

UNLIKELY INTERSECTIONS PROBLEM FOR AUTOMORPHISMS OF MARKOV SURFACES

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ABSTRACT. We study the problem of unlikely intersections for automorphisms of Markov surfaces of positive entropy. We show for certain parameters that two automorphisms with positive entropy share a Zariski dense set of periodic points if and only if they share a common iterate. Our proof uses arithmetic equidistribution for adelic line bundles over quasiprojective varieties, the theory of laminar currents and quasi-Fuchsian representation theory.

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1. INTRODUCTION

The Markov surface \mathcal{M}_D of parameter $D \in \mathbf{C}$ is the affine subvariety of \mathbf{C}^3 defined by the equation

$$x^2 + y^2 + z^2 = xyz + D \quad (1)$$

This family of surfaces has been heavily studied as they appear in different areas of mathematics (see [Can09] or [RR22] §2). We study the dynamics of polynomial automorphisms of \mathcal{M}_D , that is polynomial transformations of the ambient space \mathbf{C}^3 that preserves \mathcal{M}_D and are invertible there. A loxodromic automorphism f is an automorphism with first dynamical degree $\lambda_1 > 1$. Here the first

dynamical degree is defined as follows. Let X be a *completion* of \mathcal{M}_D , that is a projective surface with an embedding $\mathcal{M}_D \hookrightarrow X$ as an open dense subset and let H be an ample divisor on X , then

$$\lambda_1(f) := \lim_{n \rightarrow \infty} ((f^n)^* H \cdot H)^{1/n} \quad (2)$$

where \cdot is the intersection product on divisors and $(f^n)^*$ is the pull back operator induced by f^n on the Néron-Severi group of X . One shows that this definition does not depend on the choice of X nor H . Alternatively, it is known (see Theorem 1.2) that the topological entropy of f is equal to $h_{top}(f) = \log \lambda_1(f)$. Therefore, loxodromic automorphisms are the one with positive entropy.

1.1. Character varieties and the Markov surfaces. Let \mathbb{T}_1 be the once punctured torus. The fundamental group $\pi_1(\mathbb{T}_1)$ is a free group generated by two elements a and b . The commutator $[a, b] := aba^{-1}b^{-1}$ is represented by a simple loop around the puncture that follows the orientation of the surface. One can study the representation of $\pi_1(\mathbb{T}_1)$ into the affine variety $\mathrm{SL}_2(\mathbb{C})$. It is clear that

$$\mathrm{hom}(\pi_1(\mathbb{T}_1), \mathrm{SL}_2(\mathbb{C})) \simeq \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \quad (3)$$

as $\pi_1(\mathbb{T}_1)$ is a free group on two generators, therefore it is an affine variety. Define the character variety,

$$\mathcal{X} := \mathrm{hom}(\pi_1(\mathbb{T}_1), \mathrm{SL}_2(\mathbb{C})) // \mathrm{SL}_2(\mathbb{C}) \quad (4)$$

where the action of $\mathrm{SL}_2(\mathbb{C})$ is diagonal and given by conjugation and $//$ is the Geometric Invariant Theory (GIT) quotient. This is also an affine variety and we have the following result of Fricke and Klein.

Theorem 1.1 (Fricke, Klein, [Gol09]). *The algebraic variety \mathcal{X} is isomorphic to \mathbb{C}^3 . The isomorphism is given by*

$$[\rho] \in \mathcal{X} \mapsto (\mathrm{Tr}(\rho(a)), \mathrm{Tr}(\rho(b)), \mathrm{Tr}(\rho(ab))). \quad (5)$$

We will denote by $(x, y, z) = (\mathrm{Tr}(\rho(a)), \mathrm{Tr}(\rho(b)), \mathrm{Tr}(\rho(ab)))$ these are the *Fricke-Klein* coordinates. Let $\kappa: \mathcal{X} \rightarrow \mathbb{C}$ be the regular function

$$\kappa(\rho) = \mathrm{Tr}(\rho([a, b])) \quad (6)$$

where $[a, b] = aba^{-1}b^{-1}$. One can show that

$$\kappa = x^2 + y^2 + z^2 - xyz - 2 \quad (7)$$

Therefore, if $\mathcal{X}_t = \kappa^{-1}(t)$ is the relative character variety, we have

$$\mathcal{X}_t = \mathcal{M}_{t+2} \quad (8)$$

where \mathcal{M}_D is the Markov surface of parameter D .

