$ is an algebraic integer. However it should be noted that in the meromorphic setting we have in general $(f^n)^* \neq (f^*)^n$.

At least in the projective case, the p-th dynamical degree measures the exponential growth of the volume of $f^{-n}(V)$ for subvarieties $V \subset M$ of codimension p, see [15].

Remark 1.3.2. By definition $\lambda_0(f) = 1$; $\lambda_d(f)$ coincides with the topological degree of f: it is equal to the number of points in a generic fibre of f.

The topological entropy of a continuous map of a topological space is a non-negative number, possibly infinite, which gives a measure of the chaos created by the map and its iterates; for a precise definition see [16]. The computation of the topological entropy of a map is usually complicated, and requires ad hoc arguments; however, in the case of dominant self-maps of compact Kähler manifolds one can apply the following result due to Yomdin [22] and Gromov [14]:

Theorem 1.3.3 (Yomdin-Gromov)**.** *Let $f\colon M \to M$ be a dominant self-map of a compact Kähler manifold of dimension d; then the topological entropy of f is given by*

$$h_{top}(f) = \max_{p=0,\dots,d} \log \lambda_p(f).$$

1.3.1. *Concavity properties.*

Theorem 1.3.4 (Teissier-Khovanskii, see [14]). *Let X be a compact Kähler manifold of dimension d, and $\Omega := (\omega_1, \dots, \omega_k)$ be k-tuple of Kähler forms on X. For any multi-index $I = (i_1, \dots, i_k)$ let $\Omega^I = \omega_1^{i_1} \wedge \dots \wedge \omega_k^{i_k}$.*
Fix $i_3, \dots, i_k$ so that $i := \sum_{h \geq 3} i_h \leq d$, and let $I_p = (p, d-i-p, i_3, \dots, i_k)$; then the function

$$p \mapsto \log \left(\int_X \Omega^{I_p} \right)$$

is concave on the set $\{0, 1, \dots, d-i\}$.

One can use Theorem 1.3.4 to prove the following log-concavity result:

Proposition 1.3.5. *Let $f\colon X \to X$ be a dominant meromorphic self-map of a compact Kähler manifold X of dimension d. Then the function*

$$p \mapsto \log \lambda_p(f)$$

is concave on the set $\{0, 1, \dots, d\}$. In particular, if $\lambda_1(f) = 1$ then $\lambda_p(f) = 1$ for all $p = 0, \dots, d$.

This implies that the exponential growth of the norm of $(f^n)^*$ comes from an eigenvalue of one of the $f^*_{p,p}$:

Proposition 1.3.6. *Let $f\colon X \dashrightarrow X$ be a meromorphic self-map of a compact Kähler manifold X, and let*

$$r_{p,q}(f) = \lim_{n \to +\infty} \|(f^n)^*_{p,q}\|^{\frac{1}{n}};$$

here we use the convention that the norm of the identity on the null vector space is equal to 1.
Then

(1) *if $p+q = 2k$ is even, then*
$$r_{p,q}(f) \leq \lambda_k(f);$$
(2) *if $p+q = 2k+1$ is odd, then*
$$r_{p,q}(f) \leq \sqrt{\lambda_k(f)\lambda_{k+1}(f)}.$$

In particular

$$\lim_{n+\infty} \|(f^n)^*\|^{\frac{1}{n}} = \max_p \lambda_p(f)$$

.

Proof. Let us consider the linear map

$$\phi\colon H^{p,q}(X, \mathbb{C}) \to H^{p+q,p+q}(X \times X, \mathbb{C})$$
$$u \mapsto \pi_1^* u \wedge \pi_2^* \bar{u},$$

where $\pi_1, \pi_2\colon X \times X \to X$ are the two natural projections.
Remark that if $u \neq 0$, then $\phi(u) \neq 0$, and that

$$\phi \circ f = (f, f) \circ \phi,$$

where (f, f) denotes the diagonal morphism. This is evident if f is holomorphic; if not, one can consider a resolution of indeterminacies of f, which induces a resolution

of indeterminacies of (f,f). The claim then follows from the definition of the pull-back of currents.
This implies that there exists a constant $C>0$ such that

$$\|(f^n)^*_{p,q}\|^2 \leq C\|(f^n,f^n)^*_{p+q,p+q}\|.$$

In particular

$$r_{p,q}(f)^2 \leq \lambda_{p+q}(f,f).$$

Now let $\omega \in H^{1,1}(X,\mathbb{R})$ be a Kähler form on X; then $\pi_1^*\omega + \pi_2^*\omega =: \tilde{\omega} \in H^{1,1}(X\times X,\mathbb{R})$ is a Kähler form on $X\times X$.
Using the alternative definition 1.1 for dynamical degrees and denoting by $\|\alpha\| = \int_X \alpha \wedge \omega^{d-p}$ the ω-norm of a real (p,p)-form α, we get

$$\begin{aligned} r_{p,q}(f)^2 \leq \lambda_{p+q}(f,f) &= \lim_{n\to+\infty} \left\| ((f,f)^n)^* \, \tilde{\omega}^{p+q} \right\|^{\frac{1}{n}} \\ &= \lim_{n\to+\infty} \left\| ((f,f)^n)^* \, (\pi_1^*\omega + \pi_2^*\omega)^{p+q} \right\|^{\frac{1}{n}} \\ &= \lim_{n\to+\infty} \left\| ((f,f)^n)^* \sum_{h=0}^{p+q} \binom{p+q}{h} (\pi_1^*\omega)^h \wedge (\pi_2^*\omega)^{p+q-h} \right\|^{\frac{1}{n}} \\ &= \lim_{n\to+\infty} \left\| \sum_{h=0}^{p+q} \binom{p+q}{h} \pi_1^*(f^n)^*\omega^h \wedge \pi_2^*(f^n)^*\omega^{p+q-h} \right\|^{\frac{1}{n}} \\ &\leq \max\{\lambda_h(f)\lambda_{p+q-h}(f)\}. \end{aligned}$$

If $p+q=2k$ is even, then by Proposition 1.3.5 we have

$$\lambda_h(f)\lambda_{p+q-h}(f) \leq \lambda_k(f)^2,$$

which shows the first claim.

If $p+q=2k+1$ is odd, by Proposition 1.3.5 we have

$$\lambda_h(f)\lambda_{p+q-h}(f) \leq \lambda_k(f)\lambda_{k+1}(f),$$

which shows the second claim. □

Corollary 1.3.7. *Let $f\colon X\to X$ be a dominant holomorphic endomorphism of a compact Kähler manifold X. Then the following are equivalent:*

(1) $\lambda_1(f)=1$;
(2) f^* *is virtually unipotent;*
(3) $r_{p,q}(f)=1$ *for all* p,q .

Proof. The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are evident; let us show that (1) $\Rightarrow$ (2).

Since by Proposition 1.3.5 $\lambda_0(f)=\lambda_1(f)=\ldots=\lambda_d(f)=1$, Proposition 1.3.6 implies that $r_{p,q}(f)\leq 1$ for all p,q. Therefore the spectral radius of the linear automorphisms

$$f_k^*\colon H^k(X,\mathbb{Z}) \to H^k(X,\mathbb{Z})$$

is equal to 1. It follows from a lemma of Kronecker that the roots of the characteristic polynomial of f_k^* are roots of unity; therefore, some iterate of f^* has 1 as its only eigenvalue, i.e. it is unipotent. This concludes the proof. □

1.3.2. *Polynomial growth.* Suppose now that $f\colon X \to X$ is a dominant holomorphic endomorphism. The dynamical degrees measure the exponential growth of the norm of $(f^n)^*$; in the case where $\lambda_p(f) = 1$, i.e. all the eigenvalues of $f^*_{p,p}$ have modulus 1, then an easy linear algebra argument implies that

$$\|(f^n)^*_{p,p}\| \sim cn^{\mu_p(f)},$$

where $\mu_p(f) + 1$ is the maximal size of Jordan blocks of $f^*_{p,p}$. Then one can define an analogous value measuring the polynomial growth of $(f^n)^*$.

Definition 1.3.8. *Suppose that $\lambda_p(f) = 1$; the p-th polynomial dynamical degree is defined as*

$$\mu_p(f) = \lim_{n\to+\infty} \frac{\log \|(f^n)^*_{p,p}\|}{\log n}.$$

The following question is still open even for birational maps of $\mathbb{P}^k(\mathbb{C})$, $k \geq 3$.

Question 2. *Let $f\colon X \dashrightarrow X$ be a meromorphic self-map of a compact Kähler manifold such that $\lambda_1(f)$; is it true that $\|(f^n)^*\|$ grows polynomially?*

The inequalities of Tessier-Khovanskii allow to prove an equivalent of Proposition 1.3.5 and Proposition 1.3.6:

Proposition 1.3.9. *Let $f\colon X \to X$ be a holomorphic self-map of a compact Kähler manifold such that $\lambda_p(f) = 1$; then*

(1) *the function*

$$p \mapsto \mu_p(f)$$

is concave on the set $\{0,\ldots,d\}$;

(2)

$$\lim_{n\to+\infty} \frac{\log \|(f^n)^*\|}{\log n} = \max_p \mu_p(f).$$

1.4. **Generalized Hodge index theorem.** The classical Hodge index theorem asserts that if S is a compact Kähler surface, then the intersection product on $H^{1,1}(X,\mathbb{R})$ is hyperbolic, i.e. it has signature $(1, h^{1,1}(S)-1)$; this is a consequence of the Hodge-Riemann bilinear relations, which can be generalized in higher dimension in order to obtain an analogue of the classical result. We will focus on the second cohomology group, but analogue results exist for cohomology of any order (see [9]).

Let (X,ω) be a compact Kähler manifold of dimension $d \geq 2$; we define a quadratic form q on $H^2(X,\mathbb{R})$ by

$$q(\alpha,\beta) := \int_X (\alpha_{1,1}\wedge\beta_{1,1} - \alpha_{2,0}\wedge\beta_{0,2} - \alpha_{0,2}\wedge\beta_{2,0})\wedge\omega^{d-2} \quad \alpha,\beta \in H^2(X,\mathbb{R}),$$

where $\alpha_{i,j}$ (resp. $\beta_{i,j}$) denotes the (i,j)-part of α (resp. of β).
Remark that the decomposition

$$H^2(X,\mathbb{R}) = H^{1,1}(X,\mathbb{R}) \oplus (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$$

is q-orthogonal.

Theorem 1.4.1 (Generalized Hodge index theorem)**.** *Let (X,ω) be a compact Kähler manifold of dimension $d \geq 2$ and let q be defined as above. Then the restriction of q to $H^{1,1}(X,\mathbb{R})$ has signature $(1, h^{1,1}(X) - 1)$; its restriction to $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ is negative definite.*

An immediate consequence, which we will use constantly in the rest of the paper, is the following:

Corollary 1.4.2. *If $V \subset H^2(X, \mathbb{R})$ is a q-isotropic space, then $\dim V < 2$. In particular, if $u, v \in H^{1,1}(X, \mathbb{R}) \cup (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ are linearly independent classes, then the classes*

$$u \wedge u, u \wedge v, v \wedge v \in H^4(X, \mathbb{R})$$

cannot all be null. If furthermore $u \in (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$, then $u \wedge u \neq 0$. Analogously, if $u, v \in H^{1,1}(X) \cup (H^{2,0}(X) \oplus H^{0,2}(X))$ are linearly independent classes, then the classes

$$u \wedge \bar{u}, u \wedge \bar{v}, v \wedge \bar{v} \in H^4(X, \mathbb{C})$$

cannot all be null. If furthermore $u \in H^{2,0}(X) \oplus H^{0,2}(X)$, then $u \wedge \bar{u} \neq 0$.

2. The case of surfaces

Remark first that the case of automorphisms of curves is dynamically not very interesting: indeed, if the genus of the curve is $g \geq 2$, then the group of automorphism is finite; the only non-trivial dynamics arise from automorphisms of $\mathbb{P}^1$ and from automorphisms of elliptic curves (which, up to iteration, are translations), and both are well-understood.

Let us focus then on the surface case: let S be a compact Kähler surface and let $f: S \to S$ be an automorphism. By the Hodge index theorem (Theorem 1.4.1), the generalized intersection form q makes $H^2(X, \mathbb{R})$ into a hyperbolic space; furthermore, q is preserved by f, so that we may consider $g = f_2^*: H^2(X, \mathbb{R}) \to H^2(X, \mathbb{R})$ an element of $O(H^2(X, \mathbb{R}), q)$.

2.1. Automorphisms of hyperbolic spaces.

Let (V, q) be a hyperbolic vector space of dimension n and let $\|\cdot\|$ be a norm on the space $\mathcal{L}(V)$ of linear endomorphisms of V.

Definition 2.1.1. *Let $g \in O(V, q)$. We say that g is*

- loxodromic *(or* hyperbolic*) if it admits an eigenvalue of modulus strictly greater than 1;*
- parabolic *if all its eigenvalues have modulus 1 and $\|g^n\|$ is not bounded as $n \to +\infty$;*
- elliptic *if all its eigenvalues have modulus 1 and $\|g^n\|$ is bounded as $n \to +\infty$.*

In each of the cases above, simple linear algebra arguments allow to further describe the situation. For the following result see for example [13].

