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Par
Marc ABBOUD

Sur la dynamique des endomorphismes des surfaces affines

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Rapporteurs avant soutenance :

Thomas GAUTHIER	Professeur des universités à l'université Paris-Saclay
Mattias JONSSON	Professeur des universités à University of Michigan

Composition du Jury :

Examinateurs :	Jérémy BLANC	Professeur des universités à l'Université de Bâle
	Serge CANTAT	Directeur de recherche à l'Université de Rennes
	Charles FAVRE	Directeur de recherche au Centre de mathématiques Laurent Schwartz
	Thomas GAUTHIER	Professeur des universités à l'Université Paris-Saclay
	Mattias JONSSON	Professeur des universités à University of Michigan
	Juan SOUTO	Directeur de recherche à l'Université de Rennes
	Junyi XIE	Professeur au Beijing International Center for Mathematical Research
	Susanna ZIMMERMAN	Professeure des universités à l'université Paris-Saclay
Dir. de thèse :	Serge CANTAT	Directeur de recherche à l'université de Rennes
Co-dir. de thèse :	Junyi XIE	Professeur au Beijing International Center for Mathematical Research

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CHAPITRE 1

INTRODUCTION

Une variété affine X_0 sur un corps algébriquement clos $\mathbf{k}$ est un sous espace de $\mathbf{k}^N$ défini par des équations polynomiales. Un endomorphisme polynomial de X_0 est alors une transformation polynomiale de $\mathbf{k}^N$ qui préserve X_0 au sens où $f(X_0) \subset X_0$. Lorsque la dimension de X_0 vaut 2, on dira que X_0 est une surface affine. Le but de ma thèse est d'étudier le système dynamique donné par X_0 une surface affine et $f : X_0 \to X_0$ un endomorphisme polynomial de X_0. Les différentes questions que j'aborderai sont les suivantes : y a-t-il des orbites denses ou Zariski-denses ? Si l'orbite d'un point part à l'infini, peut on contrôler sa vitesse de fuite ? Y a-t-il beaucoup d'orbites périodiques ? Comment construire des mesures invariantes qui sont dynamiquement intéressantes ? Pour répondre à ces questions, j'utilise des techniques valuatives. Le système dynamique (X_0, f) induit un système dynamique $(\mathcal{V}_\infty, f_*)$ où $\mathcal{V}_\infty$ est l'espace des valuations centrées à l'infini de X_0. C'est l'étude de cette action qui sera au cœur de ce mémoire et permettra d'aborder ensuite les questions évoquées ci-dessus.

1.1 Endomorphismes

1.1.1 Degrés dynamiques

1.1.1.1 Transformations polynomiales de l'espace affine complexe

Une transformation polynomiale f de $\mathbf{C}^N$ est la donnée de N polynômes $f_i \in \mathbf{C}[x_1, \cdots, x_N]$ tels que $f = (f_1, \cdots, f_N)$. On définit le *degré* de f comme le maximum des degrés des f_i ; on le note $\deg f$. On note f^k pour le k-ième itéré de f. Lorsqu'on itère f le degré des formules de f^k croît typiquement de façon exponentielle. Il est donc naturel de considérer la quantité suivante :

$$\lambda_1(f) := \lim_k \left(\deg f^k\right)^{1/k}, \tag{1.1}$$

introduite dans [RS97], que l'on appellera le *premier degré dynamique* de f dans la suite. Les auteurs montrent que cette quantité est bien définie. On peut définir le premier degré dynamique de n'importe quelle transformation rationnelle de l'espace projectif $\mathbf{P}^N$ avec un procédé similaire.

1.1.1.2 Définitions générales

Soit X une variété projective lisse sur un corps algébriquement clos et soit d sa dimension. Pour d diviseurs de Cartier $D_1, \cdots, D_d$ de X on peut définir le produit d'intersection $D_1 \cdots D_d \in \mathbf{Z}$ (voir [Laz04]). Si $f : X \dashrightarrow X$ est une transformation rationnelle dominante, on définit pour $0 \leqslant l \leqslant d$ le l-ième degré dynamique de f par

$$\lambda_k(f) := \lim_{n\to\infty} \left((f^n)^* H^k \cdot H^{d-k}\right)^{1/n}, \tag{1.2}$$

où H est un diviseur ample de X. On peut montrer que ces quantités sont bien définies, indépendantes du choix de H. En particulier, $\lambda_0(f) = 1$. De plus, les degrés dynamiques sont des invariants birationnels : si $\varphi : X \dashrightarrow Y$ est une application birationnelle, alors

$$\lambda_l(f) = \lambda_l(\varphi \circ f \circ \varphi^{-1}), \quad \forall 0 \leqslant l \leqslant d. \tag{1.3}$$

On a que $\lambda_d(f)$ est le degré topologique de f où le degré topologique est définie comme le degré de l'extension induit par f^* sur le corps des fonctions rationnelles de X. Les inégalités de Khovanskii-Teissier (voir [Gro90], [DN05]) impliquent que la suite $(\lambda_l)_{0\leqslant l\leqslant d}$ est log-concave ; c'est à dire

$$\frac{\log \lambda_{l-1} + \log \lambda_{l+1}}{2} \leqslant \log \lambda_l, \quad \forall 1 \leqslant l \leqslant d-1. \tag{1.4}$$

En particulier, on a $\forall 1 \leqslant l \leqslant d, \lambda_1(f)^l \geqslant \lambda_k(f)$.

Soit X_0 une variété affine lisse de dimension d et $f : X_0 \to X_0$ un endomorphisme de X_0. On définit les degrés dynamiques de f de la façon suivante. Une *complétion* de X_0 est une variété projective lisse X munie d'une immersion ouverte $\iota : X_0 \hookrightarrow X$ telle que $\iota(X_0)$ est dense dans X. L'endomorphisme f induit une transformation rationnelle de X par $\widetilde{f} = \iota \circ f \circ \iota^{-1}$ et on définit les degrés dynamiques

$$\lambda_l(f) := \lambda_l(\widetilde{f}). \tag{1.5}$$

Comme les degrés dynamiques sont des invariants birationnels, cette quantité ne dépend pas du choix de la complétion X. En particulier, si $X_0 = \mathbf{k}^N$ et $X = \mathbf{P}^N_{\mathbf{k}}$ on retrouve la définition du premier degré dynamique donnée au premier paragraphe.

La connaissance de ces degrés dynamiques donne des informations sur le système dynamique. Par exemple sur $\mathbf{C}$, Dinh et Sibony ont montré dans [DS03] que pour toute transformation rationnelle $f : X \dashrightarrow X$

$$h_{\mathrm{top}}(f) \leqslant \max_{0 \leqslant l \leqslant d} \log(\lambda_l) \tag{1.6}$$

où h_{top} est l'entropie topologique de f, Gromov avait au préalable montré ce résultat pour les endomorphismes de $\mathbf{P}^N$ dans [Gro03]. Yomdin a montré dans [Yom87] l'égalité des deux membres si f est un endomorphisme. Récemment, Favre, Truong et Xie ont montré dans [FTX22] que l'inégalité (2.6) était encore valable dans le cadre non-archimédien ; cependant l'égalité n'est pas vérifiée même pour des endomorphismes.

1.1.2 Degrés dynamiques sur les surfaces projectives

Une question naturelle est de se demander quels nombres peuvent apparaître comme le premier degré dynamique d'une transformation rationnelle d'une surface projective. On peut d'abord mentionner le résultat suivant dû à Bonifant et Fornaess dans [BF00] pour $\mathbf{P}^N_{\mathbf{C}}$ et généralisé par Urech

Théorème 1.1.1 ([Ure16]). *L'ensemble*

$$\{\lambda_1(f)\} \tag{1.7}$$

où f parcourt l'ensemble des transformations rationnelles de toute variété projective lisse sur n'importe quel corps, est dénombrable.

En 2021, Bell, Diller et Jonsson ont montré dans [BDJ20] l'existence d'une transformation rationnelle $\sigma : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$ telle que $\lambda_1(\sigma)$ est transcendant. Les trois auteurs et Krieger ont montré dans [BDJ20] que cet exemple peut se généraliser pour donner un exemple de transformation birationnelle de $\mathbf{P}^N, N \geqslant 3$ avec un premier degré dynamique transcendant. Mais en dimension 2, il y a de fortes contraintes sur $\lambda_1(f)$ pour f birationnelle. Dans [DF01], Diller et Favre ont montré que le premier degré dynamique d'une transformation birationnelle d'une surface projective est un entier algébrique. Plus précisément c'est un nombre de Pisot ou de Salem. Dans [BC13], Blanc et Cantat ont obtenu les résultats suivants

Théorème 1.1.2. *Soit X une surface projective sur un corps algébriquement clos.*

1. *Soit $f : X \dashrightarrow X$ une transformation birationnelle telle que $\lambda_1(f)$ est un nombre de Salem, alors il existe une application birationnelle $\varphi : X \dashrightarrow Y$ telle que $\varphi \circ f \circ \varphi^{-1}$ est un automorphisme de Y.*

2. *Si X est rationnelle sur un corps $\mathbf{k}$, alors l'ensemble $\Lambda(X) := \{\lambda_1(f) | f \in \mathrm{Bir}(X)\} \subset \mathbf{R}$ est bien ordonné. Il est fermé si $\mathbf{k}$ n'est pas dénombrable.*

En particulier, $\Lambda(X)$ est un ordinal et Bot montre dans [Bot22] que cet ordinal est exactement ω^ω où ω est l'ordinal des entiers naturels. On ne peut cependant pas espérer obtenir une information sur les degrés des entiers algébriques obtenus. En effet Bedford, Kim et McMullen construisent dans [BK06] et [McM07] des exemples de transformations birationnelles de surfaces projectives dont le premier degré dynamique est un entier algébrique de degré arbitrairement grand. En particulier le théorème 1.1 de [McM07] établit que pour tout $d \geqslant 10$ on peut trouver une surface projective avec un automorphisme de premier degré dynamique entier algébrique de degré d.

1.1.3 Degrés dynamiques des endomorphismes des surfaces affines

Dans ma thèse je considère des endomorphismes de surfaces affines. Le premier exemple de surface affine est le plan complexe $\mathbf{C}^2$. Un endomorphisme est alors une transformation polynomiale. Même dans ce cas, le premier degré dynamique n'est pas nécessairement un entier. En effet, soit

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.8}$$

une matrice à coefficients entiers positifs tels que $ad - bc \neq 0$. Considérons la transformation monomiale suivante

$$f(x,y) = \left(x^a y^b, x^c y^d\right), \tag{1.9}$$

alors f^N est la transformation monomiale dont les monômes sont donnés par les coefficients de A^N et $\lambda_1(f)$ est égal au rayon spectral de A. Ainsi, $\lambda_1(f)$ est un entier algébrique de degré 2 car il vérifie l'équation

$$\lambda_1(f)^2 - \mathrm{Tr}(A)\lambda_1(f) + \det(A) = 0. \tag{1.10}$$

Ainsi, il existe des transformation polynomiales f du plan avec $\lambda_1(f)$ entier ou entier algébrique de degré 2. Favre et Jonsson ont montré qu'il n'y a pas d'autres possibilités.

Théorème 1.1.3. *[FJ07] Soit $f : \mathbf{C}^2 \to \mathbf{C}^2$ une transformation polynomiale dominante, alors $\lambda_1(f)$ est un entier algébrique de degré $\leqslant 2$.*

Le premier résultat de ma thèse est d'étendre ce résultat à toutes les surfaces affines et en toute caractéristique. Même si on peut trouver des surfaces affines où le monoïde des endomorphismes peut changer de façon drastique. Par exemple, Blanc et Dubouloz, dans [BD13], construisent des surfaces affines lisses avec un gros groupe d'automorphismes, bien plus riche que celui du plan affine. Bot a utilisé cette construction pour montrer l'existence de surfaces affines rationnelles lisses avec une infinité non dénombrable de formes réelles (voir [Bot23]). Le travail établi dans ma thèse montre que même si du point de vue de la structure algébrique, ces groupes sont bien plus riche ; du point de vue de la dynamique individuelle de chaque automorphisme, ce n'est pas le cas.

Théorème A. *Soit X_0 une surface affine normale sur un corps* $\mathbf{k}$ *algébriquement clos. Si $f : X_0 \to X_0$ est un endomorphisme dominant, alors $\lambda_1(f)$ est un entier algébrique de degré $\leqslant 2$.*

La preuve utilise des techniques valuatives que je décris dans la section suivante. Si la caractéristique de $\mathbf{k}$ est nulle, j'obtiens des résultats sur la dynamique de l'endomorphisme f. Je donnerai un énoncé précis dans le cas des automorphismes (voir Théorème C).

1.2 Valuations, Diviseurs à l'infini et dynamique

1.2.1 Existence d'une valuation propre

Soit A l'anneau des fonctions régulières d'une surface affine normale X_0 sur un corps algébriquement clos $\mathbf{k}$. Une *valuation* est une fonction $\nu : A \to \mathbf{R} \cup \{\infty\}$ telle que

1. $\nu(PQ) = \nu(P) + \nu(Q)$;
2. $\nu(P+Q) \geqslant \min(v(P), \nu(Q))$;
3. $\nu(0) = \infty$;
4. $\nu_{|\mathbf{k}^\times} = 0$

Deux valuations ν et μ sont *équivalentes* s'il existe $t > 0$ tel que $\nu = t\mu$. Par exemple, si X est une complétion de X_0, pour toute courbe irréductible $E \subset X$, la fonction ord_E telle que $\mathrm{ord}_E(P)$ est l'ordre d'annulation de P le long de E est une valuation. Toute valuation de la forme $\lambda \mathrm{ord}_E$ avec $\lambda > 0$ est dite *divisorielle*. Si f est un endomorphisme de X_0, alors f induit un homomorphisme d'anneaux $f^* : A \to A$. On peut alors définir le poussé en avant $f_*\nu$ d'une valuation ν par

$$f_*\nu(P) = \nu(f^*P). \tag{1.11}$$

On dit qu'une valuation est centrée à l'infini s'il existe $P \in A$ tel que $\nu(P) < 0$. Si X est une complétion de X_0 les valuations divisorielles centrées à l'infini sont exactement celles qui correspondent aux composantes irréductibles de $X \backslash X_0$. Soit $\mathcal{V}_\infty$ l'ensemble des valuations centrées à l'infini et $\widehat{\mathcal{V}}_\infty$ celui des valuations centrées à l'infini modulo équivalence. Supposons pour simplifier que f est un automorphisme de X_0, alors f_* induit une bijection de $\mathcal{V}_\infty$ et de $\widehat{\mathcal{V}}_\infty$ qui sera en fait un homéomorphisme pour une topologie que l'on décrira dans le mémoire.

Si X_0 est le plan affine complexe, alors Favre et Jonsson prouvent l'existence d'une valuation $\nu_* \in \mathcal{V}_\infty$ telle que $f_* \nu_* = \lambda_1(f)\nu_*$. Une telle valuation est appelée valuation propre de f. Pour ce faire, ils montrent dans [FJ04] que $\widehat{\mathcal{V}}_\infty$ a une structure d'arbre réel et f_* est compatible avec cette structure. L'existence de ν_* provient alors d'un théorème de point fixe sur les arbres. L'existence de cette valuation propre a un grand impact sur la dynamique de f. Elle permet notamment de trouver une bonne complétion X de $\mathbf{C}^2$ qui admet un point fixe attractif de f à l'infini. Xie utilise cette construction de valuation propre pour démontrer la conjecture des orbites Zariski-denses et la conjecture de Mordell-Lang dynamique pour les endomorphismes du plan affine ([Xie17b]). Jonsson et Wulcan utilisent ces techniques pour construire une hauteur canonique pour les endomorphismes du plan affine complexe avec petit degré topologique dans [JW12].

Théorème B. *Soit X_0 une surface affine normale sur un corps $\mathbf{k}$ algébriquement clos (de caractéristique quelconque) et f un endomorphisme dominant de X_0. Sous les hypothèses suivantes*

1. $\mathbf{k}[X_0]^\times = \mathbf{k}^\times$.
2. *Pour toute complétion X de X_0,* $\operatorname{Pic}^0(X) = 0$.
3. $\lambda_1(f)^2 > \lambda_2(f)$.

Il existe une valuation centrée à l'infini ν_*, unique à équivalence près, de f telle que

$$f_*(\nu) = \lambda_1(f)\nu_*. \tag{1.12}$$

Les techniques que j'emploie n'exploite pas la géométrie globale de $\widehat{\mathcal{V}}_\infty$ au sens où cet espace n'est plus nécessairement un arbre. Soit X une complétion de X_0, je montre qu'à toute valuation ν centrée à l'infini on peut associer un unique diviseur $Z_{\nu,X}$ de X supporté en dehors de X_0, de plus si Y est une autre complétion de X_0, il y a une compatibilité entre $Z_{\nu,X}$ et $Z_{\nu,Y}$ (voir Proposition 3.6.6). Cette construction fait intervenir l'espace de Picard-Manin de X_0. L'analyse spectrale des opérateurs f_*, f^* définit par f sur cet espace (voir [BFJ08, Can11]) permet de construire la valuation propre ν_* et de prouver son unicité. Ce procédé est similaire aux techniques de [DF21] §6.

1.2.2 Discussion des hypothèses du théorème

Les hypothèses du théorème B peuvent paraître arbitraires mais elles ne sont pas restrictives. En effet, si les hypothèses (1) ou (2) ne sont pas vérifiées, alors on peut montrer que l'endomorphisme f préserve une fibration vers une variété [1]*quasi-abélienne*. On peut décomposer la dynamique de f par cette fibration et elle devient plus simple à étudier.

Si l'hypothèse (3) n'est pas satisfaite alors on a $\lambda_1(f)^2 = \lambda_2(f)$. Notons que dans ce cas $\lambda_1(f)$ est automatiquement un entier algébrique de degré $\leqslant 2$ car $\lambda_2(f)$ est le degré topologique de f, donc un entier. Dans le cas du plan affine complexe, Favre et Jonsson arrivent à une classification des endomorphismes polynomiaux satisfaisant $\lambda_1^2 = \lambda_2$: ou bien ils préservent une fibration rationnelle, ou bien il existe une complétion X de $\mathbf{A}^2_{\mathbf{C}}$ avec au plus des singularités quotients à l'infini telle que f s'étende en un endomorphisme de X. Je m'attends à ce qu'une classification similaire existe dans le cas général, tous les exemples que j'ai étudié jusqu'à présent satisfont cette dichotomie. On peut remarquer que dans le cas inversible, une telle classification existe déjà : Par [Giz69] et [Can01], toute transformation birationnelle $\sigma : X \to X$ d'une surface projective lisse telle que $\lambda_1(\sigma) = 1$ est un automorphisme de X ou préserve une fibration rationnelle ou elliptique.

1.2.3 Énoncé du résultat dans le cas des automorphismes

En caractéristique nulle, l'existence de cette valuation propre a des conséquences sur la dynamique de f. Je prouve également pour n'importe quel endomorphisme l'existence d'une complétion X de X_0 qui admet un point fixe attractif de f à l'infini et dans le cas des automorphismes loxodromiques (c'est à dire avec $\lambda_1 > 1$), je démontre le résultat suivant

Théorème C. *Soit X_0 une surface affine normale sur $\mathbf{C}$ telle que $\mathbf{C}[X_0]^\times = \mathbf{C}^\times$. Si f est un automorphisme de X_0 tel que $\lambda_1(f) > 1$, alors il existe une complétion X de X_0 tel que*

1. *f admet un point fixe attractif $p \in X(\mathbf{C}) \setminus X_0(\mathbf{C})$ à l'infini.*
2. *Un itéré de f contracte $X \setminus X_0$ sur p.*
3. *Il existe des coordonnées analytiques locales centrées sur p telles que f est localement de la forme*

1. Une variété quasi-abélienne est un groupe algébrique X tel qu'il existe un tore algébrique T et une variété abélienne A satisfaisant la suite exacte $0 \to T \to X \to A \to 0$ de groupes algébriques.

(a)

$$f(z,w) = (z^a w^b, z^c w^d) \tag{1.13}$$

avec a,b,c,d *des entiers* $\geqslant 1$*, dans ce cas* $\lambda_1(f)$ *est le rayon spectral de* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
En particulier, $\lambda_1(f) \in \mathbf{R}\backslash\mathbf{Q}$*, c'est un entier algébrique de degré 2.*

(b) *ou bien*

$$f(z,w) = (z^a, \lambda z^c w + P(z)) \tag{1.14}$$

avec $a \geqslant 2, c \geqslant 1$ *et* $P \not\equiv 0$ *un polynôme, dans ce cas* $\lambda_1(f) = a$ *est un entier.*

4. *Les points fixes attractifs de* f *et* f^{-1} *sont distincts.*

5. *La forme normale de* f^{-1} *à son point fixe attractif est la même que celle de* f.

Les cas (3)(a) et (3)(b) sont mutuellement exclusifs au sens suivant

Théorème D. *Soit* X_0 *une surface affine normale sur* $\mathbf{C}$ *telle que* $\mathbf{C}[X_0]^\times = \mathbf{C}^\times$ *et* $f \in \mathrm{Aut}(X_0)$ *un automorphisme loxodromique. On a la dichotomie suivante :*

- *Si* $\lambda_1(f) \in \mathbf{Z}_{\geqslant 0}$*, alors pour tout automorphisme loxodromique* g *de* X_0*, on a* $\lambda_1(g) \in \mathbf{Z}_{\geqslant 0}$ *et la forme normale de* g *à son point attractif* p *est de la forme* (1.14).
- *Si* $\lambda_1(f) \notin \mathbf{Z}_{\geqslant 0}$ *alors c'est un entier algébrique de degré 2 et cela reste vrai pour tout automorphisme loxodromique* g *de* X_0*. En particulier, la forme normale de* g *à son point fixe attractif est de la forme monomiale* (2.13).

On donne deux exemples : le plan affine et la surface de Markov (voir §1.2.3.2). Les théorèmes C et D montrent qu'il suffit de comprendre ces deux exemples pour comprendre la dynamique d'un automorphisme d'une surface affine.

1.2.3.1 Le plan affine

Soit $X_0 = \mathbf{A}^2_{\mathbf{C}}$, considérons l'automorphisme

$$f(x,y) = (y + x^2, x). \tag{1.15}$$

C'est un automorphisme de *Hénon* et on a $\lambda_1(f) = 2$. On considère la complétion $X = \mathbf{P}^2_{\mathbf{C}}$ avec les coordonnées homogènes X,Y,Z telles que $x = X/Z$ et $y = Y/Z$. La transformation birationnelle induite par f possède un point fixe $p_+ = [1:0:0]$ et un point d'indétermination

$p_- = [0:1:0]$. La droite à l'infini $\{Z=0\}$ est contractée par f sur p_+ et par f^{-1} sur p_-. Prenons les coordonnées locales (u,v) en p_+ données par $u = Y/X$ et $v = Z/X$, on a

$$f(u,v) = \left(\frac{v}{1+uv}, \frac{v^2}{1+uv}\right). \tag{1.16}$$

Et il existe un changement de coordonnées analytiques telle que f a la forme normale (1.14). (Voir [Fav00] §2).

1.2.3.2 La surface de Markov

Considérons la surface de Markov $\mathcal{M}_0 \subset \mathbf{A}^3_{\mathbf{C}}$ donnée par l'équation

$$x^2+y^2+z^2 = xyz. \tag{1.17}$$

C'est une surface normale avec une singularité quotient en $(0,0,0)$. On décrira plus en détail ces propriétés dans le paragraphe 1.4. Une complétion naturelle de $\mathcal{M}_0$ est la surface projective $X \subset \mathbf{P}^3_{\mathbf{C}}$ qui est définie comme l'adhérence de Zariski de $\mathcal{M}_0$ dans $\mathbf{P}^3_{\mathbf{C}}$. L'équation de X est

$$T(X^2+Y^2+Z^2) = XYZ. \tag{1.18}$$

On voit que $X \backslash \mathcal{M}_0$ a pour équation

$$T = 0, XYZ = 0. \tag{1.19}$$

C'est donc un triangle de 3 courbes rationnelles. Par le théorème 3.1 de [Can09], si f est un automorphisme loxodromique de $\mathcal{M}_0$ algébriquement stable sur X alors f possède un point fixe attractif $p_+ \in X\backslash\mathcal{M}_0$ qui est un des sommets du triangle et un point d'indétermination $p_- \in X\backslash\mathcal{M}_0$ qui est un autre sommet du triangle. De plus, f admet une forme normale monomiale (i.e du type (2.13)) en p_+.

Remarque 1.2.1. On voit que pour toute complétion X du plan affine, le graphe dual de $X\backslash\mathbf{A}^2_{\mathbf{C}}$ est un arbre. En revanche, dans le cas de $\mathcal{M}_0$ le graphe dual de $X\backslash\mathcal{M}_0$ se rétracte sur un cercle pour toute complétion X. On montrera en fait que toute surface affine (possédant un automorphisme loxodromique) satisfait cette dichotomie. C'est cette dichotomie de la géométrie des graphes duaux qui donnent la dichotomie de la dynamique (voir Théorème 4.4.4).

1.3 Dynamique des automorphismes des surfaces affines

1.3.1 Dynamique des transformations de Hénon : Fonction de Green

Soit $\mathrm{Aut}(\mathbf{A}^2_{\mathbf{C}})$ le groupe des automorphismes polynomiaux du plan affine complexe. Les transformations affines sont des exemples de tels automorphismes. En voici un autre : soit

$$f(x,y) = (x, y + P(x)) \tag{1.20}$$

où P est un polynôme. L'automorphisme f préserve les droites d'équations $x = \alpha$ et agit par translation sur ces droites, le vecteur de translation est donnée par un polynôme en x à savoir $P(x)$. Un tel automorphisme est appelé *élémentaire*. On note E l'ensemble des automorphismes élémentaires de $\mathbf{A}^2_{\mathbf{C}}$, ces automorphismes forment un groupe isomorphe à $(\mathbf{C}[x], +)$. Le théorème de Jung ([Jun42]) affirme que $\mathrm{Aut}(\mathbf{C}^2)$ a une structure de produit amalgamé

$$\mathrm{Aut}(\mathbf{A}^2_{\mathbf{C}}) = \mathrm{Aff}(\mathbf{A}^2_{\mathbf{C}}) *_S E \tag{1.21}$$

où $S = \mathrm{Aff}(\mathbf{A}^2_{\mathbf{C}}) \cap E$.

Un automorphisme de type Hénon est un automorphisme f qui n'est conjugué ni à un élément de $\mathrm{Aff}(\mathbf{A}^2_{\mathbf{C}})$ ni à un élément de E. Ils sont caractérisés par le fait qu'il vérifie $\lambda_1(f) > 1$. Un exemple d'automorphisme de type Hénon que nous utiliserons dans la suite est le suivant

$$f(x,y) = (y + x^2, x). \tag{1.22}$$

L'extension de f à $\mathbf{P}^2$ a un point fixe à l'infini $p_+ = [1 : 0 : 0]$ et un point d'indétermination $p_- = [0 : 1 : 0]$. La droite à l'infini est contractée par f sur p_+. De même p_+ est le seul point d'indétermination de f^{-1} et p_- est un point fixe de f^{-1} sur lequel la droite à l'infini est contractée par f^{-1}. Un automorphisme h sera dit régulier si les points d'indéterminations de h et de h^{-1} sont distincts. En particulier f est régulier et tout automorphisme de type Hénon est conjugué à un automorphisme régulier [FM89]. Pour tout automorphisme de type Hénon h, $\lambda_1(h)$ est un entier que l'on notera d, en particulier $\lambda_1(f) = 2$.

On considère la norme $||(x,y)|| = \max(|x|, |y|)$ sur $\mathbf{C}^2$. Si h est un automorphisme régulier de type Hénon, on peut définir les fonctions de Green de h (voir [FM89], [BS91a] et leurs références)

$$G^+(p) := \lim_N \frac{1}{d^N} \log^+ \left|\left|h^N(p)\right|\right|, \quad G^-(p) := \lim_N \frac{1}{d^N} \log^+ \left|\left|h^{-N}(p)\right|\right| \tag{1.23}$$

où $\log^+ = \max(0, \log)$. On a alors les propriétés suivantes (voir [BS91a]).

1. G^+ est bien définie, continue et plurisousharmonique sur $\mathbf{C}^2$

2. $G^+ \circ h = dG^+$

3. La fonction $p \mapsto G^+(p) - \log^+(||p||)$ s'étend en une fonction continue sur $\mathbf{P}^2 \backslash p_-$.

4. $G^+(p) = 0$ si et seulement si l'orbite $(h^N(p))_{N \geqslant 0}$ est bornée.

La fonction G^- jouit de propriétés similaires. On peut alors définir les courants de Green $T^+ = dd^c G^+$ et $T^- = dd^c G^-$. Ce sont des $(1,1)$-courants positifs fermés. La mesure

$$\mu := T^+ \wedge T^- \tag{1.24}$$

est alors bien définie car G^+, G^- sont continues, elle est de masse totale finie et on peut supposer que c'est une mesure de probabilité. On l'appelle la *mesure d'équilibre* de h. Elle est h-invariante et son support est contenu dans l'ensemble de Julia de h.

On définit la fonction de Green suivante

$$G := \max(G^+, G^-) \tag{1.25}$$

qui satisfait les propriétés

1. G est une fonction continue, plurisousharmonique de $\mathbf{C}^2$ et est limite uniforme de

$$\max\left(\frac{1}{d^N}\log^+(||f^N(p)||), \frac{1}{d^N}\log^+(||f^{-N}(p)||)\right) \tag{1.26}$$

2. $p \mapsto G(p) - \log^+ ||p||$ s'étend en une fonction continue sur $\mathbf{P}^2$.

3. $G(p) = 0$ si et seulement si l'orbite $(f^N(p))_{N \in \mathbf{Z}}$ est bornée.

1.3.2 Dynamique des automorphismes des surfaces affines

Grâce au théorème C, je démontre le résultat suivant :

Théorème E. *Soit X_0 une surface affine normale sur* $\mathbf{C}$*, soit X une complétion de X_0 qui vérifie le théorème C. Soit $X \hookrightarrow \mathbf{P}^N$ un plongement de X qui induit un plongement $X_0 \hookrightarrow \mathbf{C}^N$ et soit* $||\cdot||$

une norme sur $\mathbf{C}^N$*. Si* f *est un automorphisme de* X_0 *tel que* $\lambda_1(f) > 1$*, la fonction de Green*

$$G^+(p) := \lim_N \frac{1}{\lambda_1^N} \log^+(||f^N(p)||) \tag{1.27}$$

vérifie les propriétés suivantes

1. G^+ *est bien définie, continue et plurisousharmonique sur* $X_0(\mathbf{C})$*.*

2. $G^+ \circ f = \lambda_1 G^+$

3. G^+ *est à croissance logarithmique (voir Proposition 5.2.5).*

4. $G^+(p) = 0$ *si et seulement si l'orbite* $(f^N(p))_{N \geqslant 0}$ *est bornée.*

On peut alors considérer la fonction $G = \max(G^+, G^-)$ qui va jouir de propriétés similaires au cas Hénon. Il y a cependant une différence majeure. En général, le maximum de deux fonctions à croissance logarithmique n'est pas à croissance logarithmique. Il y a donc une difficulté supplémentaire ici. Il s'avère que nous avons deux comportements différents : si $\lambda_1(f) \in \mathbf{Z}_{\geqslant 0}$, G est encore à croissance logarithmique et tout se passe comme dans le cas Hénon. Si $\lambda_1(f) \notin \mathbf{Z}_{\geqslant 0}$, alors G n'est pas à croissance logarithmique et donc ce n'est pas la bonne fonction à considérer, il faut alors utiliser les travaux récents de Yuan et Zhang sur les fibrés en droite adéliques sur les variétés quasiprojectives, Je serai plus précis dans la section suivante.

1.3.3 Dynamique aux places non-archimédiennes

Soit $\mathbf{K}$ un corps de nombre. Une *valeur absolue* $|\cdot|$ sur $\mathbf{K}$ est une fonction $|\cdot| : \mathbf{K} \to \mathbf{R}_+$ qui vérifie les axiomes suivants

- $|x| = 0 \Leftrightarrow x = 0$,
- $\forall x, y \in \mathbf{K}, |xy| = |x| \cdot |y|$,
- $\forall x, y \in \mathbf{K}, |x + y| \leqslant |x| + |y|$.

Deux valeurs absolues $|\cdot|_1, |\cdot|_2$ sont équivalentes si $|\cdot|_1 = |\cdot|_2^s$ pour un certain $s > 0$. Une *place* est une classe d'équivalence de valeur absolue, on note $\mathcal{M}(\mathbf{K})$ l'ensemble des places de $\mathbf{K}$. Si $|\cdot|$ est une valeur absolue de $\mathbf{K}$, on peut considérer la complétion de $\mathbf{K}$ par rapport à $|\cdot|$. Cette complétion ne dépend en fait que de la place v de $|\cdot|$, on la note $\mathbf{K}_v$. La valeur absolue $|\cdot|$ s'étend alors à $\mathbf{K}_v$ et admet une extension naturelle à $\overline{\mathbf{K}}_v$. On note $\mathbf{C}_v$ le complété de $\overline{\mathbf{K}}_v$ par rapport à

$|\cdot|$. Cette construction ne dépend que de la place v. On dit que $|\cdot|$ est *non-archimédienne* si elle vérifie l'inégalité suivante

$$\forall x, y \in \mathbf{K}, |x+y| \leqslant \max(|x|, |y|). \tag{1.28}$$

Une place v est non-archimédienne si un de ses représentants l'est. Pour toute place archimédienne v, on a $\mathbf{C}_v = \mathbf{C}$. Les énoncés du paragraphe §1.3.1 ont des analogues lorsque $\mathbf{C}$ est remplacé par un corps algébriquement clos complet non archimédien $\mathbf{C}_v$. En effet, Kawaguchi montre dans [Kaw09] que la fonction de Green d'un automorphisme de type Hénon est bien définie également dans le cas non-archimédien. Si $\mathbf{C}_v$ est non-archimédien, la fonction de Green $G = \max(G^+, G^-)$ induit un fibré en droites métrisé semipositif sur l'analytifié de Berkovich de $\mathbf{P}^2_{\mathbf{C}_v}$ que l'on note $(\mathbf{P}^2_{\mathbf{C}_v})^{an}$ (voir [Zha93] pour la définition). La mesure d'équilibre associée est une mesure positive sur $(\mathbf{P}^2_{\mathbf{C}_v})^{an}$, elle est construite dans [Cha03]. Il est à noter que des travaux plus récents de Chambert-Loir et Ducros [CD] permettent de construire les courants $T^{\pm} = dd^c G^{\pm}$ et de définir la mesure d'équilibre de la même manière que dans le cas complexe $\mu = T^+ \wedge T^-$. De plus, Lee montre dans [Lee13] que l'orbite de Galois de toute suite [2]générique de points périodiques de f est équidistribuée par rapport à la mesure $\mu = T^+ \wedge T^-$ et ce à toutes les places en utilisant le théorème d'équidistribution de Yuan dans [Yua08].

Je prouve également un analogue du théorème C dans le cas non archimédien. On définit également les fonctions G^+, G^-, G dans ce contexte. Cependant les problèmes évoquées à la fin du paragraphe 1.3.2 subsiste. Si $\lambda_1(f) \in \mathbf{Z}_{\geqslant 0}$, alors la donnée des fonctions de Green (G_v) pour chaque place v de $\mathbf{K}$ induit un fibré en droites adélique semipositif (cf [Zha93]) sur une complétion X de X_0 et le théorème d'équidistribution arithétique de Yuan s'applique.

Maintenant si $\lambda_1(f) \notin \mathbf{Z}_{\geqslant 0}$, on ne peut pas appliquer la théorie des fibrés en droite adélique sur la complétion X. Le bon point de vue est de considérer non pas une complétion de X_0 mais l'ensemble de toutes les complétions X de X_0. C'est le point de vue développé par Yuan et Zhang dans [YZ22]. Les auteurs définissent alors la notion de fibré en droites adélique associé à une variété *quasiprojective* U comme une limite de fibrés en droites adéliques sur des complétions de U. Ils démontrent dans ce contexte un théorème d'équidistribution arithmétique similaire au théorème de Yuan. Je conjecture dans mon mémoire le fait suivant (voir Conjecture F) :

Conjecture F. *La donnée de (G_v^+) et (G_v^-) pour toute place v de $\mathbf{K}$ induisent deux fibrés en droites adéliques nef f-invariant sur la variété quasiprojective X_0. En particulier, on peut définir la mesure d'équilibre μ_v de f à toute place comme la mesure de probabilité proportionelle à $dd^c G_v^+ \wedge dd^c G_v^-$ et l'orbite de Galois de toute suite générique de points périodiques de f est*

2. Une suite est générique si aucune sous suite n'est contenue dans une sous variété fermée stricte

équidistribuée par rapport à μ_v pour toute place v.

Je pense que les travaux établis dans ce mémoire et les travaux de Yuan et Zhang permettront de prouver cette conjecture à l'aide d'une construction similaire au §4 de [YZ17](voir §1.5.1).

1.3.4 Des automorphismes avec une infinité de points périodiques communs

Si X_0 une surface affine normale sur $\mathbf{K}$, un corps de nombre, et f un automorphisme loxodromique de X_0, on peut mener l'étude de la section précédente aux places archimédiennes et non-archimédiennes. On obtient ainsi une mesure d'équilibre $\mu_{f,v}$ pour f à toutes les places v de $\mathbf{K}$. Grâce aux techniques d'équidistributions arithmétiques mentionnées dans le paragraphe précédent, je démontre le résultat suivant.

Théorème G. *Soit X_0 une surface affine normale défini sur un corps de nombres $\mathbf{K}$. Si f, g sont deux automorphismes loxodromiques de X_0 tels que $\lambda_1(f) \in \mathbf{Z}_{\geqslant 0}$, les assertions suivantes sont équivalentes*

1. $\mathrm{Per}(f) \cap \mathrm{Per}(g)$ *est Zariski-dense.*
2. $\forall v \in \mathcal{M}(\mathbf{K}), \mu_{v,f} = \mu_{v,g}$
3. $\mathrm{Per}(f) = \mathrm{Per}(g)$.

Dans le cas $\lambda_1(f) \notin \mathbf{Z}_{\geqslant 0}$, en admettant la conjecture F, on a que si $\mathrm{Per}(f) \cap \mathrm{Per}(g)$ *est Zariski-dense, alors $\forall v \in \mathcal{M}(\mathbf{K}), \mu_{f,v} = \mu_{g,v}$.*

En utilisant des méthodes similaires, ce genre d'énoncé a d'abord été obtenu par Baker, DeMarco dans [BD11a] pour les endomorphismes de $\mathbf{P}^1$ de degré $\geqslant 2$ sur $\mathbf{C}$ puis a été généralisé par Yuan et Zhang pour les endomorphismes polarisables de $\mathbf{P}^m$ sur un corps de nombres dans [YZ17] et récemment dans [YZ21] sur n'importe quel corps de caractéristique nulle. Dans [CD20], Cantat et Dujardin utilisent ces mêmes outils de dynamique arithmétique pour montrer des résultats de rigidité sur les groupes d'automorphismes de surfaces projectives.

La conjecture F ne suffit pas à montrer l'égalité $\mathrm{Per}(f) = \mathrm{Per}(g)$ car la preuve utilise une version arithmétique du théorème de l'indice de Hodge qui n'a pas encore été démontré pour les fibrés en droites adéliques sur les variétés quasi-projectives (voir Théorème 5.1.20).

1.4 Un résultat de rigidité pour la surface de Markov

Dans [DF17] Dujardin et Favre montrent un résultat plus fort que celui du théorème G. Ils obtiennent que si deux automorphismes de Hénon vérifient une des assertions du théorème G, alors f et g ont des itérés communs : il existe deux entiers $M,N \in \mathbf{Z}$ tels que $f^N = g^M$. Ce résultat de rigidité ne peut pas être vrai pour toute surface affine. En effet, si $X_0 = \mathbf{C}^\times \times \mathbf{C}^\times$. Soit $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$ telle que $(\mathrm{Tr}A)^2 > 4$, alors on définit l'automorphisme

$$f_A(x,y) = (x^a y^b, x^c y^d) \tag{1.29}$$

Si $\mathbb{S}^1 = \{z \in \mathbf{C} \,|\, |z| = 1\}$, alors $\mathbb{S}^1 \times \mathbb{S}^1$ est un compact invariant par f_A. C'est l'ensemble de Julia de f_A et les points périodiques de f_A sont exactement les points $(\omega_1, \omega_2) \in \mathbb{S}^1 \times \mathbb{S}^1$ où ω_1, ω_2 sont des racines de l'unité. Ainsi, tous les automorphismes obtenus ainsi vérifie le théorème G mais n'ont pas d'itérés communs.

Soit $D \in \mathbf{C}$, on définit la surface de Markov $\mathcal{M}_D$ comme la surface dans $\mathbf{C}^3$ définie par

$$x^2 + y^2 + z^2 = xyz + D \tag{1.30}$$

Cette famille de surfaces est à la frontière de plusieurs domaines (voir [Can09]). Notamment, si $\mathbb{T}^1$ est le tore épointé, son groupe fondamental $\pi := \pi_1(\mathbb{T}^1)$ est un groupe libre à deux générateurs que l'on note a et b. On peut s'intéresser à la variété de caractères

$$\mathcal{X} := \mathrm{Hom}(\pi, \mathrm{SL}_2(\mathbf{C}))//\mathrm{SL}_2(\mathbf{C}) \tag{1.31}$$

où l'action de $\mathrm{SL}_2(\mathbf{C})$ est donnée par la conjugaison et $//$ est le quotient au sens de la théorie géométrique des invariants (GIT). On note $[a,b] := aba^{-1}b^{-1}$ le commutateur de a et b. Soit $\rho \in \mathcal{X}$, si on note $x = \mathrm{Tr}\,\rho(a), y = \mathrm{Tr}\,\rho(b), z = \mathrm{Tr}\,\rho(ab)$, alors on a que

$$\mathcal{X} \to \mathbf{A}^3_{\mathbf{C}} \tag{1.32}$$

$$\rho \mapsto (x,y,z) \tag{1.33}$$

est un isomorphisme. C'est un résultat de Fricke (voir [Gol09]). De plus on a l'égalité suivante

$$x^2 + y^2 + z^2 = xyz + \mathrm{Tr}(\rho([a,b])) + 2. \tag{1.34}$$

Autrement, dit la variété algébrique $\mathcal{X}$ est feuilleté par la famille des surfaces de Markov et la surface $\mathcal{M}_D$ représente une ligne de niveau pour la fonction régulière $\rho \mapsto \mathrm{Tr}(\rho([a,b]))$.

Théorème H. *Soit $D = 0$ ou bien $D = 2 - 2\cos(2\pi/q)$ avec $q \geqslant 2$. Si f,g sont deux automorphismes loxodromiques de $\mathcal{M}_D$, alors en admettant la conjecture F, les assertions suivantes sont équivalentes :*

1. $\mathrm{Per}(f) \cap \mathrm{Per}(g)$ *est Zariski-dense.*
2. $\mathrm{Per}(f) = \mathrm{Per}(g)$.
3. *f et g ont des itérés communs : il existe $N,M \in \mathbf{Z}$ tels que $f^N = g^M$.*

La conjecture F et le théorème G donnent l'égalité des mesures d'équilibres de f et g. Pour montrer le résultat on utilise la théorie des représentations fuchsiennes et quasi-fuchisiennes pour construire un point fixe hyperbolique $q(f)$ au bord de l'ouvert des représentations quasi-fuchsiennes dans $\mathcal{M}_D(\mathbf{C})$. Cette construction utilise le théorème de paramétrisation de Bers [Ber60], sa prolongation par Minsky [Min99] et le théorème d'hyperbolisation des variétés de dimension 3 qui fibre sur un cercle de Thurston (voir [Ota96, McM96]). On démontre ensuite grâce à des techniques de théorie des courants en géométrie complexe, notamment grâce au courant d'Ahlfors-Nevanlinna, que ce point fixe hyperbolique doit appartenir au support de $\mu_{\mathbf{C},f} = \mu_{\mathbf{C},g}$ qui est un compact invariant par le groupe $\langle f, g \rangle$. Enfin on montre que l'orbite de $q(f)$ sous g est non bornée si g n'a pas d'itérés communs avec f grâce à la théorie des laminations mesurées ce qui donne une contradiction.

1.5 Questions et compléments

1.5.1 La conjecture F

Comme établi dans cette introduction, je pense que la conjecture F doit se démontrer avec les travaux de ce mémoire. Notamment, je montre dans la proposition 5.2.5 que la fonction G^+ s'obtient par un procédé itératif à partir d'une fonction de Green de n'importe quel diviseur. Ce procédé itératif appliqué aux fibrés en droite adéliques sur X_0 doit donner un fibré en droites adéliques nef au sens [YZ22]. En effet, dans le cadre projectif si f est un endomorphisme polarisé d'une variété projective X et L un fibré en droites ample sur X tel que $f^*L = L^{\otimes d}$, alors Yuan et Zhang montrent dans [YZ17] que pour n'importe quelle extension adélique $\overline{L}$ de L, la

suite
$$\frac{1}{d^n}(f^n)^*\overline{L} \tag{1.35}$$
converge vers un fibré en droites adélique semipositif $\overline{L}_f$ tel que $f^*\overline{L_f} = d\overline{L_f}$. Au niveau des fonctions de Green ce processus itératif est le même que celui qui apparait dans la section 5.2 (voir les propositions 5.2.5 et 5.2.12), donc je m'attends à ce que tout passe dans ce cadre.

Pour obtenir le théorème G, il faudra ensuite démontrer le théorème de l'indice de Hodge arithmétique dans le cas des surfaces affines. Il me suffit d'une version affaiblie qui semble démontrable dans le cas précis qui m'intéresse.

1.5.2 les travaux de Danilov et Gizatullin

On dit qu'une surface affine X_0 est *complétable par un zigzag* s'il existe une complétion X de X_0 tel que $X \backslash X_0$ est un *zigzag*, c'est à dire une chaîne de courbes rationnelles lisses. Le plan affine est complétable par un zigzag mais pas la surface de Markov $\mathcal{M}_0$ par exemple. Dans [GD75], Danilov et Gizatullin étudient le groupe d'automorphismes des surfaces affines complétable par un zigzag. Ils montrent que ce groupe agit sur un arbre dont les sommets sont les complétions dont le bord est un zigzag. Si X_0 est complétable par un zigzag, alors son espace des valuations centrées à l'infini $\widehat{\mathcal{V}_\infty}$ est aussi un arbre sur lequel agit $\operatorname{Aut}(X_0)$. Il serait intéressant de comparer l'approche de Danilov et Gizatullin aux travaux de mon mémoire. Il est à noter que les travaux de Gizatullin (voir [Giz71b, Giz70, Giz71c]) préliminaires aux résultats de [GD75] sont également utilisé dans mon mémoire pour étudier la dynamique des automorphismes loxodromiques (voir §4.4.1).

1.5.3 Complexité dynamique vs complexité algébrique de $\operatorname{Aut}(X_0)$

J'ai démontré dans mon mémoire que l'étude de la dynamique d'un automorphisme loxodromique sur une surface affine est similaire ou bien à la dynamique d'un automorphisme de type Hénon, ou bien à un automorphisme de la surface de Markov. Cependant on sait qu'il existe des surfaces affines avec un groupe d'automorphisme bien plus compliqué que celui du plan affine par les travaux de Blanc et Dubouloz mentionnés précédemment. Prenons X_0 une telle surface, il serait intéressant d'appliquer les techniques valuatives de ce mémoire à tout un sous groupe d'automorphismes de X_0. Par exemple si f, g sont deux automorphismes loxodromiques tels que tout élément du sous groupe $\Gamma = \langle f, g\rangle$ qui n'est pas l'identité est loxodromique, que peut on dire de l'ensemble $\{\nu_*(h) : h \in \Gamma\} \subset \widehat{\mathcal{V}_\infty}$ où $\nu_*(h)$ est la valuation propre de h ? Peut on retrouver la complexité algébrique du groupe $\mathrm{Aut}(X_0)$ en utilisant des techniques valuatives ?

1.5.4 Des résultats de dynamique arithmétique utilisant des techniques valuatives

En utilisant les techniques valuatives de Favre et Jonsson pour le plan affine, Junyi Xie montrent dans [Xie17b] la conjecture des orbites Zariski dense pour les endomorphismes polynomiaux du plan affine complexe. Cette conjecture affirme qu'un endomorphisme f admet une orbite Zariski dense si et seulement si f n'admet pas de fonctions rationnelles non constantes invariantes. La preuve utilise la dynamique à l'infini provenant de l'existence de valuation propre. L'auteur montre également dans [Xie17a] la conjecture dynamique de Mordell-Lang pour les endomorphismes polynomiaux du plan affine : si $x \in \mathbf{A}^2(\mathbf{C})$ et $C \subset \mathbf{A}^2_{\mathbf{C}}$ est une courbe alors $\{n \geqslant 0 : f^n(x) \in C\}$ est une union d'un ensemble fini et d'une union finie de progressions arithmétiques.

Pour ces deux conjectures, on peut établir leur analogue dans le cas de n'importe quelle surface affine en utilisant les techniques valuatives de ce mémoire en supposant $\lambda_1^2 > \lambda_2$. Pour le cas d'égalité, Xie s'appuie sur la classification des endomorphismes vérifiant $\lambda_1^2 = \lambda_2$ établie par Favre et Jonsson. Il est donc nécessaire d'établir une telle classification en général. Pour l'instant les techniques développées dans ce mémoire ne permettent pas de traiter le cas $\lambda_1^2 = \lambda_2$. En particulier, je ne sais pas pour l'instant construire de valuations propres associées à un endomorphisme f vérifiant $\lambda_1(f)^2 = \lambda_2(f)$.

1.5.5 Fonctions de Green et hauteurs canoniques pour les petits degrés topologiques

Soit f un endomorphisme polynomial du plan affine défini sur un corps de nombre $\mathbf{K}$ tel que $\lambda_1(f) > \lambda_2(f)$. Dans [FJ11] et [JW12] Favre, Jonsson et Wulcan utilise l'existence d'une unique valuation propre de f pour construire une fonction de Green pour f à toutes les places. Jonsson et Wulcan construisent ensuite une hauteur canonique h_f associé à f qui satisfait la propriété suivante : $p \in \mathbf{A}^2(\overline{K}), h_f(p) = 0$ si et seulement si pour toute place v, $||f^n(p)||_v$ croît au plus comme μ^n avec $0 < \mu \leqslant \lambda_2 < \lambda_1$.

Il semble que cette construction doit se généraliser à toute surface affine avec les travaux de ce mémoire. La construction de fonctions de Green et d'hauteurs canoniques permettrait de prouver une version faible de l'alternative de Tits de la forme suivante : Si $f, g \in \mathrm{End}(X_0)$

satisfont $\lambda_1(f) > \lambda_2(f)$, $\lambda_1(g) > \lambda_2(g)$, alors si $h_f \neq h_g$ quitte à remplacer f et g par des itérés, le semi groupe engendré par f et g est libre. Ce résultat a été établi pour les transformations polynomiales de $\mathbf{A}^1_{\mathbf{C}}$ dans [BHPT21].

1.5.6 En dimension plus grande

Soit $d \geqslant 3$ un entier, dans [DF21] §6, Dang et Favre montrent que le degré dynamique d'une transformation polynomiale $f : \mathbf{A}^d_{\mathbf{C}} \to \mathbf{A}^d_{\mathbf{C}}$ tel que $\lambda_1(f)^2 > \lambda_2(f)$ est un *nombre* algébrique de degré $\leqslant d$. Pour se faire ils construisent une valuation propre de f centrée à l'infini à l'aide de l'analyse spectral de l'opérateur f^* sur un espace $N^1_\Sigma(\mathcal{X})$ qui est un analogue de l'espace de Picard-Manin en dimension 2. Ils utilisent ensuite l'inégalité d'Abhyankhar (voir [Abh56]) de la façon suivante : Si ν_* est une valuation propre de f, i.e $f_*\nu_* = \lambda_1\nu_*$, alors f_* induit une application linéaire sur $\Gamma_{\nu_*} \otimes \mathbf{Q}$ où Γ_{ν_*} est le groupe des valeurs de ν_*. L'inégalité d'Abhyankhar affirme que $\dim_{\mathbf{Q}} \Gamma_{\nu_*} \otimes \mathbf{Q} \leqslant d$. Ainsi, λ_1 est valeur propre d'une matrice $d \times d$ à coefficients rationnels, donc un nombre algébrique de degré $\leqslant d$.

J'affirme que la construction de la valeur propre dans le cas des surfaces affines que j'établis dans ce mémoire se généralise en dimension plus grande. En particulier, les sections 3.6 et 3.7 s'appliquent directement en toute dimension. La construction de la valuation propre provient alors d'un équivalent du théorème 4.1.16 où l'espace $\mathrm{L}^2(\mathrm{X}_0)$ doit être remplacé par son analogue $N^1_\Sigma(\mathcal{X})$. On peut alors appliquer l'inégalité d'Abhyankhar et énoncer le résultat suivant :

si X_0 est une variété affine de dimension $d \geqslant 3$ sur un corps $\mathbf{k}$ algébriquement clos de caractéristique nulle telle que

- $\mathbf{k}[X_0]^\times = \mathbf{k}^\times$;
- *Pour toute complétion X de X_0, $\mathrm{Pic}^0(X) = 0$;*

Si $f : X_0 \to X_0$ est un endomorphisme tel que $\lambda_1(f)^2 > \lambda_2(f)$, alors $\lambda_1(f)$ est un nombre *algébrique de degré $\leqslant d$.*

CHAPTER 2

INTRODUCTION

An affine variety X_0 over an algebraically closed field $\mathbf{k}$ is a subspace of $\mathbf{k}^N$ defined by polynomial equations. A polynomial endomorphism of X_0 is a polynomial transformation of $\mathbf{k}^N$ that preservers X_0 in the sense that $f(X_0) \subset X_0$. When the dimension of X_0 is 2, we say that X_0 is an affine surface. The goal of my thesis is to study the dynamical system given by an affine surface X_0 and $f : X_0 \to X_0$ a polynomial endomorphism of X_0. The different questions one can ask are: are there dense orbits or Zariski-dense orbits ? If the orbit of a point goes to infinity, can we control the speed of divergence ? Is there a lot of periodic orbits ? Can we construct interesting invariant probability measures ? To answer these questions, I use valuative techniques. The dynamical system (X_0, f) induces a dynamical system $(\mathcal{V}_\infty, f_*)$ where $\mathcal{V}_\infty$ is the space of valuations centered at infinity of X_0. The study of this dynamical system is the main goal of this memoir and it will allow to answer the questions mentioned above.

2.1 Endomorphisms

2.1.1 Dynamical degrees

2.1.1.1 Polynomial transformations of the complex affine plane

A polynomial transformation f of $\mathbf{C}^N$ is given by N polynomials $f_i \in \mathbf{C}[x_1, \cdots, x_N]$ such that $f = (f_1, \cdots, f_N)$. The *degree* of f is defined as the maximum of the degrees of the $f_i' s$; we denote it by $\deg f$. Let f^k be the k-th iterate of f. When we iterate f, the degree of the formulas of f^k must typically grow exponentially. It is therefore natural to consider the following quantity:

$$\lambda_1(f) := \lim_k \left(\deg f^k\right)^{1/k}, \tag{2.1}$$

introduced in [RS97], which we call the *first dynamical degree* of f. The authors show that this quantity is well defined. We can define the first dynamical degree of any rational transformation of the projective space $\mathbf{P}^N_{\mathbf{C}}$ with a similar definition.

2.1.1.2 General definitions

Let X be a smooth projective variety over an algebraically closed field and let d be its dimension. For d Cartier divisors $D_1, \cdots, D_d$ of X we can define the intersection product $D_1 \cdots D_d \in \mathbf{Z}$ (see [Laz04]). If $f : X \dashrightarrow X$ is a dominant rational transformation of X, we define for $0 \leqslant l \leqslant d$ the l-th dynamical degree of f by

$$\lambda_k(f) := \lim_{n \to \infty} \left((f^n)^* H^k \cdot H^{d-k} \right)^{1/n}, \tag{2.2}$$

where H is an ample divisor over X. We can show that these quantities are well defined and do not depend on the choice of H. In particular, $\lambda_0(f) = 1$. Furthermore, the dynamical degrees are birational invariants: if $\varphi : X \dashrightarrow Y$ is a birational map, then

$$\lambda_l(f) = \lambda_l(\varphi \circ f \circ \varphi^{-1}), \quad \forall 0 \leqslant l \leqslant d. \tag{2.3}$$

We have that $\lambda_d(f)$ is the topological degree of f. The Khovanskii-Teissier inequalities (see [Gro90], [DN05]) imply that the sequence $(\lambda_l)_{0 \leqslant l \leqslant d}$ is log-concave; i.e

$$\frac{\log \lambda_{l-1} + \log \lambda_{l+1}}{2} \leqslant \log \lambda_l, \quad \forall 1 \leqslant l \leqslant d-1. \tag{2.4}$$

In particular, one has $\forall 1 \leqslant l \leqslant d, \lambda_1(f)^l \geqslant \lambda_k(f)$.

Let X_0 be a smooth affine variety of dimension d and $f : X_0 \to X_0$ an endomorphism of X_0. We define the dynamical degrees of f as follows. A *completion* of X_0 is a smooth projective variety X equipped with an open immersion $\iota : X_0 \hookrightarrow X$ such that $\iota(X_0)$ is dense in X. The endomorphism f induces a dominant rational transformation of X via $\widetilde{f} = \iota \circ f \circ \iota^{-1}$ and we define the dynamical degrees

$$\lambda_l(f) := \lambda_l(\widetilde{f}). \tag{2.5}$$

As the dynamical degrees are birational invariants, these quantities do not depend on the choice of the completion X. In particular, if $X_0 = \mathbf{k}^N$ and $X = \mathbf{P}^N_{\mathbf{k}}$ we recover the definition of the first dynamical degree defined in the first paragraph.

The data of these dynamical degrees gives information on the dynamical system. For example over $\mathbf{C}$, Dinh and Sibony showed in [DS03] that for all dominant rational transformation $f : X \dashrightarrow X$

$$h_{\mathrm{top}}(f) \leqslant \max_{0 \leqslant l \leqslant d} \log(\lambda_l) \tag{2.6}$$

where h_{top} is the topological entropy of f, Gromov showed this result for endomorphisms of $\mathbf{P}^N$ in [Gro03]. Yomdin showed in [Yom87] that we have an equality if f is an endomorphism. Recently, Favre, Truong and Xie showed in [FTX22] that the inequality (2.6) still holds in the non archimedean case; however the equality does not hold even for endomorphisms.

2.1.2 Dynamical degrees on projective surfaces

A natural question is to ask what numbers can appear as the first dynamical degree of a rational transformation of a projective surface. We first mention the following result due to Bonifant and Fornaess in [BF00] for $\mathbf{P}^N_{\mathbf{C}}$ and generalised by Urech

Théorème 2.1.4 ([Ure16])**.** *The set*

$$\{\lambda_1(f)\} \tag{2.7}$$

where f runs through the set of rational transformations over every projective variety over every field, is countable.

In 2021, Bell, Diller and Jonsson showed in [BDJ20] that there exists a dominant rational transformation $\sigma : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$ such that $\lambda_1(\sigma)$ is transcendental. The authors with Krieger showed in [BDJ20] this example can be generalised to give an example of a birational transformation of $\mathbf{P}^N, N \geqslant 3$ with a transcendental first dynamical degree. However in dimension 2, there are strong constraints on $\lambda_1(f)$ for f birational. In [DF01], Diller and Favre showed that the first dynamical degree of a birational transformation of a projective surface is an algebraic integer. More precisely, it is a Salem or a Pisot number. In [BC13], Blanc and Cantat obtained the following results

Theorem 2.1.1. *Let X be a smooth projective surface over an algebraically closed field.*

(1) Let $f : X \dashrightarrow X$ be a birational transformation such that $\lambda_1(f)$ is a Salem number, then there exists a birational map $\varphi : X \dashrightarrow Y$ such that $\varphi \circ f \circ \varphi^{-1}$ is an automorphism of Y.

(2) If X is rational over a field $\mathbf{k}$, then the set $\Lambda(X) := \{\lambda_1(f) | f \in \mathrm{Bir}(X)\} \subset \mathbf{R}$ is well ordered. It is closed if $\mathbf{k}$ is not countable.

In particular, $\Lambda(X)$ is an ordinal and Bot shows in [Bot22] that this ordinal is exactly ω^ω where ω is the ordinal of the natural integers. However, we cannot hope to get an information of the degree of the algebraic numbers obtained. Indeed, Bedford, Kim and McMullen have given in [BK06] and [McM07] examples of birational transformations of projective surfaces with first dynamical degree an algebraic integer of arbitrary large degree. In particular, Theorem 1.1 of [McM07] states that for all $d \geqslant 10$ we can find a smooth projective surface with an automorphism with first dynamical degree an algebraic integer of degree d.

2.1.3 Dynamical degrees of endomorphisms of affine surfaces

In my thesis, I consider endomorphisms of affine surfaces. The first example of an affine surface is the complex affine plane $\mathbf{C}^2$. An endomorphism is then a polynomial transformation. Even in that case, the first dynamical degree is not necessarily an integer. Indeed, let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.8}$$

be a matrix with nonnegative integer coefficients such that $ad - bc \neq 0$. Consider the following monomial transformation

$$f(x,y) = \left(x^a y^b, x^c y^d\right), \tag{2.9}$$

then f^N is the monomial transformation where the monomials are given by the coefficients of A^N and $\lambda_1(f)$ is equal to the spectral radius of A. Hence, $\lambda_1(f)$ is an algebraic integer of degree 2 because it satisfies the equation

$$\lambda_1(f)^2 - \mathrm{Tr}(A)\lambda_1(f) + \det(A) = 0. \tag{2.10}$$

Thus, there exist polynomial transformations f of the affine plane with $\lambda_1(f)$ an integer or an algebraic integer of degree 2. Favre and Jonsson showed that these are the only two possibilities.

Theorem 2.1.2. *[FJ07] Let $f : \mathbf{C}^2 \to \mathbf{C}^2$ be a dominant polynomial transformation, then $\lambda_1(f)$ is an algebraic integer of degree $\leqslant 2$.*

The first result of my thesis is to extend this result to all affine surfaces, in any characteristic. Even if there are affine surfaces for which the semigroup of endomorphism can change drastically. For example, Blanc and Dubouloz, in [BD13], build smooth affine surfaces with a big group of automorphisms, much bigger than the one of the affine plane. Bot used this construction to show the existence of smooth complex rational affine surfaces with uncountably many real forms (see [Bot23]). The result in my thesis show that even though structure wise, these groups are a lot more complicated; from the point of view of the dynamics of a single element, this is not the case.

Theorem A. *Let X_0 be a normal affine surface over a field* $\mathbf{k}$*. If $f : X_0 \to X_0$ is a dominant endomorphism, then $\lambda_1(f)$ is an algebraic integer of degree $\leqslant 2$.*

The proof uses valuative techniques which I describe in the next section. If $\operatorname{char}\mathbf{k} = 0$, I also obtain results on the dynamics of f. I will give a precise statement in the case of automorphisms (see Theorem C).

2.2 Valuations, Divisors at infinity and dynamics

2.2.1 Existence of an eigenvaluation

Let A be the ring of regular functions of a normal affine surface X_0 over an algebraically closed field $\mathbf{k}$. A *valuation* is a map $\nu : A \to \mathbf{R} \cup \{\infty\}$ such that

1. $\nu(PQ) = \nu(P) + \nu(Q)$;
2. $\nu(P+Q) \geqslant \min(v(P), \nu(Q))$;
3. $\nu(0) = \infty$;
4. $\nu_{|\mathbf{k}^\times} = 0$

Two valuations ν are μ are *equivalent* if there exists $t > 0$ such that $\nu = t\mu$. For example, if X is a completion of X_0, for all irreducible curve $E \subset X$, the map ord_E defined by $\operatorname{ord}_E(P)$ being the order of vanishing of P along E is a valuation. Any valuation of the form $\lambda \operatorname{ord}_E$ with $\lambda > 0$ is called *divisorial*. If f is an endomorphism of X_0, then f induces a ring homomorphism $f^* : A \to A$. We can then define the pushforward $f_*\nu$ of a valuation ν by

$$f_*\nu(P) = \nu(f^*P). \tag{2.11}$$

We say that a valuation is *centered at infinity* if there exists $P \in A$ such that $\nu(P) < 0$. If X is a completion of X_0, the divisorial valuations centered at infinity are exactly the one corresponding to the irreducible components of $X \setminus X_0$. Let $\mathcal{V}_\infty$ the set of valuations centered at infinity and $\widehat{\mathcal{V}}_\infty$ the set of valuations centered at infinity modulo equivalence. Suppose for the sake of simplicity that f is an automorphism of X_0, then f_* induces a bijection of $\mathcal{V}_\infty$ and of $\widehat{\mathcal{V}}_\infty$ which will in fact be a homeomorphism for a topology that will be described in this memoir.

If X_0 is the complex affine plane, then Favre and Jonsson proved the existence of a valuation $\nu_* \in \mathcal{V}_\infty$ such that $f_*\nu_* = \lambda_1(f)\nu_*$. Such a valuation is called an *eigenvaluation* of f. To do so, they show in [FJ04] that $\widehat{\mathcal{V}}_\infty$ has a real tree structure and f_* is compatible with this structure. The existence of ν_* follows from a fixed point theorem on trees. The existence of this eigenvaluation has a big impact on the dynamics of f. In particular, it allows one to find a good completion X of $\mathbf{C}^2$ which admits an attracting fixed point of f at infinity. Xie uses this construction to prove the conjecture of Zariski-dense orbits and the dynamical Mordell-Lang conjecture for polynomial endomorphisms of the complex affine plane ([Xie17b]). Jonsson and Wulcan use these techniques to build canonical heights for polynomial endomorphisms of the complex affine plane with small topological degree in [JW12].

Theorem B. *Let X_0 be a normal affine surface over an algebraically closed field* $\mathbf{k}$ *(of any characteristic) and let f be a dominant endomorphism of X_0. Suppose that*

1. $\mathbf{k}[X_0]^\times = \mathbf{k}^\times$.
2. *For all completions X of X_0,* $\mathrm{Pic}^0(X) = 0$.
3. $\lambda_1(f)^2 > \lambda_2(f)$.

Then, there exists an eigenvaluation ν_*, unique up to equivalence, of f such that

$$f_*(v) = \lambda_1(f)\nu_*. \tag{2.12}$$

The techniques I use do not use the global geometry of $\widehat{\mathcal{V}}_\infty$ because it not necessarily a tree anymore. If X is a completion of X_0, I show that for any valuation ν centered at infinity, one can associate a unique divisor $Z_{\nu,X}$ of X supported outside of X_0. Furthermore if Y is another completion of X_0, there is a compatibility relation between $Z_{\nu,X}$ and $Z_{\nu,Y}$ (see Proposition 3.6.6). This construction involves the space of Eicard-Manin of X_0. The spectral analysis of the operators f_*, f^* induced by f on this space (see [BFJ08, Can11]) allows one to construct the eigenvaluation ν_* and show its uniqueness. This process is similar to the techniques of [DF21] §6.

2.2.2 Discussion of the assumptions of the Theorem

The assumptions of Theorem B may seem arbitrary but they are not restrictive. Indeed, if assumption (1) or (2) is not satisfied, then one can show that f preserves a fibration over a [1]*quasi-abelian*. We can decompose the dynamics of f with this fibration and it becomes easier

1. a quasi-abelian variety is an algebraic group such that there exists an algebraic torus T and an abelian variety A such that the sequence of algebraic groups $0 \to T \to X \to A \to 0$ is exact.

to study.

If Assumption (3) is not satisfied, then we have $\lambda_1(f)^2 = \lambda_2(f)$. In that case, $\lambda_1(f)$ is automatically an algebraic integer of degree $\leqslant 2$ because $\lambda_2(f)$ is the topological degree of f, hence an integer. In the case of the complex affine plane, Favre and Jonsson manage to classify all polynomial transformations of the complex affine plane for which $\lambda_1^2 = \lambda_2$: either they preserve a rational fibration, or there exists a completion X of $\mathbf{A}^2_{\mathbf{C}}$ with at most quotient singularities at infinity such that f extends to an endomorphism of X. I expect that such a classification should exist in general, all the examples I have studied up until now satisfy this dichotomy. One can notice that in the invertible case, such a classification exists: By [Giz69] and [Can01], every birational transformation $\sigma : X \to X$ of a smooth projective surface such that $\lambda_1(\sigma) = 1$ lifts to an automorphism or preserves a rational or elliptic fibration.

2.2.3 Statement of the theorem in the case of automorphisms

In characteristic zero, the existence of the eigenvaluation has an impact on the dynamics of f. I show that for every endomorphism, there exists a completion X of X_0 with an attracting fixed point of f at infinity. In the case of loxodromic automorphism (i.e with $\lambda_1 > 1$) I show the following

Theorem C. *Let X_0 be a normal affine surface over $\mathbf{C}$ such that $\mathbf{C}[X_0]^\times = \mathbf{C}^\times$. If f is an automorphism of X_0 such that $\lambda_1(f) > 1$, then there exists a completion X of X_0 such that*

1. *f admits an attracting fixed point $p \in X(\mathbf{C}) \setminus X_0(\mathbf{C})$ at infinity.*
2. *An iterate of f contracts $X \setminus X_0$ to p.*
3. *There exists local analytic coordinates centered at p such that f is locally of the form*
   - *(a)*
   $$f(z,w) = (z^a w^b, z^c w^d) \tag{2.13}$$
   with a,b,c,d integers $\geqslant 1$, in that case $\lambda_1(f)$ is the spectral radius of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. In particular, $\lambda_1(f) \in \mathbf{R} \setminus \mathbf{Q}$, it is an algebraic integer of degree 2.
   - *(b) or*
   $$f(z,w) = (z^a, \lambda z^c w + P(z)) \tag{2.14}$$
   with $a \geqslant 2, c \geqslant 1$ and $P \not\equiv 0$ a polynomial, in that case $\lambda_1(f) = a$ is an integer.

4. *The attracting fixed points of f and f^{-1} are distinct.*

5. *The local normal form of f^{-1} at its attracting fixed point is the same as f.*

The cases (3)(a) et (3)(b) are mutually exclusive in the following way

Theorem D. *Let X_0 be a normal affine surface over $\mathbf{C}$ such that $\mathbf{C}[X_0]^\times = \mathbf{C}^\times$ and $f \in \mathrm{Aut}(X_0)$ a loxodromic automorphism. We have the following dichotomy*

- *If $\lambda_1(f) \in \mathbf{Z}_{\geqslant 0}$, then for any loxodromic automorphism g of X_0, we have $\lambda_1(g) \in \mathbf{Z}_{\geqslant 0}$ and the local normal form of g at its attracting fixed point is given by* (2.14)*.*

- *If $\lambda_1(f) \notin \mathbf{Z}_{\geqslant 0}$ then it is an algebraic integer of degree 2 and this holds for any loxodromic automorphism g of X_0. In particular, the local normal form of g at its attracting fixed point is given by* (2.13)*.*

We give two examples: the affine plane and the Markov surface (see §2.2.3.2). Theorem C and D show that it suffices to understand these two examples to understand the dynamics of a single automorphism of an affine surface.

2.2.3.1 The affine plane

Suppose that $X_0 = \mathbf{A}^2_{\mathbf{C}}$, consider the automorphism

$$f(x,y) = (y + x^2, x). \tag{2.15}$$

It is a *Hénon* automorphism and we have $\lambda_1(f) = 2$. Let $X = \mathbf{P}^2_{\mathbf{C}}$ be a completion of $X_)$ with homogeneous coordinates X, Y, Z such that $x = X/Z$ et $y = Y/Z$. The birational transformation induced by f has fixed point $p_+ = [1:0:0]$ and an indeterminacy point $p_- = [0:1:0]$. The line at infinity $\{Z = 0\}$ is contracted by f to p_+ and by f^{-1} to p_-. Let (u, v) be the local coordinates at p_+ given by $u = Y/X$ et $v = Z/X$, one has

$$f(u,v) = \left(\frac{v}{1+uv}, \frac{v^2}{1+uv}\right). \tag{2.16}$$

And there exits a local analytic change of coordinates such that f has the normal form (2.14) (see [Fav00] §2).

2.2.3.2 The Markov surface

Consider the Markov surface $\mathcal{M}_0 \subset \mathbf{A}^3_{\mathbf{C}}$ given by the equation

$$x^2 + y^2 + z^2 = xyz. \tag{2.17}$$

It is a normal affine surface with a quotient singularity at $(0,0,0)$. We will describe in detail its properties in §2.4. A natural completion of $\mathcal{M}_0$ is the projective surface $X \subset \mathbf{P}^3_{\mathbf{C}}$ defined by the Zariski closure of $\mathcal{M}_0$ in $\mathbf{P}^3_{\mathbf{C}}$. The equation of X is

$$T(X^2 + Y^2 + Z^2) = XYZ. \tag{2.18}$$

We see that $X \backslash \mathcal{M}_0$ has equation

$$T = 0, XYZ = 0. \tag{2.19}$$

Thus it is a triangle of 3 rational curves. By Theorem 3.1 of [Can09], if f is a loxodromic automorphism of $\mathcal{M}_0$ algebraically stable over X then f admits an attracting fixed point $p_+ \in X \backslash \mathcal{M}_0$ which is one of the vertex of the triangle and one indeterminacy point $p_- \in X \backslash \mathcal{M}_0$ which is another vertex of the triangle. Furthermore, f admits a local normal form of monomial type (i.e given by (2.13)) at p_+.

Remark 2.2.1. We see that for all completion X the affine plane, the dual graph of $X \backslash \mathbf{A}^2_{\mathbf{C}}$ is a tree. However, in the case of $\mathcal{M}_0$ the dual graph of $X \backslash \mathcal{M}_0$ retracts to a circle for every completion X. We will show in fact that every affine surface (with a loxodromic automorphism) satisfies this dichotomy. It is the dichotomy of the geometry of these dual graphs that gives the dichotomy on the dynamics (see Theorem 4.4.4).

2.3 Dynamics of automorphisms of affine surfaces

2.3.1 Dynamics of Hénon maps: Green functions

Let $\mathrm{Aut}(\mathbf{A}^2_{\mathbf{C}})$ be the group of polynomial automorphisms of the complex affine plane. The affine transformations are examples of such automorphisms. Here is another example: let

$$f(x,y) = (x, y + P(x)) \tag{2.20}$$

where P is a polynomial. The automorphism f preserves the pencil of lines $x = \alpha$ and acts by translation on these lines, the vector of translation is given by $P(x)$. Such an automorphism is called *elementary*. We denote by E the set of all elementary automorphisms of $\mathbf{A}^2_{\mathbf{C}}$, they form a group isomorphic to $(\mathbf{C}[x],+))$. Jung's theorem ([Jun42]) states that $\mathrm{Aut}(\mathbf{C}^2)$ has the structure of an amalgamated product

$$\mathrm{Aut}(\mathbf{A}^2_{\mathbf{C}}) = \mathrm{Aff}(\mathbf{A}^2_{\mathbf{C}}) *_S E \tag{2.21}$$

where $S = \mathrm{Aff}(\mathbf{A}^2_{\mathbf{C}}) \cap E$.

An automorphism of Henon type is an automorphism f which is not conjugated to an element of $\mathrm{Aff}(\mathbf{A}^2_{\mathbf{C}})$ nor to an element of E. They are characterised by the condition $\lambda_1(f) > 1$. An example of automorphism of Henon type is the following which will use later on

$$f(x,y) = (y + x^2, x). \tag{2.22}$$

The extension of f to $\mathbf{P}^2$ has a fixed point at infinity $p_+ = [1:0:0]$ and an indeterminacy point $p_- = [0:1:0]$. The line at infinity is contracted by f to p_+. Analogously, p_+ is the only indeterminacy point of f^{-1} and p_- is a fixed point of f^{-1} to which the line at infinity is contracted by f^{-1}. An automorphism h is regular if the indeterminacy points of h and h^{-1} are distinct. In particular f is regular and every automorphism of Henon type can be conjugated to a regular one [FM89]. For all automorphism of Henon type h, $\lambda_1(h)$ is an integer which we denote by d, in particular $\lambda_1(f) = 2$.

Consider the norm $||(x,y)|| = \max(|x|,|y|)$ on $\mathbf{C}^2$. If h is a regular automorphism of Henon type, we can define the Green functions of h (see [FM89], [BS91a] and their references)

$$G^+(p) := \lim_N \frac{1}{d^N} \log^+ \left|\left|h^N(p)\right|\right|, \quad G^-(p) := \lim_N \frac{1}{d^N} \log^+ \left|\left|h^{-N}(p)\right|\right| \tag{2.23}$$

where $\log^+ = \max(0, \log)$. We have the following properties (see [BS91a]).

1. G^+ is well defined, continuous and plurisubharmonic over $\mathbf{C}^2$,

2. $G^+ \circ h = dG^+$,

3. the map $p \mapsto G^+(p) - \log^+(||p||)$ extends to a continuous function over $\mathbf{P}^2 \setminus p_-$.

4. $G^+(p) = 0$ if and only if the forward orbit $(h^N(p))_{N \geqslant 0}$ is bounded.

The function G^- satisfies similar properties. We define the Green currents $T^+ = dd^c G^+$ and

$T^- = dd^c G^-$. These are positive closed $(1,1)$-currents. The measure

$$\mu := T^+ \wedge T^- \tag{2.24}$$

is then well defined because G^+, G^- are continuous. It is of finite total mass, thus we can suppose that it is a probability measure. We call it the *equilibrium measure* of h. It is h-invariant and its support is contained in the Julia set of h.

We define the following Green function

$$G := \max(G^+, G^-) \tag{2.25}$$

which satisfies the following properties

1. G is continuous, plurisubharmonic over $\mathbf{C}^2$ and the uniform limit of

$$\max\left(\frac{1}{d^N}\log^+(||f^N(p)||), \frac{1}{d^N}\log^+(||f^{-N}(p)||)\right) \tag{2.26}$$

2. $p \mapsto G(p) - \log^+ ||p||$ extends to a continuous function over $\mathbf{P}^2$.

3. $G(p) = 0$ if and only if the $\mathbf{Z}$-orbit $(f^N(p))_{N\in\mathbf{Z}}$ is bounded.