1.2. Automorphism group of the Markov surfaces. The generalized mapping class group $\text{MCG}^*(\mathbb{T}_1)$ is the group of homotopy class of homeomorphism of T_1 (not necessarily orientation preserving). It contains $\text{MCG}(\mathbb{T}_1)$ as an index 2 subgroup and it acts on $\pi_1(\mathbb{T}_1)$, we have the following isomorphism:

$$\text{MCG}^* \simeq \text{Out}(\pi_1(\mathbb{T}_1)) \quad (9)$$

Furthermore,

$$\text{Out}(\pi_1(\mathbb{T}_1)) \simeq \text{GL}_2(\mathbf{Z}). \quad (10)$$

For any element $\Phi \in \text{Out}(\pi_1(\mathbb{T}_1))$, $\Phi([a, b])$ is conjugated to $[a, b]^\pm$. This implies, that the action of $\text{MCG}^*(\mathbb{T}_1)$ on \mathcal{X} preserves every \mathcal{X}_t . Now, the matrix $-\text{id}$ acts trivially, because in $\text{SL}_2(\mathbf{C})$ we have that $\text{Tr} A = \text{Tr} A^{-1}$, so for all $D \in \mathbf{C}$ we get a group homomorphism

$$\text{PGL}_2(\mathbf{Z}) \rightarrow \text{Aut}(\mathcal{M}_D) \quad (11)$$

Theorem 1.2 ([?] Theorem A and B, [ÈH74]). *Let $\Gamma^* \subset \text{PGL}_2(\mathbf{Z})$ be the subgroup of element congruent to $\text{id} \bmod 2$, then for any $D \in \mathbf{C}$,*

$$\Gamma^* \rightarrow \text{Aut}(\mathcal{M}_D) \quad (12)$$

is injective and its image is of index at most 8. Furthermore if $\Phi \in \Gamma^$ and ρ is its spectral radius, then $\log \rho = \lambda_1(f_\Phi)$ where f_Φ is the automorphism of \mathcal{M}_D induced by Φ . Furthermore, the topological entropy of f_Φ is equal to $\log \lambda_1(f_\Phi)$.*

We can describe the group homomorphism. Let $\sigma_x \in \text{Aut}(\mathcal{M}_D)$ be the automorphism

$$\sigma_x(x, y, z) = (yz - x, y, z), \quad (13)$$

If we fix the coordinates y, z , then the equation defining \mathcal{M}_D becomes a polynomial equation of degree 2 with respect to x , σ_x permutes the 2 roots of this equation. We can define σ_y, σ_z in the same way. Then, $\sigma_x, \sigma_y, \sigma_z$ generate a free group isomorphic to $(\mathbf{Z}/2\mathbf{Z}) * (\mathbf{Z}/2\mathbf{Z}) * (\mathbf{Z}/2\mathbf{Z})$ which is of finite index in $\text{Aut}(\mathcal{M}_D)$ (see [ÈH74]). The subgroup Γ^* is the free group on the three generators

$$\begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (14)$$

which correspond respectively to $\sigma_x, \sigma_y, \sigma_z$. For a more detailed description of the action of $\text{GL}_2(\mathbf{Z})$ on the character variety, see the appendix of [Gol03].