Theorem 2.1.2. *Let $g \in O^+(V, q)$, and suppose that g preserves a lattice $\Gamma \subset V$.*

- *If g is loxodromic, then it is semisimple and it has exactly one eigenvalue λ with modulus > 1 and exactly one eigenvalue λ^{-1} with modulus < 1; these eigenvalues are real and simple, so that in particular $\|g^n\| \sim c\lambda^n$. The eigenvalue λ is an algebraic integer whose conjugates over $\mathbb{Q}$ are λ^{-1} and complex numbers of modulus 1, i.e. λ is a quadratic or Salem number.*

- *If g is parabolic, then all the eigenvalues of g are roots of unity, and some iterate of g has Jordan form*
$$\begin{pmatrix} 1 & 1 & 0 & \mathbf{0} \\ 0 & 1 & 1 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{d-3} \end{pmatrix}.$$
In particular $\|g^n\| \sim cn^2$.
- *If g is elliptic, then it has finite order.*

An automorphism $f\colon S \to S$ of a compact Kähler surface S is called loxodromic, parabolic or elliptic if $g = f_2^*$ is loxodromic, parabolic or elliptic respectively. Remark that g preserves the integral lattice $H^2(X, \mathbb{Z})/(\text{torsion})$, so that Theorem 2.1.2 can be applied to g.

Remark that, if f is homotopic to the identity, then its action on cohomology is trivial. Conversely, if f acts trivially on cohomology, then some of its iterates is homotopic to the identity. More precisely:

Theorem 2.1.3 (Fujiki, Liebermann [11, 19])**.** *Let M be a compact Kähler manifold. If $[\kappa]$ is a Kähler class on M, the connected component of the identity* $\mathrm{Aut}(X)^0$ *has finite index in the group of automorphisms of M fixing $[\kappa]$.*

This implies that a surface automorphism is elliptic if and only if one of its iterates is homotopic to the identity.

2.2. **Equivariant fibrations.** In the case of surfaces, it turns out that the cohomological action of an automorphism has consequences on the following property of decomposability of its dynamics. Let $f\colon M \to M$ be an automorphism of a compact Kähler manifold M; we say that a fibration $\pi\colon M \to B$ (i.e. a surjective map with connected fibres) is *f-equivariant* if there exists an automorphism $g\colon B \to B$ such that $\pi \circ f = g \circ \pi$, i.e. the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ B & \xrightarrow{g} & B \end{array}.$$

The following theorem was stated and proved in the present form by Cantat [4], and follows from a result of Gizatullin (see [12], or [13] for a survey); see also [8] for the birational case.

Theorem 2.2.1. *Let S be a compact Kähler surface and let f be an automorphism of S.*

(1) *If f is parabolic, there exists an f-equivariant elliptic fibration $\pi\colon S \to C$; f doesn't admit other equivariant fibrations.*
(2) *Conversely, if a non-elliptic automorphism of a surface $f\colon S \to S$ admits an equivariant fibration $\pi\colon S \to C$ onto a curve, then f is parabolic. In particular, the fibration π is elliptic, and it is the only equivariant fibration.*

In other words, a non-elliptic automorphism of a surface admits an equivariant fibration if and only if its topological entropy is zero.

In higher dimension, one can ask the following question:

Question 3. *Let $f\colon X \to X$ be an automorphism of a compact Kähler manifold M. Suppose that f_2^* is virtually unipotent of infinite order. Does f admit an equivariant fibration?*

Apart from the case of surfaces, the only situation where the answer is known (and affirmative) is that of irreducible holomorphic symplectic (or hyperkähler) manifolds of deformation type $K3^{[n]}$ or generalized Kummer (see [17]); the proof uses the hyperkähler version of the abundance conjecture, which was proven in this context by Bayer and Macrí [1].

3. Automorphisms of threefolds: the unipotent case

Throughout this section let X be a compact Kähler threefold and let

$$g\colon H^*(X,\mathbb{R}) \to H^*(X,\mathbb{R})$$

be a *unipotent* linear automorphism preserving the cohomology graduation, the Hodge decomposition, the wedge-product and Poincaré's duality. In other words,

(1) $g(H^i(X,\mathbb{R})) = H^i(X,\mathbb{R})$ for all $i = 0,\dots,6$ and $g_{\mathbb{C}}(H^{p,q}(X)) = H^{p,q}(X)$ for all $p,q = 0,\dots 3$;
(2) $g(u \wedge v) = g(u) \wedge g(v)$ for all $u,v \in H^*(X,\mathbb{R})$;
(3) denoting by $g_i\colon H^i(X,\mathbb{R}) \to H^i(X,\mathbb{R})$ the restriction of g, the Poincaré's duality $H^i(X,\mathbb{R}) \cong H^{6-i}(X,\mathbb{R})^\vee$ induces an identification

$$g_i = (g_{6-i}^{-1})^\vee.$$

If $f\colon X \to X$ is an automorphism such that $\lambda_1(f) = 1$, then the linear automorphism $f^*\colon H^*(X,\mathbb{R})$ is virtually unipotent, and therefore an iterate $g = (f^N)^*$ satisfies the assumptions above.
More generally, if $f\colon X \to X$ is any automorphism and

$$f^* = g_u g_s = g_s g_u$$

is the Jordan decomposition of f^*, then by Lemma 1.2.2 the unipotent part $g = g_u$ satisfies the assumptions above.

Theorem A is thus a special case of the following:

Theorem 3.0.1. *Let X be a compact Kähler threefold and let $g\colon H^*(X,\mathbb{R}) \to H^*(X,\mathbb{R})$ be a unipotent linear automorphism preserving the cohomology graduation, the Hodge decomposition, the wedge-product and the Poincaré duality. Then*

(1) *the maximal Jordan block of g_2 (for the eigenvalue 1) has dimension ≤ 5;*
(2) *if furthermore g_2 preserves the cone $\mathcal{C} = \{v \in H^2(X,\mathbb{R})\,;\, q(v) \geq 0\}$, then its maximal Jordan block has odd dimension.*

In particular the norm of g_2^n grows as cn^k with $k \leq 4$; and if furthermore g_2 preserves the positive cone, then k is even.

Remark 3.0.2. Let $f \in \mathrm{Aut}(X)$ be an automorphism such that $\lambda_1(f) = 1$, so that, up to iterating f, $g = f^*$ satisfies the assumptions of Theorem 3.0.1. In this case, by Proposition 1.3.9, the growth of $\|g^n\|$ is the same as the maximal growths of the $\|g^n_{p,p}\|$, i.e. the growth of $\|g_2^n\|$. Furthermore, g preserves the cone $\mathcal{C}$, therefore the maximal Jordan block has odd dimension.

Proof. Let $u_1,\dots,u_k \in H^2(X,\mathbb{R})$ be a basis of a maximal Jordan block satisfying

$$g(u_1) = u_1, \qquad g(u_h) = u_h + u_{h-1} \quad \text{for } h = 2,\dots,k.$$

Since the subspaces $H^{1,1}(X,\mathbb{R})$ and $(H^{2,0}(X)\oplus H^{0,2}(X))_{\mathbb{R}}$ are g_2-invariant, we may and will suppose that $u_i \in H^{1,1}(X,\mathbb{R}) \cup (H^{2,0}(X)\oplus H^{0,2}(X))_{\mathbb{R}}$ for $i = 1,\dots,k$.

We show that the norm of g_4^n grows at least as cn^{2k-6}. Let us consider the elements $u_k \wedge u_k, u_{k-1}\wedge u_{k-1} \in H^4(X,\mathbb{R})$.
An easy linear algebra computation shows that the n-th iteration of a Jordan block is

$$\begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 1 \\ 0 & \dots & & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} P_0(n) & P_1(n) & P_2(n) & \dots & P_{k-1}(n) \\ 0 & P_0(n) & P_1(n) & \dots & P_{k-2}(n) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & P_0(n) & P_1(n) \\ 0 & \dots & & 0 & P_0(n) \end{pmatrix},$$

where $P_h(n)$ is a polynomial of degree h in n whose leading term is $n^h/h!$.
Therefore, letting $P_h = P_h(n)$,

$$\begin{aligned} &g^n(u_k\wedge u_k) = g^n(u_k)\wedge g^n(u_k) = \\ &= (P_{k-1}u_1 + P_{k-2}u_2 + P_{k-3}u_3 + \dots) \wedge (P_{k-1}u_1 + P_{k-2}u_2 + P_{k-3}u_3 + \dots) = \\ &= P_{k-1}^2(u_1\wedge u_1) + 2P_{k-1}P_{k-2}(u_1\wedge u_2) + (2P_{k-1}P_{k-3}u_1\wedge u_3 + P_{k-2}^2 u_2\wedge u_2) + \dots \end{aligned}$$

If $u_1\wedge u_1 \neq 0$ or $u_1 \wedge u_2 \neq 0$, then the norm of g_4^n would grow at least as n^{2k-3}; we can thus assume that $u_1\wedge u_1 = u_1\wedge u_2 = 0$. Thus, by Corollary 1.4.2 we have $u_2\wedge u_2 \neq 0$.
Similarly, if we had

$$2\frac{u_1\wedge u_3}{(k-1)!(k-3)!} + \frac{u_2\wedge u_2}{((k-2)!)^2} \neq 0, \tag{3.1}$$

then the norm of g_4^n would grow at least as cn^{2k-4}. We may then assume that equation 3.1 is not satisfied.
Now,

$$\begin{aligned} &g^n(u_{k-1}\wedge u_{k-1}) = g^n(u_{k-1})\wedge g^n(u_{k-1}) = \\ &= (P_{k-2}u_1 + P_{k-3}u_2 + P_{k-4}u_3 + \dots) \wedge (P_{k-2}u_1 + P_{k-3}u_2 + P_{k-4}u_3 + \dots) = \\ &\qquad\qquad = 2P_{k-2}P_{k-4}(u_1\wedge u_3) + P_{k-3}^2(u_2\wedge u_2) + \dots \end{aligned}$$

By the same argument, if

$$2\frac{u_1\wedge u_3}{(k-2)!(k-4)!} + \frac{u_2\wedge u_2}{((k-3)!)^2} \neq 0, \tag{3.2}$$

then the norm of g_4^n would grow at least as cn^{2k-6}. Since $u_2\wedge u_2 \neq 0$ and the two linear relations 3.1 and 3.2 are independent, at least one between Equation 3.1 and Equation 3.2 is satisfied.
This shows that the norm of g_4^n grows at least as cn^{2k-6}.

Now, by Poincaré duality (see Proposition 1.1.1.(3)), the norm of $g_4^n = (g_2^{-n})^\vee$ grows exactly as the norm of g_2^n. In particular

$$k-1 \geq 2k-6 \qquad \Rightarrow \qquad k \leq 5,$$

which concludes the proof. □

3.1. **Bound on the dimension of non-maximal Jordan blocks.**

Proposition 3.1.1. *Let X be a compact Kähler threefold and let $g\colon H^*(X,\mathbb{R}) \to H^*(X,\mathbb{R})$ be a unipotent linear automorphism preserving the cohomology graduation, the Hodge decomposition, the wedge-product and the Poincaré duality. Then there exists a unique Jordan block of g_2 of maximal dimension $k \le 5$ (for the eigenvalue 1); more precisely, all other Jordan blocks have dimension $\le \frac{k+1}{2}$.*

Proof. Let $v_1, \dots, v_k \in H^2(X,\mathbb{R})$ form a basis for a maximal Jordan block of g_2, and let $w_1, \dots, w_l \in H^2(X,\mathbb{R})$ form a Jordan basis for another Jordan block satisfying

$$g(v_1) = v_1, \qquad g(v_h) = v_h + v_{h-1} \quad \text{for } h = 2, \dots, k,$$

$$g(w_1) = w_1, \qquad g(w_h) = w_h + w_{h-1} \quad \text{for } h = 2, \dots, l.$$

Since the subspaces $H^{1,1}(X,\mathbb{R})$ and $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ are g_2-invariant, we may and will suppose that $v_i, w_j \in H^{1,1}(X,\mathbb{R}) \cup (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ for $i = 1, \dots, k$ and $j = 1, \dots, l$.
We will suppose that $l > 1$ (otherwise the claim is evident), and consider the action of g_4 on the classes $v_k \wedge v_k, v_k \wedge w_l, w_l \wedge w_l \in H^4(X,\mathbb{R})$.
By Corollary 1.4.2, the classes $v_1 \wedge v_1, v_1 \wedge w_1, w_1 \wedge w_1 \in H^4(X,\mathbb{R})$ cannot be all null; since $g^n v_k \sim c n^{k-1} v_1$ and $g^n w_l \sim c' n^{h-1} w_1$, this implies that $\|g_4^n\|$ grows at least as $c'' n^{2(h-1)}$. Since by Poincaré duality $\|g_2^n\|$ and $\|g_4^n\|$ have the same growth, we get

$$2(h-1) \le k-1 \qquad \Rightarrow \qquad h \le \frac{k+1}{2},$$

which concludes the proof. $\square$

3.2. **Unipotent examples on complex tori.** Examples on complex tori of dimension 3 show the optimality of Theorem 3.0.1 and Proposition 3.1.1. Let $E = \mathbb{C}/\Lambda$ be an elliptic curve, where Λ is a lattice of $\mathbb{C}$, and let

$$X := E \times E \times E = {\mathbb{C}^3}/{\Lambda^3}.$$

Every matrix $M \in \mathrm{SL}_3(\mathbb{Z})$ acts linearly on $\mathbb{C}^3$ preserving the lattice $\Lambda \times \Lambda \times \Lambda$, and therefore induces an automorphism $f\colon X \to X$. One can easily show that, if dx, dy, dz are holomorphic linear coordinates on the three factors respectively, then the matrix of $f_{1,0}^*$ with respect to the basis dx, dy, dz of $H^{1,0}(X)$ is exactly the transposed M^T. Since the wedge product of forms induces an isomorphism

$$H^{1,1}(X) \cong H^{1,0}(X) \otimes H^{0,1}(X),$$

the matrix of $f_{1,1}^*$ with respect to the basis $dx \wedge d\bar{x}, dx \wedge d\bar{y}, \dots, dz \wedge d\bar{z}$ is

$$M_{1,1} = M^t \otimes \overline{M^t} := (m_{j,i}\overline{m_{l,k}})_{i,j,k,l=1,2}.$$

Example 3.2.1. Let

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then

$$M_{1,1} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$M_{1,1}$ is unipotent and its Jordan blocks have dimension $1, 3$ and 5.

