ON THE PRIMITIVITY OF BIRATIONAL TRANSFORMATIONS OF IRREDUCIBLE SYMPLECTIC MANIFOLDS

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ABSTRACT. Let $f\colon X \dashrightarrow X$ be a bimeromorphic transformation of a complex irreducible symplectic manifold X. Some important dynamical properties of f are encoded by the induced linear automorphism f^* of $H^2(X,\mathbb{Z})$. Our main result is that a bimeromorphic transformation such that f^* has at least one eigenvalue with modulus > 1 doesn't admit any invariant fibration (in particular its generic orbit is Zariski-dense).

1. INTRODUCTION

A complex manifold is said **irreducible symplectic** if it is simply connected and the vector space of holomorphic 2-forms is spanned by a nowhere degenerate form. Irreducible symplectic manifolds form, together with Calabi-Yau manifolds and complex tori, one of the three fundamental classes of Kähler manifolds with trivial canonical bundle. We are going to denote by X an irreducible symplectic manifold and by $f\colon X \dashrightarrow X$ a bimeromorphic transformation of X.
On the second cohomology of X we can define a quadratic form, the Beauville-Bogomolov form, whose restriction to $H^{1,1}(X,\mathbb{R})$ is hyperbolic (i.e. has signature $(1, h^{1,1}(X) - 1)$) and which is preserved by the linear pull-back action f^* induced by f on cohomology; the setting is therefore similar to that of a compact complex surface, where the intersection form makes the second cohomology group into a hyperbolic lattice. In the surface case, the action of an automorphism $f\colon S \to S$ on cohomology translates into dynamical properties of f (see Paragraphs 3.3 and 3.4 for details), and we can hope to have similar results in the irreducible symplectic case.

If $g\colon M \dashrightarrow M$ is a meromorphic transformation of a compact Kähler manifold M, for $p = 0, 1, \dots, \dim(M)$ the p-th **dynamical degree** of g is

$$\lambda_p(g) := \limsup_{n\to+\infty} ||(g^n)^*_p||^{\frac{1}{n}},$$

where $(g^n)^*_p\colon H^{p,p}(M) \to H^{p,p}(M)$ is the linear morphism induced by g^n and $||\cdot||$ is any norm on the space $End(H^{p,p}(M))$. Note that in the case of an automorphism, $\lambda_p(f)$ is just the maximal modulus of eigenvalues of f^*_p.

Let $g\colon M \dashrightarrow M$ be a bimeromorphic transformation of a compact Kähler manifold. A meromorphic fibration $\pi\colon M \dashrightarrow B$ onto a compact Kähler manifold B such that $\dim B \neq 0, \dim X$ is called g-**invariant** if there exists a bimeromorphic transformation $h\colon B \dashrightarrow B$ such that $\pi \circ g = h \circ \pi$.

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$$\begin{array}{ccc} M & \overset{g}{\dashrightarrow} & M \\ \downarrow \pi & & \downarrow \pi \\ B & \overset{h}{\dashrightarrow} & B \end{array}$$

The transformation g is said to be **primitive** (see [22]) if it admits no invariant fibration.

In the surface case, an automorphism whose action on cohomology has infinite order admits an invariant fibration onto a curve if and only if all the dynamical degrees are equal to 1 (Theorem 3.8). Our main result establish an analogue of the "only if" direction.

Main Theorem. *Let X be an irreducible symplectic manifold, $f\colon X \dashrightarrow X$ a bimeromorphic transformation with at least one dynamical degree > 1. Then*

(1) *f is primitive;*
(2) *f admits at most $\dim(X) + b_2(X) - 2$ periodic hypersurfaces;*
(3) *the generic orbit of f is Zariski-dense.*

Here a hypersurface $H \subset X$ is said to be f-periodic if its strict transform $(f^n)^*H$ by some iterate of f is equal to H.

Remark 1.1. Point (2) follows from point (1) and [5, Theorem B]; point (3) follows from point (1) and [1, Theorem 4.1], but is proven here as a lemma (Lemma 4.6).

In order to prove the Main Theorem, we establish a result on the dynamics of birational transformations of projective manifolds that has its own interest.

Proposition. *Let X, B be projective manifolds, $f\colon X \dashrightarrow X, g\colon B \dashrightarrow B$ birational transformations and $\pi\colon X \to B$ a non-trivial fibration such that $\pi \circ f = g \circ \pi$. If the generic orbit of g is Zariski-dense and the generic fibre of π is of general type, then*

(1) *π is isotrivial over an open dense subset $U \subset B$;*
(2) *there exists an étale cover $U' \to U$ such that the induced fibration $X' = U' \times_U \pi^{-1}(U)$ is trivial: $X' \cong U' \times F$ for a fibre F;*
(3) *the images by the natural morphism $X' \to \pi^{-1}(U)$ of the varieties $U' \times \{pt\}$ are f-periodic; in particular the generic orbit of f is not Zariski-dense.*

Remark 1.2. Point (1) is equivalent to point (2) by [28, Proposition 2.6.10].

In Section 2 we recall the definition and main results about dynamical degrees, in the absolute and relative context; Section 3 is consecrated to irreducible symplectic manifolds, with a focus on the invariance of the Beauville-Bogomolov form under the action of a birational transformation; in Section 4 and 5 we prove the Main Theorem and the Proposition above; Section 6 presents a different approach to the proof of the Main Theorem, which allows to prove a slightly weaker version of it.

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2. DYNAMICAL DEGREES

Throughout this section M will be a compact Kähler manifold of dimension d.

2.1. **Definition and entropy.** Let $f\colon X \dashrightarrow Y$ be a dominant meromorphic map between compact Kähler manifolds; the map f is then holomorphic outside its indeterminacy locus $\mathcal{I} \subset X$, which has codimension at least 2. The closure Γ of its graph over $X \setminus \mathcal{I}$ is an irreducible analytic subset of dimension d in $X \times Y$. Let π_X, π_Y denote the restrictions to Γ of the projections from $X \times Y$ to X and to Y respectively; then π_X induces a biholomorphism $\pi_X^{-1}(X \setminus \mathcal{I}) \cong X \setminus \mathcal{I}$ and we can identify f with $\pi_Y \circ \pi_X^{-1}$.
Let α be a smooth (p,q)-form on Y; we define the pull-back of α by f as the (p,q)-current (see [8] for the basic theory of currents) on X

$$f^*\alpha := (\pi_X)_*(\pi_Y^*\alpha).$$

It is not difficult to see that if α is closed (resp. positive), then so is $f^*\alpha$, so that f induces a linear morphism between the Hodge cohomology groups. This definition of pull-back coincides with the usual one when f is holomorphic.

Remember that the p-th dynamical degree of a dominant meromorphic map $f\colon M \dashrightarrow M$ are defined as

$$\lambda_p(f) = \limsup_{n\to+\infty} ||(f^n)_p^*||^{\frac{1}{n}}.$$

Thanks to the above definition of pull-back, one can prove that

$$\lambda_p(f) = \lim_{n\to+\infty} \left(\int_M (f^n)^*\omega^p \wedge \omega^{d-p}\right)^{\frac{1}{n}}$$

for any Kähler form ω. See [12], [7] for details.

The p-th dynamical degree measures the exponential growth of the volume of $f^n(V)$ for subvarieties $V \subset M$ of dimension p [20].

Remark 2.1. By definition $\lambda_0(f) = 1$; $\lambda_d(f)$ coincides with the topological degree of f: it is equal to the number of points in a generic fibre of f.

Remark 2.2. Let f be an automorphism. Then we have $(f^n)^* = (f^*)^n$, so that $\lambda_p(f)$ is the maximal modulus of eigenvalues of the linear automorphism $f_p^*\colon H^{p,p}(M,\mathbb{R}) \to H^{p,p}(M,\mathbb{R})$; since f^* also preserves the positive cone $\mathcal{K}_p \subset H^{p,p}(M,\mathbb{R})$, a theorem of Birkhoff [2] implies that $\lambda_p(f)$ is a positive real eigenvalue of f_p^*.
It should be noted however that in the bimeromorphic setting we have in general $(f^n)^* \neq (f^*)^n$.

