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DYNAMICAL GREEN FUNCTIONS AND DISCRETE SCHRÖDINGER OPERATORS WITH POTENTIALS GENERATED BY PRIMITIVE INVERTIBLE SUBSTITUTION

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Abstract. In this paper, we set up a "dictionary" between discrete Schrödinger operators and the holomorphic dynamics on certain affine cubic surfaces, building on previous work by Cantat, Damanik and Gorodetski.

Contents

1. Set–up and Main Results

The goal of this paper is to add on previous work by Cantat [6], Damanik and Gorodetski [14, 15] (see also [9, 28] for instance) to establish a correspondence between the study of certain discrete Schrödinger operators and the holomorphic dynamics of automorphisms on certain affine cubic surfaces.

1.1. Discrete Schrödinger Operators.

1.1.1. *Left shift dynamics.* Consider the free group on two generators $\mathbf{F}_2 := \langle a, b \,|\, \emptyset \rangle$ and let $\varphi \in \mathrm{Aut}(\mathbf{F}_2)$ be a positive automorphism, *i.e* such that the images $\varphi(a)$ and $\varphi(b)$ are words in a and b — and thus do not involve the inverse a^{-1} and b^{-1}.

Using the action of $\mathrm{Aut}(\mathbf{F}_2)$ on the abelianized group $\mathrm{Ab}(\mathbf{F}_2) = \mathbb{Z}^2$, one can associate a matrix $M_\varphi \in GL_2(\mathbb{Z})$ to φ. Assume M_φ to be hyperbolic, *i.e*:

- either $\det(M_\varphi) = 1$ and $\mathrm{Tr}(M_\varphi) > 2$;
- or $\det(M_\varphi) = -1$ and $\mathrm{Tr}(M_\varphi) \neq 0$.

By replacing φ with $\varphi^2 := \varphi \circ \varphi$,which is still positive, we can restrict ourselves to the first case; this means that the spectrum of M_φ is of the form $\{\lambda, \lambda^{-1}\}$ where λ denotes a quadratic integer greater than one.

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Let Ω be the set of finite words on the generators a and b, endowed with the topology pertaining to the following distance:

$$d : (u, v) \mapsto \frac{1}{\inf\{|n| \mid u_n \neq v_n\} + 1}.$$

The initial automorphism φ extends to a substitution ι over the letters a and b which has a unique "positively infinite" invariant word $u_+ \in \{a, b\}^{\mathbb{N}}$.

Example 1.1. Let ζ be the Fibonacci substitution, given by $a \mapsto ab$ and $b \mapsto a$; its associated matrix M_ζ is given by:

$$M_\zeta := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in GL_2(\mathbb{Z})$$

and it fixes the infinite word beginning with $abaababaabaababaababaabaababaabaab\ldots$

Now consider the left shift on $\{a, b\}^{\mathbb{Z}}$:

$$\begin{aligned} T : \{a, b\}^{\mathbb{Z}} &\to \{a, b\}^{\mathbb{Z}} \\ u &\mapsto (u_{n+1})_{n \in \mathbb{Z}} \end{aligned}$$

and let W be the set of all adherent values for the sequence $(T^p u_+)_{p \geq 0}$ — in other words, it is the ω–limit set W of the T–orbit of u_+. It is well known (see for instance [12]) that there exists a unique T–invariant probability measure ν on the topological set W and that the left shift T is ergodic with respect to ν — see [23, p.58] for an outlook on uniquely ergodic maps.

1.1.2. *Discrete Schrödinger operators.* Given any word $w \in W$, one can define the following potential function:

$$\begin{aligned} v_w : \mathbb{Z} &\to \{0, 1\} \\ n &\mapsto \begin{cases} 1 \text{ if } w_n = a \\ 0 \text{ else} \end{cases}. \end{aligned}$$

Consider for any fixed $\kappa \in \mathbb{R}$ and $w \in W$ the following operator, defined on the space $\ell^2(\mathbb{Z})$ of complex–valued square–summable sequences:

$$\begin{aligned} H_{\kappa,w} : \ell^2(\mathbb{Z}) &\to \ell^2(\mathbb{Z}) \\ \xi &\mapsto (\xi_{n+1} + \xi_{n-1} + \kappa v_w(n) \xi_n)_{n \in \mathbb{Z}}. \end{aligned}$$

Remark that this operator is self–adjoint and $\|H_{\kappa,w}\| \leq 2 + |\kappa|$; therefore its spectrum $\Sigma_{\kappa,w}$ is a subset of the real interval $[-2 - |\kappa|, 2 + |\kappa|]$.

Since $H_{\kappa,w}$ is uniquely ergodic, we can apply the following result due to Kotani and Pastur [25].

Theorem 1.2 (Kotani – Pastur)**.**
There exists a compact set $\Sigma_\kappa \subset [-2 - |\kappa|, 2 + |\kappa|]$ such that $\Sigma_{\kappa,w} = \Sigma_\kappa$ for all $w \in W$.

We call the set Σ_κ the **almost–sure spectrum** of the operator $H_{\kappa,w}$ with respect to the measure ν.

Remark 1.3. If $H_{\kappa,w}$ was ergodic — non–uniquely —, one would have $\Sigma_{\kappa,w} = \Sigma_\kappa$ for ν–almost every $w \in W$, hence the colloquial name of "almost–sure spectrum".

1.1.3. *Density of states.* Let $H^N_{\kappa,w}$ be the restricted operator $H_{\kappa,w}$ to the set $\mathbb{C}^{\{-N,\dots,N\}}$ with Dirichlet boundary conditions, meaning we only consider sequences $(\xi)_n$ with $-N \leq n \leq N$ such that:

- $\xi_n = 0$ for $n \leq -N-1$;
- $\xi_n = 0$ for $n \geq N+1$.

This gives a self–adjoint endomorphism of $\mathbb{C}^{2N+1}$; as such it has real eigenvalues $\lambda_0^N, \dots, \lambda_{2N}^N$. Define the following probability measure:

$$\mu_N^\kappa := \frac{1}{2N+1} \sum_{j=0}^{2N} \delta_{\lambda_j^N}.$$

Theorem 1.4 (Avron – Simon [1])**.**

(i) *For ν–almost every $w \in W$ the sequence $(\mu_N^\kappa)_N$ weakly converges to a probability measure dk_κ on $\mathbb{C}$, called density of states;*
(ii) *for any continuous function $g : \mathbb{C} \to \mathbb{C}$:*

$$\int_{\mathbb{C}} g(E) dk_\kappa(E) = \int_{w \in W} \langle g(H_{\kappa,w}) \cdot \delta_0 \mid \delta_0 \rangle d\nu(w) \ ;$$

(iii) $\operatorname{supp}(dk_\kappa) = \Sigma_\kappa$.