Example 3.2.2. Let

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$M_{1,1} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$M_{1,1}$ is unipotent and its Jordan blocks have dimension $2, 2, 2$ and 3.

4. Automorphisms of threefolds: the semi-simple case, proof of Theorem B

Throughout this section let X be a compact Kähler threefold and let

$$g \colon H^*(X, \mathbb{R}) \to H^*(X, \mathbb{R})$$

be a *semi-simple* linear automorphism preserving the cohomology graduation, the Hodge decomposition, the wedge-product and Poincaré's duality. In other words,

(1) $g(H^i(X, \mathbb{R})) = H^i(X, \mathbb{R})$ for all $i = 0, \dots, 6$ and $g_{\mathbb{C}}(H^{p,q}(X)) = H^{p,q}(X)$ for all $p, q = 0, \dots 3$;
(2) $g(u \wedge v) = g(u) \wedge g(v)$ for all $u, v \in H^*(X, \mathbb{R})$;
(3) denoting by $g_i \colon H^i(X, \mathbb{R}) \to H^i(X, \mathbb{R})$ the restriction of g, the Poincaré's duality $H^i(X, \mathbb{R}) \cong H^{6-i}(X, \mathbb{R})^\vee$ induces an identification

$$g_i = (g_{6-i}^{-1})^\vee.$$

If $f \colon X \to X$ is an automorphism and

$$f^* = g_u g_s = g_s g_u$$

is the Jordan decomposition of f^*, then by Lemma 1.2.2 the semisimple part $g = g_s$ satisfies the assumptions above.

Let $\lambda_1 = \lambda_1(g)$ and $\lambda_2 = \lambda_2(g)$ be the dynamical degrees of g, i.e. the spectral radii of g_2 and g_4 respectively, and let Λ be the spectrum of g_2, i.e. the set of complex eigenvalues of g_2 with multiplicities; we will say that two elements $\lambda, \lambda' \in \Lambda$ are distinct if either $\lambda \neq \lambda'$ or $\lambda = \lambda'$ is an eigenvalue with multiplicity ≥ 2.

Remark 4.0.1. Let $\lambda \in \Lambda$; since the subset $S = H^{1,1}(X) \cup (H^{2,0}(X) \oplus H^{0,2}(X)) \subset H^2(X, \mathbb{C})$ is g_2-invariant, λ is the eigenvalue of an eigenvector $v \in S$. From now on, *every time we talk about eigenvectors of g_2 we pick them in S.*

The main ingredient of the proofs in the rest of this section is the following lemma.

Lemma 4.0.2. *Let $\lambda, \lambda' \in \Lambda$ be distinct elements; then*

$$\left\{ \frac{1}{|\lambda|^2}, \frac{1}{\lambda \bar{\lambda}'}, \frac{1}{|\lambda'|^2} \right\} \cap \Lambda \neq \emptyset.$$

Proof. By Remark 4.0.1, we may pick eigenvectors $v, v' \in S$ for the eigenvalues λ, λ' respectively. By Corollary 1.4.2, the wedge products

$$v \wedge \bar{v}, v \wedge \bar{v}', v' \wedge \bar{v}' \in H^4(X, \mathbb{C})$$

cannot all be null. A non-null wedge product gives rise to an eigenvector for g_4; in particular, denoting by Λ_4 the spectrum of g_4, we have

$$\left\{ |\lambda|^2, \lambda \bar{\lambda}', |\lambda'|^2 \right\} \cap \Lambda_4 \neq \emptyset.$$

Now, by assumption (3) g_4 can be identified with $(g_2^{-1})^\vee$, and in particular

$$\Lambda = \Lambda_4^{-1} = \{\lambda^{-1}\, ,\, \lambda \in \Lambda_4\}.$$

This concludes the proof. $\square$

4.1. **Structure of the algebraic group generated by g_2.** For the content of this Section we refer to [3, §8]. Let g be as above and let

$$G = \overline{\langle g_2 \rangle}^{Zar} \leq \mathrm{GL}(H^2(X, \mathbb{R}))$$

be the Zariski-closure of the group generated by g; it is a real algebraic group by [3, Proposition I.1.3]. Since the properties of preserving the Hodge decomposition is algebraic, G satisfies it; furthermore, since $\langle g \rangle$ is diagonalizable over $\mathbb{C}$ and commutative, so is G.

The Zariski-connected component of the identity G_0 of G is thus a real algebraic torus; we define $G_d \leq G_0$ as the subgroup generated by real one-parameter subgroups of G_0, and $G_a \leq G_0$ as the intersection of the kernels of real characters of G_0. Then we have the following classical result:

Proposition 4.1.1. *Let G be as above; then*

1. $G_d \cong (\mathbb{R}^*)^r$ *is the maximal split subtorus;*
2. $G_a \cong (S^1)^s$ *is the maximal anisotropic subtorus;*
3. *the product morphism $G_d \times G_a \to G_0$ is an isogeny (i.e. it is surjective and with finite kernel).*

The number $r \geq 0$ is the (real) split-rank of G; we will denote it by $r(g)$ and call it the *rank of g*; informally, $r(g)$ (respectively $s(g)$) is the number of multiplicative parameters which are necessary to describe the moduli (respectively, the arguments) of the complex eigenvalues of g_2.

4.2. **Weights of g.** Let $\lambda \in \Lambda$ be a complex eigenvalue of g_2; then the group homomorphism

$$\langle g \rangle \to \mathbb{R}^*$$
$$g^n \mapsto |\lambda|^n$$

is algebraic, and therefore can be extended to a non-trivial real character of G. Upon restriction to G_0 and pull-back to $G_d \times G_a \cong (\mathbb{R}^*)^r \times (S^1)^s$, this yields a non-trivial morphism of real algebraic groups

$$\rho_\lambda \colon (\mathbb{R}^*)^r \times (S^1)^s \to \mathbb{R}^*.$$

Since all morphisms of real algebraic groups $S^1 \to \mathbb{R}^*$ are trivial, we have

$$\rho_\lambda(x_1 \dots, x_r, \theta_1, \dots, \theta_s) = x_1^{m_1(\lambda)} \cdots x_r^{m_r(\lambda)}, \qquad m_i \in \mathbb{Z}.$$

For the sake of simplicity, we will adopt an additive notation, so that the character ρ_λ is identified with the vector $w_\lambda = (m_1(\lambda), \dots, m_r(\lambda)) \in \mathbb{R}^r$.

Definition 4.2.1. *The* weight *of the eigenvalue $\lambda \in \Lambda$ of g_2 is the vector $w_\lambda = (m_1(\lambda), \dots, m_r(\lambda)) \in \mathbb{R}^r$. We denote by W the set of all weights of eigenvalues of g_2 with multiplicities; as for the elements of Λ, we say that two elements w, w' of W are distinct if either $w \neq w'$ or $w = w'$ has multiplicity > 1.*

Remark 4.2.2. Remark that $w_\lambda = w_{\bar\lambda}$. Therefore, if λ is a non-real eigenvalue of g_2, the weight w_λ will be counted twice, once for λ and once for $\bar\lambda$.

We say that a weight $w_0 \in W$ is *maximal* for a linear functional $\alpha \in (\mathbb{R}^r)^\vee$ if $|\alpha(w_0)| = \max_{w \in W} |\alpha(w)|$. The maximal weights are exactly those belonging to the boundary of the convex hull of W.

Lemma 4.2.3. *There exist a basis $w_1, \dots, w_r$ of $\mathbb{R}^r$ and a basis $\alpha_1, \dots, \alpha_r$ of $(\mathbb{R}^r)^\vee$ such that*

1. *the w_i belong to W;*
2. *w_i is α_i-maximal for all i;*
3. *if $i > j$, then $\alpha_i(w_j) = 0$.*

Proof. Remark first that, since G is defined as the Zariski-closure of $\langle g \rangle$, the elements of W span the vector space $\mathbb{R}^r$.

We construct the adapted basis inductively. Since W is finite, there exists a maximal weight, say w_1, for a functional α_1.
Now, suppose that $w_1, \dots, w_k \in W \subset \mathbb{R}^r$ and $\alpha_1, \dots, \alpha_k \in (\mathbb{R}^r)^\vee$ are linearly independent and satisfy properties $(1) - (3)$. Pick any

$$\alpha_{k+1} \in \{\alpha \in (\mathbb{R}^r)^\vee \mid \alpha(w_1) = \ldots = \alpha(w_k) = 0\} \setminus \{0\} \subset (\mathbb{R}^r)^\vee,$$

and let $w_{k+1} \in W$ be α_{k+1}-maximal. By the condition on α_{k+1}, w_{k+1} does not belong to the span of $w_1, \dots, w_k$. This completes the proof by induction. □

In the language of weights, Lemma 4.0.2 becomes the following:

Lemma 4.2.4. *Let $w, w' \in W$ be distinct elements; then*

$$\{-2w, -w - w', -2w'\} \cap W \neq \emptyset.$$

If furthermore $w = w_\lambda$ is the weight of an eigenvalue λ of $f^*_{2,0}$ or $f^*_{0,2}$, then $-2w \in W$.

Remark 4.2.5. If $\lambda \in \Lambda$ is maximal, then $|\lambda|^{-2} \notin \Lambda$ (and in particular its weight $w_\lambda \in W$ is simple); therefore, if w, w' are maximal weights, by Lemma 4.2.4, $-w - w' \in W$.

As a preliminary result, we bound the rank of g:

Lemma 4.2.6. *The rank of g satisfies $r(g) \leq 2$.*

Proof. Let us fix bases $w_1, \dots, w_r$ and $\alpha_1, \dots, \alpha_r$ of $\mathbb{R}^r$ and $(\mathbb{R}^r)^\vee$ respectively as in Lemma 4.2.3, and suppose by contradiction that $r \geq 3$.
Since w_1, w_2 and w_3 are maximal, we have $-2w_1, -2w_2, -2w_3 \notin W$, and therefore by Lemma 4.2.4

$$-w_1 - w_2, -w_2 - w_3, -w_3 - w_1 \in W.$$

Again by maximality, we have

$$2(w_1 + w_2), 2(w_2 + w_3), 2(w_3 + w_1) \notin W,$$

so that, by Lemma 4.2.4 applied to $-w_2 - w_3, -w_3 - w_1 \in W$, we must have $w_1 + w_2 + 2w_3 \in W$. However this contradicts the α_3-maximality of w_3. $\square$

We will show later that the rank of g is < 2 if and only if its dynamical degrees $\lambda_1(g)$ and $\lambda_2(g)$ satisfy a resonance condition:

$$\lambda_1(g)^m = \lambda_2(g)^n, \qquad (m, n) \neq (0, 0);$$

see Corollary 4.3.2.

4.3. **The case $r(g) = 2$.** Recall that we denote by $\lambda_1 = \lambda_1(g)$ and $\lambda_2 = \lambda_2(g)$ the dynamical degrees of g, by Λ the spectrum of g_2 (with multiplicities) and by W the set of weights of eigenvalues $\lambda \in \Lambda$ (with multiplicities).
Throughout this section, we assume that the rank of g (i.e. the split-rank of $G = \overline{\langle g \rangle}^{Zar}$, see Section 4.1) is equal to 2; in other words, the elements of W span a real vector space of dimension 2.

Proposition 4.3.1. *Let w_1 (respectively w_2) be the weight of W associated to the eigenvalue $\lambda_1 \in \Lambda$ (respectively $\lambda_2^{-1} \in \Lambda$).*

1. *$\lambda_1^{-1}\lambda_2$ is an element of Λ, whose weight is $w_3 := -w_1 - w_2$;*
2. *w_1, w_2 and w_3 are maximal weights of W, and in particular they have multiplicity 1 in W;*
3. *for any weight $w \in W \setminus \{w_1, w_2, w_3\}$ and for any eigenvalue $\lambda \in \Lambda$ whose weight is w, $|\lambda|^{-2} \in \Lambda$ and in particular $-2w \in W$;*
4. *there exist $n_1, n_2, n_3 \geq 0$ such that, up to multiplicities,*

$$W \setminus \{0\} = \bigcup_{i=1,2,3} \left\{ \frac{w_i}{(-2)^n} \, ; \, n = 0, \dots, n_i \right\}.$$

Proof. Let us fix an adapted basis w_1, w_2 of $\mathbb{R}^2$ as in Lemma 4.2.3, and let $w_3 := -w_1 - w_2$. We show first properties (2) – (4) for these w_1, w_2, w_3, and then that, after maybe permuting indices, w_1, w_2 and w_3 are the weights of $\lambda_1, \lambda_2^{-1}, \lambda_1^{-1}\lambda_2 \in \Lambda$ respectively.

The maximality of w_1, w_2 is part of Lemma 4.2.3; since $\alpha_2(w_1) = 0$, w_3 is also α_2-maximal. Property (2) then follows from Remark 4.2.5.

Now let $\lambda \in \Lambda$ be an eigenvalue of g_2 whose weight w is different than w_1, w_2, w_3; we want to show that $|\lambda|^{-2} \in \Lambda$.