1.3. The Picard parameter $D = 4$. When $D = 4$ the trace of the commutator is equal to $\text{Tr}(\rho([a, b])) = 2$. This corresponds to reducible representations. This parameter is very special because of the following. There is a $2 : 1$ cover of S_4 by the algebraic torus $\mathbb{G}_m^2 = \mathbf{C}^\times \times \mathbf{C}^\times$ given by

$$\eta : (u, v) \in \mathbb{G}_m^2 \mapsto \left(u + \frac{1}{u}, v + \frac{1}{v}, uv + \frac{1}{uv} \right) \in S_4. \quad (15)$$

If σ is the involution on \mathbb{G}_m^2 given by $\sigma(u, v) = (u^{-1}, v^{-1})$, then S_4 is the quotient η is the quotient map. The action of the automorphism group is very explicit for the Picard parameter $D = 4$. Indeed, $\mathrm{GL}_2(\mathbf{Z})$ acts on \mathbb{G}_m^2 by monomial transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (u, v) = (u^a v^b, u^c v^d). \quad (16)$$

We have $\sigma = -\mathrm{id}$ and $\mathrm{GL}_2(\mathbf{Z})/\langle\sigma\rangle = \mathrm{PGL}_2(\mathbf{Z})$ acts on $S_4 = \mathbb{G}_m^2/\langle\sigma\rangle$. Thus, all dynamical problems on S_4 can be lifted to \mathbb{G}_m^2 . The parameter $D = 4$ is the only one where \mathcal{M}_D is a finite equivariant quotient of \mathbb{G}_m^2 . Indeed, Rebelo and Roeder in [RR22] proved that the parameter $D = 4$ is the only parameter where the Fatou set of $\mathrm{Aut}(\mathcal{M}_D)$ is empty.

1.4. Green functions. If $f : \mathbf{C}^2 \rightarrow \mathbf{C}^2$ is a Henon transformation of \mathbf{C}^2 (i.e a loxodromic automorphism of \mathbf{C}^2), it defines a birational transformation of \mathbf{P}^2 and f, f^{-1} have one indeterminacy point. Up to conjugation, we can suppose that $\mathrm{Ind}(f) \neq \mathrm{Ind}(f^{-1})$, in that case, we define the Green functions G_f^+, G_f^- as follows

$$G_f^\pm(p) = \lim_n \frac{1}{\deg(f^{\pm n})} \log^+ (||f^{\pm n}(p)||). \quad (17)$$

We have the following properties (see [BS91a]).

- (1) G_f^+ is well defined, continuous and plurisubharmonic over \mathbf{C}^2 ,
- (2) $G_f^+ \circ f = \deg(f) G_f^+$,
- (3) $G_f^+(p) = 0$ if and only if the forward orbit $(f^N(p))_{N \geq 0}$ is bounded.

The function G_f^- satisfies similar properties. We define the Green currents $T^+ = dd^c G_f^+$ and $T^- = dd^c G_f^-$. These are positive closed $(1, 1)$ -currents over \mathbf{C}^2 and the measure

$$\mu := T^+ \wedge T^- \quad (18)$$

is well defined. It is of finite mass, thus we can suppose that it is a probability measure. We call it the *equilibrium measure* of f . It is f -invariant and its support is called the *Julia set* of f .

Now if $f \in \mathrm{Aut}(\mathcal{M}_D)$ is a loxodromic automorphism, the construction of the Green functions G_f^+, G_f^- is similar as in the Henon case. Since $\mathcal{M}_D(\mathbf{C})$ embeds into \mathbf{C}^3 we can apply the same definition as in (17) by replacing $\deg(f^n)$ by $\lambda_1(f)^n$ and use the norm given by $|| (x, y, z) || = \max(|x|, |y|, |z|)$. Thus we can still define Green currents, the Julia set of f and the equilibrium measure μ_f .

1.5. Unlikely intersections problem for special parameters. The main result of this paper is

Theorem A. *Let $D = 0$ or $D = 2 \cos(\frac{2\pi}{q})$ with $q \in \mathbf{Z}_{\geq 2}$. If f and g are two loxodromic automorphisms of \mathcal{M}_D , then the following are equivalent*

- (1) $\mathrm{Per}(f) \cap \mathrm{Per}(g)$ is Zariski dense.
- (2) $\mathrm{Julia}(f) = \mathrm{Julia}(g)$.
- (3) $\mu_f = \mu_g$.

- (4) $\text{Per}(f) = \text{Per}(g)$.
- (5) $\exists N, M \in \mathbf{Z}, f^N = g^M$.