2.3.2 Dynamics of automorphisms of affine surfaces

Using theorem C, I show

Theorem E. *Let X_0 be a normal affine surface over $\mathbf{C}$, let X be a completion of X_0 that satisfy Theorem C. Let $X \hookrightarrow \mathbf{P}^N$ be an embedding of X which induces an embedding of X_0 into $\mathbf{C}^N$ and let $||\cdot||$ be a norm over $\mathbf{C}^N$. Let f be an automorphism of X_0 such that $\lambda_1(f) > 1$, the Green function*

$$G^+(p) := \lim_N \frac{1}{\lambda_1^N}\log^+(||f^N(p)||) \tag{2.27}$$

satisfies the following properties

1. *G^+ is well defined, continuous and plurisubharmonic over $X_0(\mathbf{C})$.*

2. *$G^+ \circ f = \lambda_1 G^+$*

3. *G^+ has logarithmic growth (see Proposition 5.2.5).*

4. $G^+(p) = 0$ *if and only if the forward orbit* $(f^N(p))_{N\geqslant 0}$ *is bounded.*

We can then consider the function $G = \max(G^+, G^-)$ which will satisfy similar properties as in the Henon case. There is however one major difference. In general, the maximum of two functions of logarithmic growth is not of logarithmic growth. There is a difficulty here. It turns out we get two distinct behaviour: if $\lambda_1(f) \in \mathbf{Z}_{\geqslant 0}$, G is again of logarithmic growth and everything works as in the Henon case. If $\lambda_1(f) \notin \mathbf{Z}_{\geqslant 0}$, then G is not of logarithmic growth and it is not the right function to consider. We then need to use the recent work of Yuan and Zhang on adelic line bundles over quasiprojective varieties, I will be more precise in the following section.

2.3.3 Dynamics at non archimedean places

Let $\mathbf{K}$ be a number field. An *absolute value* $|\cdot|$ over $\mathbf{K}$ is a function $|\cdot| : \mathbf{K} \to \mathbf{R}_+$ which satisfies

- $|x| = 0 \Leftrightarrow x = 0$,
- $\forall x, y \in \mathbf{K}, |xy| = |x| \cdot |y|$,
- $\forall x, y \in \mathbf{K}, |x+y| \leqslant |x| + |y|$.

Two absolute values $|\cdot|_1, |\cdot|_2$ are equivalent if $|\cdot|_1 = |\cdot|_2^s$ for some $s > 0$. A *place* is an equivalence class of absolute values, we denote by $\mathcal{M}(\mathbf{k})$ the set of places of $\mathbf{K}$. If $|\cdot|$ is an absolute value of $\mathbf{K}$, we can consider the completion of $\mathbf{K}$ with respect to $|\cdot|$. This completion depend only on the place v of $|\cdot|$, we denote it by $\mathbf{K}_v$. The absolute value $|\cdot|$ then extends to $\mathbf{K}_v$ and admits a natural extension to $\overline{\mathbf{K}}_v$. We denote by $\mathbf{C}_v$ the completion of $\overline{\mathbf{K}}_v$ with respect to $|\cdot|$. This construction depends only on the place v. We say that $|\cdot|$ is *non archimedean* if it satisfies the following inequality

$$\forall x, y \in \mathbf{K}, |x+y| \leqslant \max(|x|, |y|). \tag{2.28}$$

A place v is non archimedean if one of its representatives is. For all archimedean place v, we have $\mathbf{C}_v = \mathbf{C}$. The results of §2.3.1 have analogues when $\mathbf{C}$ is replaced by an algebraically closed complete field $\mathbf{C}_v$. Indeed, Kawaguchi showed in [Kaw09] that the Green function of an automorphism of Henon type is well defined also in the non archimedean case. If $\mathbf{C}_v$ is non archimedean, the Green function $G = \max(G^+, G^-)$ induces a semipositive adelic line bundle on the Berkovich analytification of $\mathbf{P}^2_{\mathbf{C}_v}$ which we denote by $(\mathbf{P}^2_{\mathbf{C}_v})^{an}$ (see [Zha93] for the definition). The equilibrium measure is a positive measure over $(\mathbf{P}^2_{\mathbf{C}_v})^{an}$, it is constructed in [Cha03]. It is worth noting that recent work of Chambert-Loir and Ducros [CD] allows one to construct the currents $T^{\pm} = dd^cG^{\pm}$ and to define the equilibrium measure in the same way as in the complex case $\mu = T^+ \wedge T^-$. Furthermore, Lee shows in [Lee13] that the Galois orbits of any [2]generic sequence of periodic points equidistributes with respect to the measure $\mu = T^+ \wedge T^-$ at every place. This uses the equidistributes theorem of Yuan in [Yua08].

I also show an analogue of Theorem C in the non archimedean case. We define also the functions G^+, G^-, G in that case. However, the difficulties mentioned at the end of §2.3.2 remain. If $\lambda_1(f) \in \mathbf{Z}_{\geqslant 0}$, then the data of the Green functions (G_v) for every place v of $\mathbf{K}$ induces a semipositive adelic line bundle (cf [Zha93]) over a completion X of X_0 and the arithmetic equidistribution theorem of Yuan applies.

Now, if $\lambda_1(f) \notin \mathbf{Z}_{\geqslant 0}$, we cannot apply the theory of adelic line bundles over the completion X. The right point of view is to consider not just one completion of X_0 but all of them. This is the point of view developed by Yuan and Zhang in [YZ22]. The authors define the notion of adelic line bundles over a *quasiprojective* variety U as a limit of adelic line bundles over completions of U. They show in this context an arithmetic equidistribution theorem similar to the theorem of Yuan. In my memoir, I state the following conjecture (see Conjecture F):

Conjecture F. *The data of (G_v^+) and (G_v^-) at every place v of $\mathbf{K}$ induces two nef f-invariant adelic line bundles over the quasiprojective variety X_0. In particular, we can define the equilibrium measure μ_v of f at every place as the probability measure proportional to $dd^cG_v^+ \wedge dd^cG_v^-$ and the Galois orbits of any generic sequence of periodic points of f equidistributes with respect to μ_v at every place v.*

I believe that the results established in this memoir and the work of Yuan and Zhang will allow one to prove this conjecture using similar techniques as in §4 of [YZ17](see §2.5.1).

2.3.4 Automorphisms sharing infinitely many periodic points

If X_0 is a normal affine surface over $\mathbf{K}$ a number field, and f a loxodromic automorphism of X_0, we can apply the results of the previous section at both archimedean and non archimedean places. We then get an equilibrium measure $\mu_{f,v}$ for f at every place v of $\mathbf{K}$. Using the techniques of arithmetic equidistribution mentioned in the previous paragraph, I show the following result.

Theorem G. *Let X_0 be a normal affine surface defined over a number field $\mathbf{K}$. If f, g are two loxodromic automorphisms of X_0 such that $\lambda_1(f) \in \mathbf{Z}_{\geqslant 0}$, the following are equivalent*

2. A sequence is generic if no subsequence is contained in strict closed subvariety.

1. $\mathrm{Per}(f) \cap \mathrm{Per}(g)$ *est Zariski-dense.*

2. $\forall v \in \mathcal{M}(\mathbf{K}), \mu_{v,f} = \mu_{v,g}$

3. $\mathrm{Per}(f) = \mathrm{Per}(g)$.

In the case $\lambda_1(f) \notin \mathbf{Z}_{\geqslant 0}$, admitting Conjecture F, we have that if $\mathrm{Per}(f) \cap \mathrm{Per}(g)$ *is Zariski-dense, then* $\forall v \in \mathcal{M}(\mathbf{K}), \mu_{f,v} = \mu_{g,v}$.

Using similar methods, these kind of results were first obtained by Baker, DeMarco in [BD11a] for endomorphisms of $\mathbf{P}^1$ of degree $\geqslant 2$ over $\mathbf{C}$ and then generalised by Yuan and Zhang for all polarisable endomorphisms of $\mathbf{P}^m$ over a number field in [YZ17] and recently in [YZ21] over any field of characteristic zero. In [CD20], Cantat and Dujardin use these same tools of arithmetic dynamics to show rigidity results on groups of automorphisms of projective surfaces.

Conjecture F is not enough to show the equality $\mathrm{Per}(f) = \mathrm{Per}(g)$ because the proof uses an arithmetic version of the Hodge index theorem which has not been shown yet for adelic line bundles over quasiprojective varieties (see Theorem 5.1.20).

2.4 A rigidity result for Markov surfaces

In [DF17] Dujardin and Favre show a stronger result than Theorem G. They obtain that if two automorphisms of Henon type satisfy one of the assertions of Theorem G, then f and g share common iterates: there exist integers $M, N \in \mathbf{Z}$ such that $f^N = g^M$. This rigidity result cannot be true for any affine surface. Indeed, if $X_0 = \mathbf{C}^\times \times \mathbf{C}^\times$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$ be such that $(\mathrm{Tr}A)^2 > 4$, we define the automorphism

$$f_A(x,y) = (x^a y^b, x^c y^d) \tag{2.29}$$

If $\mathbb{S}^1 = \{z \in \mathbf{C} \,|\, |z| = 1\}$, then $\mathbb{S}^1 \times \mathbb{S}^1$ is an f_A-invariant compact subset. Ii is the Julia set of f_A and the periodic points of f_A are exactly of the form$(\omega_1, \omega_2) \in \mathbb{S}^1 \times \mathbb{S}^1$ where ω_1, ω_2 are roots of unity. Hence, every automorphism of this form satisfies Theorem G but they don't share common iterates.

Let $D \in \mathbf{C}$, we define the Markov surface $\mathcal{M}_D$ as a surface in $\mathbf{C}^3$ defined by

$$x^2 + y^2 + z^2 = xyz + D \tag{2.30}$$

This family of surfaces appears in different fields of mathematics (see [Can09]). If $\mathbb{T}^1$ is the punctured torus, its fundamental group $\pi := \pi_1(\mathbb{T}^1)$ is a free group with two generators that we denote by a and b. We can look at the character variety

$$\mathcal{X} := \mathrm{Hom}(\pi, \mathrm{SL}_2(\mathbf{C}))//\mathrm{SL}_2(\mathbf{C}) \tag{2.31}$$

where the action of $\mathrm{SL}_2(\mathbf{C})$ is via conjugation and $//$ is the quotient from Geometric Invariant Theory (GIT). Denote by $[a,b] := aba^{-1}b^{-1}$ the commutator of a and b. Let $\rho \in \mathcal{X}$, if we define $x = \mathrm{Tr}\,\rho(a), y = \mathrm{Tr}\,\rho(b), z = \mathrm{Tr}\,\rho(ab)$, then we get that

$$\mathcal{X} \to \mathbf{A}^3_{\mathbf{C}} \tag{2.32}$$

$$\rho \mapsto (x,y,z) \tag{2.33}$$

is an isomorphism. This is a result of Fricke (see [Gol09]). Furthermore, we have

$$x^2 + y^2 + z^2 = xyz + \mathrm{Tr}(\rho([a,b])) + 2. \tag{2.34}$$

Thus, the algebraic variety $\mathcal{X}$ has a foliation of surfaces given by the family of the Markov surfaces and the surface $\mathcal{M}_D$ is a fiber of the regular function $\rho \mapsto \mathrm{Tr}(\rho([a,b]))$.

Theorem H. *Let $D = 0$ or $D = 2 - 2\cos(2\pi/q)$ with $q \geqslant 2$. If f, g are two loxodromic automorphisms of $\mathcal{M}_D$, then admitting Conjecture F, the following are equivalent:*

1. $\mathrm{Per}(f) \cap \mathrm{Per}(g)$ *is Zariski-dense.*
2. $\mathrm{Per}(f) = \mathrm{Per}(g)$.
3. *f and g share common iterates: there exist $N, M \in \mathbf{Z}$ such that $f^N = g^M$.*

Conjecture F and Theorem G give the equality of the equilibrium measure of f and g. To show the result we use the theory of Fuchsian and quasi-Fuchsian representations to construct a saddle fixed point $q(f)$ at the boundary of the open subset of $\mathcal{M}_D(\mathbf{C})$ consisting of quasi-Fuchsian representations. This construction uses the double parametrisation theorem of Bers in [Ber60], its extension by Minsky in [Min99] and Thurston's theorem of hyperbolisation of 3-manifolds fibering over a circle (see [Ota96, McM96]). We then use techniques of currents in complex geometry, in particular the current of Ahlfors-Nevanlinna, to show that this saddle fixed point must belong to the support of $\mu_{\mathbf{C},f} = \mu_{\mathbf{C},g}$ which is a compact subset invariant by the group $\langle f,g\rangle$. Finally, we show that if f,g do not share common iterates, then the g-orbit of $q(f)$ must be unbounded using measured laminations theory and this is a contradiction.

2.5 Questions and future projects

2.5.1 Conjecture F

As mentioned in this introduction, I believe that Conjecture F can be shown using the results of this memoir. Namely, I show in Proposition 5.2.5 that the function G^+ is obtained via an iterating process starting from the Green function of any divisor. This iterating process applied to the theory of adelic line bundles over X_0 must yield a nef adelic line bundle in the sense of [YZ22]. Indeed, in the projective setting if f is a polarised endomorphism of a projective variety X and L an ample line bundle over X such that $f^*L = L^{\otimes d}$, Yuan and Zhang show in [YZ17] that for any adelic extension $\overline{L}$ of L, the sequence

$$\frac{1}{d^n}(f^n)^*\overline{L} \tag{2.35}$$

converges to a semipositive adelic line bundle $\overline{L}_f$ such that $f^*\overline{L_f} = d\overline{L_f}$. At the level of Green functions, this iterating process is the same as the one in Section 5.2 (see Propositions 5.2.5 and 5.2.12). Hence, I expect everything to work as well in this setting.

To obtain Theorem G, we will then need to show the arithmetic Hodge index theorem in the case of affine surfaces. I only need a weaker version of this theorem that I believe should be not too hard to show.

2.5.2 The work of Danilov and Gizatullin

We say that an affine surface X_0 is *completable by a zigzag* if there exists a completion X of X_0 such that $X \setminus X_0$ is a *zigzag*, that is a chain of smooth rational curves. The affine plane is completable by a zigzag but the Markov surface $\mathcal{M}_0$ is not for example. In [GD75], Danilov and Gizatullin study the group of automorphisms of an affine surface completable by a zigzag. They show that it acts on a tree which vertices are the completions where the boundary is a zigzag. If X_0 is completable by a zigzag, then the space of valuations centered at infinity $\widehat{\mathcal{V}_\infty}$ is also a tree on which $\mathrm{Aut}(X_0)$ acts. It will be interesting to compare the approach of Danilov and Gizatullin to the approach in my memoir. Note that the work of Gizatullin (see [Giz71b, Giz70, Giz71c]) prior to [GD75] are used in my memoir to study the dynamics of loxodromic automorphisms (see §4.4.1).

2.5.3 Dynamic complexity vs algebraic complexity of $\mathrm{Aut}(X_0)$

I showed in my thesis that the study of the dynamics of a loxodromic automorphism on an affine surface is similar either to the dynamics of an automorphism of Henon type, or to the dynamics of a loxodromic automorphism of the Markov surface. However, there exists affine surfaces with a much more complicated group of automorphism as shown by Blanc and Dubouloz in [BD13]. If X_0 is such a surface it will be interesting to apply the valuative techniques of this memoir tout a subgroup of automorphism of X_0. For example, if f and g are two loxodromic automorphisms such that every element of the subgroup $\Gamma = \langle f, g \rangle$ which is not the identity is loxodromic, what can we say about the set $\{\nu_*(h) : h \in \Gamma\} \subset \widehat{\mathcal{V}_\infty}$ where $v_*(h)$ is the eigenvaluation of h ? Can we recover the algebraic complexity of $\mathrm{Aut}(X_0)$ using valuative techniques ?

2.5.4 Arithmetic dynamics result using valuative techniques

Using the valuative techniques of Favre and Jonsson for the affine plane, Junyi Xie shows in [Xie17b] the Zariski dense orbit conjecture for polynomial endomorphism of the complex affine plane. This conjecture states that any endomorphism f admits a Zariski dense orbit if and only if it does not admit a non-constant invariant rational function. The proof uses the dynamics of f at infinity using the existence of an eigenvaluation. The author also shows in [Xie17a] the dynamical Mordell-Lang conjecture for polynomial endomorphism of the affine plane: if $x \in \mathbf{A}^2(\mathbf{C})$ and $C \subset \mathbf{A}^2_{\mathbf{C}}$ is a curve, then $\{n \geqslant 0 : f^n(x) \in C\}$ is a union of a finite set and a finite union of arithmetic progressions.

For these two conjectures, we can establish their analogues for any affine surface using the valuative techniques of this memoir if $\lambda_1^2 > \lambda_2$. For the equality case, Xie uses the classification of polynomial endomorphism satisfying $\lambda_1^2 = \lambda_2$ established by Favre and Jonsson. It is therefore necessary to establish such a classification in general. For now, the techniques in this memoir do not allow to treat the case $\lambda_1^2 = \lambda_2$. In particular, I do not know how to construct an eigenvaluation for an endomorphism f satisfying $\lambda_1(f)^2 = \lambda_2(f)$.

2.5.5 Green functions and canonical heights for small topological degrees

Let f be a polynomial endomorphism of the affine plane defined over a number field $\mathbf{K}$ such that $\lambda_1(f) > \lambda_2(f)$. In [FJ11] and [JW12] Favre, Jonsson and Wulcan use the existence of a unique eigenvaluation of f to construct a Green function for f at every place. Jonsson and Wulcan construct a canonical height h_f associated to f which satisfies the following property: $p \in \mathbf{A}^2(\overline{K}), h_f(p) = 0$ if and only if for every place v, $||f^n(p)||_v$ grows at most like μ^n with $0 < \mu \leqslant \lambda_2 < \lambda_1$.

I believe that this construction can be generalised to every affine surface using the results of this memoir. The construction of such canonical heights would allow one to show the following weak version of the Tits alternative: If $f, g \in \mathrm{End}(X_0)$ satisfy $\lambda_1(f) > \lambda_2(f)$, $\lambda_1(g) > \lambda_2(g)$, then if $h_f \neq h_g$ up to replacing f and g by some iterates, the semigroup generated by f and g is free. This result has been established pour the polynomial transformations of $\mathbf{A}^1_{\mathbf{C}}$ in [BHPT21].

2.5.6 In higher dimension

Let $d \geqslant 3$ be an integer, in [DF21] §6, Dang and Favre show that any polynomial transformation $f : \mathbf{A}^d_{\mathbf{C}} \to \mathbf{A}^d_{\mathbf{C}}$ such that $\lambda_1(f)^2 > \lambda_2(f)$ is an algebraic *number* of degree $\leqslant d$. To do so, they build an eigenvaluation of f centered at infinity using the spectral analysis of f^* on the space $N^1_\Sigma(\mathcal{X})$ which is an analogue of the Picard Manin space in dimension 2. They use Abhyankhar's inequality (see [Abh56]) in the following way: If ν_* is an eigenvaluation of f, i.e $f_*\nu_* = \lambda_1 \nu_*$, then f_* induces a linear map over $\Gamma_{\nu_*} \otimes \mathbf{Q}$ where Γ_{ν_*} is the value group of ν_*. Abhyankhar's inequality states that $\dim_{\mathbf{Q}} \Gamma_{\nu_*} \otimes \mathbf{Q} \leqslant d$. Thus, λ_1 is an eigenvalue of a $d \times d$ matrix with rational coefficients, it is therefore an algebraic number of degree $\leqslant d$.

I assert that the construction of the eigenvalue in the case of affine surfaces that I establish in this memoir can be generalised in higher dimensions. In particular, Sections 3.6 and 3.7 do not use dimension 2. The construction of the eigenvaluation comes from an analog of Theorem 4.1.16 where $\mathrm{L}^2(\mathrm{X}_0)$ should be replace by its analog $N^1_\Sigma(\mathcal{X})$. We then can use Abhyankhar's inequality to obtain the following result:

if X_0 is an affine surface of dimension $d \geqslant 3$ over an algebraically closed field $\mathbf{k}$ of characteristic zero, such that

- $\mathbf{k}[X_0]^\times = \mathbf{k}^\times$;
- *For all completion X of X_0,* $\mathrm{Pic}^0(X) = 0$;

If $f : X_0 \to X_0$ is an endomorphism such that $\lambda_1(f)^2 > \lambda_2(f)$, then $\lambda_1(f)$ is an algebraic number *of degree $\leqslant d$.*

CHAPTER 3

VALUATIONS AND ALGEBRAIC GEOMETRY

3.1 Results from algebraic geometry

Let $\mathbf{k}$ be an algebraically closed field. A *variety* is an integral scheme of finite type over $\mathbf{k}$. A surface is a variety of dimension 2. An affine variety over $\mathbf{k}$ is a variety $X_0 = \operatorname{Spec} A$ with A a finitely generated $\mathbf{k}$-algebra. We will denote by $\mathbf{k}[X_0]$ the ring of regular functions of the affine variety X_0.

3.1.1 Bertini

Theorem 3.1.1 (Bertini's Theorem, [Har77])**.** *Let $X \subset \mathbf{P}^N$ be a smooth quasi-projective variety over an algebraically closed field $\mathbf{k}$. The set of hyperplanes H of $\mathbf{P}^N$ such that the intersection $H \cap X$ is a smooth irreducible subvariety of X is a dense open subset of $\mathbf{P}\Gamma(\mathbf{P}^N, \mathcal{O}(1))$.*

3.1.2 Local power series and local coordinates

Let X be a variety and $x \in X$ a closed point. We will write $\mathcal{O}_{X,x}$ for the ring of germs of regular functions at x. A *regular sequence* of $\mathcal{O}_{X,x}$ is a sequence $t_1, \cdots, t_r \in \mathcal{O}_{X,x}$ such that t_1 is not a zero divisor in $\mathcal{O}_{X,x}$ and for all $i \geqslant 2, t_i$ is not a zero divisor in $\mathcal{O}_{X,x}/(t_1, \cdots, t_{i-1})$ (see [Har77] p.184). The point x is *regular* if the local ring $\mathcal{O}_{X,x}$ is regular, i.e there exists a regular sequence of length $\dim \mathcal{O}_{X,x}$.

Theorem 3.1.2 ([Har77], Theorem 5.5A)**.** *Let R be a regular local $\mathbf{k}$-algebra of dimension n with maximal ideal $\mathfrak{m}$, then the completion of R with respect to the $\mathfrak{m}$-adic topology is isomorphic to $\mathbf{k}[[t_1, \cdots, t_n]]$ where $(t_1, \cdots, t_n)$ is a regular sequence of R.*

Let X be a surface and x a regular point of X. Then, we will say that (z, w) are *local coordinates* at x if (z, w) is a regular sequence of $\mathcal{O}_{X,x}$. If (z, w) is a regular sequence of the completion $\widehat{O_{X,x}}$ we will say that they are local *formal* coordinates. By Theorem 3.1.2, $\widehat{O_{X,x}}$ is isomorphic to $\mathbf{k}[[z,w]]$. Finally, If $\mathbf{k} = \mathbf{C}_v$, is a complete algebraically closed field (archimedean or not), we consider the local ring of germs of *holomorphic* functions at x, this is the subring of $\widehat{O_{X,x}}$ of power series with a positive radius of convergence. We denote it by $O_{X,x}^{hol}$ it is also a local ring of dimension 2, if (z,w) is a regular sequence of $O_{X,x}^{hol}$, we say that (z,w) are local *analytic coordinates*. If E,F are two germs of reduced irreducible curves at x (algebraic, analytic of formal) we will say that (z,w) are *associated* to (E,F) if $z=0$ is a local equation of E and $w=0$ is a local equation of F.

3.1.3 Boundary

Proposition 3.1.3 ([Goo69], Proposition 1 and 2)**.** *Let X_0 be an affine variety and let $\iota : X_0 \hookrightarrow X$ be an open embedding into a projective variety, then the subvariety $X \setminus X_0$ is of pure codimension 1. Furthermore, there exists a regular function P on X_0 that has poles along every component of $X \setminus X_0$.*

Set

$$\partial_X X_0 := X \setminus X_0, \tag{3.1}$$

we call it the *boundary* of X_0 in X; by Proposition 3.1.3 it is a curve when X_0 is a surface.

Theorem 3.1.4 ([Goo69])**.** *Let X be a normal proper surface and U an open dense affine subset of X (that is an open dense subset of X that is also an affine variety) such that $V := X \setminus U$ is locally factorial (each local ring is a unique factorization domain), then there exists an ample divisor H on X such that* $\operatorname{Supp} H = V$.

In fact, Goodman shows that Theorem 3.1.4 holds in higher dimension with the only difference that you may need to do some blow-ups at infinity to find an ample divisor.

3.1.4 Surfaces

Theorem 3.1.5 ([Har77] Proposition 5.3)**.** *Let $g : S_1 \to S_2$ be a birational morphism between smooth projective surfaces. Then, g is a composition of blow-ups of points and of an automorphism of S_2. Furthermore, if $h : S_1 \dashrightarrow S_2$ is a birational map, then there exists a sequence of blow-ups $\pi : S_3 \to S_1$ such that $h \circ \pi : S_3 \to S_2$ is regular and S_3 can be chosen minimal for this property.*

Proposition 3.1.6. *Let $g : S_1 \dashrightarrow S_2$ be a birational map. Let $\pi : S_3 \to S_1$ be a minimal resolution of indeterminacies of g such that the lift $h : S_3 \to S_2$ of g is regular. Then, the first curve contracted by h must be the strict transform of a curve in S_1.*

Recall the Castelnuovo criterion

Theorem 3.1.7 ([Har77] Theorem V.5.7)**.** *Let C be a curve in a projective surface S such that $C \simeq \mathbf{P}^1$ and $C^2 = -1$, then there exists a projective surface S', a birational morphism $\pi : S \to S'$ and a point $p \in S'$ such that S is isomorphic via π to the blow up of p and C is the exceptional divisor under this isomorphism.*

We will use these results for the study of automorphisms of affine surfaces as they induce birational maps. Understanding the combinatorics of the blow ups and contractions induced by the automorphism will allow us to understand their dynamics.

Our work relies heavily on the elimination of indeterminacies for rational morphism. Since we are in dimension 2, it exists in any characteristic.

Theorem 3.1.8. *Let $f : S_1 \dashrightarrow S_2$ be a dominant rational morphism between projective varieties over an algebraically closed field of any characteristic, then there exists a sequence of blow-ups $\pi : S \to S_1$ such that $f \circ \pi : S \to S_2$ is regular.*

Theorem 3.1.9 ([Cut02])**.** *Suppose* $\operatorname{char} \mathbf{k} = 0$*. Let $f : S \to S'$ be a dominant rational map between normal projective surfaces over $\mathbf{k}$. There exists blow ups $S_1 \to S$ and $S'_1 \to S'$ such that the lift $\widehat{f} : S_1 \to S'_1$ is monomial at every point. Meaning that for every closed point $p \in S_1$ there exists local coordinates (x,y) at p and local coordinates (u,v) at $f(p)$ such that $f(x,y) = (x^a y^b, x^c y^d)$.*

3.1.5 Rigid contracting germs in dimension 2 and local normal forms

Let $f : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ be the germ of a holomorphic function fixing the origin. The *critical set* $\operatorname{Crit}(f)$ of f is the set where the Jacobian of f vanishes. A germ is said to be *rigid* if the generalized critical set $\cup_{n \geqslant 0} f^{-n}(\operatorname{Crit}(f)) = \cup_{n \geqslant 1} \operatorname{Crit}(f^n)$ is a divisor with simple normal crossings (see [Fav00]).

A germ is *contracting* if there exists an open neighbourhood U of 0 such that $f(U) \Subset U$. In [Fav00], Favre classified all the rigid contracting germs in dimension 2 up to holomorphic conjugacy. There are 7 possible possibilities which we call *local normal forms*. We are interested in 3 of them that will appear in this memoir.

The first one is

$$f(x,y) = (x^a, \lambda x^c y + P(x)) \tag{3.2}$$

with $a \geqslant 2, c \geqslant 1, \lambda \in \mathbf{C}^\times$ and P is a polynomial such that $P(0) = 0$. Here the germ of curve $x = 0$ is contracted by f to the origin and f does not admit any invariant germ of curves if and only if $P \neq 0$. We have $\mathrm{Crit}(f^n) = \{x = 0\}$. This local normal form corresponds to Class 2 of Table II in [Fav00]. This is the local normal form of a Hénon map at its attracting fixed point in $\mathbf{P}^2$ (see [Fav00] §2). It will appear in the following way in this memoir. Suppose that there are local coordinates (z, w) at the origin such that f contracts $\{z = 0\}$ with an index of ramification $a \geqslant 2$, f admits no invariant curves and no other curves is contracted to the origin, then f is of the form

$$f(z,w) = (z^a \varphi(z,w), z^c w \psi_2(z,w) + \psi_1(z)) \tag{3.3}$$

with φ invertible, $\psi_1(z) \neq 0$ and $\psi_2(0,w) \neq 0$. This is true even over any field $\mathbf{k}$ of characteristic 0. If $\mathbf{k} = \mathbf{C}$, then the classification of Favre shows that (3.3) can be analytically conjugated to (3.2).

The second one is the monomial normal form

$$f(x,y) = (x^{a_{11}} y^{a_{12}}, x^{a_{21}} y^{a_{22}}) \tag{3.4}$$

with $a_{ij} \in \mathbf{Z}_{\geqslant 1}, a_{11}a_{22} - a_{12}a_{21} \neq 0$; The germ of curves $\{x = 0\}, \{y = 0\}$ are contracted to the origin. We have $\mathrm{Crit}(f^n) = \{xy = 0\}$. We can characterize the matrix A given by (a_{ij}) in the following way. The local fundamental group of $(\mathbf{C}^2, 0) \setminus \{xy = 0\}$ is isomorphic to $\mathbf{Z}^2$. The action of f_* on $\mathbf{Z}^2$ is given by the matrix A and we have that $|\det| A$ is equal to the topological degree of f. This corresponds to Class 6 of Table II of [Fav00]. It will arise in the following context, if f is a germ of holomorphic functions such that there exists local coordinates (z, w) at the origin such that both axis $\{z = 0\}$ and $\{w = 0\}$ are contracted and they are the only two germs of curves contracted. Then, f is of the following pseudomonomial form

$$f(z,w) = (z^{a_{11}} w^{a_{12}} \varphi(z,w), z^{a_{21}} w^{a_{22}} \psi(z,w)) \tag{3.5}$$

with φ, ψ invertible. Then, the classification of Favre asserts that (3.5) is analytically conjugated to (3.4).

The third one is

$$f(x,y) = (x^a y^b (1 + \varphi), \lambda y (1 + \psi)) \tag{3.6}$$

with $a \geqslant 2, b \geqslant 1, \lambda \in \mathbf{C}^\times$ and φ, ψ are germs of holomorphic function vanishing at the origin. We have that $\{y = 0\}$ is contracted to the origin. The germ $\{x = 0\}$ is f-invariant with a ramification index equal to a. We have $\mathrm{Crit}(f^n) = \{xy = 0\}$ and the origin is a noncritical fixed point of

$f_{|\{x=0\}}$. Notice that this germ is rigid but not necessarily contracting. It is contracting if and only if $|\lambda| < 1$. If the germ is contracting then the germ is conjugated to this normal form

$$f(z,w) = \left(z^a w^b, \lambda w\right) \tag{3.7}$$

with the same numbers a, b, λ as in Equation 3.6. This corresponds to Class 5 of Table II in [Fav00].

3.2 Definitions

Let $\mathbf{k}$ be an algebraically closed field of any characteristic and let X_0 be a normal affine surface over $\mathbf{k}$. We will denote by A the ring of regular functions on X_0.

3.2.1 Completions and divisors at infinity

A *completion* of X_0 is the data of a projective surface X with an open embedding $\iota : X_0 \hookrightarrow X$ such that $\iota(X_0)$ is an open dense subset of X and such that there exists an open smooth neighbourhood of $\partial_X X_0$ in X. We will say that a completion is *good* if $\partial_X X_0$ is an effective divisor with simple normal crossings, From any completion of X, one obtains a good one by a finite number of blow ups at infinity (i.e on $\partial_X X_0$) see for example [Har77] Theorem 3.9 p.391.

Let X be a completion of X_0 with the embedding $\iota_X : X_0 \to X$, we will still denote $\iota_X(X_0)$ by X_0 and we will denote by $\mathcal{O}_X(X_0)$ the subring of $\mathbf{k}(X)$ of functions $f \in \mathbf{k}(X)$ which are regular on X_0. By Proposition 3.1.3, the boundary $\partial_X X_0$ is a possibly reducible connected curve. We denote by $\mathrm{Div}(X)$ the group of divisors of X and by $\mathrm{Div}_\infty(X)$ the subgroup of divisors of X supported on $\partial_X X_0$. For $\mathbf{A} = \mathbf{Z}, \mathbf{Q}, \mathbf{R}$, we set $\mathrm{Div}(X)_\mathbf{A} := \mathrm{Div}(X) \otimes \mathbf{A}$ and $\mathrm{Div}_\infty(X)_\mathbf{A} = \mathrm{Div}_\infty(X) \otimes \mathbf{A}$. Let $E_1, \cdots, E_m$ be the irreducible components of $\partial_X X_0$ (we will call them the *prime divisors at infinity*). Any element of $\mathrm{Div}_\infty(X)_\mathbf{A}$ is of the form $D = \sum_i a_i(D) E_i$ with $a_i(D) \in \mathbf{A}$. We will write $\mathrm{ord}_{E_i}(D)$ for $a_i(D)$ of D at E_i. For a family $(D_j)_{j \in J}$ of elements of $\mathrm{Div}_\infty(X)$ the coefficients $a_i(D)$ are integers; so, using the natural order on $\mathbf{Z}$, we define the supremum $\bigvee_{j \in J} D_j$ and the infimum $\bigwedge_{j \in J} D_j$ by

$$\bigvee_j D_j = \sum_i \sup(\mathrm{ord}_{E_i}(D_j)) E_i \quad \text{and} \quad \bigwedge_j D_j = \sum_i \inf(\mathrm{ord}_{E_i}(D_j)) E_i \tag{3.8}$$

It only exists if each $(\mathrm{ord}_{E_i}(D_j))_{j \in J}$ is bounded respectively from above or from below. If $\bigwedge_j D_j$ (respectively $\bigvee_j D_j$) is well defined we say that the family (D_j) is *bounded from below (from above)*. Notice that we only define supremum and infimum for family of divisors with coefficients in $\mathbf{Z}$.

3.2.2 Morphisms between completions, Weil, Cartier divisors

Some notations If $\pi : Y \to X$ is a projective birational morphism between smooth projective surfaces and D_X is a divisor on X, we will denote by $\pi^* D_X$ the *pull-back* of D_X under π and if D_X is effective, then $\pi'(D_X)$ will be the strict transform of D_X under π. For any projective surface Z, if D_Z is a divisor on Z, we will denote by $\mathcal{O}_Z(D_Z)$ the invertible sheaf on Z associated to D_Z.

Let X_1, X_2 be two completions of X_0 with their embeddings ι_1, ι_2. There exists a unique birational map $\pi : X_1 \dashrightarrow X_2$ such that the diagram

$$\begin{tikzcd} X_1 \arrow[r, dashed, "\pi"] & X_2 \\ X_0 \arrow[u, hook, "\iota_1"] \arrow[r, equal, "\mathrm{id}"] & X_0 \arrow[u, hook, "\iota_2"'] \end{tikzcd} \tag{3.9}$$

commutes. If π is a morphism, we call it a *morphism of completions*. In that case we say that X_1 is *above* X_2. By Theorem 3.1.5, π^{-1} is a composition of blow-ups; since π is an isomorphism over X_0, the centers of these blowups are above $\partial_{X_2} X_0$. Conversely, let X be a completion of X_0 with an embedding $\iota : X_0 \hookrightarrow X$, let $\pi : Y \to X$ be the blowup of X at a point $p \in \partial_X X_0$, then Y with the embedding $\pi^{-1} \circ \iota : X_0 \to Y$ is a completion of X_0 and π is a morphism of completions. For a morphism of completions $\pi : Y \to X$, we will write $\mathrm{Exc}(\pi) \subset Y$ for the exceptional locus of π.

Lemma 3.2.1. *The system of completions of X_0 is a projective system: For any two completions X_1, X_2 of X_0 there exists a completion X_3 above X_1 and X_2.*

Proof. Let X_1, X_2 be two completions of X_0, let $\pi : X_1 \dashrightarrow X_2$ be the birational map from Diagram 3.9. By Theorem 3.1.5, there exists a sequence of blow-ups $\pi_1 : X_3 \to X_1$ such that $g = \pi_1 \circ \pi : X_3 \to X_2$ is regular. It is clear that π_1 is a morphism of completions since by definition $\iota_{X_3} =: \iota_3 = \iota_1 \circ \pi_1^{-1}$. The map g is also a morphism of completion because by construction $g = \pi \circ \pi_1$ and $\iota_2 = \pi \circ \iota_1$, therefore $\iota_3 = \pi_1^{-1} \circ \iota_1 = g^{-1} \circ \pi \circ \iota_1 = g^{-1} \circ \iota_2$ $\square$

If $\pi : X_1 \to X_2$ is a morphism of completions. We can define (see [Ful98], Section 1.4) the pushforward $\pi_* : \mathrm{Div}(X_1)_{\mathbf{A}} \to \mathrm{Div}(X_2)_{\mathbf{A}}$ and pullback $\pi^* : \mathrm{Div}(X_2)_{\mathbf{A}} \to \mathrm{Div}(X_1)_{\mathbf{A}}$ of divisors. They define group homomorphisms

$$\pi_* : \mathrm{Div}_\infty(X_1)_{\mathbf{A}} \twoheadrightarrow \mathrm{Div}_\infty(X_2)_{\mathbf{A}} \quad \text{and} \quad \pi^* : \mathrm{Div}_\infty(X_2)_{\mathbf{A}} \hookrightarrow \mathrm{Div}_\infty(X_1)_{\mathbf{A}}; \tag{3.10}$$

the map π^* is often called the *total transform*. Recall that ([Har77] Proposition 3.2 p.386)

$$\pi_* \pi^* = \mathrm{id}_{\mathrm{Div}(X_2)_{\mathbf{A}}}. \tag{3.11}$$

Let X be a completion of X_0 and $P \in A$, then $(\iota_X^{-1})^*(P) \in \mathbf{k}(X)$. We set $(\iota_X)_* := (\iota_X^{-1})^*$ and we denote by $\mathrm{div}_X(P) := \mathrm{div}((\iota_X)_* P)$ the divisor of the rational function P in X. In particular, if

$\pi : Y \to X$ is a morphism of completions above X_0, then by Diagram (3.9), one has $\iota_Y = \pi^{-1} \circ \iota_X$. Therefore $\mathrm{div}_Y(P) = \mathrm{div}((\pi^{-1} \circ \iota_X)_*(P)) = \mathrm{div}(\pi^* ((\iota_X)_*(P))) = \pi^* \mathrm{div}_X(P)$. We will write $\mathrm{div}_{\infty,X}(P) \in \mathrm{Div}_\infty(X)$ the divisor on X supported at infinity such that

$$\mathrm{div}_X(P) = D + \mathrm{div}_{\infty,X}(P)$$

where D is an effective divisor and no components of its support is in $\partial_X X_0$.

Example 3.2.2. Let $X_0 = \mathbf{A}^2 = \mathrm{Spec}\, \mathbf{k}[x,y]$ and let $P = xy$. Take the completion $\mathbf{P}^2$ of $\mathbf{A}^2$ with homogeneous coordinates X,Y,Z such that $x = X/Y$ and $y = Y/Z$. Then,

$$\mathrm{div}_{\mathbf{P}^2}(P) = \{X = 0\} + \{Y = 0\} - 2\{Z = 0\} \tag{3.12}$$

and $\mathrm{div}_{\infty,\mathbf{P}^2}(P) = -2\{Z = 0\}$. Let $\pi : X \to \mathbf{P}^2$ be the blow-up of $[1:0:0]$, we can take X to be the subscheme of $\mathbf{P}^2 \times \mathbf{P}^1$ given by the equation

$$UZ = VY \tag{3.13}$$

where U,V are the homogeneous coordinates of $\mathbf{P}^1$. Then π is the projection onto the first factor. We take the affine chart $X = 1$ in $\mathbf{P}^2$ with affine coordinates $y' = Y/X$ and $z' = Z/X$. Take the chart $U = 1$ with affine coordinate v in $\mathbf{P}^1$, then $X \cap \{X = 1\} \times \{U = 1\}$ is an affine chart of XX with coordinates v,y' and we have the relation $z' = vy'$; $y' = 0$ is a local equation of the exceptional divisor and $v = 0$ is a local equation of the strict transform of $z' = 0$.

$$\pi^*(P) = \pi^*(\frac{y'}{(z')^2}) = \frac{y'}{v^2(y')^2} = \frac{1}{v^2 y'} \tag{3.14}$$

Therefore,

$$\mathrm{div}_X(P) = \pi'\{X = 0\} + \pi'\{Y = 0\} - 2\pi'\{Z = 0\} - \widetilde{E} = \pi^*(\mathrm{div}_{\mathbf{P}^2}(P)) \tag{3.15}$$

and

$$\mathrm{div}_{\infty,X}(P) = -2\pi'\{Z = 0\} - \widetilde{E} \tag{3.16}$$

The system of completions of X_0 is a projective system by Lemma 3.2.1. Consider the system of groups $(\mathrm{Div}_\infty(X))_{\mathbf{A}}$ for X a completion of X_0 with compatibility morphisms

$$\pi_* : \mathrm{Div}_\infty(X) \to \mathrm{Div}_\infty(Y) \tag{3.17}$$

for any morphism of completions $\pi : X \to Y$. This is also a projective system of groups. We denote by $\mathrm{Weil}_\infty(\mathrm{X_0})_{\mathbf{A}}$ the projective limit of this system. Analogously, the same system of groups with π^* as compatibility morphisms is an inductive system and we denote by $\mathrm{Cartier}_\infty(\mathrm{X_0})_{\mathbf{A}}$ the inductive limit. Concretely, an element $D \in \mathrm{Weil}_\infty(\mathrm{X_0})_{\mathbf{A}}$ is a collection $D = (D_X)$ such that D_X is an element of $\mathrm{Div}_\infty(X)_{\mathbf{A}}$ for every completion X of X_0 and such that for any morphism of completions $\pi : X \to Y$, $\pi_* D_X = D_Y$; D_X is called the *incarnation* of D in X. An element of $\mathrm{Cartier}_\infty(\mathrm{X_0})_{\mathbf{A}}$ is the data of a completion X and a divisor $D \in \mathrm{Div}_\infty(X)$ where two pairs (X, D) and (X', D') are equivalent if there exists a completion Z above X and X' with morphisms of completion $\pi : Z \to X, \pi' : Z \to X'$ such that $\pi^* D = (\pi')^* D'$. We will say that $D \in \mathrm{Cartier}_\infty(\mathrm{X_0})_{\mathbf{A}}$ is *defined* over a completion X if D is the equivalence class of (X, D_X) for some $D_X \in \mathrm{Div}_\infty(X)_{\mathbf{A}}$. We have a natural inclusion

$$\varphi : \mathrm{Cartier}_\infty(\mathrm{X_0})_{\mathbf{A}} \hookrightarrow \mathrm{Weil}_\infty(\mathrm{X_0})_{\mathbf{A}} \tag{3.18}$$

defined as follows. If $(X, D) \in \mathrm{Cartier}_\infty(\mathrm{X_0})_{\mathbf{A}}$, then we need to define the incarnation $\varphi(D)_Y$ for any completion Y. First of all, set $\varphi(D)_X = D$. Then, for any completion Y, by Lemma 3.2.1, there exists a completion Z above Y and X; denote by $\pi_Y : Z \to Y$ and $\pi_Z : Z \to X$ the respective morphism of completions. We define $\varphi(D)_Y := (\pi_Y)_* \pi_X^* D$. This does not depend on the choice of Z because of Equation (3.11). In the rest of the paper, we will drop the notation $\varphi(D)$ and denote by D the image of (X, D) in $\mathrm{Weil}_\infty(\mathrm{X_0})_{\mathbf{A}}$. We equip $\mathrm{Weil}_\infty(\mathrm{X_0})_{\mathbf{A}}$ with the projective limit topology.

In the same manner we define $\mathrm{Cartier}(\mathrm{X_0})_{\mathbf{A}} := \varinjlim \mathrm{Div}(X)_{\mathbf{A}}$ and $\mathrm{Weil}(\mathrm{X_0})_{\mathbf{A}} := \varprojlim \mathrm{Div}(X)_{\mathbf{A}}$ and we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Cartier}_\infty(\mathrm{X_0})_{\mathbf{A}} \arrow[r, hook] \arrow[d] & \mathrm{Weil}_\infty(\mathrm{X_0})_{\mathbf{A}} \arrow[d, hook] \\ \mathrm{Cartier}(\mathrm{X_0})_{\mathbf{A}} \arrow[r, hook] & \mathrm{Weil}(\mathrm{X_0})_{\mathbf{A}} \end{tikzcd} \tag{3.19}$$

Remark 3.2.3. We have that $\mathrm{Cartier}_\infty(\mathrm{X_0})_{\mathbf{A}} = \mathrm{Cartier}_\infty(\mathrm{X_0}) \otimes \mathbf{A}$ but $\mathrm{Weil}_\infty(\mathrm{X_0})_{\mathbf{A}}$ is strictly larger than $\mathrm{Weil}_\infty(\mathrm{X_0}) \otimes \mathbf{A}$ when $\mathbf{A} = \mathbf{Q}, \mathbf{R}$. Indeed, let $W_1, \dots, W_r \in \mathrm{Weil}_\infty(\mathrm{X_0}), \lambda_1, \dots, \lambda_r \in \mathbf{A}$ and set $W := \sum_i \lambda_i W_i$. Then, for every completion X and for every prime divisor E at infinity in X we have

$$\mathrm{ord}_E(W_X) = \mathrm{ord}_E(\sum_i \lambda_i W_{i,X}) = \sum_i \lambda_i \, \mathrm{ord}_E(W_{i,X}) \in \mathbf{Z}\lambda_1 + \cdots + \mathbf{Z}\lambda_r \tag{3.20}$$

In particular, the group $G(W)$ generated by $(\mathrm{ord}_E(W_X))_{(X,E)}$ for all completions X and all prime divisor E at infinity in X is a finitely generated subgroup of $\mathbf{R}$. Now pick a completion X_1 and consider a sequence of blow ups $\pi_n : X_{n+1} \to X_n$ starting with X_1. Let E_n be the exceptional divisor of π_n. We still denote by E_n the strict transform of E_n in every $X_m, m \geqslant n+1$. Define the Weil divisor $W \in \mathrm{Weil}_\infty(\mathrm{X}_0)_\mathbf{A}$ such that its incarnation in X_{n+1} is $W_{X_{n+1}} = \sum_{k=1}^n \frac{1}{k} E_k$. Then, $G(W)$ is not finitely generated, therefore $W \notin \mathrm{Weil}_\infty(\mathrm{X}_0) \otimes \mathbf{A}$.

An element D of $\mathrm{Weil}_\infty(\mathrm{X}_0)_\mathbf{A}$ with $\mathbf{A} = \mathbf{Z}, \mathbf{Q}, \mathbf{R}$ is called *effective* (denoted by $D \geqslant 0$) if its incarnation in every completion X is effective; if D belongs to $\mathrm{Cartier}_\infty(\mathrm{X}_0)_\mathbf{R}$ this is equivalent to $D_X \geqslant 0$ for one completion X where D is defined. If $D_1, D_2 \in \mathrm{Weil}_\infty(\mathrm{X}_0)_\mathbf{A}$, we will write $W_1 \geqslant W_2$ for $W_1 - W_2 \geqslant 0$.

3.2.3 A canonical basis

Let X be a completion of X_0, we define $\mathcal{D}_X$ as follows. Elements of $\mathcal{D}_X$ are equivalence classes of prime divisors *exceptional above* X at infinity in completions $\pi_Y : Y \to X$ above X where two prime divisors E and E' belonging respectively to Y and Y' are equivalent if the birational map $\pi_{Y'}^{-1} \circ \pi_Y : Y \dashrightarrow Y'$ induces an isomorphism $\pi_{Y'}^{-1} \circ \pi_Y : E \to E'$. We call $\mathcal{D}_X$ the *set of prime divisors above* X. We also define $\mathcal{D}_\infty(X_0)$ as the set of equivalence classes of prime divisors at infinity modulo the same equivalence relation. We write $\mathbf{A}^{\mathcal{D}_X}$ for the set of functions $\mathcal{D}_X \to \mathbf{A}$ and $\mathbf{A}^{(\mathcal{D}_X)}$ for the subset of functions with finite support.

Proposition 3.2.4. *If X is a completion of X_0, then*

$$\mathrm{Cartier}_\infty(\mathrm{X}_0)_\mathbf{A} = \mathrm{Div}_\infty(X)_\mathbf{A} \oplus \mathbf{A}^{(\mathcal{D}_X)}, \quad \textit{and } \mathrm{Weil}_\infty(\mathrm{X}_0)_\mathbf{A} = \mathrm{Div}_\infty(X)_\mathbf{A} \oplus \mathbf{A}^{\mathcal{D}_X}. \tag{3.21}$$

This is a homeomorphism with respect to the product topology of $\mathbf{A}^{\mathcal{D}_X}$.

Proof. Following [BFJ08] Proposition 1.4, for any $E \in \mathcal{D}_X$ there exists a minimal completion X_E above X such that E is a prime divisor in X_E. We denote by $\alpha_E \in \mathrm{Cartier}_\infty(\mathrm{X}_0)$ the element $\alpha_E := (X_E, E)$. Let $E_1, \ldots, E_r$ be the prime divisor at infinity in X, then

$$(E_0, \ldots, E_r) \cup \{\alpha_E : E \in \mathcal{D}_X\} \tag{3.22}$$

is a $\mathbf{A}$-basis of $\mathrm{Cartier}_\infty(\mathrm{X}_0)_\mathbf{A}$. In the same fashion we obtain the second homeomorphism. □

Remark 3.2.5. Since for any completion X, one can find a good completion Y above X and the blow up of a good completion is still a good completion, the projective system of good completions is cofinal in the projective system of completions, so in the rest of the paper any completion that we take will be a good completion.

If $f : X_0 \to X_0$ is a dominant endomorphism, then we can define

$$f^* : \mathrm{Cartier}_\infty(\mathrm{X}_0)_{\mathbf{A}} \to \mathrm{Cartier}_\infty(\mathrm{X}_0)_{\mathbf{A}} \text{ and } f_* : \mathrm{Weil}_\infty(\mathrm{X}_0)_{\mathbf{A}} \to \mathrm{Weil}_\infty(\mathrm{X}_0)_{\mathbf{A}} \tag{3.23}$$

as follows. Let $D = (X, D_X) \in \mathrm{Cartier}_\infty(\mathrm{X}_0)_{\mathbf{A}}$. Let Y be a completion of X_0 such that the lift $F : Y \to X$ of f is regular, then we define

$$f^* D := (Y, F^* D_X) \in \mathrm{Cartier}_\infty(\mathrm{X}_0)_{\mathbf{A}}\,. \tag{3.24}$$

This does not depend on the choice of Y. If $D \in \mathrm{Weil}_\infty(\mathrm{X}_0)$, let X, Y be completions of X_0 such that the lift $F : Y \to X$ is regular, then

$$(f_* D)_X := F_* D_Y\,. \tag{3.25}$$

Again, this does not depend on the choice of Y.

3.2.4 Local version of the canonical basis

Let X be a completion and let $p \in X$ be a closed point at infinity i.e on $\partial_X X_0$. We denote by $\mathrm{Weil}(X,p)_{\mathbf{A}}$ the subspace of $\mathrm{Weil}_\infty(\mathrm{X}_0)_{\mathbf{A}}$ defined as follows: $D \in \mathrm{Weil}(X,p)_{\mathbf{A}}$ if and only if $D_X = 0$ and for every completion $\pi : Y \to X$ above X and every prime divisor E at infinity in Y, one has $E \in \mathrm{Supp}\, D_Y$ if and only if $\pi(E) = p$. We define

$$\mathrm{Cartier}(X,p)_{\mathbf{A}} = \mathrm{Weil}(X,p)_{\mathbf{A}} \cap \mathrm{Cartier}_\infty(\mathrm{X}_0)_{\mathbf{A}}\,. \tag{3.26}$$

We can define the set $\mathcal{D}_{X,p}$ of prime divisors above p as follows. We will say that a completion $\pi : Y \to X$ is *exceptional above* p if $\pi(\mathrm{Exc}(\pi)) = p$. We will write $\pi : (Y, \mathrm{Exc}(\pi)) \to (X,p)$ for such a completion. Elements of $\mathcal{D}_{X,p}$ are equivalence classes of prime divisors $E \in \mathrm{Exc}(\pi)$ for all completions $\pi : (Y, \mathrm{Exc}(\pi)) \to (X,p)$.

Proposition 3.2.6. *If X is a completion of X_0, then $\mathcal{D}_X = \bigsqcup_{p \in \partial_X X_0} \mathcal{D}_{X,p}$ and*

$$\mathrm{Cartier}(X,p)_{\mathbf{A}} = (\mathbf{A})^{(\mathcal{D}_{X,p})} \tag{3.27}$$

$$\mathrm{Weil}(X,p)_{\mathbf{A}} = (\mathbf{A})^{\mathcal{D}_{X,p}} \tag{3.28}$$

3.2.5 Supremum and infimum of divisors

Let $(D_i)_{i\in I}$ be a family of elements of $\mathrm{Weil}_\infty(\mathrm{X}_0)$ such that for all completions X, the family $(D_{i,X})$ is bounded from below, we define $\bigwedge_{i\in I} D_i$ with its incarnation in X being

$$\left(\bigwedge D_i\right)_X = \bigwedge_i D_{i,X}. \tag{3.29}$$

We have an analogous definition for $\bigvee_i D_i$ when each $(D_{i,X})$ is bounded from above.

Lemma 3.2.7. *If* $D, D' \in \mathrm{Cartier}_\infty(\mathrm{X}_0)$*, then* $D \wedge D', D \vee D' \in \mathrm{Cartier}_\infty(\mathrm{X}_0)$.

Proof. It suffices to show that $D \wedge D' \in \mathrm{Cartier}_\infty(\mathrm{X}_0)$ because $D \vee D' = -(-D \wedge -D')$. So take $D, D' \in \mathrm{Cartier}_\infty(\mathrm{X}_0)$, we have to show that $D \wedge D'$ belongs to $\mathrm{Cartier}_\infty(\mathrm{X}_0)$.

Now, it suffices to show this for D, D' effective, indeed let X be a completion such that D and D' are defined over X. Then, there exists $D_2 \in \mathrm{Div}_\infty(X)$ such that $D - D_2$ and $D' - D_2$ are effective. Indeed, take D_2 as the Cartier class determined by $D \wedge D'$ in X, Then

$$D \wedge D' = (D - D_2) \wedge (D' - D_2) + D_2. \tag{3.30}$$

Therefore, suppose D, D' are effective. Then $\mathfrak{a} = \mathcal{O}_X(-D) + \mathcal{O}_X(-D')$ is a coherent sheaf of ideals such that $\mathfrak{a}_{|X_0} = \mathcal{O}_{X_0}$, let $\pi : Y \to X$ be the blow-up along $\mathfrak{a}$. Since $\mathfrak{a}_{|X_0}$ is trivial, π is an isomorphism over X_0, therefore Y is a completion of X_0 with respect to the embedding $\iota_Y := \pi^{-1} \circ \iota_X$ and π is a morphism of completions. Then, $\mathfrak{b} := \pi^* \mathfrak{a} \cdot \mathcal{O}_Y$ is an invertible sheaf over Y trivial over X_0, so there exists a divisor $D_Y \in \mathrm{Div}_\infty(Y)$ such that $\mathfrak{b} = \mathcal{O}_Y(-D_Y)$.

Claim 3.2.8. *The Cartier class in* $\mathrm{Cartier}_\infty(\mathrm{X}_0)$ *induced by* D_Y *is* $D \wedge D'$.

We postpone the proof of this claim to the end of Section 3.3. □

Example 3.2.9. Let X be a completion that contains two prime divisors E, E' at infinity in X such that they intersect (transversely) at a point p. The sheaf of ideals $\mathfrak{a} = \mathcal{O}_X(-E) + \mathcal{O}_X(-E')$ is the ideal of regular functions vanishing at p. The blow up of $\mathfrak{a}$ is exactly the blow up $\pi : Y \to X$ at p since by universal property of the blow-up $\pi^* \mathfrak{a} = \mathcal{O}_Y(-\widetilde{E})$ where $\widetilde{E}$ is the exceptional divisor above p. If we still denote by $E, E', \widetilde{E}$ the elements they define in $\mathrm{Cartier}_\infty(\mathrm{X}_0)$, then $E \wedge E' = \widetilde{E}$.

Let X be a good completion of X_0. Let $D_1, D_2 \in \mathrm{Div}_\infty(X)$. Let E, F be two prime divisors at infinity that intersect. We say that (D_1, D_2) is *well ordered* at $E \cap F$ if

$$\mathrm{ord}_E(D_1) < \mathrm{ord}_E(D_2) \Leftrightarrow \mathrm{ord}_F(D_1) < \mathrm{ord}_F(D_2). \tag{3.31}$$

We say that (D_1, D_2) is a *well ordered* pair if it is well ordered at $E \cap F$ for every prime divisor E, F at infinity that intersect.

Lemma 3.2.10. *If $D_1 \wedge D_2$ or $D_1 \vee D_2$ is defined in X, then (D_1, D_2) is a well ordered pair.*

Proof. Suppose for example that $D_1 \vee D_2$ is defined in X and that D_1, D_2 is not a well ordered pair and let E, F be two prime divisors at infinity that intersect such that at $E \cap F, D_i = \alpha_i E + \beta_i F$ with $\alpha_1 < \alpha_2$ and $\beta_1 > \beta_2$. Then, $D_1 \vee D_2 = \alpha_2 E + \beta_1 F$. Let $\widetilde{E}$ be the exceptional divisor above $E \cap F$, then we have $\mathrm{ord}_{\widetilde{E}}(D_1 \vee D_2) = \alpha_2 + \beta_1$. But

$$\mathrm{ord}_{\widetilde{E}} D_i = \alpha_i + \beta_i < \alpha_2 + \beta_1 = \mathrm{ord}_{\widetilde{E}}(D_1 \vee D_2). \tag{3.32}$$

This is a contradiction. □

Remark 3.2.11. This is actually an equivalence, if D_1, D_2 is a well ordered pair, then $D_1 \wedge D_2$ and $D_1 \vee D_2$ is defined in X. This gives an algorithmic procedure by successive blow ups to find the minimum and maximum of two Cartier divisors.

Definition 3.2.12. Let $\mathcal{S}_\infty(X_0)$ be the semigroup of $\mathrm{Weil}_\infty(\mathrm{X}_0)$ of elements $D \in \mathrm{Weil}_\infty(\mathrm{X}_0)$ such that there exists a (potentially uncountable) family $(D_i)_{i \in I} \subset \mathrm{Cartier}_\infty(\mathrm{X}_0)$ such that

$$D = \bigvee_I D_i \tag{3.33}$$

Proposition 3.2.13. *(1)* $\mathrm{Cartier}_\infty(\mathrm{X}_0) \subset \mathcal{S}_\infty(X_0)$.

(2) For $a, b \geqslant 0$ and $D, D' \in \mathcal{S}_\infty(X_0)$, one has $aD + bD' \in \mathcal{S}_\infty(X_0)$.

(3) If $D_i \in \mathcal{S}_\infty(X_0)$ for each $i \in I$ and (D_i) is bounded from above then $\bigvee_{i \in I} D_i \in \mathcal{S}_\infty(X_0)$.

(4) If $D, D' \in \mathcal{S}_\infty(X_0)$, then $D \wedge D' \in \mathcal{S}_\infty(X_0)$.

Proof. The first assertion is trivial as for $D \in \mathrm{Cartier}_\infty(\mathrm{X}_0), D = \bigvee D$. For Property (2), let X be a completion of X_0 then $\bigvee_i aD_{i,X} + \bigvee_j bD'_{j,X} = \bigvee_{i,j}(aD_i + bD'_j)_X$. For Property (3), if $D_i = \bigvee_j D_{i,j}$, then $\bigvee_i D_i = \bigvee_{(i,j)} D_{i,j}$. Finally, the fourth assertion is a corollary of Lemma 3.2.7. □

Example 3.2.14. We have $\mathcal{S}_\infty(X_0) \subsetneq \mathrm{Weil}_\infty(\mathrm{X}_0)$. Let $X_0 = \mathbf{A}^2$ and $X = \mathbf{P}^2$. Let E_0 denote the line at infinity, a canonical divisor in $\mathbf{P}^2$ is given by $K_{\mathbf{P}^2} = -3E_0$. We can define an element

$K \in \mathrm{Weil}_\infty(\mathrm{X}_0)$ by taking for any completion Y of $\mathbf{A}^2$ the canonical divisor supported at infinity. More precisely, let Y is any completion of $\mathbf{A}^2$ above $\mathbf{P}^2$. We still denote by E_0 the strict transform of E_0 in Y. Then, K_Y is of the form

$$K_Y = -3E_0 + \sum_{E \subset \partial_X X_0, E \neq E_0} E. \tag{3.34}$$

Suppose that $K = \sup_i(D_i)$ for some $D_i \in \mathrm{Cartier}_\infty(\mathrm{X}_0)$. Let $D \in (D_i)$ such that D is defined over some completion Y and for some prime divisor $E \neq E_0$ at infinity, $\mathrm{ord}_E(D) = 1$. Then, we must have $K \geqslant D$ meaning that for any completion Z, $K_Z \geqslant D_Z$. Consider the following blow ups. Let $\pi_1 : Y_1 \to Y$ be the blow-up of a point p of E that does not belong to any other divisor at infinity. Let $\widetilde{E}$ be the exceptional divisor of π. Now let $\pi_2 : Y_2 \to Y_1$ be the blow-up at $\pi_1' E \cap \widetilde{E}$ and let $\widetilde{F}$ be the exceptional divisor of π_2. Then, $\mathrm{ord}_{\widetilde{F}}(K_{Y_2}) = 1$ but $\mathrm{ord}_{\widetilde{F}}(D_{Y_2}) = \mathrm{ord}_{\widetilde{F}}((\pi_2 \circ \pi_1)^* D) = 2$ and this is a contradiction.

3.2.6 Picard-Manin Space at infinity and its completion

Let X be a completion of X_0 and let $\mathrm{NS}(X)$ be the Néron-Severi group of X. We have a perfect pairing given by the intersection form

$$\mathrm{NS}(X)_{\mathbf{R}} \times \mathrm{NS}(X)_{\mathbf{R}} \to \mathbf{R}. \tag{3.35}$$

Recall the Hodge index theorem

Theorem 3.2.15 (Hodge Index Theorem, [Har77] Theorem 1.9 p.364)**.** *Let X be a projective surface over a smooth projective surface over an algebraically closed field. Let $\alpha \in \mathrm{NS}(X)$ and let H be an ample divisor on X. If $\alpha \cdot H = 0$, then*

$$\alpha^2 < 0. \tag{3.36}$$

In particular, the signature of the quadratic form induced by the intersection form is $(1, \rho - 1)$ where ρ is the rank of $\mathrm{NS}(X)$.

A class $\alpha \in \mathrm{NS}(X)$ is nef if for all irreducible curve $C \subset X, \alpha \cdot [C] \geqslant 0$. If $\pi : Y \to X$ is a morphism of completions we have two group homomorphisms

$$\pi_* : \mathrm{NS}(Y)_{\mathbf{A}} \to \mathrm{NS}(X)_{\mathbf{A}}, \pi^* : \mathrm{NS}(X)_{\mathbf{A}} \to \mathrm{NS}(Y)_{\mathbf{A}} \tag{3.37}$$

with the following properties

1. $\pi_* \circ \pi^* = \mathrm{id}_{\mathrm{NS}(X)_{\mathbf{A}}}$
2. $\pi^* \alpha \cdot \pi^* \beta = \alpha \cdot \beta$
3. $\pi^* \alpha \cdot \beta = \alpha \cdot \pi_* \beta$ (Projection Formula)

Furthermore, if $\pi : Y \to X$ is the blow up of one point, let $\widetilde{E}$ be the exceptional divisor, then

$$[\widetilde{E}]^2 = -1, \text{ and } \mathrm{NS}(Y)_{\mathbf{A}} = \pi^* \mathrm{NS}(X)_{\mathbf{A}} \ominus \mathbf{A} \cdot [\widetilde{E}] \tag{3.38}$$

Therefore, the system of groups $(\mathrm{NS}(X))$ with compatibility morphisms π_* is a projective system of groups and $(\mathrm{NS}(X))$ with compatibility morphisms π^* is an inductive system of groups.

Definition 3.2.16. The Picard-Manin spaces of X_0 are defined as

$$\text{Cartier-NS}(\mathrm{X}_0)_{\mathbf{A}} := \varinjlim_{X_0 \hookrightarrow X} \mathrm{NS}(X)_{\mathbf{A}}, \quad \text{Weil-NS}(\mathrm{X}_0)_{\mathbf{A}} = \varprojlim_{X_0 \hookrightarrow X} \mathrm{NS}(X)_{\mathbf{A}} \tag{3.39}$$

We equip Weil-NS$(\mathrm{X}_0)_{\mathbf{A}}$ with the topology of the projective limit. We have the same description as for Weil$_\infty(\mathrm{X}_0)$ and Cartier$_\infty(\mathrm{X}_0)$. An element of Weil-NS(X_0) is a family $\alpha = (\alpha_X)_X$ where $\alpha_X \in \mathrm{NS}(X)$ such that for all $\pi : Y \to X$, we have

$$\pi_* \alpha_Y = \alpha_X.$$

We call α_X the *incarnation* of α in X.

An element of Cartier-NS(X_0) is the data of a completion X of X_0 and a class $\alpha \in \mathrm{NS}(X)$ with the following equivalence relation: $(X, \alpha) \simeq (Y, \beta)$ if there exists a completion Z with a morphism of completion

$$\pi_Y : Z \to Y, \quad \pi_X : Z \to X$$

such that $\pi_Y^* \beta = \pi_X^* \alpha$. We say that the Cartier class is defined (by α) in X. We have a natural embedding

$$\text{Cartier-NS}(\mathrm{X}_0) \hookrightarrow \text{Weil-NS}(\mathrm{X}_0). \tag{3.40}$$

We have a pairing

$$\text{Weil-NS}(\mathrm{X}_0)_{\mathbf{R}} \times \text{Cartier-NS}(\mathrm{X}_0)_{\mathbf{R}} \to \mathbf{R} \tag{3.41}$$

given by the following: let $\alpha \in$ Weil-NS$(\mathrm{X}_0)_{\mathbf{R}}$ and $\beta \in$ Cartier-NS$(\mathrm{X}_0)_{\mathbf{R}}$; let X be a completion where β is defined i.e $\beta = (X, \beta_X)$; then

$$\alpha \cdot \beta := \alpha_X \cdot \beta_X. \tag{3.42}$$

This is well defined because if $\pi : Y \to X$ then

$$\alpha_Y \cdot \beta_Y = \alpha_Y \cdot \pi^* \beta_X = \pi_* \alpha_Y \cdot \beta_X = \alpha_X \cdot \beta_X \tag{3.43}$$

by the projection formula.

An element $\alpha \in$ Weil-NS$(\mathrm{X}_0)_{\mathbf{R}}$ is *nef* if for all completion X, α_X is nef.

Proposition 3.2.17 ([BFJ08] Proposition 1.7)**.** *The intersection pairing*

$$\text{Weil-NS}(\mathrm{X}_0)_{\mathbf{R}} \times \text{Cartier-NS}(\mathrm{X}_0)_{\mathbf{R}} \to \mathbf{R} \tag{3.44}$$

is a perfect pairing and it induces a homeomorphism $\mathrm{Weil\text{-}NS}(\mathrm{X}_0)_{\mathbf{R}} \simeq \mathrm{Cartier\text{-}NS}(\mathrm{X}_0)^*_{\mathbf{R}}$ *endowed with the weak-$*$ topology.*

Using the canonical basis of divisors introduced in §3.2.3 we have a more explicit description of the Picard Manin spaces of X_0.

Proposition 3.2.18. *Let X be a completion of X_0, then*

$$\mathrm{Cartier\text{-}NS}(\mathrm{X}_0)_{\mathbf{A}} = \mathrm{NS}(X)_{\mathbf{A}} \oplus \mathbf{A}^{(\mathcal{D}_X)}, \quad \mathrm{Weil\text{-}NS}(\mathrm{X}_0)_{\mathbf{A}} = \mathrm{NS}(X) \oplus \mathbf{A}^{\mathcal{D}_X}. \tag{3.45}$$

Moreover, the intersection product is negative definite over $\mathbf{A}^{(\mathcal{D}_X)}$ and $\{\alpha_E : E \in \mathcal{D}_X\}$ is an orthonormal basis for the quadratic form $\alpha \in \mathbf{A}^{(\mathcal{D}_X)} \mapsto -\alpha^2$.

Proof. The decomposition follows from Equation (3.38). The fact that the intersection form is negative definite follows from the existence of an ample divisor on X, the Hodge Index theorem and the projection formula. The fact that $\{\alpha_E : E \in \mathcal{D}_X\}$ is an orthonormal basis is again a consequence of the projection formula and Equation (3.38). □

3.2.6.1 The local Picard-Manin space

Let X be a completion of X_0 and let p be a point at infinity. Then, by Proposition 3.2.18 we have the canonical embeddings

$$\mathrm{Cartier}(X,p)_{\mathbf{A}} \hookrightarrow \mathrm{Cartier\text{-}NS}(\mathrm{X}_0)_{\mathbf{A}}, \quad \mathrm{Weil}(X,p)_{\mathbf{A}} \hookrightarrow \mathrm{Weil\text{-}NS}(\mathrm{X}_0) \tag{3.46}$$

Proposition 3.2.19. *If $\mathbf{A} = \mathbf{R}$, the space $\mathrm{Cartier}(X,p)_{\mathbf{R}}$ is an infinite dimensional $\mathbf{R}$-vector space and the intersection product defines a negative definite quadratic form over it. The set $\{\alpha_E : E \in \mathcal{D}_{X,p}\}$ is an orthonormal basis for the scalar product $\alpha \mapsto -\alpha^2$. Furthermore, the pairing*

$$\mathrm{Weil}(X,p)_{\mathbf{R}} \times \mathrm{Cartier}(X,p)_{\mathbf{R}} \to \mathbf{R} \tag{3.47}$$

is perfect.

3.2.6.2 The divisors supported at infinity

Fix a completion X of X_0, we have a natural linear map $\tau : \mathrm{Div}_\infty(X)_{\mathbf{R}} \to \mathrm{NS}(X)_{\mathbf{R}}$.

Proposition 3.2.20. *The intersection pairing restricted to $\tau(\mathrm{Div}_\infty(X)_{\mathbf{R}})$ is non degenerate.*

Proof. Let $D \in \tau(\mathrm{Div}_\infty(X)_{\mathbf{R}})$, suppose that $D \cdot D' = 0$ for all $D' \in \tau(\mathrm{Div}_\infty(X)_{\mathbf{R}})$. Then, by Theorem 3.1.4, there exists $H \in \mathrm{Div}_\infty(X)$ ample. We have $D \cdot H = 0$. By the Hodge index theorem, if D is not numerically equivalent to zero, then $D^2 < 0$ and this is a contradiction. □

Let $V \subset \mathrm{NS}(X)$ be the orthogonal subspace of $\tau(\mathrm{Div}_\infty(X)_{\mathbf{R}})$. Then,

$$\mathrm{NS}(X)_{\mathbf{R}} = V \oplus \tau(\mathrm{Div}_\infty(X)_{\mathbf{R}}). \tag{3.48}$$

For example if $X_0 = \mathbf{A}^2$ and $X = \mathbf{P}^2$, then $V = 0$. Since we only blow up at infinity we get

Proposition 3.2.21. *Let X_0 be an affine surface , then*

$$\text{Cartier-NS}(\mathrm{X}_0)_{\mathbf{R}} = V \oplus \tau(\mathrm{Cartier}_\infty(\mathrm{X}_0)_{\mathbf{R}}), \quad \text{Weil-NS}(\mathrm{X}_0)_{\mathbf{R}} = V \oplus \tau(\mathrm{Weil}_\infty(\mathrm{X}_0)_{\mathbf{R}}) \tag{3.49}$$

3.2.6.3 Functoriality

Let $f : X_0 \to X_0$ be a dominant endomorphism of X_0. We define f^*, f_* on the Picard-Manin spaces as follows. We first define

$$f^* : \text{Cartier-NS}(\mathrm{X}_0)_{\mathbf{R}} \to \text{Cartier-NS}(\mathrm{X}_0)_{\mathbf{R}}. \tag{3.50}$$

Let $\beta \in \text{Cartier-NS}(\mathrm{X}_0)_{\mathbf{R}}$ and let X be a completion where β is defined. Let Y be a completion of X_0 such that the lift $F : Y \to X$ is regular, then we define $f^*\beta$ as the Cartier class defined in Y by

$$f^*\beta := (Y, F^*\beta_X) \tag{3.51}$$

this does not depend on the choice of Y. Indeed, if Y' is another completion such that $F' : Y' \to X$ is well defined, then there exists a completion Z such that we have the following diagram.

$$\begin{tikzcd}
 & & Z \arrow[dl, "\pi_{Y'}"'] \arrow[dr, "\pi_Y"] & & \\
 & Y' \arrow[dl, "F'"'] \arrow[dr, "\pi'"] & & Y \arrow[dl, "\pi"'] \arrow[dr, "F"] & \\
X & & X \arrow[ll, dashed, "f"'] \arrow[rr, dashed, "f"] & & X
\end{tikzcd} \tag{3.52}$$

Then, the lift of $f : Z \dashrightarrow X$ is $F \circ \pi_Y = F' \circ \pi_{Y'}$, hence we get

$$\pi_Y^* \circ F^* = \pi_{Y'}^* \circ (F')^* \tag{3.53}$$

and the pull back of Cartier classes is well defined.

Next, we define $f_* : \text{Weil-NS}(X_0)_{\mathbf{R}} \to \text{Weil-NS}(X_0)_{\mathbf{R}}$. Let $\alpha \in \text{Weil-NS}(X_0)_{\mathbf{R}}$. Let X, Y be completions of X_0 such that the lift $F : Y \to X$ is regular, then the incarnation of $f_*\alpha$ in X is

$$(f_*\alpha)_X := F_*\alpha_Y. \tag{3.54}$$

Again, this does not depend on the choice of Y by a similar argument as for the pullback. We have the following proposition

Proposition 3.2.22 ([BFJ08] Section 2)**.** *We have the following properties.*

- *The operator f^* extends to an operator*

$$f^* : \text{Weil-NS}(X_0)_{\mathbf{R}} \to \text{Weil-NS}(X_0)_{\mathbf{R}}. \tag{3.55}$$

- *the operator f_* restricts to an operator*

$$f_* : \text{Cartier-NS}(X_0)_{\mathbf{R}} \to \text{Cartier-NS}(X_0)_{\mathbf{R}} \tag{3.56}$$

- *Let $\alpha \in \text{Weil-NS}(X_0)$, let X, Y be completions of X_0 such that the lift $f : X \dashrightarrow Y$ does not contract any curves, then*

$$(f^*\alpha)_X = (f^*\alpha_Y)_X \tag{3.57}$$

Remark 3.2.23. For a completion X, we can also define the restriction of f^* and f_* to $\text{NS}(X)$. We denote them respectively by f_X^* and $(f_X)_*$. They are defined by

$$\forall \beta \in \text{NS}(X), \quad f_X^*\beta = (f^*\beta)_X, \quad (f_X)_*\beta = (f_*\beta)_X \tag{3.58}$$

3.2.6.4 Spectral property of the first dynamical degree

Consider a completion X of X_0 and $\omega \in \text{NS}(X)$ an ample class. By the Hodge index theorem, the intersection form on $\text{Cartier-NS}(X_0) \times \text{Cartier-NS}(X_0)$ is negative definite on $\omega^\perp$. If $\alpha \in \text{Cartier-NS}(X_0)$, the projection of α on $\omega^\perp$ is $\alpha - (\alpha \cdot \omega)\omega$. Consider the quadratic form on $\text{Cartier-NS}(X_0)$ given by

$$\forall \alpha \in \text{Cartier-NS}(X_0), ||\alpha||^2 := (\omega \cdot \alpha)^2 - \frac{1}{\omega^2}(\alpha - (\alpha \cdot \omega)\omega)^2. \tag{3.59}$$

This defines a norm on Cartier-NS$(\mathrm{X}_0)_{\mathbf{R}}$ and Cartier-NS$(\mathrm{X}_0)_{\mathbf{R}}$ is not complete for this norm. We denote by $\mathrm{L}^2(\mathrm{X}_0)$ the completion of Cartier-NS$(\mathrm{X}_0)_{\mathbf{R}}$ with respect to this norm; Had we chosen a different ample class, we would have gotten an equivalent norm so the space $\mathrm{L}^2(\mathrm{X}_0)$ is independent of the choice of ω. This is a Hilbert space and we have

Proposition 3.2.24 ([BFJ08] Proposition 1.10)**.** *There is a continuous injection*

$$\mathrm{L}^2(\mathrm{X}_0) \hookrightarrow \text{Weil-NS}(\mathrm{X}_0) \tag{3.60}$$

and the topology on $\mathrm{L}^2(\mathrm{X}_0)$ *induced by* Weil-NS(X_0) *coincides with its weak topology as a Hilbert space. If* $\alpha \in$ Weil-NS(X_0) *then* α *belongs to* $\mathrm{L}^2(\mathrm{X}_0)$ *if and only if* $\inf_X(\alpha^2_{\mathbf{X}}) > -\infty$*, in which case* $\alpha^2 = \inf_X(\alpha^2_X)$*. Furthermore, the intersection product* $\cdot$ *defines a continuous bilinear form on* $\mathrm{L}^2(\mathrm{X}_0)$*.*

Remark 3.2.25. In particular, any nef class belongs to $\mathrm{L}^2(\mathrm{X}_0)$. Recall that $\alpha \in$ Weil-NS$(\mathrm{X}_0)_{\mathbf{R}}$ is nef if for every completion X, α_X is nef. The cone theorem ([Laz04] Theorem 1.4.23) states that α_X is a limit of ample classes in $\mathrm{NS}(X)_{\mathbf{R}}$, therefore $(\alpha_X)^2 \geqslant 0$ and $\alpha \in \mathrm{L}^2(\mathrm{X}_0)$.