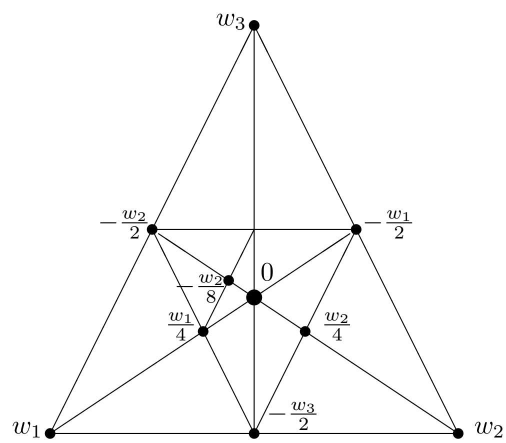

FIGURE 1. An example of the structure of $W \subset \mathbb{R}^2$ (without taking multiplicities into account) in the case $r(g) = 2$; here, with the notation of Proposition 4.3.1, $n_1 = 2, n_2 = 1, n_3 = 3$.

Suppose first that $w = 0$ (i.e. $|\lambda| = 1$), and let $\lambda' \in \Lambda$ be an eigenvalue whose weight is w_1; recall that by maximality $-2w_1 \notin W$ and w_1 is a simple weight. If $1 \notin \Lambda$, then by Lemma 4.0.2 we would have $\lambda\lambda' \in \Lambda$, which contradicts the simplicity of w_1.
Now suppose that $\alpha_2(w) \neq 0$; since $\alpha_2(w_2)$ and $\alpha_2(w_3)$ have different sign, we have either $|\alpha_2(-w - w_2)| > |\alpha_2(w_2)|$ or $|\alpha_2(-w - w_3)| > |\alpha_2(w_3)| = |\alpha_2(w_2)|$, so that by maximality $-w - w_2$ and $-w - w_3$ cannot be both weights of W. Since, again by maximality, $-2w_2, -2w_3 \notin W$, by Lemma 4.0.2 $|\lambda|^{-2} \in \Lambda$.
Finally suppose that $w \neq 0$ and $\alpha_2(w) = 0$, so that $w \in \mathbb{R}w_1$. We repeat the inductive construction of an adapted basis as in the proof of Lemma 4.2.3 starting with $w_1' := w_2$, which is maximal for $\alpha_1' := \alpha_2$; pick any non-trivial $\alpha_2' \in w_2^\perp \subset (\mathbb{R}^2)^\vee$ and $w_2' \in W$ maximal for α_2'. If we had again $\alpha_2'(w) = 0$, then $w \in \mathbb{R}w_1 \cap \mathbb{R}w_2 = \{0\}$, a contradiction; thus we can conclude as above.
This shows property (3).

Property (4) follows from property (3) by induction.

Now let us show that, after permuting indices, w_1 and w_2 are the weights of λ_1 and λ_2^{-1} respectively. Since $\lambda_2(g) = \lambda_1(g^{-1})$, it is enough to show that the weight of λ_1 is one of the w_i.
Suppose by contradiction that the weight w of λ_1 is not one of the w_i; then by property (4) there exist $k > 0$ and $i \in \{1, 2, 3\}$ such that

$$w = \frac{w_i}{(-2)^k}.$$

Since λ_1 is the spectral radius of g_2, we have $\lambda_1^4 \notin \Lambda$, so that $k = 1$; up to permuting the indices, we may suppose that $i = 1$, so that

$$2w = -w_1 = w_2 + w_3.$$

Denoting by $\lambda, \lambda' \in \Lambda$ the eigenvalues associated to w_2 and w_3, this means that

$$|\lambda\lambda'| = \lambda_1^2;$$

since λ_1 is the spectral radius of g_2, this implies that $|\lambda| = |\lambda'| = \lambda_1$, contradicting the assumption that $r(g) = 2$.
This shows that we may assume that w_1 and w_2 are the weights of the eigenvalues $\lambda_1, \lambda_2^{-1} \in \Lambda$. Since $w_3 = -w_1 - w_2$ has multiplicity 1 in W, it is associated to a real simple eigenvalue, which is $\lambda_1^{-1}\lambda_2$ by Lemma 4.0.2. This concludes the proof. □

Proposition 4.3.1 shows in particular that, if $r(g) = 2$, then λ_1 and λ_2 do not have any resonance:

$$\lambda_1(g)^m = \lambda_2(g)^n,\ m, n \in \mathbb{Z} \quad \Leftrightarrow \quad m = n = 0.$$

Conversely, if $r = 1$, since all the weights of g_2 can be interpreted as integers, λ_1 and λ_2 satisfy a non-trivial equation $\lambda_1(g)^m = \lambda_2(g)^n$. Thus we have the following:

Corollary 4.3.2. *The rank of g is equal to 2 if and only if the dynamical degrees of g do not have any resonance.*

4.4. **The case $r(g) = 1$.** Recall that we denote by $\lambda_1 = \lambda_1(g)$ and $\lambda_2 = \lambda_2(g)$ the dynamical degrees of g, by Λ the spectrum of g_2 (with multiplicities) and by W the set of weights of eigenvalues $\lambda \in \Lambda$ (with multiplicities).
Throughout all this section, we assume that the rank of g (i.e. the split-rank of $G = \overline{\langle g \rangle}^{Zar}$, see Section 4.1) is equal to 1; in other words, the elements of W span a real vector space of dimension 1. In this case the weights are equipped with a natural order: $w_\lambda > w_{\lambda'}$ if and only if $|\lambda| > |\lambda'|$; for $w \in W$ we set $|w| := \max\{w, -w\}$.

Proposition 4.4.1. *Suppose that $r = 1$ and let w_1 (respectively w_2) be the weight of W associated to the eigenvalue $\lambda_1 \in \Lambda$ (respectively $\lambda_2^{-1} \in \Lambda$). Then*

(1) *$w_3 = -w_1 - w_2$ is a weight of W;*
(2) *for any weight $w \in W \setminus \{0, w_1, w_2, w_3\}$ and for any eigenvector v with eigenvalue $\lambda \in \Lambda$ whose weight is w, we have $v \wedge \bar{v} \neq 0$, and in particular $|\lambda|^{-2} \in \Lambda$ and $-2w \in W$;*
(3) *there exist $n_1, n_2 \geq 0$ such that, up to multiplicities,*
$$W \setminus \{0\} = \bigcup_{i=1,2,3} \left\{ \frac{w_i}{(-2)^n}\ ;\ n = 0, \dots, n_i \right\};$$
(4) *if furthermore $w_2 \notin \{-2w_1, -w_1/2\}$, then w_1 and w_2 have multiplicity 1 in W.*

Remark 4.4.2. Let $g = f_s^*$, where $f \colon X \to X$ is an automorphism and f_s^* denotes the semisimple part of the induced linear automorphism $f^* \in \mathrm{GL}(H^*(X, \mathbb{R}))$. Then the assumption $w_2 \notin \{-2w_1, -w_1/2\}$ means exactly that the log-concavity inequalities

$$\sqrt{\lambda_1(f)} \leq \lambda_2(f) \leq \lambda_1(f)^2$$

are strict (see Proposition 1.3.5). If this is the case, then by Proposition 4.3.1 and 4.4.1 λ_1 and λ_2^{-1} are the only eigenvalues of f_2^* having such modulus.

Proof. After possibly replacing g by g^{-1}, we may suppose that $w_1 = |w_1| \geq |w_2| = -w_2$, so that w_1 is maximal; let $v_1, v_2 \in H^2(X, \mathbb{R})$ denote eigenvectors for the eigenvalues $\lambda_1, \lambda_2^{-1} \in \Lambda$.
Remark that, if $v \in H^2(X, \mathbb{C})$ is an eigenvector whose eigenvalue $\lambda \in \Lambda$ has weight $w \in]0, w_1[$, then by Lemma 4.0.2 applied to v and v_1 and by maximality of w_1 we have $v \wedge \bar{v} \neq 0$.

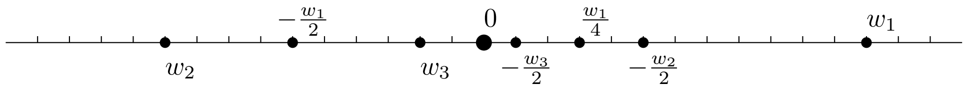

FIGURE 2. Example of weights of $W \subset \mathbb{R}$ (without taking multiplicities into account) in the case $r(g) = 1$; here, with the notation of Proposition 4.4.1, $n_1 = 2, n_2 = 1, n_3 = 1$.

Let us prove first that $\lambda_1^{-1}\lambda_2 \in \Lambda$. If $w_1 = -2w_2$, then $\lambda_1^{-1}\lambda_2 = \lambda_2^{-1}$ and the claim is evident; therefore we may suppose that $w_1 \neq -2w_2$. We observe first that, since λ_2^{-1} is the minimal modulus of eigenvalues of g_2, w_2 is the minimal weight for the natural order introduced above. Now, if we had $\lambda_1^{-1}\lambda_2 \notin \Lambda$, then by Lemma 4.0.2 we would have $\lambda_2^2 \in \Lambda$ and in particular $-2w_2 \in W$; by the above remark, this implies that $4w_2 \in W$, contradicting the minimality of w_2. This shows (1).

Now let us show that for all $\lambda \in \Lambda$ whose weight is $w \in W \setminus \{0, w_1, w_2, w_3\}$ and for all eigenvector $v \in H^2(X, \mathbb{C})$ with eigenvalue λ we have $v \wedge \bar{v} \neq 0$. The case $w > 0$ (i.e. $|\lambda| \geq 1$) has been treated above; let then $w < 0$, and suppose by contradiction that $v \wedge \bar{v} = 0$. Then by Lemma 4.0.2 we get $-w - w_2 \in W$, and since $-w - w_2 > 0$ and $-w - w_2 \neq w_1, w_3$, we also have $2w + 2w_2 \in W$; this contradicts the minimality of w_2 for the natural order, and concludes the proof of (2).

Property (3) follows from (2) by induction.

Now assume that $w_1 \neq -2w_2$ and suppose by contradiction that w_2 has multiplicity > 1 in W. Then by Lemma 4.2.4 $-2w_2 \in W$, and since $-2w_2 > 0$ we also have $4w_2 \in W$. This contradicts the minimality of λ_2 for the natural order and proves (4). $\square$

4.5. **Automorphisms of threefolds: the semisimple case, proof of Theorem C.** Let X be a compact Kähler threefold and let $g \in \mathrm{GL}(H^*(X, \mathbb{R}))$ be a *semisimple* linear automorphism preserving the Hodge decomposition, the wedge product and Poincaré's duality, and such that g and g^{-1} are defined over $\mathbb{Z}$.
We denote as usual by λ_1 and λ_2 the dynamical degrees of g (i.e. the spectral radii of g_2 and g_4 respectively), by Λ the spectrum of g_2 (with multiplicities) and by W the set of weights of g_2 (with multiplicities).
Recall that we pick all eigenvectors of g_2 inside the union of subspaces $H^{1,1}(X) \cup (H^{2,0}(X) \oplus H^{0,2}(X))$ (see Remark 4.0.1).

Let $P(T)$ be the minimal polynomial of g_2; since g_2 is defined over $\mathbb{Z}$, we have $P(T) \in \mathbb{Z}[T]$. Since g is semisimple, we can write

$$P(T) = P_1(T) \cdots P_n(T),$$

where the $P_i \in \mathbb{Z}[T]$ are distinct and irreducible over $\mathbb{Q}$. Let P_1 be the factor having $\lambda_1(g)$ as a root, and denote by $\Lambda_i \subset \Lambda$ (respectively $W_i \subset W$) the set of roots of P_i (respectively the set of weights of roots of P_i).

For $i = 1, \dots, n$ let

$$V_i = \ker P_i(g_2) \subset V := H^2(X, \mathbb{C}).$$

Since g is semisimple, we have

$$V = \bigoplus_{i=1}^{n} V_i.$$

For a polynomial $Q \in k[T]$ define

$$Q^\vee(T) = T^{\deg Q} \cdot Q(T^{-1}).$$

Poincaré's duality allows to identify $H^4(X,\mathbb{C})$ with $H^2(X,\mathbb{C})^\vee = V^\vee$; under this identification, we have $g_4 = (g_2^{-1})^\vee$, so that the minimal polynomial of g_4 is

$$P^\vee(T) = P_1^\vee(T) \cdots P_n^\vee(T).$$

Since g_4 is semisimple, we have

$$V^\vee = H^4(X,\mathbb{C}) = \bigoplus_{i=1}^n V_i^\vee.$$

Finally, let us define the bilinear map

$$\begin{aligned} \theta\colon H^2(X,\mathbb{C}) \times H^2(X,\mathbb{C}) &\to H^4(X,\mathbb{C}) \\ (u,v) &\mapsto u \wedge v. \end{aligned}$$

Remark 4.5.1. The V_i and the $V_i^\vee$ are g-invariant subspaces defined over $\mathbb{Q}$; furthermore, if the roots of some P_i are simple eigenvalues of g_2 (or, equivalently, if P_i is a simple factor of the characteristic polynomial of g_2), then V_i is minimal for such property: $\{0\}$ is the only proper subspace of V_i which is g_2-invariant and defined over $\mathbb{Q}$. The same holds for the action of g_4 on $V_i^\vee$.

The goal of this section is to describe which are the possible (moduli of) roots of a given P_i, most importantly for the factor having λ_1 as a root.

In what follows, we say for short that $\lambda, \lambda' \in \Lambda$ are conjugate if they are conjugate over $\mathbb{Q}$.