The first four equivalences are called *unlikely intersection* problems in the literature. It was first established by Baker and DeMarco for endomorphisms of \mathbf{P}^1 ([BD11]) and for polarized endomorphisms of projective varieties in characteristic zero by Yuan and Zhang in [YZ17] and [YZ21]. The first instance of this result for non-projective varieties is due to Dujardin and Favre who showed Theorem A for Hénon maps over a number field in [DF17]. It is believed that one could lower the hypothesis of the theorem requiring only $\text{Per}(f) \cap \text{Per}(g)$ to be infinite (see [DF17] Theorem D and Conjecture 3). In [CD20], Cantat and Dujardin showed a similar result for subgroups of automorphisms of projective surfaces: If $\Gamma \subset \text{Aut}(X)$ is a large subgroup of automorphisms of a projective surface then Γ cannot have a Zariski dense set of finite orbits unless X is a Kummer example, that is the quotient of an abelian surface by a finite group.

1.6. Arithmetic equidistribution. If D is algebraic, the construction of the Green functions of f can be done over any complete algebraically closed field \mathbf{C}_v such that $\mathbf{Q}(D) \hookrightarrow \mathbf{C}_v$. We write G_v^\pm and μ_v for the Green functions and the equilibrium measures respectively. For the Hénon case, the data of the line at infinity L_∞ in \mathbf{P}^2 and the data of $G_v = \max(G_v^+, G_v^-)$ for every \mathbf{C}_v defines an *adelic divisor* over \mathbf{P}^2 (see §2). Yuan's equidistribution theorem [Yua08] states that the Galois orbits of any generic sequence of periodic points equidistributes with respect to μ_v .

The equidistribution of periodic points with respect to the equilibrium measure is a key ingredient of the proof of Theorem A. However, for the Markov surface, $\max(G_v^+, G_v^-)$ will *not* define in general an adelic divisor because for every completion X of \mathcal{M}_D , $X \setminus \mathcal{M}_D$ has several irreducible components (see Lemma 2.3). To overcome this, we use Yuan and Zhang's theory of adelic divisors over quasiprojective varieties in [YZ23] where they also prove an arithmetic equidistribution theorem in this setting. This is done in §6 and §7.

1.7. For a general parameter. Theorem A cannot hold for every parameter D . Namely, the Picard surface \mathcal{M}_4 provides a counterexample. Indeed, for every monomial transformation M of \mathbb{G}_m^2 , the periodic points are given by (μ, ω) where μ, ω are roots of unity, the Julia set of M is $\mathbb{S}^1 \times \mathbb{S}^1$ and the equilibrium measure is the Lebesgue measure on the Julia set. Thus, when looking at the quotient, we have that every loxodromic monomial automorphism of \mathcal{M}_4 have the same equilibrium measure, the same Julia set and share a Zariski dense set of periodic points. We conjecture that the finite quotient of \mathbb{G}_m^2 are the only counterexample to the unlikely intersection principle. This is the affine counterpart to the Kummer example appearing in the result of Cantat and Dujardin. Using a specialization argument, we show the following result that goes in the direction of this conjecture.

Theorem B. *Let $D \in \mathbf{C}$ be transcendental and let $f, g \in \text{Aut}(\mathcal{M}_D)$ be loxodromic automorphisms. The following assertions are equivalent:*

- (1) $\text{Per}(f) = \text{Per}(g)$.
- (2) $\exists N, M \in \mathbf{Z}, f^N = g^M$.

1.8. Plan of the paper. The proof of Theorem A is split into three parts. In the first part, we construct the Green functions, Green currents and the equilibrium measure of any loxodromic automorphism of \mathcal{M}_D with D algebraic at both archimedean and non-archimedean places. We then apply Yuan-Zhang arithmetic equidistribution theorem from [YZ23] to show that two loxodromic automorphisms of \mathcal{M}_D sharing a Zariski dense set of periodic points must have the same equilibrium measure at every place.