Remark 2.3. If f is bimeromorphic we have

$$\lambda_p(f) = \lambda_{d-p}(f^{-1}).$$

Indeed, for f biregular we have

$$\int_M (f^n)^*\omega^p \wedge \omega^{d-p} = \int_M (f^{-n})^*(f^n)^*\omega^p \wedge (f^{-n})^*\omega^{d-p} = \int_M \omega^p \wedge (f^{-n})^*\omega^{d-p},$$

which proves the equality by taking the limit.
If f is only bimeromorphic, for all n we can find two dense open subsets $U_n, V_n \subset M$ such that f^n induces an isomorphism $U_n \cong V_n$; by the definition of pull-back the measures $(f^n)^*\omega^p \wedge \omega^{d-p}$ and $\omega^p \wedge (f^{-n})^*\omega^{d-p}$ have no mass on any proper closed analytic subset, so that

$$\int_M (f^n)^*\omega^p \wedge \omega^{d-p} = \int_{U_n} (f^n)^*\omega^p \wedge \omega^{d-p} = \int_{V_n} \omega^p \wedge (f^{-n})^*\omega^{d-p} = \int_M \omega^p \wedge (f^{-n})^*\omega^{d-p},$$

which proves the equality in the bimeromorphic case as well.

The main interest in the definition of dynamical degrees lies in the following theorem by Yomdin and Gromov [16].

Theorem 2.4. *If $f\colon M \to M$ is an automorphism, then the topological entropy of f is given by*

$$h_{top}(f) = \max_{p=0,\ldots,d} \log \lambda_p(f).$$

The topological entropy is a positive real number which measures the disorder created by iterations of f.
It is also possible to give a definition of topological entropy in the bimeromorphic context (see [13]), but in this situation we only have

$$h_{top}(f) \leq \max_{p=0,\ldots,d} \log \lambda_p(f).$$

2.2. **Relative setting.** Dinh, Nguyên and Truong have studied the behaviour of dynamical degrees in the relative setting ([10] and [11]). Throughout this paragraph we denote by $f\colon M \dashrightarrow M$ a meromorphic transformation of a compact Kähler manifold M of dimension d, by $\pi\colon M \dashrightarrow B$ a meromorphic fibration onto a compact Kähler manifold B of dimension k and by $g\colon B \dashrightarrow B$ a meromorphic transformation such that

$$g \circ \pi = \pi \circ f.$$

The p-th **relative dynamical degree** of f is defined as

$$\lambda_p(f|\pi) = \limsup_{n\to+\infty} \left(\int_M (f^n)^*\omega_M^p \wedge \pi^*\omega_B^k \wedge \omega_M^{d-p-k} \right)^{\frac{1}{n}},$$

where ω_M and ω_B are arbitrary Kähler forms on M and B respectively. In particular $\lambda_p(f|\pi) = 0$ for $p > d-k$.

Roughly speaking, $\lambda_p(f|\pi)$ gives the exponential growth of $(f^n)^*$ acting on the subspace of classes in $H^{p+k,p+k}(M,\mathbb{R})$ that can be supported on a generic fibre of π; if M is projective, it also represents the growth of the volume of $f^n(V)$ for p-dimensional subvarieties $V \subset \pi^{-1}(b)$ of a generic fibre of π.

Remark 2.5. Dynamical degrees and relative dynamical degrees are bimeromorphic invariants [10]. In other words, if there exist bimeromorphic maps $\phi\colon M \dashrightarrow M'$, $\psi\colon B \dashrightarrow B'$ and a meromorphic fibration $\pi'\colon M' \dashrightarrow B'$ such that $\pi' \circ \phi = \psi \circ \pi$, then

$$\lambda_p(f) = \lambda_p(\phi \circ f \circ \phi^{-1}), \qquad \lambda_q(f|\pi) = \lambda_q(\phi \circ f \circ \phi^{-1}|\pi').$$

Remark 2.6. If $F = g^{-1}(b)$ is a regular, f-invariant, non-multiple fibre, then $\lambda_p(f|\pi) = \lambda_p(f_{|F})$ for all p (see [10]).

The following theorem is due to Dinh, Nguyên and Truong [10].

Theorem 2.7. *Let M be a compact Kähler manifold, $f\colon M \dashrightarrow M$ a meromorphic transformation, $\pi\colon M \dashrightarrow B$ a meromorphic fibration and $g\colon B \dashrightarrow B$ a meromorphic transformation such that $\pi \circ f = g \circ \pi$. Then for all $p = 0, \ldots \dim(M)$*

$$\lambda_p(f) = \max_{q+r=p} \lambda_q(f|\pi)\lambda_r(g).$$

2.3. **Log-concavity.** Dynamical degrees and their relative counterparts enjoy a log-concavity property (see [23],[29], [16], [7] for the original result, [10] for the relative setting).

Proposition 2.8. *If $f\colon M \dashrightarrow M$ is a meromorphic dominant map, the sequence $p \mapsto \log\lambda_p(f)$ is concave on the set $\{0, 1, \dots, d\}$; in other words*

$$\lambda_p(f)^2 \geq \lambda_{p-1}(f)\lambda_{p+1}(f) \qquad \textit{for } p = 1, \dots, d-1.$$

Analogously, if $\pi\colon M \dashrightarrow B$ is an f-invariant meromorphic fibration, then the sequence $p \mapsto \log\lambda_p(f|\pi)$ is concave on the set $\{0, 1, \dots, \dim(M) - \dim(B)\}$.

As a consequence we have $\lambda_p \geq 1$ for all $p = 0, \dots, d$; furthermore, there exist $0 \leq p \leq p+q \leq d$ such that

$$1 = \lambda_0(f) < \cdots < \lambda_p(f) = \lambda_{p+1}(f) = \cdots = \lambda_{p+q}(f) > \cdots > \lambda_d(f). \tag{2.1}$$

3. Irreducible symplectic manifolds

We give here the basic notions and properties of irreducible symplectic manifolds (see [17], [25] for details).

Remark 3.1. Because of the non-degeneracy of σ, one can easily prove that an irreducible symplectic manifold has even complex dimension.

Throughout this section X denotes an irreducible symplectic manifold of dimension $2n$ and σ a non-degenerate holomorphic two-form on X.

Here is a list of the known examples of such manifolds that are not deformation equivalent.

(1) Let S be a $K3$ surface, i.e. a simply connected Kähler surface with trivial canonical bundle. Then the Hilbert scheme $S^{[n]} = Hilb^n(S)$, parametrizing 0-dimensional subschemes of S of length n, is a $2n$-dimensional irreducible symplectic manifold.
(2) Let T be a complex torus of dimension 2, let $\phi\colon Hilb^n(T) \to \mathrm{Sym}^n(T)$ be the natural morphism and let $s\colon \mathrm{Sym}^n(T) \to T$ be the sum morphism. Then the kernel $K_{n-1}(T)$ of the composition $s \circ \phi$ is an irreducible symplectic manifold of dimension $2n-2$, which is called a *generalized Kummer variety*.
(3) O'Grady has found two sporadic examples of irreducible symplectic manifolds of dimension 6 and 10.

An irreducible symplectic manifold is said of type $K3^{[n]}$ (respectively of type generalized Kummer) if it is deformation equivalent to $Hilb^n(S)$ for some $K3$ surface S (respectively to $K_{n-1}(T)$ for some two-dimensional complex torus T).

3.1. **The Beauville-Bogomolov form.** We can define a natural quadratic form on the second cohomology $H^2(X, \mathbb{R})$ which enjoys similar properties to the intersection form on compact surfaces; for details and proofs see [17].

Definition 3.2. *Let σ be a holomorphic two-form such that $\int(\sigma\bar\sigma)^n = 1$. The* Beauville-Bogomolov quadratic form *q_{BB} on $H^2(X, \mathbb{R})$ is defined by*

$$q_{BB}(\alpha) = \frac{n}{2}\int_X \alpha^2(\sigma\bar\sigma)^{n-1} + (1-n)\left(\int_X \alpha\sigma^n\bar\sigma^{n-1}\right)\left(\int_X \alpha\sigma^{n-1}\bar\sigma^n\right).$$

The Beauville-Bogomolov form satisfies two important properties: first the Beauville-Fujiki relation, saying that there exists a constant $c > 0$ such that

$$q_{BB}(\alpha)^n = c\int_X \alpha^{2n} \qquad \text{for all } \alpha \in H^2(X, \mathbb{R}).$$

In particular, some multiple of q_{BB} is defined over $\mathbb{Z}$.
Second, the next Proposition describes completely the signature of the form.