Remark 4.5.2. Since $P_i(0) = \pm 1$, we have

$$\prod_{\lambda \in \Lambda_i} \lambda = \pm 1, \qquad \sum_{w \in W_i} w = 0.$$

Definition 4.5.3. *Let $\lambda \in \Lambda$; we say that a weight $w \in W$ is conjugate to λ if one of the conjugates of λ has weight w.*

The main technical tool for the proofs in this section is the following basic result in Galois theory (see for example [18]).

Lemma 4.5.4. *Let $\alpha, \beta \in \overline{\mathbb{Q}}$ be two algebraic numbers. If α and β are conjugate, then there exists $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = \{\rho \in \mathrm{Aut}(\overline{Q}) \mid \rho_{|\mathbb{Q}} = \mathrm{id}_{\mathbb{Q}}\}$ such that $\rho(\alpha) = \beta$.*

Remark that, since elements of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ act as the identity on $\mathbb{Q}$, the polynomials P_i are fixed; in particular $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts by permutations on each Λ_i and on each W_i.

4.6. **The case** $r(g) = 2$**.** Let us treat first the case where the rank of g (i.e. the split-rank of $G = \overline{\langle g \rangle}^{Zar}$, see Section 4.1) is equal to 2.

We denote as usual by $w_1, w_2, w_3 \in W$ the weights of the eigenvalues of g_2

$$\alpha_1 := \lambda_1, \quad \alpha_2 := \lambda_2^{-1}, \quad \alpha_3 := \lambda_1^{-1}\lambda_2,$$

and fix non-null eigenvectors $v_1, v_2, v_3 \in H^2(X,\mathbb{R})$ for these eigenvalues.

4.6.1. *Algebraic properties of the eigenvalues.*

Lemma 4.6.1. *Let $r = 2$ and $\lambda \in \Lambda$. If one of the conjugates of λ has modulus 1, then λ is a root of unity.*

Proof. By a lemma of Kronecker, if all the conjugates of an algebraic integer λ have modulus 1, then λ is a root of unity. Therefore, we only need to show that, if a conjugate of λ has modulus 1, then λ has also modulus 1. Suppose by contradiction that this is not the case, and let μ be a conjugate of λ such that $|\mu| = 1$.
Let $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}, \mathbb{Q})$ be such that $\rho(\mu) = \lambda$; since $\mu \cdot \bar{\mu} = 1$, we have

$$\lambda \cdot \rho(\bar{\mu}) = 1,$$

so that $\rho(\bar{\mu}) = \lambda^{-1}$. In terms of weights, this means that w_λ and $w_{\lambda^{-1}} = -w_\lambda$ are both non-trivial weights of W. This contradicts Proposition 4.3.1 and concludes the proof. □

Proposition 4.6.2. *Let $r = 2$. Then for all $1 \le k \le n$ there exist $n_i = n_i(k)$, $i = 1, 2, 3$, such that, without taking multiplicities into account,*

$$W_k \subset \left\{ \frac{w_1}{(-2)^{n_1}}, \frac{w_1}{(-2)^{n_1+1}}, \frac{w_2}{(-2)^{n_2}}, \frac{w_2}{(-2)^{n_2+1}}, \frac{w_3}{(-2)^{n_3}}, \frac{w_3}{(-2)^{n_3+1}} \right\}.$$

Proof. Let $\lambda \in \Lambda_i$, and let $w = w_\lambda$ be its weight. We will prove that if a weight w' collinear to w is conjugate to λ, then

$$w' \in \left\{ -\frac{w}{2}, w, -2w \right\}.$$

The claim then follows easily.

Suppose by contradiction that $w' = w_{\lambda'} \notin \{-w/2, w, -2w\}$; remark first that by Lemma 4.6.1 w and w' are both non-trivial. By Proposition 4.3.1, after maybe swapping λ and λ', we have

$$w' = (-2)^k w, \quad k \ge 2,$$

which means that

$$\lambda\bar{\lambda} = (\lambda'\bar{\lambda}')^{(-2)^k}.$$

Now, let $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be an automorphism such that $\rho(\lambda)$ is a conjugate of λ whose weight can be written as $w_a/(-2)^{n_a}$, $a \in \{1, 2, 3\}$, with n_a maximal. Let $\alpha, \beta, \gamma, \delta$ denote the images of $\lambda, \bar{\lambda}, \lambda', \bar{\lambda}$ under ρ, and let

$$w_\alpha = \frac{w_a}{(-2)^{n_a}}, \quad w_\beta = \frac{w_b}{(-2)^{n_b}}, \quad w_\gamma = \frac{w_c}{(-2)^{n_c}}, \quad w_\delta = \frac{w_d}{(-2)^{n_d}}$$

denote their weights; here $a, b, c, d \in \{1, 2, 3\}, n_a, n_b, n_c, n_d \ge 0$ and n_a is maximal. Since

$$\alpha\beta = (\gamma\delta)^{(-2)^k},$$

in terms of weights we get

$$\frac{w_a}{(-2)^{n_a}} + \frac{w_b}{(-2)^{n_b}} = (-2)^k \frac{w_c}{(-2)^{n_c}} + (-2)^k \frac{w_d}{(-2)^{n_d}},$$

so that

$$w_a + (-2)^{n_a - n_b} w_b = (-2)^{k+n_a-n_c} w_c + (-2)^{k-n_a-n_d} w_d.$$

Let $\Gamma = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2 \subset \mathbb{R}^2$ be the lattice generated by w_1, w_2. Since $k \ge 2$ we have

$$w_a + (-2)^{n_a - n_b} w_b \equiv 0 \qquad \mod 4\Gamma,$$

which is impossible. This leads to a contradiction and concludes the proof. □

Corollary 4.6.3. *Let $r = 2$ and $\lambda \in \Lambda$. If λ is not a root of unity, then its degree over $\mathbb{Q}$ is a multiple of 3.*

Proof. Fix $\lambda \in \Lambda$ which is not a root of unity, and let P_i be the unique factor of P having λ as a root; according to Proposition 4.6.2, there exist $n_1, n_2, n_3 \geq 0$ such that the weights of conjugates of λ are elements of the set

$$\left\{ \frac{w_1}{(-2)^{n_1}}, \frac{w_1}{(-2)^{n_1+1}}, \frac{w_2}{(-2)^{n_2}}, \frac{w_2}{(-2)^{n_2+1}}, \frac{w_3}{(-2)^{n_3}}, \frac{w_3}{(-2)^{n_3+1}} \right\}.$$

Since by Remark 4.5.2 we have

$$\sum_{w \in W_i} w = 0,$$

we get

$$\left(\frac{k_1}{(-2)^{n_1}} + \frac{h_1}{(-2)^{n_1+1}} \right) w_1 + \left(\frac{k_2}{(-2)^{n_2}} + \frac{h_2}{(-2)^{n_2+1}} \right) w_2 + \left(\frac{k_3}{(-2)^{n_3}} + \frac{h_3}{(-2)^{n_3+1}} \right) w_3 = 0,$$

where the k_i and the h_i are the multiplicities of the weights in W_i. Since the only linear dependency among the w_i is $w_1 + w_2 + w_3 = 0$, this implies that there exists a constant $c \in \mathbb{Z}[1/2]$ such that

$$\frac{k_i}{(-2)^{n_i}} + \frac{h_i}{(-2)^{n_i+1}} = c, \qquad i = 1, 2, 3.$$

The equation $-2k_i + h_i = c(-2)^{n_i+1}$ implies that

$$k_i + h_i \equiv c \qquad \mod 3,$$

so that in particular $\sum_i (k_i + h_i) \equiv 0$ modulo 3. □

4.6.2. *Algebraic properties of λ_1.* Now let us focus on the factor P_1 having λ_1 as a root.

Lemma 4.6.4. *Let $r = 2$, and let P_1 be the factor of P having λ_1 as a root. If either $v_2, v_3 \in V_1$ or $v_2, v_3 \notin V_1$, then*

$$\theta(V_1 \times V_1) = V_1^\vee.$$

If either $v_2 \in V_1, v_3 \in V_i$ or $v_2 \in V_i, v_3 \in V_1$ for some $i \neq 1$, then

$$\theta(V_1 \times V_1) = V_1^\vee \oplus V_i^\vee.$$

Proof. Without loss of generality, in the second case we may assume that $i = 2$.

Let us first prove the $\subseteq$ inclusions. Denote by θ_1 the restriction of θ to $V_1 \times V_1$; let

$$\pi_1 \colon V^\vee = \bigoplus_{i=1}^n V_i^\vee \to \bigoplus_{i=2}^n V_i^\vee$$

be the projection onto the last $n - 1$ factors and

$$\pi_{1,2} \colon V^\vee = \bigoplus_{i=1}^l V_i^\vee \to \bigoplus_{i=3}^l V_i^\vee$$

be the projection onto the last $n - 2$ factors.
For $\pi \in \{\pi_1, \pi_{1,2}\}$, the subspace

$$\ker(\pi \circ \theta_1) := \{u \in V_1 \, ; \, \pi \circ \theta(u, v) = 0 \quad \text{for all } v \in V_1\} \subset V_1$$

is g_2-invariant and defined over $\mathbb{Q}$. By minimality of V_1 (see Remark 4.5.1), we then have either $\ker(\pi \circ \theta_1) = 0$ or $\ker(\pi \circ \theta_1) = V_1$. Therefore, in order to show the inclusions, we only need to prove that $v_1 \in \ker(\pi \circ \theta_1)$ ($\pi = \pi_1$ in the first case and $\pi = \pi_{1,2}$ in the second case); since g_2 is semisimple, it is enough to check that $\pi \circ \theta(v_1, v) = 0$ for all eigenvectors $v \in V_1$.

Let β be the eigenvalue associated to an eigenvector $v \in V_1$, and let $w = w_\beta$ be its weight. We distinguish the following subcases:

- if $w \notin \{w_2, w_3, -w_1/2\}$, then $-w_1 - w \notin W$, so that $v_1 \wedge v = 0$ and in particular $\pi \circ \theta(v_1, v) = 0$;
- if $w = -w_1/2$ and $v_1 \wedge v \neq 0$, then $v_1 \wedge v$ is an eigenvector with eigenvalue $\beta\lambda_1 = \bar{\beta}^{-1}$. Since β is conjugated to λ_1, so is $\bar{\beta}$, and thus $v \wedge v_1 \in V_1^\vee$ and $\pi \circ \theta(v_1, v) = 0$;
- if $w = w_2$, by simplicity of the weight w_2 we have $\beta = \alpha_2$; in particular $v_2 \in V_1$. Then $v_1 \wedge v = v_1 \wedge v_2$ is an eigenvector for g_4 with eigenvalue $\alpha_1\alpha_2 = \alpha_3^{-1}$. If also $v_3 \in V_1$, then $v_1 \wedge v \in V_1^\vee$; if $v_3 \notin V_1$, say $v_3 \in V_2$, then $v_1 \wedge v \in V_2^\vee$. In both cases, choosing the right $\pi \in \{\pi_1, \pi_{1,2}\}$ we get $\pi \circ \theta(v_1, v) = 0$;
- the case $w = w_3$ is analogous to the case $w = w_2$.

This concludes the proof of the $\subseteq$ inclusions.

Let us now prove the other inclusions $\supseteq$.

Suppose first that either $v_2, v_3 \in V_1$ or $v_2, v_3 \notin V_1$, so that $\theta(V_1 \times V_1) \subseteq V_1^\vee$. Since $\theta(V_1 \times V_1)$ is g-invariant and defined over $\mathbb{Q}$, by minimality of $V_1^\vee$ we only need to show that $\theta(V_1 \times V_1) \neq \{0\}$. Since $\dim V_1 \geq 2$, this follows from Lemma 4.0.2.

Now suppose that $v_2 \in V_1, v_3 \in V_2$, so that $\theta(V_1 \times V_1) \subset V_1^\vee \oplus V_2^\vee$. Then $v_1 \wedge v_2 \in V^\vee$ is an eigenvector with eigenvalue α_3^{-1}, so that $v_1 \wedge v_2 \in V_2^\vee$. Since $\theta(V_1 \times V_1)$ is g-invariant and defined over $\mathbb{Q}$, by minimality of $V_1^\vee$ and $V_2^\vee$ we have either

$$\theta(V_1 \times V_1) = V_2^\vee$$

or

$$\theta(V_1 \times V_1) = V_1^\vee \oplus V_2^\vee.$$

The first case contradicts Lemma 4.6.5 below, so equality must hold and the proof is complete. □

Lemma 4.6.5. *Let $r = 2$, and let P_1 be the factor of P having λ_1 as a factor. Suppose that*

$$\theta(V_1 \times V_1) \subseteq V_j^\vee$$

for some $1 \leq j \leq n$. Then $j = 1$.

Proof. Assume by contradiction that $j \neq 1$, say $j = 2$. By the $\subseteq$ inclusions in Lemma 4.6.4 (whose proof is independent on the result we want to prove here), we may then assume that $v_2 \in V_1, v_3 \in V_2$.

Let us prove first that $-w_1/2, -w_2/2 \notin W_1$. Indeed, suppose for example that $-w_1/2 \in W_1$, and let $v \in V_1$ be an eigenvector whose eigenvalue λ has weight $-w_1/2$. Then by Lemma 4.0.2 we have $v \wedge \bar{v} \neq 0$, so that $|\lambda|^2 = \lambda_1^{-1}$ is an eigenvalue of the restriction of g_4 to $\theta(V_1 \times V_1) = V_2^\vee$. This contradicts the fact that λ_1^{-1} is an eigenvalue of g_4 restricted to $V_1^\vee$, and proves that $-w_1/2 \notin W_1$; the proof for $-w_2/2$ is analogous.