Using the canonical basis of exceptional divisors we can have an explicit description of $\mathrm{L}^2(\mathrm{X}_0)$. Let $\alpha \in$ Cartier-NS(X_0) and let α_X be the incarnation of α in X. Then, since α is a Cartier class, we have for all but finitely many $E \in \mathcal{D}_X$ that $\alpha \cdot \alpha_E = 0$ and

$$\alpha = \alpha_X + \sum_{E \in \mathcal{D}_X} (\alpha \cdot \alpha_E)\alpha_E. \tag{3.61}$$

Therefore,

$$||\alpha||^2 = ||\alpha_X||^2 + \sum_{E \in \mathcal{D}_X} (\alpha \cdot \alpha_E)^2, \tag{3.62}$$

and

$$\alpha^2 = \alpha_X^2 - \sum_{E \in \mathcal{D}_X} (\alpha \cdot \alpha_E)^2 \tag{3.63}$$

Therefore, $\mathrm{L}^2(\mathrm{X}_0)$ is isomorphic to the Hilbert space

$$\mathrm{L}^2(\mathrm{X}_0) = \mathrm{NS}(X) \ominus \ell^2(\mathcal{D}_X). \tag{3.64}$$

We also have the local version of this statement

Proposition 3.2.26. *Let X be a completion of X_0 and $p \in X$ be a point at infinity. Then,*

$$\mathrm{L}^2(\mathrm{X}_0) \cap \mathrm{Weil}(X, p) = \ell^2(\mathcal{D}_{X,p}) \tag{3.65}$$

and $\{\alpha_E : E \in \mathcal{D}_{X,p}\}$ is a Hilbert basis of this space.

Proposition 3.2.27 ([BFJ08])**.** *Let f be a dominant endomorphism of X_0. The linear maps*

$$f^*, f_* : \text{Weil-NS}(\mathrm{X}_0) \to \text{Weil-NS}(\mathrm{X}_0) \tag{3.66}$$

induce continuous operators

$$f^*, f_* : \mathrm{L}^2(\mathrm{X}_0) \to \mathrm{L}^2(\mathrm{X}_0) \tag{3.67}$$

Furthermore, we have the following properties in $\mathrm{L}^2(\mathrm{X}_0)$.

(1) $(f^n)^* = (f^*)^n$*;*

(2) $\forall \alpha, \beta \in \mathrm{L}^2(\mathrm{X}_0), f^*\alpha \cdot \beta = \alpha \cdot f_*\beta$.

(3) $\forall \alpha \in \mathrm{L}^2(\mathrm{X}_0), f^*\alpha \cdot f^*\alpha = e(f)\alpha \cdot \alpha$ *where $e(f)$ is the topological degree of f.*

In particular, if $f \in \mathrm{Aut}(X_0)$ then f^* is an isometry of $\mathrm{L}^2(\mathrm{X}_0)$ viewed as an infinite dimensional hyperbolic space (see [CLC13]).

Theorem 3.2.28 ([BFJ08, DF21])**.** *Suppose that $\lambda_1(f)^2 > \lambda_2(f)$, then there exist nef classes $\theta^*, \theta_* \in \mathrm{L}^2(\mathrm{X}_0)$ unique up to multiplication by a positive constant such that*

(1) $f^*\theta^* = \lambda_1\theta^*$.

(2) $f_*\theta_* = \lambda_1\theta_*$.

(3) For all $\alpha \in \mathrm{L}^2(\mathrm{X}_0)$,

$$\frac{1}{\lambda_1^n}(f^n)^*\alpha = (\alpha \cdot \theta_*)\theta^* + {}^1O_\alpha\left(\left(\frac{\lambda_2}{\lambda_1^2}\right)^{n/2}\right), \tag{3.68}$$

$$\frac{1}{\lambda_1^n}(f^n)_*\alpha = (\alpha \cdot \theta^*)\theta_* + O_\alpha\left(\left(\frac{\lambda_2}{\lambda_1^2}\right)^{n/2}\right). \tag{3.69}$$

1. $A = O_\alpha(B)$ means that there exists a constant $C(\alpha) > 0$ such that $A \leqslant C(\alpha)B$.

In particular, for all $\alpha, \beta \in \mathrm{L}^2(\mathrm{X}_0)$,

$$\lim_n \frac{1}{\lambda_1^n}(f^n)^*\alpha\cdot\beta = (\alpha\cdot\theta_*)(\beta\cdot\theta^*). \tag{3.70}$$

Furthermore, θ^* *and* θ_* *satisfy*

$$(\theta^*)^2 = 0, \quad \theta_*\cdot\theta^* > 0 \tag{3.71}$$

We call θ^* and θ_* the *eigenclasses* of f.

Sketch of proof. We sketch here the proof for θ^*. Let X be a completion of X_0. The pull back f^* induces a linear map $f_X^* : \mathrm{NS}(X) \to \mathrm{NS}(X)$. Let ρ_X be the spectral radius of this map. We have for any ample class $w \in \mathrm{NS}(X)$ that $\rho_X = \lim_{n\to\infty}((f_X^*)^n w\cdot w)^{1/n}$. Now, f_X^* the cone C_X of nef classes in $\mathrm{NS}(X)_{\mathbf{R}}$. This is a closed convex cone with compact basis and non-empty interior. By a Perron-Frobenius type argument, there exists $\theta_X \in C_X$ such that $f_X^*\theta_X = \rho_X\theta_X$.

Now, Let (X_N) be a sequence of completions of X_0 such that $X_1 = X$ and X_{N+1} is a composition of blowups of X_N at infinity such that the lift of f to $F_N : X_{N+1} \to X_N$ is regular, we denote by $\pi_N : X_{N+1} \to X_N$ the induced morphism of completions. Let $\rho_N := \rho_{X_N}$ and $\theta_N := \theta_{X_N}$. One can show that $\lim_N \rho_N = \lambda_1$. By construction, we have that for all $N \geqslant 1$, the element $f^*\theta_N - \rho_N\theta_N \in \text{Weil-NS}(\mathrm{X}_0)_{\mathbf{R}}$ has incarnation zero in X_N, hence it tends to zero in $\text{Weil-NS}(\mathrm{X}_0)_{\mathbf{R}}$. We can normalize all θ_N such that $\theta_N\cdot w = 1$ where w is an ample class of $\mathrm{NS}(X)$. Now, the set $\{W \in \text{Weil-NS}(\mathrm{X}_0)_{\mathbf{R}} | W\cdot w = 1\}$ is a compact subset of $\text{Weil-NS}(\mathrm{X}_0)$ so the sequence (θ_N) has an accumulation point $\theta^* \in \text{Weil-NS}(\mathrm{X}_0)$ that is nef, effective and we get $f^*\theta^* = \lambda_1\theta^*$. □

3.3 Valuations

3.3.1 Valuations and completions

Our general reference for the theory of valuations is [Vaq00]. Let R be a commutative $\mathbf{k}$-algebra that is also an integral domain, a *valuation* on R is a function $\nu : R \to \mathbf{R} \cup \{\infty\}$ such that

(i) $\nu(\mathbf{k}^*) = 0$;

(ii) For all $P, Q \in R, \nu(PQ) = \nu(P) + \nu(Q)$;

(iii) For all $P, Q \in R, \nu(P+Q) \geqslant \min(\nu(P), \nu(Q))$;

(iv) $\nu(0) = +\infty$.

If I is an ideal of R, we set $\nu(I) := \min_{i \in I} \nu(i)$. If $S \subset I$ is a set of generators, then

$$\nu(I) = \min_{s \in S} \nu(s). \tag{3.72}$$

Remark 3.3.1. In [Abh56] A *valuation* can take the value $+\infty$ only at 0 but we do not require such a property. Let $\mathfrak{p}_\nu = \{a \in R : \nu(a) = \infty\}$ then $\mathfrak{p}_\nu$ is a prime ideal of R that we call the *bad ideal* of ν. If ν is a valuation on R, it defines naturally a valuation in the sense of [Abh56] on the quotient field $R/\mathfrak{p}_\nu$. Furthermore ν can be naturally extended to a valuation on the ring $R_{\mathfrak{p}_\nu}$ via the formula $\nu(p/q) = \nu(p) - \nu(q)$. In particular, if $\mathfrak{p}_\nu = \{0\}$, then ν defines a valuation over $\operatorname{Frac} R$.

Let X be a completion of X_0 and let ν be a valuation over $B := \mathcal{O}_X(X_0)$. Let $\mathfrak{p}_\nu$ be the bad ideal of ν. Consider $B_{\mathfrak{p}_\nu}$ the localization of B at $\mathfrak{p}_\nu$. Set

$$\mathcal{O}_\nu := \{x \in B_{\mathfrak{p}_\nu} : \nu(x) \geqslant 0\}. \tag{3.73}$$

This is a subring of $B_{\mathfrak{p}_\nu}$. If $\mathfrak{p}_\nu = \{0\}$, then this is the classical *valuation ring* of ν.

Lemma 3.3.2. *The subring $\mathcal{O}_\nu$ is a local ring, its maximal ideal is*

$$\mathfrak{m}_\nu := \{x \in \mathcal{O}_\nu : \nu(x) > 0\}. \tag{3.74}$$

Proof. It suffices to show that if $\nu(x) = 0$, then x is invertible in $\mathcal{O}_\nu$ but this is obvious since $\nu(x^{-1}) = -\nu(x) = 0$. □

One defines naturally a valuation ν on $C := B/\mathfrak{p}_\nu$, let L be the fraction field of C and O be the valuation ring of L with respect to ν. Then, we have the natural isomorphisms

$$L \simeq B_{\mathfrak{p}_\nu}/\mathfrak{p}_\nu \text{ and } \mathcal{O}_\nu/\mathfrak{p}_\nu \simeq O \tag{3.75}$$

Geometrically, the Zariski closure of $\mathfrak{p}_\nu$ inside X defines an irreducible closed subscheme Y of X and L is isomorphic to the field of rational functions on Y.

Two valuations ν_1, ν_2 are *equivalent* if there exists a real number $\lambda > 0$ such that $\nu_1 = \lambda\nu_2$. Let R, R' be two integral domains with a homomorphism of schemes $\varphi : \operatorname{Spec} R' \to \operatorname{Spec} R$;

it induces a ring homomorphism $\varphi^* : R \to R'$. If ν is a valuation on R' we define $\varphi_*\nu$ the *pushforward* by φ of ν by

$$\forall P \in R, \varphi_*\nu(P) = \nu(\varphi^*(P)). \tag{3.76}$$

Let $X_0 = \operatorname{Spec} A$ as in Section 3.2.2. Denote by $\mathcal{V}$ the set of valuations on A. We equip this space with the topology of weak convergence, that is the coarsest topology such that the evaluation map $\nu \in \mathcal{V} \mapsto \nu(P)$ is continuous for all $P \in A$. If f is an endomorphism of X_0, then f induces a continuous map $f_* : \mathcal{V} \to \mathcal{V}$.

Via the natural isomorphism $\iota_X^* : O_X(X_0) \to A$, every $\nu \in \mathcal{V}$ induces a valuation $(\iota_X)_*\nu$ on $O_X(X_0)$, namely

$$\forall P \in O_X(X_0), \quad (\iota_X)_*\nu(P) := \nu(\iota_X^* P). \tag{3.77}$$

We will denote $(\iota_X)_*\nu$ by ν_X for every valuation ν on A.

Remark 3.3.3. Take a morphism of completions $\pi : X_1 \to X_2$ and ν a valuation on A. Then, $(\iota_{X_2})_*\nu = (\pi^{-1} \circ \iota_{X_1})_*\nu$. In particular $\pi_*\nu_{X_2} = \nu_{X_1}$.

Remark 3.3.4. In the language of Berkovich theory, the set $\mathcal{V}$ is the Berkovich analytification of X_0 over $\mathbf{k}$ where we have endowed $\mathbf{k}$ with the trivial valuation (see [Ber12]).

Example 3.3.5 (Divisorial valuations)**.** Let X be a completion of X_0 and E be a prime divisor of X. Let ord_E be the valuation on $\mathbf{k}(X)$ such that for any $f \in \mathbf{k}(X), \operatorname{ord}_E(f)$ is the order of vanishing of f along E. Any valuation ν on A such that ν_X is equivalent to ord_E for some prime divisor E in some completion X is called a *divisorial* valuation. In that case $\mathfrak{p}_\nu = \{0\}$ and ν extends uniquely to a valuation on $\operatorname{Frac} A$. For example if $X_0 = \mathbf{A}^2$ and $X = \mathbf{P}^2$, then let L_∞ be the line at infinity, we have $\forall P \in \mathbf{k}[x,y], \operatorname{ord}_{L_\infty}(P) = -\deg(P)$. If instead we take the completion $P^1 \times \mathbf{P}^1$, decompose $\mathbf{A}^2 = \mathbf{A}^1 \times \mathbf{A}^1$ and let x,y be the affine coordinate of $\mathbf{A}^2$ each being an affine coordinate of $\mathbf{A}^1$. Let $L_x = \{\infty\} \times \mathbf{P}^1$ and $L_y = \mathbf{P}^1 \times \{\infty\}$, then

$$\forall P \in \mathbf{k}[x,y], \operatorname{ord}_{L_x}(P) = -\deg_x(P), \quad \operatorname{ord}_{L_y}(P) = -\deg_y(P) \tag{3.78}$$

where $\deg_x$ (respectively $\deg_y$) is the degree with respect to the variable x (respectively y).

Example 3.3.6 (Curve valuations)**.** Let X be a completion of X_0, let $p \in \partial_X X_0$ C be the germ of a (formal) curve at p. This means that C is defined as $\varphi = 0$ for φ in the completion $\widehat{O}_{X,p}$ of the local ring $O_{X,p}$ at p. If $\psi \in \widehat{O}_{X,p}$ is another germ of a formal curve at p, we define the intersection number at p by

$$\{\varphi = 0\} \cdot_p \{\psi = 0\} := \dim_{\mathbf{k}} \widehat{O}_{X,p} / \langle \varphi, \psi \rangle. \tag{3.79}$$

This number is equal to ∞ exactly when one of the germs divides the other. We first define a valuation $\nu_{C,p}$ on $\widehat{O}_{X,p}$ by

$$\nu_{C,p}(\psi) = \{\psi = 0\} \cdot_p C \tag{3.80}$$

Suppose φ is not divisible by the local equation of any component of $\partial_X X_0$. For any $P \in O_X(X_0)$, P can be written as $P = \psi_1^{\alpha_1} \cdots \psi_r^{\alpha_r}$ with $\psi_i \in \widehat{O}_{X,p}$ irreducible and $\alpha_i \in \mathbf{Z}$. We define

$$\nu_{C,p}(P) := \sum_i \alpha_i \nu_{C,p}(\psi_i) \in \mathbf{R} \cup \{\infty\} \tag{3.81}$$

Then $\nu_{C,p}$ is a valuation on $O_X(X_0)$. Any valuation on A such that ν_X is equivalent to $\nu_{C,p}$ is called a *curve* valuation. If ν is a valuation such that $\mathfrak{p}_\nu \neq \{0\}$, then ν is a curve valuation (see [FJ04] and Proposition 3.3.9 below). We will make the following distinction, if C is defined by $\varphi \in O_{X,p}$ we will say that $\nu_{C,p}$ is an *algebraic* curve valuation. Otherwise, we will say that it is a *formal* curve valuation.

If φ was divisible by the local equation of a component of $\partial_X X_0$, then $\nu_{C,p}$ would not define a valuation on A as some regular functions $P \in A$ would have a pole along C and $\nu(P)$ would be equal to $-\infty$.

3.3.2 Valuations over $\mathbf{k}[[x,y]]$

We recall some results about valuations from [FJ04] and [FJ07]. Let R be a regular local ring with maximal ideal $\mathfrak{m}$. We say that a valuation on R is *centered* if $v \geqslant 0$ and $v_{|\mathfrak{m}} > 0$. Here we set $R := \mathbf{k}[[x,y]]$ for our local ring. Its maximal ideal is $\mathfrak{m} := (x,y)$ we will study the set of centered valuations on R.

Proposition 3.3.7 (Proposition 2.10 [FJ04], [Spi90])**.** *Any valuation on $\mathbf{k}[x,y]$ centered at the origin extends uniquely to a centered valuation on R as follows. Let $\varphi \in R$ and let φ_n be the polynomial of degree n such that $\varphi = \lim \varphi_n$. Then,*

$$\nu(\varphi) = \lim_{n\to\infty} \min(\nu(\varphi_n), n). \tag{3.82}$$

Corollary 3.3.8. *Let R' be regular local ring of dimension 2 over $\mathbf{k}$, then the $\mathfrak{m}_{R'}$-adic completion $\widehat{R'}$ of R' is isomorphic to R. Any centered valuation on R' extends uniquely to a centered valuation on $\widehat{R'}$.*

Proof. Let (x,y) be a regular sequence of R', that is $\mathfrak{m}_{R'} = (x,y)$. It exists because R' is a regular local ring of dimension 2. Then, $\widehat{R}'$ is isomorphic to $\mathbf{k}[[x,y]]$. Let v be a centered valuation on R'. We have that $\mathbf{k}[x,y] \subset R'$, so v induces a valuation on $\mathbf{k}[x,y]$ that is centered at the origin and we can apply the previous proposition to conclude. □

Let p be a regular point on a surface X and let $R = \widehat{O_{X,p}}$ we define 4 types of valuations over R.

3.3.2.1 Divisorial valuations

A valuation ν over R is *divisorial* if there exists a sequence of blow-up $\pi : (Y, \mathrm{Exc}(\pi)) \to (X,x)$ such that ν is equivalent to $\pi_* \operatorname{ord}_E$ for some prime divisor $E \subset \mathrm{Exc}(\pi)$.

3.3.2.2 Quasimonomial valuations

Let $\pi : (Y, \mathrm{Exc}(\pi)) \to (X,x)$ be a sequence of blow-ups and let $q \in \mathrm{Exc}(\pi)$. A *monomial* valuation at q is a valuation ν on $\widehat{O_{Y,q}}$ such that there exists $s,t > 0$,

$$\nu\left(\sum_{i,j} a_{ij} x^i y^j\right) = \min\left\{si + tj : a_{ij} \neq 0\right\} \tag{3.83}$$

for some local coordinates at q. We write $\nu = \nu_{s,t}$.

A valuation over $\widehat{O_{X,p}}$ is called *quasimonomial* if there exists a sequence of blow-ups $\pi : (Y, \mathrm{Exc}(\pi)) \to (X,p)$ such that $\nu = \pi_* \nu_{s,t}$. Quasimonomial valuations split into two categories: if $s/t \in \mathbf{Q}$, one can show actually that ν is divisorial. Otherwise $s/t \in \mathbf{R} \backslash \mathbf{Q}$, ν is not divisorial and we say that it is *irrational*.

3.3.2.3 Curve valuations

Let $\varphi \in \widehat{\mathfrak{m}_p}$ be irreducible, we define ν_φ by

$$\forall \psi \in \widehat{O_{X,p}}, \quad \nu_\varphi(\psi) = \frac{\{\varphi = 0\} \cdot \{\psi = 0\}}{m(\varphi)} \tag{3.84}$$

where $m(\varphi)$ is the order of vanishing of φ at the origin. A *curve* valuation is a valuation equivalent to ν_φ for some $\varphi \in \widehat{\mathfrak{m}_p}$ irreducible.

3.3.2.4 Infinitely singular valuations

These are all the remaining valuations. They have a nice description in term of Puiseux series (see [FJ04] Section 4.1 for more details). Briefly, to any valuation ν of $\mathbf{k}[[x,y]]$, one can associated a generalized power series

$$\hat{\varphi} = \sum_j a_j x^{\beta_j} \tag{3.85}$$

with $a_j \in \mathbf{k}$ and $\beta_j \in \mathbf{Q}$. The *infinitely singular* valuations are exactly the valuations such that $\lim_j \beta_j \neq +\infty$.

Proposition 3.3.9 ([FJ04])**.** *There are four types of centered valuations on R: divisorial, irrational, curve valuations and infinitely singular valuations. The only type of valuation* ν *such that* $\mathfrak{p}_\nu = \{\nu = +\infty\} \neq 0$ *are curve valuations*

Remark 3.3.10. Instead of looking at valuations over R with values in $\mathbf{R}$, we can look at valuations with values in a totally ordered abelian group Γ, these are called *Krull valuations* (see [FJ04], section 1.3) and they have the advantage to always extend to $\operatorname{Frac} R$. We can make any curve valuation into a Krull valuation by the following procedure (see [FJ04], section 1.5.5): Let $\varphi \in \mathfrak{m}$ and consider the curve valuation ν_φ. Let $\Gamma = \mathbf{Z} \times \mathbf{Q}$ with the lexicographical order, we define $\hat{\nu}_\varphi : R \to \Gamma$ as follows. For any $\psi \in R$, there exists an integer $k \in \mathbf{N}$ such that

$$\psi = \varphi^k \hat{\psi} \tag{3.86}$$

where $\hat{\psi}$ is not divisible by φ. Set

$$\hat{\nu}(\psi) := (k, \nu_\varphi(\hat{\psi})) \tag{3.87}$$

Notice that $\nu_\varphi(\psi) = \infty \Leftrightarrow p_1(\hat{\nu}_\varphi(\psi)) > 0$ where $p_1 : \Gamma \to \mathbf{Z}$ is the projection to the first coordinate and if $\nu_\varphi(\psi) < +\infty$, then $\hat{\nu}_\varphi(\psi) = (0, \nu_\varphi(\psi))$. We will not need Krull valuations in the rest of the text. But this argument comes in handy for the proof of Proposition 3.3.18 so we state it here.

3.3.3 The center of a valuation

Let X be a completion of X_0 and let ν be a valuation on $\mathcal{O}_X(X_0)$. A *center* of ν on X is a scheme-theoretic point $p \in X$ such that $\mathcal{O}_\nu$ dominates the local ring $\mathcal{O}_{X,p}$ (i.e $\mathcal{O}_{X,p} \subset \mathcal{O}_\nu$ and $\mathfrak{m}_p \subset \mathfrak{m}_\nu$).

If such a p exists then ν induces a *centered* valuation on $\mathcal{O}_{X,p}$ (cf 3.3.2) and in particular for any open affine subset $U \subset X$ that contains p, ν induces a valuation on $\mathcal{O}_X(U)$ via the inclusion $\mathcal{O}_X(U) \subset \mathcal{O}_{X,p}$.

Lemma 3.3.11. *The center of* ν *on* X *always exists and is unique.*

Proof. Let $\mathcal{O}_\nu$ be the subring of $\mathbf{k}(X)$ where ν is $\geqslant 0$; it contains $\mathbf{k}^*$. Let $L = B_{\mathfrak{p}_\nu}/\mathfrak{p}_\nu$ and $\mathcal{O} = \mathcal{O}_\nu/\mathfrak{p}_\nu$. If p is a center of ν on X then we have the following commutative diagram of ring homomorphism

$$\mathcal{O}_{X,p} \hookrightarrow \mathcal{O}_\nu \twoheadrightarrow \mathcal{O} \hookrightarrow L \twoheadleftarrow B_{\mathfrak{p}_\nu} \ ; \tag{3.88}$$

inducing the following commutative diagram of scheme morphisms

$$\begin{tikzcd}
\operatorname{Spec} L \arrow[rrr] \arrow[d] & & & X \arrow[d] \\
\operatorname{Spec} \mathcal{O} \arrow[r] & \operatorname{Spec} \mathcal{O}_\nu \arrow[r] & \operatorname{Spec} \mathcal{O}_{X,p} \arrow[r] \arrow[ur] & \operatorname{Spec} \mathbf{k}
\end{tikzcd} \tag{3.89}$$

Since X is proper over $\mathbf{k}$ (it's a projective variety), the valuative criterion of properness ([Har77]) shows that if the center exists, then it is unique. For the existence, Let $x \in X$ be the image of the maximal ideal of $\mathcal{O}$, then x is the center of ν on X. Indeed, the image of $\operatorname{Spec} L$ is the prime ideal $\mathfrak{p}_\nu$ of $\mathcal{O}_X(X_0)$ and x belongs to its closure, therefore $\mathcal{O}_{X,x} \subset B_{\mathfrak{p}_\nu}$ and the morphism of local rings $\mathcal{O}_{X,x} \to \mathcal{O}$ shows that $\mathcal{O}_\nu$ dominates $\mathcal{O}_{X,x}$.

□

The *center* of ν on X is the center of ν_X we will denote it by $c_X(\nu)$.

Example 3.3.12. Let ν be a divisorial valuation over A and let X be a completion of X_0 such that $\nu_X \simeq \operatorname{ord}_E$ for some prime divisor E of X, then the center of ν on X is the generic point x_E of E. Indeed, the ring of regular function at the generic point of E is a discrete valuation ring since E is of codimension 1. In that case, we will identify the center with its closure and say that the center of ν on X is the prime divisor E. In fact a valuation is divisorial if and only if its center on some completion of X_0 is a prime divisor because if $c_X(\nu) = x_E$, then ν and ord_E defines the same valuation ring which is a discrete valuation ring, therefore they are equivalent.

Example 3.3.13. If ν is a curve valuation and X is a completion of X_0 such that $(\iota_X)_*\nu \simeq \nu_{C,p}$, then the center of ν on X is the closed point p.

A valuation over $A = \mathbf{k}[X_0]$ is *centered at infinity* if there exists a completion X such that $c_X(\nu) \notin X_0$.

Corollary 3.3.14. *Let $X_0 = \operatorname{Spec} A$ be a smooth affine surface, there are exactly four types of valuations centered at infinity over A: divisorial valuations, irrational valuations, curve valuations and infinitely singular valuations. If ν is a valuation such that $\mathfrak{p}_\nu \neq \{0\}$, then ν is a curve valuation.*

Proof. let ν be a valuation over A and let $c_X(\nu)$ be its center on some completion X. If $c_X(\nu)$ is a prime divisor at infinity then ν is divisorial. Otherwise, $c_X(\nu)$ is a regular point at infinity and ν induces a centered valuation over $\widehat{\mathcal{O}_{X,p}}$. The result follows from the classification of centered valuations over $\mathbf{k}[[x,y]]$ from Proposition 3.3.9. □

Definition 3.3.15.

- Let X be a good completion of X_0 and $p \in \partial_X X_0$ a point at infinity. Following [FJ04], we say that p is a *free* point if it belongs to a unique prime divisor at infinity and we say that it is a *satellite point* otherwise, i.e it is the intersection point of two prime divisors at infinity.

- Let ν be a valuation over A centered at infinity. Let $p_1 = c_X(\nu)$ be its center on X and $X_1 := X$. We define the following sequence: If p_n is a prime divisor, then the sequence stops, else p_n is a closed point of X_n and we define X_{n+1} as the blow up of p_n, then define $p_{n+1} := c_{X_{n+1}}(\nu)$. This is the *sequence of centers* of ν with respect to X.

We adopt the following convention: When we write "let $p \in E$ be a free point (at infinity)" this means that E is the unique prime divisor at infinity on which p lies. If we write "let $p = E \cap F$ be a satellite point", this means that E and F are the two prime divisors at infinity such that $p = E \cap F$ (Recall that we only work with good completions).

Proposition 3.3.16 ([FJ04], Section 6.2)**.** *Let ν be a valuation centered at infinity. Let X be a completion of X_0 and (p_n) the sequence of centers (above X) associated to ν. Then,*

(1) ν is divisorial if and only if the sequence (p_n) is finite.

(2) If ν is irrational, then (p_n) contains finitely many free points.

(3) if ν is a curve valuation, then (p_n) contains finitely many satellite points.

(4) If ν is infinitely singular, then (p_n) contains infinitely many free points.

Proof. Assertion 1 is clear since the sequence (p_n) stops if and only if p_n is a prime divisor at infinity. Assertion 2 and 4 follows from [FJ04] Theorem 6.10 and Assertion 3 follows from [FJ04] Proposition 6.12. □

3.3.4 Image of a valuation via an endomorphism

Let $f : X_0 \to X_0$ be a endomorphism of X_0, it induces a map f_* on the space of valuation $f_* : \mathcal{V} \to \mathcal{V}$ via the formula

$$\forall P \in A, \forall \nu \in \mathcal{V}, \quad f_*\nu(\varphi). \tag{3.90}$$

We will denote by $f_\bullet$ the induced map $f_\bullet : \widehat{\mathcal{V}} \to \widehat{\mathcal{V}}$.

Proposition 3.3.17 (Proposition 2.4 of [FJ07])**.** *Suppose that f is dominant, the map f_* preserves the sets of divisorial, of irrational and of infinitely singular valuations. If ν_C is a curve valuation such that f does not contract C, then $f_*\nu_C$ is a curve valuation. If f contracts C, then $f_*\nu_C$ is a divisorial valuation.*

We will use this proposition in the following context. Let X, Y be two completions of X_0 such that the lift $F : X \to Y$ of f is regular. For any point $p \in X \backslash X_0$, we have a map $F_* : \mathcal{V}_X(p) \to \mathcal{V}_Y(F(p))$ that preserves the type of the valuations. The only curve that might be contracted by F to q are the divisors at infinity; but the curve valuation that they define do not define valuations on A.

Proposition 3.3.18. *Let $f : X_0 \to X_0$ be a dominant endomorphism of topological degree λ_2. Then, every valuation v on A has at most λ_2 preimages under f_*.*

Proof. Suppose first that the valuation ν takes the value $+\infty$ only for 0. Therefore, it extends to a valuation on $K = \operatorname{Frac} A$. The extension $f^*K \hookrightarrow K$ is a finite extension of degree λ_2. The valuation v induces a valuation on f^*K and every valuation w such that $f_*w = v$ is an extension of $v_{|f^*K}$ to K. By [ZS60] Theorem 19 p.55, there are at most λ_2 extension of $v_{|f^*K}$.

If now $\mathfrak{p}_\nu = \{v = +\infty\} \neq 0$, then we know that ν is a curve valuation. By Remark 3.3.10, ν can be made into a Krull valuation $\hat{\nu}$. Since $\hat{\nu}$ is a Krull valuation, it extends to a Krull valuation over K and f_*v extends to a Krull valuation over f^*K. The same argument as above still works as [ZS60] deals with Krull valuations. □

3.4 Tree structure on the space of valuations

3.4.1 Trees

For this section, we refer to [FJ04] Section 3.1. Let $(\mathcal{T}, \leqslant)$ be a partially ordered set, a subset $\mathcal{S} \subset \mathcal{T}$ is *full* if for every $\sigma, \sigma' \in \mathcal{S}, \tau \in \mathcal{T}, \sigma \leqslant \tau \leqslant \sigma' \Rightarrow \tau \in \mathcal{S}$.

Definition 3.4.1. Let $\Lambda = \mathbf{N}, \mathbf{Q}, \mathbf{R}$. An *interval* in Λ is a subset $I \subset \Lambda$ such that for all $x, y, z \in \Lambda$, if $x \leqslant y \leqslant z$ and $x, z \in I$, then $y \in I$. If $(\mathcal{T}, \leqslant)$ be a partially ordered set, then $(\mathcal{T}, \leqslant)$ is a rooted Λ-*tree* if

(i) $\mathcal{T}$ has a unique minimal element τ_0 called the *root* of $\mathcal{T}$.

(ii) If $\tau \in \mathcal{T}$, the set $\{\sigma \in \mathcal{T} : \sigma \leqslant \tau\}$ is [2]isomorphic to an interval in Λ.

(iii) Every full, totally ordered subset of $\mathcal{T}$ is isomorphic to an interval in Λ.

A *parametrized*-Λ tree is a rooted Λ-tree $\mathcal{T}$ with a map $\alpha : \mathcal{T} \to \Lambda \cup \{\infty\}$ such that the restriction of α to any full totally ordered subset of $\mathcal{T}$ induces a bijection with an interval in Λ. The map α is called the *parametrisation*.

A rooted $\mathbf{R}$-tree is called *complete* if every increasing sequence has an upper bound.

A *subtree* $\mathcal{S}$ of a Λ-tree $\mathcal{T}$ is a subset such that $(\mathcal{S}, \leqslant_{|\mathcal{S}})$ is a Λ-tree. An *inclusion* of Λ-trees is an order preserving injection $\iota : \mathcal{S} \to \mathcal{T}$. In particular, $\iota(\mathcal{S})$ is a subtree of $\mathcal{T}$.

If $\mathcal{T}$ is an $\mathbf{R}$-tree and $\tau_1, \tau_2 \in \mathcal{T}$, then the *minimum* $\tau_1 \wedge \tau_2 \in \mathcal{T}$ exists by completeness of $\mathbf{R}$. We define the set

$$[\tau_1, \tau_2] := \{\tau \in \mathcal{T} : \tau_1 \wedge \tau_2 \leqslant \tau \leqslant \tau_1 \text{ or } \tau_1 \wedge \tau_2 \leqslant \tau \leqslant \tau_2\} \tag{3.91}$$

and we call it a *segment*. The segments $[\tau_1, \tau_2), (\tau_1, \tau_2]$ and (τ_1, τ_2) are defined similarly. A *finite* subtree of $\mathcal{T}$ is a subtree that consists of a finite union of segments in $\mathcal{T}$.

If $\mathcal{T}$ is an $\mathbf{R}$-tree, a *tangent* vector $\vec{v}$ at $\tau \in \mathcal{T}$ is an equivalence class where

$$\tau' \sim \tau'' \Leftrightarrow [\tau, \tau'] \cap [\tau, \tau''] \neq \varnothing. \tag{3.92}$$

We define the *weak* topology on $\mathcal{T}$ by the topology generated by the sets

$$U(\vec{v}) := \left\{\tau' \in \mathcal{T} : \tau' \text{ represents } \vec{v}\right\}. \tag{3.93}$$

Theorem 3.4.2 ([FJ04] Proposition 3.12)**.** *We have the following*

- *Every rooted* $\mathbf{R}$*-tree* $\mathcal{T}$ *admits a completion* $\overline{\mathcal{T}}$ *that is a complete rooted* $\mathbf{R}$*-tree.*
- *Every rooted* $\mathbf{Q}$*-tree* $\mathcal{T}_{\mathbf{Q}}$ *admits a completion* $\mathcal{T}_{\mathbf{R}}$ *into a rooted* $\mathbf{R}$*-tree, i.e there exists an order preserving injection* $\iota : \mathcal{T}_{\mathbf{Q}} \hookrightarrow \mathcal{T}_{\mathbf{R}}$ *such that*

2. isomorphic here means that there exists an order preserving bijection.

(1) *If τ_0 is the root of $\mathcal{T}_{\mathbf{Q}}, \iota(\tau_0)$ is the root of $\mathcal{T}_{\mathbf{R}}$.*

(2) *$\iota(\mathcal{T}_{\mathbf{Q}})$ is weakly dense in $\mathcal{T}_{\mathbf{R}}$*

(3) *$\mathcal{T}_{\mathbf{R}}$ is minimal for this property.*

- *If $\alpha_{\mathbf{Q}} : \mathcal{T}_{\mathbf{Q}} \to \mathbf{Q}_+$ is a parametrisation of $\mathcal{T}_{\mathbf{Q}}$, then there exists a unique parametrisation $\alpha_{\mathbf{R}}$ of $\mathcal{T}_{\mathbf{R}}$ such that $\alpha_{\mathbf{Q}} = \alpha_{\mathbf{R}} \circ \iota$.*

3.4.2 The local tree structure of the space of valuations

We denote by $\mathcal{V}_0$ the set of centered valuations on R where $R = \mathbf{k}[[x,y]]$. Define the *multiplicity valuation* $\nu_{\mathfrak{m}}$ by $\nu_{\mathfrak{m}}(\varphi) = \max\{n \geqslant 0 : \varphi \in \mathfrak{m}^n\}$. We will sometimes write $m(\varphi)$ instead of $\nu_{\mathfrak{m}}(\varphi)$. Let $\mathcal{V}_{\mathfrak{m}} \subset \mathcal{V}_0$ be the set of centered valuations on R such that $\nu(\mathfrak{m}) = 1$ and consider the following order relation on $\mathcal{V}_{\mathfrak{m}}$: $\nu \leqslant w \iff \forall \varphi \in R, \nu(\varphi) \leqslant w(\varphi)$. With this order relation $\mathcal{V}$ becomes a complete rooted $\mathbf{R}$-tree called the *valuative tree* ([FJ04] Theorem 3.14) rooted in $\nu_{\mathfrak{m}}$. The ends of $\mathcal{V}_{\mathfrak{m}}$ consist of the curve valuations and the infinitely singular ones. The interior points are all quasimonomial valuations, all divisorial valuations are branching points whereas all the irrational valuations are regular points (i.e admit only two tangent vectors). Define on $\mathcal{V}_{\mathfrak{m}}$ the following function

$$\alpha(\nu) := \sup\left\{\frac{\nu(\varphi)}{m(\varphi)} : \varphi \in \mathfrak{m}, \nu_\varphi \geqslant \nu\right\}. \tag{3.94}$$

It is called the *skewness* function (see [FJ04] §3.3)

Proposition 3.4.3 (Proposition 3.25 of [FJ04])**.** *The skewness function $\alpha : \mathcal{V}_{\mathfrak{m}} \to [1, +\infty]$ defines a parametrisation of $\mathcal{V}_{\mathfrak{m}}$. We have the following properties.*

- $\alpha(\nu) = 1 \Leftrightarrow \nu = \nu_{\mathfrak{m}}$.
- *Let $\varphi \in \mathfrak{m}$ be irreducible and let $\nu \in \mathcal{V}_{\mathfrak{m}}$, then*

$$\forall \varphi \in \mathfrak{m}, \nu(\varphi) = \alpha(\nu \wedge \nu_\varphi) m(\varphi) \tag{3.95}$$

- *If ν is divisorial, then $\alpha(\nu) \in \mathbf{Q}$*
- *if ν is irrational, then $\alpha(\nu) \in \mathbf{R} \backslash \mathbf{Q}$.*
- *If $\mathcal{V}_{\mathfrak{m},\mathrm{div}}$ is the subset of $\mathcal{V}_{\mathfrak{m}}$ consisting of the divisorial valuations, then $(\mathcal{V}_{\mathfrak{m},\mathrm{div}}, \alpha)$ is a parametrized $\mathbf{Q}$-tree.*

We can define two topologies over $\mathcal{V}_\mathfrak{m}$. The first one is the weak topology being the coarsest topology such that for all $\varphi \in R$, the evaluations map $\nu \in \mathcal{V}_\mathfrak{m} \mapsto \nu(\varphi)$ is continuous. The second is the weak topology given by the $\mathbf{R}$-tree structure on $\mathcal{V}_\mathfrak{m}$.

Proposition 3.4.4 ([FJ04], Theorem 5.1)**.** *The weak topology over $\mathcal{V}_\mathfrak{m}$ given by the evaluation maps $\nu \in \mathcal{V}_\mathfrak{m} \mapsto \nu(\varphi)$ and the weak topology induced by the tree structure of $\mathcal{V}_\mathfrak{m}$ are the same.*

Let X be a good completion of $X_0 = \operatorname{Spec} A$ and let p be a smooth point of X. Take local coordinates z, w at p, then the completion of the local ring $\mathcal{O}_{X,p}$ with respect the maximal ideal $\mathfrak{m}_p$ is isomorphic to $\mathbf{k}[[z,w]]$. Let $\mathcal{V}_X(p)$ be the set of valuations ν on A centered at p. We will denote by $\mathcal{V}_X(p;\mathfrak{m}_p)$ the subset of $\mathcal{V}_X(p)$ of valuations ν such that $\nu(\mathfrak{m}_p) = 1$. The space $\mathcal{V}_X(p;\mathfrak{m}_p)$ is an $\mathbf{R}$-tree isomorphic rooted in $\nu_{\mathfrak{m}_p}$. We make its structure precise.

Proposition 3.4.5. *The* $\mathbf{R}$*-tree* $\mathcal{V}_X(p;\mathfrak{m}_p)$ *is* not *complete.*

(1) If $p \in E$ is a free point then $\mathcal{V}_X(p;\mathfrak{m}_p)$ is isomorphic to $\mathcal{V}_\mathfrak{m} \setminus \{\nu_z\}$ where z is a local equation of E.

(2) If $p = E \cap F$ is a satellite point, then $\mathcal{V}_X(p;\mathfrak{m}_p)$ is isomorphic to $\mathcal{V}_\mathfrak{m} \setminus \{\nu_z, \nu_w\}$ where z, w are local coordinates at p with z a local equation of E and w a local equation of F.

Proof. If $p \in E$ is a free point, let z, w be local coordinates at p such that z is a local equation of E. Then, the completion of the local ring at p is isomorphic to $\mathbf{k}[[z,w]]$ by Theorem 3.1.2. Every $P \in A$ is of the form $P = \frac{\varphi}{z^a}$ with $a \geqslant 0$ and $\varphi \in \mathcal{O}_{X,p}$. Hence, a centered valuation on $\mathbf{k}[[z,w]]$ defines a valuation over A if and only if it is not the curve valuation ν_z. Hence we have an isomorphism $\mathcal{V}_X(p;\mathfrak{m}_p) \simeq \mathcal{V}_\mathfrak{m} \setminus \{\nu_z\}$.

If $p = E \cap F$ is a satellite point, then let z, w be local coordinates at p such that z is a local equation of E and w is a local equation of F. Every $P \in A$ is of the form $P = \frac{\varphi}{z^a w^b}$ where $a, b \geqslant 0$ and $\varphi \in \mathcal{O}_{X,p}$. Therefore a centered valuation on $\mathbf{k}[[z,w]]$ defines a valuation over A if and only if it is not the curve valuation ν_z or ν_w. Hence we have an isomorphism $\mathcal{V}_X(p;\mathfrak{m}_p) \to \mathcal{V}_\mathfrak{m} \setminus \{\nu_z, \nu_w\}$. □

3.4.3 The relative tree with respect to a curve $z = 0$

Let $R = \mathbf{k}[[x,y]]$ and let $\mathfrak{m}$ be the maximal ideal of R. Let $z \in \mathfrak{m}$ be irreducible such that $\nu_\mathfrak{m}(z) = 1$. One can consider the set $\mathcal{V}_z$ of centered valuations on R such that $\nu(z) = 1$; we also add the valuation ord_z to $\mathcal{V}_z$ defined by $\operatorname{ord}_z(\varphi) = \max\{n \geqslant 0 : z^n | \varphi\}$. (notice that ord_z is *not* centered, because for example if $x \neq z, \operatorname{ord}_z(x) = 0$). This is also a tree rooted in ord_z called the *relative tree* (see [FJ04] Proposition 3.61) with the order relation $\nu \leqslant_z \mu \Leftrightarrow \forall \varphi \in R, \nu(\varphi) \leqslant \mu(\varphi)$. We can define the weak topology on $\mathcal{V}_z$ being the coarsest topology such that the for all $\varphi \in R$, the evaluation map $\nu \in \mathcal{V}_z \mapsto \nu(\varphi)$ is continuous. There is also the weak topology given by the tree structure of $\mathcal{V}_z$.

Proposition 3.4.6 (Relative version of 3.4.4)**.** *The weak topology over $\mathcal{V}_z$ given by the evaluation maps $\nu \in \mathcal{V}_z \mapsto \nu(\varphi)$ and the weak topology induced by the tree structure of $\mathcal{V}_z$ are the same.*

Proposition 3.4.7 ([FJ04] Lemma 3.59)**.** *We have an onto map $N_z : \mathcal{V}_0 \to \mathcal{V}_z$ defined by*

$$\begin{aligned} N_z(\nu) &= \nu/\nu(z) \textit{ if } \nu \neq \nu_z \\ N_z(\nu_z) &= \operatorname{ord}_z . \end{aligned}$$

This map restricts to a homeomorphism $N_z : \mathcal{V}_{\mathfrak{m}} \to \mathcal{V}_z$ with respect to the weak topology. If $w \in \mathfrak{m}$ is irreducible, then the map $N_{z,w} := N_w \circ N_w^{-1} : \mathcal{V}_z \to \mathcal{V}_w$ is a homeomorphism for the weak topology.

The tree $\mathcal{V}_z$ comes with a skewness function $\alpha_z : \mathcal{V}_z \to [0, +\infty]$ and a multiplicity function $m_z(\varphi) = \nu_z(\varphi)$. The skewness is defined by

$$\alpha_z(\nu) := \sup \left\{ \frac{\nu(\psi)}{m_z(\psi)} | \psi \in \mathfrak{m}, \nu_\psi \geqslant \nu \right\} \tag{3.96}$$

Proposition 3.4.8 (Relative version of Proposition 3.4.3)**.** *The function $\alpha_z : \mathcal{V}_z \to [0, +\infty]$ defines a parametrisation of the tree $\mathcal{V}_z$. We have the following properties.*

- $\alpha_z(\nu) = 0 \Leftrightarrow \nu = \operatorname{ord}_z$.
- *Let $\varphi \in \mathfrak{m}$ be irreducible and let $\nu \in \mathcal{V}_z$, then*

$$\nu(\varphi) = \alpha_z(\nu \wedge N(\nu_\varphi)) m_z(\varphi). \tag{3.97}$$

- *If ν is divisorial or $\nu = \operatorname{ord}_z$, then $\alpha_z(\nu) \in \mathbf{Q}$*
- *If ν is irrational, then $\alpha_z(\nu) \in \mathbf{R} \backslash \mathbf{Q}$.*
- *If $\mathcal{V}_{z,\mathrm{div}}$ is the subset of $\mathcal{V}_z$ consisting of ord_z and divisorial valuations, then $\left(\mathcal{V}_{z,\mathrm{div}}, \alpha_z\right)$ is a parametrised $\mathbf{Q}$-tree.*

Proposition 3.4.9 ([FJ04], Proposition 3.65)**.** *We have the following relation*

$$\forall \nu \in \mathcal{V}_0, \quad \nu(z)^2 \alpha_z \left(\frac{\nu}{\nu(z)} \right) = \min\left(\nu(x), \nu(y)\right)^2 \alpha \left(\frac{\nu}{\min\left(\nu(x), \nu(y)\right)} \right) \tag{3.98}$$

If $w \in \mathfrak{m}$ is another irreducible element with $m(w) = 1$, then

$$\forall \nu \in \mathcal{V}_0, \nu(z)^2 \alpha_z \left(\frac{\nu}{\nu(z)} \right) = \nu(w)^2 \alpha_w \left(\frac{\nu}{\nu(w)} \right). \tag{3.99}$$

Proposition 3.4.10 ([FJ04], Lemma 3.60 and 6.47)**.** *The map $N : \mathcal{V}_\mathfrak{m} \to \mathcal{V}_z$ is not an isomorphism of trees. The two orders on $\mathcal{V}_\mathfrak{m}$ and $\mathcal{V}_z$ are compatible except on the segments $[\nu_\mathfrak{m}, \nu_z]$ and $[\mathrm{ord}_z, N(\nu_\mathfrak{m})]$ where they are reversed. More precisely,*

(1) $\forall \nu, \mu \in [\nu_\mathfrak{m}, \nu_z] \subset \mathcal{V}_\mathfrak{m}, \nu \leqslant_\mathfrak{m} \mu \Leftrightarrow N(\nu) \geqslant_z N(\mu)$.

(2) $\forall \nu_1, \nu_2 \in \mathcal{V}_z \setminus \{\mathrm{ord}_z\}, \nu_1 \leqslant_z \nu_2 \Leftrightarrow [N^{-1}(\nu_1), \nu_z] \subset [N^{-1}(\nu_2), \nu_x]$.

The situation is summed up in Figure 3.1 where we have put arrows on the branches of the tree to indicate the order.

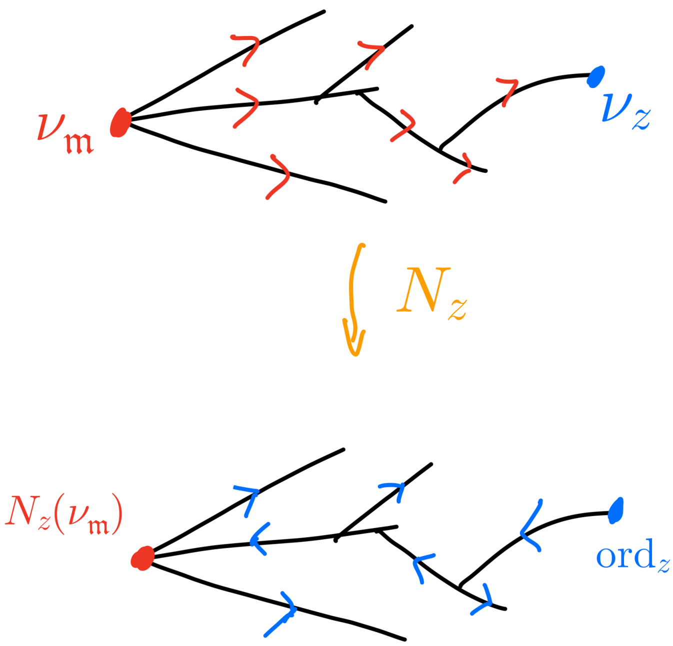


Figure 3.1: The homeomorphism between $\mathcal{V}_\mathfrak{m}$ and $\mathcal{V}_z$

We will use the relative tree in the following context. Let E be a prime divisor at infinity of some good completion X, let p be a point of E and let z,w be local coordinates at p such that $E = \{z=0\}$. The completion of the local ring at p is isomorphic to $\mathbf{k}[[z,w]]$. We define $\mathcal{V}_X(p;E)$ as follows; an element of $\mathcal{V}_X(p;E)$ is either a valuation ν on A centered at p such that $\nu(z)=1$ or the divisorial valuation ord_E. Notice that the definition of $\mathcal{V}_X(p;E)$ does not depend on the local equation $z=0$ of E because the quotient of two local equations is a regular invertible function.

Proposition 3.4.11. *Let X be a completion and let $p \in X$ be a closed point at infinity.*

(1) If $p \in E$ is a free point, then $\mathcal{V}_X(p;E)$ is isomorphic to $\mathcal{V}_z$.

(2) If $p = E \cap F$ is a satellite point. Let z,w be local coordinates at p such that z is a local equation of E and w a local equation of F then $\mathcal{V}_X(p;E)$ is isomorphic to $\mathcal{V}_z \setminus \{\nu_w\}$ and $\mathcal{V}_X(p;F)$ is isomorphic to $\mathcal{V}_w \setminus \{\nu_z\}$.

The map $N_z : \mathcal{V}_{\mathfrak{m}} \to \mathcal{V}_z$ induces a homeomorphism

$$N_{p,E} : \mathcal{V}_X(p;\mathfrak{m}_p) \to \mathcal{V}_X(p;E) \setminus \{\mathrm{ord}_E\}. \tag{3.100}$$

Furthermore, if $p = E \cap F$, then the map

$$N_{p,F} \circ N_{p,E}^{-1} : \mathcal{V}_X(p;E) \setminus \{\mathrm{ord}_E\} \to \mathcal{V}_X(p;F) \setminus \{\mathrm{ord}_F\} \tag{3.101}$$

is a homeomorphism.

Proof. If $p \in E$ is a free point. Let z,w be local coordinates at p such that z is a local equation of E. The completion of the local ring at p is isomorphic to $\mathbf{k}[[z,w]]$ by Theorem 3.1.2. For every $P \in A$, P is of the form $P = \frac{\varphi}{z^a}$ where $a \geqslant 0$ and $\varphi \in \mathcal{O}_{X,p}$. Therefore, a centered valuation on $\mathbf{k}[[z,w]]$ defines a valuation over A if and only if it is not the curve valuation ν_z. Since $\nu_z \notin \mathcal{V}_z$ we have that $\mathcal{V}_X(p;E) \simeq \mathcal{V}_z$. Call $\sigma : \mathcal{V}_X(p;E) \to \mathcal{V}_z$ the isomorphism. We define $N_{p,E}$ as follows. Recall by Proposition 3.4.7 that there is a homeomorphism $N : \mathcal{V}_{\mathfrak{m}} \to \mathcal{V}_z$ where in particular $N(\nu_z) = \mathrm{ord}_z$. Here we have that ord_z is canonically identified with ord_E and $\mathcal{V}_X(p;\mathfrak{m}_p)$ is isomorphic to $\mathcal{V}_{\mathfrak{m}} \setminus \{\nu_z\}$, call $\iota : \mathcal{V}_X(p;\mathfrak{m}_p) \to \mathcal{V}_{\mathfrak{m}} \setminus \{\nu_z\}$ the isomorphism. Define

$$N_{p,E} := \sigma^{-1} \circ N \circ \tau : \mathcal{V}_X(p;\mathfrak{m}_p) \to \mathcal{V}_X(p;E) \setminus \{\mathrm{ord}_E\}, \tag{3.102}$$

it is a homeomorphism.

If $p = E \cap F$ is a satellite point. Let (z,w) be local coordinates at p such that z is a local equation of E and w is a local equation of F. The completion of the local ring at p is isomorphic to $\mathbf{k}[[z,w]]$ by Theorem 3.1.2. Every $P \in A$ is of the form $P = \frac{\varphi}{z^a w^b}$ where $a,b \geqslant 0$ and $\varphi \in \mathcal{O}_{X,p}$. Therefore a centered valuation on $\mathbf{k}[[z,w]]$ defines a valuation over A if and only if it is not the curve valuation associated to z or w. Or ν_z does not belong to $\mathcal{V}_z$ but ν_w does. Therefore, $\mathcal{V}_X(p;E)$ is isomorphic to $\mathcal{V}_z \setminus \{\nu_w\}$. If $N_z : \mathcal{V}_\mathfrak{m} \to \mathcal{V}_z$ is the map from Proposition 3.4.7, then $N(\nu_z) = \mathrm{ord}_z$ and $N(\nu_w) = \nu_w$. Therefore, $N_w \circ N_z^{-1} : \mathcal{V}_z \to \mathcal{V}_w$ is a homeomorphism that sends ord_z to ν_z and ν_w to ord_w. Fix an isomorphism $\tau_E : \mathcal{V}_X(p;E) \to \mathcal{V}_z\{\nu_w\}$ and $\tau_F : \mathcal{V}_X(p;F) \to \mathcal{V}_w \setminus \mathcal{V}_z$. We have that the map

$$N_{p,F} \circ N_{p,E}^{-1} = \tau_F^{-1} \circ N_w \circ N_z^{-1} \circ \tau_E : \mathcal{V}_X(p;E) \setminus \{\mathrm{ord}_E\} \to \mathcal{V}_X(p;F) \setminus \{\mathrm{ord}_F\} \tag{3.103}$$

is a homeomorphism. $\square$

Proposition 3.4.12. *Let X be a completion of X_0 and let E be a prime divisor at infinity. If $p_1, p_2 \in E$ are closed points with $p_1 \neq p_2$, then $\mathcal{V}_X(p_1;E) \cap \mathcal{V}_X(p_2;E) = \{\mathrm{ord}_E\}$. Define the set $\mathcal{V}_X(E;E)$ of valuations ν such that $c_X(\nu) \in E$ and $\nu(z) = 1$ where z is a local equation of E at $c_X(\nu)$. Then*

$$\mathcal{V}_X(E;E) = \bigcup_{p \in E} \mathcal{V}_X(p;E) \tag{3.104}$$

and it has a natural structure of a rooted $\mathbf{R}$-tree rooted in ord_E. The skewness functions α_E glue together to give $\mathcal{V}_X(E;E)$ the structure of a parametrized rooted tree. Every point $p \in E$ defines a tangent vector at ord_E given by $\mathcal{V}_X(p;E) \setminus \{\mathrm{ord}_E\}$.

Furthermore, Let Y be a completion of X_0 and $q \in Y$ a closed point at infinity. Let $\pi : Z \to Y$ be the blow up of q and let $\widetilde{E}$ be the exceptional divisor of π. Then, for every $\widetilde{q} \in \widetilde{E}$, the map $\pi_\bullet : \mathcal{V}_Z(\widetilde{q};\widetilde{E}) \to \mathcal{V}_Y(q;\mathfrak{m}_q)$ is actually equal to π_* and they glue together to give a map

$$\pi_* : \mathcal{V}_Z(\widetilde{E};\widetilde{E}) \to \mathcal{V}_Y(q;\mathfrak{m}_q), \tag{3.105}$$

which is an isomorphism of trees. We have the relation $\alpha_{\mathfrak{m}_q} \circ \pi_* = 1 + \alpha_E$ and $b_{\mathfrak{m}_q} \circ \pi_* = b_E$.

We postpone the proof to the next section. If $E \simeq \mathbf{P}^1$, this tree is isomorphic to the tree of normalized valuations centered at infinity over $\mathbf{A}^2$ constructed in [FJ07], Appendix.

3.4.4 The monomial valuations centered at an intersection point at infinity

Let X be a good completion of X_0 and let E,F be two divisors at infinity that intersect at a point p. Let (x,y) be local coordinates at p such that $E=\{x=0\}$ and $F=\{y=0\}$. There are three spaces to consider: $\mathcal{V}_X(p,\mathfrak{m}_p)$, $\mathcal{V}_X(p;E)$ and $\mathcal{V}_X(p;F)$. We explain here how they are related. For $(s,t)\in[0,+\infty]^2\setminus\{(0,0),(\infty,\infty)\}$, we denote by $\nu_{s,t}$ the monomial valuation defined by

$$\nu_{s,t}\left(\sum a_{ij}x^iy^j\right)=\min\left\{si+tj|a_{ij}\neq 0\right\}. \tag{3.106}$$

Notice that $\nu_{0,1}=\mathrm{ord}_F,\nu_{1,0}=\mathrm{ord}_E,\nu_{1,\infty}=\nu_y,\nu_{\infty,1}=\nu_x$. We will denote the set of such valuation by $[\mathrm{ord}_E,\mathrm{ord}_F]$. We use this notation because of the following: $[\mathrm{ord}_E,\mathrm{ord}_F]\cap\mathcal{V}_X(p;E)$ consists of the valuations $\nu_{1,t}$ for $t\in[0,+\infty]$ and $[\mathrm{ord}_E,\mathrm{ord}_F]\cap\mathcal{V}_X(p;F)$ consists of the valuations $\nu_{s,1}$ for $s\in[0,+\infty]$. So they define segments in the respective trees. In particular we have

$$N_{p,F}\circ N_{p,E}^{-1}(\nu_{1,t})=\nu_{1/t,1},\quad \forall t\in[0,+\infty] \tag{3.107}$$

One can show with the definition of the level function α that $\alpha_E(\nu_{1,t})=t$. Therefore we show

Lemma 3.4.13. *Let ν be a monomial valuation centered at $p=E\cap F$. One has*

$$\alpha_E\left(\frac{\nu}{\nu(x)}\right)=\frac{\nu(y)}{\nu(x)}=\frac{s}{t}\ \textit{if}\ \nu=\nu_{s,t}$$
$$\alpha_F\left(\frac{\nu}{\nu(y)}\right)=\frac{\nu(x)}{\nu(y)}=\frac{t}{s}\ \textit{if}\ \nu=\nu_{s,t}$$

In particular we have that $\alpha_E\left(\frac{\nu}{\nu(x)}\right)=\alpha_F\left(\frac{\nu}{\nu(y)}\right)^{-1}$ *on* $]\mathrm{ord}_E,\mathrm{ord}_F[$.

3.4.5 Geometric interpretations of the valuative tree

Let X be a completion of X_0 and let $p\in X$ be a closed point at infinity. We consider in this section only completions above X that are exceptional above p. If $\pi:(Y,\mathrm{Exc}(\pi))\to(X,p)$ is such a completion, then we call Γ_π the dual graph which vertices consist of the exceptional divisors of π. Two exceptional divisors are linked by an edge if they intersect. The graph Γ_π is connected without cycles, it is therefore an $\mathbf{N}$-tree. We set the root of Γ_π to be the exceptional divisor $\widetilde{E}$ that appears after blowing up p.

If E is a prime divisor at infinity of X such that $p \in E$. We define the dual graph

$$\Gamma_{\pi,E} := \Gamma_\pi \cup \{E\}. \tag{3.108}$$

It is also a $\mathbf{N}$-tree. We set the root of $\Gamma_{\pi,E}$ to be E.

Lemma 3.4.14 ([FJ04], Proposition 6.2)**.** *Let $\pi : Y \to (X,p)$ be a completion exceptional above p. if $\tau : Z \to Y$ is the blow up of a point in the exceptional locus of π, then there are natural inclusions of $\mathbf{N}$-trees*

$$\Gamma_\pi \hookrightarrow \Gamma_{\pi\circ\tau}, \quad \Gamma_{\pi,E} \hookrightarrow \Gamma_{\pi\circ\tau,E}. \tag{3.109}$$

Therefore, the direct limits $\Gamma := \varinjlim_\pi \Gamma_\pi$, $\Gamma_E := \varinjlim_\pi \Gamma_{\pi,E}$ are well defined. The points of Γ are in bijection with $\mathcal{D}_{X,p}$ and $\Gamma_E = \Gamma \cup \{E\}$ and they have a structure of $\mathbf{Q}$-trees.

Lemma 3.4.15 ([FJ04] Theorem 6.9)**.** *We have a map $\pi_\bullet : \Gamma_\pi \hookrightarrow \mathcal{V}_X(p;\mathfrak{m}_p)_{\mathrm{div}}$ defined by*

$$\pi_\bullet(F) = \nu_F \tag{3.110}$$

where ν_F is the valuation equivalent to $\pi_* \operatorname{ord}_F$ that belongs to $\mathcal{V}_X(p;\mathfrak{m}_p)$. These maps are compatible with the direct limit and give a map $\Gamma \hookrightarrow \mathcal{V}_X(p;\mathfrak{m}_p)$.

Lemma 3.4.16. *We have a map $\pi_\bullet : \Gamma_{\pi,E} \hookrightarrow \mathcal{V}_{E,\mathrm{div}}$ defined by*

$$\pi_\bullet(F) = \nu_F \tag{3.111}$$

where ν_F is the valuation equivalent to $\pi_* \operatorname{ord}_F$ that belongs to $\mathcal{V}_X(p;E)$. These maps are compatible with the direct limit and give a map $\Gamma_E \hookrightarrow \mathcal{V}_X(p;E)$.

Proposition 3.4.17 ([FJ04], Lemma 6.28)**.** *Let $\pi : (Y, \operatorname{Exc}(\pi)) \to (X,p)$ be a completion exceptional above p. Let $q \in Y$ be a closed point that belongs to the exceptional component of π. Let $\widetilde{F}$ be the exceptional divisor above q.*

(1) If $q \in F$ with $F \in \Gamma_\pi$, then $\nu_{\widetilde{F}} > \nu_F$.

(2) If $q = F_1 \cap F_2$ with $F_1, F_2 \in \Gamma_\pi$, suppose that $\nu_{F_1} < \nu_{F_2}$, then $\nu_{F_1} < \nu_{\widetilde{F}} < \nu_{F_2}$.

Proposition 3.4.18 (Relative version of Proposition 3.4.17)**.** *Let $\pi : (Y, \operatorname{Exc}(\pi)) \to (X,p)$ be a completion exceptional above p. Let $q \in \operatorname{Exc}(\pi)$. Let $\widetilde{F}$ be the exceptional divisor above q.*

(1) If $q \in F$ is a free point with $F \in \Gamma_{\pi,E}$, then $\nu_{\widetilde{F}} > \nu_F$.

(2) If $q = F_1 \cap F_2$ is a satellite point with $F_1, F_2 \in \Gamma_{\pi,E}$, if $\nu_{F_1} < \nu_{F_2}$, then $\nu_{F_1} < \nu_{\tilde{F}} < \nu_{F_2}$.

(3) In particular, if $q = E \cap F$, then $\operatorname{ord}_E < \nu_{\tilde{F}} < \nu_F$.

Theorem 3.4.19 ([FJ04], Theorem 6.22)**.** *We have an isomorphism of $\mathbf{Q}$-trees*

$$\Gamma \simeq \mathcal{V}_X(p;\mathfrak{m}_p)_{\mathrm{div}}, \quad \Gamma_E \simeq \mathcal{V}_X(p;E)_{\mathrm{div}} \tag{3.112}$$

given by $F \simeq \nu_F$. We can take the completion of the $\mathbf{Q}$-trees to get the isomorphism

$$\overline{\Gamma} \simeq \mathcal{V}_X(p;\mathfrak{m}_p), \quad \overline{\Gamma}_E \simeq \mathcal{V}_X(p;E) \tag{3.113}$$

Proposition 3.4.20. *Let X be a completion of X_0 and let $p \in X$ be a closed point at infinity. Let $\mathcal{V}_*$ be either $\mathcal{V}_X(p;\mathfrak{m}_p)$ or $\mathcal{V}_X(p;E)$ for some prime divisor E at infinity such that $p \in E$. Let Γ_* be either Γ or Γ_E. Let $\pi : (Y, \operatorname{Exc}(\pi)) \to (X,p)$ be a completion exceptional above p. Let $q \in \operatorname{Exc}(\pi)$ be a closed point. The map π induces a map $\pi_* : \mathcal{V}_Y(q) \to \mathcal{V}_X(p)$.*

(1) If $q \in E_q$ is a free point with $E_q \in \Gamma_*$, then we have an inclusion map $\pi_\bullet : \mathcal{V}_Y(q;E_q) \hookrightarrow \mathcal{V}_*$. The order relation in $\mathcal{V}_Y(q;E_q)$ and $\mathcal{V}_*$ are compatible and $\pi_\bullet$ is an inclusion of trees.

(2) If $q = E_q \cap F_q$ is a satellite point with $E_q, F_q \in \Gamma_*$, then, if $\nu_{E_q} <_* \nu_{F_q}$, the order relations on $\mathcal{V}_*$ and $\mathcal{V}_Y(q;E_q)$ are compatible and $\pi_\bullet : \mathcal{V}_Y(q;E_q) \hookrightarrow \mathcal{V}_*$ is an inclusion of trees.

Proof. We only need to show that the orders are compatible on the divisorial valuations of $\mathcal{V}_Y(q;E_q)$. Therefore we show the following,

Claim 3.4.21. *For every completion $\tau : (Z, \operatorname{Exc}(\tau)) \to (Y,q)$ exceptional above q, we have the following*

1. *For all $F_1, F_2 \in \Gamma_{\tau,E_q}$,*
$$\nu_{F_1} <_* \nu_{F_2} \Leftrightarrow \nu_{F_1} <_{E_q} \nu_{F_2} \tag{3.114}$$

2. *If $F \in \Gamma_{\tau,E_q}$ satisfies $F \cap F_q \neq \emptyset$, then*
$$\nu_F < \nu_{F_q} \tag{3.115}$$

Here there is a slight abuse of notation as we denote by ν_{F_i} the image of F_i both in $\mathcal{V}_Y(q;E_q)$ and $\mathcal{V}_*$. This is done to lighten notations, we believe that it does not provide any confusion.

We prove this by induction on the number of blow ups above q. If $\tau = \mathrm{id}$, then ord_{E_q} is the root of $\mathcal{V}_Y(q;E_q)$ and $\nu_{E_q} < \nu_{F_q}$ by assumption so there is nothing to do.

Let $\tau : (Z, \mathrm{Exc}(\tau)) \to (Y, q)$ be a completion exceptional above q such that Claim (3.4.21) is true. Let $q' \in \mathrm{Exc}(\tau)$ be a closed point, let $\tau' : Z' \to Z$ be the blow up of q' and let $\widetilde{F}$ be the exceptional divisor above q'.

- If $q' \in F$ is a free point with $F \in \Gamma_{\tau,E_q}$, then by Proposition 3.4.18 we have

$$\nu_F <_{E_q} \nu_{\widetilde{F}} \tag{3.116}$$

  Now we have two possibilities.

  - If q' is also a free point with respect to Γ_*, then by Proposition 3.4.17 and 3.4.18 we also get

$$\nu_F <_* \nu_{\widetilde{F}}. \tag{3.117}$$

    Since $\widetilde{F} \cap F_q = \varnothing$, Claim 3.4.21 is shown for $\Gamma_{\tau\circ\tau',E_q}$.

  - If q' is the satellite point $F \cap F_q$, then by induction hypothesis we have $\nu_F <_* \nu_{F_q}$ and therefore $\widetilde{F} \cap F_q \neq \varnothing$ and by Proposition 3.4.17 and 3.4.18 we get

$$\nu_F <_* \nu_{\widetilde{F}} <_* \nu_{F_q} \tag{3.118}$$

    So Claim 3.4.21 is shown for $\Gamma_{\tau\circ\tau',E_q}$.

- If q' is a satellite point. Let $F_1, F_2 \in \Gamma_{\tau,E_q}$ such that $q = F_1 \cap F_2$. Suppose without loss of generality that $\nu_{F_1} <_{E_q} \nu_{F_2}$, then by the induction hypothesis we have $\nu_{F_1} <_* \nu_{F_2}$ and by Proposition 3.4.17 and 3.4.18, we get

$$\nu_{F_1} <_{E_q} \nu_{\widetilde{F}} <_{E_q} \nu_{F_2} \text{ and } \nu_{F_1} <_* \nu_{\widetilde{F}} <_* \nu_{F_2}. \tag{3.119}$$

  Since $\widetilde{F} \cap F_q = \varnothing$ we have proven Claim 3.4.21 for $\Gamma_{\tau\circ\tau',E_q}$.

□

Proof of Proposition 3.4.12. Let Y be a completion of X_0 and let $q \in Y$ be a closed point at infinity. Let $\pi : Z \to Y$ be the blow up of q. Let $\widetilde{E}$ be the exceptional divisor and let $\widetilde{q} \in \widetilde{E}$ be a closed point. Apply Proposition 3.4.20 with $\mathcal{V}_* = \mathcal{V}_Y(q;\mathfrak{m}_q)$. The map $\pi_\bullet : \mathcal{V}_Z(\widetilde{q};\widetilde{E}) \to \mathcal{V}_Y(q;\mathfrak{m}_q)$ is an inclusion of trees. There exists local coordinates z,w at q and x,y at p such that $\pi(z,w) = (z,zw)$ where z is a local equation of $\widetilde{E}$. We therefore get

$$\nu(z) = 1 \Leftrightarrow \min(\pi_*\nu(x), \pi_*\nu(y)) = 1. \tag{3.120}$$

Hence, $\pi_\bullet = \pi_*$ and $\pi_*(\mathrm{ord}_{\widetilde{E}}) = \nu_{\mathfrak{m}_q}$. Therefore we can glue these maps to obtain an isomorphism of trees

$$\pi_* : \mathcal{V}_Z(\widetilde{E};\widetilde{E}) \to \mathcal{V}_Y(q;\mathfrak{m}_q) \tag{3.121}$$

We get the relation on the skewness functions by Proposition 3.4.28 which will be proven in the next section. □

3.4.6 Properties of skewness

We have two valuative tree structures. We describe some properties of the skewness function for these two structures and how they behave after blowing up. Fix a completion X, let $p \in X$ be a closed point at infinity and let E be a prime divisor at infinity in X such that $p \in E$. In accordance with the notations of the previous section, set $\Gamma = \mathcal{D}_{X,p}$ and $\Gamma_E = \mathcal{D}_{X,p} \cup \{E\}$.

Definition 3.4.22. If $F \in \Gamma$ is a prime divisor above p, we define the *generic multiplicity* $b(F)$ inductively as follows.

- $b(\widetilde{E}) = 1$ where $\widetilde{E}$ is the exceptional divisor above p.
- If $q \in F$ is a free point with $F \in \Gamma$, then $b(\widetilde{F}) = b(F)$ where $\widetilde{F}$ is the exceptional divisor above q.
- If $q = F_1 \cap F_2$ is a satellite point with $F_1, F_2 \in \Gamma$, then $b(\widetilde{F}) = b(F_1) + b(F_2)$.

If $\nu \in \mathcal{V}_X(p;\mathfrak{m}_p)$ is divisorial then we define $b(\nu) := b(E)$ where E is the center of ν in some completion above X.

Definition 3.4.23. If $F \in \Gamma_E$, we define the *relative generic multiplicity* $b_E(F)$ inductively as follows.

- $b_E(E) = 1$.
- If $q \in F$ is a free point with $F \in \Gamma_E$, then $b_E(\widetilde{F}) = b_E(F)$.
- If $q = F_1 \cap F_2$ is a satellite point with $F_1, F_2 \in \Gamma_E$, then $b_E(\widetilde{F}) = b_E(F_1) + b_E(F_2)$.

If $\nu \in \mathcal{V}_X(p;E_z)$ is divisorial, then we set $b_E(\nu) := b_E(F)$ where F is the center of ν in some completion above X.

Figure 3.2 sums up the definition of the generic multiplicity.

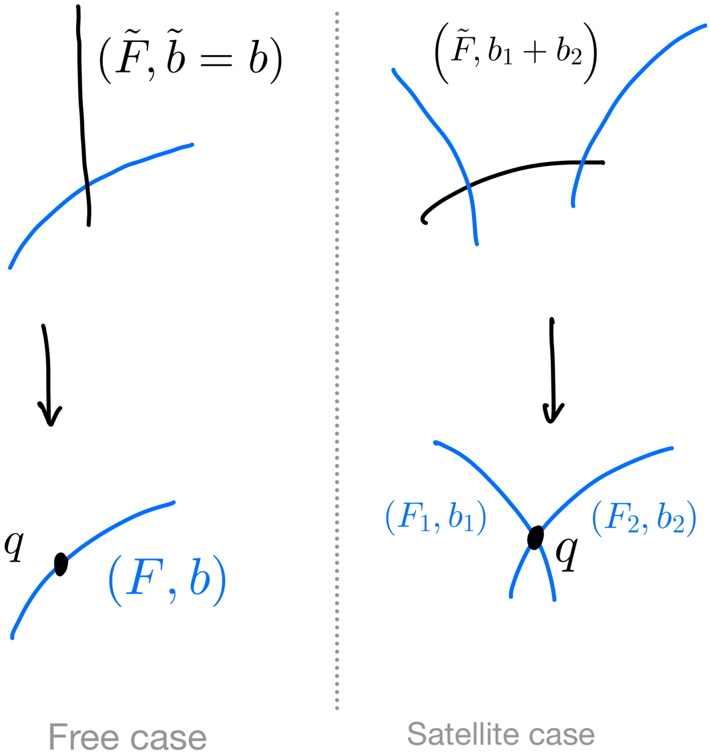


Figure 3.2: Algorithm for computing the generic multiplicity

The term *generic multiplicity* is justified by the following proposition.

Proposition 3.4.24 ([FJ04] Proposition 6.26)**.** *Let $\nu \in \mathcal{V}_X(p;\mathfrak{m}_p)$ be divisorial, let $E \in \Gamma$ be the center of ν over some completion $\pi : Y \to X$ above X. Then,*

$$\pi_* \operatorname{ord}_E(\mathfrak{m}_p) = b(\nu) \tag{3.122}$$

Proposition 3.4.25 (Relative version of Proposition 3.4.24)**.** *If $\nu \in \mathcal{V}_X(p;E)$ is divisorial, let F be the center of ν over some completion $\pi : Y \to X$ above X. Then,*

$$\pi_* \operatorname{ord}_F(z) = b_E(F) \tag{3.123}$$

where $z \in \mathcal{O}_{X,p}$ is a local equation of E. This means that $\operatorname{ord}_F(\pi^* E) = b_E(F)$.

From now on we write $\mathcal{V}_*$ for either $\mathcal{V}_X(p;\mathfrak{m}_p)$ and $\mathcal{V}_X(p;E)$ and we write α_*, b_* for the skewness function and the generic multiplicity function associated to the tree structure.

For a valuation $\nu \in \mathcal{V}_*$, we define the *approximating sequence* of ν as follows, set $\nu_0 = \nu_*$ the root of $\mathcal{V}_*$ and let p_n be the sequence of centers above X associated to ν. Let E_n be the exceptional divisor above p_n. Set $\nu_n = \frac{1}{b_*(E_n)} \operatorname{ord}_{E_n}$, if ν is quasimonomial (ν_n) is the approximating sequence of ν. If ν is a curve valuation or infinitely singular we define the approximating sequence of ν as the subsequence of ν_n where $c_{X_n}(\nu)$ is a free point (at infinity).

Proposition 3.4.26 ([FJ04] Theorem 6.9, Theorem 6.10 and Lemma 3.32)**.** *Let $\nu \in \mathcal{V}_*$ and let ν_n be its approximating sequence*

- *the sequence $\nu_n := \frac{1}{b_n} \operatorname{ord}_{E_n}$ converges weakly towards ν.*
- $\alpha_*(\nu) = \lim_n \alpha_*(\nu_n)$.

We will say that two divisorial valuations ν, ν' are *adjacent* if there exists a completion Y above X such that the centers of ν and ν' are both prime divisors and they intersect.

Proposition 3.4.27 ([FJ04], Corollary 6.39)**.** *Let $\nu, \nu' \in \mathcal{V}_*$. Assume $\nu < \nu'$ and that they are adjacent, then*

$$\alpha_*(\nu') - \alpha_*(\nu) = \frac{1}{b_*(\nu) b_*(\nu')} \tag{3.124}$$

Proposition 3.4.28 ([FJ04], Theorem 6.51)**.** *Let $\pi : Y \to X$ be a completion above X and let $q \in E_q$ be a free point of Y such that $\pi(E_q) = p$. By Proposition 3.4.20, $\pi_\bullet : \mathcal{V}_Y(q;E_q) \to \mathcal{V}_*$ is an inclusion of trees.*

(1) The normalization of $\pi_* \operatorname{ord}_{E_q}$ (to get a valuation in $\mathcal{V}_*$) is

$$\nu_{E_q} = \frac{1}{b_*(E_q)} \pi_* \operatorname{ord}_{E_q}. \tag{3.125}$$

(2)

$$\forall \nu \in \mathcal{V}_Y(p;E), \quad \alpha_*(\pi_\bullet \nu) = \alpha_*(\nu_{E_q}) + \frac{1}{b_*(E_q)^2} \alpha_{E_q}(\nu) \tag{3.126}$$

$$b_*(\pi_\bullet \nu) = b_*(E_q) b_{E_1}(\nu) \tag{3.127}$$

Proof. It suffices to show this formula for every divisorial valuation $\nu \in \mathcal{V}_Y(q;E_q)$ by Proposition 3.4.26. We prove the result by induction on the number of blow-ups above q. Namely we show the following

Claim 3.4.29. *For every completion $\tau : (Z, \mathrm{Exc}(\tau)) \to (Y, q)$ exceptional above q, for every $F \in \Gamma_{\tau,E_q}$,*

$$b_*(F) = b_{E_q}(F) b_*(E_q) \tag{3.128}$$

$$\alpha_*(\nu_F) = \alpha_*(\nu_{E_q}) + \frac{1}{b_*(E_q)} \alpha_{E_q}(\nu_F) \tag{3.129}$$

If $\tau = \mathrm{id} : Y \to Y$, then $\Gamma_{\tau,E_q} = \{E_q\}$. We have by definition that $b_{E_q}(E_q) = 1, \alpha_{E_q}(\mathrm{ord}_{E_q}) = 0$. Therefore Equations (3.128) and (3.129) holds.

Suppose the claim to be true for a completion $\tau : (Z, \mathrm{Exc}(\tau)) \to (Y, q)$ exceptional above q. Let $\tau' : Z' \to Z$ be the blow up of a closed point $q' \in \mathrm{Exc}(\tau)$. Let $\tilde{E}$ be the exceptional divisor above q'.