Remark 1.5. It is standard to then define the **integrated density of states** as the repartition function of the probability measure dk_κ:

$$k_\kappa : (E \in \mathbb{R}) \mapsto \int_{-\infty}^{E} dk_\kappa.$$

1.1.4. *Lyapunov exponent.* A hypothetical eigenvalue–eigenvector pair (E, ξ) for $H_{\kappa,w}$ should satisfy the equation:

$$\forall n \in \mathbb{Z}, \quad \xi_{n+1} + \xi_{n-1} + \kappa v_w(n) \xi_n = E \xi_n \ , \tag{1.1}$$

that is:

$$\forall n \in \mathbb{Z}, \quad \begin{pmatrix} \xi_{n+1} \\ \xi_n \end{pmatrix} = M^E_{n,\kappa,w} \begin{pmatrix} \xi_n \\ \xi_{n-1} \end{pmatrix} \tag{1.2}$$

where:

$$M^E_{n,\kappa,w} := \begin{pmatrix} E - \kappa v_w(n) & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{C}),$$

i.e $M^E_{n,\kappa,w}$ is equal to one of the two matrices:

$$M^E_\kappa(a) := \begin{pmatrix} E - \kappa & -1 \\ 1 & 0 \end{pmatrix}, \quad M^E_\kappa(b) := \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}.$$

Consider the **Lyapunov exponent**:

$$\gamma_\kappa(E) := \limsup_{N \to \infty} \frac{1}{N} \int_W \log \left\| \prod_{n=0}^{N} M^E_{n,\kappa,w} \right\| d\nu(w).$$

By Osseledet's Theorem, this quantity is well defined and:

$$\limsup_{N \to \infty} \log \left\| \prod_{n=0}^{N} M^E_{n,\kappa,w} \right\|$$

is ν–almost surely constant equal to $\gamma_\kappa(E)$.

Theorem 1.6 (see[12] and [8])**.**
The Lyapunov exponent is a non–negative function such that :

(*i*)
$$\gamma_\kappa(E) = \int_{\Sigma_\kappa} \log|E - E'| dk_\kappa(E');$$

(*ii*)
$$dd^c \gamma_\kappa = 2\pi dk_\kappa; \tag{1.3}$$

(*iii*) *the almost–sure spectrum satisfies* $\Sigma_\kappa = \{\gamma_\kappa = 0\}$.

Proof. The first item is the Thouless formula — see [8, p.340] — and thus, since $dd^c \log|z - z_0| = 2\pi\delta_{z_0}$, one obtains property (*ii*). The third result is a theorem due to Ishii, Kotani and Pastur — see [12] for an overview.

□

1.1.5. *Green function for the almost–sure spectrum.* First, recall the following definition. Let U be an open set in $\mathbb{C}$ such that its complement $\mathbb{C} \setminus U$ is a compact set. A function $g_U : U \to (0, \infty)$ is a **Green function** for the domain U — alternatively, for the compact $\mathbb{C} \setminus U$ — if:

($G1$) g_U is harmonic;
($G2$) the following limit exists:
$$\lim_{z\to\infty} (g_U(z) - \log|z|);$$
($G3$) for all $\xi \in \partial U$, one has:
$$\lim_{z\to\xi} g_U(z) = 0.$$

Remark 1.7.

(1) If U is such an open subset of $\mathbb{C}$ then its Green function, if it exists, is unique — see [22, p.182]. Moreover, one can replace ($G2$) with $g_U(z) - \log|z| = O(1)$ at infinity.
(2) If U as a Green function, there exists a positive real number C such that
$$g_U(z) = \log|z| - \log(C) + o(1) \text{ as } z \text{ goes to infinity.}$$
C is called the **capacity** of the compact set $\mathbb{C} \setminus U$. For more details on set capacities, see [26, p.132].
(3) The measure $dd^c g_U$ is called the **equilibrium measure** of the compact set $\mathbb{C} \setminus U$.

Consider the open set $U := \mathbb{C} \setminus \Sigma_\kappa$; it satisfies $\partial U = \Sigma_\kappa$. We then have the following result, which is well known to experts — see for example [15, p.979], remark (g).

Proposition 1.8.

(*i*) *The Lyapunov exponent* γ_κ *is the Green's function for the domain* U*.*
(*ii*) *The density of states is the equilibrium measure of* Σ_κ*.*
(*iii*) *The capacity* $\mathrm{Cap}(\Sigma_\kappa)$ *of the almost–sure spectrum is one.*

Proof. The Thouless formula shows that $\gamma_\kappa : U \to (0,\infty)$ satisfies condition $(G1)$; moreover, for $E \in \mathbb{C}$:

$$\begin{aligned}\gamma_\kappa(E) - \log|E| &= \int_{\Sigma_\kappa} \log|E - E'|dk_\kappa(E') - \log|E| \\ &= \int_{\Sigma_\kappa} \log\left|1 - \frac{E'}{E}\right| dk_\kappa(E') \\ &\xrightarrow[E\to\infty]{} 0,\end{aligned}$$

where the final line follows from the preceding because the function $\log|1 - E'/E|$ converges uniformly towards zero on the compact support Σ_κ of dk_κ. Therefore condition $(G2)$ holds. Finally, one checks $(G3)$ using Theorem 1.6. Thus (i) and (ii) hold, using 1.3 and since $\gamma_\kappa(E) - \log|E| \xrightarrow[E\to\infty]{} 0$, one immediately gets (iii).

□

1.2. **Holomorphic Dynamics.**

1.2.1. *Character variety of the free group on two generators.* Let us fix a generating set $\{a, b\}$ of the free group $\mathbf{F}_2$ and consider the algebraic quotient $\chi(\mathbf{F}_2)$ of:

$$\mathrm{Rep}(\mathbf{F}_2) := \mathrm{Hom}(\mathbf{F}_2, SL_2(\mathbb{C})) \cong SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$$

under $SL_2(\mathbb{C})$–conjugacy. The variety $\chi(\mathbf{F}_2)$ is isomorphic to $\mathbb{C}^3$ with the following projection map:

$$\begin{aligned}\chi : \mathrm{Rep}(\mathbf{F}_2) &\mapsto \mathbb{C}^3 \\ \rho &\mapsto (x, y, z) = (\mathrm{Tr}(\rho(a)), \mathrm{Tr}(\rho(b)), \mathrm{Tr}(\rho(ab))).\end{aligned}$$