Now let us prove that $w_3/(-2)^n \notin W_1$ for $n \geq 2$. Suppose by contradiction that $v \in V_1$ is an eigenvector whose eigenvalue λ has weight $-w_3/(-2)^n$, $n \geq 2$. Then by Lemma 4.0.2 $v\wedge\bar{v}\neq 0$ is a non-trivial eigenvector with eigenvalue $\mu=|\lambda|^2$; since $\theta(V_1\times V_1)=V_2^\vee$, μ is conjugated to $\mu'=\alpha_3^{-1}$, and these two algebraic integers satisfy an algebraic equation

$$\mu^{(-2)^N}=\mu' \qquad N\geq 1.$$

Using Lemma 4.5.4 it is not hard to see that for this to happen we need to have $\mu=\mu'=1$, a contradiction. This proves that $w_3/(-2)^n\notin W_1$ for $n\geq 2$.

Now, by Proposition 4.6.2 this implies that, up to multiplicities,

$$W_1\subset\left\{w_1,w_2,-\frac{w_3}{2}\right\}.$$

This however contradicts the equation

$$\sum_{w\in W_1} w=0.$$

The claim is then proven. □

Remark 4.6.6. Lemma 4.6.4 and 4.6.5 still hold if one permutes α_1,α_2 and α_3; the proofs are completely analogous.

Proposition 4.6.7. *Let $r(g)=2$; then the conjugates of $\lambda_1(g)$ have all modulus belonging to the following set:*

$$\left\{\lambda_1,\lambda_2^{-1},\lambda_1^{-1}\lambda_2,\sqrt{\lambda_1^{-1}},\sqrt{\lambda_2},\sqrt{\lambda_1\lambda_2^{-1}}\right\}.$$

More accurately, up to permuting the eigenvalues $\alpha_1=\lambda_1,\alpha_2=\lambda_2^{-1},\alpha_3=\lambda_1^{-1}\lambda_2\in\Lambda$, one of the following is true:

(1) *α_1,α_2 and α_3 are all cubic algebraic integers without real conjugates;*
(2) *α_1 is a cubic algebraic integer without real conjugates; α_2 and α_3 are conjugate to one another, and their other conjugates are pairs of conjugate complex numbers with modulus $\alpha_1^{-1/2},\alpha_3^{-1/2},\alpha_3^{-1/2}$ ($k,k+1$ and $k+1$ pairs respectively, $k\geq 0$);*
(3) *α_1,α_2 and α_3 are conjugate, and their other conjugates are pairs of conjugate complex numbers with modulus $\alpha_1^{-1/2},\alpha_3^{-1/2},\alpha_3^{-1/2}$ (k pairs for each module, $k\geq 0$).*

Proof. Thanks to Lemma 4.6.4 and Remark 4.6.6, up to permutation of indices only three situations are possible:

(1) α_1,α_2 and α_3 are not mutually conjugate. In this case, denoting by P_i the factor of P having α_i as a factor, Lemma 4.6.4 implies that

$$\theta(V_i\times V_i)=V_i^\vee.$$

Then $W_i=\{w_i,-w_i/2\}$; indeed, suppose by contradiction that $w=w_j/(-2)^n\in W_i$ for some $j\neq i$, and let $v\in V_i$ be an eigenvector whose eigenvalue has weight w. Then by Lemma 4.0.2, either $n=0$ or $v\wedge\bar{v}\neq 0$, so that $-2w\in W_i$. By a recursive argument, this proves that $w_j\in W_i$, which contradicts the assumption that α_j and α_i are not conjugate. Therefore, by Proposition 4.6.2,

$$W_i\subseteq\left\{w_i,-\frac{w_i}{2}\right\}.$$

Since $\sum_{w\in W_i} w = 0$, the multiplicity of $-w_i/2$ must be 2, which implies that the conjugates of α_i are two conjugate complex numbers. This concludes the proof of case (1).

(2) α_2 and α_3 are conjugate, while α_1 is not. The above proof shows that α_1 is cubic without real conjugates. Let P_1 (respectively P_2) be the factor of P having α_1 (respectively α_2 and α_3) as a root; by Lemma 4.6.4 and Remark 4.6.6 we have
$$\theta(V_2 \times V_2) = V_1^\vee \oplus V_2^\vee.$$
Suppose by contradiction that $w = w_1/(-2)^n \in W_2$ for some $n \geq 2$, and let $v \in V_2$ be an eigenvector whose eigenvalue λ has weight w. Then by Lemma 4.0.2 $v \wedge v \neq 0$, so that $|\lambda|^{-2} \in \Lambda_1 \cup \Lambda_2$; since α_1 is a non-trivial power of $|\lambda|^{-2}$, these two numbers cannot be conjugate, therefore $|\lambda|^{-2} \in \Lambda_2$. Inductively, this shows that $\beta = \alpha_1^{-1/2} \in \Lambda_1$ is conjugate to α_2 and α_3. We can write
$$\beta^2 = \alpha_2\alpha_3;$$
taking $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\rho(\beta) = \alpha_1$ leads to a contradiction. Therefore $w_1/(-2)^n \notin W_2$ for $n \geq 2$ and $n = 0$.
By Proposition 4.6.2 this implies that, up to multiplicities,
$$W_2 \subset \left\{w_2, w_3, -\frac{w_1}{2}, -\frac{w_2}{2}, -\frac{w_3}{2}\right\}.$$
The equation
$$\sum_{w\in W_2} w = 0$$
implies that the multiplicities of $-w_1/2, -w_2/2, -w_3/2$ are $h, h+2, h+2$ respectively for some $h \geq 0$. Since α_2 cannot be conjugate to $\alpha_2^{-1/2}$, $h = 2k$ is even, which concludes the proof of case (2).

(3) α_1, α_2 and α_3 are conjugate. Then, by Proposition 4.6.2, up to multiplicities,
$$W_2 \subset \left\{w_1, w_2, w_3, -\frac{w_1}{2}, -\frac{w_2}{2}, -\frac{w_3}{2}\right\}.$$
The equation
$$\sum_{w\in W_2} w = 0$$
implies that the multiplicities of $-w_1/2, -w_2/2, -w_3/2$ are alle equal to h for some $h \geq 0$. Since α_1 cannot be conjugate to $\alpha_1^{-1/2}$, $h = 2k$ is even, which concludes the proof of case (3).

□

4.7. **The case $r(g) = 1$.** Let us now suppose that the rank of g (i.e. the split-rank of $G = \overline{\langle g \rangle}^{Zar}$, see Section 4.1) is equal to 2. Recall that in this case the weights are equipped with a natural order: $w_\lambda > w_{\lambda'}$ if and only if $|\lambda| > |\lambda'|$; for $w \in W$ we set $|w| := \max\{w, -w\}$.

Denote as usual by $w_1, w_2 \in W$ the weights of the eigenvalues $\lambda_1, \lambda_2^{-1} \in \Lambda$ respectively.

Lemma 4.7.1. *Suppose that $r = 1$ and let $\lambda \neq \lambda_1$ be a real conjugate of λ_1; then $\lambda = \lambda_1^{-1}$. In this case $\lambda_1 = \lambda_2$.*

Proof. Since $r = 1$, there exist integers m, n, not both equal to 0, such that

$$\lambda^m = \lambda_1^n.$$

Suppose that $|m| \geq |n|$ (the case $|n| \geq |m|$ is proven in the same way) and let $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be such that $\rho(\lambda_1) = \mu$, where μ is a conjugate of λ_1 whose weight has maximal modulus. Denoting by w and w' the weights of μ and $\rho(\lambda_1)$ respectively, the above equation implies that

$$mw = nw';$$

by maximality of $|w|$ we get $|m| = |n|$, so that $\lambda = \lambda_1^{-1}$ as claimed.

In order to prove that in this case $\lambda_1 = \lambda_2$, it suffices to apply Proposition 4.4.1: if this were not the case, since $\lambda_1^{-1} \in \Lambda$, then either $\lambda_1^{-1} = \lambda_1^{-1}\lambda_2$, a contradiction, or $\lambda_1^2 \in \Lambda$, contradicting the maximality of λ_1. □

Proposition 4.7.2. *Suppose that $r = 1$. Then*

- *either λ_1 and λ_2 are both cubic without real conjugates;*
- *or $\lambda_2 = \lambda_1 = \lambda$, λ and λ^{-1} are conjugate, and all of their other conjugates are pairs of conjugate complex numbers of modulus $\sqrt{\lambda}, 1$ or $\sqrt{\lambda^{-1}}$ (k, k' and k pairs respectively, $k, k' \geq 0$).*

Proof. Denote by P_1 the factor of P having λ_1 as a root.

Case 1: $\lambda_2 \notin \{\sqrt{\lambda_1}, \lambda_1, \lambda_1^2\}$. We show first that

$$\theta(V_1 \times V_1) = V_1^\vee.$$

Since λ_1 is a simple eigenvalue of g_2, V_1 is minimal among the g-invariant subspaces defined over $\mathbb{Q}$ (see Remark 4.5.1); therefore, as in the proof of Lemma 4.6.4, we only need to show that

$$v_1 \wedge v \in V_1^\vee$$

for all eigenvectors $v \in V_1$. Let $v \in V_1$ be an eigenvector with eigenvalue λ and let $w = w_\lambda$, and let as usual $w_3 := -w_1 - w_2$.

- If $w \notin \{w_1, -w_1/2, w_2, w_3\}$, then $v_1 \wedge v = 0$. Indeed, if this were not the case, then $-w_1 - w \in W$, and the assumption and Proposition 4.4.1 imply that
$$w > \min\left\{-\frac{w_1}{2}, \frac{w_2}{4}, w_3\right\} = -\frac{w_1}{2}.$$
Therefore
$$-w_1 - w < -\frac{w_1}{2},$$
which implies that $-w_1 - w = w_2$, i.e. $w = w_3$, contradicting the assumption.
- If $w = -w_1/2$, i.e. $\lambda\bar{\lambda} = \lambda_1^{-2}$, then $v_1 \wedge \bar{v}$ is an eigenvector with eigenvalue $\bar{\lambda}^{-1}$; since λ and $\bar{\lambda}$ are conjugate, this implies that $v_1 \wedge v \in V_1^\vee$.
- If $w = w_2$ or $w = w_3$, then $\lambda = \lambda_2^{-1}$ or $\lambda = \lambda_1^{-1}\lambda_2$ is a real conjugate of λ_1; but then by Lemma 4.7.1 we have $\lambda = \lambda_1^{-1}$, which contradicts the assumptions on λ_2. Thus this case cannot occur.
- Finally, if $w = w_1$ then $\lambda = \lambda_1$ and since $-2w_1 \notin W$ we have $v_1 \wedge v = 0$.

We have showed that $\theta(V_1 \times V_1) = V_1^\vee$.

Now let us show that λ_1 is cubic without real conjugates. Since

$$\sum_{w \in W_1} w = 0$$

and since w_1 has multiplicity 1 in W, we only need to show that the conjugates of λ_1 have weight $-w_1/2$. Let λ be a conjugate of λ_1 with weight w and let v be an eigenvector for λ.

- If we had $w = w_2$, then by simplicity of such weight we have $\lambda = \lambda_2^{-1}$, contradicting Lemma 4.7.1 since $\lambda_2 \neq \lambda_1$.
- If we had $w = 0$, a conjugate λ of λ_1 would satisfy
$$\lambda\bar{\lambda} = 1.$$
Applying $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\rho(\lambda) = \lambda_1$, we would have that λ_1^{-1} is a conjugate of λ_1, so that $\lambda_1 = \lambda_2$, a contradiction.
- If $w = w_3$, since $\lambda_1^{-1}\lambda_2 \neq \lambda_1, \lambda_1^{-1}$ Lemma 4.7.1 implies that $\lambda \notin \mathbb{R}$. Lemma 4.0.2 applied to v and v_1 implies that $v \wedge \bar{v} \neq 0$: indeed otherwise we would have $v_1 \wedge \bar{v}_1 \neq 0$ (contradicting the minimality of $\lambda_2^{-1} > \lambda_1^{-2}$) or $v_1 \wedge \bar{v} \neq 0$ (contradicting the simplicity of the weight w_2).
Therefore, since $\theta(V_1, V_1) = V_1^\vee$, we have
$$|\lambda|^{-2} = \lambda_1^2\lambda_2^{-2} \in \Lambda_1$$
and by Lemma 4.7.1 either $\lambda_1^2\lambda_2^{-2} = \lambda_1$, i.e. $\lambda_1 = \lambda_2^2$, a contradiction or $|\lambda|^{-2} = \lambda_1^{-1} \in \Lambda$ and $\lambda_1 = \lambda_2$, again a contradiction.
- Finally, if $w \notin \{0, w_1, -w_1/2, w_2, w_3\}$, then by Proposition 4.4.1 $v \wedge \bar{v} \neq 0$ would be an eigenvector with (real) eigenvalue $|\lambda|^2$. Since $\theta(V_1 \times V_1) = V_1^\vee$, this implies that $|\lambda|^{-2} \in \Lambda_1$; by Lemma 4.7.1 we would have $|\lambda|^2 = \lambda_1^{\pm 1}$, a contradiction.

This shows that λ_1 is cubic without real conjugates; the proof for λ_2 is completely analogous.