If $q' \in F$ is a free point with $F \in \Gamma_{\tau,E}$, then q' is also a free point with respect to $\Gamma_{*,\pi\circ\tau}$ because $q \in Y$ is a free point. Therefore by definition

$$b_*(\tilde{E}) = b_*(F), \quad b_{E_q}(\tilde{E}) = b_{E_q}(F) \tag{3.130}$$

So Equation (3.128) is true for $\tilde{E}$ by induction. Now, by Proposition 3.4.27

$$\alpha_*(\nu_{\tilde{E}}) = \alpha_*(\nu_F) + \frac{1}{b_*(F) b_*(E_q)}, \quad \alpha_{E_q}(\nu_{\tilde{E}}) = \alpha_{E_q}(\nu_F) + \frac{1}{b_{E_q}(\tilde{E}) b_{E_q}(F)} \tag{3.131}$$

By induction, Equation (3.129) is true for $\tilde{E}$.

If $q' = F_1 \cap F_2$ is a satellite point with $F_1, F_2 \in \Gamma_{\tau,E_q}$, then

$$b_*(\tilde{E}) = b_*(F_1) + b_*(F_2), \quad b_{E_q}(\tilde{E}) = b_{E_q}(F_1) + b_{E_q}(F_2) \tag{3.132}$$

So by induction Equation (3.128) holds for $\tilde{E}$. Suppose without loss of generality that $\nu_{F_1} < \nu_{F_2}$ both in $\mathcal{V}_*$ and $\mathcal{V}_Y(q; E_q)$. This is possible by Proposition 3.4.20. By Proposition 3.4.27

$$\alpha_*(\nu_{\tilde{E}}) = \alpha_*(\nu_{F_1}) + \frac{1}{b_*(F_1) b_*(\tilde{E})}, \quad \alpha_{E_q}(\nu_{\tilde{E}}) = \alpha_{E_q}(\nu_{F_1}) + \frac{1}{b_{E_q}(F_1) b_{E_q}(\tilde{E})}. \tag{3.133}$$

Therefore, Equation (3.129) holds for $\tilde{E}$. And the claim is shown by induction. □

Proposition 3.4.30. *Let ν be a valuation over A centered at infinity. Let X be a completion of X_0 and let E be a prime divisor of X at infinity such that $\tilde{\nu} \in \mathcal{V}_X(E; E)$ for some valuation $\tilde{\nu}$ equivalent to ν. If $\alpha_E(\tilde{\nu}) < +\infty$, then for every completion Y of X_0 if $\tilde{\nu} \in \mathcal{V}_Y(F, F)$ for some* *prime divisor F at infinity in Y, then $\alpha_F(\tilde{\nu}) < +\infty$.*

Proof. If ν is quasimonomial, this is immediate as for any prime divisor E at infinity and any closed point $p \in E$, we have that $\alpha_E(\nu) < +\infty$ for $\nu = \mathrm{ord}_E$ or ν quasimonomial centered at p. If ν is a curve valuation, then $\alpha_E(\nu) = +\infty$ for any prime divisor E of any completion X such that $c_X(\nu) \in E$. So it remains to show the result for ν an infinitely singular valuation.

We show that if $\pi : Y \to X$ is a completion above X, then $\alpha_{E'}(\nu) < +\infty \Leftrightarrow \alpha_E(\nu) < +\infty$ where E' is a prime divisor of Y at infinity such that some multiple of ν belongs to $\mathcal{V}_Y(E', E')$. Let $p = c_X(\nu)$ and $q = c_Y(\nu)$. Since ν is infinitely singular, by Proposition 3.3.16 there exists a completion $\tau : (Z, \mathrm{Exc}(\tau)) \to (Y, q)$ exceptional above q such that $c_Z(\nu)$ is a free point q' lying over a unique prime divisor F at infinity. We apply Proposition 3.4.28. We have that

$$\alpha_E(\nu) = \alpha_E(\nu_F) + \frac{1}{b_E(F)^2}\alpha_F(\nu) \tag{3.134}$$

$$\alpha_{E'}(\nu) = \alpha_{E'}(\nu_F) + \frac{1}{b_{E'}(F)^2}\alpha_F(\nu) \tag{3.135}$$

Thus $\alpha_E(\nu) < +\infty \Leftrightarrow \alpha_F(\nu) < +\infty \Leftrightarrow \alpha_{E'}(\nu) < +\infty$. $\square$

Proposition 3.4.31 ([FJ04] Proposition 6.35)**.** *Let $\pi : (Y, \mathrm{Exc}(\pi)) \to (X, p)$ be a completion exceptional above p. Let $q = E \cap F \in \mathrm{Exc}(\pi)$ be a satellite point with $E, F \in \Gamma_{*,\pi}$. Define $\nu_E = \frac{1}{b_*(E)}\pi_* \mathrm{ord}_E$ and $\nu_F = \frac{1}{b_*(F)}\pi_* \mathrm{ord}_F$. Let z, w be local coordinates at q associated to $(E,\ F)$. Let $\nu_{s,t}$ be the monomial valuation centered at q such that $\nu(z) = s$ and $\nu(w) = t$. Then, the map π_* induces a homeomorphism from the set $\{\nu_{s,t} | s, t \geqslant 0, s b_*(E) + t b_*(F) = 1\}$ and $[\nu_E, \nu_F] \subset \mathcal{V}_*$ for the weak topology.*

Furthermore, the skewness function is given by

$$\alpha_*(\pi_* \nu_{s,t}) = \alpha(\nu_E) + \frac{t}{b(E)} \tag{3.136}$$

3.5 Different topologies over the space of valuations

3.5.1 The weak topology

Let X_0 be an affine surface and let $\mathcal{V}_\infty$ be the space of valuations centered at infinity. We define $\widehat{\mathcal{V}_\infty}$ to be the space of valuations centered at infinity modulo equivalence and $\eta : \mathcal{V}_\infty \to \widehat{\mathcal{V}_\infty}$ the quotient map. We define the weak topology over $\mathcal{V}_\infty$ as follows. A basis for the topology is given by

$$\left\{\nu \in \mathcal{V}_\infty : t < \nu(P) < t'\right\} \tag{3.137}$$

for some $t, t' \in \mathbf{R}, P \in A$. A sequence ν_n of $\mathcal{V}_\infty$ converges towards ν if and only if for every $P \in A$, the sequence $\nu_n(P)$ converges towards $\nu(P)$. We define the weak topology over $\widehat{\mathcal{V}_\infty}$ to be the thinnest topology such that $\eta : \mathcal{V}_\infty \to \widehat{\mathcal{V}_\infty}$ is continuous with respect to the weak topology.

Proposition 3.5.1. *Let X be a completion of X_0. Let $\nu \in \mathcal{V}_\infty$ and (ν_n) a sequence of elements of $\mathcal{V}_\infty$. Suppose that $\nu_n \to \nu$ with respect to the weak topology. Then,*

- *If $c_X(\nu) = p$ is a closed point at infinity, then for all n large enough $c_X(\nu_n) = p$.*
- *If $c_X(\nu) = E$ is a prime divisor at infinity, then for all n large enough $c_X(\nu_n) \in E$.*

Proof. Suppose first that $c_X(\nu) = p$ is a closed point at infinity. Let (x, y) be local coordinates at p. By definition of the center we have $\nu(x), \nu(y) > 0$. We can find $P_1, P_2, Q_1, Q_2 \in \mathcal{O}_X(X_0)$ such that $x = P_1/Q_1, y = P_2/Q_2$ and such that $\nu(Q_1), \nu(Q_2) \neq \infty$. Indeed by Lemma 3.3.11, $\mathcal{O}_{X,p}$ is a subring of $\mathcal{O}_X(X_0)_{\mathfrak{p}_\nu}$ where $\mathfrak{p}_\nu = \{\nu = +\infty\}$. Now, we have that $\nu_n(P_i) \to \nu(P_i)$ and $\nu_n(Q_i) \to \nu(Q_i)$ as $n \to \infty$, therefore for all n large enough

$$\nu_n(x), \nu_n(y) > 0. \tag{3.138}$$

Thus, for all n large enough $c_X(\nu_n) = p$.

If $c_X(\nu) = E$, then $\nu = \lambda \operatorname{ord}_E$ for some $\lambda > 0$. Let U be an open affine subset of X such that $U \cap E \neq \emptyset$. Let z be a local equation of E over U. Similarly, we can write $z = P/Q$ with $\nu(Q) \neq \infty$. Since $\nu_n(P) \to \nu(Q)$ and $\nu_n(Q) \to \nu(Q)$, we get that $\nu_n(z) \to \nu(z) > 0$. Therefore for n large enough, $\nu_n(z) > 0$ and therefore $c_X(\nu_n) \in E$. □

Proposition 3.5.2. *Let X be a completion and let $p \in X$ be a closed point at infinity. Let $\nu \in \mathcal{V}_X(p)$ and $\nu_n \in \mathcal{V}_X(p)$. Then, $\nu_n \to \nu$ weakly if and only if for every $\varphi \in \mathcal{O}_{X,p}, \nu_n(\varphi) \to \nu(\varphi)$.*

Proof. Indeed, every $\varphi \in O_{X,p}$ can be written as $\varphi = \frac{P}{Q}$ with $\nu(Q) \neq \infty$. This shows one implication. Conversely, every $P \in A$ is of the form $\frac{\varphi}{\psi}$ where $\varphi, \psi \in O_{X,p}$. Furthermore, if $p \in E$ is a free point then $\psi = u^a$ where $a \in \mathbf{Z}_{\geqslant 0}$ and u is a local equation of E. If $p = E \cap F$ is a satellite point, then $\psi = u^a v^b$ where uv is a local equation of $E \cup F$. Now since ν_n and ν are valuations over A, they cannot be the curve valuations associated to a prime divisor at infinity. Therefore, for all $n, \nu_n(\psi) \neq \infty$ and $\nu(\psi) \neq \infty$. This shows the other implication. □

Proposition 3.5.3. *Let X be a completion of X_0 and let $p \in X$ be a closed point. Let E be a prime divisor at infinity in X such that $p \in X$. Let $\eta_p : \mathcal{V}_X(p) \to \mathcal{V}_X(p;E)$ be the natural map defined by $\eta_p(\nu) = \frac{\nu}{\nu(z)}$ where $z \in O_{X,p}$ is a local equation of E. Let (ν_n) be a sequence of $\mathcal{V}_X(p)$ and let $\nu \in \mathcal{V}_X(p)$. If $\nu_n \to \nu$ for the weak topology of $\mathcal{V}_\infty$, then $\eta_p(\nu_n) \to \eta_p(\nu)$ for the weak topology of $\mathcal{V}_X(p;E)$.*

Proof. If $\nu_n \to \nu$ for the weak topology, then, $\nu_n(z) \to \nu(z)$ by Proposition 3.5.2. Therefore $\eta_p(\nu_n) \to \eta_p(\nu)$, again by Proposition 3.5.2. This shows the first implication. □

Theorem 3.5.4. *Let X be a completion of X_0. The weak topology on $\widehat{\mathcal{V}_\infty}$ is the topology induced by the open subsets $\mathcal{V}_X(E;E)$ for all prime divisor E at infinity.*

Proof. Let X be a completion at infinity and let E be a prime divisor at infinity. Let $\mathcal{V}_X(E)$ be the set of valuations ν over A such that $c_X(\nu) \in E$ (this includes $c_X(\nu) = E$, i.e $\nu = \mathrm{ord}_E$). We have that

$$\mathcal{V}_X(E) = \{\mathrm{ord}_E\} \cup \bigcup_{p \in E} \mathcal{V}_X(p). \tag{3.139}$$

Let $U_1, \cdots, U_r$ be a finite open affine cover of E such that for every $i = 1, \cdots, r$ there exists $z_i \in O_X(U_i)$ a local equation of E. Then, every z_i is of the form $z_i = P_i/Q_i$ with $P_i, Q_i \in A$. Then,

$$\mathcal{V}_X(E) = \bigcup_i \{\nu(Q_i) < +\infty, \nu(P_i) - \nu(Q_i) > 0\} \tag{3.140}$$

and, it follows that $\mathcal{V}_X(E)$ is an open subset of $\mathcal{V}_\infty$. Set $\widehat{\mathcal{V}_\infty}(p) := \eta(\mathcal{V}_X(p))$. Define a map $\sigma_p : \widehat{\mathcal{V}_\infty}(p) \to \mathcal{V}_X(p;E) \setminus \{\mathrm{ord}_E\} \subset \mathcal{V}_X(p)$ by

$$\sigma_p([\nu]) = \eta_p(\nu) \tag{3.141}$$

where η_p is the map from Proposition 3.5.3 and $[\nu]$ is the class of ν in $\widehat{\mathcal{V}_\infty}$. By Proposition 3.5.3, σ_p is a continuous section of $\eta_{|\mathcal{V}_X(p)} : \mathcal{V}_X(p) \to \widehat{\mathcal{V}_\infty}(p)$. Still by Proposition 3.5.3, the map $\sigma_p : [\mathrm{ord}_E] \cup \widehat{\mathcal{V}_\infty}(p) \to \mathcal{V}_X(p;E)$ extended by $\sigma_p([\mathrm{ord}_E]) = \mathrm{ord}_E$ is also a continuous section of $\eta : \{\lambda \mathrm{ord}_E : \lambda > 0\} \cup \mathcal{V}_X(p) \to \{[\mathrm{ord}_E]\} \cup \widehat{\mathcal{V}_\infty}(p)$. These maps σ_p glue together to give a continuous section $\sigma_E : \widehat{\mathcal{V}_\infty}(E) \to \mathcal{V}_X(E;E) \subset \mathcal{V}_X(E)$ of $\eta : \mathcal{V}_X(E) \to \widehat{\mathcal{V}_\infty}(E)$.

To finish the proof we need to understand the behaviour of σ_F, σ_E on

$$\widehat{\mathcal{V}_\infty}(E) \cap \widehat{\mathcal{V}_\infty}(F) = \widehat{\mathcal{V}_\infty}(p) \tag{3.142}$$

for $p = E \cap F$ where E, F are two prime divisors at infinity. By Proposition 3.4.11, we have that the map $N_{p,F} \circ N_{p,E}^{-1} : \mathcal{V}_X(p;E) \backslash \{\mathrm{ord}_E\} \to \mathcal{V}_X(p;F) \backslash \{\mathrm{ord}_F\}$ is a homeomorphism and we have

$$(\sigma_F)_{|\widehat{\mathcal{V}_\infty}(p)} = (N_{p,F} \circ N_{p,E}^{-1}) \circ (\sigma_E)_{|\widehat{\mathcal{V}_\infty}(p)} \tag{3.143}$$

□

3.5.2 The strong topology

Let $R = \mathbf{k}[[x,y]]$ and let $\mathfrak{m} = (x,y)$. Let $\mathcal{V}_*$ be the valuative tree with either the normalization by $\mathfrak{m}$ or with respect to a curve z. We will write α_* for the skewness function over $\mathcal{V}_*$. We consider a stronger topology on $\mathcal{V}_*$. Let $\mathcal{V}_*^{qm}$ be the subset of quasimonomial valuations. We define the following distance

$$d(\nu_1, \nu_2) = \alpha(\nu_1) - \alpha(\nu_1 \wedge \nu_2) + \alpha(\nu_2) - \alpha(\nu_1 \wedge \nu_2). \tag{3.144}$$

The topology induced by this distance is called the *strong* topology.

Proposition 3.5.5 ([FJ04] Proposition 5.12)**.** *We have the following*

- *The strong topology is stronger than the weak topology.*
- *The closure of $\mathcal{V}_*^{qm}$ with respect to the strong topology is the subspace of $\mathcal{V}_*$ consisting of valuations of finite skewness.*

Proposition 3.5.6. *Let $R = \mathbf{k}[[z,w]]$ and let $\mathcal{V}_\mathfrak{m}, \mathcal{V}_z, \mathcal{V}_w$ be the three valuation trees. Let $\mathcal{V}'_\mathfrak{m}, \mathcal{V}'_z, \mathcal{V}'_w$ be the three subtrees of valuations of finite skewness. Then, the maps*

$$N_z : \mathcal{V}'_\mathfrak{m} \to \mathcal{V}'_z \backslash \{\mathrm{ord}_z\}, \quad N_w \circ N_z^{-1} : \mathcal{V}'_z \to \mathcal{V}'_w \tag{3.145}$$

are homeomorphisms with respect to the strong topology.

This follows from Proposition 3.4.9.

Let $\mathcal{V}''_\infty$ be the subset of $\mathcal{V}_\infty$ of valuations of finite skewness, this set is well defined thanks to Proposition 3.4.30. We define the *strong topology* on $\mathcal{V}''_\infty$ as follows. First define the strong topology on $\widehat{\mathcal{V}_\infty}' := \eta(\mathcal{V}''_\infty)$ using the notations from the proof of Theorem 3.5.4. Consider the map $\sigma_E : \widehat{\mathcal{V}_\infty}' \cap \widehat{\mathcal{V}_\infty}(E) \to \mathcal{V}_X(E;E)'$. We define the strong topology on $\widehat{\mathcal{V}_\infty}' \cap \widehat{\mathcal{V}_\infty}(E)$ as the coarsest topology such that σ_E is continuous for the strong topology on $\mathcal{V}_X(E;E)'$. This defines a topology on $\widehat{\mathcal{V}_\infty}'$ thanks to Proposition 3.5.6.

Corollary 3.5.7. *Let* ν *be a valuation centered at infinity, let* X *be a completion of* X_0 *and let* (ν_n) *be the approximating sequence of* ν *from Proposition 3.4.26. If* $\nu \in \mathcal{V}''_\infty$*, then* $\eta(\nu_n)$ *converges towards* $\eta(\nu)$ *with respect to the strong topology.*

Proof. Let $p = c_X(\nu)$ and we can suppose that $\nu_n, \nu \in \mathcal{V}_X(p;E)$ for some prime divisor E at infinity with $p \in E$. Then, we have $\nu_n \leqslant \nu$ for all n and $\alpha(\nu_n) \to \alpha(\nu)$. Therefore

$$d(\nu_n, \nu) = \alpha(\nu) - \alpha(\nu_n) \xrightarrow[n \to \infty]{} 0 \tag{3.146}$$

$\square$

3.6 Valuations as Linear forms

As done in [JM12], we can view valuations on X_0 as

- linear forms with values in $\mathbf{R}$ over the space of integral Cartier Divisors over X supported at infinity
- as real-valued functions over the set of coherent fractional ideal sheaves of X co-supported at infinity.

We recall how to do so. For a divisor D, we denote by $H^0(X, O_X(D))$ the set of global sections of the line bundle $O_X(D)$ and

$$\Gamma(X, O_X(D)) = \left\{h \in \mathbf{k}(X)^\times : D + \operatorname{div}(h) \geqslant 0\right\}. \tag{3.147}$$

3.6.1 Valuations as linear forms over $\operatorname{Div}_\infty(X)$

Lemma 3.6.1. *Let $D \in \operatorname{Div}(X)$ such that the negative part (if any) of D is supported in $\partial_X X_0$. For any point $p \in X$, there exists an open neighbourhood U of p such that a local equation of D on U is of the form $\varphi = P \cdot \psi$ with $P \in O_X(X_0)$ and $\psi \in O_X(U)$.*

Proof. Let $\varphi \in \mathbf{k}(U')^* = \mathbf{k}(X)^*$ be a local equation of D where U' is an open subset of X containing p.

Let H be an effective divisor such that the linear system $|H|$ is base point free and such that $\operatorname{Supp}(H) = \partial_X X_0$. There exists an integer $n \geqslant 1$ such that $D + nH \geqslant 0$. Pick P general in $\Gamma(X, O_X(nH)) \subset O_X(X_0)$, then $\operatorname{div} P = Z_P - nH$ with $Z_P \geqslant 0$ and $p \notin \operatorname{Supp} Z_P$ because we chose P general and $|nH|$ is basepoint free, in particular P restricts to a regular function over X_0. Set $\psi := \varphi/P$, one has

$$\operatorname{div}\left(\psi_{|U}\right) = D_{|U} + nH_{|U} - Z_{P|U}. \tag{3.148}$$

Set $U = U' \setminus \operatorname{Supp} Z_P$, then $\operatorname{div}(\psi)_{|U'} \geqslant 0$, i.e $\psi \in O_X(U)$ and we are done. $\square$

Corollary 3.6.2. *If D is a divisor such that the negative part (if any) of D is at infinity and ν is a valuation on A, then for all small enough affine open subsets $U \subset X$ containing $c_X(\nu)$,*

$$\Gamma(U, O_X(-D)) \subset O_X(X_0)_{\mathfrak{p}_{\nu_X}} \tag{3.149}$$

and ν_X can be extended to $\Gamma(U, O_X(-D))$.

Proof. If U is small enough, then $\Gamma(U, O_X(-D))$ is the $O_X(U)$-module generated by φ where φ is a local equation of D. Now, by Lemma 3.6.1, φ is of the form $\varphi = P \cdot \psi$ where $P \in O_X(X_0)$ and $\psi \in O_X(U)$. By definition we have $O_X(X_0) \subset O_X(X_0)_{\mathfrak{p}_{\nu_X}}$ and for all affine open neighbourhood U of $c_X(\nu), O_X(U) \subset O_X(X_0)_{\mathfrak{p}_{w_X}}$ by the proof of Lemma 3.3.11. □

Let D be divisor of X supported at infinity and let $\varphi \in \mathbf{k}(X)$ be a local equation of D at $c_X(\nu)$. Then we set

$$L_{\nu,X}(D) := \nu_X(\varphi). \tag{3.150}$$

This is well defined because by Corollary 3.6.2 because by definition there exists an open affine neighbourhood U of $c_X(\nu)$ such that $\varphi \in \Gamma(U, O_X(-D))$. This does not depend on the choice of the local equation because if ψ is another local equation of D, then $\frac{\varphi}{\psi}$ is a regular invertible function near $c_X(\nu)$ and $\nu_X(\varphi/\psi) = 0$.

Lemma 3.6.3. *Let ν be a valuation over A and let X be a completion of X_0, then for all $D \in \mathrm{Div}_\infty(X)_\mathbf{R}$, $L_{\nu,X}(D) < \infty$.*

Proof. It suffices to show Lemma 3.6.3 for D an integral divisor supported at infinity in X. We can apply corollary 3.6.2 to D and $-D$, therefore if φ is a local equation of D, we have that both $\iota_X^*(\varphi)$ and $\iota_X^*(1/\varphi)$ belong to $A_{\mathfrak{p}_\nu}$ and this means that $\nu_X(\varphi) < \infty$. □

Remark 3.6.4. We can in fact define $L_{\nu,X}$ at any divisor D on X such that the negative part of D is supported at infinity but it could happen that $L_{\nu,X}(D)$ is infinite. For example, let $X_0 = \mathbf{A}^2, X = \mathbf{P}^2$. Let ν be the curve valuation centered at $[1:0:0]$ associated to the curve $y = 0$, then

$$L_{\nu,\mathbf{P}^2}(\{Y = 0\} - \{Z = 0\}) = \nu(Y/Z) = +\infty. \tag{3.151}$$

Example 3.6.5. If X is a completion of X_0, let E be a prime divisor at infinity. Let $D \in \mathrm{Div}_\infty(X)$. Recall that we have defined in Section 3.2.1 that $\mathrm{ord}_E(D)$ is the weight of D along E, then

$$L_{\mathrm{ord}_E}(D) = \mathrm{ord}_E(D). \tag{3.152}$$

Indeed, at the generic point of E, a local equation of D is $z^{\mathrm{ord}_E(D)}\varphi$ where z is a local equation of E and φ is regular not divisible by z.

Proposition 3.6.6. *If ν is a valuation over A, and X is a completion of X_0 then*

(1) $L_{\nu,X}(0_{\mathrm{Div}_\infty(X)}) = 0$.

(2) *For any* $D, D' \in \mathrm{Div}_\infty(X), L_{\nu,X}(D+D') = L_{\nu,X}(D) + L_{\nu,X}(D')$*, and* $L_{\nu,X}(mD) = mL_{\nu,X}(D)$ *for all* $m \in \mathbf{Z}$.

(3) *If* $D \geqslant 0$*, then* $L_{\nu,X}(D) \geqslant 0$ *and* $L_{\nu,X}(D) > 0 \Leftrightarrow c_X(\nu) \in \mathrm{Supp}\, D$*. In particular, if* ν *is not centered at infinity then* $L_\nu = 0$*.*

(4) *If* $P \in O_X(X_0)$*, then* $\nu_X(P) = L_{\nu,X}(\mathrm{div}\, P)$*.*

(5) *If* Y *is another completion of* X_0 *above* X*, and* $\pi : Y \to X$ *is the morphism of completions over* X_0*, then* $L_{\nu,X}(D) = L_{\nu,Y}(\pi^* D)$*.*

Thus, we can extend $L_{\nu,X}$ to $\mathrm{Div}_\infty(X)_\mathbf{R}$ by linearity:

$$L_{\nu,X} : \mathrm{Div}_\infty(X)_\mathbf{R} \to \mathbf{R}. \tag{3.153}$$

Proof. The first assertion is trivial as 1 is a local equation of the trivial divisor. The second assertion follows from the fact that if φ, ψ are local equations of D and D' respectively, then $\varphi\psi$ is a local equation of $D + D'$ and $1/\varphi$ is a local equation of $-D$. For the third one, suppose D is an integral divisor. If D is effective and f is a local equation at $c_X(\nu)$, then f is regular at p and by definition of the center $\nu(f) \geqslant 0$, now if $c_X(\nu)$ belongs to $\mathrm{Supp}\, D$, then f vanishes at $c_X(\nu)$; thus, $\nu(f) > 0$. If on the other hand $c_X(\nu) \notin \mathrm{Supp}\, D$, then f is invertible at the center of ν_X and $\nu_X(f) = 0$. The fourth assertion follows from f being a local equation of $\mathrm{div}(f)$ and the fact that f has no pole over X_0. Finally, if $f \in \mathbf{k}(X)$ is a local equation of D at $c_X(\nu)$, then $\pi^* f$ is a local equation of $\pi^* D$ at $c_Y(\nu)$ and by Remark 3.3.3, one has $\nu_X(f) = \nu_Y(\pi^* f)$. □

Proposition 3.6.7. *Let* $f : X_0 \to X_0$ *be a dominant endomorphism of* X_0*. Let* Y, X *be two completions of* X_0 *such that the lift* $F : Y \to X$ *of* f *is regular. Then,*

$$F(c_Y(v)) = c_X(f_* v) \text{ and } \forall D \in \mathrm{Div}_\infty(X), L_{f_* v, X}(D) = L_{v,Y}(F^* D) \tag{3.154}$$

Proof. Let $p = c_Y(\nu)$ and $q = c_X(f_*\nu)$. Then, F induces a local ring homomorphism

$$F^* : O_{X,q} \to O_{Y,p}$$

Now, for any $\varphi \in O_{X,q}$, there exists $P, Q \in A$ such that $\varphi = \frac{P}{Q}$. Therefore,

$$F^* \varphi = \frac{f^* P}{f^* Q}$$

and therefore $f_*\nu(\varphi) = \nu(F^*\varphi) > 0$. Therefore, $q = c_X(f_*\nu)$.

Now, to show the second result. If g is a local equation of D at the center of ν_X, then F^*g is a local equation of F^*D at the center of ν_Y. Since $\pi_*\nu_Y = \nu_X$, one has

$$\nu_Y(F^*g) = \nu_X((F\circ\pi^{-1})^*g) = \nu_X(f^*g) = (f_*\nu)_X(g) \tag{3.155}$$

and this shows the result. □

3.6.2 Valuations as real-valued functions over the set of fractional ideals co-supported at infinity in X

An *ideal* of X is a sheaf of ideals of $\mathcal{O}_X$ and a *fractional ideal* is a coherent sub-$\mathcal{O}_X$-module of the constant sheaf $\mathbf{k}(X)$. Let $\mathfrak{a}$ be a fractional ideal of X, we say that $\mathfrak{a}$ is *co-supported* at infinity if $\mathfrak{a}_{|X_0} = \mathcal{O}_{X_0}$. For example, for any divisor $D \in \mathrm{Div}(X)$, $\mathcal{O}_X(D)$ is a fractional ideal of X and if $D \in \mathrm{Div}_\infty(X)$ then $\mathcal{O}_X(D)$ is co-supported at infinity.

Proposition 3.6.8. *Let $\mathfrak{a}$ be a fractional ideal of X co-supported at infinity and let $p \in X$, for any finite system $(f_1,\cdots,f_r)$ of local generators of $\mathfrak{a}$ at p there exists an open neighbourhood U of p such that $f_{i|U}$ is of the form*

$$f_i = F_i g_i \tag{3.156}$$

with $F_i \in \mathcal{O}_X(X_0)$ and $g_i \in \mathcal{O}_X(U)$.

Proof. Pick U' an open neighbourhood containing p. Since f_i is regular over X_0, we have $\mathrm{div}\, f_i = D^+ - D_1^- - D_2^-$ where D^+, D_1^- and D_2^- are effective divisors such that $\mathrm{Supp}\, D_1^- \subset \partial_X X_0$ and $D^-_{2\,|U'} = 0$. By Lemma 3.6.1 there exists an open neighbourhood $U_i \subset U'$ of p such that $(D^+ - D_1^-)_{|U_i} = \mathrm{div}\, F_i g_i'$ with $F_i \in \mathcal{O}_X(X_0)$ and $g_i' \in \mathcal{O}_X(U_i)$. Therefore, there exists $g_i'' \in \mathcal{O}_X(U_i)$ such that $f_i = F_i g_i' g_i''$. Set $U = \cap U_i$ and $g_i = g_i' g_i''$.

□

Corollary 3.6.9. *Let $\mathfrak{a}$ be a fractional ideal co-supported at infinity and let ν be a valuation over A, then for all affine open neighbourhood of $c_X(\nu)$, $\Gamma(U,\mathfrak{a}) \subset \mathcal{O}_X(X_0)_{\mathfrak{p}_{\nu_X}}$ and ν_X is defined over $\Gamma(U,\mathfrak{a})$.*

If ν is a valuation over A, then we define $L_{\nu,X}(\mathfrak{a})$ as

$$L_{\nu,X}(\mathfrak{a}) := \min_f \nu_X(f). \tag{3.157}$$

where the f runs over the germs of sections of $\mathfrak{a}$ at $c_X(\nu)$. This makes sense by Corollary 3.6.9.

Proposition 3.6.10. *If ν is a valuation over A, then*

(1) $L_{\nu,X}(\mathcal{O}_X) = 0$.

(2) If $\mathfrak{a}, \mathfrak{b}$ are two fractional ideals of X co-supported at infinity, then

$$L_{\nu,X}(\mathfrak{a}\cdot\mathfrak{b}) = L_{\nu,X}(\mathfrak{a}) + L_{\nu,X}(\mathfrak{b}) \text{ and } L_{\nu,X}(\mathfrak{a}+\mathfrak{b}) = \min(L_{\nu,X}(\mathfrak{a}), L_{\nu,X}(\mathfrak{b})) \tag{3.158}$$

(3) If $f_1, \cdots, f_r \in \mathbf{k}(X)$ is a set of local generators of $\mathfrak{a}$ at $c_X(\nu)$, then

$$L_{\nu,X}(\mathfrak{a}) = \min(\nu_X(f_1), \cdots, \nu_X(f_r)). \tag{3.159}$$

(4) If $D \in \mathrm{Div}(X)$ is a divisor, then $L_{\nu,X}(D) = L_{\nu,X}(\mathcal{O}_X(-D))$.

(5) If Y is another completion of X_0 above X, and $\pi : Y \to X$ is the morphism of completions over $\mathbf{X}_0$, then $\widetilde{\mathfrak{a}} := \pi^*\mathfrak{a}\cdot\mathcal{O}_Y$ is a fractional ideal over Y and $L_{\nu,X}(\mathfrak{a}) = L_{\nu,Y}(\widetilde{\mathfrak{a}})$.

Proof. The first assertion is trivial since 1 is a local generator of the trivial sheaf. For Assertion (2), notice that if $(f_1, \dots, f_r)$ are local generators of $\mathfrak{a}$ at $c_X(\nu)$ and $(g_1, \dots, g_s)$ local generators of $\mathfrak{b}$ at $c_X(\nu)$ then $(f_i g_j)_{i,j}$ is a set of local generators of $\mathfrak{a}\cdot\mathfrak{b}$ at $c_X(\nu)$ and $(f_1, \dots, f_r, g_1, \dots, g_s)$ is a set of local generators of $\mathfrak{a}+\mathfrak{b}$ at $c_X(\nu)$, so Assertion (2) follows from Assertion (3). To show Assertion (3), let $f_1, \cdots, f_r$ be local generators of $\mathfrak{a}$ at $c_X(\nu)$. This implies that $\mathfrak{a}_{c_X(\nu)} = f_1\mathcal{O}_{c_X(\nu)} + f_2\mathcal{O}_{c_X(\nu)} + \cdots + f_r\mathcal{O}_{c_X(\nu)}$. Since ν is nonnegative on $\mathcal{O}_{c_X(\nu)}$ by definition of the center, the assertion follows. For assertion 5, if $f_1, \cdots, f_r$ are local generators of $\mathfrak{a}$, then $\pi^*f_1, \cdots, \pi^*f_r$ are local generators of $\widetilde{\mathfrak{a}}$ at $c_Y(\nu)$ and the result follows since $\pi_*\nu_Y = \nu_X$. Assertion (4) follows from the fact $\mathcal{O}_X(-D)$ is locally generated by an equation of D and Assertion (5) follows from the fact that if $(f_1, \cdots, f_r)$ are local generators of $\mathfrak{a}$ at $c_X(\nu)$ then $(\pi^*f_1, \cdots, \pi^*f_r)$ are local generators of $\widetilde{\mathfrak{a}}$ at $c_Y(\nu)$. $\square$

Proposition 3.6.11. *If ν is a valuation over A and $\mathfrak{a}$ is a fractional ideal co-supported at infinity, then $L_{\nu,X}(\mathfrak{a}) < \infty$.*

Proof. Take $f_1, \cdots, f_r$ local generators of $\mathfrak{a}$ at p the center of ν on X. The proof of Lemma 3.6.1 shows that there exists an affine open neighbourhood U of p such that $f_{i|U} = h_i g_i$ with $h_i \in A$ and $g_i \in \mathcal{O}_X(U)$ and such that f_i^{-1} can be put into the same form. This shows that for all i, $\nu(f_i) < \infty$. $\square$

Remark 3.6.12. The same definition would allow one to define $L_{\nu,X}(\mathfrak{a})$ for any fractional ideal such that $\mathfrak{a}$ is a sheaf of ideals of X_0 but we have to allow infinite values. In particular, $L_{\nu,X}(\mathfrak{a})$ is defined for any sheaf of ideals over X.

3.6.3 Valuations centered at infinity

Recall that a valuation ν over A is *centered at infinity*, if ν does not admit a center on X_0. We denote by $\mathcal{V}_\infty$ the set of valuations over A centered at infinity.

Lemma 3.6.13. *Let ν be valuation over A. The following assertions are equivalent.*

(1) ν is centered at infinity.

(2) There exists $P \in A$ such that $\nu(P) < 0$.

(3) For any completion X of X_0 and any effective divisor H in X such that $\operatorname{Supp} H = \partial_X X_0$, one has $L_{\nu,X}(H) > 0$.

(4) There exists a completion X of X_0 and an effective divisor $H \in X$ with $\operatorname{Supp} H = \partial_X X_0$ such that $L_{\nu,X}(H) > 0$.

Proof. We will show the following implications $2 \Rightarrow 1 \Rightarrow 3 \Rightarrow 4$. Then, we will show that $1 \Rightarrow 2$ and finally that $4 \Rightarrow 2$.

$2 \Rightarrow 1 \Rightarrow 3 \Rightarrow 4$ If there exists a regular function P over X_0 such that $\nu(P) < 0$ then the center of ν cannot be a point of X_0 because P is regular at every point of X_0. This shows $2 \Rightarrow 1$, then if ν is centered at infinity, take a completion X of X_0, let E be a prime divisor at infinity in X such that $c_X(\nu) \in E$. Then, since H is effective and $E \in \operatorname{Supp} H$, $L_{\nu,X}(H) \geqslant \nu(E) > 0$ by Proposition 3.6.6 (1). This shows $1 \Rightarrow 3$ and $3 \Rightarrow 4$ is clear.

$1 \Rightarrow 2$ Conversely, suppose ν is centered at infinity and fix a closed embedding $X_0 \hookrightarrow \mathbf{A}^N$ for some integer N. Let X be the Zariski closure of X_0 in $\mathbf{P}^N$ with homogeneous coordinates $x_0, \cdots, x_N$ such that $\{x_0 = 0\}$ is the hyperplane at infinity. The surface X might not be smooth so it is not necessarily a completion of X_0 but it still is proper and the center p of ν on X belongs to $\{x_0 = 0\} \cap X$. Let $1 \leqslant i \leqslant N$ be an integer such that p belongs to the open subset $\{x_i \neq 0\}$. Then, the rational function $P := \frac{x_i}{x_0}$ is a regular function on X_0 and $1/P$ vanishes at p. Therefore, $\nu(P) < 0$.

$4 \Rightarrow 1$ Suppose that ν is not centered at infinity, i.e the center of ν belongs to X_0. Then, for any completion X and for any divisor $D \in \mathrm{Div}_\infty(X)$, one has $L_{\nu,X}(D) = 0$ by Proposition 3.6.6 (1) since $c_X(\nu) \notin \mathrm{Supp}\, D$. □

This lemma shows that being centered at infinity is a property that can be tested on only one completion X_0.

Corollary 3.6.14. *The space $\mathcal{V}_\infty$ is an open subset of $\mathcal{V}$.*

Proof. We have by Lemma 3.6.13 that

$$\mathcal{V}_\infty = \bigcup_{P \in A} \{\nu(P) < 0\}. \tag{3.160}$$

Therefore, it is a union of open subsets.

□

3.6.3.1 Topologies over the set of valuations centered at infinity

Let X be a completion of X_0. Call σ the coarsest topology such that the evaluation maps $\varphi_f : \nu \in \mathcal{V}_\infty \mapsto \nu(f)$ are continuous for all $f \in A$ and τ the coarsest topology such that the evaluation maps $\psi_{\mathbf{A}} : \nu \in \mathcal{V}_\infty \mapsto L_\nu(\mathbf{A})$ are continuous for all fractional ideals $\mathbf{A}$ of X such that $\mathbf{A}_{|X_0}$ is a sheaf of ideals over X_0. Recall that we allow in both cases infinite values.

Proposition 3.6.15. *[JM12] These two topologies on $\mathcal{V}$ are the same.*

Proof. First if $f \in A$, then $\nu(f) = L_\nu((f))$ where (f) is the fractional ideal generated by f. So σ is finer than τ. For the other way, Let H be an ample divisor supported at infinity and let $\mathbf{A}$ be a fractional ideal co-supported at infinity. There exists an integer $n > 0$ such that $\mathbf{A} \otimes O_X(nH)$ and $O_X(nH)$ are generated by global sections $(f_1, \cdots, f_r)$ and $(g_1, \cdots, g_s)$ respectively. Notice that for all i, j, the rational functions f_i, g_j belong to $O_X(X_0)$. Now, we have that $L_\nu(\mathbf{A}) = L_\nu(\mathbf{A} \otimes O_X(nH) \otimes O_X(-nH))$, therefore

$$L_\nu(\mathbf{A}) = \min_{i,j} \left(\nu \left(\frac{f_i}{g_j} \right) \right) = \min_{i,j} \left(\nu(f_i) - \nu(g_j) \right)$$

Therefore, τ is finer than σ and the result is shown.

□

3.6.3.2 Valuations centered at infinity as linear forms over $\mathrm{Cartier}_\infty(\mathrm{X}_0)$

Definition 3.6.16. Let ν be a valuation over A. Let $D \in \mathrm{Cartier}_\infty(\mathrm{X}_0)$ and X be a completion of X_0 such that D is defined by D_X. We define

$$L_\nu(D) := L_{\nu,X}(D_X). \tag{3.161}$$

This does not depend on the choice X and defines a linear map on $\mathrm{Cartier}_\infty(\mathrm{X}_0)$ by Proposition 3.6.6 and $L_\nu(D) < +\infty$ by Lemma 3.6.3. Notice that $L_\nu = 0$ if and only if ν is not centered at infinity.

Proposition 3.6.17. *If ν is a valuation on A centered at infinity then L_ν is a linear form* $\mathrm{Cartier}_\infty(\mathrm{X}_0) \to \mathbf{R}$ *and satisfies*

(1) If $D \geqslant 0$, then $L_\nu(D) \geqslant 0$.

(2) For $D, D' \in \mathrm{Cartier}_\infty(\mathrm{X}_0), L_\nu(D \wedge D') = \min(L_\nu(D), L_\nu(D'))$.

We will say that an element of $\mathrm{Hom}(\mathrm{Cartier}_\infty(\mathrm{X}_0), \mathbf{R})$ *that satisfies these 2 properties satisfies property (+).*

Proof. Assertion 1 follows from Proposition 3.6.6 (3). We show the second assertion. Take $D, D' \in \mathrm{Cartier}_\infty(\mathrm{X}_0)$ and X a completion of X_0 such that D, D' are defined by their incarnation D_X, D'_X. By Claim 3.2.8 (that we prove in the next section), we know that there exists a completion Y along with a morphism of completions $\pi : Y \to X$ such that $D \wedge D'$ is the Cartier class determined by some divisor D_Y in Y such that $\pi^*(\mathcal{O}_X(-D_X) + \mathcal{O}_X(-D'_X)) \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D_Y)$. Using Proposition 3.6.10, it follows that

$$\begin{aligned}
L_\nu(D \wedge D') &= L_{\nu,Y}(D_Y) \\
&= L_{\nu,Y}(\mathcal{O}_Y(-D_Y)) \quad 3.6.10(4) \\
&= L_{\nu,X}(\mathcal{O}_X(-D_X) + \mathcal{O}_X(-D'_X)) \quad 3.2.8 \\
&= \min(L_{\nu,X}(\mathcal{O}_X(-D_X)), L_{\nu,X}(\mathcal{O}_X(-D'_X))) \quad 3.6.10(2) \\
&= \min(L_\nu(D), L_\nu(D')) \quad 3.6.10(4)
\end{aligned}$$

For the third assertion, let X be a completion of X_0, by Theorem 3.1.4 there exists an ample divisor $H \in \mathrm{Div}_\infty(X)$ such that $H \geqslant 0$ and $\mathrm{Supp}\, H = \partial_X X_0$. We get that $c_X(\nu) \in \mathrm{Supp}\, H$ (whether it is a prime divisor or a closed point) and therefore by Proposition 3.6.6 item (3), we get $L_\nu(H) > 0$. $\square$

Proposition 3.6.18. *Let ν be a valuation over A and $f : X_0 \to X_0$ a dominant endomorphism, then for all $D \in \mathrm{Cartier}_\infty(X_0)$,*

$$L_{f_*\nu}(D) = L_\nu(f^*D) = (f_*L_\nu)(D) \tag{3.162}$$

Proof. Let X be a completion of X_0 where D is defined, then f induces a dominant rational map $f : X \to X$. Let $\pi : Y \to X$ be a projective birational morphism such that the lift $F : Y \to X$ is regular. Since f is an endomorphism of X_0 we can suppose that π is the identity over X_0, hence Y is a completion of X_0 and π is a morphism of completions. Now, if φ is a local equation of D near the center of ν_X, then $F^*\varphi$ is a local equation of F^*D near the center of ν_Y. Since $\pi_*\nu_Y = \nu_X$, one has

$$\nu_Y(F^*g) = \nu_X((F \circ \pi^{-1})^*g) = \nu_X(f^*g) = (f_*\nu)_X(g) \tag{3.163}$$

$\square$

We equip $\mathrm{Hom}(\mathrm{Cartier}_\infty(X_0), \mathbf{R})$ with the weak-$\star$ topology, that is the coarsest topology such that the map $L \in \mathrm{Hom}(\mathrm{Cartier}_\infty(X_0), \mathbf{R}) \mapsto L(D)$ is continuous for all $D \in \mathrm{Cartier}_\infty(X_0)$. We extend L_ν to $\mathrm{Cartier}_\infty(X_0)_{\mathbf{R}}$ by linearity.

Proposition 3.6.19. *The map $\nu \in \mathcal{V}_\infty \mapsto L_\nu \in \mathrm{Hom}(\mathrm{Cartier}_\infty(X_0), \mathbf{R})$ is a continuous embedding.*

Proof. For the injectivity, let $\nu, \omega \in \mathcal{V}_\infty$ such that $\nu \neq \omega$. First, if $w = t\nu$ with $t > 0$, then since $L_\nu \neq 0$, we have $L_\nu \neq L_w$. Otherwise, there exists a completion X such that $c_X(\nu) \neq c_X(\omega)$. If the centers are both prime divisors at infinity then it is clear that $L_\nu \neq L_w$. If $c_X(\nu)$ is a point, let $\widetilde{E}$ be the exceptional divisor above it. Then, by Proposition 3.6.6, $L_\nu(\widetilde{E}) > 0$, but $L_w(\widetilde{E}) = 0$.

By definition, to show continuity we have to show that for all $D \in \mathrm{Cartier}_\infty(X_0)$, the map $\nu \in \mathcal{V}_\infty \mapsto L_\nu(D)$ is continuous. Let X be a completion where D is defined, then by Proposition 3.6.6 $L_\nu(D) = L_\nu(\mathcal{O}_X(-D))$ and by Proposition 3.6.15 the map $\nu \in \mathcal{V}_\infty \mapsto L_\nu(\mathcal{O}_X(-D))$ is continuous. $\square$

Proposition 3.6.20. *Let X be a completion of X_0 and $p \in X$ a closed point at infinity. Let $\nu \in \mathcal{V}_X(p; \mathfrak{m}_p)$. If E is a prime divisor of X at infinity such that $p \in E$, then*

$$1 \leqslant L_\nu(E) \leqslant \alpha(\nu) \tag{3.164}$$

Proof. Let $z \in O_{X,p}$ be a local equation of E, z is irreducible and we have $L_\nu(E) = \nu(z)$. We have that $z \in \mathfrak{m}_p$, therefore $\nu(z) \geqslant \nu(\mathfrak{m}_p) = 1$. This shows the first inequality. For the second one, let ν_z be the curve valuation associated to z. It does not define a valuation over $\mathbf{k}[X_0]$ but it defines a valuation over $O_{X,p}$ by Proposition 3.4.3, we get

$$\nu(z) = \alpha(\nu_z \wedge \nu) \leqslant \alpha(\nu) \tag{3.165}$$

□

3.6.3.3 Special look at divisorial valuations centered at infinity

Lemma 3.6.21. *Let X be a completion of X_0 and let E be a prime divisor at infinity. One has $L_{\mathrm{ord}_E}(E) = 1$ and for any prime divisor $F \neq E$ in X, $L_{\mathrm{ord}_E}(F) = 0$.*

Furthermore, if $\pi : Y \to X$ is some blow-up of X, and $\pi'(E)$ the strict transform of E by π, then

$$\pi_* \operatorname{ord}_{\pi'(E)} = \operatorname{ord}_E. \tag{3.166}$$

Proof. The first assertion follows from Proposition 3.6.6 (3). We show the second assertion. It suffices to show it when π is the blow-up of one point of X. Let $D = aE + \sum_{F \neq E} \operatorname{ord}_F(D)F$, then $\pi^* D$ is of the form

$$\pi^* D = a\pi'(E) + b\widetilde{E} + \sum_{F \neq E} a_F(D)\pi'(F) \tag{3.167}$$

where $\widetilde{E}$ is the exceptional divisor of π. Therefore $\operatorname{ord}_{\pi'(E)}(\pi^*(D)) = a = \operatorname{ord}_E(D)$. □

Proposition 3.6.22. *Let ν be a divisorial valuation, then L_ν can be extended naturally to* $\mathrm{Weil}_\infty(\mathrm{X}_0)$ *in a compatible way with the definition of L_ν over* $\mathrm{Cartier}_\infty(\mathrm{X}_0)$.

Proof. Take $W \in \mathrm{Weil}_\infty(\mathrm{X}_0)$. Since ν is divisorial, there exists a completion X of X_0 that contains a prime divisor E at infinity such that $(\iota_X)_* \nu = \lambda \operatorname{ord}_E$. We set

$$L_\nu(W) := L_{\nu,X}(W_X) \tag{3.168}$$

This does not depend on the completion X. To show this, it suffices to show that we get the same result if we blow up one point of X. So, let $\pi : Y \to X$ be the blow up of one point of X_0 at infinity. Then, by Lemma 3.6.21, $\nu_Y = \lambda \operatorname{ord}_{\pi'(E)}$ and $\operatorname{ord}_{\pi'(E)}(W_Y) = \operatorname{ord}_E(\pi_* W_Y) = \operatorname{ord}_E(W_X)$.

If $D \in \mathrm{Cartier}_\infty(\mathrm{X}_0)$, then this is compatible with the previous definition of $L_\nu(D)$ because if D is defined over X, there exists a completion $\pi : Y \to X$ such that the center of ν on Y is a prime divisor at infinity and by Proposition 3.6.6 (5) $L_{\nu,Y}(\pi^* D) = L_{\nu,X}(D)$. $\square$

Remark 3.6.23. Recall that we have defined in Section 3.2.1 the set $\mathcal{D}_\infty(X_0)$ as the set of equivalence classes of prime divisors at infinity modulo the following equivalence relations : $(X_1, E_1) \sim (X_2, E_2)$ if $\pi = \iota_2 \circ \iota_1^{-1} : X_1 \dashrightarrow X_2$ satisfies $\pi(E_1) = E_2$. Lemma 3.6.21 shows that it makes sense to define ord_E for $E \in \mathcal{D}_\infty(X_0)$ and that ord_E is defined over $\mathrm{Weil}_\infty(\mathrm{X}_0)$.

Proposition 3.6.24. *Let $W, W' \in \mathrm{Weil}_\infty(\mathrm{X}_0)$, then $W'' = W \wedge W'$ if and only if for any divisorial valuation $E \in \mathcal{D}_\infty(X_0)$,*

$$\mathrm{ord}_E(W'') = \min(\mathrm{ord}_E(W), \mathrm{ord}_E(W')). \tag{3.169}$$

Proof. This is immediate as for any completion X,

$$W_X = \sum_{E \in \partial_X X_0} \mathrm{ord}_E(W) \cdot E. \tag{3.170}$$

$\square$

We can now show that the minimum of two Cartier divisors is still a Cartier divisor.

Proposition 3.6.25. *Let X be a completion of X_0, let $D, D' \in \mathrm{Div}_\infty(X)$ be two effective divisor and let $\mathfrak{a}$ be the sheaf of ideals $\mathfrak{a} = \mathcal{O}_X(-D) + \mathcal{O}_X(-D')$. Then, $D \wedge D'$ is the Cartier divisor defined by $\pi^* \mathfrak{a}$ where π is the blow up of $\mathfrak{a}$.*

Notice that $\mathfrak{a}$ is not locally principle only at satellite points, so π is a sequence of blow-ups of satellite points. This shows the Claim 3.2.8.

Proof of Claim 3.2.8. Define the sheaf of ideals $\mathfrak{a} = \mathcal{O}_X(-D) + \mathcal{O}_X(-D')$ and let $\pi : Y \to X$ be the blow up of $\mathfrak{a}$. There exists a Cartier divisor D_Y on Y such that $\mathfrak{b} = \mathcal{O}_Y(-D_Y) = \pi^* \mathfrak{a} \cdot \mathcal{O}_Y$. We show that $D_Y = D \wedge D'$ in $\mathrm{Cartier}_\infty(\mathrm{X}_0)$. By Proposition 3.6.24, we only need to show that for any divisorial valuation $\nu, L_{\nu,Y}(D_Y) = \min(L_{\nu,X}(D), L_{\nu,X}(D'))$, but by Proposition 3.6.10 we have the following equalities

$$L_{\nu,Y}(D_Y) = L_{\nu,Y}(\mathfrak{b}) = L_{\nu,X}(\mathfrak{a}) = \min(L_{\nu,X}(D), L_{\nu,X}(D')) \tag{3.171}$$

$\square$

3.6.4 Local divisor associated to a valuation

Let X be a completion of X_0 and let $p \in X$ be a closed point at infinity. Let ν be a valuation centered at p. We know by Section 3.6.3.2 that ν induces a linear form L_ν on $\mathrm{Cartier}_\infty(\mathrm{X}_0)_{\mathbf{R}}$. By restriction, it induces a linear form $L_{\nu,X,p}$ on $\mathrm{Cartier}(X,p)_{\mathbf{R}}$. Now by Proposition 3.2.19, the pairing

$$\mathrm{Weil}(X,p)_{\mathbf{R}} \times \mathrm{Cartier}(X,p)_{\mathbf{R}} \to \mathbf{R} \tag{3.172}$$

induced by the intersection product is perfect. Thus, there is a unique $Z_{\nu,X,p} \in \mathrm{Weil}(X,p)_{\mathbf{R}}$ such that

$$\forall D \in \mathrm{Cartier}(X,p)_{\mathbf{R}}, \quad Z_{\nu,X,p} \cdot D = L_{\nu,X,p}(D) \tag{3.173}$$

Example 3.6.26. If $\widetilde{E}$ is the exceptional divisor above p, then $Z_{\mathrm{ord}_{\widetilde{E}},X,p} = -\widetilde{E}$.

Proposition 3.6.27. *For any valuation $\nu \in \mathcal{V}_X(p)$, we have $Z_{\nu,X,p} \in \mathrm{Cartier}(X,p)$ if and only if ν is divisorial. Furthermore, $Z_{\nu,X,p}$ is defined over any completion such that the center of ν is a prime divisor at infinity. Furthermore, for any $E \in \mathcal{D}(X,p), Z_{\mathrm{ord}_E,X,p} \in \mathrm{Cartier}(X,p)_{\mathbf{Q}}$.*

Proof. Let $E \in \mathcal{D}_{X,p}$, for every $W \in \mathrm{Weil}(X,p), \mathrm{ord}_E(W) = \mathrm{ord}_E(W_Y)$ where Y is a completion exceptional above p by Proposition 3.6.22. Let $E, E_1, \cdots, E_r$ be the component of $\partial_Y X_0$ that are exceptional above p. The intersection form is non degenerate on

$$V := \mathbf{Q}E \oplus \left(\bigoplus_i \mathbf{Q}E_i \right). \tag{3.174}$$

Let L be the restriction of ord_E to V, by duality there exists a unique $Z \in V$ such that for all $W \in V, W \cdot Z = L(W) = \mathrm{ord}_E(W)$. This implies that $Z = Z_{\mathrm{ord}_E,X,p}$. Conversely, if ν is a valuation such that $Z_{\nu,X,p} \in \mathrm{Cartier}(X,p)$ then let Y be a completion where $Z_{\nu,X,p}$ is defined. If $c_Y(\nu)$ is a point at infinity, then let $\widetilde{E}$ be the exceptional divisor above $c_Y(\nu)$. Then, we must have $Z_{\nu,X,p} \cdot \widetilde{E} > 0$ but it is equal to 0, this is a contradiction. □

Proposition 3.6.28. *The embedding $\mathcal{V}_X(p;\mathfrak{m}_p) \hookrightarrow \mathrm{Weil}(X,p)_{\mathbf{R}}$ is continuous with respect to the weak topology.*

Proof. This is a direct consequence of Proposition 3.6.19 and Proposition 3.5.2. □

Thus, For all completion $\pi : Y \to X$, for all $E \in \Gamma_\pi$, we can consider $Z_{\mathrm{ord}_E,X,p}$ as an element of $\mathrm{Div}_\infty(Y)_{\mathbf{R}}$.

Proposition 3.6.29. *Let $\pi : (Y, \mathrm{Exc}(\pi)) \to (X, p)$ be a completion exceptional above p. Let ν be a valuation such that $c_X(\nu) = p$. Suppose that $c_Y(\nu)$ is a point at infinity. Consider $\mathcal{V}_X(p;\mathfrak{m}_p)$ with its generic multiplicity function b.*

(1) If $c_Y(\nu) \in E$ is a free point with $E \in \Gamma_\pi$, then the incarnation of $Z_{\nu,X,p}$ in Y is

$$(Z_{\nu,X,p})_Y = L_\nu(E) Z_{\mathrm{ord}_E,X,p} \tag{3.175}$$

Moreover if $\nu \in \mathcal{V}_X(p;\mathfrak{m}_p)$, then $L_\nu(E) = \frac{1}{b(E)}$.

(2) If $c_Y(\nu) = E \cap F$ is a satellite point with $E, F \in \Gamma_\pi$, then

$$(Z_{\nu,X,p})_Y = L_\nu(E) Z_{\mathrm{ord}_E,\nu,p} + L_\nu(F) Z_{\mathrm{ord}_F,X,p} \tag{3.176}$$

Moreover if $\nu \in \mathcal{V}_X(p;\mathfrak{m}_p)$, then $L_\nu(E)b(E) + L_\nu(F)b(F) = 1$.

Furthermore, if $q \neq c_Y(\nu)$ and $\tau : Z \to Y$ is the blow up of q then

$$(Z_{\nu,X,p})_Z = \tau^*(Z_{\nu,X,p})_Y \tag{3.177}$$

Proof. For any prime divisor E at infinity of Y, $L_\nu(E) > 0 \Leftrightarrow c_Y(\nu) \in E$ by item (3) ofProposition 3.6.6. Therefore, if $c_Y(\nu) \in E$ is a free point with $E \in \Gamma_\pi$, then for $F \in \Gamma_\pi, L_\nu(F) \neq 0 \Leftrightarrow F = E$, hence

$$(L_\nu)_{|\mathrm{Div}_\infty(Y)_\mathbf{R}} = (L_\nu(E))(L_{\mathrm{ord}_E})_{|\mathrm{Div}_\infty(Y)_\mathbf{R}}. \tag{3.178}$$

by definition (see Equation (3.152)). This shows the result if $c_Y(\nu)$ is a free point. Now, if $c_Y(\nu) = E \cap F$ is a satellite point with $E, F \in \Gamma_\pi$, then for all prime divisors F' of Y at infinity $L_\nu(F') > 0 \Leftrightarrow F' = E$ or $F' = F$. We therefore have

$$(L_\nu)_{|\mathrm{Div}_\infty(Y)_\mathbf{R}} = (L_\nu \cdot E)(L_{\mathrm{ord}_E})_{|\mathrm{Div}_\infty(Y)_\mathbf{R}} + (L_\nu \cdot F)(L_{\mathrm{ord}_F})_{|\mathrm{Div}_\infty(Y)_\mathbf{R}}. \tag{3.179}$$

This shows the result in the satellite case.

If $\nu \in \mathcal{V}_X(p;\mathfrak{m}_p)$. Let $\tau : Z \to X$ be the blow up of p. We know then that $L_\nu(\widetilde{E}) = 1$ where $\widetilde{E}$ is the exceptional divisor above p by Proposition 3.4.12. Let $b_{\widetilde{E}}$ be the generic multiplicity function of the tree $\mathcal{V}_Z(\widetilde{E};\widetilde{E})$. We have for every prime divisor F exceptional above p that $\mathrm{ord}_F(\widetilde{E}) = b_{\widetilde{E}}(F)$ again by Proposition 3.4.12. In the free point case, we get

$1 = L_\nu(\widetilde{E}) = L_\nu(b_{\widetilde{E}}(E)E)$ by Proposition 3.6.6 (3) and (5). In the satellite point case, we get

$$1 = L_\nu(\widetilde{E}) = L_\nu(b_{\widetilde{E}}(E)E + b_{\widetilde{E}}(F)F) \tag{3.180}$$

again by Proposition 3.6.6 (3) and (5).

For the last assertion, if $\widetilde{F}$ is the exceptional divisor above q, we have

$$(Z_{\nu,X,p})_Z = \tau^*(Z_{\nu,X,p})_Y - (Z_{\nu,X,p} \cdot \widetilde{F})\widetilde{F}. \tag{3.181}$$

Since $c_Z(\nu) \notin \widetilde{F}$, we have $L_\nu(\widetilde{F}) = 0$ by Proposition 3.6.6 (3). □

From now on let b be the generic multiplicity function of $\mathcal{V}_X(p;\mathfrak{m}_p)$ and for any prime divisor $E \in \mathcal{D}_{X,p} = \Gamma$, set $\nu_E = \frac{1}{b(E)}\operatorname{ord}_E$.

Proposition 3.6.30. *Let $\pi : (Y, \operatorname{Exc}(\pi)) \to (X,p)$ be a completion exceptional above p. Let $q \in \operatorname{Exc}(\pi)$ be a closed point. Let $\tau : Z \to Y$ be the blow up of q and let $\widetilde{E}$ be the exceptional divisor above q.*

(1) If $q \in E$ is a free point with $E \in \Gamma_\pi$, then

$$Z_{\nu_{\widetilde{E}},X,p} = \tau^*(Z_{\nu_E,X,p}) - \frac{1}{b(\widetilde{E})}\widetilde{E} \in \operatorname{Div}_\infty(Z)_{\mathbf{Q}} \tag{3.182}$$

(2) If $q = E \cap F$ is a satellite point with $E, F \in \Gamma_\pi$, then

$$Z_{\nu_{\widetilde{E}},X,p} = \frac{b(E)}{b(E)+b(F)}\tau^* Z_{\nu_E,X,p} + \frac{b(F)}{b(E)+b(F)}\tau^* Z_{\nu_F,X,p} - \frac{1}{b(\widetilde{E})}\widetilde{E} \in \operatorname{Div}_\infty(Z)_{\mathbf{Q}} \tag{3.183}$$

Proof. If $q \in E$ is a free point with $E \in \Gamma_\pi$, we have by Proposition 3.6.29 that the incarnation of $Z_{\operatorname{ord}_{\widetilde{E}},X,p}$ in Y is

$$\tau_*(Z_{\operatorname{ord}_{\widetilde{E}},X,p}) = Z_{\operatorname{ord}_E,X,p} \tag{3.184}$$

because $\operatorname{ord}_{\widetilde{E}}(E) = 1$. Therefore

$$Z_{\operatorname{ord}_{\widetilde{E}},X,p}\tau^* Z_{\operatorname{ord}_E,X,p} + \lambda\widetilde{E} \tag{3.185}$$

with $\lambda \in \mathbf{R}$. Since $Z_{\operatorname{ord}_{\widetilde{E}},X,p} \cdot \widetilde{E} = 1$, we get $\lambda = -1$. Now, by the definition of the generic multiplicity, we have $b(\widetilde{E}) = b(E)$. Therefore,

$$Z_{\nu_{\widetilde{E}},X,p} = \tau^* Z_{\nu_E,X,p} - \frac{1}{b(\widetilde{E})}\widetilde{E} \tag{3.186}$$

If $q = E \cap F$ is a satellite point with $E, F \in \Gamma_\pi$, then $b(\widetilde{E}) = b(E) + b(F)$. Note that $\mathrm{ord}_{\widetilde{E}}(E) = \mathrm{ord}_{\widetilde{E}}(F) = 1$. We have by Proposition 3.6.29

$$\tau_* Z_{\mathrm{ord}_{\widetilde{E}},X,p} = Z_{\mathrm{ord}_E,X,p} + Z_{\mathrm{ord}_F,X,p} \tag{3.187}$$

and since $\mathrm{ord}_{\widetilde{E}}(\widetilde{E}) = 1$, we get

$$Z_{\mathrm{ord}_{\widetilde{E}},X,p} = \tau^* Z_{\mathrm{ord}_E,X,p} + \tau^* Z_{\mathrm{ord}_F,X,p} - \widetilde{E}. \tag{3.188}$$

Therefore,

$$Z_{\nu_{\widetilde{E}},X,p} = \frac{b(E)}{b(E)+b(F)}\tau^* Z_{\mathrm{ord}_E,X,p} + \frac{b(F)}{b(E)+b(F)}\tau^* Z_{\mathrm{ord}_F,X,p} - \frac{1}{b(\widetilde{E}))}\widetilde{E}. \tag{3.189}$$

□

Theorem 3.6.31. *Let $\nu, \nu' \in \mathcal{V}_X(p; \mathfrak{m}_p)$, then*

$$Z_{\nu,X,p} \cdot Z_{\nu',X,p} = -\alpha(\nu \wedge \nu') \tag{3.190}$$

Proof. We show by induction the

Claim 3.6.32. *For every completion $\pi : (Y, \mathrm{Exc}(\pi)) \to (X, p)$ exceptional above p, for all $E \in \Gamma_\pi$, for all $\nu \in \mathcal{V}_X(p; \mathfrak{m}_p)$,*

$$Z_{\nu_E,X,p} \cdot Z_{\nu,X,p} = -\alpha(\nu_E \wedge \nu) \tag{3.191}$$

First if $\pi : Y \to X$ is the blow up of p with exceptional divisor $\widetilde{E}$. Recall that $\pi_* \mathrm{ord}_{\widetilde{E}} = \nu_{\mathfrak{m}_p}$ then $Z_{\mathrm{ord}_{\widetilde{E}},X,p} = -E$ and

$$Z_{\mathrm{ord}_{\widetilde{E}},X,p} \cdot Z_{\nu,X,p} = Z_{\nu,X,p} \cdot (-\widetilde{E}) = \cdot L_\nu(-\widetilde{E}). \tag{3.192}$$

By definition, $\nu(\mathfrak{m}_p) = 1$ and $\pi^* \mathfrak{m}_p = \mathcal{O}_Y(-\widetilde{E})$. Therefore, by Proposition 3.6.10, we get $Z_{\mathrm{ord}_{\widetilde{E}},X,p} \cdot Z_{\nu,X,p} = -1 = -\alpha(\nu_{\mathfrak{m}_p} \wedge \nu)$.

Suppose that $\pi : (Y, \mathrm{Exc}(\pi)) \to (X, p)$ is a completion exceptional above p for which the claim holds. Let $q \in Y$ be a closed point at infinity, let $\tau : Z \to Y$ be the blow up of q and let $\widetilde{E}$ be the exceptional divisor. Let $\nu \in \mathcal{V}_X(p;\mathfrak{m}_p)$, we show that $Z_{\nu,X,p} \cdot Z_{\nu_{\widetilde{E}},X,p} = -\alpha(\nu \wedge \nu_{\widetilde{E}})$. We divide the proof in 2 different cases.

Case 1: $q \in E$ is a free point with $E \in \Gamma_\pi$ In that case $\nu_{\widetilde{E}} > \nu_E$ by Proposition 3.4.17. We also have $b(\widetilde{E}) = b(E)$ and $Z_{\nu_{\widetilde{E}},X,p} = Z_{\nu_E,X,p} - \frac{1}{b(\widetilde{E})}\widetilde{E}$ by Proposition 3.6.30. If $c_Y(\nu) \neq (q)$ (this includes the case where $c_Y(\nu)$ is a prime divisor at infinity. Then, $\nu \wedge \nu_{\widetilde{E}} = \nu \wedge \nu_E$. We have by Proposition 3.6.30 that $Z_{\nu_{\widetilde{E}},X,p} = \tau^*(Z_{\nu_E,X,p}) - \frac{1}{b(\widetilde{E})}\widetilde{E}$. Since $Z_{\nu,X,p} \cdot \widetilde{E} = 0$, we get

$$Z_{\nu,X,p} \cdot Z_{\nu_{\widetilde{E}},X,p} = Z_{\nu,X,p} \cdot Z_{\nu_E,X,p}. \tag{3.193}$$

This is equal to $-\alpha(\nu \wedge \nu_E)$ by induction and therefore it is equal to $-\alpha(\nu \wedge \nu_{\widetilde{E}})$.

If $c_Y(\nu) = q$, then $c_Z(\nu) \in \widetilde{E}$. We either have $\nu_{\widetilde{E}} \leqslant \nu$ or $\nu_E < \nu \wedge \nu_{\widetilde{E}} < \nu_{\widetilde{E}}$.

1. If $\nu \geqslant \nu_{\widetilde{E}}$, then $\nu \wedge \nu_{\widetilde{E}} = \nu_{\widetilde{E}}$ and $c_Z(\nu)$ is either $\widetilde{E}$ or a free point on $\widetilde{E}$. In both cases by Proposition 3.6.29, the incarnation of $Z_{\nu,X,p}$ in Z is $Z_{\nu_{\widetilde{E}},X,p}$. Therefore

$$Z_{\nu,X,p} \cdot Z_{\nu_{\widetilde{E},X,p}} = (Z_{\nu_{\widetilde{E}},X,p})^2 = (Z_{\nu_E,X,p})^2 - \frac{1}{b(\widetilde{E})^2}. \tag{3.194}$$

   By induction $(Z_{\nu_E,X,p})^2 = -\alpha(\nu_E)$ and $\alpha(\nu_{\widetilde{E}}) = \alpha(\nu_E) + \frac{1}{b(\widetilde{E})^2}$ by Proposition 3.4.27, so the claim is shown in that case.

2. If $\nu_E < \nu \wedge \nu_{\widetilde{E}} < \nu_{\widetilde{E}}$. Then, $\nu \wedge \nu_E$ is a monomial valuation centered at $E \cap \widetilde{E}$ (we still denote by E the strict transform of E in Z). Therefore, by Proposition 3.4.31 there exists $s,t > 0$ such that $sb(E) + tb(\widetilde{E}) = 1$ and $\nu \wedge \nu_{\widetilde{E}} = \nu_{s,t}$ is the monomial valuation with weight s,t with respect to local coordinates associated to E and $\widetilde{E}$ respectively. By Proposition 3.6.29, we have

$$(Z_{\nu,X,p})_Z = sZ_{\mathrm{ord}_E,X,p} + tZ_{\mathrm{ord}_{\widetilde{E}},X,p} = sb_E Z_{\nu_E,X,p} + tb_{\widetilde{E}} Z_{\nu_{\widetilde{E}},X,p}. \tag{3.195}$$

   Therefore,

$$Z_{\nu,X,p} \cdot Z_{\nu_{\widetilde{E}},X,p} = sb(E) Z_{\nu_E,X,p} \cdot Z_{\nu_{\widetilde{E}},X,p} + tb(\widetilde{E})(Z_{\nu_{\widetilde{E}},X,p})^2. \tag{3.196}$$

   By induction and the previous case this is equal to $-b(E)(s\alpha(\nu_E) + t\alpha(\nu_{\widetilde{E}}))$. By Proposition 3.4.27, we have $\alpha(\nu_{\widetilde{E}}) = \alpha(\nu_E) + \frac{1}{b(E)^2}$. Therefore, we get

$$-b(E)\left(s\alpha(\nu_E) + t\alpha(\nu_{\widetilde{E}})\right) = -\alpha(\nu_E) - \frac{t}{b(E)} \tag{3.197}$$

and this is equal to $-\alpha(\pi_* \nu_{s,t})$ by Proposition 3.4.31.

Case 2: $q = E_1 \cap E_2$ **is a satellite point** We can suppose without loss of generality that $\nu_{E_1} < \nu_{E_2}$. In that case we get $\nu_{E_1} < \nu_{\widetilde{E}} < \nu_{E_2}, b(\widetilde{E}) = b(E_1) + b(E_2)$ and

$$Z_{\nu_{\widetilde{E}},X,p} = \frac{b(E_1)}{b(E_1)+b(E_2)} Z_{\nu_{E_1},X,p} + \frac{b(E_2)}{b(E_1)+b(E_2)} Z_{\nu_{E_2},X,p} - \frac{1}{b(\widetilde{E})}\widetilde{E} \tag{3.198}$$

by Proposition 3.6.30.

If $c_Y(\nu) \neq q$, then $\nu \wedge \nu_{\widetilde{E}} \leqslant \nu_{E_1}$ or $\nu \wedge \nu_{\widetilde{E}} \geqslant \nu_{E_2}$ and we get

$$Z_{\nu,X,p} \cdot Z_{\nu_{\widetilde{E}},X,p} = \frac{b(E_1)}{b(E_1)+b(E_2)}(Z_{\nu,X,p} \cdot Z_{\nu_{E_1},X,p}) + \frac{b(E_2)}{b(E_1)+b(E_2)}(Z_{\nu,X,p} \cdot Z_{\nu_{E_2},X,p}). \tag{3.199}$$

By induction, this is equal to $-\frac{b(E_1)}{b(E_1)+b(E_2)}\alpha(\nu_{E_1} \wedge \nu) - \frac{b(E_2)}{b(E_1)+b(E_2)}\alpha(\nu_{E_2} \wedge \nu)$.

If $\nu \wedge \nu_{\widetilde{E}} \leqslant \nu_{E_1}$, then $\nu \wedge \nu_{E_2} = \nu \wedge \nu_{\widetilde{E}} \nu \wedge \nu_{E_1}$ and the quantity in Equation (3.199) is equal to $-\alpha(\nu \wedge \nu_{\widetilde{E}})$.

If $\nu \wedge \nu_{\widetilde{E}} \geqslant \nu_{E_2}$, then $\nu > \nu_{\widetilde{E}}$ and $\nu \wedge \nu_{\widetilde{E}} = \nu_{\widetilde{E}}$. In that case $\nu \wedge \nu_{E_1} = \nu_{E_1}$ and $\nu \wedge \nu_{E_2} \nu_{E_2}$. Therefore, the quantity in Equation (3.199) is equal to

$$-\frac{b(E_1)}{b(E_1)+b(E_2)}\alpha(\nu_{E_1}) - \frac{b(E_2)}{b(E_1)+b(E_2)}\alpha(\nu_{E_2}). \tag{3.200}$$

By Proposition 3.4.27, $\alpha(\nu_{E_2}) = \alpha(\nu_{E_1}) + \frac{1}{b(E_1)b(E_2)}$, so we get

$$Z_{\nu,X,p} \cdot Z_{\nu_{\widetilde{E}},X,p} = -\alpha(\nu_{E_1}) - \frac{1}{b(E_1)(b(E_1)+b(E_2))} = -\alpha(\nu_{E_1}) - \frac{1}{b(E_1)b(\widetilde{E})} \tag{3.201}$$

and this is equal to $-\alpha(\nu_{\widetilde{E}})$ again by Proposition 3.4.27.

If $c_Y(\nu) = q$, then $c_Z(\nu) \in \widetilde{E}$. We have that $\nu_{E_1} < \nu \wedge \nu_{\widetilde{E}} < \nu_{E_2}$. Therefore either $\nu = \nu_{\widetilde{E}}$ or $c_Z(\nu) \text{n} \widetilde{E}$ is a free point and $\nu \wedge \nu_{\widetilde{E}}$ is a monomial valuation centered at $E_1 \cap \widetilde{E}$ or $E_2 \cap \widetilde{E}$. We show again the claim by induction in an analogous way as in Case 1. We have thus shown the claim by induction.

To show the Proposition, let $\nu, \nu' \in \mathcal{V}_X(p; \mathfrak{m}_p)$. If $\nu \neq \nu'$, then there exists a completion $\pi : (Y, \mathrm{Exc}(pi)) \to (X, p)$ exceptional above p such that $c_Y(\nu) \neq c_Y(\nu')$. Then, we have that

$$Z_{\nu,X,p} \cdot Z_{\nu',X,p} = (Z_{\nu,X,p})_Y \cdot (Z_{\nu',X,p})_Y \tag{3.202}$$

If ν' is infinitely singular or a curve valuation, we can suppose that $c_Y(\nu')$ is a free point lying over a unique prime divisor E at infinity. Then, $\nu' > \nu_E$ and $\nu' \wedge \nu = \nu' \wedge \nu_E$. Furthermore, the incarnation of $Z_{\nu,X,p}$ in Y is exactly $Z_{\nu_E,X,p}$ by Proposition 3.6.29. Therefore,

$$Z_{\nu,X,p} \cdot Z_{\nu',X,p} = Z_{\nu,X,p} \cdot Z_{\nu_E,X,p}. \tag{3.203}$$

This is equal to $-\alpha(\nu \wedge \nu_E) = -\alpha(\nu \wedge \nu')$ by the Claim.

If ν' is irrational, then we can suppose that $c_Y(\nu') = E_1 \cap E_2$ for E_1, E_2 two prime divisors at infinity. Suppose without loss of generality that $\nu_{E_1} < \nu_{E_2}$. By Proposition 3.4.31, we have that $\nu' = \pi_* \nu_{s,t}$ for some $s, t > 0$ such that $sb(E_1) + tb(E_2) = 1$ and $\alpha(\nu') = \alpha(\nu_{E_1}) + \frac{t}{b(E_1)}$. Furthermore, by Proposition 3.6.29, the incarnation of $Z_{\nu',X,p}$ in Y is

$$(Z_{\nu',X,p})_Y = sb(E_1) Z_{\nu_{E_1},X,p} + tb(E_2) Z_{\nu_{E_2},X,p}. \tag{3.204}$$

And we have

$$Z_{\nu,X,p} \cdot Z_{\nu',X,p} = sb(E_1)(Z_{\nu,X,p} \cdot Z_{\nu_{E_1},X,p}) + tb(E_2)(Z_{\nu,X,p} \cdot Z_{\nu_{E_2},X,p}). \tag{3.205}$$

Either $\nu \wedge \nu' = \nu \wedge \nu_{E_1}$ or $\nu \wedge \nu' = \nu'$. If $\nu \wedge \nu' = \nu \wedge \nu_{E_1}$, then we also have $\nu \wedge \nu_{E_2} = \nu \wedge \nu_{E_1}$. The quantity in Equation (3.205) is then equal to

$$-sb(E_1)\alpha(\nu \wedge \nu_{E_1}) - tb(E_2)\alpha(\nu \wedge \nu_{E_2}) = \alpha(\nu \wedge \nu_{E_1}) = -\alpha(\nu \wedge \nu'). \tag{3.206}$$

If $\nu \wedge \nu' = \nu'$, then $\nu \wedge \nu_{E_1} = \nu_{E_1}$ and $\nu \wedge \nu_{E_2} = \nu_{E_2}$. The quantity in Equation (3.205) is then equal to

$$-sb(E_1)\alpha(\nu_{E_1}) - tb(E_2)\alpha(\nu_{E_2}) = -\alpha(\nu_{E_1}) - \frac{t}{b(E_1)} = -\alpha(\nu'). \tag{3.207}$$

To get the last two equalities we use Proposition 3.4.27 and 3.4.31.