The second part is to apply the method of Bedford, Lyubich and Smillie in [BLS93] to show that in $\mathcal{M}_D(\mathbb{C})$ every saddle periodic point of a loxodromic automorphism is in the support of the equilibrium measure. We use the theory of laminar and strongly approximable currents from [Duj05] and apply techniques from [Duj04].

The third part is to construct a "special" saddle periodic point $q(f)$ which has the following property: the orbit of $q(f)$ under any loxodromic automorphism g that does not share a common iterate with f is unbounded. To construct $q(f)$ we use the theory of quasi Fuchsian representation, the simultaneous uniformization theorem of Bers and Thurston's hyperbolisation theorem for 3-fold fibering over a circle (see [McM96]). This third part is where the hypothesis on the parameter D is used. The specific values of D give an interpretation of \mathcal{M}_D as representation of the fundamental group of the punctured torus ($D = 0$) or of an orbifold obtained from a genus 1 torus with a singular point of index q ($D = 2 - 2\cos(\frac{2\pi}{q})$).

Acknowledgments. This work was done during my PhD thesis. I would like to thank my PhD advisors Serge Cantat and Junyi Xie for their guidance. I also thank Juan Souto for answering questions about quasi Fuchsian representation theory and Xinyi Yuan for our discussions on adelic divisors. I thank Seung uk Jang for pointing out typos and small mistakes in an earlier version of this paper. Part of this paper was written during my visit at Beijing International Center for Mathematical Research which I thank for its welcome. Finally, I thank the France 2030 framework programme Centre Henri Lebesgue ANR-11-LABX-0020-01 and European Research Council (ERCGOAT101053021) for creating an attractive mathematical environment.

2. ADELIC DIVISORS OVER QUASIPROJECTIVE VARIETIES

2.1. Berkovich spaces. For a general reference on Berkovich spaces, we refer to [Ber12]. Let \mathbf{C}_v be a complete algebraically closed field with respect to an absolute value $|\cdot|_v$. Let $X_{\mathbf{C}_v} = \text{Spec } A$ be an affine \mathbf{C}_v -variety, the *Berkovich analytification* $X_{\mathbf{C}_v}^{\text{an}}$ of $X_{\mathbf{C}_v}$ is the set of multiplicative seminorms over A that extends $|\cdot|_v$. It is a locally ringed space with a contraction map

$$c : X_{\mathbf{C}_v}^{\text{an}} \rightarrow X_{\mathbf{C}_v} \quad (19)$$

defined as follows, if $x \in X_{\mathbf{C}_v}^{\text{an}}$, then

$$c(x) = \ker(x) = \{a \in A : |a|_x = 0\} \quad (20)$$

where $|\cdot|_x$ is the seminorm associated to x . If $X_{\mathbf{C}_v}$ is a \mathbf{C}_v -variety then $X_{\mathbf{C}_v}^{\text{an}}$ is defined by a glueing process using affine charts and we have the contraction map $c : X_{\mathbf{C}_v}^{\text{an}} \rightarrow X_{\mathbf{C}_v}$. In particular, if $\phi \in$

$\mathbf{C}_v(X_{\mathbf{C}_v})$ is a rational function with divisor $\text{div}(\phi)$, for $x \in X_{\mathbf{C}_v}^{\text{an}} \setminus (\text{Supp } D)_{\mathbf{C}_v}^{\text{an}}$, we define $|\phi|(x) := |c^* \phi|_x$. If $X_{\mathbf{C}_v}$ is proper (e.g projective), then $X_{\mathbf{C}_v}^{\text{an}}$ is compact. If $p \in X_{\mathbf{C}_v}(\mathbf{C}_v)$ is a closed point, then the fiber $c^{-1}(p)$ consists of a single point $|\cdot|_p$ defined by

$$|a|_p = |a(p)|_v \quad (21)$$

where $a(p) = a \bmod p$, this uses the fact that the local field at p is \mathbf{C}_v . We thus have an embedding

$$\iota_0 = c^{-1} : X_{\mathbf{C}_v}(\mathbf{C}_v) \hookrightarrow X_{\mathbf{C}_v}^{\text{an}} \quad (22)$$