Moreover, if one enforces the condition $\mathrm{Tr}([\rho(a), \rho(b)]) = D - 2 \in \mathbb{C}$ one obtains an affine cubic surface $\mathcal{S}_D$, the equation of which is (see [6, 7] for details):

$$x^2 + y^2 + z^2 = xyz + D.$$

Let φ be an element of $\mathrm{Aut}(\mathbf{F}_2)$; then the following defines an automorphism of the surface $\mathcal{S}_D$:

$$f : \chi(\rho) \mapsto \chi(\rho \circ \varphi^{-1}).$$

Since the group $\mathrm{Aut}(\mathbf{F}_2)$ acts on $\mathrm{Ab}(\mathbf{F}_2) = \mathbb{Z}^2$ one can set

$$M_f = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in GL_2(\mathbb{Z})$$

to be the matrix corresponding to φ^{-1} and if $A := \rho(a)$, $B := \rho(b)$ for some $\rho \in \mathrm{Rep}(\mathbb{T}_1^2)$ then:

$$f(\chi(\rho)) = ((\mathrm{Tr}(A^pB^q), \mathrm{Tr}(A^rB^s), \mathrm{Tr}(A^pB^qA^rB^s)). \tag{1.4}$$

This gives us an action of $GL_2(\mathbb{Z})$ on $\mathcal{S}_D$ whose kernel contains $\pm I_2$; therefore $PGL_2(\mathbb{Z})$ acts on the surface $\mathcal{S}_D$.

Using (1.4) and Fricke–Klein's formulas, one sees that f is a polynomial automorphism of $\mathcal{S}_D$; in the following, we will denote by $\mathcal{B}$ be the subgroup of $\mathrm{Aut}(\mathcal{S}_D)$ formed by such mappings f. We will say that an automorphism $f \in \mathcal{B}$ is hyperbolic if one of the next two conditions holds:

- either $\det(M_f) = 1$ and $\mathrm{Tr}(M_f) > 2$;
- or $\det(M_f) = -1$ and $\mathrm{Tr}(M_f) \neq 0$.

Example 1.9. For the Fibonacci substitution ζ, consider the automorphism f associated with ζ. Then:

$$M_f = M_{\zeta^{-1}} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

Since $\det(M_f) = -1$ and $\mathrm{Tr}(M_f) = -1 \neq 0$ the morphism f is in fact hyperbolic. It is given by:

$$f(x, y, z) = (y, xy - z, x).$$

Denote by $\overline{\mathcal{S}_D}$ the compactified surface:

$$w(x^2 + y^2 + z^2) = xyz + w^3 D,$$

where $[x : y : z : w]$ are homogeneous coordinates on the projective space $\mathbb{P}^3$. Its intersection with the plane at infinity $\{w = 0\}$ is equal to the "triangle at infinity" $\Delta = \{xyz = 0\}$. Thus $\mathrm{Aut}(\mathcal{S}_D)$ embeds into the birational transformations of $\overline{\mathcal{S}_D}$. The dynamics at infinity of the hyperbolic elements in $\mathcal{B}$ is quite rich, as we will see throughout this paper; first, we have the following result.

Proposition 1.10 (see [6, 7, 18])**.**
Let $f \in \mathcal{B}$ be a hyperbolic automorphism. Then f extends to a birational transformation of $\overline{\mathcal{S}_D}$ and:

(i) *f has a unique indeterminacy point v_- which is either* $[1 : 0 : 0 : 0]$, $[0 : 1 : 0 : 0]$ *or* $[0 : 0 : 1 : 0]$*;*
(ii) *the mapping f contracts $\Delta \setminus \{v_-\}$ onto the indeterminacy point v_+ of f^{-1};*
(iii) *up to conjugacy by an element of $\mathcal{B}$, one can assume v_+ to be distinct from v_-.*

Remark 1.11. Èl'Huti [18] gave a detailed description of the automorphism group $\mathrm{Aut}(\mathcal{S}_D)$; in particular, he proved that $\mathcal{B}$ has finite index in $\mathrm{Aut}(\mathcal{S}_D)$.

1.2.2. *Main theorem on dynamical Green functions.* Fix a hyperbolic automorphism $f \in \mathcal{B}$ for which $v_+ \neq v_-$ and denote by λ the spectral radius of M_f. We now try to understand the escape rate at infinity in the unbounded orbits under f. First, a theorem by Dloussky [17] combined with work by Cantat [6] — see also [19] — yields the following result, which will be essential to our study of the dynamics of f at infinity.

Proposition 1.12 ([6])**.**
There exists a matrix $N_f \in GL_2(\mathbb{Z})$ with non–negative entries which is conjugate to M_f in $PGL_2(\mathbb{Z})$, an open neighbourhood U of v_+ in $\overline{\mathcal{S}_D}$ and a biholomorphism $\psi_f^+ : \mathbb{D} \times \mathbb{D} \to U$ such that:

(i) $\psi_f^+(0, 0) = v_+$*;*
(ii) *for all $(u, v) \in \mathbb{D}^* \times \mathbb{D}^*$ one has:*

$$\psi_f^+((u, v)^{N_f}) = f(\psi_f^+(u, v)),$$

where $(u, v)^{N_f}$ denotes the monomial action of N_f on the pair (u, v), i.e *if*

$$N_f = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

then $(u, v)^{N_f} = (u^p v^q, u^r v^s)$.

As a consequence, if $m \in \mathcal{S}_D$ has unbounded forward orbit under f, then $f^n(m)$ goes to v_+ at infinity.

Before stating our main result regarding dynamical Green functions, let us set a few conventions:

- define the **filled Julia set** $K^+(f)$ as follows:
$$K^+(f) := \{m \in \mathcal{S}_D \mid \exists M > 0,\, \forall n \geq 0,\, \|f^n(m)\| \leq M\}\,,$$
where $\|.\|$ denotes the standard euclidean norm on $\mathbb{C}^3$;
- set $\alpha, \beta \in \mathbb{R}_+^*$ to be the coordinates of the projection of the vector $\begin{pmatrix}1\\1\end{pmatrix}$ on the eigenline for N_f associated with the maximal eigenvalue of M_f — and so of N_f —.

Theorem A (Dynamical Green function)**.**
Let $f \in \mathcal{B}$ be a hyperbolic element and let $m \in \mathcal{S}_D$. Then the following quantity is well defined:
$$G_f^+ : m \mapsto \lim_{n\to\infty} \frac{1}{\lambda^n} \log^+ \|f^n(m)\|\,,$$
and:

(i) *the function G_f^+ is pluriharmonic (resp. plurisubharmonic) on the complement of the filled Julia set $K^+(f)$ in $\mathcal{S}_D$ (resp. on $\mathcal{S}_D$) and takes non–negative values;*
(ii) *the zero set of G_f^+ is $K^+(f)$;*
(iii) *the following relation holds:*
$$G_f^+ \circ f = \lambda G_f^+; \tag{1.5}$$
(iv) *if $m = \psi_f^+(u,v) \in \psi_f^+(\mathbb{D}^* \times \mathbb{D}^*)$, then:*
$$G_f^+(m) = -\alpha \log|u| - \beta \log|v| \tag{1.6}$$
(v) *the function G_f^+ is locally Hölder–continuous.*