Finally, if $\nu = \nu'$, we need to show that $(Z_{\nu,X,p})^2 = -\alpha(\nu)$. We know the result if ν is divisorial. We use approximating sequence to conclude in general. If ν is infinitely singular or a curve valuation. Let (X_n, p_n) be the sequence of infinitely near points associated to ν. The approximating sequence of ν (Proposition 3.4.26) is the subsequence $\nu_n = \frac{1}{b(E_n)} \operatorname{ord}_{E_n}$ where p_n is a free point lying over a unique prime divisor E_n at infinity. We have that $\alpha(\nu_n) \to \alpha(\nu)$ and the incarnation of $Z_{\nu,X,p}$ in X_n is $Z_{\nu_n,X,p}$. Therefore,

$$(Z_{\nu,X,p})^2 = \lim_n (Z_{\nu_n,X,p})^2 = -\lim_n \alpha(\nu_n) = -\alpha(\nu) \tag{3.208}$$

If ν is irrational, then let (X_n, p_n) be the sequence of infinitely near points associated to ν. For every n large enough, $p_n = E_n \cap F_n$ for E_n, F_n two prime divisors at infinity. Suppose that for all $n, \nu_{E_n} < \nu_{F_n}$. Then, we have $\nu_{E_n} < \nu < \nu_{F_n}$, $\alpha(\nu_{E_n}) \to \alpha(\nu), \alpha(\nu_{F_n}) \to \alpha(\nu)$ and $b(E_n) \to +\infty, b(F_n) \to +\infty$. We have by Proposition 3.6.29 that the incarnation of $Z_{\nu,X,p}$ in X_n is

$$s_n b(E_n) Z_{\nu_{E_n},X,p} + t_n b(F_n) Z_{\nu_{F_n},X,p} \tag{3.209}$$

for some $s_n, t_n > 0$ such that $s_n b(E_n) + t_n b(F_n) = 1$. We have

$$(Z_{\nu,X,p})^2 = \lim_n (s_n b(E_n) Z_{\nu_n,X,p} + t_n b(F_n) Z_{\nu_{F_n,X,p}})^2 \tag{3.210}$$

$$= \lim_n -s_n^2 b(E_n)^2 \alpha(\nu_{E_n}) - 2 s_n t_n b(E_n) b(F_n) \alpha(\nu_{E_n}) - t_n^2 b(F_n)^2 \alpha(\nu_{F_n}) \tag{3.211}$$

Therefore we get

$$\lim_n -\alpha(\nu_{E_n}) \leqslant (Z_{\nu,X,p})^2 \leqslant \lim_n -\alpha(\nu_{F_n}). \tag{3.212}$$

Hence $(Z_{\nu,X,p})^2 = -\alpha(\nu)$. □

Corollary 3.6.33. *If $\nu \in \mathcal{V}_X(p; \mathfrak{m}_p)$, then $Z_{\nu,X,p} \notin \operatorname{Weil}(X,p)_{\mathbf{Q}}$ if and only if ν is irrational.*

Proof. If ν is divisorial, let $E \in \mathcal{D}_{X,p}$ such that ν is equivalent to ord_E. Then,

$$Z_{\nu,X,p} = \frac{1}{b(E)} Z_{\operatorname{ord}_E,X,p} \in \operatorname{Weil}(X,p)_{\mathbf{Q}} \tag{3.213}$$

by Proposition 3.6.27. If ν is infinitely singular or a curve valuation, let μ be any divisorial valuation. We have that $\mu \wedge \nu$ must be a divisorial valuation, therefore by Theorem 3.6.31 we have

$$Z_\mu \cdot Z_\nu = -\alpha(\nu \wedge \mu) \in \mathbf{Q}. \tag{3.214}$$

Hence $Z_{\nu,X,p} \in \operatorname{Weil}(X,p)_{\mathbf{Q}}$.

If ν is irrational, then for all $\mu \geqslant \nu$ divisorial we have $\alpha(\mu \wedge \nu) = \alpha(\nu) \in \mathbf{R} \backslash \mathbf{Q}$. Therefore, $Z_{\nu,X,p} \notin \operatorname{Weil}(X,p)$. □

Proposition 3.6.34. *Let X be a completion, let $p \in X$ be a closed point at infinity. If (ν_n) is a sequence of $\mathcal{V}_X(p;\mathfrak{m}_p)$ such that $\alpha(\nu_n) < +\infty$ for all n and $\nu \in \mathcal{V}_X(p;\mathfrak{m}_p)$, then $\nu_n \to \nu$ for the strong topology if and only if $\mathrm{Z}_{\nu_n,X,p} \to \mathrm{Z}_{\nu,X,p}$ for the strong topology of $\mathrm{L}^2(\mathrm{X}_0)$.*

Proof. This all comes from Theorem 3.6.31 as

$$\left|(Z_{\nu,X,p} - Z_{\nu_n,X,p})^2\right| = |-\alpha(\nu) + 2\alpha(\nu \wedge \nu_n) - \alpha(\nu_n)| \tag{3.215}$$

$$= |\alpha(\nu) - \alpha(\nu \wedge \nu_n) + \alpha(\nu_n) - \alpha(\nu \wedge \nu_n)|. \tag{3.216}$$

□

3.7 From linear forms to valuations

Suppose now that we have an element L of $\mathrm{Hom}(\mathrm{Cartier}_\infty(\mathrm{X}_0), \mathbf{R})$ satisfying property (+), we want to construct a valuation $\nu_L : A \to \mathbf{R} \cup \{\infty\}$ centered at infinity such that $L_{\nu_L} = L$.

First we extend L to $\mathcal{S}_\infty(X_0)$ (see Definition 3.2.12) by setting

$$\text{If } D = \bigvee_i D_i \text{ with } D_i \in \mathrm{Cartier}_\infty(\mathrm{X}_0), \quad L(D) := \sup_i L(D_i). \tag{3.217}$$

Proposition 3.7.1. *This definition does not depend on the representation of D as a supremum $D = \bigvee_i D_i$ with $D_i \in \mathrm{Cartier}_\infty(\mathrm{X}_0)$.*

Proof. If $D = \bigvee_{i \in I} D_i = \bigvee_{j \in J} D'_j$. Let $j \in J$ be an index and X a completion such that D'_j is defined on X. Let $\varepsilon > 0$ and let H be an effective divisor such that $\mathrm{Supp}(H) = \partial_X X_0$. There exists an index $i \in I$ such that $D_i + \varepsilon H \geqslant D'_j$, since otherwise we would get $D + \varepsilon H \leqslant D'_j \leqslant D$. Therefore we have by property $(+)$ item (1)

$$L(D'_j) \leqslant L(D_i) + \varepsilon L(H) \leqslant \sup_k L(D_k) + \varepsilon L(H). \tag{3.218}$$

Letting ε go to 0, we get $\sup_j L(D'_j) \leqslant \sup_k L(D_k)$ and the result holds by symmetry. $\square$

Proposition 3.7.2. *We have the following properties: for $D, D' \in \mathcal{S}_\infty(X_0)$*

(1) $L(D + D') = L(D) + L(D')$.

(2) $L(D \wedge D') = \min(L(D), L(D'))$.

(3) If $D \geqslant 0$, then $L(D) \geqslant 0$.

Proof. For (1), write

$$\begin{aligned} L(D + D') &= \sup_{(i,j) \in I \times J} L(D_i + D'_j) \\ &= \sup_{i \in I} L(D_i) + \sup_{j \in J} L(D'_j) = L(D) + L(D') \end{aligned}$$

For (2), let $D = \bigvee_i D_i$ and $D' = \bigvee_j D'_j$ be two elements of $\mathcal{S}_\infty(X_0)$. Then,

$$D \wedge D' = \bigvee_{i,j} D_i \wedge D'_j \tag{3.219}$$

and

$$L(D \wedge D') = \sup_{i,j} \min(L(D_i), L(D'_j)) \tag{3.220}$$
$$= \min(\sup_i L(D_i), \sup_j L(D'_j)) \tag{3.221}$$
$$= \min(L(D), L(D')). \tag{3.222}$$

For (3), if $D = 0$, then $L(D) = 0$. Otherwise, $D > 0$ and there exists a Cartier divisor D_i defined in some completion X of X_0 such that $D_X \geqslant D_i \geqslant 0$ and therefore

$$L(D) \geqslant L(D_i) \geqslant 0. \tag{3.223}$$

□

Recall the notations of Section 3.2.2. Define

$$w(P) := (\mathrm{div}_{\infty,X}(P))_X. \tag{3.224}$$

Proposition 3.7.3. *For $P \in A$, $w(P)$ defines an element of* $\mathrm{Weil}_\infty(\mathrm{X}_0)$*, moreover if one identifies for any completion X the divisor* $\mathrm{div}_{\infty,X}(P) \in \mathrm{Div}_\infty(X)$ *with its image in* $\mathrm{Cartier}_\infty(\mathrm{X}_0)$*, then*

$$w(P) = \bigvee_X \mathrm{div}_{\infty,X}(P). \tag{3.225}$$

Thus, $w(P)$ defines an element of $\mathcal{S}_\infty(X_0)$.

Proof. To prove both assertions it suffices to show that if X is a completion of X_0 and Y is the blow up of some point at infinity, then $\pi_* \mathrm{div}_{\infty,Y}(P) = \mathrm{div}_{\infty,X}(P)$ and $\pi^* \mathrm{div}_{\infty,X}(P) \leqslant \mathrm{div}_{\infty,Y}(P)$. Let $\widetilde{E}$ be the exceptional divisor of π and let $E_1, \dots, E_r$ be the prime divisors in $\partial_X X_0$. Since P is regular over X_0, $\mathrm{div}_X(P)$ is of the form

$$\mathrm{div}_X(P) = D + \sum_{i=1}^r a_i E_i \tag{3.226}$$

where D is an effective divisor such that no irreducible component of its support is one of the E_i's; by definition $\mathrm{div}_{\infty,X}(P) = \sum_{i=1}^r a_i E_i$. Then, $\mathrm{div}_Y(P)$ is of the form

$$\mathrm{div}_Y(P) = \mathrm{div}_Y(P \circ \pi) = \pi^* \mathrm{div}_X(P) = \pi'(D) + b\widetilde{E} + \sum_{i=1}^{r} a_i \pi'(E_i) \tag{3.227}$$

for some $b \in \mathbf{Z}$. So $\mathrm{div}_{\infty,Y}(P) = b\widetilde{E} + \sum_{i=1}^{r} a_i \pi'(E_i)$ and we get $\pi_*(\mathrm{div}_{\infty,Y}(P)) = \mathrm{div}_{\infty,X}(P)$ as $\pi_*(\widetilde{E}) = 0$, This shows that $w(P)$ is an element of $\mathrm{Weil}_\infty(\mathrm{X}_0)$.

To show that $\pi^* \mathrm{div}_{\infty,X}(P) \leqslant \mathrm{div}_{\infty,Y}(P)$ we have to be more precise about the coefficient b. We can write $b = c + d$, where $\pi^* D = \pi'(D) + d\widetilde{E}$ and $\pi^* \mathrm{div}_{\infty,X}(P) = c\widetilde{E} + \sum_i a_i \pi'(E_i)$. Since, D is effective, we have $d \geqslant 0$ and the result follows. □

We define

$$\nu_L(P) := L(w(P)). \tag{3.228}$$

Remark 3.7.4. The class $w(P)$ is not in general a Cartier class. Indeed, take $X_0 = \mathbf{A}^2, X = \mathbf{P}^2$ with homogeneous coordinates $[x : y : z]$ such that $\{z = 0\}$ is the line at infinity. Consider $P = y/z \in \mathbf{k}(\mathbf{P}^2)$. Define a sequence of blow ups X_i by $X_0 = \mathbf{P}^2, E_0 = \{z = 0\}$ and $\pi_{i+1} : X_{i+1} \to X_i$ the blow up of the intersection point of the strict transform of $\{y = 0\}$ in X_i and E_i, where E_i is the exceptional divisor in X_i. Let C_y be the strict transform of $\{y = 0\}$ in any the X_i. We still denote by E_i its strict transform in every $X_j, j \geqslant i$. Then,

$$\begin{aligned}
\mathrm{div}_{\mathbf{P}^2}(P) &= C_y - E_0 \\
\mathrm{div}_{X_1}(P) &= C_y - E_0 \\
\mathrm{div}_{X_2}(P) &= C_y + E_2 - E_0 \\
\mathrm{div}_{X_3}(P) &= C_y + 2E_3 + E_2 - E_0
\end{aligned}$$

and by induction, we get for all $k \geqslant 2$

$$\mathrm{div}_{X_k}(P) = C_y + \sum_{j=2}^{k} (j-1)E_j - E_0. \tag{3.229}$$

Therefore, for all $k \geqslant 2$

$$\begin{aligned}\pi_{k+1}^* \operatorname{div}_{\infty,X_k}(P) &= (k-1)E_{k+1} + \sum_{j=2}^{k}(j-1)E_j - E_0 \\ &\neq kE_{k+1} + \sum_{j=2}^{k}(j-1)E_j - E_0 = \operatorname{div}_{\infty,X_{k+1}}(P).\end{aligned}$$

Thus, $w(P)$ is not a Cartier class.

Proposition 3.7.5. *The function ν_L is a valuation on A centered at infinity.*

Proof. We first show that ν_L is in fact a valuation

1. For any $\lambda \in \mathbf{k}^*$ and for any completion X of X_0, $\operatorname{div}_X(\lambda) = 0$ so $\nu_L(\lambda) = 0$.
2. If $f, g \in A$, then $\operatorname{div}_X(fg) = \operatorname{div}_X(f) + \operatorname{div}_X(g)$. So, $w(fg) = w(f) + w(g)$ and by Proposition 3.7.2 $\nu_L(fg) = \nu_L(f) + \nu_L(g)$.
3. Let $f, g \in A, f \neq -g$, then $\operatorname{div}_X(f+g) \geqslant \operatorname{div}_X(f) \wedge \operatorname{div}_X(g)$, therefore
$$w(f+g) \geqslant w(f) \wedge w(g) \tag{3.230}$$
and by Proposition 3.7.2 $\nu_L(f+g) \geqslant \min(\nu_L(f), \nu_L(g))$.

If $L \neq 0$, there exists a completion X and a prime divisor E at infinity such that $L(E) > 0$. By Theorem 3.1.4, there exists $H \in \operatorname{Div}_\infty(X)$ ample such that $H \geqslant 0, \operatorname{Supp} H = \partial_X X_0$. We have by item (1) of (+) that $L(H) \geqslant L(E) > 0$. To show that ν_L is centered at infinity, it suffices to show that $L_{\nu_L}(H) > 0$. Up to replacing H by one of its multiples (which does not change the hypothesis $L(H) > 0$), we can suppose that H is very ample and that it induces an embedding $\tau : X \hookrightarrow \mathbf{P}^N$ such that $\tau(H)$ is the intersection of $\tau(X)$ with the hyperplane $\{x_0 = 0\}$. By Bertini's theorem, we can find a hyperplane $M = \{\sum_i \lambda_i x_i = 0\} \neq \{x_0 = 0\}$ such that $M \cap \tau(X)$ is a smooth irreducible subvariety C in X satisfying

1. The intersection of C with any divisor at infinity of X is transverse.
2. If ν_L is not divisorial, the center of ν_L is not contained in C.

Indeed, by Bertini theorem, the set U_X of hyperplanes H such that $H \cap X$ is a smooth irreducible curve is an open dense subset. Let $E_1, \cdots, E_n$ be the primes at infinity in X. Applying Bertini theorem to E_i yields an open subset U_i of hyperplanes that meet E_i transversally. Finally, if the center of ν_L is a subvariety Y of codimension $\geqslant 2$, then the set of hyperplanes that contain Y is a closed nowhere dense subset of $\mathbf{P}(\Gamma(\mathbf{P}^n, \mathcal{O}(1)))$ because $|H|$ is base point free, so its complementary is a non-empty open subset U_Y. Now, $U_1 \cap \cdots \cap U_n \cap U_Y$ is an open subset that intersects U_X since it is dense, we then choose M in the intersection. Define

$$P = \sum_{i=0}^{N} \lambda_i \frac{x_i}{x_0} \tag{3.231}$$

Then, P is a regular function over X_0 such that $\operatorname{div}_X(P) = C - H$ and $1/P$ is a local equation of H at the center of ν_L (even if ν_L is divisorial). Hence,

$$L_{\nu_L}(H) = \nu_L(1/P) = \sup_Y (L(\operatorname{div}_{\infty,Y}(1/P)) \geqslant L(H) > 0. \tag{3.232}$$

□

In Section 3.6, we have constructed a map

$$L : \mathcal{V}_\infty \to \operatorname{Hom}(\operatorname{Cartier}_\infty(\mathrm{X}_0), \mathbf{R})_{(+)}; \tag{3.233}$$

here, we have constructed a map

$$\nu : \operatorname{Hom}(\operatorname{Cartier}_\infty(\mathrm{X}_0), \mathbf{R})_{(+)} \to \mathcal{V}_\infty \tag{3.234}$$

where $\operatorname{Hom}(\operatorname{Cartier}_\infty(\mathrm{X}_0), \mathbf{R})_{(+)}$ are the linear forms over $\operatorname{Cartier}_\infty(\mathrm{X}_0)$ that satisfy property (+). We shall prove that they are mutual inverse in Section 3.8 (this result is not needed in this memoir).

3.8 Proof that ν and L are mutual inverses

Set $\mathcal{M} := \mathrm{Hom}(\mathrm{Cartier}_\infty(\mathrm{X}_0), \mathbf{R})_{(+)}$. In Section 3.6, we have defined $L : \nu \in \mathcal{V}_\infty \mapsto L_\nu \in \mathcal{M}$ and $\nu : L \in \mathcal{M} \mapsto \nu_L \in \mathcal{V}_\infty$. The goal is to show that these two maps are inverse of each other.

Proposition 3.8.1. *For all valuation $\nu \in \mathcal{V}_\infty$ and for all $P \in \mathcal{O}_X(X_0), \nu(P) = L_\nu(w(P))$.*

Proof. Let X be a completion of X_0. We have seen that $\mathrm{div}_{\infty,X}(P) = \mathrm{div}_X(P) - D$ where D is an effective divisor not supported in $\partial_X X_0$. Therefore,

$$L_{\nu,X}(\mathrm{div}_{\infty,X}(P)) = \nu(P) - L_{\nu,X}(D) \leqslant \nu(P) \tag{3.235}$$

Taking the supremum over X, we get $L_\nu(w(P)) \leqslant \nu(P)$.

To show the other inequality, take a valuation ν centered at infinity and let X be a completion of X_0. Up to further blow ups of point at infinity, we can suppose that $D := \mathrm{div}_X(P)$ is a divisor in X with simple normal crossing on $\partial_X X_0$. Let $E_1, \cdots, E_r$ be the prime divisors at infinity of X. Then, D is of the form

$$D = \sum_{i=1}^{r} a_i E_i + \sum_{j \in J} b_j F_j \tag{3.236}$$

for some prime divisors F_j not supported at infinity. Let p be the center of ν on X, there are two cases.

1. For all $j \in J, p \notin F_j$, in that case for all $j \in J, L_{\nu,X}(F_j) = 0$ and $\nu(P) = L_{\nu,X}(\mathrm{div}_{\infty,X}(P))$. Therefore, $\nu(P) \leqslant L_\nu(w(P))$ and they are equal.

2. There exist a unique $j \in J$ and a unique i such that $p = E_i \cap F_j$. The uniqueness comes from the fact that D is a divisor with simple normal crossing. We denote them respectively by E and F. Then, we construct a sequence of blow up of points $\pi_i : \overline{X_{i+1}} \to \overline{X_i}$ such that π_i is the blow-up of the center of ν in X_i and $X_0 = X$. We still denote by F the strict transform of F in any of these blow-ups. There are two possibilities:

    (a) Either there exists a number k such that the center of ν in X_k does not belong to F (This includes the case where ν is divisorial, in that case the center becomes a prime divisor and there are no more blow-ups to be done). In that case, we are back in case 1 and $\nu(P) = \nu_{X_k}(\mathrm{div}_{\infty,X_k}(P)) \leqslant L_\nu(w(P))$ and we get the desired equality.

    (b) Or for all $k \geqslant 0$, the center of ν in X_k belongs to F, in that case ν is the curve valuation associated to F at p and $\nu(P) = +\infty$. We show that $\nu_{X_k}(\mathrm{div}_{\infty,X_k}(P)) \to +\infty$ using the following result.

Lemma 3.8.2. *In case 2.(b), set $E_0 = E$ and for $k \geqslant 1$, $\widetilde{E}_k$ the exceptional divisor in X_k above $c_{X_{k-1}}(\nu)$, then $L_{\nu,X_0}(E) = L_{\nu,X_k}(E_k)$ for all k and the divisor $\operatorname{div}_{X_k}(P)$ is of the form*

$$\operatorname{div}_{X_k}(P) = (a+kb)\widetilde{E}_k + bF + D'_k \tag{3.237}$$

where $a = \operatorname{ord}_E(P) > 0$, $b = \operatorname{ord}_F(P) > 0$ and $c_{X_{k+1}}(\nu)$ does not belong to the support of D'_k.

Proof. First, since we are in case 2b and we have supposed that $\operatorname{Supp}\operatorname{div}_X(P)$ is with simple normal crossings, we have that for all $k \geqslant 0$ the center of ν in X_k is the intersection point $p_k := \widetilde{E}_k \cap F$.

We proceed by induction on k. If $k = 0$ then the result is true as $X_0 = X$ and $c_X(\nu) = E \cap F$. Suppose the result true for a given index $k \geqslant 0$, then when we blow up p_k, p_{k+1} is the intersection point of $\widetilde{E}_{k+1}$ and F so it does not belong to $\pi'_k(\widetilde{E}_k)$ therefore $L_{\nu,X_{k+1}}(\pi'_k(\widetilde{E}_k)) = 0$. By induction we have $\nu_{X_k}(\widetilde{E}_k) = L_{\nu,X_0}(E)$, and we know that

$$L_{\nu,X_k}(\widetilde{E}_k) = L_{\nu,X_{k+1}}(\pi_k^*\widetilde{E}_k) = L_{\nu,X_{k+1}}(\pi'_k(\widetilde{E}_k) + \widetilde{E}_{k+1}) = L_{\nu,X_{k+1}}(\widetilde{E}_{k+1}) \tag{3.238}$$

so this shows the first assertion. Now, by induction $\operatorname{div}_{X_k}(P)$ is of the form

$$\operatorname{div}_{X_k}(P) = (a+kb)\widetilde{E}_k + bF + D'_k \tag{3.239}$$

Now, since $p_k = \widetilde{E}_k \cap F$ and $p_k \notin \operatorname{Supp} D'_k$, one has

$$\operatorname{div}_{X_{k+1}(P)} = \pi_k^* \operatorname{div}_{X_k}(P) = (a+(k+1)b)\widetilde{E}_{k+1} + bF + (a+kb)\pi'_k(\widetilde{E}_k) + \pi'_k(D'_k). \tag{3.240}$$

Since $p_{k+1} \notin \pi'_k(\widetilde{E}_k)$, the support of the divisor $D'_{k+1} := \pi'_k(D'_k) + (a+kb)\pi'_k(\widetilde{E}_k)$ does not contain p_{k+1} and we are done. □

Using this lemma we see that

$$L_{\nu,X_k}(\operatorname{div}_{\infty,X_k}(P)) = (a+kb)L_{\nu,X_0}(E) \xrightarrow[k\to\infty]{} +\infty \tag{3.241}$$

Therefore $L_\nu(w(P)) = +\infty$ and since $\nu(P) \geqslant L_\nu(w(P))$ we have that $\nu(P) = +\infty$

□

To show that $L \circ \nu = \mathrm{id}_{\mathcal{M}}$ we need some technical lemmas.

Proposition 3.8.3. *Let $L \in \mathcal{M}$ and X be a completion of X_0. If there exists two divisors E, E' at infinity in X such that $L(E), L(E') > 0$, then E and E' must intersect.*

Proof. Suppose that E and E' do not intersect, then the sheaf of ideals $\mathfrak{a} = \mathcal{O}_X(-E) \oplus \mathcal{O}_X(-E')$ is trivial, $\mathfrak{a} = \mathcal{O}_X$. From Proposition 3.6.25, we get $E \wedge E' = 0$. Thus $L(E \wedge E') = 0$. But $L(E \wedge E') = \min(L(E), L(E')) > 0$ and this is a contradiction. □

Corollary 3.8.4. *Let X be a completion of X_0, suppose there exists two prime divisors at infinity E, F such that $L(E), L(F) > 0$. Then, let $\widetilde{E}$ be the exceptional divisor above $p = E \cap F$, one has $L(\widetilde{E}) > 0$.*

Proof. Let $\pi : Y \to X$ be the blow up of p and suppose that $L(\widetilde{E}) = 0$. Since $\pi^* E = \pi'(E) + \widetilde{E}$ and $\pi^* F = \pi'(F) + \widetilde{E}$, one has $L(\pi'(E)) > 0$ and $L(\pi'(F)) > 0$ but the two divisors no longer meet and this is a contradiction. □

Proposition 3.8.5. *Let X be a completion of X_0, there are two possibilities*

(1) There exist a unique closed point p in X at infinity such that if $\widetilde{E}$ is the exceptional divisor above p, one has $L(\widetilde{E}) > 0$. We call this point the center *of L in X.*

(2) If no point satisfy this property, then there exists a unique divisor at infinity E in X such that $L(E) > 0$. In that case we call E the center *of L in X.*

and we have the following properties

(a) Let E be a prime divisor at infinity in X. If the center of L on X is a point p, then $p \in E \Leftrightarrow L(E) > 0$.

(b) If Y is a completion of X_0 above X, then the center of L in Y belongs to the inverse image of the center of X.

Proof. Suppose there are two points p_1, p_2 satisfying this property on X. Let π_i be the blow up of p_i in X, we have commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & \\ X_1 \arrow[dr, "\pi_1"'] & & X_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

where on the left side we first blow up p_1 then we blow up the strict transform of p_2 and the other way around on the right. Now let $\widetilde{E}_1, \widetilde{E}_2$ be the exceptional divisors above p_1 and p_2 respectively in X_1 and in X_2 and suppose that $L(\widetilde{E}_1), L(\widetilde{E}_2) > 0$. Then, since p_1 does not belong to $\widetilde{E}_2$ and p_2 does not belong to $\widetilde{E}_1$, we have that $L(\widetilde{E}_1) = L(\tau_1^* \widetilde{E}_1) = L(\tau_1'(\widetilde{E}_1)) > 0$ and $L(\tau_2'(\widetilde{E}_2)) > 0$. But in Y the prime divisors $\tau_1'(\widetilde{E}_1)$ and $\tau_2'(\widetilde{E}_2)$ do not intersect and that contradicts Proposition 3.8.3.

Now, if E, F are two divisors at infinity such that $L(E), L(F) > 0$, Lemma 3.8.4 shows that $E \cap F$ must be the center of L on X. Hence if no point of X is the center of L there is only one prime divisor at infinity E such that $L(E) > 0$.

To show assertion (a), suppose that the center of L on X is a point p and let π be the blow up of p. If $p \in E$, then $\pi^*(E) = \pi'(E) + \widetilde{E}$ and $L(E) = L(\pi^* E) \geqslant L(\widetilde{E}) > 0$. If $L(E) > 0$ then p must belong to E otherwise $\widetilde{E}$ and E would not intersect and this contradicts Proposition 3.8.3.

We now assertion (b), we only need to show it for the blow up of a point $\pi : Y \to X$. Suppose first that the center of L on X is a (closed) point p. If we blow up another point than p, then it is clear that the center of L on Y is the point $\pi^{-1} p$ as the order of the blow ups does not matter in that case.

Suppose now that we blow up p, then the exceptional divisor $\widetilde{E}$ verifies $L(\widetilde{E}) > 0$, if the center of L on Y is a prime divisor then it must be $\widetilde{E}$. If it is a point then it must belong to $\widetilde{E}$ by assertion (a).

If the center of L on X is a prime divisor E, then for any blow up $\pi : Y \to X$ of a point of X, we show that the center of L on Y is $\pi'(E)$. The exceptional divisor $\widetilde{E}$ verifies $L(\widetilde{E}) = 0$ and $\pi'(E)$ is the only prime divisor of Y such that $L(\pi'(E)) > 0$. Thus, if the center of L on Y is not a point, it must be $\pi'(E)$. If the center of L on Y is a point q, then it must belong to $\pi'(E)$ by assertion (a). If q is not the intersection point $\pi'(E) \cap \widetilde{E}$, then it is the strict transform of a point $p \in E$ and in that case p was the center of L in X this is a contradiction. If $q = \widetilde{E} \cap \pi'(E)$, then $L(\widetilde{E}) > 0$ by assertion (a) and this is also a contradiction. Therefore, the center of L on Y cannot be a point, it is $\pi'(E)$. □

We say that L is *divisorial* if there exists a completion X of X_0 such that the center of L on X is a prime divisor at infinity.

Proposition 3.8.6. *The map ν sends divisorial valuations to divisorial elements of $\mathcal{M}$ and the map L sends divisorial functions to divisorial valuations.*

Proof. The fact that divisorial valuations induce divisorial functions on Cartier divisors is clear. Suppose that L is a divisorial function and let X be a completion such that the center of L in

X is a prime divisor E at infinity. Then, for all completion $\pi : Y \to X$ above X, the center of L on Y is the strict transform of E by Proposition 3.8.5 and $L(E) = L(\pi'(E))$. Therefore, let ν be the divisorial valuation on A such that $\nu_X = \mathrm{ord}_E$ and let $P \in \mathcal{O}_{X_0}(X_0)$, then for all completion Y above X, we have by Proposition 3.8.5

$$L(\mathrm{div}_{\infty,Y}(P)) = L(\pi'(E))\,\mathrm{ord}_E(\mathrm{div}_Y(P)) = L(E)\nu(P). \tag{3.242}$$

Therefore $\nu_L(P) = L(E)\nu(P)$ and it is a divisorial valuation. □

Proposition 3.8.7. *One has* $L \circ \nu = \mathrm{id}_{\mathcal{M}}$.

Proof. We can assume that L and ν_L are not divisorial. Let X be a completion of X_0, we will show first that if $H \in \mathrm{Div}_\infty(X)$ is an effective divisor such that $|H|$ is base point free and $\mathrm{Supp} H = \partial_X X_0$, then $\nu_L(H) = L(H)$. Pick f generic in $H^0(X, \mathcal{O}_X(H))$. We have that $\mathrm{div}\, f = Z_f - H$ with Z_f effective, $\mathrm{Supp} Z_f$ does not contain any divisor at infinity and the center of ν_L and the center of L do not belong to $\mathrm{Supp} Z_f$. Thus, f defines a regular function over X_0, $1/f$ is a local equation of H at the center of ν_L and we have

$$\nu_L(f) = \sup_Y L(\mathrm{div}_{\infty,Y}(f)) \tag{3.243}$$

Now, by our assumptions on f we have

Lemma 3.8.8. *For all Y above X, $\mathrm{div}_Y(f)$ is of the form $Z_{f,Y} + \mathrm{div}_{\infty,Y}(f)$ where $Z_{f,Y}$ is effective, supported on X_0 and $\mathrm{Supp} Z_{f,Y}$ does not contain the center of L. Furthermore, we have $L(\mathrm{div}_{\infty,Y}(f)) = L(\mathrm{div}_{\infty,X}(f))$.*

Proof. This is true for $Y = X$. We proceed by induction. Let Y be a completion above Y where the lemma is true and let $\pi : Y_1 \to Y$ be a blow up of Y at a point p. If p is not the center of L then the lemma is clearly true over Y_1, if p is the center of L over Y then since p does not belong to $\mathrm{Supp} Z_{f,Y}$ we have

$$\mathrm{div}_{f,Y_1} = \pi'(Z_{f,Y}) + \pi^*(\mathrm{div}_{\infty,Y}(f)) \tag{3.244}$$

and the lemma is true since $Z_{f,Y_1} = \pi'(Z_{f,Y})$ and $\mathrm{div}_{\infty,Y_1}(f) = \pi^*(\mathrm{div}_{\infty,Y}(f))$. □

Using this lemma we conclude that $\nu_L(f) = L(\mathrm{div}_{\infty,X}(f)) = -L(H)$. Therefore,

$$\nu_L(H) = \nu_L(1/f) = L(H). \tag{3.245}$$

Now take any divisor $D \in \mathrm{Div}_{\infty}(X)$. There exists an integer $n \geqslant 1$ such that $D + nH$ is effective and $|D + nH|$ is base-point free. Therefore,

$$\nu_L(D) = \nu_L(D + nH) - \nu_L(nH) = L(D + nH) - L(nH) = L(D). \tag{3.246}$$

□

CHAPTER 4

EIGENVALUATIONS AND DYNAMICS AT INFINITY

4.1 Dynamics when $A^\times = \mathbf{k}^\times$ and $\mathrm{Pic}^0(X_0) = 0$

4.1.1 The structure of the Picard-Manin space of X_0

From Section 3.2.6 we have linear maps

$$\tau : \mathrm{Cartier}_\infty(\mathrm{X}_0)_\mathbf{R} \to \mathrm{Cartier\text{-}NS}(\mathrm{X}_0)_\mathbf{R}, \quad \tau : \mathrm{Weil}_\infty(\mathrm{X}_0)_\mathbf{R} \to \mathrm{Weil\text{-}NS}(\mathrm{X}_0)_\mathbf{R}. \tag{4.1}$$

For this section we suppose that $X_0 = \mathrm{Spec} A$ is a normal affine surface over an algebraically closed field $\mathbf{k}$ such that

1. $A^\times = \mathbf{k}^\times$;
2. For all completion X of $X_0, \mathrm{Pic}^0(X) = 0$.

It suffices to test the second condition on one completion of X_0 as the Albanese variety of a projective variety is a birational invariant. We will make an abuse of notations and write $\mathrm{Pic}^0(X_0) = 0$ for the second hypothesis.

If these two conditions are satisfied, the finite dimensional subspace $\mathrm{Div}_\infty(X)$ embeds into $\mathrm{NS}(X)$. Indeed, consider the composition

$$\mathrm{Div}_\infty(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X), \tag{4.2}$$

the first map is injective since $A^\times = \mathbf{k}^\times$ and the second is an isomorphism because $\mathrm{Pic}^0(X) = 0$. Therefore the maps τ are injective and we have the orthogonal decomposition

$$\mathrm{Weil\text{-}NS}(\mathrm{X}_0)_\mathbf{R} = \mathrm{Weil}_\infty(\mathrm{X}_0)_\mathbf{R} \ominus V \tag{4.3}$$

where V is a finite-dimensional vector space(this decomposition also holds over $\mathbf{Q}$); in fact let X be a completion of X_0, then V is the orthogonal of $\mathrm{Div}_\infty(X)$ in $\mathrm{NS}(X)$.

4.1.1.1 The intersection form at infinity

Proposition 4.1.1. *Let X be a completion of X_0, then*

- $\mathrm{Div}_\infty(X)$ *embeds into* $\mathrm{NS}(X)$ *and the intersection form is non degenerate on* $\mathrm{Div}_\infty(X)$.

- *The perfect pairing* $\text{Cartier-NS}(\mathrm{X}_0) \times \text{Weil-NS}(\mathrm{X}_0) \to \mathbf{R}$ *induces a pairing*

$$\mathrm{Cartier}_\infty(\mathrm{X}_0) \times \mathrm{Weil}_\infty(\mathrm{X}_0) \to \mathbf{R} \tag{4.4}$$

  that is also perfect.

- $\mathrm{Weil}_\infty(\mathrm{X}_0)$ *is isomorphic, as a linear topological vector space, to* $\mathrm{Cartier}_\infty(\mathrm{X}_0)^*$ *endowed with the weak-$*$ topology.*

Proof. Everything follows from Propositions 3.2.20 and 3.2.17 and that $\tau : \mathrm{Div}_\infty(X) \hookrightarrow \mathrm{NS}(X)$ is injective. □

Corollary 4.1.2. *The subspace* $\mathrm{Hom}(\mathrm{Cartier}_\infty(\mathrm{X}_0), \mathbf{R})_{(+)}$ *is a closed subspace of* $\mathrm{Weil}_\infty(\mathrm{X}_0)$ *with the weak-$\star$ topology.*

Proof. All the conditions that elements of $\mathrm{Hom}(\mathrm{Cartier}_\infty(\mathrm{X}_0), \mathbf{R})_{(+)}$ have to satisfy are closed conditions. Indeed, we have

$$\mathrm{Hom}(\mathrm{Cartier}_\infty(\mathrm{X}_0), \mathbf{R})_{(+)} = C_1 \cap C_2 \tag{4.5}$$

where

$$C_1 = \bigcap_{D \geqslant 0} \{L(D) \geqslant 0\} \tag{4.6}$$

$$C_2 = \bigcap_{D, D' \in \mathrm{Cartier}_\infty(\mathrm{X}_0)} \left\{ L(D \wedge D') = \min(L(D), L(D')) \right\}. \tag{4.7}$$

□

4.1.1.2 A continuous embedding of $\mathcal{V}_\infty$ into $\mathrm{Weil}_\infty(\mathrm{X}_0)$

From Proposition 4.1.1, we get the immediate corollary.

Corollary 4.1.3. *For any valuation ν centered at infinity, there exists a unique $Z_\nu \in \mathrm{Weil}_\infty(\mathrm{X}_0)$ such that for all $D \in \mathrm{Cartier}_\infty(\mathrm{X}_0), L_\nu(D) = Z_\nu \cdot D$.*

Corollary 4.1.4. *A valuation ν is divisorial if and only if Z_ν belongs to $\mathrm{Cartier}_\infty(\mathrm{X}_0)$. In particular, for any prime divisor E at infinity, $Z_{\mathrm{ord}_E} \in \mathrm{Cartier}_\infty(\mathrm{X}_0)_{\mathbf{Q}}$. The embedding*

$$\nu \in \mathcal{V}_\infty \mapsto Z_\nu \in \mathrm{Weil}_\infty(\mathrm{X}_0) \tag{4.8}$$

is a continuous map for the weak topology.

Proof. If ν is divisorial, then there exists a completion X such that the center of ν is a prime divisor E at infinity. For every $W \in \mathrm{Weil}_\infty(\mathrm{X}_0), L_{\mathrm{ord}_E}(W) = L_{\mathrm{ord}_E,X}(W_X)$, by Proposition 3.6.22. By non-degeneracy of the intersection pairing on $\mathrm{Div}_\infty(X)_{\mathbf{Q}}$, there exists $Z \in \mathrm{Div}_\infty(X)_{\mathbf{Q}}$ such that for all $D \in \mathrm{Div}_\infty(X)_{\mathbf{Q}}, L_{\mathrm{ord}_E,X}(D) = Z \cdot D$. It follows that Z_{ord_E} is the Cartier class defined by Z, hence it is an element of $\mathrm{Cartier}_\infty(\mathrm{X}_0)_{\mathbf{Q}}$.

Conversely, if $Z_\nu \in \mathrm{Cartier}_\infty(\mathrm{X}_0)$, let X be a completion where Z_ν is defined. The center of ν over X cannot be a closed point p; otherwise let $\widetilde{E}$ be the exceptional divisor above p, we would have $L_\nu(\widetilde{E}) > 0$, but $Z_\nu \cdot \widetilde{E} = 0$.

Now to show the continuity of the map of the Corollary, it suffices by Proposition 4.1.1 to show that for any $D \in \mathrm{Cartier}_\infty(\mathrm{X}_0)$, the map $\nu \in \mathcal{V}_\infty \mapsto Z_\nu \cdot D$ is continuous, but this follows immediately from $Z_\nu \cdot D = L_\nu(D)$ and Proposition 3.6.19. □

Proposition 4.1.5. *Let ν be a valuation centered at infinity and X a completion of X_0 such that $c_X(\nu) \in E$ is a free point. Then, the incarnation of Z_ν in X is*

$$Z_{\nu,X} = (Z_\nu \cdot E) Z_{\mathrm{ord}_E}. \tag{4.9}$$

If $c_X(\nu) = E \cap F$ is a satellite point, then

$$Z_{\nu,X} = (Z_\nu \cdot E) Z_{\mathrm{ord}_E} + (Z_\nu \cdot F) Z_{\mathrm{ord}_F}. \tag{4.10}$$

Furthermore, if $\pi : Y \to X$ is the blow up of a point at infinity $p \neq c_X(\nu)$, then

$$Z_{\nu,Y} = \pi^* Z_{\nu,X}. \tag{4.11}$$

Proof. If $c_X(\nu) \in E$ is a free point. For any $D \in \mathrm{Div}_\infty(X)$, one has $D = \sum_F L_{\mathrm{ord}_F}(D)F$, therefore by Proposition 3.6.6 (2) and (3) $L_\nu(D) = L_{\mathrm{ord}_E}(D)L_\nu(E)$. Since $(Z_\nu \cdot E) = L_\nu(E)$, we get the result. The proof is similar for the case $c_X(\nu) = E \cap F$.

For the last assertion, if $\widetilde{E}$ is the exceptional divisor of $\pi : Y \to X$, then by definition

$$Z_{\nu,Y} = \pi^* Z_{\nu,X} - (Z_\nu \cdot \widetilde{E})\widetilde{E} \tag{4.12}$$

However, since $c_X(\nu) \neq p$, we have that $c_Y(\nu) \notin \widetilde{E}$ and therefore $Z_\nu \cdot \widetilde{E} = 0$ by Proposition 3.6.6. □

Recall that in §3.6.4, we have defined for a point p at infinity in a completion X the local divisor $Z_{\nu,X,p}$ for every valuation ν centered at p. The divisor is defined by duality via the following property

$$\forall D \in \mathrm{Cartier}(X, p), \quad L_\nu(D) = Z_{\nu,p,X} \cdot D. \tag{4.13}$$

Corollary 4.1.6. *Let X be a completion of X_0 and let ν be a valuation centered at infinity.*

- *If $p := c_X(\nu) \in E$, then*
$$Z_\nu = (Z_\nu \cdot E)Z_{\mathrm{ord}_E} + Z_{\nu,X,p} \tag{4.14}$$

- *If $p := c_X(\nu) = E \cap F$ is a satellite point, then*

$$Z_\nu = (Z_\nu \cdot E)Z_{\mathrm{ord}_E} + (Z_\nu \cdot Z_{\mathrm{ord}_F})Z_{\mathrm{ord}_F} + Z_{\nu,X,p} \tag{4.15}$$

In particular, $Z_\nu \in \mathrm{L}^2(\mathrm{X}_0)$ if and only if ν is quasimonomial or there exists a completion X and a closed point $p \in X$ at infinity such that $c_X(\nu) = p$ and $\alpha(\tilde{\nu}) < +\infty$ where $\tilde{\nu}$ is the valuation equivalent to ν such that $\tilde{\nu} \in \mathcal{V}_X(p; \mathfrak{m}_p)$.

Proof. We have that

$$Z_\nu = Z_{\nu,X} + Z' \tag{4.16}$$

where $Z' \in \mathrm{Weil}_\infty(\mathrm{X}_0)$ is exceptional above X. Now, for every divisor D exceptional above X, we have

$$L_\nu(D) = Z_\nu \cdot D = Z' \cdot D. \tag{4.17}$$

If D is exceptional above a point $q \neq p$, then $L_\nu(D) = 0$ by Proposition 3.6.6 as $q \neq c_X(\nu)$. Therefore, we get that $Z' = Z_{\nu,X,p}$.

Now, we have $Z_\nu \in \mathrm{L}^2(\mathrm{X}_0) \Leftrightarrow (Z_\nu)^2 < -\infty$. Replace ν by the equivalent valuation such that $\nu \in \mathcal{V}_X(p;\mathfrak{m}_p)$, then by Theorem 3.6.31 $(Z_{\nu,X,p})^2 = -\alpha(\nu)$ and therefore

$$(Z_\nu)^2 = (Z_{\nu,X})^2 - \alpha(\nu). \tag{4.18}$$

This shows the result. □

Corollary 4.1.7. *Let $\nu \in \mathcal{V}_\infty$, then up to normalisation $Z_\nu \in \mathrm{Weil}_\infty(\mathrm{X}_0)_{\mathbf{Q}}$ if and only ν is not irrational.*

Proof. First, if ν is divisorial, the result follows from Corollary 4.1.4. Then, if ν is infinitely singular or a curve valuation. Then, there exists a completion X such that $c_X(\nu)$ is a free point $p \in E$. Then, replace ν by its equivalent valuation such that $\nu \in \mathcal{V}_X(p;\mathfrak{m}_p)$. Let (z,w) be local coordinates at p such that $z = 0$ is a local equation of E. Then, $Z_\nu(E) = \nu(z) = \alpha(\nu \wedge \nu_z) \in \mathbf{Q}$ because $\nu \wedge \nu_z$ has to be a divisorial valuation. Therefore, by Corollary 3.6.33 and Proposition 4.1.5, we get that $Z_\nu \in \mathrm{Weil}_\infty(\mathrm{X}_0)_{\mathbf{Q}}$.

Finally, if ν is irrational then let X be a completion such that $c_X(\nu) = E \cap F$ is a satellite point. Then, $Z_{\nu,X} = sZ_{\mathrm{ord}_E} + tZ_{\mathrm{ord}_F}$ with $s/t \notin \mathbf{Q}$ by Proposition 4.1.5. It is clear that no multiple of $Z_{\nu,X}$ can be in $\mathrm{Div}_\infty(X)_{\mathbf{Q}}$. □

Corollary 4.1.8. *Let $\mathcal{V}''_\infty$ be the subspace of $\mathcal{V}_\infty$ consisting of $\nu \in \mathcal{V}_\infty$ such that $Z_\nu \in \mathrm{L}^2(\mathrm{X}_0)$, then*

$$\mathcal{V}''_\infty \hookrightarrow \mathrm{L}^2(\mathrm{X}_0) \tag{4.19}$$

is a continuous embedding for the strong topology. Furthermore, it is a homeomorphism onto its image.

Proof. Let X be a completion of X_0. Let ν_n be a sequence of $\mathcal{V}''_\infty$ converging towards $\nu \in \mathcal{V}''_\infty$ for the strong topology. We treat two cases, whether ν is associated to a prime divisor of X or ν is centered at a closed point $p \in X$ at infinity.

If ν is centered at a closed point p at infinity, then since ν_n converges strongly towards ν then it converges also weakly, therefore for n big enough, ν_n is centered at p by Proposition 3.5.1. We can replace each ν_n and ν by their representative such that $\nu_n, \nu \in \mathcal{V}_X(p;\mathfrak{m}_p)$. Then

- If $p \in E$ is a free point,

$$Z_{\nu_n} = (Z_{\nu_n} \cdot E) Z_{\mathrm{ord}_E} + Z_{\nu_n,X,p} \tag{4.20}$$

- If $p = E \cap F$ is a satellite point, then

$$Z_{\nu_n} = (Z_{\nu_n} \cdot E) Z_{\mathrm{ord}_E} + (Z_{\nu_n} \cdot F) Z_{\mathrm{ord}_F} + Z_{\nu_n, X, p} \tag{4.21}$$

and we have similar formulas for Z_ν. Now the incarnation of Z_{ν_n} in X converges towards the incarnation of Z_ν in X in both the free and the satellite case by weak convergence. Let $||\cdot||$ be any norm over $\mathrm{NS}(X)_{\mathbf{R}}$, then

$$||Z_\nu - Z_{\nu_n}||^2_{\mathrm{L}^2(\mathrm{X}_0)} \asymp ||Z_{\nu,X} - Z_{\nu_n,X}||^2 - (Z_{\nu,X,p} - Z_{\nu_n,X,p})^2 \tag{4.22}$$

where $f \asymp g$ means that there exists constants $A, B > 0$ such that $Ag \leqslant f \leqslant Bg$. By Proposition 3.6.34, we have that $||Z_\nu - Z_{\nu_n}||^2_{\mathrm{L}^2(\mathrm{X}_0)} \to 0$.

If $\nu \simeq \mathrm{ord}_E$ for some prime divisor E at infinity in X, then for all n large enough, $c_X(\nu_n) \in E$. We can suppose that $\nu = \mathrm{ord}_E$ and for all n $\nu_n(E) > 0$, i.e $\nu, \nu_n \in \mathcal{V}_X(E)$ and $Z_{\nu_n} \cdot E \to 1$ as $n \to \infty$. We show that

$$\frac{Z_{\nu_n}}{Z_{\nu_n} \cdot E} \xrightarrow[n \to +\infty]{} Z_{\mathrm{ord}_E} \tag{4.23}$$

in $\mathrm{L}^2(\mathrm{X}_0)$. We can replace ν_n by its equivalent valuation such that $\nu_n \in \mathcal{V}_X(p_n, \mathfrak{m}_{p_n})$ where $p_n = c_X(\nu_n)$. Then, we have that $Z_{\nu_n,X}/Z_{\nu_n} \cdot E$ converges towards Z_{ord_E} in $\mathrm{NS}(X)_{\mathbf{R}}$ by weak convergence. It suffices to show

$$\frac{(Z_{\nu_n,X,p})^2}{(Z_{\nu_n} \cdot E)^2} \to 0 \tag{4.24}$$

but this is equal to

$$-\frac{\alpha_{\mathfrak{m}_{p_n}}(\nu_n)}{\nu_n(E)^2} = -\frac{\alpha_E(\nu_n)}{\nu(E)^2} \xrightarrow[n \to +\infty]{} 0 \tag{4.25}$$

by Theorem 3.6.31 and Proposition 3.4.9 so we are done.

Finally, to show the homeomorphism, we have to show that if $Z_{\nu_n} \to Z_\nu$ in $\mathrm{L}^2(\mathrm{X}_0)$, then ν_n converges strongly towards ν. Let X be a completion of X_0. Suppose first that $c_X(\nu)$ is a point at infinity. Let $\widetilde{E}$ be the exceptional divisor above $c_X(\nu)$, we have $Z_\nu \cdot \widetilde{E} > 0$, therefore for all n large enough $Z_{\nu_n} \cdot \widetilde{E} > 0$ and $c_X(\nu_n) = c_X(\nu) =: p$. Now, we can suppose that $\nu_n, \nu \in \mathcal{V}_X(p; \mathfrak{m}_p)$, it suffices to show that $\nu_n \to \nu$ for the strong topology of $\mathcal{V}_X(p; \mathfrak{m}_p)$ and this is a direct consequence of Proposition 3.6.34.

If $c_X(\nu) = E$ a prime divisor at infinity, then for all n large enough, $Z_{\nu_n} \cdot E > 0$. Suppose that $\nu = \mathrm{ord}_E$ and $\nu_n \in \mathcal{V}_X(E)$. We have that $Z_{\nu_n,X}/Z_\nu \cdot E \to Z_{\mathrm{ord}_E}$ in $\mathrm{NS}(X)_{\mathbf{R}}$. We need to show that $\alpha_E(\frac{\nu_n}{\nu_n(E)}) \to 0$. We can suppose that $\nu_n \in \mathcal{V}_X(p_n, \mathfrak{m}_{p_n})$ where $p_n = c_X(\nu_n)$, then by Proposition

3.4.9,

$$\alpha_E\left(\frac{\nu_n}{\nu_n(E)}\right) = \frac{\alpha_{\mathfrak{m}_{p_n}}(\nu_n)}{\nu_n(E)^2}. \tag{4.26}$$

Thus, by Proposition 3.4.9 and Theorem 3.6.31

$$\alpha_E\left(\frac{\nu_n}{\nu_n(E)}\right) = \left|\frac{Z^2_{\nu_n,X,p_n}}{(Z_{\nu_n}\cdot E)^2}\right| \xrightarrow[n\to+\infty]{} 0. \tag{4.27}$$

□

Corollary 4.1.9. *If ν is a curve valuation, then Z_ν is a Weil class satisfying $Z_\nu^2 = -\infty$.*

Proof. Let X be a completion of X_0, let $p = c_X(\nu)$ and replace ν by the valuation equivalent to ν such that $\nu \in \mathcal{V}_X(p;\mathfrak{m}_p)$. We have by Corollary 4.1.6 that

$$Z_\nu = Z_{\nu,X} + Z_{\nu,X,p}. \tag{4.28}$$

Therefore, by Theorem 3.6.31

$$(Z_\nu)^2 = Z^2_{\nu,X} + (Z_{\nu,X,p})^2 = Z^2_{\nu,X} - \alpha(\nu) = -\infty \tag{4.29}$$

because $\alpha(\nu) = -\infty$ for any curve valuation ν (see [FJ04] Lemma 3.32). □

4.1.2 Endomorphisms

Proposition 4.1.10. *Let f be an endomorphism of X_0 and let X,Y be completions of X_0 such that the lift $F : X \to Y$ of f is regular. Let $p \in X$ be a closed point and $q := F(p) \in Y$. Then,*

- *$f_*\mathcal{V}_X(p) \subset \mathcal{V}_Y(q)$.*
- *f_* preserves the set of divisorial, irrational and infinitely singular valuations.*
- *If ν_C is a curve valuation centered at infinity and such that $f_*\nu_C$ is still centered at infinity, then $f_*\nu_C$ is also a curve valuation.*

Proof. The map F induces a local ring homomorphism $F^* : \widehat{\mathcal{O}_Y(q)} \to \widehat{\mathcal{O}_X(p)}$. Let ν be a valuation centered at p. For $\varphi \in \mathcal{O}_Y(q), f_*\nu(\varphi) = \nu(F^*\varphi) \geqslant 0$ and for $\psi \in \mathfrak{m}_{Y,q}, f_*\nu(\psi) = \nu(F^*\psi) > 0$. Therefore $f_*\nu$ is centered at q. The fact that f_* preserves the type of valuations is shown in Proposition 3.3.17. It only remains to show the statement for curve valuations. Let $p = c_X(\nu_C)$

and $q = c_Y(f_*\nu_C)$. We have that $F(p) = q$. By Proposition 3.3.17 $f_*\nu_C$ is not a curve valuation only if it is contracted by F. But the only germ of holomorphic curve at p that can be contracted by F is the germ of a prime divisor E at infinity on which p lies, and the curve valuation associated to E does not define a valuation on A. So, $f_*\nu_C$ is a curve valuation. □

Example 4.1.11. It might happen that $f_*\nu$ is not centered at infinity even though ν is; if this is the case then f is not proper. For example, let $X_0 = \mathbf{A}^2$ with affine coordinates (x,y) and consider the completion $\mathbf{P}^2$ with homogeneous coordinates $[X : Y : Z]$. We have the relation $x = X/Z, y = Y/Z$. Consider the chart $X \neq 0$ with affine coordinates $y' = Y/X$ and $z' = Z/X$. Define v_t to be the monomial valuation centered at $[1 : 0 : 0]$ such that $\nu_t(y') = 1$ and $\nu_t(z') = t$ with $t > 0$. Let $P = \sum_{i,j} a_{ij}x^iy^j \in \mathbf{k}[x,y]$, we have that $\nu_t(P) = \min\left\{j + (j-i)t | a_{ij} \neq 0\right\}$. Now take the map $f : (x,y) \in \mathbf{A}^2 \mapsto (xy,y)$, f contracts the curve $\{y = 0\}$ to the point $(0,0)$ in $\mathbf{A}^2$, hence it is not proper. For any polynomial $P = \sum_{i,j} a_{ij}x^iy^j, f^*P = \sum_{i,j} a_{ij}x^iy^{i+j}$. We get

$$\nu_{1,t}(f^*P) = \min_{i,j}\left\{i + j(t+1) | a_{ij} \neq 0\right\}. \tag{4.30}$$

The center of $f_*\nu_t$ is $[0 : 0 : 1]$ and $f_*\nu_t$ is the monomial valuation centered at $[0 : 0 : 1]$ such that $\nu_t(x) = 1, \nu_t(y) = t + 1$.

Lemma 4.1.12 (Proposition 3.2 of [FJ07])**.** *Let $f : X_0 \to X_0$ be a dominant endomorphism and let X,Y be completions of X_0. Let $F : X \to Y$ be the lift of f, let p be a closed point of X at infinity and $\mathcal{V}_X(p)$ be the set of valuations on A centered at p. Then, F is defined at p if and only if $f_*\mathcal{V}_X(p)$ does not contain any divisorial valuation associated to a prime divisor (not necessarily at infinity) of Y. If F is defined at p, then $F(p)$ is the unique point q such that $f_*\mathcal{V}_X(p) \subset \mathcal{V}_Y(q)$.*

Proof. If $\hat{f}$ is defined at p, then let $q = \hat{f}(p)$, we have that $f_*\mathcal{V}_X(p) \subset \mathcal{V}_Y(q)$ by Proposition 4.1.10.

Conversely, If p is an indeterminacy point of $\hat{f}$. Let $\pi : Z \to X$ be a completion above X such that the lift $F : Z \to Y$ is regular. Then, $F(\pi^{-1}(p))$ contains a prime divisor E' of Y. Let E be a prime divisor at infinity in Z above p such that $F(E) = E'$, then $F_* \operatorname{ord}_E = f_*(\pi_* \operatorname{ord}_E) = \lambda \operatorname{ord}_{E'}$ for some constant $\lambda > 0$ and $\operatorname{ord}_{E'} \in f_*\mathcal{V}_X(p)$. □

Proposition 4.1.13. *Let ν be a valuation over A and let $f : X_0 \to X_0$ be a dominant endomorphism, then*

- $f_*Z_\nu = Z_{f_*\nu} \mod \mathrm{Cartier}_\infty(\mathrm{X}_0)^\perp$.

- *If f is proper then f_* preserves $\mathrm{Weil}_\infty(\mathrm{X}_0)$ and $f_* Z_\nu = Z_{f_*\nu}$.*

Proof. Indeed, let $D \in \mathrm{Cartier}_\infty(\mathrm{X}_0)$, then

$$f_* Z_\nu \cdot D = Z_\nu \cdot f^* D = L_\nu(f^* D) = L_{f_*\nu}(D) = Z_{f_*\nu} \cdot D. \tag{4.31}$$

Therefore, we get that $Z_{f_*\nu} - f_* Z_\nu$ belongs to $\mathrm{Cartier}_\infty(\mathrm{X}_0)^\perp$. If f is proper, then $\mathrm{Weil}_\infty(\mathrm{X}_0)$ is f_*-stable and $f_* Z_\nu \in \mathrm{Weil}_\infty(\mathrm{X}_0)$, thus $Z_{f_*\nu} = f_* Z_\nu$. □

Example 4.1.14. Suppose that $P(x)$ and $Q(x)$ are two rational fractions of degree two and E in $\mathbf{P}^1 \times \mathbf{P}^1$ defined by the equation

$$y^2 - P(x)y + Q(x) = 0. \tag{4.32}$$

if P, Q are general, then E is smooth and irreducible and it is an elliptic curve in $\mathbf{P}^2$. Let $X = \mathbf{P}^1 \times \mathbf{P}^1$ and $X_0 = X \backslash E$. We have $\mathrm{Pic}^0(X_0) = 0$ because it is a rational surface and $A^\times = \mathbf{k}^\times$ because $X \backslash X_0$ consists of a single irreducible curve. We have $Z_{\mathrm{ord}_E} = \frac{1}{8} E$. Consider the projection $\mathrm{pr}_1 : X \to \mathbf{P}^1$ to the first coordinates. Each fiber of π_1 is isomorphic to $\mathbf{P}^1$ and generically it has two intersection points with E. Let x_0, x_1, x_2, x_3 be the four roots of the discriminant $\delta = P(x)^2 - 4Q(x)$. Then, $\mathrm{pr}_1^{-1}(x_i)$ has only one intersection point with E. Consider the following selfmap of X_0

$$f(x, y) = \left(x, \frac{y^2 - Q(x)}{2y - P(x)} \right). \tag{4.33}$$

It preserves the fibers of pr_1 and it acts as $z \mapsto z^2$ in each fiber where the points 0 and ∞ of $\mathbf{P}^1$ are the intersection point of the fiber with E. See Figure 4.1. There are exactly 4 indeterminacy points on X, they are the points (x_i, y_i) where x_i is one of the roots of Δ and $y_i \in \mathbf{P}^1$ is such that $(x_i, y_i) \in E$.

Let $C_0 = \{x_0\} \times \mathbf{P}^1$. Then, $\mathrm{Cartier}_\infty(\mathrm{X}_0)^\perp = \mathbf{R} \cdot (4C_0 - E)$ because $C_0 \cdot E = 2$ and $E^2 = 8$ and $\rho(X) = 2$.

The endomorphism f is not proper, indeed we have in $\mathrm{NS}(X)$, $f_* E = E + 4C_0$. Since $f^* E$ is of the form $f^* E = 2E + \ldots$, we have $f_* \mathrm{ord}_E = 2 \mathrm{ord}_E$. And we get

$$f_* Z_{\mathrm{ord}_E} = \frac{1}{8} E + \frac{1}{2} C_0 \tag{4.34}$$

$$= \frac{1}{8} E + \frac{1}{8}(4C_0 - E) + \frac{1}{8} E \tag{4.35}$$

$$= 2 Z_{\mathrm{ord}_E} + \frac{1}{8}(4C_0 - E) \tag{4.36}$$

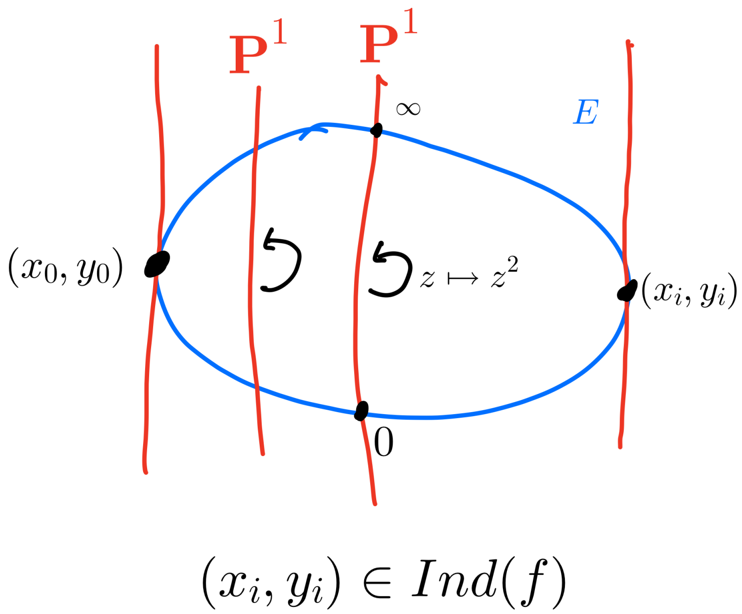


Figure 4.1: The endomorphism f on X_0

4.1.3 Existence of Eigenvaluations

Recall from Theorem 3.2.28 that there exists unique nef classes $\theta^*, \theta_* \in \mathrm{L}^2(\mathrm{X}_0)$ up to normalization such that $f^*\theta^* = \lambda_1\theta^*$ and $f_*\theta_* = \lambda_1\theta^*$.

Proposition 4.1.15. *If* $A^\times = \mathbf{k}^\times$ *and* $\mathrm{Pic}^0(X_0) = 0$*, then* $\theta^* \in \mathrm{Weil}_\infty(\mathrm{X}_0) \cap \mathrm{L}^2(\mathrm{X}_0)$ *and is effective.*

Proof. We have that $\text{Weil-NS}(\mathrm{X}_0) = V \oplus \mathrm{Weil}_\infty(\mathrm{X}_0)$ where V is a finite dimensional vector space. Furthermore, $\mathrm{Weil}_\infty(\mathrm{X}_0)$ is f^*-invariant as f is an endomorphism of X_0. In the proof of Theorem 3.2.28, for every completion X we can consider the cone $C'_X \subset \mathrm{Div}_\infty(X)_\mathbf{R}$ of nef, effective divisors supported at infinity. By Theorem 3.1.4, there exists an ample effective divisor $H \in \mathrm{Div}_\infty(X)$ such that $\mathrm{Supp}\, H = \partial_X X_0$. Therefore, C'_X is a closed convex cone with compact basis and non-empty interior, the Perron-Frobenius type argument shows that there exists $\theta_X \in C'_X$ such that $f_X^*\theta_X^* = \rho_X\theta_X$ and the rest of the proof is unchanged. $\square$

Theorem 4.1.16. *Let* $X_0 = \mathrm{Spec}\, A$ *be an irreducible normal affine surface such that* $A^\times = \mathbf{k}^\times$ *and* $\mathrm{Pic}^0(X_0) = 0$*. Let* f *be a dominant endomorphism such that* $\lambda_1(f)^2 > \lambda_2(f)$*, then there* *exists a unique valuation ν_* centered at infinity up to equivalence satisfying*

$$\forall P \in A, \nu_*(P) \leqslant 0 \tag{4.37}$$

$$f_*\nu_* = \lambda_1(f)\nu_* \tag{4.38}$$

$$Z_{\nu_*}^2 > -\infty \tag{4.39}$$

In particular, there exists $w \in \mathrm{Cartier}_\infty(\mathrm{X}_0)^\perp$ such that $\theta_* = w + Z_{\nu_*}$. Furthermore, ν_* is not a curve valuation.

We call ν_* the *eigenvaluation* of f.

Proof. By Theorem 3.2.28, there exists nef classes $\theta_*, \theta^* \in \mathrm{L}^2(\mathrm{X}_0)$ that satisfy

1. $f^*\theta^* = \lambda_1\theta^*$
2. $f_*\theta_* = \lambda_1\theta_*$
3. $\forall \alpha \in \mathrm{L}^2(\mathrm{X}_0), \frac{1}{\lambda_1^n}(f^n)^*\alpha \to (\theta_* \cdot \alpha)\theta^*$

Let X be a completion of X_0. Write the decomposition $\theta_* = w + Z$ with $w \in \mathrm{Div}_\infty(X)^\perp$ and $Z \in \mathrm{Weil}_\infty(\mathrm{X}_0)_\mathbf{R} \cap \mathrm{L}^2(\mathrm{X}_0)$. Let E be a prime divisor at infinity in X such that $Z_{\mathrm{ord}_E} \cdot \theta^* > 0$, it exists because θ^* is effective and nef. Then, Item (3) and the continuity of the intersection product in $\mathrm{L}^2(\mathrm{X}_0)$ imply that for all $D \in \mathrm{Cartier}_\infty(\mathrm{X}_0)$,

$$Z_{\mathrm{ord}_E} \cdot \left(\frac{1}{\lambda_1^n}(f^n)^*D\right) \to (Z_{\mathrm{ord}_E} \cdot \theta^*)(\theta_* \cdot D) = (Z_{\mathrm{ord}_E} \cdot \theta^*)(Z \cdot D) \tag{4.40}$$

Now, set $\nu_n := \frac{1}{\lambda_1^n}(f^n)_* \mathrm{ord}_E$. Equation (4.40) shows that Z_{ν_n} converges towards Z in $\mathrm{Weil}_\infty(\mathrm{X}_0)$. But, for all n, Z_{ν_n} belongs to $\mathrm{Hom}(\mathrm{Cartier}_\infty(\mathrm{X}_0), \mathbf{R})_{(+)}$ which is a closed set of $\mathrm{Weil}_\infty(\mathrm{X}_0)$ by Corollary 4.1.2. Therefore, $Z \in \mathrm{Hom}(\mathrm{Cartier}_\infty(\mathrm{X}_0), \mathbf{R})_{(+)}$ and it defines a valuation ν_* by Proposition 3.7.5. From the relation $f_*\theta_* = \lambda_1\theta_*$ we get that $f_*\nu_* = \lambda_1\nu_*$.

Using the decomposition $\theta_* = w + Z_{\nu_*}$ we have

$$0 \leqslant \theta_*^2 = \omega^2 + Z_{\nu_*}^2 \tag{4.41}$$

Therefore we get $Z_{\nu_*}^2 \neq -\infty$ and by Corollary 4.1.9, ν_* is not a curve valuation.

Now to show the uniqueness of ν_*, if ν is another valuation satisfying Equations (4.37), (4.38), (4.39), then for all $D \in \mathrm{Cartier}_\infty(X_0)$, Item (3) implies

$$Z_\nu \cdot D = \frac{1}{\lambda_1^n} Z_\nu \cdot (f^n)^* D \xrightarrow[n\to\infty]{} (Z_\nu \cdot \theta^*)(\theta_* \cdot D) \tag{4.42}$$

Since $\nu \neq 0$, we get $Z_\nu \cdot \theta^* > 0$. And then $\nu = \nu_*$ up to a scalar factor. □

Corollary 4.1.17. *With the hypothesis of Theorem 4.1.16. The dynamical degree $\lambda_1(f)$ is an algebraic integer of degree $\leqslant 2$. More precisely,*

- *If ν_* is divisorial or infinitely singular, then $\lambda_1 \in \mathbf{Z}_{>1}$.*
- *If ν_* is irrational, then λ_1 is an algebraic integer of degree 2, in particular $\lambda_1 \notin \mathbf{Q}$.*

Proof. By Theorem 4.1.16 f admits an eigenvaluation ν_* satisfying Equations (4.37), (4.38), (4.39). We know that ν_* cannot be a curve valuation, so there are three cases. It can either be a divisorial valuation, an irrational one or an infinitely singular one. Hence, $\nu_*(P) = \infty \Leftrightarrow P = 0$ and it defines a valuation over $K = \mathrm{Frac} A$. Let $G = \nu(K^\times)$ be the value group of ν_*. The value group of $f_*\nu_*$ is a subgroup of G, hence f_* induces a Z-linear map $f_* : G \to G$.

1. If ν_* is divisorial, then G is isomorphic to $\mathbf{Z}$. Since $f_*\nu_* = \lambda_1 v_*$ we get that λ_1 is an integer.
2. If ν_* is irrational, then G is isomorphic to $\mathbf{Z}^2$. Since $f_*\nu_* = \lambda_1\nu_*$, λ_1 is an eigenvalue of a 2×2 matrix with integer coefficients. Therefore, it is a quadratic integer.
3. If ν_* is infinitely singular. We will show in Proposition 4.2.3 below, the following.

   Claim 4.1.18. *There exists a completion X of X_0 such that $p := c_X(\nu) \in E$ is a free point at infinity , the lift $f : X \to X$ is defined at p, $f(p) = p$ and f contracts E to p.*

   Suppose the claim is true. Let (z, w) be local coordinates at p such that $z = 0$ is a local equation of E, f^*z is of the form $z^a\Phi(z, w)$ where Φ is a unit. Then,

   $$\lambda_1 L_{\nu_*}(E) = L_{f_*\nu_*}(E) = L_{\nu_*}(f^*E) = aL_{\nu_*}(E). \tag{4.43}$$

   Since $L_{\nu_*}(E) > 0$ we get $\lambda_1 = a$ and it is an integer.

□

4.2 Local normal forms

From now on we suppose $\operatorname{char}\mathbf{k} = 0$ and that X_0 is an affine surface. Since everything is defined over a finitely generated field over $\mathbf{Q}$, we can suppose that $\mathbf{k}$ is a subfield of $\mathbf{C}_v$, which is a complete algebraically closed field. We show that the existence of this eigenvaluation allows one to find an attracting fixed point at infinity and a local normal form at this fixed point.

Theorem 4.2.1. *Let $X_0 = \operatorname{Spec} A$ be an irreducible normal affine surface over a complete algebraically closed field $\mathbf{C}_v$. Let f be a dominant endomorphism of X_0 such that $\lambda_1^2 > \lambda_2$. Suppose that $\operatorname{Pic}^0(X_0) = 0$ and $A^\times = \mathbf{k}^\times$ then*

(1) If ν_* is infinitely singular or irrational, there exists a completion X such that the lift $f : X \to X$ is defined at $c_X(\nu_*)$, $f(c_X(\nu_*)) = c_X(\nu_*)$ and f defines a rigid contracting germ of holomorphic function at $c_X(\nu_*)$ with no f-invariant germ of curves at $c_X(\nu_*)$. Furthermore, there exists an open (euclidian) f-invariant neighbourhood U^* of $c_X(\nu_*)$ such that $f(U^*) \Subset U^*$. We have the following local normal form:

(a) If ν_* is infinitely singular, $c_X(\nu_*) \in E$ is a free point and f has the local normal form (3.3) *and* (3.2) *if $\mathbf{C}_v = \mathbf{C}$ with $\{x = 0\}$ a local equation of E $\lambda_1 = a \in \mathbf{Z}_{\geqslant 2}$.*

(b) If ν_* is irrational, $c_X(\nu_*) = E \cap F$ is a satellite point. The local normal form is pseudomonomial (3.5) *with (x,y) associated to (E,F). If $\mathbf{C}_v = \mathbf{C}$ it is monomial* (3.4) *The dynamical degree λ_1 is the spectral radius of the matrix $\left(a_{ij}\right)$. It is an algebraic integer of degree 2; in particular $\lambda_1 \notin \mathbf{Q}$.*

(2) If ν_* is divisorial, then there exists a completion such that $c_X(\nu_*)$ is a prime divisor E at infinity. In that case, E is f-invariant and $\lambda_1 \in \mathbf{Z}_{\geqslant 2}$ is such that $f_X^* E = \lambda_1 E + D$ where $D \in \operatorname{Div}_\infty(X)$ and $E \notin \operatorname{Supp} D$.

(a) Up to replacing f by some iterate, there exists a noncritical fixed point $p \in E$ of $f_{|E}$, $p = E \cap E_0$ is a satellite point, $f : X \dashrightarrow X$ is defined at p, $f(p) = p$ and f is a rigid germ (not necessarily contracting) at p with E the only f-invariant germ of curves at p. The local normal form of f at p is (3.6) *with (x,y) associated to (E,E_0) and $\lambda_1 = a$.*

(b) The curve E is an elliptic curve and $f_{|E}$ is a translation by a non-torsion element.

In particular, the dynamical degree of f is an algebraic number of degree $\leqslant 2$, and if it is not an integer then the eigenvaluation ν_* of f is irrational and the normal form is monomial.

We will call (2)b the *elliptic* case. The rest of this section is devoted to the proof of Theorem 4.2.1, we will prove the Theorem page 163.

To prove the theorem we need to understand the dynamics of f_* on the space of valuations.

Proposition 4.2.2. *Let $\nu \in \mathcal{V}_\infty$ such that $Z_\nu \in \mathrm{L}^2(\mathrm{X}_0)$. If $Z_\nu \cdot \theta^* > 0$, then $\frac{1}{\lambda_1^n} f_*^n \nu$ strongly converges towards $(Z_\nu \cdot \theta^*)\nu_*$.*

Proof. This is a direct consequence of Equation (3.69) and Corollary 4.1.8. □

We will use this to show that f admits a fixed point at infinity on some completion and that f contracts a divisor at infinity there.

For the rest of Section 4.2, we suppose that we are in the conditions of Theorem 4.1.16.

4.2.1 Attractingness of ν_*, the infinitely singular case

In this section we show the following

Proposition 4.2.3. *Let* **k** *be an algebraically closed field (of any characteristic). If the eigenvaluation ν_* is infinitely singular, then there exists a completion X of X_0 such that*

(1) $p := c_X(\nu_*) \in E$ is a free point at infinity.

(2) $f_* \mathcal{V}_X(p) \subset \mathcal{V}_X(p)$;

(3) f contracts E to p.

(4) Let $f_\bullet : \mathcal{V}_X(p; \mathfrak{m}_p) \to \mathcal{V}_X(p; \mathfrak{m}_p)$, then for all $\nu \in \mathcal{V}_X(p; \mathfrak{m}_p)$, $f_\bullet^n \nu \to \nu_*$.

Furthermore, the set of completions Y above X that satisfy these 3 properties is cofinal in the set of all completions above X.

Let X be a completion of X_0 such that $c_X(\nu_*)$ is a free point $p_X \in E_X$. Such a completion X exists and there are infinitely many of them above X by Proposition 3.3.16. Let Y be a completion above X such that $c_Y(\nu_*)$ on Y is a free point $p_Y \in E_Y$ such that the diagram

$$\begin{tikzcd} & Y \arrow[dl, "\pi"'] \arrow[dr, "F"] & \\ X \arrow[rr, dashed, "f"] & & X \end{tikzcd}$$

commutes, where F is regular and $F(p_Y) = p_X$, such a completion Y exists by Proposition 3.3.16. Let x, y be coordinates at p_X such that $x = 0$ is a local equation of E_X and z, w be coordinates at p_Y such that $z = 0$ is a local equation for E_Y. We use the notations of Section 3.4. We have that $f_* \mathcal{V}_Y(p_Y) \subset \mathcal{V}_X(p_X)$ by Lemma 4.1.12. We define $F_\bullet : \mathcal{V}_Y(p_Y; E_Y) \mapsto \mathcal{V}_X(p_X, \mathfrak{m}_{p_X})$ as follows:

$$\forall \nu \in \mathcal{V}_Y(p_Y; E_Y), \quad F_\bullet(\nu) := \frac{F_* \nu}{\min(\nu(F^* x), \nu(F^* y))}. \tag{4.44}$$

Similarly, we define

$$\forall \nu \in \mathcal{V}_Y(p_Y; E_Y), \quad \pi_\bullet(\nu) := \frac{\pi_* \nu}{\min(\nu(\pi^* x), \nu(\pi^* y))}. \tag{4.45}$$

By Proposition 3.4.20 item (1), the map $\pi_\bullet : \mathcal{V}_Y(p_Y; E_Y) \to \mathcal{V}_X(p_X; \mathfrak{m}_{p_X})$ is an inclusion of trees and allows one to view $\mathcal{V}_Y(p_Y; E_Y)$ as a subtree of $\mathcal{V}_X(p_X; \mathfrak{m}_{p_X})$.

See Figure 4.2. The tree $\mathcal{V}_X(p_X, \mathfrak{m}_{p_X})$ is in black with its root $\nu_{\mathfrak{m}_{p_X}}$ in blue, the tree $\mathcal{V}_Y(p_Y; E_Y)$ is in orange with its root ord_{E_Y} in red. One can see how $\pi_\bullet$ maps homeomorphically $\mathcal{V}_Y(p_Y; E_Y)$ to a subtree of $\mathcal{V}_X(p_X, \mathfrak{m}_{p_X})$.

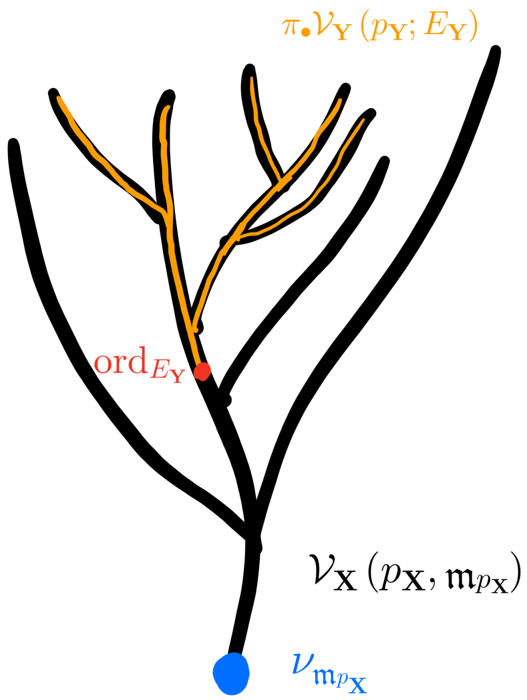


Figure 4.2: The embedding $\pi_\bullet$

Remark 4.2.4. Since the orders $\leqslant_{\mathfrak{m}_{p_X}}$ and $\leqslant_{E_Y}$ are compatible on $\mathcal{V}_Y(p_Y;E_Y)$ and $\pi_\bullet\mathcal{V}_Y(p_Y;E_Y)$ we will not write $\pi_\bullet$ or $\leqslant_{E_Y}$ when no confusion is possible to avoid heavy notations.

By Proposition 3.4.28, we have the following relation

$$\alpha_{\mathfrak{m}_{p_X}}(\pi_\bullet\mu) = \alpha_{\mathfrak{m}_{p_X}}(\pi_\bullet \operatorname{ord}_{E_Y}) + b(E_Y)^{-2}\alpha_{E_Y}(\mu) \tag{4.46}$$

where b is the generic multiplicity function of the tree $\mathcal{V}_X(p;\mathfrak{m}_p)$ and α is the skewness function defined in §3.4. Indeed, with the notation of Proposition 3.4.28, $\nu_{E_Y} = \pi_\bullet \operatorname{ord}_{E_Y}$.