Proposition 3.3. *The Beauville-Bogomolov form has signature $(3, b_2(X)-3)$ on $H^2(X,\mathbb{R})$. More precisely, the decomposition $H^2(X,\mathbb{R}) = H^{1,1}(X,\mathbb{R}) \oplus \left(H^{2,0}(X) \oplus H^{0,2}(X)\right)_{\mathbb{R}}$ is orthogonal with respect to q_{BB}, and q_{BB} has signature $(1, h^{1,1}(X)-1)$ on $H^{1,1}(X,\mathbb{R})$ and is positive definite on $\left(H^{2,0}(X) \oplus H^{0,2}(X)\right)_{\mathbb{R}}$.*

Remark 3.4. For a divisor $D \in Div(X)$, we define $q_{BB}(D) := q_{BB}(c_1(\mathcal{O}_X(D)))$.
If D is effective and without fixed components, then $q_{BB}(D) \geq 0$. Indeed, let D' be an effective divisor linearly equivalent to D and with no components in common with D. We have

$$q_{BB}(D) = \frac{n}{2} \int_{D \cap D'} (\sigma\bar{\sigma})^{n-1},$$

where each irreducible component of the intersection $D \cap D'$ is counted with its multiplicity. The integral on the right hand side is non-negative because σ is a holomorphic form.
If furthermore D is ample, then by Beauville-Fujiki relation $q_{BB}(D) > 0$.

3.2. **Bimeromorphic maps between irreducible symplectic manifolds.** A bimeromorphic map $f\colon M \dashrightarrow M'$ between compact complex manifolds is an isomorphism in codimension 1 if there exist dense open subsets $U \subset M$ and $U' \subset M'$ such that

(1) $\mathrm{codim}(X \setminus U) \geq 2, \mathrm{codim}(X' \setminus U') \geq 2$;
(2) f induces an isomorphism $U \cong U'$.

A **pseudo-automorphism** of a complex manifold X is a bimeromorphic transformation which is an isomorphism in codimension 1.

Proposition 3.5 (Proposition 21.6 and 25.14 in [17])**.** *Let $f\colon X \dashrightarrow X'$ be a bimeromorphic map between irreducible symplectic manifolds. Then f is an isomorphism in codimension 1 and induces a linear isomorphism $f^*\colon H^2(X',\mathbb{Z}) \xrightarrow{\sim} H^2(X,\mathbb{Z})$ which preserves the Beauville-Bogomolov form.*
In particular, the group of birational transformation of an irreducible symplectic manifold X coincides with its group of pseudo-automorphisms and acts by isometries on $H^2(X,\mathbb{Z})$.

3.3. **Isometries of hyperbolic spaces.** Proposition 3.3 establishes a parallel between the dynamics of automorphisms of compact Kähler surfaces and that of bimeromorphic transformations of irreducible symplectic manifolds: in both cases the map on the manifold induces an isomorphism at the level of the integral cohomology group $H^2(X,\mathbb{Z})$ preserving a non-degenerate quadratic form (the intersection form in the surface case and the Beauville-Bogomolov form in the irreducible symplectic one). By Hodge's index theorem, the intersection form on the Picard group of a surface S has signature $(1, \rho(S)-1)$, which leads to a classification of automorphisms of surfaces as loxodromic, parabolic or elliptic depending on their action on the hyperbolic lattice $NS_{\mathbb{Z}}(S)$ (see [6]).
Analogously if X is an irreducible symplectic manifold, the restriction of the Beauville-Bogomolov form to $H^{1,1}(X,\mathbb{R})$ has signature $(1, h^{1,1}(X)-1)$. Since $H^{1,1}(X,\mathbb{R})$ is invariant by the action of a bimeromorphic transformation $f\colon X \dashrightarrow X$, and since the two lines $\mathbb{C}\sigma$ and $\mathbb{C}\bar{\sigma}$ are also invariant (the action of f^* being given by multiplication by a complex number of modulus 1), we can also classify bimeromorphic transformations of irreducible symplectic manifolds depending on their action on $H^2(X,\mathbb{Z})$ as follows.

Definition 3.6. *Let $f\colon X \dashrightarrow X$ a bimeromorphic transformation of an irreducible symplectic manifold (respectively, an automorphism of a compact Kähler surface) and denote by f_1^* the linear automorphism of $H^{1,1}(X,\mathbb{R})$ induced by f. We say that f is*

- loxodromic *if f_1^* admits an eigenvalue of modulus strictly greater than* 1 *(or, equivalently, if $\lambda_1(f) > 1$);*
- parabolic *if all the eigenvalues of f_1^* have modulus* 1 *and $||(f^n)_1^*||$ is not bounded as $n \to +\infty$;*
- elliptic *if $||(f^n)_1^*||$ is bounded as $n \to +\infty$.*

In each of the cases above, simple linear algebra arguments allow to further describe the situation.
Denote by $\mathcal{C}_{\geq 0} \subset H^{1,1}(X,\mathbb{R})$ (respectively $\mathcal{C}_0 \subset H^{1,1}(X,\mathbb{R})$) the positive (resp. null) cone for the Beauville-Bogomolov (repsectively, intersection) form q:

$$\mathcal{C}_{\geq 0} = \{\alpha \in H^{1,1}(X,\mathbb{R}) | q(\alpha) \geq 0\},$$

$$\mathcal{C}_0 = \{\alpha \in H^{1,1}(X,\mathbb{R}) | q(\alpha) = 0\}.$$

$\mathcal{C}_0$ is called the *isotropic cone* for the Beauville-Bogomolov form.

For a proof of the following result, see [27] (for irreducible symplectic manifolds) and [6] (for surfaces).

Theorem 3.7. *Let $f\colon X \dashrightarrow X$ a bimeromorphic transformation of an irreducible symplectic manifold (respectively, an automorphism of a compact Kähler surface).*

- *If f is loxodromic, then f_1^* has exactly one eigenvalue with modulus > 1 and exactly one eigenvalue with modulus < 1; these eigenvalues are real, simple and they are the inverse of each another; their eigenspaces are contained in $\mathcal{C}_0$, they are the only f_1^*-invariant lines in $\mathcal{C}_{\geq 0}$ and they are not defined over $\mathbb{Z}$.*
- *If f is parabolic, then all eigenvalues of f_1^* are roots of unity; the Jordan form of f^* has exactly one non-trivial Jordan block, which is of dimension* 3 *(in other words $||(f^n)_1^*||$ has quadratic growth); for every $\alpha \in H^{1,1}(X,\mathbb{R})$, $(f^n)_1^*\alpha/n^2$ converges to a class contained in $\mathcal{C}_0$, which (for every α outside a proper subspace of $H^{1,1}(X,\mathbb{R})$) spans the only f_1^*-invariant line of $\mathcal{C}_{\geq 0}$.*
- *If f is elliptic, then some iterate of f_1^* is equal to the identity.*

3.4. **The parabolic case.** In the case of surfaces, an automorphism being of parabolic type has a clear geometric interpretation (see [14], [15], or [9] for the birational case).

Theorem 3.8. *Let S be a compact Kähler surface; an automorphism $f\colon S \to S$ is of parabolic type if and only if there exists an f-invariant fibration $\pi\colon S \to C$ onto a nonsingular compact curve C.*

We could expect the situation to be similar in the irreducible symplectic context; indeed, Hu, Keum and Zhang have proved a partial analogue to Theorem 3.8, see [22]:

Theorem 3.9. *Let X be a $2n$-dimensional projective irreducible symplectic manifold of type $K3^{[n]}$ or of type generalized Kummer and let $f \in Bir(X)$ be a bimeromorphic transformation which is not elliptic; f is parabolic if and only if it admits a rational Lagrangian invariant fibration $\pi\colon X \dashrightarrow \mathbb{P}^n$ such that the induced transformation on $\mathbb{P}^n$ is biregular, i.e. there exists $g \in Aut(\mathbb{P}^n)$ such that $\pi \circ f = g \circ \pi$.*

The hard direction is to exhibit an invariant fibration for a parabolic transformation. The Main Theorem generalizes the converse, proving that the dynamics of a loxodromic transformation is too complicated to expect an invariant fibration.