Case 2: $\lambda_2 \in \{\sqrt{\lambda_1}, \lambda_1^2\}$. Up to replacing g by g^{-1}, we may assume that $\lambda_2 = \sqrt{\lambda_1}$; let us show that λ_1 and λ_2 are both cubic without real conjugates.
Remark that by Proposition 4.4.1, up to multiplicities
$$W \setminus \{0\} = \left\{ \frac{w_1}{(-2)^n}, n = 0, \dots, N \right\}.$$
Let us show first that λ_1 is cubic without real conjugates. Since
$$\sum_{w \in W_1} w = 0$$
and since w_1 has multiplicity 1 in W, we only need to show that $W_1 \subset \{w_1, -w_1/2\}$. Let $w \in W_1$ be the weight of an eigenvalue $\lambda \in \Lambda_1$.

- If $w = 0$, we show as in Case 1 that $\lambda_1 = \lambda_2$, a contradiction.
- If $w = w_1/(-2)^n$ and $n \geq 2$, then we argue as in the proof of Proposition 4.6.2 to obtain a contradiction: indeed in this case
$$(\lambda\bar{\lambda})^k = \lambda_1, \qquad 2|k.$$
Applying $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\rho(\lambda_1)$ has weight $w_1/(-2)^n$ with n maximal and letting w' and w'' be the weights of $\rho(\lambda)$ and $\rho(\bar{\lambda})$ respectively, we would have
$$k(w' + w'') = \frac{w_1}{(-2)^n},$$
a contradiction modulo $\mathbb{Z}w_1/(-2)^n$.

Therefore $W_1 \subset \{w_1, -w_1/2\}$ and thus λ_1 is cubic without real conjugates.

Now let us prove that λ_2 is also cubic without real conjugates; this is equivalent to λ_2^{-1} being cubic without real conjugates. Let P_2 be the factor of P having λ_2^{-1} as a root. Since $\lambda_2 = \sqrt{\lambda_1}$, λ_2 has degree 3 or 6 over $\mathbb{Q}$; the same proof as above and the simplicity of the weight w_1 show that

$$W_2 \subset \left\{w_2, -\frac{w_2}{2}\right\}.$$

Since

$$\sum_{w \in W_2} w = 0,$$

if λ_2 had degree 6 then the multiplicity of the weight w_2 in W_2 would be equal to 2, contradicting the fact that λ_2^{-1} is a real eigenvalue with weight w_2. Therefore λ_2 is cubic, and by Lemma 4.7.1 it doesn't have any real conjugate.

Case 3: $\lambda_1 = \lambda_2$. Suppose that λ_1 is not a cubic algebraic integer without real conjugates. Denote by P_1 the factor of P having λ_1 as a root; since

$$\sum_{w \in W_1} w = 0,$$

and since the weight $w_1 \in W$ has multiplicity 1, λ_1 is *not* cubic without real conjugates if and only if some conjugate λ of λ_1 has weight $w \notin \{w_1, -w_1/2\}$. Let us prove first that in this case λ_1 and λ_1^{-1} are conjugate. We distinguish the following sub-cases:

- $w = -w_1$. Then, since λ_1^{-1} is the only eigenvalue with weight $-w_1$, λ_1 and λ_1^{-1} are conjugate.
- $0 < |w| < w_1/2$. Since the rank r is equal to 1, λ and λ_1 satisfy an equation
$$(\lambda\bar{\lambda})^m = \lambda_1^n,$$
and since $|w| < w_1/2$ we have $|n| < |m|$. By Lemma 4.5.4 there exists $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\rho(\lambda) = \lambda_1$; let $\lambda' = \rho(\bar{\lambda}), \lambda'' = \rho(\lambda_1)$, and let w', w'' be their weights respectively. Then the above equation implies that
$$mw_1 + mw' = nw'' \quad \Leftrightarrow \quad w_1 = -w' + \frac{n}{m}w''. \tag{4.1}$$
This implies that either $w' = w_2$ or $w'' = w_2$; indeed, if this were not the case, by Proposition 4.4.1 we would have
$$|w'|, |w''| \leq \max\left\{\frac{|w_1|}{2}, \frac{|w_2|}{2}, |w_1 + w_2|\right\} = \frac{|w_1|}{2};$$
this would contradict equation 4.1 because $|n/m| < 1$.
We have shown that w_2 is a conjugate weight of λ_1; since λ_1^{-1} is the only eigenvalue with weight w_2, this means that λ_1 and λ_1^{-1} are conjugate as claimed.
- $w = 0$. Then we show as in Case 1 that λ_1 is conjugate to $\lambda_2^{-1} = \lambda_1^{-1}$.
- $w = w_1/2$. We may assume that we don't fall in one of the cases above, i.e. that
$$W_1 \subset \left\{w_1, \frac{w_1}{2}, -\frac{w_1}{2}\right\}.$$
We show that $\theta(V_1 \times V_1) = V_1^\vee$; in order to do that it suffices to show that $v_1 \wedge v \in V_1^\vee$ for every eigenvector $v \in V_1$ (see the proof of Lemma 4.6.4). If

the eigenvalue μ of the eigenvector v has weight w_1 or $w_1/2$, then $v_1 \wedge v = 0$; if the weight is $-w_1/2$, then $v_1 \wedge v$ is an eigenvector with eigenvalue $\bar{\mu}^{-1}$, hence $v_1 \wedge v \in V_1^\vee$. Therefore $\theta(V_1 \times V_1) = V_1^\vee$ as claimed.
Now let $v \in V_1$ be an eigenvector with eigenvalue of weight $w_1/2$; by Proposition 4.4.1, $v \wedge \bar{v} \neq 0$, so that $-w_1 \in W_1$. This shows that we fall in one of the above cases, and in particular λ_1 and λ_1^{-1} are conjugate.

We have shown that λ_1 and λ_1^{-1} are conjugate. By Lemma 4.7.1 there are no other real conjugates, therefore in order to complete the proof we only need to show that

$$\pm \frac{w_1}{2^n} \notin W_1 \qquad \text{for } n \geq 2.$$

This can be proven exactly as in Case 2. □

4.8. **Examples on tori.** In this section we provide examples of automorphisms of compact complex tori of dimension 3 which show that (almost) all of the sub-cases of Proposition 4.6.7 and 4.7.2 can actually occur. For more examples see [21, 20].

Lemma 4.8.1. *Let $P \in \mathbb{Z}[T]$ be a monic polynomial of degree $2n$ all of whose roots are distinct and non-real and such that $P(0) = 1$. Then there exists a compact complex torus X of dimension n and an automorphism*

$$f\colon X \to X$$

such that the characteristic polynomial of the linear automorphism $f_1^*\colon H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C})$ is equal to P.

Proof. Let

$$P(T) = T^{2n} + a_{2n-1}T^{2n-1} + \ldots + a_1 T + 1 \in \mathbb{Z}[T]$$

be any polynomial.
We will prove first that there exists a linear diffeomorphism f of the real torus $M = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ such that the induced linear automorphism $f_1^* \in \mathrm{GL}(H^1(M, \mathbb{R}))$ has characteristic polynomial P. Indeed, the companion matrix

$$A = A(P) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & -1 \\ 1 & 0 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & 0 & \ldots & 0 & -a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & 0 & 1 & 0 & -a_{2n-2} \\ 0 & \cdots & 0 & 0 & 1 & -a_{2n-1} \end{pmatrix}$$

has characteristic polynomial P; since $A \in \mathrm{SL}_{2n}(\mathbb{Z})$, the induced linear automorphism f of $\mathbb{R}^{2n}$ preserves the lattice $\mathbb{Z}^{2n}$ and so does its inverse. Hence, A induces a linear automorphism, which we denote again by f:

$$f\colon \mathbb{R}^{2n}/\mathbb{Z}^{2n} \to \mathbb{R}^{2n}/\mathbb{Z}^{2n}.$$

Let dx_i be a coordinate on the i-th factor of $M = \mathbb{R}^{2n}/\mathbb{Z}^{2n} = (\mathbb{R}/\mathbb{Z})^{2n}$. In the basis $dx_1, \ldots, dx_{2n}$ of $H^1(X, \mathbb{R})$, the matrix of f_1^* is exactly the transposed A^T; in particular, the characteristic polynomial of f_1^* is equal to P.

In order to conclude the proof, we will show that, if the roots of P are all distinct and non-real, then M can be endowed with a complex structure J such that f is holomorphic with respect with the structure J. Let

$$\beta_1, \bar\beta_1, \dots, \beta_n, \bar\beta_n \in \mathbb{C} \setminus \mathbb{R}$$

be the roots of P, and let

$$V_i = \ker(f - \beta_i I)(f - \bar\beta_i I) \subset \mathbb{R}^{2n},$$

where we have identified f with the linear automorphism of $\mathbb{R}^{2n}$ induced by the matrix A.
The V_i are planes such that

$$\mathbb{R}^{2n} = \bigoplus_{i=1}^{n} V_i.$$

The restriction of f to V_i is diagonalizable with eigenvalues β_i and $\bar\beta_i$; therefore there exists a unique complex structure J_i on V_i such that, with respect to a holomorphic coordinate z_i on $V_i \cong \mathbb{C}$, the action of f is the multiplication by β_i:

$$f|_{V_i}(z_i) = \beta_i z_i.$$

The complex structures J_i induce a complex structure on $\mathbb{R}^{2n}$; by canonically identifying $\mathbb{R}^{2n}$ with the tangent space at any point of M, we get an almost-complex structure on M. It is not hard to see that J is integrable, and that f is holomorphic with respect to J. This concludes the proof. $\square$

Let us apply Lemma 4.8.1 to the three-dimensional case: fix a monic polynomial $P \in \mathbb{Z}[T]$ of degree 6 such that $P(0) = 1$, and suppose that its roots

$$\beta_1, \beta_2, \beta_3, \beta_4 = \bar\beta_1, \beta_5 = \bar\beta_2, \beta_6 = \bar\beta_3$$

are all distinct and non-real.
By Lemma 4.8.1, there exists a 3-dimensional complex torus $X = \mathbb{C}^3/\Lambda$ and an automorphism $f\colon X \to X$ such that the induced linear automorphism $f_1^*\colon H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C})$ has characteristic polynomial P. Remark that the proof of the Lemma shows something more precise: the restriction of f_1^* to $H^{1,0}(X)$ (respectively to $H^{0,1}(X)$) is diagonalizable with eigenvalues $\beta_1, \beta_2, \beta_3$ (respectively $\bar\beta_1, \bar\beta_2, \bar\beta_3$).

Since for a complex torus the wedge-product of forms induces isomorphisms

$$H^{2,0}(X) \cong \textstyle\bigwedge^2 H^{1,0}(X), \quad H^{1,1}(X) \cong H^{1,0}(X) \otimes H^{0,1}(X), \quad H^{0,2}(X) \cong \textstyle\bigwedge^2 H^{0,1}(X),$$

the eigenvalues of $f_2^* \in \mathrm{GL}(H^2(X,\mathbb{R}))$ are exactly the 15 numbers $\beta_i\beta_j$, $1 \le i < j \le 6$. If $|\beta_1| \ge |\beta_2| \ge |\beta_3|$, then

$$\alpha_1 := \lambda_1 = |\beta_1|^2, \quad \alpha_2 := \lambda_2^{-1} = |\beta_3|^2, \quad \alpha_3 := \lambda_1^{-1}\lambda_2 = |\beta_2|^2.$$

Let

$$Q(T) = \prod_{1 \le i < j \le 6} (T - \beta_i\beta_j).$$

Then Q is the characteristic polynomial of f_2^*, and in particular $Q \in \mathbb{Z}[T]$. Let

$$K_P := \mathbb{Q}(\beta_i)_{1 \le i \le 6} \supset K_Q := \mathbb{Q}(\beta_i\beta_j)_{1 \le i < j \le 6}.$$

We are interested in the irreducible factors of Q over $\mathbb{Z}$; assuming that all the roots of Q are distinct, the irreducible factors of Q are in $1:1$ correspondence with the orbits of the action of $\mathrm{Gal}(K_Q/\mathbb{Q})$ on the set of roots $R_Q = \{\beta_i\beta_j\}$. Since each element of $\mathrm{Gal}(K_Q/\mathbb{Q})$ can be extended to an element of $\mathrm{Gal}(K_P/\mathbb{Q})$, we consider instead the orbits under the action of

$$G := \mathrm{Gal}(K_P/\mathbb{Q});$$

G acts by permuting the roots of P, and thus it can be seen as a subgroup of $\mathcal{S}_6$. Under this identification, the action of G on R_Q is given by the natural action of (subgroups of) $\mathcal{S}_6$ on the set

$$S := \{\{i, j\} \subset \{1, 2, 3, 4, 5, 6\}\}.$$

Therefore, as long as we know how $\mathrm{Gal}(K_P/\mathbb{Q})$ permutes the roots of P, we can deduce the number and the degrees of the irreducible factors of Q. This is a classical problem in Galois theory (see [6]), and programs like Magma allow to easily compute this action.

Example 4.8.2. Let $P(T) = T^6 - T^5 + T^3 - T^2 + 1$; then $G = S_6$ acts transitively on S. This means that Q is irreducible, and thus $\alpha_1, \alpha_2, \alpha_3$ are conjugate; their other conjugates are six pairs of complex conjugates, two of modulus $1/\sqrt{\alpha_1}$, two of modulus $1/\sqrt{\alpha_2}$ and two of modulus $1/\sqrt{\alpha_3}$.
This realizes subcase 1 of Proposition 4.6.7 with $k = 2$.