Lemma 4.2.5. *There exists $\nu \in \mathcal{V}_Y(p_Y;E_Y)$ such that $\nu < \nu_*$ and for all $\mu \geqslant \nu$,*

$$\min(\mu(F^*x),\mu(F^*y)) = \lambda_1. \tag{4.47}$$

I.e set $U = \{\mu \geqslant \nu\}$, we have $F_\bullet = \frac{F_*}{\lambda_1}$ over U. In particular, $F_\bullet$ is order preserving over U and $F_\bullet([\nu,\nu_*]) \subset \left[\nu_{\mathfrak{m}_{p_X}},\nu_*\right]$.

Proof. Using Proposition 3.4.3, we see that the map $\nu \mapsto \min(\nu(f^*x,f^*y))$ is locally constant outside a finite subtree of $\mathcal{V}_Y(p_Y;E_{p_Y})$. Indeed, one has $f^*x = \prod_i \psi_i$ with ψ_i irreducible and therefore

$$\begin{aligned}\nu(f^*x) &= \sum_i \nu(\psi_i) &(4.48)\\ &= \sum_i \alpha_{E_Y}(\nu\wedge\nu_{\psi_i})m_{E_Y}(\psi_i) \quad \text{by Proposition 3.4.3.} &(4.49)\end{aligned}$$

Let S_x be the finite subtree consisting of the segments $[\operatorname{ord}_{E_Y},\nu_{\psi_i}]$, then the map $\mu \mapsto \mu(f^*x))$ is locally constant outside of S_x. Let S be the maximal finite subtree of $\mathcal{V}_Y(p_Y;E_{p_Y})$ such that the evaluation maps on f^*x, f^*y and z are locally constant outside of S. Since ν_* is an infinitely singular valuation it does not belong to S and these three evaluation maps are constant on the open connected component V of $\mathcal{V}_Y(p_Y;E_{p_Y})\backslash S$ containing ν_*. Since $f_*\nu_* = \lambda_1\nu_*$, we have $f_{\bullet|V} = \frac{f_*}{\lambda_1}$ and the map $F_\bullet$ is order preserving on V. Following Remark 4.2.4, the two orders $\leqslant_{\mathfrak{m}_{p_X}}$ and $\leqslant_{E_Y}$ agree on V. Let $\nu \in [\operatorname{ord}_{E_Y},\nu_*]\cap V$ be a divisorial valuation, $F_\bullet$ sends the segment $[\nu,\nu_*] \subset \mathcal{V}_Y(p_Y;E_Y)$ inside the segment $[\nu_{\mathfrak{m}_{p_X}},\nu_*] \subset \mathcal{V}_X(p_X;\mathfrak{m}_{p_X})$. Notice that $U := \{\mu \geqslant \nu\} \subset V$ so the valuation ν satisfies Lemma 4.2.5. $\square$

Proposition 4.2.6 ([FJ07], Theorem 3.1)**.** *Let ν be as in Lemma 4.2.5. For $t \in [\alpha_{E_Y}(\nu),\alpha_{E_Y}(\nu_*)]$, let ν_t be the unique valuation in $[\nu,\nu_*]$ such that $\alpha_{E_Y}(\nu_t) = t$. Then, there exists a divisorial valuation $\nu' \in [\nu,\nu_*]$ such that the map*

$$t \in [\alpha_{E_Y}(\nu'), \alpha_{E_Y}(\nu_*)] \mapsto \alpha_{\mathfrak{m}_{p_X}}(F_\bullet \nu_t) \tag{4.50}$$

is an affine function of t with nonnegative coefficients.

Proof. Let $\nu_1, \nu_2 \in \mathcal{V}_Y(p_Y; E_Y)$ be such that $\nu < \nu_1 < \nu_2 < \nu_*$. Since $F_\bullet$ is order preserving on $U = \{\mu \geqslant \nu\}$ one has that $F_\bullet$ maps $[\nu_1, \nu_2]$ homeomorphically to $[F_\bullet\nu_1, F_\bullet\nu_2]$. Let $\psi \in \widehat{O_{X,p_X}}$ be irreducible such that $\nu_\psi > F_\bullet\nu_2$, then by Proposition 3.4.3, for all $\mu \in [\nu_1, \nu_2]$ one has

$$\alpha_{\mathfrak{m}_{p_X}}(F_\bullet\mu) = \frac{F_\bullet\mu(\psi)}{m_{p_X}(\psi)} = \frac{\mu(f^*\psi)}{m_{p_X}(\psi)\lambda_1} \tag{4.51}$$

Now let $\psi_1, \cdots, \psi_r \in \widehat{O_{Y,p_Y}}$ be irreducible (not necessarily distinct) such that $f^*\psi = \psi_1 \cdots \psi_r$. One has,

$$\mu(f^*\psi) = \sum_i \mu(\psi_i) = \sum_i \alpha_{E_Y}(\mu \wedge \nu_{\psi_i}) m_{E_Y}(\psi_i). \tag{4.52}$$

Take one of the ψ_i and call it ψ_0, we shall study the map $\mu \in [\nu_1, \nu_2] \mapsto \alpha_{E_Y}(\mu \wedge \nu_{\psi_0})$. Let $\mu_0 = \nu_2 \wedge \nu_{\psi_0}$, this map is equal to α_{E_Y} on $[\nu_1, \mu_0]$ and constant equal to $\alpha_{E_Y}(\mu_0)$ on $[\mu_0, \nu_2]$. Therefore, the map $\mu \in [\nu_1, \nu_2] \mapsto \mu(f^*\psi)$ is a piecewise affine function with nonnegative coefficients of $\alpha_{E_Y}(\mu)$. The points on $[\nu_1, \nu_2]$ where this map is not smooth are exactly the valuations $\nu_* \wedge \nu_{\psi_i}$ and there are at most λ_2 of them by Proposition 3.3.18. Therefore the map $\mu \mapsto \nu(f^*\psi)$ is an affine function of α_{E_Y} with nonnegative coefficients on the segment $[\mu', \nu_*]$ for any $\mu' < \nu_*$ close enough to ν_*. □

As a corollary of the proof, we get the following proposition.

Proposition 4.2.7. *Let $\nu \in \mathcal{V}_Y(p_Y; E_Y)$ be as in Proposition 4.2.6, let $\nu_0 \in [\nu, \nu_*]$ and let $\psi \in \widehat{O}_{X,p}$ be irreducible such that $\nu_\psi > f_\bullet\nu_0$. Then, for all $\varphi \in \widehat{O_{Y,p_Y}}$ such that $f_\bullet\nu_\varphi = \nu_\psi$, one has two possibilities:*

(1) Either $\nu_\varphi > \nu_0$.

(2) or $\nu_0 \wedge \nu_\varphi = \nu_* \wedge \nu_\varphi \leqslant \nu$.

I.e the configuration of Figure 4.3 cannot occur.

Proof. The map $\mu \in [\nu, \nu_0] \mapsto \alpha_{m_{p_X}}(F_\bullet\mu)$ is a smooth affine function of $\alpha_{E_Y}(\mu)$. If (1) and (2) were not satisfied, then we would get $\nu_\varphi \wedge \nu_* \in [\nu, \nu_*]$ and this would contradict the smoothness of the map $\mu \in [\nu, \nu_*] \mapsto \alpha_{m_{p_X}}(F_\bullet\mu)$ □

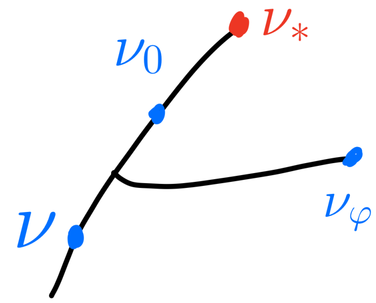


Figure 4.3: Configuration which is not possible

Lemma 4.2.8. *Let ν be as in Proposition 4.2.6. If $\mu \in [\nu, \nu_*]$ is sufficiently close to ν_*, then $F_\bullet \mu > \mu$ and $F_\bullet(\{\mu' \geqslant \mu\}) \Subset U(\overrightarrow{v})$ where $\overrightarrow{v}$ is the tangent vector at μ defined by ν_* and $U(\overrightarrow{v})$ is its associated open subset.*

We sum up Lemma 4.2.8 in Figure 4.4

Proof. Let $U = \{\mu \geqslant \nu\}$. Recall that $F_\bullet$ is order preserving over U. We first notice that if every $\mu \in [\nu, \nu_*]$ close enough to ν_* satisfies $F_\bullet \mu > \mu$, it is clear that $F_\bullet \{\mu' \geqslant \mu\} \Subset U(\overrightarrow{v})$. Indeed, let $\mu' \geqslant \mu$ and set $\mu_0 := \mu' \wedge \nu_* \geqslant \mu$. Then, $F_\bullet \mu' \geqslant F_\bullet \mu_0 > \mu_0$. In particular, $F_\bullet \mu' \wedge \nu_* > \mu' \wedge \nu_* \geqslant \mu$.

Secondly, by Proposition 4.2.6, the map $t \in [\alpha_{E_Y}(\nu), \alpha_{E_Y}(\nu_*)] \mapsto \alpha_{m_{p_X}}(\nu_t)$ is affine and we know that it is non decreasing.

Lemma 4.2.9. *Let $a : \mathbf{R} \to \mathbf{R}$ be a non-decreasing non constant affine function that admits a fixed point t_0. If there exists $s < t_0$, $a(s) > s$ then the slope of a is < 1 and for all $t < t_0$, $a(t) > t$.*

Proof of Lemma 4.2.9. We can suppose that $t_0 = 0$ by a linear change of coordinate. Then, $a(t)$ is of the form

$$a(t) = \alpha t \tag{4.53}$$

with $\alpha > 0$. Now, if $s < 0$ satisfies $a(s) > s$, this means that $0 < \alpha < 1$ and therefore for all $t < 0$, $a(t) > t$. □

We show that there exists $\mu \in [\nu, \nu_*]$ such that $F_\bullet \mu > \mu$. If not, then for all $\mu \in [\nu, \nu_*[$, $F_\bullet \mu \leqslant \mu$. Under such an assumption, we show the following

Claim For all $\mu' \geqslant \nu$ we have $F_\bullet \mu' \wedge \nu_* \leqslant \mu' \wedge \nu_*$.

Suppose that the claim is false and let μ' be a valuation that contradicts this statement. It is clear that μ' does not belong to $[\nu, \nu_*]$. Pick $\nu_0 \in [\nu, \nu_*]$ such that $\nu \leqslant \mu' \wedge \nu_* < \nu_0 < F_\bullet \mu' \wedge \nu_*$.

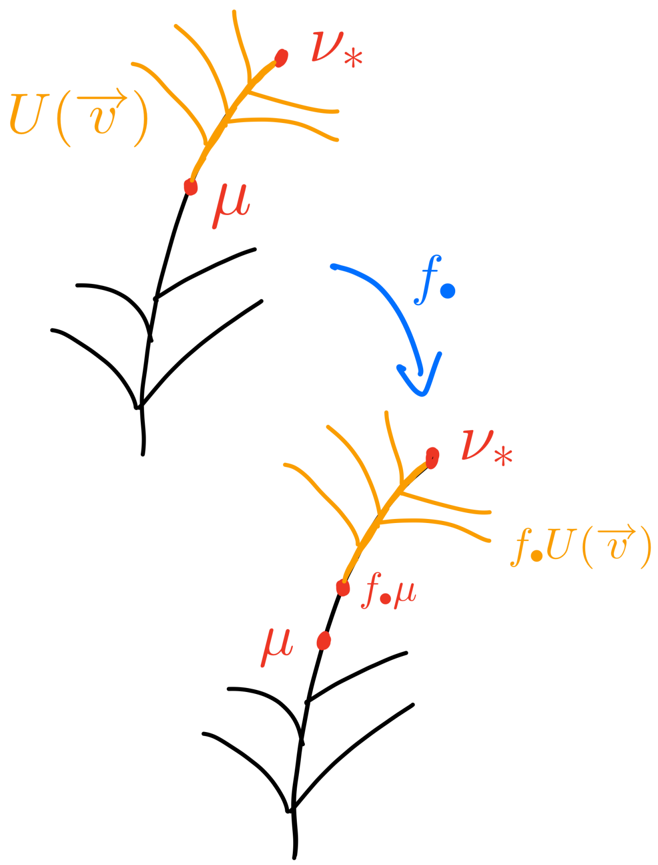


Figure 4.4: An $f_\bullet$-invariant open subset of $\mathcal{V}_\infty$, infinitely singular case

Let $\varphi \in \widehat{\mathcal{O}}_{Y,p_Y}$ be such that $\nu_\varphi > \mu'$ and let $\psi \in \widehat{\mathcal{O}}_{X,p}$ be such that $f_\bullet \nu_\varphi = \nu_\psi$. Since f is order preserving we get that $\nu_\psi > F_\bullet \mu' \geqslant F_\bullet \mu' \wedge \nu_* > \nu_0$, therefore $\nu_\psi > F_\bullet \nu_0$. But then φ contradicts Proposition 4.2.7 since $\nu_\varphi \wedge \nu_0 = \mu' \wedge \nu_0 \in [\nu, \nu_0]$. So the claim is shown.

Now, pick ω divisorial such that $Z_\omega \cdot \theta^* > 0$ by Proposition 4.2.2 the sequence $\frac{1}{\lambda_1^n} f_*^n \omega$ converges towards $(Z_\omega \cdot \theta^*)\nu_*$. Hence, there exists an integer $N_0 > 0$ such that for all $N \geqslant N_0$, $f_*^N \nu \in \mathcal{V}_Y(p_Y)$, replace ω by $f_*^{N_0}\omega$ and normalize it such that $\omega \in \mathcal{V}_Y(p_Y, E_Y)$. We can suppose up to choosing a larger N_0 that $\omega > \nu$. In that case $F_\bullet^N \omega$ converges towards ν_* but by the claim, $\forall N \geqslant 0, F_\bullet^N \omega \wedge \nu_* \leqslant \omega \wedge \nu_*$ which is a contradiction.

Therefore, there exists a valuation $\mu \in [\nu, \nu_*[$ such that $F_\bullet \mu > \mu$. $\square$

Proposition 4.2.10. *With the notations from Lemma 4.2.8, we have $F_\bullet(U(\overrightarrow{v})) \Subset U(\overrightarrow{v})$ and for all $\mu' \in U(\overrightarrow{v})$,*

$$F_\bullet^n \mu' \xrightarrow[n \to +\infty]{} \nu_* \tag{4.54}$$

for the weak topology.

Proof. For every μ' in $U(\overrightarrow{v})$, write $\widetilde{\mu'} = \mu' \wedge \nu_*$. By the proof of Lemma 4.2.8, $F_\bullet^n(\mu') \to \nu_*$ for the strong topology. Therefore, $F_\bullet^n \mu' \wedge \nu_* \geqslant F_\bullet^n(\widetilde{\mu'}) \to \nu_*$ and $F_\bullet^n \mu'$ converges weakly towards ν_* because for all $\varphi \in \mathcal{O}_{Y,p}$ irreducible, we have

$$F_\bullet^n(\mu')(\varphi) = \alpha_{E_Y}(F_\bullet^n \mu' \wedge \nu_\varphi) m_{E_Y}(\varphi). \tag{4.55}$$

For n large enough we have $F_\bullet^n \mu' \wedge \nu_* \geqslant \nu_* \wedge \nu_\varphi$, hence $F_\bullet^n \mu' \wedge \nu_\varphi = \nu_* \wedge \nu_\varphi$ and

$$F_\bullet^n(\mu')(\varphi) = \alpha_{E_Y}(\nu_* \wedge \nu_\varphi) m_{E_Y}(\varphi) = \nu_*(\varphi) \tag{4.56}$$

□

Proof of Proposition 4.2.3. Let ν be as in Proposition 4.2.6. Let ν_n be the approximating sequence of ν_* (see Proposition 3.4.26). We have for n large enough $\nu_n \in [\nu, \nu_*]$ and ν_n satisfies Lemma 4.2.8. Set $\mu = \nu_n$ for some n large enough and let Z be a completion such that $c_Z(\mu) = E$ and $c_Z(\nu_*) =: p \in E$ is a free point. The open subset $U(\overrightarrow{v})$ associated to the tangent vector at μ defined by ν_* is exactly the image of $\mathcal{V}_Z(p)$ in $\mathcal{V}_Y(p_Y; E_Y)$. By Proposition 4.2.10, $F_\bullet U(\overrightarrow{v}) \Subset U(\overrightarrow{v})$, this means that $f_* \mathcal{V}_Y(p) \subset \mathcal{V}_Y(p)$. By Lemma 4.1.12, f is defined at $p, f(p) = p$ and since $F_\bullet \mu > \mu$, we get f contracts E to p. We have that for every $\mu \in \mathcal{V}_Z(p; \mathfrak{m}_p), f_\bullet^n \mu \to \nu_*$ also by Proposition 4.2.10.

The statement about cofinalness follows from the fact that the sequence of infinitely near points associated to ν_* contains infinitely many free points, so for every completion X of X_0, there exists a completion above it where the center of ν_* is a free point at infinity. □

4.2.2 Attractingness of ν_*, the irrational case

Suppose now that $\operatorname{char} \mathbf{k} = 0$, this is necessary as we will use Theorem 3.1.9 in this paragraph. Suppose now that ν_* is an irrational valuation. There exists a completion X such that the center of ν_* on X and on any completion above X is the intersection of two divisors at infinity E, F. We still write $f : X \dashrightarrow X$ for the lift of f.

Let $X_1 = X$ and for all $n \geqslant 1$, let X_{n+1} be the blow up of X_n at $c_{X_n}(\nu_*)$. (The center of ν_* is always a point since ν_* is not divisorial). Let $p_n = c_{X_n}(\nu_*)$ and E_n, F_n be the divisors at infinity in X_n such that $p_n = E_n \cap F_n$. A consequence of Theorem 3.1.9 is

Proposition 4.2.11. *There exist integers $N \geqslant M$ such that the lift $\widehat{f} : X_N \to X_M$ is regular at $p_N := c_{X_N}(\nu_*)$ and such that $\widehat{f}$ is monomial at p_N in the coordinates that have E_N, F_N and E_M, F_M for axis respectively.*

Proof. Apply Theorem 3.1.9 to $f : X \dashrightarrow X$. There exist completions Y, Z above X such that the lift $F : Y \to Z$ of f is regular and monomial at every point. Let $N_Y = \max\{N : Y \text{ is above } X_N\}$ and define N_Z in the same way. By construction, the morphism of completions $\pi : Y \to X_{N_Y}$ consists of blow up of points that are not p_{N_Y}. The same holds for $\tau : Z \to X_{N_Z}$. This shows that the lift $f : X_{N_Y} \dashrightarrow X_{N_Z}$ is defined at p_{N_Y}. We therefore have that $f(p_{N_Y}) = p_{N_Z}$ because $f_*(\nu_*) = \lambda_1 \nu_*$ and f is monomial at p_{N_Y} in the coordinates that have $E_{N_Y, F_{M_Y}}$ and E_{N_Z}, F_{N_Z} for axis respectively by Theorem 3.1.9. We set $M = N_Z$. If $N_Y < M$, we keep blowing up p_{N_Y} until $N_Y \geqslant M$. This does not change the result because in local coordinates the blow up is given by a monomial map $\pi(u, v) = (uv, v)$ where u and v are local equation of the prime divisors at infinity to which the center of ν_* belong. □

Using this we show

Proposition 4.2.12. *There exists a completion Y such that*

(1) The lift $\widehat{f} : Y \to Y$ is defined at $p = c_Y(\nu_*)$;

(2) $\widehat{f}(p) = p$;

(3) If E, F are the two divisors at infinity such that $p = E \cap F$, then E and F are both contracted to p by $\widehat{f}$.

(4) Define $f_\bullet : \mathcal{V}_Y(p; \mathfrak{m}_p) \to \mathcal{V}_Y(p; \mathfrak{m}_p)$. For all $\mu \in \mathcal{V}_Y(p; \mathfrak{m}_p), f_\bullet^n mu \to \nu_*$ for the weak topology of $\mathcal{V}_Y(p; \mathfrak{m}_p)$.

Furthermore, If Z is a completion above Y, then (1)-(4) remain true.

Proof. Let $N \geqslant M$ given by Proposition 4.2.11. We still write $f : X_N \dashrightarrow X_M$ for the lift of f and $\pi : X_N \to X_M$ for the composition of blow ups. Let x, y be local coordinates at p_N such that $E_N = \{x = 0\}$ and $F_N = \{y = 0\}$ and let z, w be local coordinates at p_M such that $E_M = \{z = 0\}$ and $F_M = \{w = 0\}$. Both maps f and π are monomial at p_N with respect to these coordinates. Write

$$f(x, y) = (x^a y^b, x^c y^d). \tag{4.57}$$

Consider the tree $\mathcal{V}_{X_M}(p_M; E_M)$ with its order $<_M$, its skewness function α_M and the generic multiplicity function b_M. This tree is rooted in ord_{E_M} and F_M defines the end ν_w that we denote by ν_{F_M}. Let $\nu_{E_N} = \frac{1}{b_M(E_N)} \mathrm{ord}_{E_N}, \nu_{F_N} = \frac{1}{b_M(F_N)} \mathrm{ord}_{F_N}$. Suppose without loss of generality that $\nu_{E_N} <_M \nu_{F_N}$. Consider the tree $\mathcal{V}_{X_N}(p_N; E_N)$ with its order $<_N$ and skewness function α_N. We have by Proposition 3.4.20 item (2) that the map $\pi_\bullet : \mathcal{V}_{X_N}(p_N; E_N) \to \mathcal{V}_{X_M}(p_M; E_M)$ is an inclusion of trees. Hence, the orders $<_M, <_N$ are compatible and $\mathcal{V}_{X_N}(p_N; E_N)$ is naturally a subtree of $\mathcal{V}_{X_M}(p_M; E_M)$ via the map $\pi_\bullet$. We also have the map $f_\bullet : \mathcal{V}_{X_N}(p_N; E_N) \to \mathcal{V}_{X_M}(p_M; E_M)$. The root of $\mathcal{V}_{X_N}(p_N; E_N)$ is ord_{E_N} and F_N defines the end ν_y in $\mathcal{V}_{X_N}(p_N; E_N)$ that we also denote by ν_{F_N}. We have that $\mathrm{ord}_{E_N} <_N \nu_* <_N \nu_{F_N}$. Using Equation (4.57), we can write

$$\forall \nu \in \mathcal{V}_{X_N}(p_N; E_N), \quad f_\bullet(\nu) = \frac{f_* \nu}{a + b\nu(y)}. \tag{4.58}$$

Now, both maps $f_\bullet$ and $\pi_\bullet$ send the segment $[\mathrm{ord} E_N, \nu_{F_N}]$ into the segment $[\mathrm{ord}_{E_M}, \nu_{F_M}]$ via a Möbius transformation. Indeed, if $\nu_{1,t} \in \mathcal{V}_{X_N}(p_N; E_N)$ is a monomial valuation at p_N, then $f_* \nu_{1,t} = \nu_{a+bt, c+td}$ and one has by Lemma 3.4.13 and Equation (4.58)

$$\alpha_M(f_\bullet \nu_{1,t}) = \alpha_M\left(\nu_{1, \frac{c+td}{a+tb}}\right) = \frac{c + \alpha_N(\nu_{1,t}) d}{a + \alpha_N(\nu_{1,t}) b} = M_f(\alpha_N(\nu_{1,t})) \tag{4.59}$$

Where M_f is the Möbius transformation associated to the matrix $\begin{pmatrix} d & c \\ b & a \end{pmatrix}$. We can do the same process with the map $\pi_\bullet$ to get a Möbius transformation represented by a matrix M_π. Set M to be the Möbius transformation $M_f \circ M_\pi^{-1}$.

Lemma 4.2.13. *The Möbius map M is loxodromic with an attracting fixed point $t_* = \alpha_M(\pi_\bullet \nu_*)$ and the multiplier of M at t_* is $\leqslant \sqrt{\frac{\lambda_2}{\lambda_1^2}} < 1$.*

In particular, for every $\nu_1, \nu_2 \in \mathcal{V}_{X_N}(p_N; E_N)$ close enough to ν_* such that $\nu_1 < \nu_* < \nu_2$, $f_\bullet([\nu_1, \nu_2]) \Subset [\pi_\bullet \nu_1, \pi_\bullet \nu_2]$.

Proof of Lemma 4.2.13. First of all, M cannot be of finite order. Indeed, for every $\nu \in [\nu_{E_N}, \nu_{E_M}]$ sufficiently close to ν_*, we have $Z_\nu \cdot \theta^* > 0$ since $\theta^* \cdot \theta_* = 1$. So $f_\bullet^n \nu \to \nu_*$ by Proposition 4.2.2.

We know that $M(t_*) = t_*$ and we want to show that $|M'(t_*)| < 1$. The only way that the proposition is not true is if t_* is a parabolic fixed point of M. This means up to reversing the orientation that t_* is attracting for $t < t_*$ sufficiently close to t_* and t_* is repelling for $t > t_*$ sufficiently close to t. In particular, there exists t' such that the segment $[t', t_*]$ is sent strictly into itself, so we can iterate M on it, and there exist two constant $c_1, c_2 > 0$ such that $\frac{c_1}{n} \leqslant |M^n(s) - t_*| \leqslant \frac{c_2}{n}$. We will show that we have actually an exponential speed of convergence and this leads to a contradiction. Let ν be the valuation centered at p_N such that $\alpha_M(\pi_\bullet \nu) = t'$, we can suppose that ν is divisorial up to shrinking $[t', t_*]$. Since $f_\bullet^n \nu \to \nu_*$, we have $Z_\nu \cdot \theta^* > 0$. We have by Equation (3.68)

$$\frac{1}{\lambda_1^k}(f_*^k Z_\nu) \cdot E_M = (\theta_* \cdot E_M)(Z_\nu \cdot \theta^*) + O\left(\left(\frac{\lambda_2}{\lambda_1^2}\right)^{k/2}\right) \tag{4.60}$$

$$\frac{1}{\lambda_1^k}(f_*^k Z_\nu) \cdot F_M = (\theta_* \cdot F_M)(Z_\nu \cdot \theta^*) + O\left(\left(\frac{\lambda_2}{\lambda_1^2}\right)^{k/2}\right). \tag{4.61}$$

Using Lemma 3.4.13 we get that

$$\left|M^k(\alpha_M(\pi_\bullet \nu)) - t_*\right| = \left|\frac{f_*^k Z_\nu \cdot F_M}{f_*^k Z_\nu \cdot E_M} - \frac{\theta_*(F_M)}{\theta_*(E_M)}\right| = O\left(\left(\frac{\lambda_2}{\lambda_1^2}\right)^{k/2}\right). \tag{4.62}$$

Therefore the speed of convergence is exponential and this shows that $|M'(t_*)| < 1$. □

End of Proof of Proposition 4.2.12. By Lemma 4.2.13, pick $\nu_1, \nu_2 \in \mathcal{V}_{X_N}(p_N; E_N)$ divisorial sufficiently close to ν_* such that

$$\mathrm{ord}_{E_N} <_N \nu_1 <_N \nu_* <_N \nu_2 <_N \nu_{F_N} \tag{4.63}$$

and

$$f_\bullet([\nu_1, \nu_2]) \Subset [\pi_\bullet \nu_1, \pi_\bullet \nu_2]. \tag{4.64}$$

Let $U_N = \{\nu : \nu_1 < \nu \wedge \nu_{F_N} < \nu_2\} \subset \mathcal{V}_{X_N}(p_N; E_N)$. It is clear that $\nu_{F_N} \notin U_N$. Let $\psi \in \widehat{\mathcal{O}_{X_M, p_M}}$ be such that $\nu_\psi >_M f_\bullet([\nu_1, \nu_2])$. Let $\psi_1, \cdots, \psi_r \in \widehat{\mathcal{O}_{X_N, p_N}}$ be irreducible such that $f^* \psi = \psi_1 \cdots \psi_r$. We can shrink the segment $[\nu_1, \nu_2]$ to make sure that none of the ψ_i belong to U_N (see Figure 4.5). If this is the case, then for all $\mu \in U_N$, set $\widetilde{\mu} = \mu \wedge \nu_2$, then for all i

$$\mu \wedge \nu_{\psi_i} = \widetilde{\mu} \wedge \nu_{\psi_i} \tag{4.65}$$

and

$$\mu \wedge \nu_{F_N} = \widetilde{\mu} \wedge \nu_{F_N}. \tag{4.66}$$

Now, for all $\mu \in U_N$, by Equation (4.58) and Proposition 3.4.3

$$(f_\bullet \mu)(\psi) = \frac{\mu(f^* \psi)}{a + b\mu(y)} = \frac{\sum_k \alpha_N(\mu \wedge \nu_{\psi_k} m(\psi_k))}{a + b\mu(y)}. \tag{4.67}$$

By Equations (4.65) and (4.66), we get

$$(f_\bullet \mu)(\psi) = (f_\bullet \widetilde{\mu})(\psi). \tag{4.68}$$

This means that

$$\forall \mu \in U_N, \quad \alpha_M((f_\bullet \mu) \wedge \nu_\psi) = \alpha_M((f_\bullet \widetilde{\mu}) \wedge \nu_\psi). \tag{4.69}$$

In particular, $f_\bullet(U_N) \Subset \pi_\bullet(U_N)$. So we can iterate $f_\bullet$ on U_N.

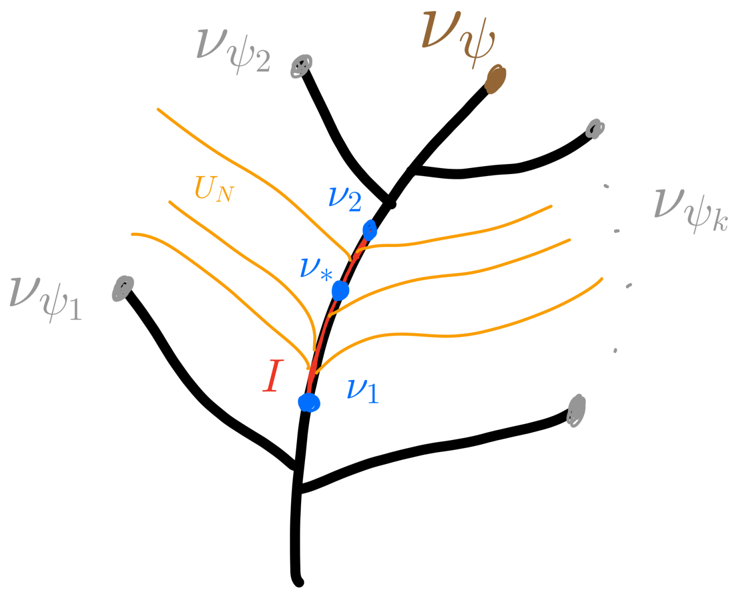


Figure 4.5: An $f_\bullet$-invariant open subset of $\mathcal{V}_\infty$, irrational case

Proposition 4.2.14. *For every $\mu \in U_N, f_\bullet^n \mu \to \nu_*$ for the weak topology.*

Proof. Let $\mu \in U_N$ and let $\widetilde{\mu} := \mu \wedge \nu_2$. We have $f_\bullet^n \widetilde{\mu} \to \nu_*$ for the strong topology by Lemma 4.2.13. By equation (4.66), we have $f_\bullet^n \mu \wedge \nu_2 = f_\bullet^n \widetilde{\mu} \wedge \nu_2$. Therefore for $\varphi \in \mathcal{O}_{X_N, p_N}$ irreducible and for n large enough, $\mathcal{F}_\bullet^n \mu \wedge \nu_\varphi = f_\bullet^n \widetilde{\mu} \wedge \nu_\varphi$. Therefore,

$$f_\bullet^n \mu(\varphi) = \alpha_N(f_\bullet^n \mu \wedge \nu_\varphi) m_N(\varphi) \tag{4.70}$$

$$= \alpha_N(f_\bullet^n \widetilde{\mu} \wedge \nu_\varphi) m_N(\varphi) \tag{4.71}$$

$$= f_\bullet^n \widetilde{\mu}(\varphi) \xrightarrow[n \to +\infty]{} \nu_*(\varphi). \tag{4.72}$$

$\square$

Now pick a completion X above X_N such that for $i = 1,2$, the center of ν_i is a prime divisor E_i at infinity such that E_1 and E_2 intersect at a unique point p. We have $c_X(\nu_*) = p$. The open set $U_N \in \mathcal{V}_{X_n}(p_N, E_N)$ is the image of $\mathcal{V}_X(p)$. Since $f_\bullet U_N \Subset \pi_\bullet(U_N)$, this shows that $f_* \mathcal{V}_X(p) \subset \mathcal{V}_X(p)$. Therefore by Lemma 4.1.12 the lift $f : X \dashrightarrow X$ is defined at p, $f(p) = p$ and since $f_\bullet$ contracts the segment $[\nu_1, \nu_2]$ we have that f contracts E_1 and E_2 to p. We have for every $\mu \in \mathcal{V}_X(p; \mathfrak{m}_p), f_\bullet^n \mu \to \nu_*$ by Proposition 4.2.14.

If Y is a completion above X, then $c_Y(\nu_*) = F_1 \cap F_2$ where F_i is a prime divisor at infinity because ν_* is irrational. The segment $[\nu_{F_1}, \nu_{F_2}]$ is a subsegment of $[\nu_{E_1}, \nu_{E_2}]$ and the same proof applies. This shows that Y satisfies also Proposition 4.2.12. □

4.2.3 Attractingness of ν_*, the divisorial case

Suppose that ν_* is divisorial and let X be a completion such that the center of ν_* on X is a prime divisor E at infinity. Since $f_* \operatorname{ord}_E = \lambda_1 \operatorname{ord}_E$ we have that f induces a rational selfmap of E.

Lemma 4.2.15. *Either there exists an integer $N > 0$ such that f^N admits a noncritical fixed point on E, or E is an elliptic curve and $f_{|E}$ is a translation by a non-torsion element of E.*

Proof. The rational transformation f induces a rational selfmap on E. If E is rational, then $E \simeq \mathbf{P}^1$ and it admits a noncritical fixed point. If E is of general type, then some iterate of f induces the identity on E. Finally, if E is an elliptic curve, then E is isomorphic to $\mathbf{C}/\Lambda$ for some lattice Λ, f lifts to a map $F : z \in \mathbf{C} \mapsto az + b$. If $a = 1$, then F is a translation. Otherwise F and hence $f_{|E}$ admits a noncritical fixed point. □

Suppose $\operatorname{char} \mathbf{k} = 0$ and $\mathbf{k} = \mathbf{C}$. In the case where $f_{|E}$ is not a translation by a non-torsion element on an elliptic curve, f defines a holomorphic fixed point germ at p and we can proceed as in [FJ07] §5.2 to show that there exists a completion X that contains a prime divisor E_0 at infinity such that $p = E \cap E_0$ and $f_\bullet$ maps the segment of monomial valuations $[\nu_E, \nu_{E_0}]$ strictly into itself. Here is how to proceed.

Set $X_0 = X, p_0 = p$. Define the sequence of completions (X_n) as follows: $\pi_n : X_{n+1} \to X_n$ is the blow up of X_n at p_n and p_{n+1} is the intersection point of the strict transform of E with the exceptional divisor of π_{n+1}. We still denote by E its strict transform in every X_n. For every n, we have $f_{|E}(p_n) = p_n$ and if $f : X_n \dashrightarrow X$ is defined at p_n, we have $f(p_n) = p$. We apply Theorem 3.1.9 to get

Proposition 4.2.16. *There exists integers $N \geqslant M$ such that the lift $f : X_N \dashrightarrow X_M$ is defined at p_N, $f(p_N) = p_M$. Furthermore, there exists local coordinates $(x,y),(z,w)$ respectively at*

p_N, p_M such that $x = 0$ and $z = 0$ are local equations of the strict transform of E in X_N and X_M respectively and f is monomial in these coordinates.

The proof is the same as in Proposition 4.2.11.

Proposition 4.2.17. *If ν_* is divisorial, there exists a completion X such that*

(1) $c_X(\nu_*)$ is a prime divisor E at infinity.

(2) E intersects another prime divisor E_0 at infinity.

(3) Up to replacing f by an iterate, $f : X \dashrightarrow X$ is defined at p, $f(p) = p$.

(4) p is a noncritical fixed point of $f_{|E}$.

(5) f leaves E invariant and contracts E_0 to p.

(6) Define $f_\bullet : \mathcal{V}_X(p;E) \to \mathcal{V}_X(p;E)$, then for all $\mu \in c\mathcal{V}_X(p;E), f_\bullet^n \mu \to \mathrm{ord}_E$ for the weak topology.

If $\pi : (Y, \mathrm{Exc}(\pi)) \to (X, p)$ is a completion exceptional above p, then all the item above remain true in Y.

Proof. Let $N \geqslant M$ be as in Proposition 4.2.16. Let $F : X_N \dashrightarrow X_M$ be the lift of f. We can suppose that $N \geqslant M$ and denote by $\pi : X_N \to X_M$ the morphism of completions. We therefore have a map $f_\bullet : \mathcal{V}_Y(p_N, E) \to \mathcal{V}_X(p_M, E)$. Again, the tree $\mathcal{V}_Y(p_N, E)$ is a subtree via the map $\pi_\bullet$ and they are both rooted at the divisorial valuation ord_E.

Let $(x,y), (z,w)$ be the local coordinates at p_N and p_M respectively given by Proposition 4.2.16. We have that $x = 0$ is a local equation of E in X_N and $z = 0$ is a local equation of E in X_M.

$$f(x,y) = \left(x^a y^b, x^c y^d\right). \tag{4.73}$$

Since we know that E is not contracted by f we actually have $c = 0$. We can therefore write

$$\forall \nu \in \mathcal{V}_{X_N}(p_N; E), \quad f_\bullet(\nu) = \frac{f_* \nu}{a + b\nu(y)}. \tag{4.74}$$

(Recall from §3.4 that $\mathcal{V}_{X_N}(p_n; E)$ is defined by the normalization $\nu(E) = 1$). We have

$$f_\bullet[\mathrm{ord}_E, \nu_y] \subset [\mathrm{ord}_E, \nu_w] \tag{4.75}$$

and the map is given by the following formula

$$f_\bullet \nu_{1,s} = \nu_{1,\frac{sd}{a+sb}}. \tag{4.76}$$

As in the irrational case, we can consider the matrix M_f and M_π and study the type of the Möbius transformation induced by $M_\pi^{-1} \circ M_f$. Since ord_E is a fixed point, we show that it is not repelling on the segment $[\mathrm{ord}_E, \nu_y]$.

Let $\nu_0 \in [\mathrm{ord}_E, \nu_w]$ be a divisorial valuation. We have $f_\bullet([\mathrm{ord}_E, \nu_0]) \subset [\mathrm{ord}_E, \nu_w]$. Let $U_N = \{\mu : \mathrm{ord}_E \leqslant \mu \wedge \nu_y < \nu_0\} \subset \mathcal{V}_{X_N}(p_N; E)$. It is clear that $\nu_y \notin U_N$. Let $\psi \in \widehat{O_{X_M,p_M}}$ be irreducible such that $\nu_\psi > f_\bullet([\mathrm{ord}_E, \nu_0])$. Let $\psi_1, \cdots, \psi_r, \in \widehat{O}_{X_N,p_N}$ be irreducible such that $f^*\psi = \psi_1 \cdots \psi_r$. Up to shrinking the segment $[\mathrm{ord}_E, \nu_0]$ we can suppose that none of the ν_{ψ_i} belong to U_N (See Figure 4.6). If this is the case, then for all $\mu \in U_N$, set $\widetilde{\mu} = \mu \wedge \nu_0$, then for all i

$$\mu \wedge \nu_{\psi_i} = \widetilde{\mu} \wedge \nu_{\psi_i}, \quad \mu \wedge \nu_y = \widetilde{\mu} \wedge \nu_y. \tag{4.77}$$

Now, for all $\mu \in U_N$, by Equation (4.74) and Proposition 3.4.3

$$(f_\bullet \mu)(\psi) = \frac{\mu(f^*\psi)}{a + b\mu(y)} = \frac{\sum_k \alpha_N(\mu \wedge \nu_{\psi_k}) m(\psi_k)}{a + b\mu(y)}. \tag{4.78}$$

By Equation (4.77), we get

$$(f_\bullet \mu)(\psi) = (f_\bullet \widetilde{\mu})(\psi). \tag{4.79}$$

This means that

$$\forall \mu \in U_N, \quad \alpha_M((f_\bullet \mu) \wedge \nu_\psi) = \alpha_M((f_\bullet \widetilde{\mu}) \wedge \nu_\psi). \tag{4.80}$$

If $\nu \in \mathcal{V}_\infty$ is divisorial such that $Z_\nu \cdot \theta^* > 0$, then $\frac{1}{\lambda_1^n} f_*^n \nu \to \nu_*$ by Proposition 4.2.2. Then, there exists $N_0 \geqslant 1$ such that for $n \geqslant N_0$, $\frac{1}{\lambda_1^n} f_*^n \nu \in U_N$. Replace ν by $\frac{1}{\lambda_1^{N_0}} f_*^{N_0}(\nu)$. If ord_E was a repelling fixed point, then we could not have $f_\bullet^n \nu \to \nu_*$ by Equation (4.77) and (4.80). Therefore, we can pick ν_0 such that $f_\bullet[\mathrm{ord}_E, \nu_0] \Subset \pi_\bullet[\mathrm{ord}_E, \nu_0]$. In that case $f_\bullet(U_N) \Subset \pi_\bullet(U_N)$. So we can iterate $f_\bullet$ on U_N.

Proposition 4.2.18. *For all $\mu \in U_N$, $f_\bullet^n \mu \to \mathrm{ord}_E$ for the weak topology.*

Proof. The proof is similar to the proof of Proposition 4.2.14. Let $\mu \in U_N$ and set $\widetilde{\mu} = \mu \wedge \nu_0$. Since ord_E is an attracting fixed point for $f_\bullet$ and $f_\bullet[\mathrm{ord}_E, \nu_0] \Subset [\mathrm{ord}_E, \nu_0]$, we have $f_\bullet^n \widetilde{\mu} \to \mathrm{ord}_E$ for the strong topology. Then, by Equation (4.80), $f_\bullet^n \mu \wedge \nu_0 = f_\bullet^n \widetilde{\mu}$. Let $\varphi \in O_{X_N,p_N}$ be irreducible

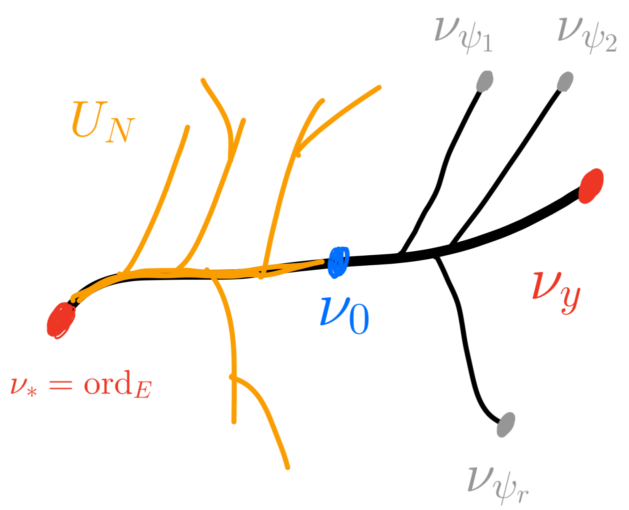


Figure 4.6: An $f_\bullet$-invariant open subset of $\mathcal{V}_\infty$, divisorial case

such that φ is not a local equation of E, then for n large enough

$$f_\bullet^n\mu(\varphi) = \alpha_E(f_\bullet^n\mu \wedge \nu_\varphi)m_E(\varphi) \tag{4.81}$$

$$= \alpha_E(f_\bullet^n\widetilde{\mu} \wedge \nu_\varphi)m_E(\varphi) \tag{4.82}$$

$$= \alpha_E(f_\bullet^n\widetilde{\mu})m_E(\varphi) \xrightarrow[n\to+\infty]{} 0 \tag{4.83}$$

□

Let E_0 be the divisor associated to the divisorial valuation ν_0 and let Z be a completion such that $c_Z(\nu_0)$ is the divisor E_0 and such that $E_0 \cap E$ is a point p. Then, the open subset U_N corresponds to $\mathcal{V}_Z(p)$ and we have $f_*\mathcal{V}_Z(p) \subset \mathcal{V}_Z(p)$. By Lemma 4.1.12, we have that the lift $\widehat{f}: Z \to Z$ is regular at p, $\widehat{f}(p) = p$ and since we know that $f_\bullet\nu_0 < \nu_0$ and $f_*\,\mathrm{ord}_E = \lambda_1(f)\,\mathrm{ord}_E$ we have that $\widehat{f}$ contracts E_0 at p, E is f-invariant and for all $\mu \in \mathcal{V}_Z(p;E), f_\bullet^n\mu \to \nu_*$ by Proposition 4.2.18.

If $\pi : (Z', \mathrm{Exc}(\pi)) \to (Z, p)$ is a completion exceptional above p, then $\mathrm{Exc}(\pi)$ is a tree of rational curves, let E_0' be the irreducible component of $\mathrm{Exc}(\pi)$ that intersect the strict transform of E. Then E_0' corresponds to a divisorial valuation ν_0' such that $\mathrm{ord}_E = \nu_* < \nu_0' < \nu_0$ and all the proofs above apply so Proposition 4.2.17 holds also for Z'. □

Lemma 4.2.19. *When ν_* is divisorial, $\lambda_1 \leqslant \lambda_2$, with equality if and only if $f_{|E} : E \to E$ has degree 1.*

Proof. Let X be a completion such that the center of ν_* is a prime divisor E at infinity. Since

$f_*\nu_* = \lambda_1\nu_*$, we have that $f^*E = \lambda_1 E + R$ where R denotes an effective divisor supported at infinity. Now, we also have $f_*E = dE + R'$. From the equality $f_* \circ f^* = \lambda_2 \,\mathrm{id}$, we get that $\lambda_1 d \leqslant \lambda_2$. In particular, $\lambda_1 \leqslant \lambda_2$. □

4.2.4 Local normal form of f

We are now ready to proof Theorem 4.2.1.

Proof of Theorem 4.2.1. Suppose ν_* is infinitely singular. From Proposition 4.2.3, there exists a completion X such that $c_X(\nu_*) =: p \in E$ is a free point, $f : X \dashrightarrow X$ is defined at p and $f_*(\mathcal{V}_X(p)) \Subset \mathcal{V}_X(p)$. We need to show that the germ of holomorphic functions induced by f at p is contracting and rigid. It is clear that $E \in \mathrm{Crit}(f)$ (Recall the notations from §3.1.5). If $\mathrm{Crit}(f)$ admits another irreducible component, it induces a curve valuation in $\mathcal{V}_X(p)$, we can blow up p to get another completion above X satisfying Proposition 4.2.3 such that $\mathrm{Crit}(f)$ does not admit any other component than E. Thus, f is rigid at p it remains to show that it is contracting. Let (x,y) be local coordinates at p such that $x = 0$ is a local equation of E. Since $\nu_*(E) > 0$ and $f_*\nu_* = \lambda_1\nu_*$ we get that $f^*x = x^{\lambda_1}\varphi$ with $\varphi \in \mathcal{O}_{X,p}^\times$ and $\lambda_1 \geqslant 2$. Now, since E is contracted by f, we get that $f^*y = x\psi$ with $\psi \in \mathcal{O}_{X,p}$ but since f is dominant we have $\psi \in \mathfrak{m}_p$. Hence, we get that

$$f(x,y) = (x^{\lambda_1}\varphi, x\psi) \tag{4.84}$$

with $\varphi \in \mathcal{O}_{X,p}^\times$ and $\psi \in \mathfrak{m}_p$. Consider the norm $||(x,y)|| = \max(|x|,|y|)$ associated to the coordinates x,y and let U^* be the ball of center p and radius $\varepsilon > 0$. If $\varepsilon > 0$ is small enough, then U^* is f-invariant and $f(U^*) \Subset U^*$, so f is contracting at p. Finally, there are no f-invariant germ of curves at p. Indeed, if $\varphi \in \widehat{\mathcal{O}_{X,p}}$ is f-invariant, then $f_\bullet\nu_\varphi = \nu_\varphi$. But we have by Proposition 4.2.3 that $f_\bullet^n\nu_\varphi \to \nu_*$ and this is a contradiction. Thus, we get that f has the local normal form of (3.3) with $a = \lambda_1$. If $\mathbf{k} = \mathbf{C}$, Looking at the classification of the rigid contracting germs in dimension 2, we see that f is in Class 4 of Table 1 in [Fav00] hence of type (3.2) thus there exists local analytic coordinates (z,w) at p

$$\widehat{f}(z,w) = (z^a, \lambda z^c w + P(z)) \tag{4.85}$$

where $a \geqslant 2, c \geqslant 1, \lambda \in \mathbf{C}^\times$ and P is a polynomial such that $P(0) = 0$. Since E is the only germ of curve contracted by f (all the other germs of analytic curves are contained in X_0 they cannot be contracted to p by f since f is an endomorphism of X_0), we have that $z = 0$ is a local equation of E. We infer $\nu_*(z) = \nu_*(E) > 0$ and therefore

$$\lambda_1 \nu_*(z) = f_* \nu_*(z) = \nu_*(z^a) = a \cdot \nu_*(z); \tag{4.86}$$

thus $\lambda_1 = a \in \mathbf{Z}_{\geqslant 0}$. Furthermore, since f does not have any invariant germ of analytic curve, we get that $P \not\equiv 0$.

Suppose now that ν_* is irrational, by Proposition 4.2.12, there exists a completion X of X_0 such that the lift $f : X \dashrightarrow X$ is defined at $p = c_X(\nu_*)$, X contains two divisors at infinity E, F such that $p = E \cap F$ and $\widehat{f}$ contracts both E and F at p. It remains to show that f is contracting and rigid at p. First we can suppose up to further blow ups that $\mathrm{Crit}(f) \cap X_0 = \varnothing$. Therefore f is rigid, now since both E, F are contracted to p, f is contracting. Finally, there are no f-invariant germs of curves at p since for all $\mu \in \mathcal{V}_X(p; \mathfrak{m}_p), f_\bullet^n \mu \to \nu_*$ by Proposition 4.2.12. Let (z, w) be local coordinates at p associated to (E, F). We have that f is of the pseudomonomial form

$$f(z, w) = \left(z^a w^b \varphi, z^c w^d \psi \right). \tag{4.87}$$

with $\varphi, \psi \in \mathcal{O}_{X,p}^\times$ and $a, b, c, d \geqslant 1$ since E, F are contracted to p. Notice that $f_* \operatorname{ord}_E = \nu_{a,b}$ and $f_* \operatorname{ord}_F = \nu_{c,d}$. Consider the segment of monomial valuations I centered at p inside $\mathcal{V}_X(p; \mathfrak{m}_p)$ we have that $f_\bullet : I \to I$ is injective, therefore (a, b) is not proportional to (c, d). Furthermore the open subset U^* corresponding to the ball of radius $\varepsilon > 0$ is f-invariant for $\varepsilon > 0$ small enough and $f(U^*) \Subset U^*$. In that case, we show that $\lambda_1(f)$ is the spectral radius of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, hence an algebraic integer of degree 2. Indeed, $\nu_* = \nu_{s,t}$ where (s, t) is an eigenvector of A for the eigenvalue λ_1. Since ν_* is irrational, we have $s/t \notin \mathbf{Q}$ and therefore $\lambda_1 \notin \mathbf{Q}$. Now, when we iterate f, we get that f^n is pseudomonomial with monomials given by the matrix A^n, hence we get

$$\lambda_1^n \begin{pmatrix} \nu_*(z) \\ \nu_*(w) \end{pmatrix} = A^n \begin{pmatrix} s \\ t \end{pmatrix} \tag{4.88}$$

If $\mathbf{k} = \mathbf{C}$, then f is in the class 6 of Table 1 of [Fav00]. Hence it can be made monomial and there exists local analytic coordinates x, y at p such that

$$f(x, y) = (x^a y^b, x^c y^d) \tag{4.89}$$

It is clear that (x, y) is associated to (E, F) since these are the only two germs of curves contracted by f.

Now finally, suppose that ν_* is divisorial. Take a completion X as in Proposition 4.2.17. Let $p = E \cap E_0$ with $\nu_* = \mathrm{ord}_E$. The lift $f : X \dashrightarrow X$ is defined at p. Up to further blow-ups we can suppose that $\mathrm{Crit}(f) \cap X_0 = \varnothing$. Therefore, $\mathrm{Crit}(f) \subset E \cup E_0$ which is totally invariant as $f_* \mathcal{V}_X(p) \Subset \mathcal{V}_X(p)$ so f is rigid at p. There are no f-invariant germs of curves apart from E at p since for all $\mu \in \mathcal{V}_X(p;E), f_\bullet^n \mu \to \mathrm{ord}_E$ by Proposition 4.2.17. Let (x,y) be local coordinates at p associated to (E,E_0). Since $f_* \mathrm{ord}_E = \lambda_1 \mathrm{ord}_E$ with $\lambda_1 \geqslant 2$ we have $f^* x = x^{\lambda_1} \varphi$ with $\varphi \in O_{X,p}$. Since no germ of curve is sent to E apart from E_0, we have that up to multiplying x by a constant that $f^* x = x^{\lambda_1} y^b (1+\varphi)$ with $\varphi \in O_{X,p}$. Then, E_0 is contracted to p so $f^* y = y^c \psi$ with $\psi \in O_{X,p}^\times$ and $c = 1$ since p is a noncritical fixed point of $f_{|E}$. Hence, in these coordinates the local normal form of f is (3.6):

$$\widehat{f}(x,y) = \left(x^a y^b (1+\varphi), \lambda y(1+\psi)\right) \tag{4.90}$$

with $a = \lambda_1 \geqslant 2, b \geqslant 1, \lambda \in \mathbf{C}^\times$ and $\varphi(0) = \psi(0) = 0$. $\square$

4.3 General case

In this section, we extend Theorem A to the general case, without assuming $A^\times = \mathbf{k}^\times$ or $\mathrm{Pic}^0(X_0) = 0$. We rely on the universal property of the quasi-Albanese variety (see [Ser01]), as well as on the geometric properties of subvarieties of quasi-abelian varieties (see [Abr94]).

4.3.1 Quasi-Albanese variety and morphism

Let G be an algebraic group over $\mathbf{k}$ with $\mathbf{k}$ algebraically closed. We say that G is a *quasi-abelian* variety if there exists an algebraic torus $T = \mathbb{G}_m^r$, an abelian variety A, and an exact sequence of $\mathbf{k}$-algebraic groups

$$0 \to T \to G \to A \to 0. \tag{4.91}$$

Theorem 4.3.1 (see [Ser01], Théorème 7)**.** *Let X be a variety over $\mathbf{k}$, then there exists a quasi-abelian variety G and a morphism $q : X \to G$ such that for any quasi-abelian variety G' and any morphism $\varphi : X \to G'$ there exists a unique morphism $g : G \to G'$ and a unique $b \in G'$ such that*

$$\varphi = g \circ q.$$

Moreover, g is the composition of a homomorphism $L_g : G \to G'$ of algebraic groups and a translation $T_g : G' \to G'$ by some element $b \in G'$.

Such a G is unique up to (a unique) isomorphism. It is called the *quasi-Albanese variety* of X and it will be denoted by $\mathrm{QAlb}(X)$; the universal morphism $q : X \to \mathrm{QAlb}(X)$ is “the” *quasi-Albanese morphism* (it is unique up to post-composition with an isomorphism of G).

Proposition 4.3.2. *Let X_0 be an affine variety. Then $\mathbf{k}[X_0]^\times = \mathbf{k}^\times$ and $\mathrm{Pic}^0(X_0) = 0$ if and only if $\mathrm{QAlb}(X_0) = 0$.*

Proof. Let $G = \mathrm{QAlb}(X_0)$ and $q : X_0 \to G$ be a quasi-Albanese morphism. Let

$$0 \to T \to G \xrightarrow{\pi} A \to 0. \tag{4.92}$$

be an exact sequence, as in Equation (4.91). Let X be a completion of X_0 such that $\pi \circ q$ extends to a regular map $\pi \circ q : X \to A$.

Assume $\mathbf{k}[X_0]^\times = \mathbf{k}^\times$ and $\mathrm{Pic}^0(X_0) = 0$. Then, $\pi \circ q(X_0)$ is a point in A, and composing q with a translation of G, we can assume that this point is the neutral element of A. Then,

$q(X_0) \subset T$, so q is a regular map from X_0 to an algebraic torus, and $\mathbf{k}[X_0]^\times = \mathbf{k}^\times$ implies that $q(X_0)$ is a point. This shows that $\mathrm{QAlb}(X_0)$ is a point.

Now, suppose that $\mathbf{k}[X_0]^\times \neq \mathbf{k}^\times$, then any non-constant invertible function $X_0 \to \mathbf{k}^\times$ provides a dominant morphism to a 1-dimensional torus, so $\dim(\mathrm{QAlb}(X_0)) \geqslant 1$ by the universal property. And if $\mathrm{Pic}^0(X_0) \neq 0$, the Albanese morphism also shows that $\dim(\mathrm{QAlb}(X_0)) \geqslant 1$. This concludes the proof. $\square$

In the following, we show that if X_0 is an irreducible normal affine surface with non-trivial quasi-Albanese variety and f is a dominant endomorphism of X_0, then $\lambda_1(f)$ is a quadratic integer. See Proposition 4.3.6 below. We will rely on the following result.

Theorem 4.3.3 (Theorem 3 of [Abr94]). *Let Q be a quasi-abelian variety and let V be a closed subvariety of Q. Let K be the maximal closed subgroup of Q such that $V + K = V$. Then, the variety V/K is of general type.*

4.3.2 Dynamical degree in presence of an invariant fibration

Proposition 4.3.4 (Stein Factorization). *Let X, S be projective varieties and let $f : X \to X$ be a rational transformation. Suppose that there exists $\varphi : X \to S$ and $g : S \to S$ such that the following diagram commutes,*

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, "\varphi"] & X \arrow[d, "\varphi"] \\ S \arrow[r, "g"] & S \end{tikzcd}$$

Then there exists a variety $\widetilde{S}$ and morphisms $\psi : X \to \widetilde{S}$, $\pi : \widetilde{S} \to S$ such that

- $\varphi = \pi \circ \psi$,
- *π is finite and ψ has connected fibers*
- *there exists a rational transformation $\widetilde{g} : \widetilde{S} \dashrightarrow \widetilde{S}$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, "\psi"] & X \arrow[d, "\psi"] \\ \widetilde{S} \arrow[r, dashed, "\widetilde{g}"] \arrow[d, "\pi"] & \widetilde{S} \arrow[d, "\pi"] \\ S \arrow[r, "g"] & S \end{tikzcd}$$

commutes.

Proof. The existence of $\widetilde{S}$ along with π and ψ is due to Stein Factorization theorem: It is known that one can take $\widetilde{S} = \operatorname{Spec}_S \varphi_* O_X$ where Spec_S is the relative Spec; that is for every affine open subset U of S, one has

$$\pi^{-1}(U) \simeq \operatorname{Spec} O_X(\varphi^{-1}(U)). \tag{4.93}$$

Now to construct $\widetilde{g}$, take affine open subsets U and V of S such that $U \subset g^{-1}(V)$. Suppose also that $\varphi^{-1}(U)$ and $\varphi^{-1}(V)$ do not contain any indeterminacy of f. To construct

$$\widetilde{g}_{|\pi^{-1}(U)} : \pi^{-1}(U) \to \pi^{-1}(V) \tag{4.94}$$

we use the map $f^* : O_X(\varphi^{-1}(V)) \to O_X(\varphi^{-1}(U))$ induced by f; this is well defined since $\varphi^{-1}(U) \subset f^{-1}(\varphi^{-1}(V))$. It is clear that $\psi \circ f = \widetilde{g} \circ \psi$. □

Proposition 4.3.5. *Let S be a quasiprojective surface and f be a dominant endomorphism of S. Suppose there exists a quasiprojective curve C with a dominant morphism $\pi : S \to C$ and an endomorphism $g : C \to C$ such that $\pi \circ f = g \circ \pi$. Then, the first dynamical degree of f is an integer.*

Proof. Let X be a completion of S; f extends to a rational transformation of X. We can also suppose that C is a projective curve, and then we apply Theorem 4.3.4 to suppose also that π has connected fibers.

Let P be a general point of C and H an ample divisor of X. We have by [DN11, Tru15] that

$$\lambda_1(f) = \max\left(\lambda_1(g), \lambda_1(f_{|\pi}))\right) \tag{4.95}$$

where $\lambda_1(g)$ is the integer given by the topological degree of g and

$$\lambda_1(f_{|\pi}) := \lim_n \left(H \cdot (f^n)_* \pi^{-1}(P)\right)^{1/n}. \tag{4.96}$$

Since C is a curve and π is dominant we have that π is flat ([Har77] Proposition III.9.7) so for any point $P \in C$,

- $\pi^{-1}(P)$ is an irreducible curve C_P and the topological degree of $f : C_P \to C_{g(P)}$ is an integer d that does not depend on P
- $d \cdot d_{\text{top}}(g) = \lambda_2(f)$.

Indeed, consider the following 0-cycle in $S \times S$:

$$\alpha(P) = (\pi_1^* C_P) \cdot (\pi_2^* H) \cdot \Gamma_f \tag{4.97}$$

where $\pi_1, \pi_2 : S \times S \to S$ are the two projections and Γ_f is the graph of f. The degree of $\alpha(P)$ is

$$\deg \alpha = (H \cdot C_{g(P)}) \cdot \deg(f : C_P \to C_{g(p)}). \tag{4.98}$$

Now, since C is a curve the morphism $\pi \circ \pi_1 : S \times S \to C$ is flat, therefore $\deg(\alpha(P))$ does not depend on P ([Ful98] §20.3) and since π is flat, the intersection number $(H \cdot C_P)$ does not depend on P either. Therefore, $\deg(f : C_P \to C_{g(P)})$ is an integer d independent of P. Hence, we infer

$$\lambda_1(f_{|\pi}) = \lim_n \left(H \cdot (f^n)_* \pi^{-1} P\right) = d \cdot \lim_n \left(H \cdot \pi^{-1} P\right)^{1/n} = d \tag{4.99}$$

and we get that $\lambda_1(f)$ is the integer $\max(d, \lambda_1(g))$. □

4.3.3 Dynamical degree when the quasi-Albanese variety is non-trivial

The goal of this section is to show the following proposition.

Proposition 4.3.6. *Let X_0 be an irreducible normal affine surface and f a dominant endomorphism of X_0. Suppose that* $\mathrm{QAlb}(X_0)$ *is non-trivial, then $\lambda_1(f)$ is an integer or a quadratic integer.*

Set $Q_0 = \mathrm{QAlb}(X_0)$ and let $q : X_0 \to Q_0$ be a quasi-Albanese morphism. Let $V = \overline{q(X_0)}$ be the closure of the image of X_0 in Q_0. By the universal property, there exists an endomorphism g of Q_0 such that

$$q \circ f = g \circ q \tag{4.100}$$

$$g(z) = L_g(z) + b_g \tag{4.101}$$

for some algebraic homomorphism $L_g : Q_0 \to Q_0$ and some translation $z \mapsto z + b_g$ (here, we denote the group law by addition). In particular $g_{|V}$ defines a regular endomorphism of $q(X_0)$ and since f is dominant, so is $g_{|V}$. As in Theorem 4.3.3, set $K = \{x \in Q_0 \, ; \, x + V = V\}$. Then, denote by $\pi_V : V \to V/K$ the canonical projection onto the quotient.

Proposition 4.3.7. *There exists an endomorphism $g' : V/K \to V/K$ such that $g' \circ \pi_V = \pi_V \circ g_{|V}$.*

Proof. We have to show that $g_{|V}$ is compatible with the quotient map. Take $v \in V$ and $k \in K$. Since $v + k \in V$, $g(v+k) \in V$. Now,

$$g(v+k) = L_g(v+k) + b_g = L_g(v) + L_g(k) + b_g = g(v) + L_g(k). \tag{4.102}$$

Thus, $L_g(k) + g(V) \subset V$. Taking the closure and knowing that $g_{|V}$ is dominant, we have $L_g(k) + V = V$. Therefore, $L_g(k) \in K$ and $g_{|V}$ is compatible with the quotient modulo K. □

Case $\dim V/K = 2$ – In that case, the map $\pi_V \circ q : X \to V/K$ is generically finite. Since V/K is of general type, g' has finite order: there is some positive integer n such that $(g')^n = Id_{V/K}$. Thus, f is also a finite order automorphism, and $\lambda_1(f) = 1$.

Case $\dim V/K = 1$ – In that case $\pi_V \circ q$ induces a fibration of X_0 over a curve of general type and we conclude that $\lambda_1(f)$ is an integer by Proposition 4.3.5 .

Case $\dim V/K = 0$ – This means that V is equal to K up to translation. Therefore, by the universal property of the quasi-Albanese variety, $K = V = Q_0$ and $q : X_0 \to Q_0$ is dominant.

If $\dim Q_0 = 1$, then f preserves a fibration over a curve and Proposition 4.3.5 implies again that $\lambda_1(f)$ is an integer.

Suppose now that $\dim Q_0 = 2$. Then q is generically finite, so that $\lambda_1(f) = \lambda_1(g)$. A priori, there are three possibilities.

The first case is when Q_0 is a 2-dimensional multiplicative torus. In that case, g is a monomial endomorphism: in coordinates, $g(x,y) = (\alpha x^a y^b, \beta x^c y^d)$ for some α, β in $\mathbf{k}^\times$ and some integers (a,b,c,d) with $ad - bc \neq 0$; then, $\lambda_1(g)$ is the spectral radius of the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4.103}$$

Thus, it is the maximum of the moduli $|\lambda|$, $|\lambda'|$ of the eigenvalues of this matrix and, as such, it is an algebraic integer of degree $\leqslant 1$.

The second possibility is that Q_0 is an extension of an elliptic curve A by a one dimensional torus $\mathbb{G}_m$; then, the projection $Q_0 \to A$ is g-equivariant, and Proposition 4.3.5 implies that $\lambda_1(g)$ is an integer.

The third and last possibility is that Q_0 is an abelian surface. Let X be a (good) completion of X_0 such that q extends to a regular morphism $q_X : X \to Q_0$. Pulling back a regular 2-form by q, we see that the Kodaira dimension of X is non-negative. If it is equal to 2, a positive iterate of f is the identity, so $\lambda_1(f) = 1$. If it is equal to 1, the Kodaira-Iitaka fibration gives an f-invariant fibration and Proposition 4.3.5 implies that $\lambda_1(g)$ is an integer. Thus, we may assume that the Kodaira dimension of X vanishes. Since the dimension of the Albanese variety of X is 2, the classification of surfaces implies that X is a blow-up of its Albanese variety Q_0, and q_X is the inverse of this blow-up. In particular, q_X is a birational morphism, it is one-to-one on the complement of its exceptional locus $\mathrm{Exc}(q_X)$.

Set $B = q_X(\partial_X X_0)$. Since $\partial_X X_0$ supports an ample divisor, B is a curve ($\partial_X X_0$ cannot be contracted by q_X).

Let p be an indeterminacy point of $f : X \dashrightarrow X$ and C be the total transform of p by f. Since C is a union of rational curves and abelian surfaces do not contain rational curves, $q_X(C)$ is a point. Moreover, this point must be contained in B. Thus, $g(B) = q_X(f(\partial_X X_0))$ is contained in B, and B is g-invariant. Also, since X_0 does not contain any complete curve, each component of $\mathrm{Exc}(q_X)$ must intersect $\partial_X X_0$, and $q_X(\mathrm{Exc}(q_X)) \subset B$.

Composing q by a translation we may assume that B contains the neutral element o of Q_0. Let B_0 be an irreducible component of B containing o. Then, some positive iterate g^n of g preserves B_0. If the genus of B_0 is $\geqslant 2$, $g^n_{B_0}$ has finite order and B_0 generates the group Q_0, so g has finite order, so does f, and $\lambda_1(f) = 1$. Thus, we can now assume that the genus of each component of B is 1, each component being a translate of some elliptic curve.

If B is irreducible, the quotient map $Q_0 \to Q_0/B$ is g-equivariant and we conclude again by Proposition 4.3.5. If there is an irreducible component B_0 of B with $g(B_0) = B_0 + b$ for some $b \in Q_0$, we conclude in the same way.

Now, we can assume that B is reducible and $g(B_0)$ is not a translate of B_0. There is an integer $n \geqslant 1$ such that the curve B_0 is periodic of period n, i.e. $B_0, B_1 := g(B_0), ..., B_{n-1} = g^{n-1}(B_0)$ are pairwise distinct, and $g^n(B_0) = B_0$. Taking some further iterate g^{nm}, and changing the position of the neutral element, we can suppose that o is a point of intersection of B_0 and B_1 and that $g^{nm}(o) = o$. Let d denote the degree of g^n along B_0; since g maps B_0 to B_1, d is also the degree of g^n along B_1. If $d = 1$, then f and g have $\lambda_1 = \lambda_2 = 1$.

Let us now assume $d \geqslant 2$ and derive a contradiction. On B_0, the pre-images $(g^{nm}_{|B_0})^{-k}(z)$ form a dense subset of B_0; the same is true for B_1. The homomorphism $B_0 \times B_1 \to Q_0$ given by addition is an isogeny, so the preimages of any point of Q_0 under the action of g form a dense subset of Q_0. Let z be a point in $\mathrm{Exc}(q_X)$, then its preimages should be dense, but this would imply that f maps some point in the interior of X_0 into the boundary $\partial_X X_0$. This contraction concludes the proof.

4.4 The automorphism case

Here we suppose that X_0 is an irreducible normal affine surface that admits a loxodromic automorphism over a field of characteristic zero. In this situation, we can actually deduce a lot more from the result of Section 4.1. In particular one can first check that X_0 has to be rational, see [DF01] Table 1 Class 5. So the condition $\mathrm{Pic}^0(X_0)$ is automatically satisfied. We change the notation for this section, we will denote θ^* and θ_* by θ^+ and θ^- respectively. So that $(f^{\pm 1})^*\theta^{\pm} = \lambda_1\theta^{\pm}$. By Proposition 4.1.15 and Theorem 4.1.16, we get that

- $\theta^+, \theta^- \in \mathrm{Weil}_\infty(\mathrm{X}_0) \cap \mathrm{L}^2(\mathrm{X}_0)$ and they are both effective.
- $\theta^+ = Z_{\nu_-}$ and $\theta^- = Z_{\nu_+}$ where ν_+ is the eigenvaluation of f and ν_- the eigenvaluation of f^{-1}.

Proposition 4.4.1. *Let $X_0 = \mathrm{Spec} A$ be a rational affine surface such that $A^\times = \mathbf{k}^\times$ and let f be a loxodromic automorphism of X_0, then*

(1) The eigenvaluations ν_+, ν_- of f and f^{-1} respectively are of the same type.

(2) If $\lambda_1 \in \mathbf{Z}_{\geqslant 0}$, then ν_+ and ν_- are infinitely singular.

(3) If $\lambda_1 \in \mathbf{R} \setminus \mathbf{Z}_{\geqslant 0}$ then ν_+ and ν_- are irrational.

Proof. If the eigenvaluation was divisorial, then we would get by Lemma 4.2.19 that $\lambda_1 \leqslant \lambda_2$ and this is absurd because $\lambda_1 > 1$, f being loxodromic. The dichotomy of the type of eigenvaluation follows from Theorem 4.2.1 and the fact that $\lambda_1(f) = \lambda_1(f^{-1})$. □

Corollary 4.4.2. *In that case, the nef eigenclasses θ^- and θ^+ verify*

$$(\theta^-)^2 = (\theta^+)^2 = 0$$

and in any completion X of X_0 one has $(\theta^{\pm}_X)^2 > 0$.

Proof. The equalities $(\theta^-)^2 = (\theta^+)^2 = 0$ come from Theorem 3.2.28 (3.71). Since the eigenvaluations are not divisorial, θ^- and θ^+ are not Cartier divisors by Corollary 4.1.4 therefore for any completion X of X_0, $(\theta^{\pm}_X)^2 > 0$. Indeed, if $(\theta^{\pm}_X)^2 = 0$ then since $\theta^{\pm}$ is nef, we would get $\theta^{\pm}_X = \theta^{\pm}$. □

Let X be a completion of X_0. We have a simple criterion to check whether a divisor at infinity is contracted thanks to Proposition 4.2.2.

Proposition 4.4.3. *Let E be a prime divisor at infinity in a completion X of X_0. If $Z_{\mathrm{ord}_E} \cdot \theta^- > 0$ then there exists $N > 0$ such that f^N contracts E to the point $c_X(\nu_+)$.*

4.4.1 Gizatullin's work on the boundary and applications

In [Giz71a], Gizatullin considers *minimal completions* of affine surface. That is a completion X of X_0 minimal with respect to the following property:

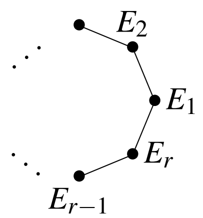
- The boundary $\partial_X X_0$ does not have three prime divisors that intersect at the same point.
- If $\partial_X X_0$ has a singular irreducible component then $\partial_X X_0$ consists only of one irreducible curve with at most one nodal singularity.

For such a completion $\iota : X_0 \hookrightarrow X$, Gizatullin defines the curve $E(\iota)$ as the union of the irreducible components E of $\partial_X X_0$ that are contracted by an automorphism of X_0 (the automorphism depends on E).

We call a *zigzag* a chain of rational curves. That is a sequence $(E_1, \cdots, E_r)$ of rational curves such that $E_i \cdot E_{i+1} = 1, i = 1, \cdots, r-1$ and for all i, j such that $|i-j| \geqslant 2, E_i \cdot E_j = 0$. In particular the dual graph with respect to the E_i's is of the form

We will write $E_1 \rhd E_2 \rhd \cdots \rhd E_r$ for the zigzag defined by $(E_1, \cdots, E_r)$.

A *cycle* of rational curves is a sequence $(E_1, \cdots, E_r)$ of rational curves such that $E_i \cdot E_{i+1} = 1$ and $E_1 \cdot E_r = 1$. The dual graph with respect to the E_i's is of the form



Theorem 4.4.4. *Let $X_0 = \operatorname{Spec} A$ be an irreducible normal affine surface such that $A^\times = \mathbf{k}^\times$ and $\operatorname{Pic}^0(X_0) = 0$. Suppose that X_0 admits an automorphism f with $\lambda_1(f) > 1$. If X is a minimal completion of X_0, one has $E(\iota) = \partial_X X_0$. Furthermore we have two mutually excluding cases*

(1) $\lambda_1(f)$ is an integer and in that case $E(\iota)$ is a zigzag.

(2) $\lambda_1(f)$ is irrational and $E(\iota)$ is a cycle of rational curves.

Furthermore, there exists a completion Y with two distinct points $p_+, p_- \in \partial_Y X_0$ and an integer $N > 0$ such that

- *$f^{\pm 1}(p_\pm) = p_\pm$.*
- *$f^{\pm N}$ contracts $\partial_Y X_0$ to $p_\pm$.*
- *$f^{\pm N}$ has a normal form at $p_\pm$ given by Theorem 4.2.1, it pseudomonomial or monomial in the cycle case and of type* (3.2) *or* (3.3) *in the zigzag case.*
- *In the cycle case, this set of properties remains true if we blow up p_+ or p_-.*
- *In the zigzag case, the set of completions above Y that satisfy these properties is cofinal in the set of all completions above Y.*

The normal form of f at $p_\pm$ is monomial in the cycle case and of the form of Theorem 4.2.1 case (3) in the zigzag case.

This shows Theorem C. We will prove Theorem 4.4.4 in §4.4.2 and 4.4.3. We end this section with some technical result that will be useful in the proof of Theorem 4.4.4.

Lemma 4.4.5. *Let X be a completion of X_0 and let E be a prime divisor at infinity such that $Z_{\mathrm{ord}_E} \cdot \theta^+ = 0$ and E intersects some prime divisor in the support of θ^+_X , then $c_X(\nu_+)$ belongs to E.*

Proof. Since θ^+ is effective and $\mathrm{ord}_E(\theta^+) = 0$ we get $\theta^+ \cdot E > 0$ since E intersects the support of θ^+. This implies by Proposition 3.6.6 that $c_X(\nu_+)$ belongs to E. □

Lemma 4.4.6. *Let Y be a completion of X_0 and E a prime divisor at infinity of Y such that $Z_{\mathrm{ord}_E} \cdot \theta^+ > 0$. If $p \in E \backslash \{c_X(\nu_+)\}$, then for any divisorial valuation ν such that $c_X(\nu) = p$, one has $Z_\nu \cdot \theta^+ > 0$.*

Proof. Let Z be the blow up of Y at p. Then, $\theta^+_Z = (\pi^* \theta^+_Y) + c\widetilde{E}$ for some $c \in \mathbf{R}$. Since the center of ν_+ is not on $\widetilde{E}$, one has $\theta^+_Z \cdot \widetilde{E} = 0$, hence $c = 0$. Now whether p is a free point on E or a satellite point, we have $Z_{\mathrm{ord}_{\widetilde{E}}} \cdot \theta^+ \geqslant Z_{\mathrm{ord}_E} \cdot \theta^+ > 0$. □

Lemma 4.4.7. *Let Y be a completion of X_0 such that the center of ν_+ is the intersection of two prime divisors at infinity F_1, F_2. Then, $Z_{\mathrm{ord}_{F_1}} \cdot \theta^+ > 0$ or $Z_{\mathrm{ord}_{F_2}} \cdot \theta^+ > 0$.*

Proof. Recall that θ^+ is nef and effective. Suppose that $Z_{\mathrm{ord}_{F_i}} \cdot \theta^+ = 0$ for $i = 1, 2$ and let $\widetilde{E}$ be the exceptional divisor above p_+. Let $\pi : Z \to Y$ be the blow-up at p_+. Then we have

$$\theta^+_Z = \pi^*(\theta^+_Y) + c\widetilde{E}$$

for some $c \in \mathbf{R}$. This implies $\theta^+ \cdot \widetilde{E} = -c > 0$ because p_+ was the center of ν_+ on Y, therefore $c < 0$. But $Z_{\mathrm{ord}_{\widetilde{E}}} \cdot \theta^+_Z = (Z_{\mathrm{ord}_{F_1}} + Z_{\mathrm{ord}_{F_2}})\theta^+_Y + c = c < 0$ and this contradicts the fact that θ^+ is effective. □

Proposition 4.4.8. *For any completion Y such that $c_Y(\nu_+)$ is a free point, we have*

$$\operatorname{Supp}\theta^+_Y = \partial_Y X_0. \tag{4.104}$$

Hence, if $\nu_\pm$ is an infinitely singular valuation, then for any completion Z, there exists an integer $N > 0$ such that $f^{\pm N}(\partial_Z X_0) = p_\pm$.