4. Proof of the main results

Throughout this section, $f\colon X \dashrightarrow X$ denotes a loxodromic bimeromorphic transformation of an irreducible symplectic manifold X, $\pi\colon X \dashrightarrow B$ a meromorphic invariant fibration onto a Kähler manifold B such that $0 < \dim B < \dim X$ and $g\colon B \dashrightarrow B$ the induced transformation of the base.

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, dashed, "\pi"] & X \arrow[d, dashed, "\pi"] \\ B \arrow[r, dashed, "g"] & B \end{tikzcd}$$

The results in this Section are largely inspired by [1].

4.1. Meromorphic fibrations on irreducible symplectic manifolds.

We collect here some useful facts about the fibration π.

Remark 4.1. If B is Kähler, then it is projective. Indeed, if B wasn't projective, by Kodaira's projectivity criterion and Hodge decomposition

$$H^2(B, \mathbb{C}) = H^{2,0}(B) \oplus H^{1,1}(B) \oplus H^{0,2}(B),$$

we would have $H^{2,0}(B) \neq \{0\}$, meaning that B carries a non-trivial holomorphic 2-form σ_B. Since the indeterminacy locus of π has codimension at least 2, the pull-back $\pi^*\sigma_B$ can be extended to a global non-trivial 2-form on X which is not a multiple of σ, contradicting the hypothesis on X.

Here we use the same conventions as in [1]: let $\eta\colon \tilde{X} \to X$ be a resolution of the indeterminacy locus of π (see [31]), and let $\nu\colon \tilde{X} \to B$ be the induced holomorphic fibration, whose generic fibre is bimeromorphic to that of π.

$$\begin{tikzcd} \tilde{X} \arrow[d, "\eta"] \arrow[dr, "\nu"] & \\ X \arrow[r, dashed, "\pi"] & B \end{tikzcd}$$

The pull-back π^*D of an effective divisor $D \in Div(B)$ is defined as

$$\pi^*D = \eta_*\nu^*D,$$

where η_* is the pushforward as cycles. The pull-back induces linear morphisms $Pic(B) \to Pic(X)$ and $NS(B) \to NS(X)$, and is compatible with the pull-back of smooth forms defined in Section 2.

Now let $H \in Pic(B)$ be an ample class, and let $L = \pi^*H$. The pull-back of the complete linear system $|H|$ is a linear system $U \subset |L|$, whose associated meromorphic fibration is exactly π. In particular, L has no fixed component, and by Remark 3.4 we have $q_{BB}(L) \geq 0$.

Let $NS(B) \subset H^{1,1}(B, \mathbb{R})$ denote the Neron-Severi group with real coefficients of B. The following Lemma is essentially proven in [1].

Lemma 4.2. *The restriction of the Beauville-Bogomolov form to the pull-back $\pi^*NS(B)$ is not identically zero if and only if the generic fibre of π is of general type. If this is the case, then X is projective.*

Proof. Remark first that, since the generic fibre of ν is bimeromorphic to the generic fibre of π and the Kodaira dimension is a bimeromorphic invariant, the generic fibre of π is of general type if and only if the generic fibre of ν is.

As a second remark, by [26] if there exists a big line bundle on a compact Kähler manifold X, then X is projective.

Suppose that the generic fibre of π is of general type. Let H be an ample divisor on B and let $L = \pi^*H$. By [1][Theorem 2.3] we have $\kappa(X, L) = \dim(B) + \kappa(F)$, where F is the generic fibre of ν; we conclude that L is big (and in particular X is projective). We can thus write $L = A + E$ for an ample divisor A and an effective divisor E on X. Now, if q denotes the Beauville-Bogomolov form, we have

$$q(L) = q(L, A) + q(L, E) \geq q(L, A) = q(A, A) + q(A, E) \geq q(A, A) > 0,$$

where the first and second inequalities are consequences of L and A being without fixed components and the last one follows directly from Remark 3.4 . This proves the "if" direction.

Now assume that the restriction of q_{BB} to $\pi^*NS(B)$ is not identically zero. Since ample classes generate $NS_{\mathbb{R}}(B)$, there exists an ample line bundle $H \in Pic(B)$ such that, denoting $L = \pi^*H$, $q(L) \neq 0$; furthermore, L is without fixed components, so that $q(L) > 0$ by Remark 3.4. It follows by[3][Theorem 4.3.i] that L is big (thus X is projective), and so is η^*L since η is a birational morphism. Therefore, the restriction $\eta^*L|_F$ to a generic fibre of ν is also big (see [24][Corollary 2.2.11]). Now we have

$$\eta^*L = \nu^*H + \sum a_i E_i \qquad \text{for some } a_i \geq 0,$$

where the sum runs over all the irreducible components of the exceptional divisor of η. The adjunction formula leads to

$$K_F = K_{\tilde{X}}|_F + \det N^*_{F/\tilde{X}} = K_{\tilde{X}}|_F = \sum e_i E_i|_F \qquad \text{for some } e_i > 0,$$

since the conormal bundle $N^*_{F/\tilde{X}}$ is trivial.
This implies that, for some $m > 0$, the divisor $mK_F - \eta^*L|_F$ is effective because $\nu^*H|_F$ is trivial. Thus

$$\kappa(F) \geq \kappa(F, \eta^*L|_F) = \dim(F),$$

meaning that F is of general type. This proves the "only if" direction. □

Corollary 4.3. *If the generic fibre of ν is not of general type, then $\pi^*NS(B) \subset H^{1,1}(X, \mathbb{R})$ is a line contained in the isotropic cone $\mathcal{C}_0$.*

Proof. By Lemma 4.2, $\pi^*NS(B)$ is contained in the isotropic cone. The pull-back L of an ample line bundle on B is effective and non-trivial, so that its numerical class is also non-trivial; thus $\pi^*NS(B)$ cannot be trivial. To conclude it suffices to remark that $\pi^*NS(B)$ is a linear subspace of $H^{1,1}(X, \mathbb{R})$, and the only non-trivial subspaces contained in the isotropic cone are lines. □

4.2. **Density of orbits.** The following theorem was proven in [1].

Theorem 4.4. *Let X be a compact Kähler manifold and let $f\colon X \dashrightarrow X$ be a dominant meromorphic endomorphism. Then there exists a dominant meromorphic map $\pi\colon X \dashrightarrow B$ onto a compact Kähler manifold B such that*

1. $\pi \circ f = \pi$;
2. *the general fibre X_b of π is the Zariski closure of the orbit by f of a generic point of X_b.*

Lemma 4.5. *Let $\phi\colon X \dashrightarrow Y$, $\psi\colon Y \dashrightarrow Z$ be meromorphic maps between compact complex manifolds. If ϕ is an isomorphism in codimension 1, then for all $D \in Div(Z)$*

$$(\psi \circ \phi)^*D = \phi^*\psi^*D.$$

Proof. Let $U \subset X$, $V \subset Y$ two open sets such that ϕ induces an isomorphism $U \cong V$ and such that $\operatorname{codim}(X \setminus U) \geq 2, \operatorname{codim}(Y \setminus V) \geq 2$. It is easy to see that, for every effective divisor $D_Y \in Div(Y)$, we have an equality $\phi^* D_Y = \overline{\phi|_U^*(D_Y \cap V)}$; therefore the equality is true for every divisor in $Div(Y)$.
Up to shrinking V to some other open subset whose complement has codimension at least 2, we can suppose that ψ is regular on V; therefore the composition $\psi \circ \phi$ is regular on U, and since the complement of U has codimension ≥ 2 in X, for all $D \in Div(Z)$ we have

$$(\psi \circ \phi)^* D = \overline{(\psi \circ \phi)|_U^* D} = \overline{\phi|_U^*(\psi|_V^* D)} = \overline{\phi|_U^*(\psi^* D \cap V)} = \phi^* \psi^* D,$$

where the third equality follows again from the fact that the complement of V has codimension at least 2 in Y. This proves the claim. □

Let us prove point (3) of the Main Theorem.