Example 1.13. In the Fibonacci case, M_f is conjugate to
$$N_f = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} \text{ in } PGL_2(\mathbb{Z}).$$
The eigenvalues of M_f — and so of N_f — are
$$\phi := \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \overline{\phi} := \frac{1-\sqrt{5}}{2}.$$
and the corresponding eigenlines for N_f are spanned by $\begin{pmatrix}\phi\\1\end{pmatrix}$ and $\begin{pmatrix}\overline{\phi}\\1\end{pmatrix}$. Thus, since:
$$\begin{pmatrix}1\\1\end{pmatrix} = \frac{1-\overline{\phi}}{\sqrt{5}}\begin{pmatrix}\phi\\1\end{pmatrix} + \frac{\phi-1}{\sqrt{5}}\begin{pmatrix}\overline{\phi}\\1\end{pmatrix},$$
one has $\alpha = \dfrac{1-\overline{\phi}}{\sqrt{5}}\phi = \dfrac{\phi-1}{\sqrt{5}}$ and $\beta = \dfrac{1-\overline{\phi}}{\sqrt{5}}$. Moreover, we have in this case $v_+ = [0:1:0:0]$.

1.3. **Applications to Discrete Schrödinger Operators.**

1.3.1. *Schrödinger curve.* Consider the following cubic surface in $\mathbb{C}^3$, for some fixed $\kappa \in \mathbb{R}$:

$$(\mathcal{S}_{4+\kappa^2}) \qquad x^2 + y^2 + z^2 = xyz + 4 + \kappa^2 \quad ;$$

this is a connected smooth — $if \kappa \neq 0$ — affine surface, containing what we call its Schrödinger curve:

$$\begin{aligned} s : \mathbb{C} &\to \mathcal{S}_{4+\kappa^2} \\ E &\mapsto (E - \kappa, E, E(E - \kappa) - 2). \end{aligned}$$

Remark 1.14. The function s is in fact the trace map associated with the matrices $M^E_{n,\kappa,w}$. Namely, one has $s(E) = (\mathrm{Tr}(M^E_\kappa(a)), \mathrm{Tr}(M^E_\kappa(b)), tr(M^E_{\kappa,}(b)M^E_\kappa(a)))$.

Starting from our automorphism $\varphi \in \mathrm{Aut}(\mathbf{F}_2)$ (cf. 1.1.1) with associated substitution ι, we obtain a polynomial automorphism f of $\mathcal{S}_{4+\kappa^2}$ associated with φ^{-1} (cf. 1.2.1); one can then explicitly compute it using the formula $f(\chi(\rho)) = \chi(\rho \circ \varphi)$ and so its restriction to the Schrödinger curve is:

$$\forall E \in \mathbb{C}, \quad f(s(E)) = (\mathrm{Tr}(M^E_\kappa(\iota(a))), \mathrm{Tr}(M^E_\kappa(\iota(b))), \mathrm{Tr}(M^E_\kappa(\iota(ab)))),$$

where, if $u = (u_1, \dots, u_n) \in \{a, b\}^n$, then:

$$M^E_\kappa(u) := \prod_{i=0}^{n-1} M^E_\kappa(u_{n-i}).$$

Since φ is hyperbolic, f is a hyperbolic automorphism of $\mathcal{S}_{4+\kappa^2}$. We then have the following result [11] — see also some earlier work by Sütő [28, 29].

Proposition 1.15 (Damanik [11])**.**
If f is the polynomial automorphism of $\mathcal{S}_{4+\kappa^2}$ associated with a positive hyperbolic substitution ι on two letters, then the almost–sure spectrum Σ_κ satisfies:

$$\Sigma_\kappa = s^{-1}(K^+(f)).$$

1.3.2. *"Dictionary" Between Holomorphic Dynamics and Schrödinger Operators.* We now move on to our second result. Since the subgroup $\mathcal{B}$ has finite index in $\mathrm{Aut}(\mathcal{S}_{4+\kappa^2})$ (cf. remark 1.11) and f has infinite order we can suppose, up to replacing it with some iterate f^{n_0} that $f \in \mathcal{B}$; thus, we will be able to exploit theorem A to obtain the following result.

Theorem B.
Let ι be a positive hyperbolic substitution over the letters a and b and let $f \in \mathcal{B}$ be the associated automorphism of $\mathcal{S}_{4+\kappa^2}$. Then for $E \in \mathbb{C}$:

$$\gamma_\kappa(E) = \frac{1}{\alpha + \beta} G^+_f(s(E)),$$

where $\alpha, \beta \in \mathbb{R}^*_+$ are the same as in Theorem A.

Remark 1.16. Proposition 1.15 was mostly a qualitative one, concerning the boundedness of the orbit alone. Here, using our Theorem A, we get tools to estimate the escape rate at infinity thus obtaining a more quantitative result.

This, combined with previous work by Cantat, Damanik and Gorodetski, allows us to work out the following "dictionary".

Discrete Schrödinger operators	**Holomorphic dynamics on $\mathcal{S}_{4+\kappa^2}$**
Almost–sure spectrum Σ_κ	Julia set $K^+(f)$
Lyapunov exponent γ_κ	Dynamical Green's function G_f^+
Density of states dk_κ	Green's current T_f^+
Thouless formula	$T_f^+ = dd^c G_f^+$
γ_κ and k_κ Hölder–continuous near Σ_κ	G_f^+ locally Hölder–continuous
Avron & Simon Theorem 1.4	Convergence to T_f^+

More precisely, one goes from the right-hand side of this table to the left by taking pull-backs with the Schrödinger curve $s\colon \mathbb{C} \to S_{4+\kappa^2}$; for instance, the first line is Damanik's Proposition 1.15, and the second is our Theorem B. Similarly, the Hölder continuity of γ_κ corresponds to the Hölder continuity of G_f^+ —obtained in Theorem A; we shall see in Section 3.2.1 that it implies directly Hölder continuity of the integrated density of states. The last line of this table is less precise: this is explained in paragraph 3.2.3.