and we write $X_{\mathbf{C}_v}(\mathbf{C}_v)$ for its image. It is a dense subset of $X_{\mathbf{C}_v}^{\text{an}}$. If the reader is not familiar with Berkovich spaces, it is enough for this paper to think of $X_{\mathbf{C}_v}^{\text{an}}$ as being equal to $X_{\mathbf{C}_v}(\mathbf{C}_v)$ (this is the case in particular when $\mathbf{C}_v = \mathbf{C}$). If $\phi : X_{\mathbf{C}_v} \rightarrow Y_{\mathbf{C}_v}$ is a morphism of varieties, then there exists a unique morphism

$$\phi^{\text{an}} : X_{\mathbf{C}_v}^{\text{an}} \rightarrow Y_{\mathbf{C}_v}^{\text{an}} \quad (23)$$

such that the diagram

$$\begin{array}{ccc} X_{\mathbf{C}_v}^{\text{an}} & \xrightarrow{\phi^{\text{an}}} & Y_{\mathbf{C}_v}^{\text{an}} \\ \downarrow & & \downarrow \\ X_{\mathbf{C}_v} & \xrightarrow{\phi} & Y_{\mathbf{C}_v} \end{array} \quad (24)$$

commutes. In particular, if $X_{\mathbf{C}_v} \subset Y_{\mathbf{C}_v}$, then $X_{\mathbf{C}_v}^{\text{an}}$ is isomorphic to $c_Y^{-1}(X_{\mathbf{C}_v}) \subset Y_{\mathbf{C}_v}^{\text{an}}$.

2.2. Places and restricted analytic spaces. Let \mathbf{K} be a number field. A *place* v of \mathbf{K} is an equivalence class of absolute values over \mathbf{K} . If v is archimedean then there is an embedding $\sigma : \mathbf{K} \hookrightarrow \mathbf{C}$ such that any absolute value representing v is of the form $|x| = |\sigma(x)|_{\mathbf{C}}^t$ with $0 < t \leq 1$. In that case we will write $|\cdot|_v$ for the absolute value with $t = 1$ and we write $\mathbf{C}_v = \mathbf{C}$. If v is non-archimedean (we also say that v is *finite*) it lies over a prime p then we write $|\cdot|_v$ for the absolute value of \mathbf{K} representing v such that $|p|_v = \frac{1}{p}$. Every finite place v is of the form

$$v(P) = \#(O_{\mathbf{K}}/\mathfrak{m})^{-\text{ord}_{\mathfrak{m}}(P)} \quad (25)$$

for $P \in \mathbf{K}$ where \mathfrak{m} is a maximal ideal of $O_{\mathbf{K}}$. Let \mathbf{K}_v be the completion of \mathbf{K} with respect to $|\cdot|_v$, we denote by \mathbf{C}_v the completion of the algebraic closure of \mathbf{K}_v with respect to $|\cdot|_v$. We write $M(\mathbf{K})$ for the set of places of \mathbf{K} . If $V \subset M(\mathbf{K})$, we write $V[\text{f}]$ for the subset of finite places in V and $V[\infty]$ for the archimedean ones.

If v is archimedean, then define $O_v = \mathbf{C}_v$. If v is non-archimedean, we define O_v as the ring of elements of absolute values ≤ 1 and κ_v as the residue field

$$\kappa_v := O_v/\mathfrak{m}_v. \quad (26)$$

where \mathfrak{m}_v is the maximal ideal of elements of absolute value < 1 .