Lemma 4.6. *Let $f \colon X \dashrightarrow X$ be a bimeromorphic loxodromic transformation of an irreducible symplectic manifold. Then the generic orbit of f is Zariski-dense.*

Proof. If the claim were false, then by Theorem 4.4 we could construct a commutative diagram

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{id} & B \end{array}$$

where π is a meromorphic map whose general fibre X_b coincides with the Zariski-closure of the f-orbit of a generic point of X_b. Remark 4.1 applies in the case where the fibres are not connected; therefore the base B is projective.
Now, remark that f^* acts as the identity on the space $\pi^* NS(B) \subset NS(X)$, which is defined on $\mathbb{Q}$: indeed, for $v \in NS(B)$, we have

$$f^* \pi^* v = (\pi \circ f)^* v = (id_B \circ \pi)^* v = \pi^* v,$$

where the first equality follows from Lemma 4.5; by Theorem 3.7, the Beauville-Bogomolov form is negative definite on $\pi^* NS(B)$.
Now, let $H \in Pic(B)$ be an ample line bundle and let $L = \pi^* H$. We have seen in 4.1 (again the hypothesis on fibres being connected was irrelevant) that L is a numerically non-trivial line bundle such that $q_{BB}(L) \geq 0$, contradiction. This proves the claim.

□

4.3. **The key lemma.** The following key lemma, together with the Proposition in Section 1, implies the Main Theorem.

Lemma 4.7 (Key lemma)**.** *Let X be an irreducible symplectic manifold, $f \colon X \dashrightarrow X$ a loxodromic bimeromorphic transformation and $\pi \colon X \dashrightarrow B$ a meromorphic f-invariant fibration onto a compact Kähler manifold. Then X is projective and the generic fibre of π is of general type.*

Proof. Let $g \colon B \dashrightarrow B$ be a bimeromorphic transformation such that $g \circ \pi = \pi \circ f$.
Let us define

$$V := Span\,\{(h \circ \pi)^* NS_{\mathbb{R}}(B) | h \colon B \dashrightarrow B \text{ birational transformation}\} \subset NS_{\mathbb{R}}(X).$$

The linear subspace V is clearly defined over $\mathbb{Q}$. Since the pull-back by π of an ample class is numerically non-trivial, we also have $V \neq \{0\}$.

Furthermore, V is f^*-invariant: if $v = (h \circ \pi)^* w$ for some $w \in NS(B)$ and for some birational transformation $h\colon B \dashrightarrow B$, then

$$f^* v = f^*(h \circ \pi)^* w = (h \circ \pi \circ f)^* w = (h \circ g \circ \pi)^* w = (\tilde{h} \circ \pi)^* w,$$

where $\tilde{h} = h \circ g\colon B \dashrightarrow B$ is a birational transformation and the second equality follows from Lemma 4.5.

Now suppose that the generic fibre of π is not of general type; we are first going to show that V is contained in the isotropic cone $\mathcal{C}_0 = \{v \in H^{1,1}(X, \mathbb{R}) | q_{BB}(v) = 0\}$. The generic fibre of the meromorphic fibration $h \circ \pi$ is bimeromorphic to that of π. By Lemma 4.2 we know that $(h \circ \pi)^* NS_{\mathbb{R}}(B)$ is contained in the isotropic cone for all birational transformations $h\colon B \dashrightarrow B$. We just need to show that for all birational transformations of B onto itself h_i, h_j and for all $w_i, w_j \in NS_{\mathbb{R}}(B)$ we have

$$q_{BB}((h_i \circ \pi)^* w_i, (h_j \circ \pi)^* w_j) = 0.$$

Let $h = h_j \circ h_i^{-1}$, and let $\rho\colon \tilde{B} \to B$ be a resolution of the indeterminacy locus of h; denote by $\tilde{h}\colon \tilde{B} \to B$ the induced holomorphic transformation, and let $\tilde{\pi} = \rho^{-1} \circ h_i \circ \pi\colon X \dashrightarrow \tilde{B}$; $\tilde{\pi}$ is a meromorphic fibration onto the birational model $\tilde{B}$, whose generic fibre is bimeromorphic to that of g. Finally, let $\eta\colon \tilde{X} \to X$ be a resolution of singularities of $\tilde{\pi}$ and let $\nu\colon \tilde{X} \to B$ be the induced holomorphic map.

$$\begin{tikzcd}
\tilde{X} \arrow[dd, "\eta"'] \arrow[dr, "\nu"] & & \\
& \tilde{B} \arrow[d, "\rho"] \arrow[dr, "\tilde{h}"] & \\
X \arrow[ur, dashed, "\tilde{\pi}"] \arrow[r, dashed, "h_i \circ \pi"] & B \arrow[r, dashed, "h"] & B
\end{tikzcd}$$

Now it is clear that $\eta\colon \tilde{X} \to X$ is a resolution of singularities of both $h_i \circ \pi$ and $h_j \circ \pi = h \circ h_i \circ \pi$. Therefore

$$(h_i \circ \pi)^* w_i = \eta_* \nu^* \rho^* w_i = \tilde{\pi}^* \rho^* w_i \in \tilde{\pi}^* NS(\tilde{B})$$

and

$$(h_j \circ \pi)^* w_j = \eta_* \nu^* \tilde{h}^* w_j = \tilde{\pi}^* \tilde{h}^* w_j \in \tilde{\pi}^* NS(\tilde{B}).$$

Since the fibres of $\tilde{\pi}$ are not of general type, it suffices to apply Lemma 4.2 to the fibration $\tilde{\pi}\colon X \dashrightarrow \tilde{B}$ to conclude that $q_{BB}((h_i \circ \pi)^* w_i, (h_j \circ \pi)^* w_j) = 0$. This proves that V is contained in the isotropic cone.

Now the only non trivial vector subspaces of $NS_{\mathbb{R}}(X)$ contained in the isotropic cone are lines; by Theorem 3.7, V is then an f^*-invariant line contained in the isotropic cone and not defined over $\mathbb{Q}$. But this contradicts the definition of V. We have thus proved that the generic fibre of π is of general type.

In order to prove that X is projective it suffices to apply the last part of Lemma 4.2.

□

By [31, Corollary 14.3]) we know that the group of birational transformations of a variety of general type is finite. Therefore, we expect the dynamics of f on the fibres to be simple.

4.4. **Relative Iitaka fibration.** Before giving the proof of the Main Theorem, we are going to recall the basic results about the relative Iitaka fibration. We will follow the approach of [30] with some elements from [31]. See also [19], [18].

Let X be a smooth projective variety, and suppose that some multiple of K_X has some non trivial section. Recall that, for $m > 0$ divisible enough, the rational map

$$\begin{aligned}\phi_{|mK_X|}\colon X &\dashrightarrow \mathbb{P}H^0(X, mK_X)^\vee\\ p &\mapsto \{s \in \mathbb{P}H^0(X, mK_X) | s(p) = 0\}\end{aligned}$$

has connected fibres. Moreover the rational map $\phi_{|mK_X|}$ eventually stabilize to a rational fibration that we call *canonical fibration* of X.