2. Dynamical Green Functions

2.1. **Preliminary Computations.**

2.1.1. *Geometry of $\mathcal{S}_D$ at infinity.* In order to measure the escape rate at infinity of a point with unbounded orbit, we will now study the behaviour of $\log \|m\|$ when $m = (x, y, z) \in \mathcal{S}_D$ goes to v_+, where $\|\cdot\|$ denotes the euclidean norm on $\mathbb{C}^3$. For the sake of clarity, suppose — our problem being symmetric with respect to x, y and z — that v_+ is the point $[0 : 0 : 1 : 0]$; in a neighbourhood of v_+, $\mathcal{S}_D$ can be seen, using the chart $\{z \neq 0\}$, as the surface:

$$(X^2 + Y^2 + 1)W = XY + DW^3, \tag{2.1}$$

where $X := x/z$, $Y := y/z$ and $W := w/z$. Equivalently, this can be written as follows:

$$W = XY + DW^3 + W^2(AX + BY + C) + W(X^2 + Y^2). \tag{2.2}$$

Using these new coordinates (X, Y, W), v_+ corresponds to the point at origin $(0, 0, 0)$ and one has:

$$\begin{aligned}
\log \|m\| =& \frac{1}{2} \log \left(\left| \frac{X}{W} \right|^2 + \left| \frac{Y}{W} \right|^2 + \frac{1}{|W|^2} \right) \\
=& -\frac{1}{2} \log(|W|^2) + \frac{1}{2} \log(|X|^2 + |Y|^2 + 1) \\
=& -\frac{1}{2} \log(|XY + DW^3 + W^2(AX + BY + C) + W(X^2 + Y^2)|^2) \\
&+ \frac{1}{2} \log(|X|^2 + |Y|^2 + 1).
\end{aligned}$$

Using Taylor's approximation one gets:

$$\log \|m\| = -\log(|XY|) + g(X, Y, W),$$

where g is bounded in a neighbourhood of $(0, 0, 0)$. Now, for m close enough to v_+ one can apply the biholomorphism ψ_f^+ to get $(u, v) := {\psi_f^+}^{-1}(m)$ and use the following lemma.

Lemma 2.1.
There exists a germ of bounded function h such that for all $(u,v) \in \mathbb{D}^* \times \mathbb{D}^*$:

$$\log \|\psi_f^+(u,v)\| = -\log|uv| + h(u,v).$$

Proof. Using Taylor's theorem at the origin one gets:

$$\psi_f^+(u,v) = v_+ + L(u,v) + R(u,v),$$

where L is the linear part of ψ_f^+ at the origin and R is a smooth bounded function on $\mathbb{D} \times \mathbb{D}$ such that $R(u,v) = O(\|(u,v)\|^2)$. Since ψ_f^+ is a conjugacy between the dynamics of f and N_f and since f (resp. N_f) only contracts the axes $\{X=0\}$ and $\{Y=0\}$ (resp. $\{u=0\}$ and $\{v=0\}$) on the origin then $L = d\psi_f^+(0,0)$ must be of the form

$$\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & r_1 \\ r_2 & 0 \end{pmatrix}.$$

Therefore, there exists a bounded function h on $\mathbb{D} \times \mathbb{D}$ such that:

$$\log \|\psi_f^+(u,v)\| = -\log(|uv|) + h(u,v), \tag{2.3}$$

□

2.1.2. *Estimate at infinity.* Since M_f is hyperbolic, one can assume — replacing M_f with $M_{f^2} = M_f^2$ — that it has eigenvalues λ and λ^{-1}, with λ a real number greater than one. Now consider the following quantity, for $n \geq 0$ and m with unbounded forward orbit, chosen sufficiently close to v_+ — *i.e* in ${\psi_f^+}^{-1}(\mathbb{D} \times \mathbb{D})$:

$$\frac{1}{\lambda^n} \log \|f^n(m)\|.$$

Let $(u_n, v_n) := (u,v)^{N_f^n}$; using the previous lemma one gets:

$$\frac{1}{\lambda^n} \log \|f^n(m)\| = -\frac{1}{\lambda^n} \log(|u_n v_n|) + \frac{1}{\lambda^n} h(u_n, v_n).$$

Since $\frac{1}{\lambda^n} h(u_n,v_n) \xrightarrow[n\to\infty]{} 0$, we want to understand the behaviour at infinity of the following quantity:

$$-\frac{1}{\lambda^n} \log(|u_n v_n|).$$

Lemma 2.2.
The following estimate holds, as n goes to infinity:

$$\frac{1}{\lambda^n} \log |u_n v_n| \xrightarrow[n\to\infty]{} (\alpha \log|u| + \beta \log|v|), \tag{2.4}$$

where $\alpha, \beta \in \mathbb{R}_+^*$ are the coordinates of the projection of the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ on the eigenline for N_f associated with λ.

Proof. Since $(u,v) \in \mathbb{D}^* \times \mathbb{D}^*$ one can set $(e^s, e^t) := (u,v)$ with:

$$s, t \in \{z \in \mathbb{C} \,|\, \Re(z) < 0, \Im(z) \in (-\pi, \pi]\}.$$

Then it is just a matter of describing the behaviour of $|uv| = |e^{s+t}| = e^{\Re(s+t)}$ under N_f, which acts linearly on the coordinates (s,t). A computation thus yields:

$$\frac{1}{\lambda^n} \log |u_n v_n| \xrightarrow[n\to\infty]{} (\alpha \Re(s) + \beta \Re(t)) = (\alpha \log|u| + \beta \log|v|).$$

$\square$

Using lemmas 2.1 and 2.2, one gets the following estimate:

$$\frac{1}{\lambda^n}\log\|f^n(m)\| \xrightarrow[n\to\infty]{} -(\alpha\log|u|+\beta\log|v|). \tag{2.5}$$

Note that this only holds for m sufficiently near v_+, *i.e* for (u,v) in $\mathbb{D}^*\times\mathbb{D}^*$.

2.2. **Proof of Theorem A.** First remark that if $m\in K^+(f)$ then it is clear that $G_f^+(m)$ is well defined and equal to 0.

Now consider $m\notin K^+(f)$; up to replacing m with some $f^{n_0}(m)$, one can assume that m is sufficiently near v_+ so that one can set $(u,v) := {\psi_f^+}^{-1}(m)$ and $(u_n,v_n) := (u,v)^{N_f^n}$. Since $f^n(m)\xrightarrow[n\to\infty]{} v_+$, for n large enough $\log^+\|f^n(m)\| = \log\|f^n(m)\|$. Applying the estimate (2.5) then yields, using the same notations as before:

$$\frac{1}{\lambda^n}\log\|f^n(m)\| \xrightarrow[n\to\infty]{} -(\alpha\log|u|+\beta\log|v|)$$

We have thus proved that G_f^+ is well defined and that (ii) and (iv) hold. Moreover, the estimate (2.5) implies that:

$$\forall(u,v)\in\mathbb{D}^*\times\mathbb{D}^*,\qquad G_f^+\circ\psi_f^+(u,v) = -\alpha\log|u|-\beta\log|v|. \tag{2.6}$$

(i) Let H be a compact set in $\mathcal{S}_D$, $m\in H$ and $n,p\geq 0$. If $m\in K^+(f)$ then we clearly have uniform boundedness. Else, $f^n(m)\xrightarrow[n\to\infty]{} v_+$ and so for n large enough $m=\psi_f^+(u,v)$ with $(u,v)\in\mathbb{D}^*\times\mathbb{D}^*$ and $\|f^n(m)\|>1$, *ergo* $\log^+\|f^n(m)\| = \log\|f^n(m)\|$ and:

$$\left|\frac{1}{\lambda^{n+p}}\log^+\|f^{n+p}(m)\| - \frac{1}{\lambda^n}\log^+\|f^n(m)\|\right| = \frac{1}{\lambda^{n+p}}\left|\log\|f^{n+p}(m)\| - \lambda^p\log\|f^n(m)\|\right|$$