Let X be a variety over \mathbf{K} . For every place v of K , define $X_v := X \times_{\mathbf{K}} \text{Spec } C_v$. Similarly, if D is an \mathbf{R} -divisor over X then we denote by D_v its image under the base change. We write X_v^{an} for the Berkovich analytification of X_v . We also define the global Berkovich analytification of X as

$$X^{\text{an}} := \bigsqcup_v X_v^{\text{an}}. \quad (27)$$

Comparing to [YZ23], this space is called the *restricted analytic space* of X by Yuan and Zhang. If V is a set of places, we also define

$$X_V^{\text{an}} := \bigsqcup_{v \in V} X_v^{\text{an}}. \quad (28)$$

In particular, we define

$$X^{\text{an}}[\mathbf{f}] := \bigsqcup_{v \in M(\mathbf{K})[\mathbf{f}]} X_v^{\text{an}}, \quad X^{\text{an}}[\infty] := \bigsqcup_{v \in M(\mathbf{K})[\infty]} X_v^{\text{an}} \quad (29)$$

If \mathcal{X} is a variety over $O_{\mathbf{K}}$, we write \mathcal{X}_v for the base change

$$\mathcal{X}_v = \mathcal{X} \times_{O_{\mathbf{K}}} \text{Spec } O_v. \quad (30)$$

Similarly, if \mathcal{D} is an \mathbf{R} -divisor over \mathcal{X} , we denote by \mathcal{D}_v its image under the base change.

2.3. Adelic divisors over a projective variety. Let \mathbf{K} be a number field and X a projective variety over \mathbf{K} . A divisor on X is a formal finite sum $\sum_i a_i E_i$ where $a_i \in \mathbf{Z}$ and E_i is an irreducible subvariety of codimension 1. An \mathbf{R} -divisor on X is a sum

$$D = \sum a_i D_i \quad (31)$$

where $a_i \in \mathbf{R}$ and D_i are divisors on X .

Models, horizontal and vertical divisors. If $D = \sum_i a_i D_i$ is an \mathbf{R} -divisor on X , a *model* of (X, D) is the data of $(\mathcal{X}, \mathcal{D})$ where \mathcal{X} is a normal projective variety over $O_{\mathbf{K}}$ with generic fiber X and $\mathcal{D} = \sum_i a_i \mathcal{D}_i$ is an \mathbf{R} -divisor on \mathcal{X} such that $\mathcal{D}_{i|X} = D_i$. There are two types of divisors on a projective variety \mathcal{X} over $O_{\mathbf{K}}$: *horizontal divisors* which irreducible components are the closure of prime divisors over the generic fiber and *vertical divisors* which irreducible components do not intersect the generic fiber. If D is a divisor on X , we will still write D for the horizontal divisor it induces over \mathcal{X} . Every divisor \mathcal{D} over \mathcal{X} can be uniquely split into a sum $\mathcal{D} = D_{\text{hor}} + \mathcal{D}_{\text{vert}}$ of an horizontal divisor and a vertical one. In particular, $D_{\text{hor}} = \mathcal{D} \times_{\text{Spec } O_{\mathbf{K}}} \text{Spec } \mathbf{K}$ is the generic part of \mathcal{D} .

Green functions. A *Green function* of D is a continuous function $g : X^{\text{an}} \setminus (\text{Supp } D)^{\text{an}} \rightarrow \mathbf{R}$ such that for any point $p \in (\text{Supp } D)^{\text{an}}$, if z_i is a local equation of D_i at p then the function

$$g + \sum_i a_i \log |z_i| \quad (32)$$

extends locally to a continuous function at p . For any place $v \in M(\mathbf{K})$, we write g_v for $g|_{X_v^{\text{an}}}$, $g[\mathbf{f}] = g|_{X^{\text{an}}[\mathbf{f}]}$ and $g[\infty] = g|_{X^{\text{an}}[\infty]}$. For the archimedean places v we add the extra condition that g_v must be invariant under complex conjugation. That is if \bar{v} is the conjugate place of v , then $g_{\bar{v}}(z) = g_v(\bar{z})$ for $z \in X(\mathbf{C})$. Notice that compared to [Mor16] or [YZ23], our definition of Green functions differs by a factor 2.