Example 4.8.3. Let $P(T) = T^6 - 3T^5 + 4T^4 - 2T^3 + T^2 - T + 1$; then $G = \langle(1\,3\,4)(2\,6\,5), (1\,4\,3), (1\,6)(2\,3)(4\,5)\rangle$. The action of G on S has two orbits, of cardinality 9 and 6 respectively; one can check that the roots of Q are all distinct, so that α_1 and α_2 are not both cubic, and that $\alpha_1 \neq \alpha_2^{-1}$, so that by Proposition 4.7.2 we have $r = 2$. By Proposition 4.6.7, the only possibility is that α_1, α_2 and α_3 are conjugate of degree 9; their other conjugates are three pairs of complex conjugates, of modulus $1/\sqrt{\alpha_1}, 1/\sqrt{\alpha_2}$ and $1/\sqrt{\alpha_3}$ respectively.
This realizes subcase 1 of Proposition 4.6.7 with $k = 1$.

Example 4.8.4. Let $P(T) = T^6 + T^5 + 2T^4 - T^3 + 2T^2 - 3T + 1$; then $G = \langle(1\,2\,3)(4\,5\,6), (1\,4\,5)(2\,3\,6), (2\,4)(3\,5), (1\,5\,6\,3)\rangle$. The action of G on S has two orbits, of cardinality 12 and 3 respectively. One can check that α_1, α_2 and α_3 are not all conjugate, so that we are in case 2 of Proposition 4.6.7: after permuting the indices, α_1 is cubic without real conjugates; α_2 and α_3 are conjugate and their other conjugates are 5 pairs of complex conjugates, one of modulus $1/\sqrt{\alpha_1}$, two of modulus $1/\sqrt{\alpha_2}$ and two of modulus $1/\sqrt{\alpha_2}$. Remark that, after possibly replacing f by f^{-1} (which replaces P by $P^\vee$), we may assume that λ_1 is not cubic without real conjugates.
This realizes subcase 2 of Proposition 4.6.7 with $k = 1$.

Example 4.8.5. Let $P(T) = T^6 + T^5 + 4T^4 + T^3 + 2T^2 - 2T + 1$; then $G = \langle(1\,2\,5)(3\,6\,4), (1\,3)(2\,4)(5\,6)\rangle$. The action of G on S has three orbits of cardinality 3 and one of cardinality 6. Then λ_1 and λ_2 are both cubic without real conjugates:

- if $r = 1$, since it is easy to prove that $\lambda_1 \neq \lambda_2$, this follows from Proposition 4.7.2 ;
- if $r = 2$, it can be proven easily that α_1, α_2 and α_3 are not all conjugate, and that all the other eigenvalues of f_2^* are non-real. Therefore, the α_i are all contained in (distinct) orbits of cardinality 3, meaning that they are cubic without real conjugates.

However it is unclear whether $r = 1$ or $r = 2$; if one could prove that $r = 2$, this would realize subcase 3 of Proposition 4.6.7.

Example 4.8.6. Let $P(T) = T^6 + T^4 - 2T^3 + T^2 - T + 1$ and let $\beta_1, \bar{\beta}_1, \beta_2, \bar{\beta}_2, \beta_3, \bar{\beta}_3$ be the roots of P, with $|\beta_1| \geq |\beta_2| \geq |\beta_3|$; then one finds out that

$$|\beta_1| = |\beta_2|^{-2} = |\beta_3|^{-2}.$$

In particular $\lambda_1 = \lambda_2^2$, so that $r = 1$. By Proposition 4.7.2, λ_1 and λ_2 are both cubic without real conjugates.
This realizes subcase 1 of Proposition 4.7.2.

Remark 4.8.7. In order to realize subcase 2 of Proposition 4.7.2 it would suffice to exhibit an irreducible polynomial $P \in \mathbb{Z}[T]$ such that $P(0) = 1$ and exactly two roots of P have modulus equal to 1.

5. Automorphisms of threefolds: the mixed case

In this last section, we deal with the case of an automorphism $f\colon X \to X$ of a compact Kähler threefold X such that the action in cohomology $f_2^*\colon H^2(X,\mathbb{R}) \to H^2(X,\mathbb{R})$ is neither (virtually) unipotent nor semisimple. As we saw in Section 2, this situation is not possible in the surface case; in the threefold case, we manage to give some constraints but not to completely exclude this situation. However, due to restriction on the dimension, no examples can be produced on complex tori, and to the best of my knowledge no examples are known at all.

Conjecture 4. *Let $f\colon X \to X$ be an automorphism of a compact Kähler threefold. Then $f_2^*\colon H^2(X,\mathbb{R}) \to H^2(X,\mathbb{R})$ is either semisimple or virtually unipotent.*

Proposition 5.0.1. *Let X be a compact Kähler threefold and let $f\colon X \to X$ be an automorphism such that $\lambda_1(f) > 1$ and f_2^* is not semisimple. Then*

(1) *$\lambda_2(f) \in \{\sqrt{\lambda_1(f)}, \lambda_1(f)^2\}$; in particular, $r(f) = 1$ and $\lambda_1 = \lambda_1(f)$ and $\lambda_2 = \lambda_2(f)$ are both cubic without real conjugates;*
(2) *if $\lambda_2 = \lambda_1^2$, then the eigenvalue λ_1 has a unique non-trivial Jordan block whose dimension is $m \leq 3$; the other eigenvalues having non-trivial Jordan blocks have modulus $1/\sqrt{\lambda_1}$, and their non-trivial Jordan blocks have dimension at most $m-1$;*
(3) *analogously, if $\lambda_2 = \sqrt{\lambda_1}$, then the eigenvalue λ_2^{-1} has a unique non-trivial Jordan block whose dimension is $m \leq 3$; the other eigenvalues having non-trivial Jordan blocks have modulus $\sqrt{\lambda_2}$, and their non-trivial Jordan blocks have dimension at most $m-1$.*

In what follows denote by $g = f^*\colon H^*(X,\mathbb{R}) \to H^*(X,\mathbb{R})$ the linear automorphism induced by f, and by g_i the restriction of g to $H^i(X,\mathbb{R})$. We will denote by $\lambda_i = \lambda_i(f)$ the dynamical degrees and we will assume that

$$\lambda_2 \geq \lambda_1 > 1;$$

the case $\lambda_1 \geq \lambda_2$ follows from the previous one by replacing f by f^{-1}.

Lemma 5.0.2. *If $\lambda_2 \neq \lambda_1^2$, then λ_1 has no non-trivial Jordan block for g_2. If $\lambda_2 = \lambda_1^2$, then g_2 has at most one non-trivial Jordan block for the eigenvalue λ_1, whose dimension is at most 3. In either case, g_2 does not have non-trivial Jordan blocks for other eigenvalues of modulus λ_1.*

Proof. Suppose first that $\lambda_2 \neq \lambda_1^2$; then, by Theorem B, w_1 is a simple weight of (the semisimple part of) g. Therefore in particular λ_1 has no non-trivial Jordan block.

Now suppose that $\lambda_2 = \lambda_1^2$. We prove first that the Jordan blocks for the eigenvalue λ_1 have dimension at most 3. Suppose by contradiction that there exists a Jordan block of dimension at least 4; then, as in the proof of Theorem 3.0.1, we may pick $u_1, u_2, u_3, u_4 \in H^{1,1}(X,\mathbb{R}) \cup (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ such that

$$g(u_1) = \lambda_1 u_1, \qquad g(u_i) = u_{i+1} + \lambda_1 u_i \quad i = 1, 2, 3.$$

Considering

$$g^n(u_4 \wedge u_4), g^n(u_3 \wedge u_3) \in H^4(X,\mathbb{R}),$$

and applying Lemma 4.0.2 as in the proof of Theorem 3.0.1, we obtain a class $v \in H^4(X,\mathbb{R})$ such that

$$\|g_4^n v\| \sim cn^k \lambda_1^{2n} = cn^k \lambda_2^n \qquad \text{for some } k \geq 1.$$

This means that the eigenvalue λ_2 has a non-trivial Jordan block for g_4, and since $g_4 = (g_2^{-1})^\vee$, the eigenvalue λ_2^{-1} has a non-trivial Jordan block for g_2. This however would imply that $\lambda_1 = \lambda_2^2 = \lambda_1^4$, a contradiction.
This proves that Jordan blocks of g_2 for the eigenvalue λ_1 have dimension at most 3.

Now let us prove that there exists a unique non-trivial Jordan block of g_2 for the eigenvalue λ_1. Suppose by contradiction that we can find linearly independent elements $u_1, u_2, v_1, v_2 \in H^{1,1}(X,\mathbb{R}) \cup (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ such that

$$g(u_1) = \lambda_1 u_1, \quad g(u_2) = u_1 + \lambda_1 u_2, \qquad g(v_1) = \lambda_1 v_1, \quad g(v_2) = v_1 + \lambda_1 v_2.$$

Then, considering

$$g^n(u_2 \wedge u_2), \quad g^n(u_2 \wedge v_2), \quad g^n(v_2 \wedge v_2)$$

and applying Lemma 4.0.2 to the classes u_1 and v_1, we get as before a class $v \in H^4(X,\mathbb{R})$ such that

$$\|g_4^n v\| \sim cn^k \lambda_1^{2n} = cn^k \lambda_2^n \qquad \text{for some } k \geq 1,$$

which yields a contradiction. This concludes the proof. □

Proof of Proposition 5.0.1. Let $\lambda \in \Lambda$ be an eigenvalue of g_2 with weight w such that g_2 has a non-trivial Jordan block for λ of dimension $k > 1$. As in the proof of Theorem 3.0.1, we can take a Jordan basis $u_1, \ldots, u_k \in H^{1,1}(X) \cup (H^{2,0}(X) \oplus H^{0,2}(X))$ such that

$$g(u_1) = \lambda u_1, \qquad g(u_{i+1}) = u_i + \lambda u_{i+1} \quad i = 1, \ldots, k-1.$$

Suppose that $\lambda_2 \geq \lambda_1$, so that by Lemma 5.0.2 applied to f^{-1} the eigenvalue λ_2 has no non-trivial Jordan block. Let as usual w_1, w_2, w_3 the weights of the eigenvalues $\alpha_1 = \lambda_1, \alpha_2 = \lambda_2^{-1}, \alpha_3 = \lambda_1^{-1}\lambda_2 \in \Lambda$.

We distinguish the following cases:

- $w \notin \{0, w_1, w_2, w_3\}$: by Propositions 4.3.1.(3) and 4.4.1.(2) we have $u_1 \wedge \bar{u}_1 \neq 0$. In particular,

  $$g^n(u_k \wedge \bar{u}^k) \sim cn^{2k-2}|\lambda|^{2n}(u_1 \wedge \bar{u}_1),$$

  which means that g_4 has a Jordan block of dimension $\geq 2k-1$ for the eigenvalue $|\lambda|^2$. Since $g_4 = (g_2^{-1})^\vee$, g_2 has a Jordan block of dimension $\geq 2k-1 > k$ for the eigenvalue $|\lambda|^{-2} \in \Lambda$;

- $w = 0$; take $\lambda \in \Lambda$ with weight 0 such that the dimension k of its maximal Jordan block is maximal, and let $v \in H^{1,1}(X, \mathbb{R}) \cup (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ be an eigenvector for the eigenvalue λ_2.
  Since $v \wedge v = 0$, by Lemma 4.0.2 we have either $u_1 \wedge v \neq 0$ or $u_1 \wedge \bar{u}_1 \neq 0$. In the first case, considering $g^n(u_k \wedge v)$ we obtain a non-trivial Jordan block for an eigenvalue λ' of weight $-w_2$; this implies that $w_1 = -w_2$, and by Proposition 4.4.1 the weight w_1 is simple, contradicting the existence of a non-trivial Jordan block. In the second case, considering $g^n(u_k \wedge \bar{u}_k)$ we obtain a Jordan block of dimension $\geq 2k - 1 > k$ for the eigenvalue $|\lambda|^{-2} = 1$; since 1 has weight 0, this contradicts maximality. Therefore, this case cannot occur;
- $w = w_1$: in this case, by Lemma 5.0.2 we have $\lambda_2 = \lambda_1^2$, $\lambda = \lambda_1$ and $k \leq 3$;
- $w = w_2$: by Lemma 5.0.2 applied to f^{-1} this case cannot occur;
- $w = w_3$: let $v \in H^{1,1}(X, \mathbb{R}) \cup (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ be an eigenvector for the eigenvalue λ_2^{-1}. Since $v \wedge v = 0$, by Lemma 4.0.2 we have either $v \wedge \bar{u}_1 \neq 0$ or $u_1 \wedge \bar{u}_1 \neq 0$.
  In the first case we obtain a non-trivial Jordan block for an eigenvalue of weight w_1; by Lemma 5.0.2 we have then $\lambda_2 = \lambda_1^2$, thus $w_3 = w_1$ and, again by Lemma 5.0.2, $\lambda = \lambda_1$.
  In the second case, we get a non-trivial Jordan block for the eigenvalue $|\lambda|^{-2}$.

The above computation show that, if g_2 has a non-trivial Jordan block of dimension k for the eigenvalue $\lambda \in \Lambda$, then either $\lambda = \lambda_1$, in which case $\lambda_2 = \lambda_1^2$, or $|\lambda| \neq 1$ and there is a Jordan block of dimension $> k$ for the eigenvalue $|\lambda|^{-2}$.
By Proposition 4.3.1 and 4.4.1, one proves inductively that g_2 admits a non-trivial Jordan block for the eigenvalue λ_1. By Lemma 5.0.2, such block has dimension at most 3; the claim follows from the fact that, as we proved above, the dimension of a non-trivial Jordan block of $\lambda \neq \lambda_1$ is strictly smaller than that of a non-trivial Jordan block of $|\lambda|^{-2}$. □

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