Remark 4.8. If $f\colon X \dashrightarrow X$ is a bimeromorphic transformation of X, the pull-back of forms induces a linear automorphism $f^*\colon H^0(X, mK_X) \to H^0(X, mK_X)$. For example, for $m = 1$ a section $\sigma \in H^0(X, K_X)$ is a holomorphic d-form ($d = \dim X$); f is defined on an open set $U \subset X$ such that $X \setminus U$ has codimension at least 2. Therefore by Hartogs theorem the pull-back $f|_U^*\sigma$ can be extended to X. It is easy to see that the construction is invertible and induces a linear automorphism of $\mathbb{P}H^0(X, mK_X)^\vee$ which commutes with the Iitaka fibration:

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, dashed, "\phi_{|mK_X|}"] & X \arrow[d, dashed, "\phi_{|mK_X|}"] \\ \mathbb{P}H^0(X, mK_X)^\vee \arrow[r, "\tilde{f}"] & \mathbb{P}H^0(X, mK_X)^\vee \end{tikzcd}$$

The above construction can be generalized to the relative setting: let $\pi\colon X \to B$ be a regular fibration onto a smooth projective variety B, and let $K_{X/B} = K_X \otimes \pi^* K_B^{-1}$ be the relative canonical bundle.
For some fixed positive integer $m > 0$ (divisible enough), let $\mathcal{S} = \pi_*(mK_{X/B})^\vee$. $\mathcal{S}$ is a coherent sheaf over B; therefore one can construct (generalizing the construction of the projective bundle associated to a vector bundle, see [31] for details) the algebraic projective fibre space

$$\eta\colon \operatorname{Proj}(\mathcal{S}) \to B$$

associated to $\mathcal{S}$, which is a projective scheme (a priori neither reduced nor irreducible) Y over B. Its generic geometric fibre Y_b over a generic point $b \in B$ is canonically isomorphic to $\mathbb{P}H^0(X_b, mK_{X_b})^\vee$. The Iitaka morphisms $\phi_b\colon X_b \dashrightarrow \mathbb{P}H^0(X_b, mK_{X_b})^\vee$ induce a rational map $\phi\colon X \dashrightarrow Y$ over B.
The *relative canonical fibration* of X with respect to π is

$$\begin{aligned}\phi\colon X &\dashrightarrow Y\\ x \in X_b &\mapsto \big[\{s \in H^0(X_b; mK_{X_b}) | s(x) = 0\}\big] \in Y_b.\end{aligned}$$

It can be shown that, for m divisible enough:

- ϕ stabilizes to a certain rational fibration;
- the image by ϕ of the generic fibre $X_b = \pi^{-1}(b)$ of π is contained inside the fibre $\eta^{-1}(b)$ of the natural projection $\eta\colon Y \to B$;
- the restriction of ϕ to a generic fibre X_b is birationally equivalent to the canonical fibration of X_b.

Remark 4.9. The construction in Remark 4.8 can also be generalized to the relative setting: let $f\colon X \dashrightarrow X$ and $g\colon B \dashrightarrow B$ be birational transformations such that $\pi \circ f = g \circ \pi$.

For a generic $b \in B$ define

$$\begin{aligned} \tilde{f}|_{Y_b} \colon \mathbb{P}H^0(X_b, mK_{X_b})^\vee &\dashrightarrow \mathbb{P}H^0(X_{g(b)}, mK_{X_{g(b)}})^\vee \\ [s^*] &\mapsto \left\{ [s] \in \mathbb{P}H^0(X_{g(b)}, mK_{X_{g(b)}}) | s^*(f^* s) = 0 \right\}. \end{aligned}$$

These are well defined linear automorphisms because, for a fibre X_b of π not contained in the indeterminacy locus of f, the restriction $f \colon X_b \dashrightarrow X_{g(b)}$ is a birational map, and thus induces a linear isomorphism

$$f^* \colon H^0(X_{g(b)}, mK_{X_{g(b)}}) \to H^0(X_b, mK_{X_b}).$$

Furthermore the $\tilde{f}_{X_b}$ can be glued to a birational transformation $\tilde{f} \colon Y \dashrightarrow Y$ such that $\eta \circ \tilde{f} = g \circ \eta$.

Now suppose the generic fibre of π is of general type. Since the restriction of ϕ to a generic fibre of g is birational onto its image and the images of fibres are disjoint, ϕ itself must be birational onto its image; denote by Z the closure of the image of ϕ and let $f_Z = \phi \circ f \circ \phi^{-1} \colon Z \dashrightarrow Z$ be the birational transformation induced by f.
By the above Remark, f_Z is the restriction of the birational transformation $\tilde{f} \colon Y \dashrightarrow Y$. In particular f_Z induces an isomorphism between generic fibres of $\eta|_Z$.

$$\begin{tikzcd} X \arrow[loop above, "f"] \arrow[r, dashed, "\phi"] \arrow[d, dashed, "\pi"] & Z \arrow[loop above, "f_Z"] \arrow[r, hook] \arrow[dl, dashed] & Y \arrow[loop above, "\tilde{f}"] \arrow[dll, dashed, "\eta"] \\ B \arrow[loop left, "g"] & & \end{tikzcd}$$

4.5. Proof of the Main Theorem.

Lemma 4.10. *Let X, B be projective manifolds, $f \colon X \dashrightarrow X$ and $g \colon B \dashrightarrow B$ birational transformations and $\pi \colon X \to B$ a fibration such that $\pi \circ f = g \circ \pi$.*

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, "\pi"] & X \arrow[d, "\pi"] \\ B \arrow[r, dashed, "g"] & B \end{tikzcd}$$

Assume that the generic fibre of π is of general type and that the generic orbit of g is Zariski-dense. Then all the fibres over a non-empty Zariski open subset of B are isomorphic.

Proof. Denote as before

$$\phi \colon X \dashrightarrow Y$$

the relative Iitaka fibration. We are going to identify X with its birational model $\overline{\phi(X)}$. Let $F = \pi^{-1}(b_0)$ be the fibre of π over a point b_0 whose orbit is Zariski-dense in B, and let

$$\mathfrak{I} := \mathfrak{Isom}_B(X, F \times B)$$

be the B-scheme of isomorphisms over B between X and $F \times B$; the fibre $\mathfrak{I}_b$ parametrizes the isomorphisms $X_b \cong F$. We can realize $\mathfrak{I}$ as an open subset of the Hilbert scheme $\mathfrak{Hilb}_B(X \times_B (B \times F))$ by identifying a morphism $X_b \to F$ with its graph in $X_b \times F$. Therefore,

$$\mathfrak{I} = \coprod_{P \in \mathbb{Q}[\lambda]} \mathfrak{I}^P,$$

where the fibre $\mathfrak{I}_b^P$ is the (a priori non irreducible and non reduced) quasi-projective scheme of (graphs of) isomorphisms $X_b \xrightarrow{\sim} F$ having fixed Hilbert polynomial $P(\lambda)$; such polynomials are calculated with respect to the restriction to the fibre $X_b \times F$ of a fixed line bundle L on $X \times_B (B \times F)$ relatively very ample over B. We shall fix

$$L = H_Y|_X \boxtimes_B H_F,$$

where H_Y is a very ample line bundle on Y and H_F is a very ample line bundle on F.
Now, the pull-back of forms by f induces a linear isomorphism

$$\tilde{f}_b \colon \mathbb{P}H^0(X_b, mK_{X_b})^\vee \xrightarrow{\sim} \mathbb{P}H^0(X_{g(b)}, mK_{X_{g(b)}})^\vee$$

between fibres of $\eta\colon Y \xrightarrow{\sim} B$, which restricts to an isomorphism $X_b \to X_{g(b)}$; under the canonical identification of fibres of η with $\mathbb{P}^N$, $H_Y|_{Y_b} \cong \mathcal{O}_{\mathbb{P}^N}(d)$ (meaning that the section $H_Y|_{Y_b}$ has degree d) for some $d > 0$ independent of the fibre. Under the identification, the action of $\tilde{f}_b$ is linear, so that $\tilde{f}_b^*(H_Y|_{Y_{g(b)}})$ also has degree d on $\mathbb{P}^N$. In particular we have

$$\tilde{f}_b^*(H_Y|_{X_{g(b)}}) = H_Y|_{X_b}.$$

Now take any isomorphism $X_{b_0} \xrightarrow{\sim} F$, which we can identify with its graph $\Gamma \subset X_{b_0} \times F$; the image of Γ by the isomorphism $\tilde{f}_{b_0} \times id_F \colon X_{b_0} \times F \xrightarrow{\sim} X_{g(b_0)} \times F$ is the graph Γ' of an isomorphism $X_{g(b_0)} \xrightarrow{\sim} F$. Furthermore, since $(\tilde{f}_{b_0} \times id_F)^*(L|_{X_{g(b_0)} \times F}) = L|_{X_{b_0} \times F}$, Γ' has the same Hilbert polynomial as Γ. Iterating this reasoning we find that for some $P \in \mathbb{Q}[\lambda]$ the image of the natural morphism $\psi\colon \mathfrak{I}^P \to B$ is Zariski-dense.
By Chevalley's theorem ([21, Theorem 3.16]) we also know that $\psi(\mathfrak{I}^P)$ is constructible; since every constructible Zariski-dense subset of an irreducible scheme contains a dense open set [21, Proof of Theorem 3.16], we have $X_b \cong F$ for all b in an open dense subset of B. This concludes the proof. $\square$

Proof of the Proposition in Section 1. By Lemma 4.10, all the fibres over a dense open subset $U \subset B$ are isomorphic, which shows (1). By [28, Proposition 2.6.10], there exists an étale cover $\epsilon\colon U' \to U$ such that the induced fibration $X'_{U'} := X \times_U U'$ is trivial:

$$X'_{U'} \cong U' \times F.$$

This shows (2).