Since we just proved that there exists a constant $C_m = (\alpha\log|u|+\beta\log|v|)$ depending only on the orbit of m such that $\log\|f^n(m)\| = C_m\lambda^n+\lambda^n\varepsilon_m(n)$, with $\varepsilon_m(n)\xrightarrow[n\to\infty]{} 0$ so:

$$\left|\frac{1}{\lambda^{n+p}}\log^+\|f^{n+p}(m)\| - \frac{1}{\lambda^n}\log^+\|f^n(m)\|\right| = \frac{1}{\lambda^{n+p}}\left|\varepsilon_m(n+p)-\varepsilon_m(n)\right|.$$

As $|\varepsilon_m(n)|\xrightarrow[n\to\infty]{} 0$, then for all positive η and $m\in H$, there exists $N_m\in\mathbb{N}$ such that:

$$\forall n\geq N_m,\quad |\varepsilon_m(n)|\leq|\varepsilon_m(N_m)|<\eta$$

hence:

$$\left|\frac{1}{\lambda^{n+p}}\log^+\|f^{n+p}(m)\| - \frac{1}{\lambda^n}\log^+\|f^n(m)\|\right| \leq 2|\varepsilon_m(N_m)|.$$

Since C_m and $\log\|f^n(m)\|$ are continuous with respect to m (cf. (iv)),

$$m\mapsto\varepsilon_m(N_m) = \lambda^{-N_m}(\log\|f^{N_m}(m)\|-C_m)$$

is continuous. Using the compactness of H, there exists $m_0\in H$ such that:

$$\sup_{m\in H}\varepsilon_m(N_m) = \varepsilon_{m_0}(N_{m_0})$$

where:

$$0\leq 2|\varepsilon_m(N_m)|\leq 2|\varepsilon_{m_0}(N_{m_0})|<2\eta.$$

The sequence defining G_f^+ thus converges uniformly on all compact subsets in $\mathcal{S}_D$ and so the limit function inherits the pluri(sub)harmonic properties of its terms.
(iii) This stems from the fact that if $m \in \mathcal{S}$ then:

$$\frac{1}{\lambda^n} \log^+ \|f^n(f(m))\| = \frac{1}{\lambda^n} \log^+ \|f^{n+1}(m)\| = \lambda \left(\frac{1}{\lambda^{n+1}} \log^+ \|f^{n+1}(m)\| \right).$$

(v) Here we adapt work by Fornaess and Sibony [20]. Since G_f^+ is $\mathcal{C}^1$ outside any neighbourhood of $K^+(f)$ it is Hölder–continuous there. Now let $z_1 \in \mathcal{S}_D$ and $z_0 \in K^+(f)$ be such that:

$$d(z_1, K^+(f)) = \|z_1 - z_0\|.$$

If $z_1 \in K^+(f)$, there is nothing to show. Else, note that by definition of the filled Julia set there exists $R_0 > 0$ such that:

$$\forall n \in \mathbb{N}, \quad \|f^n(z_0)\| \leq R_0.$$

Let us consider a positive real number $R \geq R_0 + 1$ and set:

$$N := \min\{n \geq 0 \,|\, \|f^n(z_1)\| > R\} < \infty;$$

thus:

$$\begin{aligned}
|\, \|f^N(z_1)\| - \|f^N(z_0)\| \,| &\leq \|f^N(z_1) - f^N(z_0)\| \\
&\leq \sup_{\|z\| \leq R} \|df(z)\| \|f^{N-1}(z_1) - f^{N-1}(z_0)\| \text{ car } \|f^{N-1}(z_1)\| \leq R \\
&\;\;\vdots \\
&\leq (\sup_{\|z\| \leq R} \|df(z)\|)^N \|z_1 - z_0\| \\
&\leq (\sup_{\|z\| \leq R} \|df(z)\|)^N d(z_1, K^+(f)).
\end{aligned}$$

Hence, if one sets:

$$H(R) := \sup_{\|z\| \leq R} \|df(z)\|$$

one gets:

$$\begin{aligned}
1 \leq R - R_0 &\leq |\, \|f^N(z_1)\| - \|f^N(z_0)\| \,| \\
&\leq H(R)^N d(z_1, K^+(f))
\end{aligned}$$

thus $H(R)^N d(z_1, K^+(f)) \geq 1$. Setting $\gamma := \dfrac{\log(\lambda)}{\log(H(R))}$ one has:

$$\frac{1}{\lambda^N} \leq d(z_1, K)^\gamma. \tag{2.7}$$

Using (iii) one gets:

$$\begin{aligned}
G_f^+(z_1) &= \frac{1}{\lambda^N} G_f^+ \circ f^N(z_1) \\
&\leq \frac{1}{\lambda^N} \sup_{\|z\| \leq R} G_f^+ \circ f(z) \text{ car } \|f^{N-1}(z_1)\| \leq R \\
&\leq d(z_1, K^+(f))^\gamma \sup_{\|z\| \leq R} G_f^+ \circ f(z) \text{ par (2.7).}
\end{aligned}$$

Let:

$$C := \sup_{\|z\|\leq R} G_f^+ \circ f(z)$$

one then has, *in fine*:

$$G_f^+(z_1) \leq C d(z_1, K^+(f))^\gamma \tag{2.8}$$

for any point $z_1 \in \mathcal{S}_D$.

□

Remark 2.3. Using the notations of paragraph 2.1.2, we can estimate the local coordinates (X, Y) around v_+ as follows (up to a permutation of u and v in the linear part):

$$(X, Y) = (r_1 u, r_2 v) + R(u, v).$$

Therefore, we have, as m goes to v_+:

$$G_f^+(m) = -\alpha \log |X| - \beta \log |Y| - \log |r_1^\alpha r_2^\beta| + o(1). \tag{2.9}$$

Remark 2.4. Replacing f with its inverse f^{-1}, one can define the negative dynamical Green function:

$$G_f^- := \lim_{n\to\infty} \frac{1}{\lambda^n} \log^+ \|f^{-n}(m)\|.$$

Our main result extends to this function.

2.3. **Corollaries.** We can now consider the closed positive current [6] associated with G_f^+, namely:

$$T_f^+ := dd^c G_f^+ = 2i\partial\bar{\partial} G_f^+$$

which satisfies the following:

$$f^* T_f^+ = \lambda T_f^+$$

and has support in the Julia set $J^+(f) := \partial K^+(f)$.