Proof. Let E be the unique prime divisor at infinity such that $c_Y(\nu_+) \in E$. If $\operatorname{Supp}\theta^+_Y \neq \partial_Y X_0$, there a prime divisor F at infinity such that $Z_{\mathrm{ord}_F} \cdot \theta^+ = 0$ and $F \cap \operatorname{Supp}\theta^+_Y \neq \varnothing$. By Lemma 4.4.5, we have $F = E$; therefore $Z_{\mathrm{ord}_E} \cdot \theta^+ = 0$. But we have that $\theta^+_Y = \lambda Z_{\mathrm{ord}_E}$ for some $\lambda > 0$ by Proposition 4.1.5. So $(\theta^+_Y)^2 = 0$, but this is absurd by Corollary 4.4.2.

For the second assertion, assume that $\nu_\pm$ is an infinitely singular valuation. Let Z be a completion of X_0. Then, by Proposition 3.3.16, there exists a completion Y above Z such that $c_Y(\nu_\pm)$ is a free point. The first assertion shows that $\operatorname{Supp}\theta^\pm_Y = \partial_Y X_0$ and so the same is true for $\operatorname{Supp}\theta^\pm_Z$. The fact that some iterate of $f^{\pm 1}$ contracts the boundary on $p_\pm$ follows from Proposition 4.4.3. □

4.4.2 Proof of Theorem 4.4.4, the cycle case

In that case it was already proven by Gizatullin that $\partial_X X_0 = E(\iota)$.

Proposition 4.4.9 ([ÈH74, CdC19]). *Let X be projective surface and U an open subset of X such that $X \backslash U$ is a cycle of rational curves. Assume that $X \backslash U$ is not an irreducible curve with one nodal singularity. Let g be an automorphism of U, then the indeterminacy points of g can only be intersection points of two components of the cycle.*

Corollary 4.4.10. *In the cycle case, the eigenvaluation of a loxodromic automorphism must be irrational and therefore λ_1 is an algebraic integer of degree 2, in particular it is irrational.*

Proof. Proposition 4.4.9 shows that for any completion X of X_0, $p_+ = c_X(\nu_+)$ is a satellite point at infinity. Indeed, since θ^+ is nef, its incarnation in X cannot be 0. Therefore, there exists a prime divisor E at infinity such that $Z_{\mathrm{ord}_E} \cdot \theta^+ > 0$ because θ^+ is effective. Therefore, by Proposition 4.4.3, E must be contracted by f^N to p_+ so it must be an indeterminacy point of f^{-N}. Proposition 3.3.16 shows that the eigenvaluations $\nu_\pm$ are irrational. □

Proof of Theorem 4.4.4. Corollary 4.4.10 shows the first part of the theorem. We get the normal form at $p_\pm$ by blowing up the center of $\nu_\pm$ enough times. Since these are always intersection points of two prime divisors at infinity we can suppose that $\partial_Y X_0$ is still a cycle.

It remains to show that $\partial_Y X_0$ is contracted by some iterate of f and f^{-1}. Suppose that there exists a prime divisor E that is not contracted to p_+ by any iterate of f. In particular $Z_{\mathrm{ord}_E} \cdot \theta^+ = 0$ by Proposition 4.4.3. By Lemma 4.4.5, we have that E contains $c_Y(\nu_-)$ and f^{-1} contracts E to p_-. And by Lemma 4.4.7 and Corollary 4.4.10 we have that E is the unique prime divisor at infinity that satisfy this property. Either f contracts E to a satellite point $p \neq p_+$ of the boundary or f is sent to a prime divisor at infinity. Indeed, we cannot have $f(E) = E$, otherwise E is f-invariant but this contradicts that f^{-1} contracts E. If E is contracted, it cannot be contracted to p_- because it is not an indeterminacy point of f^{-1}. Therefore, we have that the center of $f_* \mathrm{ord}_E$ is either another prime divisor at infinity or a satellite point at infinity that is not the center of ν_+. In both case, we get $f_* Z_{\mathrm{ord}_E} \cdot \theta^+ > 0$ by Lemma 4.4.6 and this is a contradiction. □

4.4.3 Proof of Theorem 4.4.4, the zigzag case

4.4.3.1 Some technical lemmas about zigzags

We will say following [GD75, BD11b] that a zigzag Z is *standard* if it is of the form

$$Z = F \rhd E \rhd Z' \tag{4.105}$$

where $F^2 = 0, E^2 \leqslant -1$ and Z is a *negative* zigzag meaning that every component of Z' has self-intersection $\leqslant -2$. Any zigzag can be put to a standard form via blow-up of points and contractions of (-1)-curves (see [GD75], §1.7)

Following [BD11b], an *almost standard* zigzag is a zigzag $Z = B_1 \rhd B_2 \rhd \cdots \rhd B_r$ such that

1. There exists a unique irreducible component B_k such that $(B_k)^2 \geqslant 0$.

2. There exists at most one component B_l such that $(B_l)^2 = -1$ and in that case we must have $l = k \pm 1$.

We need to state some technical results for the proof of Theorem 4.4.4, we will need to apply them to a quasiprojective surface which is not necessarily affine. If U is a quasiprojective surface, a completion of U is defined in the same way as the completion of an affine surface. All the results in this Section rely heavily on Proposition 3.1.6 and the Castelnuovo criterion.

Lemma 4.4.11 (Proposition 3.1.3 of [BD11b]). *Let U be a quasiprojective surface and X a completion of U such that $X \backslash U$ is an almost standard zigzag that has no component of self intersection -1. Let B_k be the unique irreducible component of nonnegative self-intersection of $X \backslash U$. Let g be an automorphism of U, then*

(1) g has at most one indeterminacy point q on X.

(2) q has to be on B_k (if it exists).

(3) If B_k is not on the boundary of the zigzag then q must be the intersection point of B_k with B_{k+1} or B_{k-1}.

Proof. Suppose that g has an indeterminacy point, then g^{-1} also has one and g has to contract a curve of the zigzag. Let $\pi : Y \to X$ be the minimal resolution of indeterminacies of g and let $\widetilde{g}$ be the lift of g. Then, the first curve contracted by $\widetilde{g}$ has to be the strict transform of B_k. So g has at least one indeterminacy point on B_k.

There cannot be any indeterminacy point q outside of B_k because otherwise it belongs to components that have self-intersection $\leqslant -2$ and since the zigzag $X \backslash U$ contains no (-1)-curve any exceptional divisor above q has to be contracted by g so q is not an indeterminacy point.

Suppose that B_k is not on the boundary and that the indeterminacy point p of g is not an intersection point. Then, the map π factorizes through the blow-up of p and after contracting the strict transform of B_k, we get at infinity three prime divisors that intersect at the same point. But this is a contradiction because $\widetilde{g}$ consists only of blow ups of point at infinity and $X \backslash U$ does not have three divisors that intersects at the same point.

Finally, there cannot be more than one indeterminacy point on X. Suppose the contrary and let p_1, p_2 be two indeterminacy points, they both belong to B_k. Let E_1, E_2 be two exceptional divisor above p_1 and p_2 in Y respectively. They cannot be contracted by $\widetilde{g}$ because Y is the minimal resolution of singularities of g. Therefore, their strict transform is either a (-1)-curve or a curve with nonnegative self intersection. But this is absurd because $X \backslash U$ does not contain any (-1)-curve and has only one curve of nonnegative self-intersection. □

Corollary 4.4.12. *Let X be a completion of U such that $X \backslash U$ is an almost standard zigzag Z and let f be an automorphism of U. Suppose that f has an indeterminacy point that is a free point on B_k, then one of the two sides of Z can be contracted so that B_k becomes a boundary component of the zigzag.*

Proof. Suppose that B_k is not a boundary component of the zigzag and that f has an indeterminacy point that is a free point on B_k. Then, by Lemma 4.4.11, B_{k-1} or B_{k+1} has to be a

(-1)-curve, suppose it is B_{k+1}. We contract it and we obtain an almost standard zigzag and f still has an indeterminacy point that is a free point on B_k. If B_k is on the boundary we are done, otherwise the only (-1)-curve is the strict transform of B_{k+2} and we keep contracting until B_k becomes a boundary component of the zigzag. □

Lemma 4.4.13. *Let U be a quasiprojective variety and X a completion of U such that $X \backslash U$ is a zigzag of type $(-m_1, \cdots, -m_k, -1, -1, -m_{k+1}, \cdots, -m_r)$ such that for all $i, m_i \geqslant 2$. Let f be an automorphism of U. Then the intersection point of the two (-1)-curves cannot be an indeterminacy point of f.*

If the zigzag is of type $(-1, -2, \ldots, \underbrace{-2}_{F}, \underbrace{-1}_{E}, -m_{k+1}, \cdots, -m_r)$ with $m_i \geqslant 2$, then $F \cap E$ cannot be an indeterminacy point of f.

Proof. Let $\pi : Z \to X$ be a minimal resolution of indeterminacy of $f : X \to X$ and let $\widetilde{f} : Z \to X$ be the lift of f. The first curve contracted by $\widetilde{f}$ must be the strict transform of one of the prime divisors at infinity of X. But if the intersection of the (-1)-curves is an indeterminacy point of f, then all the strict transforms of the prime divisors at infinity of X have self-intersections $\leqslant -2$ and this is a contradiction.

If $X \backslash U$ is a zigzag Z of type $(-1, -2, \cdots, -2, -1, -m_{k+1}, \cdots, -m_r)$, suppose that $F \cap E$ is an indeterminacy point of f, then the first curve contracted by $\widetilde{f}$ must be the strict transform of the (-1)-curve on the left of the zigzag. So we can start by contracting it and we get a zigzag Z' of type $(-1, -2, \cdots, \underbrace{-2}_{F}, \underbrace{-1}_{E}, -m_{k+1}, \cdots, -m_r)$ and of size $\#Z - 1$. We can repeat this process until we get a zigzag of the form $(\underbrace{-1}_{F}, \underbrace{-1}_{E}, -m_{k+1}, \cdots, -m_r)$ and we have that $F \cap E$ cannot be an indeterminacy point of f by the previous case, this is a contradiction. □

Lemma 4.4.14. *Let f be an automorphism of X_0 and let X be a minimal completion of X_0 in the sense of Gizatullin. Then, f defines an automorphism of $U = (E(\iota))^c \subset X$, the complement of $E(\iota)$, i.e the birational map $f : X \dashrightarrow X$ does not have any indeterminacy point on U.*

Proof. Suppose that f admits an indeterminacy point p on some component E_1 of $\partial_X X_0$ with $p \notin E(\iota)$. Let $\pi : Y \to X$ be a minimal resolution of indeterminacies for f and let $F : Y \to X$ be the lift of f. The fiber $\pi^{-1}(p)$ contains at least one (-1)-curve and we claim that none of the irreducible components of $\pi^{-1}(p)$ can be contracted by F, indeed since E_1 is not contracted, one can only contract (-1)-curves of $\pi^{-1}(p)$ but that would contradict the minimality of Y. Therefore, the fiber $\pi^{-1}(p)$ is not affected by F and neither are the self-intersections in the fiber. This would imply that $\partial_X X_0$ contains some (-1)-curves that can be contracted and this contradicts the minimality of X.

$\square$

Corollary 4.4.15. *Let $X_{\min}$ be a minimal completion of the affine surface X_0. The centers $c_{X_{\min}}(\nu_\pm)$ must belong to $E(\iota)$.*

We will apply all the results of this section with $U = (E(\iota))^c \subset X_{\min}$ where $X_{\min}$ is a minimal completion of X_0.

4.4.3.2 Elementary links between almost standard zigzags

From now on $U = (E(\iota))^c \subset X_{\min}$ where $X_{\min}$ is a minimal completion of the affine surface X_0. All the results of §4.4.3.1 will be applied to the following situation. If X is a completion of U (hence of X_0) and f is a loxodromic automorphism of X_0, then some positive iterate of f contracts a component of $X \backslash U$ to $c_X(\nu_+)$. Thus, $c_X(\nu_+)$ is an indeterminacy point of some positive iterate of f^{-1} on X.

Proposition 4.4.16. *Let X be a completion of U such that $X \backslash U$ is an almost standard zigzag, then one can find a completion Y of U with a birational map $\varphi : X \to Y$ that is an isomorphism above U such that*

(1) $Y \backslash U$ is also an almost standard zigzag.

(2) Let $\widetilde{X}$ be the blow up of X at $c_X(\nu_+)$, then the lift $\varphi : \widetilde{X} \dashrightarrow Y$ is defined at $c_{\widetilde{X}}(\nu_+)$ and is a local isomorphism there.

Proof. Let B the unique irreducible component of $X \backslash U$ of nonnegative self intersection.

Case: B is on the boundary $X \backslash U$ is a zigzag of the form $B \rhd E \rhd Z$ where $B^2 \geqslant 0, E^2 \leqslant -1$ and Z is a negative zigzag.

- **$c_X(\nu_+)$ is a free point on B** If $E^2 = -1$, we blow up $c_X(\nu_+)$ and then contract the strict transform of E. Let Y be the new projective surface obtained, it satisfies the proposition.

  Suppose $E^2 < -1$, If $B^2 > 0$ we blow up $B \cap E$ to obtain a new zigzag $B \rhd E' \rhd Z'$ which is still almost standard. We keep blowing up the strict transform of B with the second component of the zigzag until $B^2 = 0$. After all these blowups, let X' be the newly obtained projective surface, we have that $X' \backslash U$ is an almost standard zigzag of the form $B \rhd E \rhd Z$

where $B^2 = 0, E^2 = -1$ and Z is a negative zigzag. We blow up $c_{X'}(\nu_+)$ and let $\widetilde{E}$ be the exceptional divisor, by Lemma 4.4.13, the center of ν_+ cannot be the intersection point of $\widetilde{E}$ and the strict transform of B, therefore it is a free point of $\widetilde{E}$ and we can contract the strict transform of B. We call Y the new obtained surface it satisfies the proposition.

- $c_X(\nu_+)$ **is the satellite point** $B \cap E$ We blow up $B \cap E$ and call $\widetilde{E}$ the exceptional divisor. If $B^2 > 0$ in X, then we still have an almost standard zigzag and we call Y the new obtained surface. If $B^2 = 0$ in X, then by Lemma 4.4.13 is a free point of $\widetilde{E}$ and we can contract the strict transform of B, we call Y the newly obtained surface.

Case: B **is not on the boundary**

- $c_X(\nu_+)$ **is a free point of** B By Corollary 4.4.12, one of the two sides of $X \backslash U$ is contractible, so we contract it and call X_1 the newly obtained surface, we can now apply the proof of the boundary case to find Y.

- $c_X(\nu_+)$ **is the satellite point** $B \cap E$ We can suppose up to contraction that if $X \backslash U$ contains a (-1)-component, it must be E. We start by blowing up $c_X(\nu_+)$ and let $\widetilde{E}$ be the exceptional divisor.

  - If $B^2 > 0$ in X, then we still have an almost standard zigzag and we call Y the newly obtained surface.
  - If $B^2 = 0$ in X, then by Lemma 4.4.13 the center of ν_+ cannot be the intersection of $\widetilde{E}$ and the strict transform of B where $\widetilde{E}$ is the exceptional divisor. So we can contract the strict transform of B and we get an almost standard zigzag and we call Y the newly obtained surface.

□

Corollary 4.4.17. *If $\partial_X X_0$ is a zigzag, the eigenvaluation ν_+ cannot be irrational, hence it is infinitely singular and λ_1 is an integer. Furthermore, $U = X_0$.*

Proof. It suffices to show that the sequence of centers of ν_+ contains infinitely many free points. If not, we can apply Proposition 4.4.16 finitely many times so that we get a completion X of X_0 such that $X \backslash U$ is an almost standard zigzag and the center of ν_+ is always a satellite point. We show that this leads to a contradiction.

Case 1: $c_X(\nu_+) = B \cap E$ with E a component of $X\backslash U$ We can suppose after contractions and blow ups that $B^2 = 0$. We will show that we can suppose that B is a boundary component of the zigzag. The zigzag $X\backslash U$ is of the form $Z_1 \lhd B \rhd E \rhd Z$. Denote by $(m_1, \cdots, m_r)$ the type of Z_1.

- **Case** $m_1 \geqslant 2$ Blow up $B \cap E$ and call $\widetilde{E}$ the exceptional divisor. The center of ν_+ has to be $B \cap \widetilde{E}$ or $\widetilde{E} \cap E$, but it cannot be $B \cap \widetilde{E}$ by Lemma 4.4.13. So we can contract the strict transform of B. We get a new zigzag of the form $Z_1' \lhd B' \rhd Z'$ with $m_1' = m_1 - 1$ and $\#Z_1' = \#Z_1$.

- **Case** $m_1 = 1$ call E_1 the first component of Z_1. Blow up $B \cap E$. The center of ν_+ is either $B \cap \widetilde{E}$ or $\widetilde{E} \cap E$. Either way, we can contract the strict transform of E_1. We get a zigzag of the form $Z_1' \lhd B \rhd \widetilde{E} \rhd E \rhd Z$ where $\#Z_1' = \#Z_1 - 1$.

We can apply this procedure recursively, it stops because the sequence $(\#Z_1, m_1)$ is strictly decreasing for the lexicographical order. And we never blow down a curve that contains the center of ν_+ nor do we blow down a curve to the center of ν_+.

Now that we have that B is a boundary component, we can suppose that $X\backslash U$ is a 1-standard zigzag. Call E the (-1)-component of $X\backslash U$, we will show that $Z_{\nu_+} \cdot E = +\infty$. Indeed, blow up $B \cap E$ and let $\widetilde{E}$ be the exceptional divisor. By Lemma 4.4.13, the center of ν_+ has to be $\widetilde{E} \cap E$. If we blow up the center of ν_+ again we can still apply Lemma 4.4.13, so the center of ν_+ is always the intersection point of the strict transform of E with the exceptional divisor. This implies that ν_+ is the curve valuation associated to the curve E and this is absurd.

Case 2: $c_X(\nu_+) = B \cap C$ with C a component of $\partial_X X_0$ but $C \cap U \neq \varnothing$. This means that $c_X(\nu_+)$ belongs to no other component of $X\backslash U$ than B. Using Lemma 4.4.11 we can contract one of the two sides of the zigzag so that B is a boundary component of the zigzag $X\backslash U$, we can furthermore suppose that $X\backslash U$ has no (-1)-component. Call m the self intersection of the component next to B in the zigzag, we have by assumption $m \leqslant -2$.

- **Case** $B^2 > 0$ let X' be the blow up of $B \cap C$ and let $\widetilde{E}$ be the exceptional divisor. Then, since the strict transform of B has nonnegative self intersection $X'\backslash U$ is an almost standard zigzag. We must have that $c_{X'}(\nu_+) \in \widetilde{E}$ and by Lemma 4.4.11 $c_{X'}(\nu_+)$ must be $B \cap \widetilde{E}$ and we are back in Case 1. This leads to a contradiction.

- **Case** $B^2 = 0$ Let E be the component on $X\backslash U$ next to B (if it exists). Let X' be the blow up of $B \cap C$ and let $\widetilde{E}$ be the exceptional divisor. By Lemma 4.4.13, $c_{X'}(\nu_+)$ cannot be $B \cap \widetilde{E}$

so it has to be $\widetilde{E} \cap C$. Let X'' be the blow down of the strict transform of B. The strict transform of $\widetilde{E}$ has nonnegative self-intersection and $X'' \backslash U$ is an almost standard zigzag and $c_{X''}(\nu_+) = \widetilde{E} \cap C$. Rename $\widetilde{E}$ by B in X''. If $E^2 = m$ in X, then the strict transform of E in X'' satisfies $E^2 = m+1$. We repeat this procedure until $E^2 = -1$. We then blow down E and we end up back in the case $B^2 > 0$ and this leads to a contradiction.

The last case to treat is if $X \backslash U$ is a zigzag containing only B with $B^2 = 0$. We will show in that case that $\nu_+(C) = +\infty$ which is a contradiction. Indeed, let X' be the blow up of $B \cap C$ and let $\widetilde{E}$ be the exceptional divisor. Then, by Lemma 4.4.13, $c_{X'}$ cannot be $B \cap \widetilde{E}$ so it must be $\widetilde{E} \cap C$. Let X'' be the blow up of $\widetilde{E} \cap C$ and let $\widetilde{E}^{(2)}$ be the exceptional divisor. Again, by Lemma 4.4.13, $c_{X''}(\nu_+) = \widetilde{E}^{(2)} \cap C$. By induction, we see that the centers of ν_+ must always belong to the strict transform of C in every blow up, this implies that ν_+ is the curve valuation associated to C and this is absurd.

Thus, ν_+ is not irrational. Hence, by Proposition 4.4.1 ν_+ is an infinitely singular valuation, so we get that $U = X_0$ by Proposition 4.4.8. $\square$

4.4.4 A summary and applications

We sum up the content of Theorem 4.4.18 in Figure 4.7 and 4.8

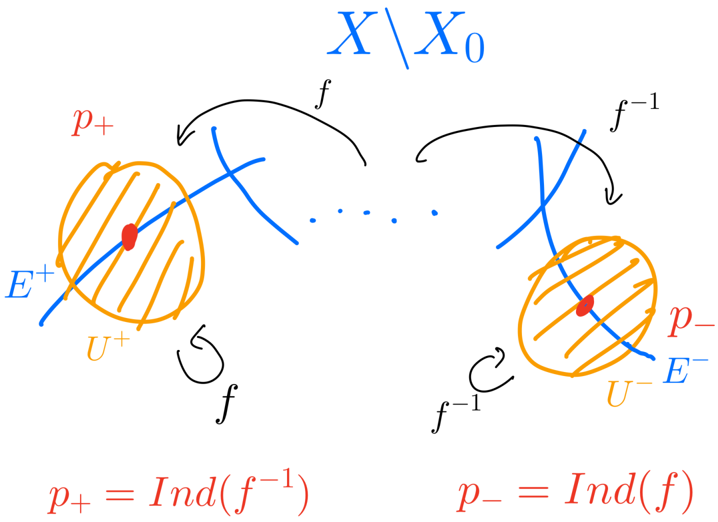


Figure 4.7: Dynamics at infinity of f when $\lambda_1(f) \in \mathbf{Z}_{\geqslant 0}$

Theorem 4.4.18. *Let $X_0 = \operatorname{Spec} A$ be a normal affine surface defined over an algebraically closed complete field $\mathbf{C}_v$ such that $A^\times = \mathbf{C}_v^\times$ and $\operatorname{Pic}^0(X_0) = 0$. Let f be a loxodromic automorphism of X_0. Then, there exists two unique (up to normalization) distinct valuations centered at ν_+, ν_- such that $f_*^{\pm 1}(\nu_\pm) = \lambda_1 \nu_\pm$. Let $\theta^- = Z_{\nu_+}$ and $\theta^+ = Z_{\nu_-}$. We have that θ^+, θ^- are nef, effective and satisfy the following relations*

$$f^*\theta^+ = \lambda_1 \theta^+, \quad f^*\theta^- = \frac{1}{\lambda_1}\theta^- \tag{4.106}$$

$$f_*\theta^+ = \frac{1}{\lambda_1}\theta^+, \quad f_*\theta^- = \lambda_1 \theta^-. \tag{4.107}$$

Furthermore we have the following intersection relations: $(\theta^+)^2 = \left(\theta^-\right)^2 = 0$ and $\theta^+ \cdot \theta^- = 1$.

We can find a completion X of X_0 such that if $p_+ := c_X(\nu_+), p_- := c_X(\nu_-)$, then

(1) $p_+ \neq p_-$.

(2) some positive iterate of $f^{\pm 1}$ contracts $\partial_X X_0$ to $p_\pm$.

(3) $f^\pm$ is defined at $p_\pm, f^\pm = p_\pm$ and $p_\mp$ is the unique indeterminacy point of $f^\pm$.

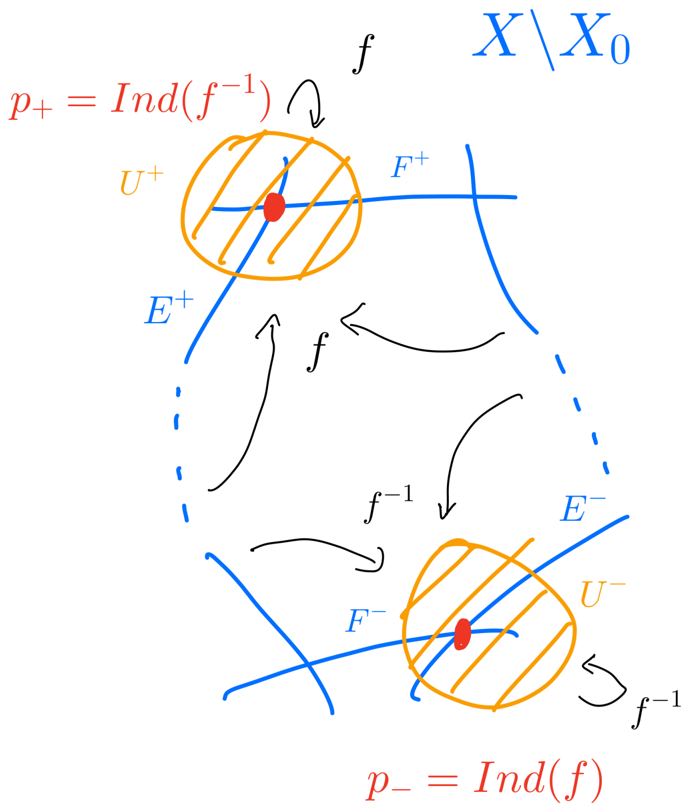


Figure 4.8: Dynamics at infinity of f when $\lambda_1(f) \in \mathbf{R}\backslash\mathbf{Q}$

(4) *There exists an open neighbourhood $U^\pm$ of $p_\pm$ in $X(\mathbf{C}_v)$ and local coordinates at $p_\pm$ such that $f^\pm_{|U^\pm}$ has a local normal form of (pseudo)monomial type* (3.4) *or (*(3.5)*) if $\lambda_1(f) \notin \mathbf{Z}_{\geqslant 0}$ or of type* (3.2) *or* (3.3) *if $\lambda_1(f) \in \mathbf{Z}_{\geqslant 0}$.*

(5) *For all prime divisor E^+ of X at infinity such that $p_+ \in E^+$,*

$$\operatorname{ord}_{E^+}(\theta^+) > \operatorname{ord}_{E^+}(\theta^-) \tag{4.108}$$

(6) *For all prime divisor E^- of X at infinity such that $p_- \in E^-$,*

$$\operatorname{ord}_{E^-}(\theta^-) > \operatorname{ord}_{E^-}(\theta^+) \tag{4.109}$$

(7) *If $\lambda_1 \in \mathbf{Z}_{\geqslant 0}$, then $(\theta^+_X, \theta^-_X) \in \operatorname{Div}_\infty(X)_\mathbf{Q}$ is a well ordered pair (cf §3.2.5).*

Proof. Any completion provided by Theorem 4.4.4 satisfies item (1)-(4). Fix X such a completion, we show that there exists a completion above X that satisfy (1)-(6) by successively blowing up the centers of ν_+ and ν_-.

Lemma 4.4.19. *There exists a completion Y above X such that for all completion Y' above Y, for all prime divisor E^+ of Y' at infinity such that $c_{Y'}(\nu_+) \in E^+$,*

$$\operatorname{ord}_{E^+}(\theta^+) > \operatorname{ord}_{E^+}(\theta^-) \tag{4.110}$$

Proof of Lemma 4.4.19. Recall that $\theta^+ = Z_{\nu_-}$ and $\theta^- = Z_{\nu_+}$. Let $p_+ = c_X(\nu_+)$ and replace ν_+ (and θ^+) by their multiple such that $\nu_+ \in \mathcal{V}_X(p_+; \mathfrak{m}_{p_+})$. Let X_n be the sequence of completions defined by $X_0 = X$ and $\pi_n : X_{n+1} \to X_n$ is the blow up of X_n at $c_{X_n}(\nu_+)$. Define also the morphism of completions $\tau_n := \pi_0 \circ \pi_1 \circ \cdots \circ \pi_n : X_{n+1} \to X$. Since $c_X(\nu_+) \neq c_X(\nu_-)$, we have that for all $n, c_{X_n}(\nu_+) \neq c_{X_n}(\nu_-)$. By Proposition 4.1.5 Equation (4.11), we have that for all $n, \theta^+_{X_{n+1}} = \pi_n^* \theta^+_{X_n}$ since $c_{X_n}(\nu_+) \neq c_{X_n}(\nu_-)$, hence $\theta^+_{X_{n+1}} = \tau_n^* \theta^+_X$. Let E_n be the exceptional divisor of $\pi_n : X_{n+1} \to X_n$. Notice that

$$\forall n \geqslant 0, \quad c_X((\tau_n)_* \operatorname{ord}_{E_n}) = c_X(\nu_+). \tag{4.111}$$

We have by Proposition 3.4.26 that the sequence $\nu_n := \frac{1}{b(E_n)} \operatorname{ord}_{E_n}$ converges strongly towards ν_+. Therefore, by Corollary 4.1.8, we have

$$Z_{\nu_n} \to \lambda \theta^- \tag{4.112}$$

where $\lambda > 0$ such that $\lambda \nu_+ \in \mathcal{V}_X(p; \mathfrak{m}_p)$. This convergence is with respect to the strong topology of $\mathrm{L}^2(\mathrm{X}_0)$, therefore we can intersect both sides with θ^-, to get

$$Z_{\nu_n} \cdot \theta^- \to 0. \tag{4.113}$$

This means that $\operatorname{ord}_{E_n}(\theta^-) = o(b(E_n))$ when $n \to \infty$. Now, we evaluate $Z_{\nu_n} \cdot \theta^+$. Since for all n, $\theta^+_{X_{n+1}} = \tau_n^* \theta^+_X$, we get

$$Z_{\nu_n} \cdot \theta^+ = Z_{\nu_n, X_{n+1}} \cdot \theta^+_{X_{n+1}} = (\tau_n)_* Z_{\nu_n, X_{n+1}} \cdot \theta^+_X = Z_{\nu_n, X} \cdot \theta^+_X. \tag{4.114}$$

If $c_X(\nu_+) \in E$ is a free point, then by Equation (4.111) and Proposition 4.1.5

$$Z_{\nu_n, X} = (Z_{\nu_n} \cdot E) Z_{\operatorname{ord}_E}. \tag{4.115}$$

By Proposition 3.6.20, we have that $L_{\nu_n}(E) \geqslant 1$. Hence, we get $\frac{1}{b(E_n)} \operatorname{ord}_{E_n}(\theta^+) \geqslant \operatorname{ord}_E(\theta^+) > 0$.

If $c_X(\nu_+)$ is a satellite point, i.e $c_X(\nu_+) = E \cap F$ where E, F are two prime divisors of X at infinity, then we get by Equation (4.111), Proposition 4.1.5 and Proposition 3.6.20 that

$$\frac{1}{b(E_n)} \operatorname{ord}_{E_n}(\theta^+) \geqslant \operatorname{ord}_E(\theta^+) + \operatorname{ord}_F(\theta^+) \tag{4.116}$$

and the lemma is proven. □

Let Y be a completion above X given by Lemma 4.4.19. By the last assertion of Theorem 4.4.4, there exists a completion Y' above Y that satisfy conditions (1)-(4) and Lemma 4.4.19 shows that Y' satisfies also conditions (5) and (6). Suppose now that $\lambda_1 \in \mathbf{Z}_{\geqslant 0}$, then the eigenvaluations ν_+ and ν_- are infinitely singular, therefore up to normalisation $\theta^+, \theta^- \in \operatorname{Weil}_\infty(\mathrm{X}_0)_{\mathbf{Q}}$ by Corollary 4.1.7 and $c_X(\nu_+), c_X(\nu_-)$ are free points at infinity. Let Y be a completion above X such that $\theta_X^+ \vee \theta_X^-$ is defined in Y. By Proposition 3.6.25 ,the morphism of completions $\pi : Y \to X$ is a composition of blow ups of satellite points. Therefore, by Proposition 4.1.5, $\theta_Y^\pm = \pi^* \theta_X^\pm$ and conditions (1)-(6) still holds in Y. □

Proposition 4.4.20. *Let X_0 be a normal affine surface defined over $\mathbf{C}_v$. If f is a loxodromic automorphism of X_0, then, there are no f-invariant algebraic curves in X_0.*

Proof. If $\dim \operatorname{QAlb}(X_0) = 2$, then X_0 is a finite ramified cover of $\mathbb{G}_m^2$. It suffices to show the result for the loxodromic automorphisms of $\mathbb{G}_m^2$. Any monomial automorphism of $\mathbb{G}_m^2$ does not admit invariant curves, so the result follows.

If $\dim \operatorname{QAlb}(X_0) = 1$, then every automorphism of X_0 preserves a fibration over a curve, hence it cannot be loxodromic.

Finally, if $\dim \operatorname{QAlb}(X_0) = 0$, let X be a completion of X_0 given by Theorem 4.4.18. Suppose that $C \subset X_0$ is an algebraic curve invariant by f. Let $\overline{C}$ be the closure of C in X. We must have $\{p_+, p_-\} \cap (\overline{C} \cap \partial_X X_0) \neq \varnothing$. Indeed, $\overline{C} \cap \partial_X X_0$ is not empty so let p be a point in it. If $p \notin \{p_+, p_-\}$, then f is defined at p and $f(p) = p_+$. Since $\overline{C}$ is f-invariant, we get $p_+ \in \overline{C}$. This means that C defines a germ of an analytic curve at p_+ that is invariant by f but this is not possible by Theorem 4.2.1. □

Corollary 4.4.21. *If X_0 is a normal affine surface defined over a number field K and f is a loxodromic automorphism of X_0, then all periodic points of f are defined over $\overline{K}$.*

Proof. Suppose there exists $p \in X_0(\mathbf{C}) \backslash X_0(\overline{K})$ such that $f^N(p) = p$. Let $G := \operatorname{Gal}(\mathbf{C}/\overline{Q})$, then for all $q \in G \cdot p$, we have $f^N(q) = q$. Since $p \notin X_0(\overline{K})$, the orbit $G \cdot p$ is infinite and its Zariski closure $\overline{G \cdot p} \subset X_0 \times \operatorname{Spec} \mathbf{C}$ has dimension > 0. If $\dim \overline{G \cdot p} = 2$, then $f^N = \mathrm{id}$ and this is impossible because f is loxodromic. If $\dim \overline{G \cdot p} = 1$, then $C = \overline{G \cdot p}$ is an f^N-invariant curve of $X_0 \times \operatorname{Spec} \mathbf{C}$. This is impossible by Proposition 4.4.20. $\square$

Corollary 4.4.22. *Let X_0 be a normal affine surface defined over $\mathbf{C}_v$ such that* $\mathrm{QAlb}(X_0) = 0$. *Let f be a loxodromic automorphism of X_0 and let X be a completion of X_0 from Theorem 4.4.18. If $p \in X_0(\mathbf{C}_v)$, we have two possibilities.*

1. *The forward f-orbit of p is bounded.*

2. *$(f^n(p))_{n \geqslant 0}$ converges towards p_+.*

Proof. Suppose that $(f^n(p))_n$ is not bounded. Since $X(\mathbf{C}_v)$ is compact, $(f^n(p))$ has an accumulation point $q \in \partial_X X_0$. Let U_+ be the open neighbourhood of p_+ given by Theorem 4.4.18. We must have $q \in \{p_+, p_-\}$. Otherwise, since $f(q) = p_+$, if $f^{N_0}(p)$ is sufficiently close to q, then for all $N \geqslant N_0 + 1, f^N(p) \in U_+$ and q cannot be an accumulation point. Suppose that $q = p-$. Let (x, y) be the local coordinates at p_- over U^- given by Theorem 4.4.18. Consider the norm $\max(|x|, |y|)$ over U^-. Looking at the normal form of f, for any $\varepsilon > 0$ small enough, the ball $B(p_-, \varepsilon)$ of center p_- and radius ε, with respect to this norm, is f^{-1}-invariant and we have $f^{-1}B(p_-, \varepsilon) \Subset B(p_-, \varepsilon)$. Therefore if $f^{N_0}(p) \in B(p_-, \varepsilon)$, we have $p \in B(p_-, \varepsilon)$. Letting $\varepsilon \to 0$ we get $p = p_-$ and this is a contradiction. Therefore, the only accumulation point of $(f^N(p))_N$ is p_+ and it is the limit of this sequence.

$\square$

CHAPTER 5

GREEN FUNCTIONS AND DYNAMICS OF LOXODROMIC AUTOMORPHISMS OF AFFINE SURFACES

5.1 Berkovich spaces, Adelic divisors and line bundles

5.1.1 Berkovich spaces

Let $\mathbf{k}$ be a complete field with a multiplicative norm $|\cdot|$. We recall the definition and main properties of Berkovich spaces, for a reference see [Ber12]. If X is scheme over $\mathbf{k}$, we will write X^{an} or $(X/\mathbf{k})^{an}$ the Berkovich analytification of X.

Definition 5.1.1. (i) If $X = \operatorname{Spec} A$ where A is a k-algebra, then X^{an} is the set of multiplicative seminorms on A extending the norm on $\mathbf{k}$. For every $x \in X^{an}$ we have a seminorm $|\cdot|_x : A \to \mathbf{R}_+$. We will write $|P|_x$ as $|P(x)|$. The topology on X^{an} is the coarsest topology such that the evaluation maps $|f| : X^{an} \to \mathbf{R}$ are continuous. This is the weak topology of simple convergence.

(ii) If X is covered by an open affine cover $\{\operatorname{Spec} A_i\}$, then X^{an} is defined to be the union of the $(\operatorname{Spec} A_i)^{an}$ glued in a canonical way. X^{an} has a locally ringed space structure.

If $X = \operatorname{Spec} A$, for any $x \in X^{an}$, the seminorm $|\cdot|_x$ induces a norm over $A/\ker|\cdot|_x$. We can take the fraction field of $A/\ker|\cdot|_x$ and complete it with respect to the norm induced by $|\cdot|_x$. This defines the *residue field* of x which we denote by H_x.

We have a functoriality property, If $f : X \to Y$ is a morphism of $\mathbf{k}$-schemes, then it induces a continuous map $f^{an} : X^{an} \to Y^{an}$

Proposition 5.1.2 (Topological properties of the Berkovich space)**.** *(1) If X is separated and of finite type over $\mathbf{k}$, then X^{an} is Hausdorff.*

(2) If X is of finite type over $\mathbf{k}$*, then X^{an} is locally compact.*

(3) If X is projective over $\mathbf{k}$*, then X^{an} is compact.*

If $\mathbf{k} = \mathbf{C}$ equipped with the usual norm, then if X is a scheme of finite type over $\mathbf{C}$, $X^{an} = X(\mathbf{C})$.

Contraction map There is a natural contraction map $c : X^{an} \to X$ defined as follows. Suppose $X = \operatorname{Spec} A$, then if $x \in X^{an}$ the kernel of $|\cdot|_x : A \to H_x \to \mathbf{R}$ is a prime ideal of A, we let $c(x)$ be this prime ideal. If $p \in X$ is a closed point, then $c^{-1}(p)$ consists of a unique point and we have a natural embedding $X(\overline{\mathbf{k}}) \hookrightarrow X^{an}$. Indeed, let $x \in c^{-1}(p)$, then x induces a norm on the field $\kappa(p) := O_{X,p}/\mathfrak{m}_p$, but $\kappa(p)$ is a finite extension of $\mathbf{k}$ so there exists a unique extension of the norm of $\mathbf{k}$ to $\kappa(p)$.

The reduction map Suppose that $\mathbf{k}$ is a complete valued non-archimedean field. We write $\mathbf{k}^\circ$ for its valuation ring and $\mathbf{k}^{\circ\circ}$ for the maximal ideal of $\mathbf{k}^\circ$. Let X be a projective scheme over $\mathbf{k}$ and let $\mathscr{X}$ be a *model* of X. That is a projective $\mathbf{k}^\circ$-scheme $\mathscr{X}$ such that the generic fiber $\mathscr{X}_\eta$ is isomorphic to X. We denote by $\mathscr{X}_o$ the special fiber of $\mathscr{X}$. There exists a canonical reduction map $r_{\mathscr{X}} : X^{an} \to \mathscr{X}_o$ defined as follows. Recall that we have the contraction map $c : X^{an} \to X \simeq \mathscr{X}_\eta$. For every $x \in \mathbf{X}^{an}$, let $\xi := c(x)$, we have a non-archimedean norm on the residue field $\mathbf{k}(\xi)$ induced by x. Let R_ξ be the valuation ring of $\mathbf{k}(\xi)$ with respect to the norm x. There is a map $\operatorname{Spec} R_\xi \to \operatorname{Spec} \mathbf{k}^\circ$ induced by $\mathbf{k}^\circ \to R_\xi$. By the valuative criterion for properness, there exists a unique lift in the following diagram

$$\begin{tikzcd} \operatorname{Spec} \mathbf{k}(\xi), \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\ \operatorname{Spec} R_\xi \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \mathbf{k}^\circ \end{tikzcd} . \tag{5.1}$$

We define $r_{\mathscr{X}}(x)$ as the image of the closed point of $\operatorname{Spec} R_\xi$ in $\mathscr{X}_\circ$.

5.1.2 Green functions

Now until the end of this memoir, $\mathbf{C}_v$ will be an algebraically closed complete field. If X is a scheme over $\mathbf{C}_v$, then X^{an} will be the Berkovich analytification of $\mathbf{C}_v$.

Definition 5.1.3. Let X be a completion of X_0 and $D = \sum_i a_i E_i \in \mathrm{Div}(X)_{\mathbf{R}}$. A (continuous) *Green function* of D is a continuous function $g : X^{an} \backslash (\mathrm{Supp}\, D)^{an} \to \mathbf{R}$ such that for any finite open affine cover $X = \bigcup_j U_j$ if h_i^j is a local equation of E_i over U_j, the function

$$g + \sum_i a_i \log \left| h_i^j \right| \tag{5.2}$$

extends to a continuous function over U_j^{an}.

Proposition 5.1.4. *Two Green functions of the same* $\mathbf{R}$*-divisor* D *differ by a bounded continuous function*

Proof. If g_1, g_2 are two Green functions of D, $g_1 - g_2$ can be extended to a continuous function over X^{an}. Since X is projective, X^{an} is compact and the function $g_1 - g_2$ is bounded. □

Proposition 5.1.5. *Let* $D \in \mathrm{Div}(X)_{\mathbf{R}}$ *be an effective divisor, then any Green function of* D *is bounded from below.*

Proof. Let g be any Green function of D. Write $D = \sum_i a_i E_i$ where $a_i \in \mathbf{R}$ and E_i is a prime divisor. Let $x \in (\mathrm{Supp}\, D)^{an}$ and let h_i be a local equation of E_i at $c(x)$. By definition, the function $g + \sum_i a_i \log |h|_i$ extends to a continuous function at x. Since D is effective, $a_i > 0 \forall i$ and $\sum_i a_i \log |h|_i \to -\infty$ at x. This means that there exists an open neighbourhood $U_x \subset X^{an}$ of x such that $g_{|U_x \backslash (\mathrm{Supp}\, D)^{an}} \geqslant 0$. Since $\mathrm{Supp}\, D$ is a closed curve, $(\mathrm{Supp}\, D)^{an}$ is compact so we can cover it by a finite number of such open subset U_x. We have therefore constructed an open neighbourhood V of the curve $(\mathrm{Supp}\, D)^{an}$ over which g is $\geqslant 0$. Now, the complement of V is a closed compact subset of $X^{an} \backslash (\mathrm{Supp}\, D)^{an}$, therefore g is bounded over it. We get that g is bounded from below over $X^{an} \backslash (\mathrm{Supp}\, D)^{an}$. □

Proposition 5.1.6. *Let* X, Y *be two projective varieties and let* $\varphi : Y \to X$ *be a surjective morphism. Let* $D \in \mathrm{Div}(X)_{\mathbf{R}}$*, let* g_D *be a Green function of* D *and let* $g_{\varphi^* D}$ *be a Green function of* $\varphi^* D$*, then*

$$g_D \circ \varphi^{an} - g_{\varphi^* D} \tag{5.3}$$

defines a continuous (bounded) function over Y^{an}.

Proposition 5.1.7. *Let* X *be a completion of* X_0*. Let* $D_1, D_2 \in \mathrm{Div}_\infty(X)_{\mathbf{R}}$ *and let* g_1, g_2 *be Green functions of* D_1 *and* D_2 *respectively. Suppose that* (D_1, D_2) *is a well ordered pair. Then,* $\max(g_1, g_2)$ *is a Green function of* $\max(D_1, D_2)$.

Proof. Let $D = \max(D_1, D_2)$ and $x \in \mathrm{Supp}(D)^{an}$. We have that $c(x)$ is either a closed point on $\mathrm{Supp}\, D$ or the generic point of one of the irreducible components of $\mathrm{Supp}\, D$.

If $c(x) = \eta_E$ is the generic point of an irreducible component of $\mathrm{Supp}\, D$ or if $c(x) \in E$ is a free point. Set $\alpha_i = \mathrm{ord}_E(D_i), i = 1, 2$. Then, if z is a local equation of E at $c(x)$ there exists a continuous function ψ_i defined locally at x such that $g_i + \alpha_i \log|z| = \psi_i$. If $\alpha_1 = \alpha_2 = \alpha$, then

$$\max(g_1, g_2) + \alpha \log|z| = \max(\psi_1, \psi_2) \tag{5.4}$$

which is continuous. If $\alpha_1 < \alpha_2$, then

$$\max(g_1, g_2) + \alpha_1 \log|z| = \max(\psi_1, \psi_2 + (\alpha_1 - \alpha_2)\log|z|). \tag{5.5}$$

Since $\alpha_1 - \alpha_2 > 0$, this is equal to ψ_1 on the open neighbourhood $\left\{\log|z| < \frac{\max|\psi_2|}{\alpha_1 - \alpha_2}\right\}$ of x, so it extends to a continuous function at x.

If $c(x) = E \cap F$ is a satellite point where $E, F \in \mathrm{Supp}\, D$, set $\alpha_i = \mathrm{ord}_E(D_i), \beta_i = \mathrm{ord}_F(D_i)$. Let z, w be local equations of E, F at $c(x)$ respectively. There exist two continuous functions ψ_1, ψ_2 locally defined at x such that $g_i + \alpha_i \log|z| + \beta_i \log|w| = \psi_i$. If $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$, then

$$\max(g_1, g_2) + \alpha \log|z| + \beta \log|w| = \max(\psi_1, \psi_2) \tag{5.6}$$

which is continuous.

If $\alpha_1 > \alpha_2$, since (D_1, D_2) is a well ordered pair, we have $\beta_1 \geqslant \beta_2$. Therefore,

$$\max(g_1, g_2) + \alpha_1 \log|z| + \beta_1 \log|w| = \max(\psi_1, \psi_2 + (\alpha_1 - \alpha_2)\log|z| + (\beta_1 - \beta_2)\log|w|) \tag{5.7}$$

Since $\alpha_1 - \alpha_2 > 0$ and $\beta_1 - \beta_2 \geqslant 0$, the right hand side is equal to ψ_1 on the open neighbourhood $\{(\alpha_1 - \alpha_2)\log|z| < \max|\psi_2|\}$ of x, so it extends to a continuous function at x. □

Suppose $\mathbf{C}_v$ is non-archimedean. A *model Green function* of D is any Green function of the following form. Let $\mathscr{X}$ be a projective variety over $\mathrm{Spec}\, O_v$ such that $X = \mathscr{X} \otimes_{O_v} \mathrm{Spec}\, \mathbf{C}_v$ and let $\mathscr{D}$ be a Cartier divisor of $\mathscr{X}$ such that $\mathscr{D} \otimes_{O_v} \mathrm{Spec}\, \mathbf{C}_v = D$. We say that $(\mathscr{X}, \mathscr{D})$ is a *model* of (X, D). We define the function $g_{(\mathscr{X}, \mathscr{D})} : (X \setminus \mathrm{Supp}\, D)(\mathbf{C}_v)$ as follows. Let $r_{\mathscr{X}} : X(\mathbf{C}_v) \to \mathscr{X}_o$ be the reduction map defined in Section 5.1. Let $x \in X^{an} \setminus (\mathrm{Supp}\, D)^{an}$. Let h be a local equation of $\mathscr{D}$ at $r(x)$. By definition, we have a local ring homomorphism $O_{\mathscr{X}, r_{\mathscr{X}}(x)} \to R_{c(x)}$ where $R_{c(x)}$ is the valuation ring of the residue field $\kappa(c(x))$ equipped with the non-archimedean norm induced by x, in particular we can define $|h(x)|$. We define $g_{(\mathscr{X}, \mathscr{D})}(x) = -\log|h(x)|$. This does not depend on the choice of h because the quotient of two local equations of $\mathscr{D}$ is an invertible element of $O_{\mathscr{X},r_{\mathscr{X}}(x)}$ hence it has norm 1. If $D = \sum_i a_i D_i$ is an $\mathbf{R}$-divisor of X with D_i Cartier divisors, then a model Green function is a function $g_{(\mathscr{X},\mathscr{D})} = \sum_i a_i g_{(\mathscr{X},\mathscr{D}_i)}$ where $(\mathscr{X},\mathscr{D}_i)$ is a model of (X,D_i).

A model Green function $g_{(\mathscr{X},\mathscr{D})}$ is said to be *semipositive* if $\mathscr{D}$ is nef over $\mathscr{X}$.

Example 5.1.8. Let $h \in \mathbf{C}_v(X)^\times$ be a rational function, then $\log|h|$ is a model Green function of $\operatorname{div}(h)$. Indeed, let $\mathscr{X}$ be a model of X and consider the principal divisor $\operatorname{div}(h)$ on $\mathscr{X}$ as h defines a rational function on $\mathscr{X}$.

Proposition 5.1.9. *Let $X = \mathbf{P}^N_{\mathbf{C}_v}$ with homogeneous coordinates $T_0,\cdots,T_N$. Consider the affine chart $\{T_0 \neq 0\}$ with affine coordinates $t_1,\cdots,t_n$. Then the function*

$$g(x) = \log^+ \max(|t_1(x)|,\cdots,|t_n(x)|) \tag{5.8}$$

If $\mathbf{C}_v$ is non-archimedean, g is a semipositive model Green function for the divisor $\{T_0 = 0\}$. If $\mathbf{C}_v = \mathbf{C}$, then g is a psh Green function of $\{T_0 = 0\}$.

Proof. Take $\mathscr{X} = \mathbf{P}^N_{O_v}$ with homogeneous coordinates $T_0,\cdots,T_N$ and set $\mathscr{D} = \{T_0 = 0\}$, then $(\mathscr{X},\mathscr{D})$ is a model of X,D where $D = \{T_0 = 0\} \in \operatorname{Div}(X)$. $\square$

Proposition 5.1.10. *Let X,Y be $\mathbf{C}_v$-projective scheme with a morphism $\varphi : Y \to X$. Let D be a $\mathbf{R}$-divisor on X. Let $(\mathscr{X},\mathscr{D})$ be a model of (X,D) and suppose that there is a model $\mathscr{Y}$ of Y and a morphism $\Phi : \mathscr{Y} \to \mathscr{X}$ extending φ. Then,*

$$g_{(\mathscr{Y},\Phi^*\mathscr{D})} = g_{(\mathscr{X},\mathscr{D})} \circ \varphi \tag{5.9}$$

and it is a model Green function of $\varphi^* D$. Furthermore, $g_{(\mathscr{X},\mathscr{D})}$ is semipositive, then $g_{\mathscr{X},\mathscr{D}} \circ \varphi$ also is.

Corollary 5.1.11. *Let X be a projective variety and let D be an integral effective divisor on X such that $O_X(D)$ is generated by global sections. Let s_D be the global section defining D and let $s_1,\cdots,s_n$ be global sections of $O_X(D)$ such that $(s_D,s_1,\cdots,s_n)$ generates $O_X(D)$. Then, the function*

$$\forall x \in (X \setminus \operatorname{Supp} D)(\mathbf{C}_v), \quad g_{s_1,\cdots,s_n}(x) := \log^+ \max\left(\left|\frac{s_1}{s_D}(x)\right|,\cdots,\left|\frac{s_n}{s_D}(x)\right|\right) \tag{5.10}$$

is a semipositive model Green function of D, If $\mathbf{C}_v = \mathbf{C}$, then it is a psh Green function of D.

Proof. If $\mathbf{C}_v = \mathbf{C}$ the statement is clear, so we treat only the non-archimedean case. Consider the morphism $\varphi : X \to \mathbf{P}^N$ induced by the sections $s_D, s_1, \cdots, s_n$ such that $\varphi^* X_0 = s_D$. Then, $g_{s_1,\cdots,s_n}$ is the pull back of the model Green function defined in 5.1.9 which is semipositive. By Proposition 5.1.10 it is a semipositive model Green function of D. □

Proposition 5.1.12. *Every* $\mathbf{R}$*-divisor* $D \in \mathrm{Div}(X)_{\mathbf{R}}$ *admits a Green function.*

Proof. We can suppose that D is effective. Let H be an ample divisor on X. Let m be a large enough integer such that $O_X(mH+D)$ and $O_X(mH)$ are both generated by global sections. Let g_1 be a Green function of $mH+D$ and g_2 be a Green function of mH both provided by Corollary 5.1.11. Then, $g_1 - g_2$ is a Green function of D. □

5.1.3 Adelic divisors

Definition 5.1.13. Let X be a projective variety over a number field $\mathbf{K}$. An *adelic* $\mathbf{R}$*-divisor* over X is the data $\left(D, (g_v)_{v \in \mathcal{M}(\mathbf{K})}\right)$ where D is an $\mathbf{R}$-divisor over X and for each place v of $\mathbf{K}$, g_v is a Green function of the divisor $D_{\mathbf{C}_v} := D \otimes \mathrm{Spec}\, \mathbf{C}_v$ over $X_{\mathbf{C}_v}$ such that there exists an open subset U of $\mathrm{Spec}\, O_{\mathbf{K}}$ such that there is a model $(\mathscr{X}_U, \mathscr{D}_U)$ of (X, D) over U and for all $v \in U$, g_v is the model Green function induced by

$$(\mathscr{X}_U \times \mathrm{Spec}_U\, O_v, \mathscr{D}_U \times_U \mathrm{Spec}\, O_v) \quad \text{(coherence condition)} \tag{5.11}$$

An adelic $\mathbf{R}$-divisor is *semipositive* if

- For every archimedean place, g_v is a plurisubharmonic function and $c_1(D_{\mathbf{C}}, g_v)$ is a positive current.
- For every non-archimedean place v, $(D_{\mathbf{C}_v}, g_v)$ is semipositive.

To an adelic $\mathbf{R}$-divisor $\overline{D} = (D, (g_v)_v)$ we can associate a *height function* defined as follows

$$\forall p \in (X \setminus \mathrm{Supp}\, D)(\overline{\mathbf{K}}), \quad h_D(p) := \frac{1}{\deg q} \sum_v \sum_{q \in \mathrm{Gal}(\overline{\mathbf{K}}/\mathbf{K}) \cdot p} n_v g_v(q) \tag{5.12}$$

where n_v is an integer that depend only on the place v.

5.1.4 Metrics over line bundles

Let X be a projective variety over $\mathbf{C}_v$ and let X^{an} be its Berkovich analytification. Since X^{an} is a locally ringed space, we can define line bundles on X^{an}. If L is a line bundle over X we define

$L^{an} := c^*L$ the *analytification* of L where $c : X^{an} \to X$ is the contraction map. Let L be a line bundle over X^{an} a *metric* over L is the data for every $x \in X^{an}$ of a $\mathbf{C}_v$-norm over the stalk L_x of L at x.

Let L be a line bundle over X. A *model* metric of L^{an} is a metric defined as follows. Let $(\mathscr{X}, \mathscr{L})$ be a model of (X,L) over $\mathbf{C}_v$. That is $\mathscr{X}$ is a model of X and $\mathscr{L}$ is a line bundle over $\mathscr{X}$ such that $\mathscr{L} \otimes \operatorname{Spec} \mathbf{C}_v \simeq L^n$ for some integer $n \geqslant 1$. Let $x \in X^{an}$ and let $s \in L^{an}_x$. Then, there exists $s' \in L_{c(x)}$ such that $s = c^*s'$. Now, let $\tilde{s}$ be a local generator of $\mathscr{L}$ at $c(x) \in X \subset \mathscr{X}$. We have that there exists a germ of regular function φ at $c(x)$ such that $(s')^n = \varphi\tilde{s}$. We set

$$||s(x)|| = |\varphi(x)|^{1/n}. \tag{5.13}$$

Example 5.1.14. Let $X = \mathbf{P}^N_{\mathbf{C}_v}$ with homogeneous coordinates $T_0, \cdots, T_N$ and let $L = O(1)$. Consider the model $\mathscr{X} = \mathbf{P}^N_{O_v}$ with the same homogeneous and the line bundle $\mathscr{L} = O(1)_{\mathscr{X}}$ over $\mathscr{X}$. The line bundle L (and $\mathscr{L}$) is generated by the global sections induced by the $T_i's$. Let $x \in X^{an}$, suppose that $r_{\mathscr{X}}(x) \in \{T_i \neq 0\}$ this means that $c(x) \in \{T_i \neq 0\}$ and $\max_{j\neq i} \left|\frac{T_j}{T_i}(x)\right| \leqslant 1$ (indeed, $\frac{T_j}{T_i}$ defines a germ of regular function at $r_{\mathscr{X}}(x)$). Consider s_i the global section of L induced by T_i. Then, it is also a section of $\mathscr{L}$, therefore

$$\begin{aligned} ||T_i(x)|| &= 1 && (5.14)\\ &= \frac{1}{\max\left(\left|\frac{T_0}{T_i}(x)\right|, \cdots, \left|\frac{T_{i-1}}{T_i(x)}\right|, 1, \left|\frac{T_{i+1}}{T_i}(x)\right|, \cdots, \left|\frac{T_n}{T_i}(x)\right|\right)} && (5.15)\\ &= \frac{|T_i(x)|}{\max\left(|T_0(x)|, \cdots, |T_n(x)|\right)} && (5.16) \end{aligned}$$

In particular, consider the global section T_0 and consider the affine space $\{T_0 \neq 0\} \simeq \mathfrak{a}^n$ with homogeneous coordinates $t_1, \cdots, t_n$. Then, we have the Green function of $\{T_0 = 0\}$ given by

$$g(x) = -\log||T_0(x)|| = \log^+ \max(|t_1(x)|, \cdots, |t_n(x)|) \tag{5.17}$$

Which is the model Green function from Proposition 5.1.9.

A model metric is said to be *semipositive* if for every vertical curve $\mathscr{C}$ in $\mathscr{X}$, $\deg_{\mathscr{C}} \mathscr{L} \geqslant 0$.

Proposition 5.1.15. *Let L be a line bundle over X and let $s \in H^0(X,L)$. Then, the function $x \mapsto -\log||(c^*s)(x))||$ is a Green function of* $\operatorname{div}(s)$. *Conversely, if $D \in \operatorname{Div}(X)$ and g is a Green* *function of D. Then, we can define a metric on $O_X(D)^{an}$ by setting*

$$\forall x \in X^{an} \backslash (\mathrm{Supp}\, D)^{an}, \quad ||(c^* s_D)(x)|| = e^{-g(x)} \tag{5.18}$$

5.1.5 Adelic line bundles

Let X be a projective variety over a number field $\mathbf{K}$, an *adelic line bundle* $\overline{L}$ is the data of a line bundle L over X and a collection of metrics $\{||\cdot||_v\}\, v \in \mathcal{M}(\mathbf{K})$ such that there exists an open subset $U \subset \mathrm{Spec}\, O_{\mathbf{K}}$ and a model $(\mathscr{X}_U, \mathscr{L}_U)$ of (X, L) over U such that for every place $v \in U$, the metric $||\cdot||_v$ is the metric induced by the model $(\mathscr{X}_U \times \mathrm{Spec}\, O_v, \mathscr{L}_U \times \mathrm{Spec}\, O_v)$.

An adelic line bundle is *semipositive* if

- For every archimedean place v, $c_1(L_{\mathbf{C}}, ||\cdot||_v)$ is a positive current.
- For every non archimedean place v, the metric $||\cdot||_v$ on L_v is a uniform limit of semipositive model metric on L_v.

It is *integrable* if it is the difference of two semipositive adelic line bundles.

To an adelic line bundle $\overline{L}$, we can associate a *height* function $h_{\overline{L}}$ defined for all closed subvarieties of X defined by the following formula

$$h_{\overline{L}}(Z) = \frac{(\overline{L}_Z)^{\dim Z + 1}}{\deg_Z(L_{|Z})}. \tag{5.19}$$

In particular, if $s \in H^0(X, L)$ is a global section of L, then for all $p \in (X \backslash \mathrm{Supp}\, \mathrm{div}(s))(\overline{\mathbf{K}})$,

$$h_{\overline{L}}(x) = \frac{1}{\deg x} \sum_v \sum_{y \in \mathrm{Gal}(\overline{\mathbf{K}}/\mathbf{K}) \cdot x} -n_v \log ||s(y)||_{\overline{L}_v} \tag{5.20}$$

which is exactly the height function associated to the adelic divisor $(\mathrm{div}(s), -\log ||s||_{\overline{L}_v})$ (see (5.12)).

5.1.6 Chambert-Loir measure

Let X be a projective variety over $\mathbf{C}_v$ of dimension d, let $\overline{L_1}, \cdots, \overline{L_d}$ be integrable metrized line bundles, then Chambert-Loir constructed in [Cha03] a measure

$$c_1(\overline{L_1}) \cdots c_1(\overline{L_d}) \tag{5.21}$$

defined over X^{an}. Here are the main properties of this measure:

Proposition 5.1.16. *If for every i, there exists e_i such that $\overline{L_i}^{e_i}$ is induced by a model $(\mathscr{X}, \mathscr{L}_i)$. Then, let $\mathsf{X}_1, \cdots, \mathsf{X}_l$ be the irreducible components of the special fiber X of $\mathscr{X}$ and let L_i be the restriction of $\mathscr{L}_i$ to the special fiber of $\mathscr{X}$. For each j, there exists a unique point $\xi_j \in X^{an}$ such that $r(\xi_j)$ is the generic point of X_j, we have*

$$c_1(\overline{L_1}) \cdots c_1(\overline{L_d}) = \frac{1}{e_1 \cdots e_d} \sum_j \left(c_1(\mathsf{L}_1) \cdots c_1(\mathsf{L}_d) | \mathsf{X}_j \right) \delta_{\xi_j} \tag{5.22}$$

This is in fact how the measure is defined for the model case.

Proposition 5.1.17 ([Cha03]). *Let X be a projective variety over $\mathbf{C}_v$ of dimension d. Let $\overline{L_1}, \cdots, \overline{L_d}$ be semipositive metrized line bundles, then for any sequences $(\overline{L_{i,n}})_n$ of semipositive model metrics of L_i converging to $\overline{L_i}$ one has that the measures*

$$c_1(\overline{L_{1,n}}) \cdots c_1(\overline{L_{n,d}}) \tag{5.23}$$

converges to a measure independent of the choices of the sequences. We denote this measure $c_1(\overline{L_1}) \cdots c_d(\overline{L_d})$. Furthermore, it has total mass

$$\int_{X^{an}} c_1(\overline{L_1}) \cdots c_d(\overline{L_d}) = c_1(L_1) \cdots c_1(L_d) \tag{5.24}$$

In particular, we write $\mu_{\overline{L}} := \frac{1}{c_1(L)^d} c_1(\overline{L}) \cdots c_1(\overline{L})$, we call it the *equilibrium* measure of $\overline{L}$, it is a probability measure by Proposition 5.1.17. If $\overline{L}$ is an adelic line bundle over a projective variety X over a number field $\mathbf{K}$, we write $\mu_{\overline{L},v}$ for the equilibrium measure of $\overline{L}_v$.

5.1.7 Equidistribution

Let (x_n) be a sequence of $X(\overline{K}) \subset X(\overline{\mathbf{C}}_v)$ and let μ_v be a measure on $(X_{\mathbf{C}_v})^{an}$. We say that the Galois orbit of (x_n) is equidistributed with respect to μ_v if the sequence of measures

$$\delta(x_n) := \frac{1}{\deg(x_n)} \sum_{x \in Gal(\overline{\mathbf{K}}/\mathbf{K}) \cdot x_n} \delta_x \tag{5.25}$$

weakly converges towards μ, where δ_x is the Dirac measure at x.

We say that a sequence of points (x_n) of $X(\overline{\mathbf{K}})$ is *generic* if no subsequence of (x_n) is contained in a strict subvariety of X. In particular, a generic sequence is Zariski dense.

Lemma 5.1.18. *Let X be a projective variety over a number field $\mathbf{K}$ and let (x_n) be a Zariski dense sequence of $X(\overline{K})$, then one can extract a generic subsequence of (x_n).*

Proof. The set of strict irreducible subvarieties of X is countable because $\mathbf{K}$ is a number field. Let $(Y_q)_{q\in\mathbf{N}}$ be the set of strict irreducible subvarieties of X. We construct a generic subsequence $(x'_q)_{q\in\mathbf{N}}$ as follows. Set $Y'_q = \bigcup_{k\leqslant q} Y_k$. This is a strict subvariety of X, since (x_n) is Zariski dense, there exists an integer $n(q)$ such that $x_{n(q)} \notin Y'_q$. We set $x'_q = x_{n(q)}$. The sequence (x'_q) is a subsequence of (x_n) which is clearly generic. □

Theorem 5.1.19 (Yuan-Zhang equidistribution theorem, [YZ22])**.** *Let X be a projective variety and let $\overline{L}$ be a semipositive adelic line bundle over X such that $\deg_X(L) > 0$. Let $(x_n) \in X(\overline{\mathbf{K}})$ be a generic sequence such that $\lim_n h_{\overline{L}}(x_n) \to h_{\overline{L}}(X)$, then at every place v the Galois orbit of the sequence (x_n) is equidistributed with respect to the equilibrium measure $\mu_{\overline{L},v}$.*

5.1.8 Intersection of line bundles

Let X be a projective variety over $\mathbf{C}_v$ of dimension d and let $\overline{L}_0, \cdots, \overline{L}_d$ be integrable line bundles over X. Then, there exists an intersection number

$$\overline{L}_0 \cdots \overline{L}_d \tag{5.26}$$

with the following properties:

1. It is multilinear.

2. If s is a global rational section of L_0, then

$$\overline{L}_0 \cdots \overline{L}_d = \left(\overline{L}_1 \cdots \overline{L}_d | \operatorname{div}(s)\right) - \int_{X^{an}} \log ||s|| \, c_1(\overline{L}_1) \cdots c_1(\overline{L}_d) \tag{5.27}$$

Theorem 5.1.20 (Arithmetic Hodge index theorem, [YZ17])**.** *Let X be a projective surface over some complete algebraically closed field $\mathbf{C}_v$. Let D be a big, nef and effective divisor on X and let $\overline{L}$ be a semipositive metrized line bundle such that $L = O_X(D)$. If $(M, ||\cdot||)$ is an integrable metrized line bundle such that $M = O_X$, then*

$$\overline{M}^2 \cdot \overline{L} \leqslant 0. \tag{5.28}$$

Furthermore, if $\overline{M}$ is $\overline{L}$ bounded, we have equality if and only if $||\cdot||$ is constant.

As in [YZ17], we get the following corollary

Corollary 5.1.21 (Calabi Theorem)**.** *Let D be a big, nef and effective divisor over a projective surface X over $\mathbf{C}_v$. Let g_1, g_2 be two semipositive Green functions of D such that $c_1(D,g_1)^2 = c_1(D,g_2)^2$, then $g_1 - g_2$ is constant over X.*

Proof. Let $\overline{L_i}$ be the metrized line bundle such that $L_i = O_X(D)$ and the metric on $\overline{L_i}$ is induced by g_i. Consider $\overline{M} = \overline{L_1} - \overline{L_2}$. Let $f = g_1 - g_2$, then

$$\overline{M} \cdot \overline{L_1}^2 = -\int_{X^{an}} f c_1(\overline{L_1})^2 = -\int_{X^{an}} f c_1(\overline{L_2})^2 = \overline{M} \cdot \overline{L_2}^2. \tag{5.29}$$

Hence we get

$$\overline{M}^2 \cdot (\overline{L_1} + \overline{L_2}) = 0 \tag{5.30}$$

Now, $\overline{M}$ is $(\overline{L_1} + \overline{L_2})$-bounded so by the equality case in the Arithmetic hodge index theorem we get that $g_1 - g_2$ is constant. $\square$

Proof of Arithmetic Hodge index theorem. The only part not shown by Yuan and Zhang is the equality part in the case where we only suppose that D is big, nef and effective and not ample. So we prove only the second assertion. Suppose that $\overline{M}^2 \cdot \overline{L} = 0$. Following the proof of [YZ17], We have the following result

Lemma 5.1.22 ([YZ17], Lemma 2.5)**.** *For any integrable line bundle $\overline{M'}$ such that $M' = O_X$, we have*

$$\overline{M}^2 \cdot \overline{M'} = 0 \tag{5.31}$$

In particular, it implies that $c_1(\overline{M})^2 = 0$. Indeed, since $\overline{M}^2 \cdot \overline{M}' = 0$, this means that

$$\int_{X^{an}} g' c_1(\overline{M})^2 = 0 \tag{5.32}$$

where $g' = \log ||1||_{\overline{M'}}$. So this holds for any model metric of the trivial line bundle. Now, by a result of Gubler or [Mor16] Theorem 3.3.3, the set of model metric of the trivial line bundle is dense in the set of all real-valued continuous function over X^{an} for the topology of uniform convergence so we get $c_1(\overline{M})^2 = 0$.

Lemma 5.1.23. *For all curve $C \subset X$,*

$$\overline{M}^2_{|C} = 0 \tag{5.33}$$

Proof. We first show that there exists an integer $m \geqslant 1$ such that $O_X(mD)$ has a section that vanishes along C. Indeed, if C is not in the support of D, since D is big , by [Laz04] Proposition 2.2.6, there exists an integer $m \geqslant 1$ such that $H^0(X, O_X(mD - C) \neq \emptyset$. Therefore, we can find a section $s \in H^0(X, mD)$ such that s vanishes along C. If C is in the support of D, there is a global section $s \in H^0(X, O_X(D))$ such that $\operatorname{div}(s) = D$, in particular it vanishes along C.

Write $\operatorname{div}(s) = \sum_i a_i C_i$ with $a_i > 0$. We get

$$0 = \overline{M}^2 \cdot \overline{D} = \sum_i a_i \overline{M}^2_{|C_i} - \int_{X^{an}} \log ||s||_{\overline{L}} c_1(\overline{M})^2. \tag{5.34}$$

By Lemma 5.1.22, we get $0 = \sum_i a_i (\overline{M}_{|C_i})^2$. By the arithmetic Hodge index theorem in the case of curves, every term in the sum is nonpositive, hence there are all equal to 0. Since C is one of the C_i we get the result. □

Now, the equality case when X is a curve is shown in [YZ17] and therefore we get that for every curve $C \subset X$, $g_{|C^{an}}$ is constant where $g = \log ||1||_{\overline{M}}$. We are going to show that g is constant. The set of rational points $X(\mathbf{C}_v)$ is Zariski dense in X^{an} so it suffices to show that g is constant on this subset. Let $p, q \in X(\mathbf{C}_v)$ be two closed point, it suffices to show that there exists a connected curve of X containing p and q. If we embed X in some projective space $\mathbf{P}^N$ we get by [Har77] Chapter III Corollary 7.9 that for every hyperplane $H, H \cap X$ is connected. Therefore, if H is a hyperplane containing p and q, $H \cap X$ is connected curve C that contains p and q and we get the result. □

5.2 Definition of the Green functions

Definition 5.2.1. Fix a completion X of X_0 that satisfy Theorem 4.4.18. Let $D \in \mathrm{Div}_\infty(X)_{\mathbf{R}}$ and let $G(D)$ be a Green function of D. Recall that f is a fixed loxodromic automorphism of X_0. We define the sequence of continuous functions over $X_0(\mathbf{C}_v)$

$$G^+_{n,D} := \frac{1}{\lambda_1^n} G(D) \circ (f)^n \tag{5.35}$$

$$G^-_{n,D} := \frac{1}{\lambda_1^n} G(D) \circ (f)^{-n} \tag{5.36}$$

In the following we are going to state all the results for the sequence $G^+_{n,\bullet}$ as everything is analogous for $G^-_{n,\bullet}$.

Remark 5.2.2. The choice of the Green function $G(D)$ is not canonical but by Proposition 5.1.4, the limit process we are going to apply will not depend on this choice.

Proposition 5.2.3. *For any effective* $\mathbf{R}$*-divisor* $R \in \mathrm{Div}_\infty(X)_{\mathbf{R}}$. *The function*

$$G(R) \circ f^{an} - G(f_X^* R) \tag{5.37}$$

extends to a continuous function over $X^{an} \backslash p_-$.

Proof. Let $\pi : Y \to X$ be a completion above $p_- \in X$ such that the lift $F : Y \to X$ is regular. By definition, $f_X^* R = \pi_* F^* R$. Now, by Proposition 5.1.6, we have that $G(R) \circ F^{an}$ is a Green function of $F^* R$ over Y^{an}. Now, π induces an isomorphism $\pi : Y \backslash \mathrm{Exc}(\pi) \to X \backslash p_-$. Let $q \in \partial_X X_0 \backslash p_-$, let ψ be a local equation of $f_X^* R$ at q. By definition, $F^* R - \pi^* f_X^* R$ is π-exceptional, therefore $\pi^* \psi$ is a local equation of $F^* R$ at $\pi^{-1} q$. Thus, the function

$$G(R) \circ F + \log |\pi^* \psi| \tag{5.38}$$

extends to a continuous function at $\pi^{-1}(q)$ and therefore $G(R) \circ f + \log |\psi|$ extends to a continuous function at q. $\square$

5.2.1 The Green function of θ_X^+

Start with the following lemma

Lemma 5.2.4. *Let $\pi : Y \to X$ be a birational morphism between smooth projective surfaces. Let $D \in \mathrm{Div}(Y)_{\mathbf{R}}$, suppose that D is effective and nef, then*

$$\pi^*\pi_* D \geqslant D \tag{5.39}$$

Proof. If $\pi = \mathrm{id}$ then the lemma is true. Suppose $\pi = \pi' \circ \tau$ where $\tau : Y \to X'$ is the blow up of a point. Let $D \in \mathrm{Div}(Y)_{\mathbf{R}}$ be nef and effective, then

$$\pi^*\pi_* D = \tau^*(\pi')^*\pi'_*\tau_* D. \tag{5.40}$$

By induction, we get $(\pi')^*\pi'_*(\tau_* D) \geqslant \tau_* D$ because $\tau_* D$ is nef and effective. Therefore, it suffices to show that $\tau^*\tau_* D \geqslant D$. Let $p \in X'$ be the center of τ, write $D' = \tau_* D = \sum_i a_i C_i + R$ with $p \in C_i$ and $p \notin \mathrm{Supp} R$. Let $\widetilde{E}$ be the exceptional divisor above p, then

$$\tau^* D' = \tau' R + \sum_i a_i \tau'(C_i) + (\sum_i a_i m_i)\widetilde{E} \tag{5.41}$$

where m_i is the multiplicity of C_i at p and

$$D = \tau' R + \sum_i a_i \tau'(C_i) + \delta \widetilde{E}. \tag{5.42}$$

Since D is nef, we have $D \cdot \widetilde{E} \geqslant 0$. Hence,

$$D \cdot \widetilde{E} = -\delta + \sum_i a_i m_i \geqslant 0 \tag{5.43}$$

and $\delta \leqslant \sum_i a_i m_i$ which shows the result. □

Proposition 5.2.5. *The sequence (G^+_{n,θ^+_X}) converges uniformly over any compact of $X_0(\mathbf{C}_v)$ to a continuous function $G^+_{\theta^+_X}$ that satisfy the following properties*

(1) $G^+_{\theta^+_X} \circ f = \lambda_1 G^+_{\theta^+_X}$.

(2) $\left\{ G^+_{\theta^+_X} > 0 \right\} = \bigcup_{n \geqslant 0} f^{-n}(U_+ \setminus \partial_X X_0)$.

(3) $\forall p \in X_0(\mathbf{C}_v), G^+_{\theta^+_X}(p) \geqslant 0$ *and* $G^+_{\theta^+_X}(p) = 0$ *if and only if the forward f-orbit of p is bounded.*

(4) *If $\mathbf{C}_v = \mathbf{C}$, then G_+ is a plurisubharmonic function over $X_0(\mathbf{C})$, it is pluriharmonic on the set $\{G_+ > 0\}$.*

(5) *The function $G^+ - G(\theta_X^+)$ extends to a continuous function h over $(X \backslash p_-)(\mathbf{C}_v)$ which is bounded from above.*

(6) *The sequence $\left(G^+_{n,\theta_X^+} - G(\theta_X^+)\right)$ converges uniformly to h over any compact subset of $X(C_v) \backslash p_-$.*

Proof. By Proposition 5.2.3, the function

$$\Psi := \frac{1}{\lambda_1} G(\theta_X^+) \circ f - G(\theta_X^+) \tag{5.44}$$

extends to a continuous function over $X(C_v) \backslash p_-$. We first show that Ψ is bounded from above. Let $\pi : Y \to X$ be a morphism of completions such that the lift $F : Y \to X$ is regular. We have $\lambda_1 \theta_X^+ = f_X^* \theta_X^+ = \pi_* F^* \theta_X^+$. By Lemma 5.2.4, we get that there exists an effective divisor $R \in \mathrm{Div}_\infty(Y)$ such that $\lambda_1 \pi^* \theta_X^+ = F^* \theta_X^+ + R$. By Proposition 5.1.6, we get that

$$\Psi = -G(R) + O(1). \tag{5.45}$$

And by Proposition 5.1.5, we get that Ψ is bounded from above. Set $G := G(\theta_X^+), G_n^+ := G^+_{n,\theta_X^+}$. In particular, $G_0^+ = G$. We have

$$G_n^+ = \frac{1}{\lambda_1^n} G \circ f^n = \frac{1}{\lambda_1^{n-1}} \left(\frac{1}{\lambda_1} G \circ f \right) \circ f^{n-1} \tag{5.46}$$

$$= \frac{1}{\lambda_1^{n-1}} (\Psi + G) \circ f^{n-1} \tag{5.47}$$

$$= \frac{1}{\lambda_1^{n-1}} \Psi \circ f^{n-1} + G_{n-1}^+ \tag{5.48}$$

By induction we get

$$G_n^+ = G_0^+ + \sum_{k=0}^{n-1} \frac{1}{\lambda_1^k} \Psi \circ f^k \tag{5.49}$$

So, for all $n \geqslant 0, G_n^+ - G_0^+ = G_n^+ - G$ extends to a continuous function over $X \backslash p_-$ which is bounded from above since ψ is. Now, let U_- be a small open neighbourhood of p_-. Since p_- is a super attracting fixed point of f^{-1} we can suppose that $f^{-1} U_- \subset U_-$ so that $W := X \backslash U_-$ is

f-invariant. The function $|\psi|$ is bounded by a constant M and therefore

$$\sup_W \frac{1}{\lambda^n}\psi \circ f^n \leqslant \frac{M}{\lambda_1^n} \tag{5.50}$$

In particular, G_n^+ converges uniformly over $W \cap X_0(\mathbf{C}_v)$ to a continuous function $G^+_{\theta_X^+}$ and $G^+_{\theta_X^+} - G_0^+ = G^+_{\theta_X^+} - G$ extends to continuous bounded from above function over $X \backslash p_-$. This shows (5) and (6).