Now suppose that the generic fibre F is of general type. This implies that the group $G := \mathrm{Aut}(F)$ is finite; therefore, for any $x \in F$, we can define the subvariety

$$W^x := U' \times G \cdot x \subset X'_{U'}.$$

We are going to show that the image of W^x by the cover $\epsilon_X \colon X'_{U'} \to X_U$ is f-invariant.

Remark that the fibration $\pi_U \colon X_U \to U$ is locally trivial in the euclidean topology. Let $\{U_i\}_{i \in I}$ be a covering of U by euclidean open subsets such that the restriction of the fibration to each X_{U_i} is trivial: there exist biholomorphisms $X_{U_i} \cong U_i \times F$. Then the subvarieties

$$V_i^x := U_i \times G \cdot x \subset X_{U_i}$$

patch together to algebraic subvarieties V^x of X_U which are exactly the images of the W^x. Now we will prove that the varieties V^x are f-invariant. Let $p \in X_U$ be a point where f is defined and such that g is defined on $\pi(p)$, and let $i \in I$ be such that $p \in X_{U_i}$; up to shrinking U_i, we can suppose that $g(U_i) \subset U_j$. By an identification $X_{U_i} \cong U_i \times F, X_{U_j} \cong U_j \times F$, we can write $f(x, y) = (g(x), h(x, y))$; here, for all x in on open dense subset of U_i, the continuous map $b \mapsto h(x, \bullet) \in \mathrm{Bir}(F)$ is well defined. Since $\mathrm{Bir}(F)$ is a finite (hence discrete) group, h doesn't depend on x, which shows that all the varieties V^x are f-invariant.

Now remark that the varieties $\epsilon_X(U' \times G \cdot \{x\})$ are the disjoint union of varieties of type $\epsilon_X(U' \times \{y\})$; since the first are f-invariant, the latter must be f-periodic, which concludes the proof.

□

Proof of the Main Theorem, point (1). Let X be an irreducible symplectic manifold, $f\colon X \dashrightarrow X$ a birational loxodromic transformation, and suppose by contradiction that f is imprimitive: there exist thus a meromorphic fibration $\pi\colon X \dashrightarrow B$ and a bimeromorphic transformation $g\colon B \dashrightarrow B$ such that $\pi \circ f = g \circ \pi$.
By Lemma 4.7, X is projective and the generic fibre of π is of general type. However, we also know by Lemma 4.6 that the generic orbit of f is Zariski-dense; therefore, by the Proposition in Section 1, the generic fibre of π cannot be of general type, a contradiction. □

5. Invariant subvarieties

Let X be a compact complex manifold. If $f\colon X \to X$ is an automorphism, we say that a subvariety $W \subset X$ is *invariant* if $f(W) = W$, or, equivalently, if $f^{-1}(W) = W$. We say that $W \subset X$ is *periodic* if it is invariant for some positive iterate f^n of f.
Now let $f\colon X \dashrightarrow X$ be a pseudo-automorphism of X (i.e. a bimeromorphic transformation which is an isomorphism in codimension 1). We say that a hypersurface $W \subset X$ is invariant if the strict transform f^*W of W is equal to W (as a set); since f and f^{-1} don't contract any hypersurface, this is equivalent to $f(W) = W$ (here $f(W)$ denotes the analytic closure of $f|_U(W \cap U)$, where $U \subset X$ is the maximal open set where f is well defined). We say that a hypersurface is periodic if it is invariant for some positive iterate of f.

The following Theorem is a special case of [5][Theorem B].

Theorem 5.1. *Let $f\colon X \dashrightarrow X$ be a pseudo-automorphism of a compact complex manifold X. If f admits at least $\dim(X)+b_2(X)-1$ invariant hypersurfaces, then it preserves a non-constant meromorphic function.*

Proof of the Main Theorem, point (2). Let $f\colon X \dashrightarrow X$ be a loxodromic bimeromorphic transformation of an irreducible symplectic manifold X (which is a pseudo-automorphism by 3.5).
Suppose that f admits more than $\dim(X) + b_2(X) - 2$ periodic hypersurfaces; then some iterate of f satisfies the hypothesis of Theorem 5.1. Therefore f^n preserves a non-constant meromorphic function $\pi\colon X \to \mathbb{P}^1$, and, up to considering the Stein factorization of π, we can assume that π is an f^n-invariant fibration onto a curve. This contradicts point (1) of the Main Theorem. □

The following example shows that we cannot hope to obtain an analogue of point (2) of the Main Theorem for higher codimensional subvarieties.

Example 5.2. Let $f\colon S \to S$ be a loxodromic automorphism of a $K3$ surface S, and let $X = Hilb^n(S)$. Then X is an irreducible symplectic manifold and f induces a loxodromic automorphism f_n of X. By point (2) of the Main Theorem, f_n admits only a finite number of invariant hypersurfaces. However f admits infinitely many periodic points ([4],[6]); if x is a periodic point in S, then (the image in X of) $\{x\}^p \times S^{n-p}$ is a periodic subvariety of codimension $2p$.

Thus we have showed the following Proposition.

Proposition 5.3. *For all integers $0 < p \leq n$, there exist a $2n$-dimensional projective irreducible symplectic manifold X and a loxodromic automorphism $f\colon X \to X$ admitting infinitely many periodic subvarieties of codimension $2p$.*

6. Appendix: An alternative approach to the Main Theorem

In this section we describe a different approach to the proof of the Main Theorem which doesn't require the Proposition in Section 1. The result we obtain is actually slightly weaker than the Main Theorem; however this approach allows to prove point (2) and (3), as well as point (1) for automorphisms.

We have already seen in Proposition 3.7 that the first dynamical degree of a bimeromorphic transformation $f\colon X \dashrightarrow X$ is either 1 or an algebraic integer λ whose conjugates over $\mathbb{Q}$ are λ^{-1} and some complex numbers of modulus 1 (so that λ is a quadratic or a Salem number). In the case of automorphisms, the following Proposition from Verbitsky [32] allows to completely describe all the other dynamical degrees as well.

Proposition 6.1. *Let X be an irreducible symplectic manifold of dimension $2n$ and let $SH2(X, \mathbb{C}) \subset H^*(X, \mathbb{C})$ be the subalgebra generated by $H^2(X, \mathbb{C})$. Then we have an isomorphism*

$$SH^2(X, \mathbb{C}) = \operatorname{Sym}^* H^2(X, \mathbb{C})/\langle \alpha^{n+1} | q_{BB}(\alpha) = 0\rangle$$

.

The following Corollary is due to Oguiso [27].