Corollary A.1.
Let $f \in \mathcal{B}$ be a hyperbolic element and $m \in \mathcal{S}_D$. Then there exists a neighbourhood U of v_+ in $\overline{\mathcal{S}_D}$ such that:

$$dd^c G^+_{f\,|U} = -2\pi \left(\alpha \int_{X=0} + \beta \int_{Y=0} \right),$$

where $\alpha, \beta \in \mathbb{R}_+^*$ and (v_+, X, Y) are the same as in Theorem A.

Proof. Let $U := {\psi_f^+}^{-1}(\mathbb{D}^* \times \mathbb{D}^*)$; then using (2.6) and (2.9) one gets:

$$dd^c G^+_{f\,|U} = dd^c(-\alpha \log |u| - \beta \log |v|) = dd^c(-\alpha \log |X| - \beta \log |Y|).$$

The result then follows from the Lelong–Poincaré lemma.

□

3. From Holomorphic Dynamics to Schrödinger Operators

3.1. **Proof of Theorem B.** Consider the function:

$$\begin{aligned} g : \mathbb{C} \setminus \Sigma_\kappa &\to (0, \infty) \\ E &\mapsto G_f^+\left(s\left(E\right)\right) \quad ; \end{aligned}$$

our aim is to show that it is — up to a multiplicative constant — the Green's function of the domain $U := \mathbb{C} \setminus \Sigma_\kappa$, thus proving the theorem. Since G_f^+ is psh, condition $(G1)$ holds and $(G3)$ is a direct consequence of Damanik's result (Proposition 1.15).

Using Fricke–Klein's formulas and relation (1.4), one shows using induction that f contracts the triangle at infinity Δ on the point $v_+ = [0:0:1:0]$. Using (2.9), one then gets

$$g(E) = \alpha \log|x| - \beta \log|y| - \log|C| + o(1) \text{ as } E \text{ goes to infinity,}$$

where $C \in \mathbb{C}$ and $s\left(E\right) = [x : y : 1 : 1]$. One also has:

$$s([E:t]) = [Et - t^2\kappa : Et : E^2 - Et\kappa - 2t^2 : t^2],$$

hence, using the chart $\{z \neq 0\}$:

$$\begin{aligned} s(E) &= s\left([E:1]\right) \\ &= \left(\frac{E-\kappa}{E^2 - E\kappa - 2}, \frac{E}{E^2 - E\kappa - 2}, \frac{1}{E^2 - E\kappa - 2}\right) \\ &= \left(\frac{1}{E}\left(\frac{1-\kappa/E}{1-\kappa/E-2/E^2}\right), \frac{1}{E}\left(\frac{1}{1-\kappa/E-2/E^2}\right), \frac{1}{E^2 - E\kappa - 2}\right) \end{aligned}$$

Thus the following limit exists:

$$\lim_{E\to\infty} g(E) - (\alpha+\beta)\log|E|.$$

□

Remark 3.1. Using Proposition 1.8, one has:

$$\lim_{E\to\infty} g(E) - (\alpha+\beta)\log|E| = -\log \operatorname{Cap}(\Sigma_\kappa) = 0.$$

3.2. **Consequences.** Theorem B yields a few interesting corollaries, further detailing the entwining between certain dynamical invariants and discrete Schrödinger operators.

3.2.1. *Hölder continuity — see also* [10, 16].

Corollary B.2.
One has the following results:

(i) $s^*(dd^c G_f^+) = 2\pi(\alpha+\beta)dk_\kappa$;
(ii) *the functions* γ_κ *and* k_κ *are Hölder–continuous near* Σ_κ*, with the same Hölder exponent* τ*;*
(iii) *the density of states does not charge sets with Hausdorff dimension less than* τ*. In particular, the Hausdorff dimension of the almost–sure spectrum is strictly positive.*

Proof. The first assertion follows from (1.3). To prove property (ii), we reproduce an argument from [27]. Using Theorem A, G_f^+ is locally Hölder–continuous near $K^+(f)$; since $s(\mathbb{C}) \cap K^+(f) = \Sigma_\kappa$ is a compact set, that property is global near the almost–sure spectrum and so γ_κ is Hölder–continuous near Σ_κ. Denote by τ the exponent of Hölder continuity.

To show that k_κ is Hölder continuous, consider two real numbers $E_2 > E_1$. Let M be the middle point of the segment $[E_1, E_2]$ and $R = |E_2 - E_1|/2$ be the distance from M to E_1. Denote by $D(r) \subset \mathbb{C}$ the disk of radius r centred at M. Let $\psi\colon \mathbb{C} \to \mathbb{R}_+$ be a smooth function which is equal to 1 on $D(R)$ and equal to 0 on $\mathbb{C} \setminus D(2R)$, and whose partial derivatives of order 1 and 2 are bounded from above by $100R^{-2}$ (such a function exists, see [21]). Then,

$$\begin{aligned}
|k_\kappa(E_2) - k_\kappa(E_1)| &= \int_{[E_2,E_1]} dk_\kappa(E) \\
&\leq \int_{D(R)} dd^c(\gamma_\kappa - \gamma_\kappa(M)) \\
&\leq \int_{D(3R)} \psi \cdot dd^c(\gamma_\kappa - \gamma_\kappa(M)) \\
&\leq \int_{D(3R)} dd^c\psi \cdot (\gamma_\kappa - \gamma_\kappa(M)) \\
&\leq C^{st} R^\tau \operatorname{Area}(D(3R)) R^{-2} \\
&\leq 9\pi C^{st} |E_2 - E_1|^\tau
\end{aligned}$$

for some uniform constant C^{st} because γ_κ is Hölder continuous (with exponent τ) on a neighbourhood of Σ_κ.

The same proof shows that dk_κ does not charge any closed subset of $\mathbb{C}$ whose Hausdorff dimension is less than τ (see [27]).

□

3.2.2. *Hausdorff dimension of the density of states.* Once we know that γ_κ is equal to $(\alpha + \beta)^{-1} G_f^+ \circ s$, we can generalize the first results of Damanik and Gorodetski concerning the Hausdorff dimension of the density of states — proved in [15] for the Fibonacci substitution. Doing this, we obtain an alternative (but almost equal) proof of some of the results of May Mei (see [24]).