Proof of (1): This follows from $G^+_{n,\theta_X^+} \circ f = \lambda_1 G_{n+1,\theta_X^+}$.

Proof of (2) and (3): Since $G(\theta_X^+)(p) \to +\infty$ when $p \to p_+$ we can replace U^+ by a smaller f-invariant subset such that $(G^+_{\theta_X^+})_{|U^+ \cap \partial_X X_0} > 0$. By (1), we get

$$\bigcup_{n \geqslant 0} f^{-n}(U^+ \backslash \partial_X X_0) \subset \left\{ G^+_{\theta_X^+} > 0 \right\} \tag{5.51}$$

. To get the other inclusion, we use Corollary 4.4.22. Let $p \in X_0(\mathbf{C}_v)$. If $(f^n(p))_{n \geqslant 0}$ is bounded, then $G^+_{\theta_X^+}(p) = 0$. If not, then by Corollary 4.4.22 we have that $f^n(p) \xrightarrow[n \to +\infty]{} p_+$ so for n large enough $f^n(p) \in U^+$ and by (1), $G^+(p) = \frac{1}{\lambda_1^n} G^+(f^n(p)) > 0$.

Proof of (4): We will show in Proposition 5.2.12 that G^+ is locally the uniform limit of a sequence of psh functions, so G^+ is plurisubharmonic. We show the pluriharmonicity over $\left\{ G^+_{\theta_X^+} > 0 \right\}$ we only need to show by (1) and (2) that $G^+_{\theta_X^+}$ is pluriharmonic over $U^+ \cap \partial_X X_0$. We have that U^+ is f-invariant. Let (u, v) be local analytic coordinates at p_- such that if $p_+ \in E$ is a free point, then $u = 0$ is a local equation of the E; and if $p_+ = E \cap F$ is a satellite point then $uv = 0$ is a local equation of $E \cup F$ at p_+.

In the free case, we have that $G(\theta_X^+) = \alpha \log |u| + \log |\varphi|$ where φ is an invertible holomorphic function over U^+, then $(f^n)^* u = u^{\lambda_1^n} \psi_n$ where ψ_n is an invertible holomorphic function over U^+ and $(f^n)^* \circ \varphi$ is still an invertible holomorphic function over U^+, therefore

$$\frac{1}{\lambda_1^n} G(\theta_X^+) \circ f^n = \alpha \log |u| + \frac{\alpha}{\lambda_1^n} \log |\psi_n| + \frac{1}{\lambda_1^n} \log |(f^n)^* \varphi| \tag{5.52}$$

over $U^+ \cap X_0$. Since u does not vanish on $U^+ \cap X_0$ we have that G^+ is a uniform limit of pluriharmonic functions over $U^+ \cap X_0$ so it is pluriharmonic.

□

5.2.2 The Green function for any divisor not supported on E_+ and F_+

Let $R \in \mathrm{Div}_\infty(X)_\mathbf{R}$ be an effective divisor such that $\mathrm{Supp} R \cap \{E^+, F^+\} = \varnothing$ where E_+, F_+ are the prime divisor at infinity on which $c_X(\nu_+)$ lies. If it is a free point, we use the convention that $E_+ = F_+$.

Proposition 5.2.6. *For any such* $\mathbf{R}$*-divisor* R*, the function*

$$G(R) \circ f \tag{5.53}$$

extends to a continuous function over $X(C_v) \backslash p_-$*.*

Proof. For any E in the support of R, we have $f_X^*(E) = 0$, therefore by Proposition 5.2.3, we have that $G(E) \circ f$ extends to a continuous function over $X(C_v) \backslash p_-$. □

Corollary 5.2.7. *For any such* $\mathbf{R}$*-divisor, the sequence* $G^+_{n,R}$ *converges uniformly to the zero function over any compact subset of* $X \backslash p_-$*.*

Proof. Any compact subspace of X_0 is a subset of $X \backslash U^-$ for some open neighbourhood U^- of p_-. We can shrink U^- such that $f^{-1}(U^-) \Subset U^-$. Therefore, $W := X \backslash U^-$ is f-invariant. The function $G(R) \circ f$ is continuous over W by Proposition 5.2.6. Now W is compact, therefore $|G(R) \circ f|$ is bounded over W. We get

$$\sup_W \left| \frac{1}{\lambda_1^n} G(R) \circ f^n \right| \leqslant \frac{1}{\lambda_1^n} \sup_W |G(R) \circ f| \to 0 \tag{5.54}$$

□

5.2.3 The Green function for D^-

Proposition 5.2.8. *If we are in the cycle case, there exists* $D^- \in \mathrm{Div}_\infty(X)_\mathbf{R}$ *such that*

$$f_X^* D^- = \frac{1}{\lambda_1} D^-. \tag{5.55}$$

Proof. Write $\theta_X^+ = \alpha^+ E_+ + \beta^+ F_+ + \cdots$. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix associated to the normal form of f at p_+. From $f_X^* \theta_X^+ = \lambda_1 \theta_X^+$ we get that the vector (α^+, β^+) is an eigenvector of ${}^tM$ for the eigenvalue λ_1. Since $\det M = \lambda_2(f) = 1$ there exists $\alpha^-, \beta^- \in \mathbf{R}$ such that (α^-, β^-)

is an eigenvector of ${}^tM$ for the eigenvalue $1/\lambda_1$. Define $D := \alpha^- E_+ + \beta^- F_+$. We have

$$f_X^* D = \frac{1}{\lambda_1} D + R \tag{5.56}$$

with $E_+, F_+ \notin \operatorname{Supp} R$, therefore $f_X^* R = 0$. Set $D^- := D + \lambda_1 R$, then

$$f_X^* D^- = f_X^* D = \frac{1}{\lambda_1} D + R = \frac{1}{\lambda_1} (D + \lambda_1 R) = \frac{1}{\lambda_1} D^- \tag{5.57}$$

□

Lemma 5.2.9. *One has* $\theta^- \cdot D^- = 0$.

Proof. We have $\theta^- \cdot f_X^* D^- = \lambda_1 \theta^- \cdot D^-$ because θ^- is associated to the eigenvaluation of f. On the other hand,

$$\theta^- \cdot f_X^* D^- = \frac{1}{\lambda_1} \theta^- \cdot D^-. \tag{5.58}$$

Since $\lambda_1 > 1$, we get $\theta^- \cdot D^- = 0$.

□

Lemma 5.2.10. *The family* $(\theta_X^+, D^-) \cup (E; E \notin \{E_+, F_+\})$ *is a basis of* $\operatorname{Div}_\infty(X)_{\mathbf{R}}$.

Proof. Since the length of the family is the dimension of $\operatorname{Div}_\infty(X)$, we only need to show that the family is free. Suppose that

$$\lambda \theta_X^+ + \mu D^- + R = 0 \tag{5.59}$$

with $\lambda, \mu \in \mathbf{R}$ and $R \in \operatorname{Div}_\infty(X)_{\mathbf{R}}, E_+, F_+ \notin \operatorname{Supp} R$. Since $\theta_X^- \cdot \theta_X^+ = \theta^+ \cdot \theta^- = 1$ and $\theta^- \cdot R = 0$, by intersecting Equation (5.59) with θ_X^- we get

$$\lambda = 0 \tag{5.60}$$

Now, write $D^- = \alpha^- E_+ + \beta^- F_+$. From the proof of Proposition 5.2.8, we have either $\alpha^- \neq 0$ or $\beta^- \neq 0$ since the vector (α^-, β^-) is an eigenvector for an invertible 2×2 matrix. Suppose for example that $\alpha^- \neq 0$, then intersecting Equation (5.59) with $Z_{\mathrm{ord}_{E_+}}$, we get

$$\mu \alpha^- = 0 \tag{5.61}$$

and therefore $\mu = 0$. It remains that $R = 0$ and the result is proven. □

Proposition 5.2.11. *The sequence* $\left(G^+_{n,D^-}\right)$ *converges uniformly to zero over any compact subset of* $X_0(\mathbf{C}_v)$*. Moreover the sequence* $\left(G^+_{n,D_-} - \frac{1}{\lambda_1^{2n}}G(D^-)\right)$ *converges uniformly to the zero function over any compact subspace of* $X(C_v)\backslash p_-$*.*

Proof. Set $G := G(D^-)$ and $G^+_n := G^+_{n,D^-}$. From $f_X^* D^- = \frac{1}{\lambda_1}D^-$ and Proposition 5.2.3 we get that the function

$$\Psi = G(D^-)\circ f - \frac{1}{\lambda_1}G(D^-) \tag{5.62}$$

extends to a continuous function over $X\backslash p_-$. By an analogous computation as before we get

$$\begin{aligned} G^+_n &= \frac{1}{\lambda_1^n}\left(G(D^-)\circ f\right)\circ f^{n-1} && (5.63)\\ &= \frac{1}{\lambda_1^n}\left(\Psi + \frac{1}{\lambda_1}G(D^-)\right)\circ f^{n-1} && (5.64)\\ &= \frac{1}{\lambda_1^n}\Psi\circ f^{n-1} + \frac{1}{\lambda_1^2}G^+_{n-1} && (5.65)\end{aligned}$$

By induction, we get

$$G^+_n = \sum_{k=0}^{n}\left(\frac{1}{\lambda_1^{n+k}}\Psi\circ f^{n-k+1}\right) + \frac{1}{\lambda_1^{2n}}G(D^-) \tag{5.66}$$

Take a small open neighbourhood U_- of p_- such that $W := X\backslash U_-$ is f-invariant. For any compact subset $K\subset X_0(\mathbf{C}_v)\cap W$, we get

$$\sup_K\left|G^+_n\right| \leqslant \frac{1}{\lambda_1^n}\cdot\sup_W|\Psi|\cdot\left(\frac{\lambda_1}{\lambda_1-1}\right) + \frac{1}{\lambda_1^{2n}}\sup_K\left|G(D^-)\right| \to 0 \tag{5.67}$$

□

5.2.4 The Green function for any divisor

Proposition 5.2.12. *Let H be an* $\mathbf{R}$*-divisor supported at infinity, then the sequence* $(G^+_{n,H})$ *of continuous function over* $X_0(\mathbf{C}_v)$ *converges uniformly locally to the function* $(H\cdot\theta^-)G^+_{\theta^+_X}$*. Moreover, there exists a real number t such that the sequence*

$$\left(G^+_{n,H} - (H\cdot\theta^-)G(\theta^+_X) + \frac{t}{\lambda_1^{2n}}G(D^-)\right)_n \tag{5.68}$$

converges to a continuous function over $(X \backslash p_-)(\mathbf{C}_v)$ uniformly over any compact subspace of $X(C_v) \backslash p_-$.

Proof. If we are in the cycle case, let D^- be the divisor from Proposition 5.2.8. If we are in the zigzag case, set $D^- = 0$. By Lemma 5.2.10, we can write

$$H = (H \cdot \theta^-)\theta_X^+ + \mu D^- + R. \tag{5.69}$$

with $E_+, F_+ \notin \operatorname{Supp} R$. Therefore, we get that for all $n \geqslant 0$

$$G_{n,H}^+ = (H \cdot \theta^-)G_{n,\theta_X^+}^+ + \mu G_{n,D^-}^+ + G_{n,R_X}^+ \tag{5.70}$$

By Propositions 5.2.5, 5.2.7 and 5.2.11, $G_{n,H}^+$ converges uniformly locally to $(H \cdot \theta^-)G_{\theta_X^+}^+$ and we also get the result on the uniform convergence over any compact subset of $X \backslash p_-$ □

Corollary 5.2.13. *The function $G_{\theta_X^+}$ is plurisubharmonic over $X_0(\mathbf{C})$.*

Proof. Let H be a very ample divisor supported at infinity, then $H \cdot \theta^- > 0$ and by Proposition 5.2.12 $(H \cdot \theta^-)G_{\theta_X^+}$ is uniformly locally the limit of $\frac{1}{\lambda_1^n} G(H) \circ f^n$, now since H is very ample, it is globally generated so by Corollary 5.1.11 we can suppose that $G(H)$ is plurisubharmonic over $X_0(\mathbf{C})$. Then, for all $n \geqslant 0$, $\frac{1}{\lambda_1^n} G(H) \circ f^n$ is plurisubharmonic, so $G_{\theta_X^+}$ also is. □

5.2.5 An invariant adelic divisor

Lemma 5.2.14. *The $\mathbf{R}$-divisor $D = \max(\theta_X^+, \theta_X^-)$ is big, nef and effective.*

Proof. It is obvious that D is effective since θ_X^+ and θ_X^- both are. For every prime divisor E at infinity, set $a_\pm(E) = \operatorname{ord}_E(\theta_X^\pm)$. Let E be a prime divisor at infinity, then since X is a good completion

$$\theta_X^\pm \cdot E = a_\pm(E)E^2 + \sum_{|F \cap E| = 1} a_\pm(F). \tag{5.71}$$

And,

$$D \cdot E = \max(a_+(E), a_-(E))E^2 + \sum_{F \neq E} \max(a_+(F), a_-(F)) \tag{5.72}$$

If for example $a_+(E) \geqslant a_-(E)$, we get

$$D \cdot E \geqslant a_+(E)E^2 + \sum_{F \neq E} a_+(F) \geqslant \theta_X^+ \cdot E \geqslant 0 \tag{5.73}$$

Therefore, D is nef. Since the intersection form is non-degenerate over $\mathrm{Div}_\infty(X)$ there must exist a prime divisor E at infinity such that $D \cdot E > 0$, therefore $D^2 > 0$ and D is big. $\square$

Set $G^+ := G^+_{\theta_X^+}$ and $G^- := G^-_{\theta_X^-}$.

Proposition 5.2.15. *Suppose that $\lambda_1(f)$ is an integer. Let $G := \max(G^+, G^-)$, then*

(1) G is a continuous function over $X_0(\mathbf{C}_v)$.

(2) If $\mathbf{k} = \mathbf{C}$, then G is plurisubharmonic on $X_0(\mathbf{C})$, it is pluriharmonic on $\{G > 0\}$.

(3) $G(p) = 0$ if and only if the orbit of p under $f^{\mathbf{Z}}$ is bounded. In particular, $\{G = 0\}$ is a compact subset of $X_0(\mathbf{C}_v)$.

(4) The function $G - G(\max(\theta_X^+, \theta_X^-))$ extends to a continuous function Ψ over X.

(5) Set $G_n = \max\left(G^+_{n,\theta_X^+}, G^-_{n,\theta_X^-}\right)$, then the sequence of continuous functions

$$\left(G_n - \max\left(G(\theta_X^+), G(\theta_X^-)\right)\right)_n \tag{5.74}$$

converges uniformly to Ψ over X^{an}.

Proof. (1) is immediate as both G^+ and G^- are continuous over $\mathbf{X}_0(\mathbf{C}_v)$.

(2) is also direct as the maximum of two plurisubharmonic (resp. pluriharmonic) is plurisubharmonic (resp. pluriharmonic)

(3): $G(p) = 0 \Leftrightarrow G^+(p) = G^-(p) = 0$ so the forward and backward orbit of p under f has to be bounded.

(4)-(5): On a small open neighbourhood U^- of p_- we have $G(\theta_X^+) \leqslant G(\theta_X^-)$ because, by our assumption, if $c_X(v_-) \in E$, then $\mathrm{ord}_E(\theta^-) > \mathrm{ord}_E(\theta^+)$. Now, by Proposition 5.2.5, there exists a constant $M > 0$ such that over $U^- \cap X_0$,

$$G^+(p) \leqslant G(\theta_X^+)(p) + M \tag{5.75}$$

$$G^-(p) \geqslant G(\theta_X^-)(p) - M. \tag{5.76}$$

We can shrink U^- even more such that on $U^-, G(\theta_X^-) > G(\theta_X^+) + 10M$ because of the weights of θ_X^+, θ_X^- at the prime divisor at infinity on which p_- lies. Therefore, $G^- > G^+$ over $U^- \cap X_0$ and $G = G^-$ on $U^- \cap X_0$. Therefore, $G - G(\max(\theta_X^+, \theta_X^-))$ extends to a continuous function at p_-. The same assertion holds at p_+. This shows (4). Now to show (5), set $W = X \setminus \left(U^+ \cup U^-\right)$. We have by Proposition 5.2.5 that $G^+_{n,\theta_X^+} - G(\theta_X^+)$ converges uniformly to $G^+ - G(\theta_X^+)$ over $W \cup U^+$ and that $G^- n, \theta_X^- - G(\theta_X^-)$ converges uniformly to $G^- - G(\theta_X^-)$ over $W \cup U^-$. We therefore get that $\max(G^+_{n,\theta_X^+}, G^-_{n,\theta_X^-})$ converges uniformly towards G over W. Now since $G^+ > G^-$ over $U_+ \cap X_0$ and $G^- > G^+$ over $U^- \cap X_0$ the convergence is uniform over $X(\mathbf{C}_v) = W \cup U^+ \cup U^-$. This shows (5). □

Proposition 5.2.16. *Let X_0 be an affine surface over a number field $\mathbf{K}$, let f be a loxodromic automorphism of X_0 with $\lambda_1(f) \in \mathbf{Z}_{\geqslant 0}$ and let X be as in Theorem 4.4.18. If $G_v = \max(G_v^+, G_v^-)$, then $\left(\max(\theta_X^+, \theta_X^-), (G_v)_{v \in \mathcal{M}(\mathbf{K})}\right)$ is a semipositive $\mathbf{Q}$-adelic divisor over X. In particular, the corresponding adelic line bundle $\overline{L}$ satisfies the hypothesis of Theorem 5.1.19.*

Proof. We show the semipositivity. Let v be a place of K, replace X by X_v and set $D^\pm = \theta_X^\pm$ and $D := \max(D^+, D^-)$. Since θ_X^+ and θ_X^- are both big and nef and their support supports an ample divisor there exists an integer m such that $O_X(m\theta_X^+)$ and $O_X(m\theta_X^-)$ are generated by global sections. Set $s_{mD^+}, s_1^+, \cdots, s_r^+$ and $s_{mD^-}, s_1^-, \cdots, s_t^-$ be a set of global sections generating $O_X(mD^+), O_X(mD^-)$ and let $P_i = \frac{s_i^+}{s_{D^+}}, Q_i = \frac{s_i^-}{s_{D^-}}$ be the induced regular functions over X_0. Then by Corollary 5.1.11, the function

$$G(D^+_{\mathbf{C}_v}) := \frac{1}{m} \log^+ \max(|P_1|_v, \cdots, |P_r|_v) \tag{5.77}$$

is a semipositive model Green function of $D^+_{\mathbf{C}_v}$. The same holds for $D^-_{\mathbf{C}_v}$ with the Q_i's instead of the P_i's.

Claim 5.2.17. *For every $n \geqslant 0$, the line bundle $O_X(m\lambda_1^n \max(D^+, D^-))$ is globally generated by*

$$\left((f^n)^* P_1, \cdots, (f^n)^* P_r, (f^{-n})^* Q_1, \cdots, (f^{-n})^* Q_s\right) \tag{5.78}$$

viewed as elements of $\Gamma(X, O_X(m\lambda_1^n D))$.

The claim along with Corollary 5.1.11 shows that for every $n \geqslant 0$,

$$\max\left(G(D^+_{\mathbf{C}_v})_v \circ (f^{an})^n, G(D^-_{\mathbf{C}_v})_v \circ (f^{an})^{-n}\right) \tag{5.79}$$

is a semipositive model Green function of $D_{\mathbf{C}_v}$ which converges uniformly to G_v, so G_v is semipositive.

Proof. Proof of the Claim First of all, since $O_X(D^+)_{|X_0} = O_{X_0}$ we have that $\cap_i P_i^{-1}(0) = \varnothing$ and this remains true for $P_i \circ f^n$ since f is an automorphism. So it is clear that this set of global sections generate $O_X(m\lambda_1^n D)_{|X_0}$. Now, take a point at infinity $q \in \partial_X X_0$, we want to show that this set of global sections generates $O_X(m\lambda_1^n D)$ at q. First suppose that $q \neq c_X(\nu_+), c_X(\nu_-)$. We are going to suppose that q is a satellite point because this is the harder case. So, suppose $q = E \cap F$ with E, F two prime divisors at infinity. Let (z,w) be local coordinates at p associated to E and F. Both f and f^{-1} are defined at q. Since $(f_X^n)^*\theta_X^+ = \lambda_1\theta_X^+$, the fractional ideal $\langle (f^n)^*P_1, \ldots, (f^n)^*P_r \rangle$ is locally generated at q by

$$z^{-m\lambda_1^n \operatorname{ord}_E(D^+)} \cdot w^{-m\lambda_1^n \operatorname{ord}_F(D^+)}. \tag{5.80}$$

In the same way, $\left\langle (f^{-n})^*Q_1, \cdots, (f^{-n})^*Q_s) \right\rangle$ is locally generated at q by

$$z^{-m\lambda_1^n \operatorname{ord}_E(D^-)} \cdot w^{-m\lambda_1^n \operatorname{ord}_F(D^-)}. \tag{5.81}$$

Now, $O_X(m\lambda_1^n D)$ is locally generated at q by

$$z^{-m\lambda_1^n \max\left(\operatorname{ord}_E(D^+), \operatorname{ord}_F(D^+)\right)} \cdot w^{-m\lambda_1^n \max\left(\operatorname{ord}_F(D^+), \operatorname{ord}_F(D^-)\right)} \tag{5.82}$$

Since D^+, D^- is a well ordered pair we have that

$$\left((f^n)^*P_1, \cdots, (f^n)^*P_r, (f^-n)^*Q_1, \cdots, (f^{-n})^*Q_s\right) \tag{5.83}$$

generates $O_X(m\lambda_1^n D)$ at q. □

Now suppose for example that $q = c_X(\nu_-) = p_-$ the indeterminacy point of f. Since we have supposed that $\lambda_1(f) \in \mathbf{Z}$, we have that p_- is a free point at infinity. Let E be the unique prime divisor at infinity over which p_- lies and let z be a local equation of E. Then for every i , we have locally at p_- $(f^n)^*P_i = z^{-m\lambda_1^n \operatorname{ord}_E(D^+)}\varphi_i$ where φ_i is a regular non invertible function because f is not defined at p_-. However, f^{-1} is defined at p_-, therefore the fractional ideal $\left((f^{-n})^*Q_1, \cdots, (f^{-n})^*Q_r\right)$ is locally generated by $z^{-m\lambda^n \operatorname{ord}_E(D^-)}$. Since we have $\operatorname{ord}_E(D^-) > \operatorname{ord}_E(D^+)$ we get that the fractional ideal

$$\left\langle (f^n)^*P_1, \cdots (f^n)^*P_r, (f^{-n})^*Q_1, \cdots, (f^{-n})^*Q_s \right\rangle \tag{5.84}$$

is locally generated by $z^{-m\lambda_1^n \operatorname{ord}_E(D^-)}$ so it is equal to $O_X(m\lambda_1^n D)_{p_-}$ at p_-.

We show that the coherence condition is satisfied. Let $\mathscr{X}$ be a model of X over $\operatorname{Spec} O_{\mathbf{K}}$, f and f^{-1} induce birational transformations on $\mathscr{X}$. There exists an open subset U of $\operatorname{Spec} O_{\mathbf{K}}$

such that if we set $\mathscr{X}_U = \mathscr{X} \times_{O_{\mathbf{K}}} U$.

1. The indeterminacy locus of $f_U : \mathscr{X}_U \dashrightarrow \mathscr{X}_U$ does not contain vertical components.
2. We have $\overline{\{p_+\}}_{|U} \cap \overline{\{p_-\}}_{|U} = \varnothing$.
3. The horizontal divisors D_U^+ and D_U^- induced by D^+ and D^- over $\mathscr{X}_U$ are big and nef.

For every $v \in U$ set $\mathscr{X}_v = X_U \times_U \operatorname{Spec} O_v$, $\mathscr{D}_v^{\pm} = D_U^{\pm}$ and $f_v^{\pm} = f_U \times \operatorname{Spec} O_v$.

Claim 5.2.18. *For every $v \in U$, we have $\forall x \in (X_{\mathbf{C}_v})^{an} \backslash (\operatorname{Supp} D_v^{\pm})$, if $r_{\mathscr{X}_\nu}(x) \neq r_{\mathscr{X}_\nu}(p_\mp)$, then*

$$\frac{1}{\lambda_1} g_{(\mathscr{X}_v, \mathscr{D}_v^{\pm})}((f_v^{\pm})^{an}(x)) = g_{(\mathscr{X}_v, \mathscr{D}_v^{\pm})}(x) \tag{5.85}$$

Proof of the claim. We have that $f_v^{\pm}$ defines a regular endomorphism of $\mathscr{X}_\nu \backslash \{p_\mp, r_{\mathscr{X}_v}(p_\mp)\}$ by condition (1) and (2). Recall that $r_{\mathscr{X}_\nu}$ is anti continuous so the set $V^- := \{r_{\mathscr{X}_\nu} = r_{\mathscr{X}_\nu}(p_-)\}$ is an open subset of $(X_{\mathbf{C}_v})^{an}$. Since p_- is fixed by f_v^{-1}, $r_{\mathscr{X}_\nu}(p_-)$ also is and therefore V^- is $(f_v^{-1})^{an}$-invariant. Therefore, the complement of V^- is f_v^{an}-invariant. Let $x \in X^{an} \backslash V^-$, Let ξ be a local equation of $\mathscr{D}_v^+$ at $r_{\mathscr{X}_\nu}(f_v^{an}(x))$ and ψ a local equation of $\mathscr{D}_v^+$ at $r_{\mathscr{X}_v}(x)$. From $f_v^* \mathscr{D}_v^+ = \lambda_1 \mathscr{D}_\nu^+$ over $\mathscr{X}_v \backslash \{p_-, r_{\mathscr{X}_v}(p_-)\}$ we get that there exists an invertible regular function at $r_{\mathscr{X}_v}(x)$ such that

$$f_v^* \xi = u \cdot \psi^{\lambda_1} \tag{5.86}$$

Since u is invertible, we have $|u(x)| = 1$ and the claim is shown. □

To show the coherence condition we show that on the open subset $V^-, g_{(\mathscr{X}_\nu, \mathscr{D}_\nu^+)} \leqslant g_{(\mathscr{X}_\nu, \mathscr{D}_\nu^-)}$ and this is immediate as $p_- \in E_-$ is a free point and therefore the only irreducible component of $\mathscr{D}_\nu^{\pm}$ on which $r_{\mathscr{X}_v}(p_-)$ lies is the closure of E_- in $\mathscr{X}_\nu$, since $\operatorname{ord}_{E_-}(\theta_X^-) > \operatorname{ord}_{E_-}(\theta_X^+)$ the result is proven.

Finally, let $\overline{L}$ be the associated semipositive adelic line bundle. To show that $\overline{L}$ satisfies the hypothesis of Theorem 5.1.19 it suffices to show that $\deg_X(L) > 0$ but this is equal to D^2 with $D = \max(\theta_X^+, \theta_X^-)$. By Lemma 5.2.14, D is big and nef therefore $D^2 > 0$ (see [Laz04] Theorem 2.2.16). □

Remark 5.2.19. If $\lambda_1(f) \in \mathbf{R} \backslash \mathbf{Q}$, then we can still define $G = \max(G^+, G^-)$, however since θ_X^+ and θ_X^- are $\mathbf{R}$-divisors, in general they are not a well ordered pair and G is not the Green function of any $\mathbf{R}$-divisor. In fact, the right way to look at G^+ and G^- is to consider *adelic line bundles* over the quasi-projective variety X_0 (see [YZ22]). Roughly speaking an adelic line bundle over a quasi-projective variety U is a limit of model adelic line bundles over completions X of U

that satisfy a compatibility condition over U. The process is very similar to the construction of $\mathrm{Weil}_\infty(\mathrm{X}_0)$ or $\mathrm{L}^2(\mathrm{X}_0)$. In particular, Yuan and Zhang showed the equidistribution theorem for this generalized class of adelic line bundles. We conjecture the following result.

Conjecture 5.2.20. *Suppose f is a loxodromic automorphism of X_0 and $\lambda_1(f) \notin \mathbf{Z}_{\geqslant}0$. The Green functions G^+ and G^- induce two nef adelic line bundles $\overline{L}^+$ and $\overline{L}^-$ on the quasiprojective variety X_0 in the sense of [YZ22] such that*

1. $f^*\overline{L^+} = \lambda_1 \overline{L^+}$
2. $(f^{-1})^*\overline{L^-} = \lambda_1 \overline{L^-}$
3. *If $\overline{L} := \frac{1}{2}(\overline{L^+} + \overline{L^-})$, then $\overline{L}$ satisfies the hypothesis of Theorem 5.1.19.*
4. *At the archimedean places, the equilibrium measure of $\frac{1}{2}(\overline{L^+} + \overline{L^-})$ is $dd^cG^+ \wedge dd^cG^-$.*

As explained in the previous remark, I believe that the work done in this memoir and the work of Yuan and Zhang will be sufficient to prove this Conjecture, with a construction similar to [YZ17] Section 4.

Using Proposition 5.2.16 or assuming Conjecture 5.2.20 we can consider the canonical height $h_{\overline{L}}$ associated to f. From Proposition 5.2.15 (3) and Proposition 5.2.5 (3) it follows that if $p \in X_0(\overline{K})$ is periodic, then $h_{\overline{L}}(p) = 0$.

For the last two propositions of this section, we assume $\lambda_1(f) \in \mathbf{Z}$. We assume that they will be true for $\lambda_1(f) \notin \mathbf{Z}_{\geqslant 0}$ once Conjecture 5.2.20 is established.

Proposition 5.2.21 (Northcott property for heights for affine surfaces)**.** *Let $d, B > 0$, the set*

$$\left\{ p \in X_0(\overline{K}) | \deg p \leqslant d, h_{\overline{L_f}}(p) \leqslant B \right\} \tag{5.87}$$

is finite.

Proof. Let $D = \max(\theta_X^+, \theta_X^-)$, then D is big, nef and effective by Lemma 5.2.14 and we know that $\mathrm{Supp}\, D = \partial_X X_0$. Let $H \in \mathrm{Div}_\infty(X)$ be an ample divisor such that $\mathrm{Supp}\, H = \partial_X X_0$. Then, for $m \geqslant 1$ large enough, there exists an effective $\mathbf{Q}$-divisor N such that

$$D = \frac{1}{m} H + N. \tag{5.88}$$

Now we have by the well known properties of heights [Sil86] that

$$h_D = h_{\overline{L_f}} = (1/m) h_H + h_N + O(1) \tag{5.89}$$

and since N is effective, we have $h_N \geqslant O(1)$ over $X_0(\overline{\mathbf{K}})$ (see [Sil86]), therefore

$$h_{\overline{L_f}} \geqslant (1/m)h_H + O(1) \tag{5.90}$$

and the result follows from Northcott Theorem [Sil86] which states that since H is ample, for all $d, B > 0$ the set

$$\left\{p \in X(\overline{\mathbf{K}}) \,|\, \deg p \leqslant d, h_H(p) \leqslant B\right\} \tag{5.91}$$

is finite. □

Proposition 5.2.22. *For any $p \in X_0(\overline{K})$ we have*

$$h_{\overline{L_f}}(p) = 0 \Leftrightarrow p \textit{ is periodic} \tag{5.92}$$

Proof. We look at the sequence $\left(h_{\overline{L_f}}(f^n(p))\right)$. We have $h_{\overline{L_f}}(p) = 0$ if and only if for every place $v, G_v(q) = 0$ for all points q in the Galois orbit of p, this is equivalent to saying that $f^{\mathbf{Z}}(q)$ is bounded for all places v. This means that $h_{\overline{L_f}}(f^n(p)) = 0$ for all n, since the points $(f^n)(p)$ all have the same degree, we get that this sequence is finite by Proposition 5.2.21 □

5.3 Periodic points and equilibrium measure

5.3.1 Equidistribution of periodic points

Let X_0 be a normal affine surface defined over a number field $\mathbf{K}$ and let $f \in \mathrm{Aut}(X_0)$ be a loxodromic automorphism. Let X be a completion as in Theorem 4.4.18. For any place $v \in \mathcal{M}(\mathbf{K})$, let G_v^+, G_v^-, G_v be the Green functions of f defined in Section 5.2. Let $\overline{L}_f$ be the adelic line bundle induced by these Green functions. If $\lambda_1(f) \in \mathbf{Z}_{\geqslant 0}$, then this comes from Proposition 5.2.16 and if $\lambda_1(f) \notin Z$, then we use Conjecture 5.2.20. We have for every place the equilibrium measure $\mu_{\overline{L_f},v}$.

If v is archimedean, then we can apply the dd^c operator to our plurisubharmonic functions. Namely the equilibrium measure is proportional to

$$(dd^c G)^2 = dd^c G_+ \wedge dd^c G_- \tag{5.93}$$

which is well-defined via the work of Bedford and Diller in [BD05], indeed the condition of Bedford and Diller is satisfied because every iterate of f has indeterminacy point either p_+ or p_-. The measure μ is f-invariant thanks to Proposition 5.2.5. In addition, Dujardin showed in [Duj04] that over $X_0(\mathbf{C})$ the periodic points of f equidistributes with respect to μ.

Theorem 5.3.1. *If (p_n) is a generic sequence of $X_0(\overline{\mathbf{K}})$ of periodic points of f, then for every place v of $\mathbf{K}$ the Galois orbit of (p_n) is equidistributed with respect to the measure $\mu_{\overline{L_f},v}$.*

Proof. We apply Yuan-Zhang's equidistribution theorem to the adelic line bundle $\overline{L_f}$. We need to show that the sequence $h_{\overline{L_f}}(p_n)$ converges to $h_{\overline{L_f}}(X)$. Since the points p_n are periodic, this bounds to show that $h_{\overline{L}_f}(X) = 0$. To do that we apply Theorem 5.3.3 of [YZ22]. Namely, let

$$e(X,(D,G)) := \sup_{U \subset X} \inf_{p \in U} h_{\overline{L_f}}(p) \tag{5.94}$$

this quantity is called the *essential minimum* of (D,G). Here, since we suppose that we have a generic sequence of periodic points, we get $e(X,(D,G)) = 0$. Theorem 5.3.3 of [YZ22] states that

$$e(X,(D,G)) \geqslant h_f(X) \tag{5.95}$$

Therefore we get $h_f(X) = 0$ and Yuan's equidistribution theorem gives the desired result. □

For any place v (archimedean or not), we have that

$$\operatorname{Supp}\mu_{f,v} \subset \{G_v = 0\} = K_v. \tag{5.96}$$

If $\lambda_1(f) \in \mathbf{Z}$, we characterize the set $\{G_v = 0\}$ with the measure μ_v.

Theorem 5.3.2 (Extension of [DF17] Lemma 6.3)**.** *If $\lambda_1(f) \in \mathbf{Z}$, then for any $P \in O(X_0)$, one has*

$$\sup_{\operatorname{Supp}\mu_v} |P|_v = \sup_{K_v} |P|_v \tag{5.97}$$

In analogy with the case of the affine plane, we can say that K_v is the polynomial convex hull of $\operatorname{Supp}\mu_v$.

Proof. Let $D = \max(\theta_X^+, \theta_X^-)$, let a be an integer such that $aD \geqslant \operatorname{div}_{\infty,X}(P)$ and let C_0 be a constant such that $\log \frac{|P|_v}{C_0} \leqslant 0$ over $\operatorname{Supp}\mu_v$. Then, the functions aG_v and $\widetilde{G}_v = \max(aG_v, \log \frac{|P|_v}{C_0+\varepsilon})$ are both semipositive (or psh if v is archimedean) Green functions of the divisor aD. Now, on an open neighbourhood V of $\operatorname{Supp}\mu_v$ we have $\widetilde{G}_v = aG_v$ and we get that

$$(c_1(aD, \widetilde{G}_v)_{|V})^2 = (c_1(aD, aG_v)_{|V})^2 \tag{5.98}$$

by [DF17] Appendix A.2 (in loc. cit. the result is stated for ample divisors but the proof works for big and nef divisors). Since the two measures $c_1(aD, \widetilde{G})^2$ and $c_1(aD, aG_v)^2$ are positive and have total mass $a^2D^2 > 0$ we conclude that they are equal. Therefore, by the arithmetic Hodge index theorem we get that $aG_v - \widetilde{G}$ is a constant, since they coincide on $\operatorname{Supp}\mu_v$ we get $\widetilde{G} = aG$ and therefore $\log \frac{|P|_v}{C_0+\varepsilon} \leqslant 0$ over K_v. Letting $\varepsilon \to 0$ yields the result.

□

5.3.2 A rigidity theorem

Theorem 5.3.3. *Let X_0 be a normal affine surface over a number field $\mathbf{K}$ such that $\mathbf{K}[X_0]^\times = \mathbf{K}^\times$ and let f, g be two loxodromic automorphisms of X_0 such that $\lambda_1(f) \in \mathbf{Z}_{\geqslant 1}$, then the following assertions are equivalent*

(1) $\operatorname{Per}(f) \cap \operatorname{Per}(g)$ *is Zariski dense.*

(2) $\mu_{f,v} = \mu_{g,v}, \forall v \in \mathcal{M}(\mathbf{K})$.

(3) $K_{f,v} = K_{g,v}, \forall v \in \mathcal{M}(\mathbf{K})$.

(4) $\mathrm{Per}(f) = \mathrm{Per}(g)$.

If $\lambda_1(f) \notin \mathbf{Z}$*, assuming Conjecture 5.2.20, we have that if* $\mathrm{Per}(f) \cap \mathrm{Per}(g)$ *is Zariski dense, then for every place* $v \in \mathcal{M}(\mathbf{K}), \mu_{f,v} = \mu_{g,v}$.

Proof. We apply the results of Section 5.3.1. Let $\mu_{f,v}$, $\mu_{g,v}$ be the equilibrium measure of f and g at every place. Let (p_n) be a Zariski dense sequence of $\mathrm{Per}(f) \cap \mathrm{Per}(g)$. By Lemma 5.1.18 We can suppose that (p_n) is generic. We can apply Theorem 5.3.1 to f and g with the sequence (p_n). Therefore, we get for all places $v \in \mathcal{M}(\mathbf{K})$ that $\mu_{f,v} = \mu_{g,v}$. If $\lambda_1(f) \notin \mathbf{Z}$ we are done.

Otherwise, let $G_{v,f}$ and $G_{v,g}$ be the Green functions of f and g respectively at every place v of $\mathbf{K}$, $K_{v,f} := \left\{G_{v,f} = 0\right\}, K_{v,g} := \{G_{v,g} = 0\}$ and let h_f, h_g be the respective canonical height of f and g. by Theorem 5.3.2 we get that $K_{v,f} = K_{v,g}$ for any place v. Therefore, the canonical heights h_f, h_g have the same set of points of height 0. By Proposition 5.2.22, we get that $\mathrm{Per}(f) = \mathrm{Per}(g)$. □

5.3.3 A stronger rigidity result for the Markov Surface

Assuming Conjecture 5.2.20 we show the following result.

Theorem 5.3.4. *Let* M_D *be the Markov surface of parameter* D*. Suppose that* $D = 0$ *or* $D = -2 + 2\cos\left(\frac{2\pi}{q}\right)$ *with* $q \in \mathbf{Z}_{\geqslant 1}$*. Let* f, g *be two loxodromic automorphism of* M_D *defined over a number field* $\mathbf{K}$*. Then, the following assertions are equivalent*

(1) $\mathrm{Per}(f) \cap \mathrm{Per}(g)$ *is Zariski dense.*

(2) $\exists N, M \in \mathbf{Z}, f^N = g^M$.

The proof relies on the following proposition.

Proposition 5.3.5. *Suppose* $D = 0$ *or* $D = -2 + 2\cos\left(\frac{2\pi}{q}\right)$ *and let* $f \in \mathrm{Aut}(\mathcal{M}_D)$ *be a loxodromic automorphism. If* v *is an archimedean place, then* f *admits a periodic saddle fixed point* $q(f) \in \mathcal{M}_D(\mathbf{C})$ *such that*

(1) $q(f) \in \mathrm{Supp}(\mu_{f,v})$

(2) If $g \in \mathrm{Aut}(\mathcal{M}_D)$ *is loxodromic such that* f *and* g *do not share a common iterate, then* $(g^n(q(f)))$ *is unbounded.*

Assuming the proposition, suppose that f,g share a Zariski dense subset of periodic points, then by Theorem 5.3.3 we have equality of the equilibrium measures of f and g at every place so in particular at every archimedean place. Fix v one of them. Suppose that f and g do not share a common iterate, then $(g^n(q(f)))_n$ is unbounded. Let $\mu = \mu_{f,v} = \mu_{g,v}$. Since $\mathrm{Supp}\,\mu = \mathrm{Supp}\,\mu_{f,v} = \mathrm{Supp}_{g,v}$, we have that $\mathrm{Supp}\,\mu$ is a compact subset of $\mathcal{M}_D(\mathbf{C})$ invariant by f and g. Since $q(f) \in \mathrm{Supp}\,\mu_{f,v} = \mathrm{Supp}\,\mu$ we get that $(g^n(q(f))) \subset \mathrm{Supp}\,\mu$ which is a contradiction.

To construct $q(f)$ we use Quasi-Fuchsian representation theory.

5.3.4 Character varieties and the Markov surface

Let $\mathbb{T}_1$ be the once punctured torus. The fundamental group $\pi_1(\mathbb{T}_1)$ is a free group generated by two elements a and b. The commutator $[a,b] := aba^{-1}b^{-1}$ is represented by a simple loop around the puncture that follows the orientation of the surface. One can study the representation of $\pi_1(\mathbb{T}_1)$ into $\mathrm{SL}_2(\mathbf{C})$. It is clear that

$$\mathrm{Hom}(\pi_1(\mathbb{T}_1), \mathrm{SL}_2(\mathbf{C})) \simeq \mathrm{SL}_2(\mathbf{C}) \times \mathrm{SL}_2(\mathbf{C}) \tag{5.99}$$

as $\pi_1(\mathbb{T}_1)$ is a free group on two generators, therefore it is an algebraic variety. We are interested in the Character variety,

$$\mathcal{X} := \mathrm{Hom}(\pi_1(\mathbb{T}_1), \mathrm{SL}_2(\mathbf{C}))//\mathrm{SL}_2(\mathbf{C}) \tag{5.100}$$

where the action of $\mathrm{SL}_2(\mathbf{C})$ is diagonal and given by conjugation and $//$ is the Geometric Invariant Theory (GIT) quotient. This is also an algebraic variety and we have the following result of Fricke and Klein.

Theorem 5.3.6 (Fricke, Klein, [Gol09])**.** *The algebraic variety $\mathcal{X}$ is isomorphic to $\mathbb{A}^3_{\mathbf{C}}$. The isomorphism is given by*

$$[\rho] \in \mathcal{X} \mapsto (\mathrm{Tr}(\rho(a)), \mathrm{Tr}(\rho(b)), \mathrm{Tr}(\rho(ab))). \tag{5.101}$$

We will denote by $(x,y,z) = (\mathrm{Tr}(\rho(a)), \mathrm{Tr}(\rho(b)), \mathrm{Tr}(\rho(ab)))$ these are the *Frick-Klein* coordinates.

Let $K = [a,b]$ and let $\kappa : \mathcal{X} \to \mathbf{C}$ be the regular function

$$\kappa(\rho) = \mathrm{Tr}(\rho(K)). \tag{5.102}$$

One can show that

$$\kappa = x^2 + y^2 + z^2 - xyz - 2 \tag{5.103}$$

Therefore, if $\mathcal{X}_t = \kappa^{-1}(t)$ is the relative character variety, we have

$$\mathcal{X}_t = \mathcal{M}_{t+2} \tag{5.104}$$

where $\mathcal{M}_D$ is the Markov surface of parameter D.

The generalized mapping class group $\mathrm{MCG}^*(\mathbb{T}_1)$ is the group of homotopy class of homeomorphism of T_1 not necessarily orientation preserving. It contains $\mathrm{MCG}(\mathbb{T}_1)$ as an index 2 subgroup and it acts on $\pi_1(\mathbb{T}_1)$, we have the following isomorphism:

$$\mathrm{MCG}^* \simeq \mathrm{Out}(\pi_1(\mathbb{T}_1)) \tag{5.105}$$

Furthermore,

$$\mathrm{Out}(\pi_1(\mathbb{T}_1)) \simeq \mathrm{GL}_2(\mathbf{Z}) \tag{5.106}$$

and the action on F_2 is as follows, if $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$, then

$$M \cdot a = a^{m_{11}} b^{m_{12}} \tag{5.107}$$

$$M \cdot b = a^{m_{21}} b^{m_{22}}. \tag{5.108}$$

For any element $\varphi \in \mathrm{Out}(\pi_1(\mathbb{T}_1)), \varphi([a,b])$ is conjugated to $[a,b]^{\pm}$. This implies, that the action of $\mathrm{MCG}^*(\mathbb{T}_1)$ on $\mathcal{X}$ preserves every $\mathcal{X}_t$. Now, the matrix *id* acts trivially, because in $\mathrm{SL}_2(\mathbf{C})$ we have that $\mathrm{Tr}A = \mathrm{Tr}A^{-1}$, so for all $D \in \mathbf{C}$ we get a group homomorphism

$$\mathrm{PGL}_2(\mathbf{Z}) \to \mathrm{Aut}(\mathcal{M}_D) \tag{5.109}$$

Theorem 5.3.7 ([CL07] Theorem A, [ÈH74])**.** *Let* $\Gamma^* \subset \mathrm{PGL}_2(\mathbf{Z})$ *be the subgroup of element congruent to* $\mathrm{id} \mod 2$*, then for any* $D \in \mathbf{C}$*,*

$$\Gamma^* \to \mathrm{Aut}(\mathcal{M}_D) \tag{5.110}$$

is injective and its image is of index at most 8.

We can describe the group homomorphism. Let $\sigma_x \in \mathrm{Aut}(\mathcal{M}_D)$ be the automorphism

$$\sigma_x(x,y,z) = (yz - x, y, z), \tag{5.111}$$

If we fix the coordinates y, z, then the equation defining $\mathcal{M}_D$ becomes a polynomial equation of degree 2 with respect to x, σ_x permutes the 2 roots of this equation. We can define σ_y, σ_z in the same way. Then, $\sigma_x, \sigma_y, \sigma_z$ generate a free group isomorphic to $(\mathbf{Z}/2\mathbf{Z}) * (\mathbf{Z}/2\mathbf{Z}) * (\mathbf{Z}/2\mathbf{Z})$ which is of finite index in $\mathrm{Aut}(\mathcal{M}_D)$ (see [ÈH74]). The subgroup $\Gamma*$ is the free group on the three generators

$$\begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.112}$$

which correspond respectively to $\sigma_x, \sigma_y, \sigma_z$.

5.3.5 Fuchsian and Quasi-Fuchsian representation

A *Fuchsian* group is a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbf{R})$. A *Quasi-Fuchsian* group is a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbf{C})$ such that its limit set in $\widehat{\mathbf{C}} := \mathbf{P}^1(\mathbf{C})$ is a Jordan curve. Let S be a compact surface of negative Euler characteristic. We say that a representation $\rho : \pi_1(S) \to \mathrm{SL}_2(\mathbf{C})$ is Fuchsian (resp, Quasi-Fuchsian) if $\overline{\rho}(S) \subset \mathrm{PSL}_2(\mathbf{C})$ is Fuchsian (resp. Quasi-Fuchsian).

Let $\mathrm{Teich}(S)$ be the Teichmuller space of S, that is the set of complete finite hyperbolic metrics over S. Every point of $\mathrm{Teich}(S)$ induces a Fuchsian representation of S. We can actually parametrize the set of Quasi-Fuchsian representations of S using $\mathrm{Teich}(S)$ by the double uniformization theorem of Bers.

Theorem 5.3.8 ([Ber60])**.** *There is a biholomorphic map*

$$\mathrm{Bers} : \mathrm{Teich}(S) \times \overline{\mathrm{Teich}(S)} \to \mathrm{QF}(S) \tag{5.113}$$

where $\overline{\mathrm{Teich}(S)}$ *is the Teichmuller space with its reversed orientation.*

Using this theorem, one can apply an iterative process to find a fixed point in the character variety of S.

Theorem 5.3.9 ([McM96])**.** *Let S be a compact surface of negative Euler characteristic. Let* $(X,Y) \in \mathrm{Teich}(S) \times \overline{\mathrm{Teich}(S)}$*, let* $\varphi \in \mathrm{Mod}(S)$ *be pseudo-Anosov, then the sequence*

$$\mathrm{Bers}(\varphi^n(X), \varphi^{-n}(Y)) \tag{5.114}$$

has an accumulation point $\rho_\infty : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$. *Furthermore,*

(1) ρ_∞ *is discrete and faithful.*

(2) The limit set of $\rho_\infty(\pi_1(S))$ *is the whole boundary* $\mathbb{S}^2$ *of* $\mathbb{H}^3$.

(3) ρ_∞ *is a fixed point of* φ *and* φ *is conjugated to an isometry* α *of* $\widetilde{M}_\varphi = \mathbb{H}^3/\rho_\infty(\pi_1(S))$.

(4) The group of isometries of M_∞ *is discrete and* α *is of infinite order.*

(5) The mapping torus M_φ *is isomorphic as an hyperbolic manifold to* $\widetilde{M}_\varphi / < \alpha >$.

(6) The subgroup generated by α *of the group of isometries of* $\widetilde{M}_\varphi$ *is of finite index.*

5.3.6 The surface $\mathcal{M}_0$ and a Theorem of Minsky

We are interested in this section with the Markov surface $\mathcal{M}_0$ that is when $\mathrm{Tr}(K) = -2$, therefore $\rho(K)$ is a parabolic Möbius transformation. The real points $\mathcal{M}_0(\mathbf{R})$ consist of an isolated point $(0,0,0)$ and four diffeomorphic connected components that are given by the signs of x and y. We will denote by $\mathcal{M}_0(\mathbf{R})^+$ the connected component such that $x, y > 0$. It is known that $\mathrm{Teich}(\mathbb{T}_1)$ ($\mathbb{T}_1$ the punctured torus) is isomorphic to the upper half plane $\mathbb{H}^+$ and we make this identification from now on. The action of $\mathrm{Mod}(\mathbb{T}_1)$ on $\mathrm{Teich}(\mathbb{T}_1)$ is conjugated to the usual action of $\mathrm{PSL}_2(\mathbf{Z})$ by isometries on $\mathbb{D}$.

Any point in $\mathrm{Teich}(\mathbb{T}_1)$ gives rise to a representation $\overline{\rho} : \pi_1(\mathbb{T}_1) \to \mathrm{PSL}_2(\mathbf{R})$ which can be lifted to four distinct representations $\rho : \pi_1(\mathbb{T}_1) \to \mathrm{SL}_2(\mathbf{R})$. The cusp condition gives the condition $\mathrm{Tr}(\rho(a,b)) = -2$ (because $\mathrm{Tr} = 2$ corresponds to reducible representations). Therefore, we get an embedding of $\mathrm{Teich}(\mathbb{T}_1)$ into the 4 different connected component of $\mathcal{M}_0(\mathbf{R}) \backslash (0,0,0)$. We will restrict our attention to the embedding $\mathrm{Teich}(\mathcal{T}_1) \hookrightarrow \mathcal{M}_0(\mathbf{R})^+$. The set $\mathcal{M}_0(\mathbf{R})^+$ is made of (conjugacy class of) Fuchsian representations. Let $\mathrm{DF}_0 \subset \mathcal{M}_0(\mathbf{C})$ be the subset of discrete and faithful representation of $\pi_1(\mathbb{T}_1)$. Then DF_0 has four different connected components, one of them contains $\mathcal{M}_0(\mathbf{R})^+$. We denote it by DF_0^+ and we denote by QF_0^+ the set of Quasi-Fuchsian representation inside DF_0^+. In fact, QF_0^+ is the interior of DF_0^+ (see [Min02]). We can identify $\mathrm{Teich}(T_1)$ with the upper half plane $\mathbb{H}^+$ and $\mathrm{Teich}(\overline{\mathbb{T}}_1)$ with the lower half plane $\mathbb{H}^-$. The group $\mathrm{PSL}_2(\mathbf{Z})$ acts on $\mathbf{P}^1(\mathbf{C})$ via Möbius transformation. It preserves $\mathbb{H}^+, \mathbb{H}^-$ and $\mathbf{P}^1(\mathbf{R})$. In particular, the mapping class group $\mathrm{MCG}(\mathbb{T}_1) = \mathrm{SL}_2(\mathbf{Z})$ acts on $\mathbf{P}^1(\mathbf{C})$ and we can conjugate this action to the action on $\mathcal{M}_0(\mathbf{R})^+$ via the Bers mapping. Namely, let $\Phi \in \mathrm{MCG}(\mathbb{T}_1)$ and let

$f_\Phi \in \mathrm{Aut}(\mathcal{M}_0)$ induced by the map from Equation (5.109). We have for every $(s,t) \in \mathbb{H}^+ \times \mathbb{H}^-$,

$$\mathrm{Bers}(\Phi(s,t)) = f_\Phi(\mathrm{Bers}(s,t)) \tag{5.115}$$

Theorem 5.3.9 is not applicable directly as $\mathbb{T}_1$ is not compact. However, Minsky showed that the Bers mapping can be extended to almost all the boundary of $\mathrm{Teich}(\mathbb{T}_1) \times \mathrm{Teich}(\overline{\mathbb{T}}_1)$. The boundary of $\mathbb{H}^+$ is $\mathbf{P}^1(\mathbf{R})$. We denote by Δ the diagonal in $\partial\mathrm{Teich}(\mathbb{T}_1) \times \partial\mathrm{Teich}(\overline{\mathbb{T}}_1)$.

Theorem 5.3.10 ([Min99])**.** *The Bers mapping extend to a continuous bijection*

$$Bers : \overline{\mathrm{Teich}(\mathbb{T}_1) \times \mathrm{Teich}(\overline{\mathbb{T}}_1)} \backslash \Delta \to \mathrm{DF}^+ \tag{5.116}$$

In particular, let $\Phi \in \mathrm{SL}_2(\mathbf{Z}) = \mathrm{MCG}(\mathbb{T}_1)$ be a loxodromic element and let f_Φ be its associated automorphism over $\mathcal{M}_0$. The isometry Φ has a repulsive fixed point $\alpha(\Phi)$ on $\mathbf{P}^1(\mathbf{R})$ and an attractive one $\omega(\Phi)$. By Minsky's theorem, this gives two unique fixed point

$$p(\Phi) = \mathrm{Bers}((\alpha(\Phi), \omega(\Phi))), \quad q(\Phi) = \mathrm{Bers}((\omega(\Phi), \alpha(\Phi))) \tag{5.117}$$

of f_Φ in $\mathrm{DF}^+ \setminus \mathrm{QF}^+$.

5.3.7 Construction of the saddle fixed point $q(f)$

Suppose first that $D = 0$. Up to taking an iterate of f we can suppose that there exists a loxodromic element $\Phi_f \in \mathrm{SL}_2(\mathbf{Z})$ such that $f = f_{\Phi_f}$. Denote by $p(f) = p(f_{\Phi_f})$ and $q(f) = q(f_{\Phi_f})$ the fixed point constructed using Minsky theorem. These two fixed point are saddle fixed points by [McM96] Corollary 3.19. The fixed point $q(f)$ corresponds to a representation $\rho_\infty : F_2 \to \mathrm{PSL}_2(\mathbf{C})$, one can show that ρ_∞ also satisfies Theorem 5.3.9 even though the punctured torus is not compact.

Suppose now that $D = 2 - 2\cos\frac{2\pi}{q}$. Following [McM96] §3.7, let S be the orbifold obtained from a genus 1 torus with a singular point of index q. The fundamental group of S is

$$\pi_1(S) = \langle a, b | [a,b]^q = 1 \rangle \tag{5.118}$$

The modular class group $\mathrm{Mod}(S)$ of S is also $\mathrm{SL}_2(\mathbf{Z})$. Let $\Phi_f \in \mathrm{SL}_2(\mathbf{Z})$ be an element of $\mathrm{Mod}(S)$ associated to f.

There exists a smooth (real) surface $\widetilde{S}$ with a map $\widetilde{S} \to S$ which is a finite characteristic covering. In particular, Φ_f lifts to $\widetilde{S}$ and defines an element of $\mathrm{Mod}(\widetilde{S})$ that we denote by $\widetilde{\Phi}_f$. Apply Theorem 5.3.9 to $(\widetilde{S}, \widetilde{\Phi}_f)$, there exists a faithful and discrete representation $\widetilde{\rho}_\infty : \pi_1(\widetilde{S}) \to \mathrm{PSL}_2(\mathbf{C})$. Let $\widetilde{M}_\infty = \mathbb{H}^3/\widetilde{\rho}_\infty(\pi_1(\widetilde{S}))$, the group of isometries of $\widetilde{M}_\infty$ contains the subgroup generated by $\widetilde{\Phi}_f$. The quotient $\widetilde{M}_\infty/ < \widetilde{\Phi}_f >$ is the mapping torus $M_{\widetilde{\Phi}_f}$ of $\widetilde{\Phi}_f$ which is a finite cover of the mapping torus M_{Φ_f}. By Mostow rigidity theorem, the covering group can be realized by isometries, therefore the hyperbolic structure on $M_{\widetilde{\Phi}_f}$ descends to a hyperbolic structure on the mapping torus M_{Φ_f}, which yields a fixed point ρ_∞ of f in $\mathcal{M}_D$ that we denote by $q(f)$. By [McM96] Corollary 3.19, $q(f)$ is a saddle fixed point.

5.3.8 Saddle periodic points are in the support of the equilibrium measure

Theorem 5.3.11. *Let f be a loxodromic automorphism of the Markov surface. Every periodic saddle point of f is in the support of the measure μ_f.*

This shows item (1) of Proposition 5.3.5. This theorem, stated in [Can01], follows directly from the work of Dinh and Sibony in [DS13], which extends [BS91b], and an argument of [BLS93] for Hénon type automorphisms of the complex affine plane. We do not provide a detailed proof, our goal in this section is only to describe the type of techniques and arguments used in [BLS93, DS13].

5.3.8.1 Green functions and bounded orbits

First, let us summarize some of the properties of the function $G_f^+ : X_0(\mathbf{C}) \to \mathbf{R}_{\geqslant 0}$

(a) $\{G_f^+ = 0\}$ coincides with the set $K^+(f)$ of points with a bounded forward orbit;

(b) G_f^+ is plurisubharmonic, and is pluriharmonic on the set $\{G_f^+ > 0\}$;

(c) the set $K^+(f)$ is closed in $X_0(\mathbf{C})$, its closure in $X(\mathbf{C})$ coincides with $K^+(f) \cup p_-$;

(d) locally, near every point $q \neq p_-$ of $\partial_X X_0$,

$$G_f^+(x) = -\sum_i a_i \log(|s_i(x)|) + u(x) \tag{5.119}$$

where the functions $s_i(x)$ are holomorphic equations of the boundary components containing q, the real numbers $a_i \geqslant 0$ are the weight of θ_X^+, and $u(x)$ is a continuous (pluriharmonic) function.

(e) there is an open neighborhood U^- of p_- in $X(\mathbf{C})$ such that $f^{-1}(U^-) \Subset U^-$ and U^- is contained in the basin of attraction of p_- for the backward dynamics; there is an open neighborhood U^+ of p_+ with similar properties for f instead of f^{-1};

(f) If q is a saddle periodic point, its stable manifold $W^s(q)$ is contained in $K^+(f)$; in fact, the proof of Proposition 5.1 in [BS91a] shows that $W^s(q)$ is contained in the boundary of $K^+(f)$;

(g) f does not preserve any algebraic curve $C_0 \subset X_0(\mathbf{C})$.

In particular, if S is a closed positive current supported by $\overline{K(f)} = \overline{K^+(f) \cap K^-(f)}$, then its support does not intersect the open set U^-.

5.3.8.2 Rigidity of $\overline{K^+(f)}$ and equidistribution of stable manifolds

The properties (a) to (g) are sufficient to apply the arguments of Sections 4, 5, 6 of [DS13]. More precisely, one first obtains Theorem 6.6 of [DS13], because its proof relies only on the above properties and general results concerning closed positive currents (in particular Corollary 3.13 of [DS13]). [1]

Then, one gets directly the following fact (which corresponds to a weak version of Theorem 6.5 of [DS13], with the same proof):

Theorem 5.3.12. *The set $\overline{K^+(f)}$ (resp. $\overline{K^-(f)}$) supports a unique closed positive current, namely $T_f^+ = dd^cG_f^+$ (resp. T_f^-) up to multiplication by a positive constant.*

This rigidity results provides automatic equidistribution theorems for $(1,1)$ positive currents. We shall need the following specific application.

If q is a saddle periodic point of f, then its stable manifold $W^s(q)$ is biholomorphic to the complex line [2]. Denote by $\xi \colon \mathbf{C} \to W^s(q) \subset X_0(\mathbf{C})$ a one to one holomorphic parametrization of $W^s(q)$; ξ is an entire holomorphic curve. To such a curve, one can associate a family of currents of mass 1, constructed as follows. One fixes a Kähler form κ on $X(\mathbf{C})$ and one measures lengths, areas and volumes with respect to this form. For instance, if $\mathbb{D}_r \subset \mathbf{C}$ is the disk of

1. The only changes in this proof are that (1) $\mathbf{P}^2(\mathbf{C})$ should be replaced by $X(\mathbf{C})$ and the line at infinity by $\partial_X X_0$; and (2) the function $\log(1+ \parallel z \parallel^2)^{1/2}$ should be replaced by a smooth Green function associated to the $\mathbf{R}$-divisor θ_X^+, as in Definition 5.1.3.

2. Indeed, it is a Riemann surface, it is homeomorphic to $\mathbf{R}^2$, and f acts on it as a contraction fixing q, so $W^s(q)$ cannot be a disk and Riemann uniformization theorem says that it is a copy of $\mathbf{C}$

radius r centered at the origin, then

$$Area(\xi(\mathbb{D}_r)) = \int_{\xi(\mathbb{D}_r)} \kappa = \int_{\mathbb{D}_r} \xi^*\kappa \tag{5.120}$$

is the area of the image of $\mathbb{D}_r$ by ξ. Averaging with respect to dr/r, one introduces the function

$$N(R) = \int_{t=0}^{R} Area(\xi(\mathbb{D}_t))\frac{dt}{t}. \tag{5.121}$$

Now, for each disk $\mathbb{D}_r$, one can consider the current of integration over $\xi(\mathbb{D}_r)$: to a smooth form α of type $(1,1)$, this current $\{\xi(\mathbb{D}_r)\}$ associates the number

$$\langle\{\xi(\mathbb{D}_r)\}|\alpha\rangle = \int_{\xi(\mathbb{D}_r)} \alpha = \int_{\mathbb{D}_r} \xi^*\alpha. \tag{5.122}$$

Taking averages with respect to the weight dr/r one obtains the following family of currents, parametrized by a radius $R > 0$:

$$\langle N_\xi(R)|\alpha\rangle = \frac{1}{N(R)}\int_{t=0}^{R}\langle\{\xi(\mathbb{D}_r)\}|\alpha\rangle\,\frac{dt}{t} \tag{5.123}$$

$$= \frac{1}{N(R)}\int_{t=0}^{R}\int_{\xi(\mathbb{D}_R)}\alpha\,\frac{dt}{t}. \tag{5.124}$$

The normalization by $1/N(R)$ assures that the mass $\langle N_\xi(R)|\kappa\rangle$ is equal to 1 for every $R > 0$. From an inequality of Ahlfors, and from the compactness of the space of positive currents of mass 1, there are sequences of radii (R_n) such that $N_\xi(R_n)$ converges to a closed positive current S. A priori, such a closed positive current S depends on the choice of the sequence R_n; if there is a unique closed positive current S that can be obtained as such a limit, one says that there is a unique Ahlfors-Nevanlinna current (namely S) associated to ξ.

Corollary 5.3.13 (Proposition 4.10, Corollary 4.11 [DS13])**.** *Let q be a saddle periodic point of f. Let $\xi\colon \mathbf{C} \to X_0(\mathbf{C})$ be a holomorphic parametrization of the stable manifold of f. Then, there is a unique Ahlfors-Nevanlinna current associated to ξ, and this current is equal to T_f^+.*

Here is another similar consequence of [DS13]: *Given any algebraic curve $C_0 \subset X_0$, the sequence of currents $\lambda(f)^{-n}\{(f^n)^*C_0\}$ converges towards a positive multiple of T_f^+ as n goes to $+\infty$* (see Corollary 6.7 of [DS13]). Thus, T_f^+ can be approximated by a sequence of currents of integration on algebraic curves of a fixed genus (properly renormalized); in this context, one can apply the theory of strongly approximable laminar currents, as developed by Dujardin (see [Can14, Duj04] for an introduction).

5.3.8.3 Laminarity, Pesin theory and consequence

The measure $\mu_f = T_f^+ \wedge T_f^-$ is an ergodic measure of positive (and maximal) entropy for f, and tools from Pesin theory can be used to describe the dynamics of f with respect to this measure. In particular, in our setting, one can apply the work of Bedford, Lyubich, and Smillie in [BLS93] or the work of Dujardin in [Duj04].

First, the laminar structure of T_f^+ is compatible with Pesin theory; the second one is that μ_f has a local product structure. Taken together, these facts imply that one can find holomorphic bi-disks $V \simeq \mathbb{D} \times \mathbb{D}$ in $X_0(\mathbf{C})$ and transverse laminations $\mathcal{L}^s$ and $\mathcal{L}^u$ of V, the leaves of $\mathcal{L}^s$ being horizontal graphs, the leaves of $\mathcal{L}^u$ being vertical graphs, such that

(a) it makes sense to restrict T_f^+ (resp. T_f^-) to the support $\mathrm{Supp}(\mathcal{L}^s)$ of $\mathcal{L}^s$ (resp. on $\mathrm{Supp}(\mathcal{L}^u)$);

(b) the restriction is given by the current of integration on the leaves of $\mathcal{L}^s$ (resp. $\mathcal{L}^u$) averaged by a transversal measure μ_V^+ (resp. μ_V^-); in other words, if $\mathcal{L}^u(w)$ is a leaf of $\mathcal{L}^u$, μ_V^+ induces a positive measure on $\mathrm{Supp}(\mathcal{L}^s) \cap \mathcal{L}^u(w)$ and if α is a smooth form supported by a compact subset of V, then

$$\langle T_f^+ | \alpha \rangle = \int_{z \in \mathcal{L}^u(w)} \langle \{\mathcal{L}^s(z)\} | \alpha \rangle \, d\mu_V^+(z).$$

(c) in restriction to $\mathrm{Supp}(L^+) \cap \mathrm{Supp}(L^-)$, the measure μ_f is given by the product of the currents, i.e. by the Dirac masses at the points of intersection of the leaves, weighted by $d\mu_V^+ \otimes d\mu_V^+$;

(d) for μ_f almost every point $x \in \mathrm{Supp}(L^+) \cap \mathrm{Supp}(L^-)$, the leaf $\mathcal{L}^s(x)$ is a piece of stable manifold, and $\mathcal{L}^u(x)$ is a piece of unstable manifold.

Then, one can apply the following argument, taken from Section 9 of [BLS93]. Pick a saddle periodic point q of f, take a small neighborhood W of q, and consider its stable manifold, parametrized by $\xi \colon \mathbf{C} \to W^s(q)$. Since the Ahlfors-Nevanlinna current of ξ coincides with T_f^+, each disk of $\mathcal{L}^s$ is a limit of disks $\xi(D_i)$, for some topological disks $D_i \subset \mathbf{C}$. Since the laminations $\mathcal{L}^u$ and $\mathcal{L}^s$ intersect transversally, one finds a disk $\xi(D_i)$ that intersects $\mathcal{L}^u$ transversally. Then, if one applies f^N with N large, the preimages of $\xi(D_i) \cap \mathcal{L}^u$ approach the point q, and the inclination lemma implies that the images of the leaves of $\mathcal{L}^u$ are (very large) disks which, in the neighborhood W of q, converge towards $W^u(q)$ (in the C^1 topology). Doing the same with the unstable manifold $W^u(q)$ and the dynamics of f^{-N}, one pull back $\mathcal{L}^s$ near q. On the other hand, T_f^+ and T_f^- are eigencurrents for f. Thus, one sees that T_f^+ and T_f^- give mass to two transversal laminations of W. And this implies that μ_f gives a positive mass to W. Since this work for any neighborhood of q, this point is in the support of μ_f. Thus, Theorem 5.3.11 is proven.

5.3.9 The sequence $(g^n(q(f)))$ is unbounded

Suppose $D = 0$ we can consider S as the flat torus $T = \mathbf{R}^2/\mathbf{Z}^2$ with a puncture at the origin, i.e. $S = T\backslash\{o\}$, or as a complete hyperbolic surface X of finite area (we fix such a hyperbolic structure, it corresponds to some point X in the Teichmuller space $Teich(S) \simeq \mathbb{D}$).

An element f of $Out^+(F_2)$ is pseudo-Anosov if the corresponding matrix $A_f \in \mathrm{SL}_2(\mathbf{Z})$ has $\mathrm{Tr}(A_f)^2 \geqslant 4$. In that case, the matrix has two eigenvalues $\lambda(f) > 1$ and $1/\lambda(f) < 1$ and the mapping class is represented by a linear automorphism of the torus T (fixing the origin o) with stable and unstable linear foliations. In the hyperbolic surface X, these foliations give rise to two measured laminations F_- and F_+ (by geodesic lines). If $C \subset S$ is a closed curve (represented by some geodesic in X), one can define two intersection numbers $i(C,F_+)$ and $i(C,F_-)$; they depend only on the free homotopy class of C. The product $j(C) = i(C,F_+)i(C,F_-)$ is f-invariant, because f stretches F_+ by a dilatation factor $\lambda(f) > 1$, and contracts F_- by $1/\lambda(f)$; if C is not homotopic to a loop around the puncture $j(C)$ is strictly positive (any closed geodesic is transverse to F_+ and F_-).

If $D = 2-2\cos(2\pi/q)$, let S be the genus one torus with an orbifold singularity of order q. We have seen that there exists a characteristic finite covering $\widetilde{S} \to S$ with $\widetilde{S}$ a compact surface of negative Euler characteristic. We let $X = \mathbb{H}^2/\Gamma$ be a hyperbolic surface homeomorphic to $\widetilde{S}$ (i.e $X \in \mathrm{Teich}(\widetilde{S})$). If $f \in \mathrm{Out}^+(F_2)$ is pseudo-Anosov then it lifts to a pseudo-Anosov $\widetilde{f} \in \mathrm{Mod}(X) = \mathrm{Out}^+(F_2)$ pseudo-Anosov also. In that case, there exist two measured laminations F_+ and F_- over $\widetilde{S}$ (the stable and the unstable one) and by Proposition 1.5.1 of [Ota96]. We have that for any geodesic $\gamma \in \widetilde{S}$,

$$\frac{(\widetilde{f})^{\pm i}(\gamma)}{\ell(\widetilde{f})^{\pm i}(\gamma)} \xrightarrow[i\to+\infty]{} F_{\pm} \tag{5.125}$$

in the sense of measured laminations. (This also holds in the case $D = 0$). Here ℓ is the length induced by the hyperbolic structure from the quotient $\mathbb{H}^2/\Gamma$ so $\ell(\widetilde{f})_*^{\pm i}(\gamma)$ grows like $\lambda(\widetilde{f})^i$. We also have that $j(\gamma) = i(\gamma, F_+)i(\gamma, F_-)$ is f-invariant as $i(\widetilde{f}_*(\gamma), F_\pm) = \lambda(\widetilde{f})^{\mp 1} i(\gamma, F_\pm)$ and if γ is a geodesic, then $j(\gamma) > 0$. To unify the notations we will still denote by f the lift $\widetilde{f}$ of f to X.

Lemma 5.3.14. *If f and g are two loxodromic elements of $Out^+(F_2) \simeq SL_2(\mathbf{Z})$ generating a non-elementary subgroup of $SL_2(\mathbf{Z})$, then given any geodesic $\gamma \subset X$, $j(g^n(\gamma))$ goes to $+\infty$ as n goes to $+\infty$.*

Proof. Let G_+ and G_- be the unstable and stable laminations associated to g in X. Since f and g generate a non-elementary subgroup of $GL_2(\mathbf{Z})$, G_+ is transverse to both F_+ and F_- (equivalently, the four fixed points of A_f and A_g on $\mathbf{P}^1(\mathbf{R})$ are distinct). Thus, by Equation (5.125) $j(g^n(C)) \simeq \lambda(g)^n i(G_+, F_+) i(G_-, F_-)$ by continuity of the intersection number (see [Ota96] p.151). □

Lemma 5.3.15. *Let f and g be two loxodromic elements of $Out^+(F_2) \simeq SL_2(\mathbf{Z})$ generating a non-elementary subgroup of $SL_2(\mathbf{Z})$. Let $\gamma \subset X$ be a geodesic, and let $[\gamma]$ be its free homotopy class. Then the sequence $g^n[\gamma]$ intersects each orbit of f only finitely many times.*

Proof. This follows from the previous lemma and the fact that $j(\cdot)$ is f-invariant so it is constant in each orbit of f. □

Recall the definition of M_{Φ_f}, $\widetilde{M}_{\Phi_f}$, ρ_∞ and α_f from Theorem 5.3.9 (here we consider $f \in \mathrm{Mod}(\widetilde{S})$ if we are in the orbifold case). In M_{Φ_f}, the number of simple closed geodesics of length $\leqslant L$ is finite (for every $L > 0$); thus, in $\widetilde{M}_{\Phi_f}$, given any upper bound L, there are only finitely many homotopy classes of simple closed curves *up to the action of $f^{\mathbf{Z}}$* (Note that, since α_f acts by isometry, each closed geodesic $C \subset \widetilde{M}_f$ gives rise to infinitely many geodesics $\alpha_f^n(C)$ with the exact same length).

Proof of Proposition 5.3.5 item (2) Fix a generator a in $\pi_1(S)$ where S is either the punctured torus or the genus 1 torus with an orbifold singularity of index q. Set k to be the degree of the finite cover $\widetilde{S} \to S$ in the orbifold case and $k = 1$ otherwise. The element a^k gives rise to a closed geodesic A in $\widetilde{M}_{\Phi_f}$. From these preliminaries and the previous lemma, the sequence of homotopy classes $g^n(a^k)$ correspond to a sequence of closed geodesics in $\widetilde{M}_{\Phi_f}$, with length going to infinity because f acts by isometry on $\widetilde{M}_{\Phi_f}$.

Now, $g^n(a^k)$ corresponds to a (conjugacy class of a) matrix $\rho_\infty(g^n(a^k)) \in SL_2(\mathbf{C})$, and the trace of this matrix is related to the length of the geodesic by a simple formula; in particular, the fact that the length goes to infinity implies that the modulus of the trace goes to $+\infty$. Since for any matrix $A \in \mathrm{SL}_2(\mathbf{C}), \mathrm{Tr}A^k$ is a polynomial in $\mathrm{Tr}A$ we get that $\mathrm{Tr}(\rho_\infty(g^n(a)))$ goes to infinity. This implies that the orbit of $q(f)$ under the action of g on $\mathcal{M}_D(\mathbf{C})$ is discrete, going to infinity.

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COLLEGE DOCTORAL BRETAGNE | MATHS, TELECOMS INFORMATIQUE, SIGNAL SYSTEMES, ELECTRONIQUE

Université de Rennes

Titre : Sur la dynamique des endomorphismes des surfaces affines

Mot clés : Système dynamique, dynamique arithmétique, valuations, géométrie algébrique

Résumé : Une variété affine X_0 sur un corps algébriquement clos $\mathbf{k}$ est un sous-espace de $\mathbf{k}^N$ défini par des équations polynomiales. Un endomorphisme polynomial f de X_0 est alors une transformation polynomiale de $\mathbf{k}^N$ qui préserve X_0 au sens où $f(X_0) \subset X_0$. Lorsque la dimension de X_0 vaut 2, on dira que X_0 est une surface affine. Le but de ma thèse est d'étudier le système dynamique donné par X_0 une surface affine et $f : X_0 \to X_0$ un endomorphisme polynomial de X_0. Les différentes questions que j'aborderai sont les suivantes : y'a-t-il des orbites denses ou Zariski-denses ? Si l'orbite d'un point part à l'infini, peut-on contrôler sa vitesse de fuite ? Y'a-t-il beaucoup d'orbites périodiques ? Comment construire des mesures invariantes qui sont dynamiquement intéressantes ? Pour répondre à ces questions, j'utilise des techniques valuatives. Le système dynamique (X_0, f) induit un système dynamique $(\mathcal{V}_\infty, f_*)$ où $\mathcal{V}_\infty$ est l'espace des valuations centrées à l'infini de X_0. C'est l'étude de cette action qui sera au coeur de ce mémoire et permettra d'aborder ensuite les questions évoquées ci-dessus.

Title: On the dynamics of endomorphisms of affine surfaces

Keywords: Dynamical systems, arithmetic dynamics, valuations, algebraic geometry

Abstract: An affine variety X_0 over an algebraically closed field $\mathbf{k}$ is a subspace of $\mathbf{k}^N$ defined by polynomial equations. A polynomial endomorphism of X_0 is a polynomial transformation of $\mathbf{k}^N$ that preservers X_0 in the sense that $f(X_0) \subset X_0$. When the dimension of X_0 is 2, we say that X_0 is an affine surface. The goal of my thesis is to study the dynamical system given by an affine surface X_0 and $f : X_0 \to X_0$ a polynomial endomorphism of X_0. The different questions one can ask are: are there dense orbits or Zariski-dense orbits ? If the orbit of a point goes to infinity, can we control the speed of divergence ? Is there a lot of periodic orbits ? Can we construct interesting invariant probability measures ? To answer these questions, I use valuative techniques. The yynamical system (X_0, f) induces a dynamical system $(\mathcal{V}_\infty, f_*)$ where $\mathcal{V}_\infty$ is the space of valuations centered at infinity of X_0. The study of this dynamical system is the main goal of this memoir and it will allow to answer the questions mentionned above.