Corollary 6.2. *Let $f\colon X \to X$ be an automorphism of an irreducible symplectic manifold of dimension $2n$. Then for $p = 0, 1, \ldots, n$*

$$\lambda_p(f) = \lambda_{2n-p}(f) = \lambda_1(f)^p.$$

Proof. By Proposition 6.1 the cup-product induces an injection

$$\operatorname{Sym}^p H^2(X, \mathbb{C}) \hookrightarrow H^{2p}(X, \mathbb{C})$$

for $p = 1, \ldots, n$.
Let $v_1 \in H^2(X, \mathbb{C})$ be an eigenvector for the eigenvalue $\lambda = \lambda_1(f)$. Then $v_p := v_1^p \in H^{2p}(X, \mathbb{C})$ is a non-zero class for $p = 1, \ldots, n$ and $f^*v_p = (f^*v_1)^p = \lambda^p v_p$. This implies that $\lambda_p(f) \geq \lambda_1(f)^p$, and we must have equality by log-concavity (Proposition 2.8). This proves the result for $p = 0, 1, \ldots, n$.
Now by Remark 2.3 we have $\lambda_{2n-p}(f) = \lambda_p(f^{-1})$. Applying what we have just proved to f^{-1} we obtain

$$\lambda_{2n-1}(f) = \lambda_1(f^{-1}) = \lambda_n(f^{-1})^{1/n} = \lambda_n(f^{-1})^{1/n} = \lambda_1(f)$$

and thus, for $p = 0, \ldots, n$,

$$\lambda_{2n-p}(f) = \lambda_p(f^{-1}) = \lambda_1(f^{-1})^p = \lambda_1(f)^p,$$

which concludes the proof. □

Lemma 6.3. *Let X be a smooth projective variety, $f\colon X \dashrightarrow X$ a birational transformation of X, $\pi\colon X \dashrightarrow B$ a rational f-invariant fibration onto a smooth projective variety B. If the generic fibre of π is of general type, then all the relative dynamical degrees $\lambda_p(f|\pi)$ are equal to 1 (for $p = 0, \ldots, \dim(X) - \dim(B)$).*

Proof. Since the Kodaira dimension and the relative dynamical degrees are bimeromorphic invariants (Remark 2.5), up to considering a resolution of the indeterminacy locus of π, we can suppose that π is regular.
Let

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\pi"] & Y := \mathrm{Proj}(\pi_* K_{X/Y}^{\otimes m}) \arrow[dl, "\eta"] \\ B & \end{tikzcd}$$

be the Iitaka fibration. Since ϕ is birational onto its image, denoting $Z \subset Y$ the closure of $\phi(X)$, the claim is equivalent to $\lambda_p(f_Z|\eta_Z) = 1$, where η_Z denotes the restriction of η to Z and $f_Z = \phi \circ f \circ \phi^{-1} \colon Z \dashrightarrow Z$.
The construction of Remark 4.9 provides a birational transformation $\tilde{f} \colon Y \dashrightarrow Y$ extending f_Z.

$$\begin{tikzcd} X \arrow[loop above, "f"] \arrow[r, dashed, "\phi"] \arrow[d, "\pi"] & Z \arrow[loop above, "f_Z"] \arrow[r, hook] \arrow[dl] & Y \arrow[loop above, "\tilde{f}"] \arrow[dll, "\eta"] \\ B \arrow[loop left, "g"] & & \end{tikzcd}$$

Now we will prove that if $\lambda_p(\tilde{f}|\eta) = 1$ then $\lambda_p(f_Z|\eta_Z) = 1$. Let $H_Y \in Pic(Y)$ and $H_B \in Pic(B)$ be ample classes; therefore $H_Y|_Z$ is an ample class on Z. The map

$$H^{2n,2n}(Y, \mathbb{R}) \to \mathbb{R}$$
$$\alpha \mapsto \int_Y \alpha \wedge c_1(H_Y)^{\dim(Y)-\dim(X)} = \alpha \cdot H_Y^{\dim(Y)-\dim(X)}$$

is linear and strictly positive (except on 0) on the closed positive cone $\mathcal{K}_{2n} \subset H^{2n,2n}(Y, \mathbb{R})$. Since $\alpha \mapsto \alpha \cdot [Z]$ is linear too, we can define

$$M = \max_{\alpha \in \mathcal{K}_{2n} \setminus \{0\}} \frac{\alpha \cdot [Z]}{\alpha \cdot H_Y^{\dim(Y)-\dim(X)}} \geq 0.$$

Now

$$\lambda_p(f_Z|\eta_Z) = \lim_{n \to +\infty} \left((\tilde{f}^n)^* H_Y^p \cdot \eta^* H_B^{\dim(B)} \cdot H_Y^{2n-p-\dim(B)} \cdot [Z] \right)^{\frac{1}{n}} \leq$$

$$\lim_{n \to +\infty} \left(M (\tilde{f}^n)^* H_Y^p \cdot \eta^* H_B^{\dim(B)} \cdot H_Y^{\dim(Y)-p-\dim(B)} \right)^{\frac{1}{n}} =$$

$$\lim_{n \to +\infty} \left((\tilde{f}^n)^* H_Y^p \cdot \eta^* H_B^{\dim(B)} \cdot H_Y^{\dim(Y)-p-\dim(B)} \right)^{\frac{1}{n}} = \lambda_p(\tilde{f}|\eta) = 1,$$

and since all relative dynamical degrees are ≥ 1 (Proposition 2.8) we have $\lambda_p(f_Z|\eta_Z) = 1$.

Now all is left to prove is that $\lambda_p(\tilde{f}|\eta) = 1$. There exists $k > 0$ such that $\eta^* H_B^{\dim(B)} \equiv_{num} k[F]$, where $[F]$ is the numerical class of a fibre F of η. We have

$$\lambda_p(\tilde{f}|\eta) = \lim_{n \to +\infty} \left((\tilde{f}^n)^* H_Y^p \cdot \eta^* H_B^{\dim(B)} \cdot H_Y^{\dim(Y)-p-\dim(B)} \right)^{\frac{1}{n}} =$$

$$\lim_{n \to +\infty} \left((\tilde{f}^n)^* H_Y^p \cdot k[F] \cdot H_Y^{\dim(Y)-p-\dim(B)} \right)^{\frac{1}{n}} =$$

$$\lim_{n \to +\infty} \left(\left((\tilde{f}^n)^* H_Y \right) |_F^p \cdot H_Y|_F^{\dim(Y)-p-\dim(B)} \right)^{\frac{1}{n}}.$$

For each fibre we have a canonical identification $F \cong \mathbb{P}^N$, and by this identification $H_Y|_F \cong \mathcal{O}_{\mathbb{P}^N}(d)$, meaning that the hyperplane section $H_Y|_F$ is defined by an equation of degree d. Under the identification, the action of $\tilde{f}$ from one fibre to another is linear, so that $\left((\tilde{f^n})^* H_Y\right)|_F$ is also defined by an equation of degree d on $\mathbb{P}^N$. This means that

$$\lambda_p(\tilde{f}|\eta) = \lim_{n\to+\infty}(d^{\dim(F)})^{\frac{1}{n}} = 1$$

as we wanted to show. This concludes the proof.

□

The following Proposition is a weaker version of point (1) of the Main Theorem.

Proposition 6.4. *Let $f\colon X \dashrightarrow X$ be a loxodromic transformation of an irreducible symplectic manifold X of dimension $2n$, and let*

$$1 = \lambda_0(f) < \cdots < \lambda_{p_0}(f) = \cdots = \lambda_{p_0+k}(f) > \cdots > \lambda_{2n}(f) = 1$$

be its dynamical degrees.
If $\pi\colon B \dashrightarrow B$ is an f-invariant meromorphic fibration, then $\dim(B) \geq 2n-k$. In particular, if f is an automorphism (or, more generally, if all the consecutive dynamical degrees of f are distinct), then it is primitive.

Proof. Let $g\colon B \dashrightarrow B$ be a birational transformation such that $g\circ\pi = \pi\circ f$.

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, dashed, "\pi"] & X \arrow[d, dashed, "\pi"] \\ B \arrow[r, dashed, "g"] & B \end{tikzcd}$$

We know by Lemma 4.7 that the generic fibre of π is of general type; by Lemma 6.3 this implies that all the relative dynamical degrees $\lambda_p(f|\pi)$ are equal to 1. By Theorem 2.7 we then have

$$\lambda_p(f) = \max_{p-\dim(F)\leq q\leq p} \lambda_q(g),$$

where $\dim(F) = \dim(X) - \dim(B)$ is the dimension of a generic fibre.
Let $q \in \{0, 1, \ldots, \dim(B)\}$ be such that $\lambda_q(g)$ is maximal. Then

$$\lambda_q(f) = \lambda_{q+1}(f) = \cdots = \lambda_{q+\dim(F)}(f) = \lambda_q(g),$$

meaning that $k \geq \dim(F) = 2n - \dim(B)$. This concludes the proof.

□

Remark 6.5. Since in the Theorem we have $k \leq 2n-1$, the base of an invariant fibration cannot be a curve. Therefore Proposition 6.4 implies point (2) of the Main Theorem.

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