Theorem 3.2 (Damanik, Gorodetski, Mei)**.** *Let φ be a positive and hyperbolic automorphism of the free group $\mathbf{F}_2$. Let $H_{\kappa,w}$ be the corresponding family of discrete Schrödinger operators. For small coupling factors $0 < \kappa < \kappa_0$, the density of states dk_κ is of exact dimension* $\dim(\kappa)$*, i.e. for dk_κ-almost every real number E,*

$$\lim_{\epsilon \to 0} \frac{\log dk_\kappa[E - \epsilon, E + \epsilon]}{|\log(\epsilon)|} = \dim(\kappa).$$

Moreover,

1. $\dim(\kappa)$ *is a $\mathcal{C}^\infty$-smooth function of $\kappa \in (0, \kappa_0)$;*
2. $\lim_{\kappa \to 0} \dim(\kappa) = 1$*;*
3. $\dim(\kappa) < \mathrm{Haus} - \mathrm{Dim}(\Sigma_\kappa) < 1$ *for $\kappa \in (0, \kappa_0)$;*
4. $\dim(\kappa)$ *coincides with the infimum of the Hausdorff dimension* $\mathrm{Haus} - \mathrm{Dim}(S)$ *of all measurable sets S such that $dk_\kappa(S) = 1$.*

The proof is due to Damanik, Gorodetski, and Mei. Let us explain how one can relate its proof to Theorem A and Theorem B:

a.– The dynamics of f on the intersection of its filled Julia sets $K^+(f) \cap K^+(f^{-1})$ is *uniformly hyperbolic*, the filled Julia set $K^+(f)$ is the support of a lamination by holomorphic curves, and the current T_f^+ is a current of integration on this lamination with respect to a transverse measure μ_f^+ — see [6]).

b.– The Schrödinger curve s is *transverse to the lamination* of $K^+(f)$ if the coupling factor is sufficiently small. This is proved in [14]; it follows from the transversality for $\kappa = 0$ and a study of the bifurcation from $\kappa = 0$ to $\kappa > 0$.

c.– There exists κ_0' such that, for $0 < \kappa < \kappa_0'$, *there are two saddle periodic points $p(\kappa)$ and $q(\kappa)$ of f on $\mathcal{S}_{4+\kappa^2}$ with distinct multipliers.*

To prove this, take a periodic point p on $\mathcal{S}_4$ which is not a singular point of $\mathcal{S}_4$. Deform it into a family of periodic points $p(\kappa)$ for $-\kappa_1(p) \leq \kappa \leq \kappa_1(p)$. Do the same for a second periodic point q: it can be deformed into $q(\kappa)$ for $\kappa_1(q) < q < \kappa_1(q)$. If the multipliers of $p(\kappa)$ and $q(\kappa)$ are equal for a sequence of parameters $\kappa_n > 0$ converging to 0, they are equal for all κ because they are analytic functions of κ. In particular, q can be analytically deformed along the interval $[-\kappa_1(p), 0]$. Thus, if the assertion was not satisfied, there would exist $\kappa_1 > 0$ such that all periodic points of f on $\mathcal{S}_4$ (distinct from the singularities) could be analytically deformed to saddle periodic points of the same period for $\kappa_1(p) < \kappa < 0$. This would contradict the fact that the topological entropy of f on $\mathcal{S}_{4-\epsilon}(\mathbb{R})$ is strictly less than $\log(\lambda)$ for $\epsilon > 0$, a property that implies that most periodic points of f on $\mathcal{S}_{4-\epsilon}(\mathbb{C})$ are not real — see [6].

With these three remarks in hand, one can then copy the proof given by Damanik and Gorodetski in [15].

3.2.3. *Convergence theorems.* From [6] and [4] (see also [27], [3]) one gets the following convergence theorem. *Let f be a hyperbolic automorphism of the surface $\mathcal{S}_D$. Let T be a positive current and ψ a smooth non-negative function with compact support which vanishes in a neighbourhood of the support of ∂T. Then, the sequence of currents*

$$\frac{1}{\lambda^n}(f^n)^*(\psi T)$$

converges towards a multiple cT_f^+, with $c = \langle T_f^- | \psi T \rangle$. For instance, T can be the current of integration on an algebraic curve $C \subset \mathcal{S}_D$.

Our goal is to explain, heuristically, why this result is similar to Avron-Simon convergence theorem for the density of states (see Theorem 1.4).

Consider the restriction $H_{\kappa,w}^N$ of the Schrödinger operator to some interval $[0, N] \subset \mathbb{Z}$. If $(u(0), \ldots, u(N))$ is an eigenfunction of $H_{\kappa,w}^N$ with eigenvalue E, then $(u(2), u(1))$ is obtained from $(u(1), u(0))$ by the linear action of the matrix $M_\kappa^E(w(0))$, $\ldots$, and $(u(N), u(N-1))$ is obtained from $(u(1), u(0))$ by the action of the product $M_\kappa^E(w(N-2)) \ldots M_\kappa^E(w(0))$.

Now, restrict the study to $w = u_+$, the infinite ι-invariant word, and to intervals $[0, \ell(n)]$, where $\ell(n)$ is the length of the word $\iota^n(a)$. When n goes to infinity, $\ell(n)$ behaves approximately like λ^n. With such a choice, the trace of the product

$M^E_\kappa(w(\ell(n)-2))\dots M^E_\kappa(w(0))$ is equal to the first coordinate of $f^n(s(E))$. Thus, if

$$\Lambda := \{E \mid \mathrm{Tr}(M^E_\kappa(\iota^n(a))) = 2\},$$

then

$$\Lambda = \{E \mid s(E) \in (f^n)^{-1}(C_2)\}$$

where C_2 is the algebraic curve $C_2 = \{(x,y,z) \in \mathcal{S}_D | x = 2\}$. In other words, Λ corresponds to the intersection of the algebraic curve $s(\mathbb{C})$ with the algebraic curve $f^{-n}(C_2)$; it contains approximately λ^n points, and the convergence theorem for currents tells us, roughly, that the average measure on these λ^n points converges towards $s^*(T^+_f)$, up to some multiplicative factor.

On the other hand, the trace of a matrix $M \in SL(2,\mathbb{R})$ is 2 if and only if 1 is an eigenvalue of M. Thus, a complex number E is in Λ if and only if there is an eigenvector $(u(0),\dots,u(\ell(n)))$ of $H^{\ell(n)}_{\kappa,w}$ with eigenvalue E such that

$$(\star) \quad \begin{pmatrix} u(\ell(n)) \\ u(\ell(n)-1) \end{pmatrix} = \begin{pmatrix} u(\ell(1)) \\ u(0) \end{pmatrix};$$

these are mixed boundary conditions — not the usual Dirichlet conditions, as in [1]. Thus, the convergence theorem for currents implies a convergence theorem for the density of states of $H^N_{\kappa,w}$ with the boundary conditions $(\star)$. Changing the curve C_2 into another algebraic curve — for instance $x = 3$ —, one gets different boundary conditions.

To sum up, Avron-Simon convergence theorem corresponds to the convergence theorem towards T^+_f, with the following differences: One only gets convergence along subsequences (one has to take $N = \ell(n)$), the boundary conditions are not the classical ones, but one gets convergence theorems which are valid in $\mathcal{S}_D$ (not only along the Schrödinger curve) and work for all positive currents.

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