If $(\mathcal{X}, \mathcal{D})$ is a model of (X, D) , then for every finite place v , $(\mathcal{X}_v, \mathcal{D}_v)$ induces a Green function of D_v over X_v^{an} as follows. We have the reduction map (see [Ber12])

$$r_{\mathcal{X}_v} : X_v^{\text{an}} \rightarrow (\mathcal{X}_v)_{\mathbf{K}_v}. \quad (33)$$

Let $x \in X_v^{\text{an}} \setminus (\text{Supp } D)_v^{\text{an}}$, and let z_i be a local equation of $\mathcal{D}_{i,v}$ at $r_{\mathcal{X}_v}(x)$, we define

$$g_{(\mathcal{X}_v, \mathcal{D}_v)}(x) := - \sum_i a_i \log |z_i|(x). \quad (34)$$

It does not depend on the choice of the local equations z_i because an invertible regular function ϕ satisfies $|\phi|(x) = 1$ (recall that here v is non-archimedean).

Adelic divisors. An *adelic* divisor \bar{D} on X is the data of $\bar{D} = (D, g)$ where D is an \mathbf{R} -divisor on X and g is a continuous Green function of D over X^{an} such that

- (1) There exists an open subset $V \subset \text{Spec } O_{\mathbf{K}}$ and a model $(\mathcal{X}_V, \mathcal{D}_V)$ of (X, D) over V such that for every finite place v lying over V

$$g_v = g_{(\mathcal{X}_v, \mathcal{D}_v)} \quad (35)$$

- (2) For any finite place not lying over V and all the archimedean ones, the Green function g_v is the uniform limit of model Green functions of D over X_v^{an} .

A *model adelic* divisor on X is an adelic divisor (D, g) on X such that $g[\mathbf{f}]$ is induced by a model $(\mathcal{X}, \mathcal{D})$. It is *vertical* if \mathcal{D} is vertical, in particular $D = 0$.

Following [Zha93], a model adelic divisor is *semipositive* if $g[\infty]$ is plurisubharmonic (psh) and \mathcal{D} is nef over \mathcal{X} . An adelic divisor $\bar{D} = (D, g)$ is *semipositive* if $g[\infty]$ is psh and there exists a sequence of semipositive model adelic divisors $\bar{\mathcal{D}}_n$ such that $(\mathcal{X}_n, \mathcal{D}_n)$ is a model of (X, D) and the sequence

$$g[\mathbf{f}] - g_{(\mathcal{X}_n, \mathcal{D}_n)} \quad (36)$$

converges uniformly to zero over X^{an} . An adelic divisor is *integrable* if it is the difference of two semipositive adelic divisors. We say that an adelic divisor $\bar{D} = (D, g)$ is *effective* if $g \geq 0$ in particular this implies that D is an effective divisor. We write $\bar{D} \geq \bar{D}'$ if $\bar{D} - \bar{D}'$ is effective. We say that \bar{D} is *strictly effective* if $\bar{D} \geq 0$ and $g[\infty] > 0$.

Lemma 2.1. *Every ample divisor H on X admits a strictly effective semipositive model.*

Proof. Let $\iota : X \hookrightarrow \mathbf{P}_{\mathbf{K}}^N$ be a projective embedding such that $H = \{T_0 = 0\} \cap \iota(X)$ where T_0, \dots, T_N are the homogeneous coordinates of $\mathbf{P}_{\mathbf{K}}^N$. We can find a model \mathcal{X} such that ι extends to a regular

map

$$\bar{\iota}: \mathcal{X} \rightarrow \mathbf{P}_{O_K}^N \quad (37)$$

then the divisor $\mathcal{D} = \bar{\iota}^* \{T_0 = 0\}$ is a model of H and a nef divisor over \mathcal{X} (Here T_0 is a homogeneous coordinate of $\mathbf{P}_{O_K}^N$). For every archimedean place v , g_v is given by

$$g_v(p) = 1 + \log^+ \max(|t_1(p)|, \dots, |t_N(p)|) \quad (38)$$

where $t_i := \frac{T_i}{T_0}$. This is (one plus) the pull back of the *Weil* metric over \mathbf{P}^N (see e.g [Cha11]). \square

Proposition 2.2 (Lemma 3.3.3 of [YZ23]). *If $\overline{\mathcal{D}} = (D, g)$ is a model adelic divisor, then $\overline{\mathcal{D}}$ is effective if and only if $g[\infty] \geq 0$ and \mathcal{D} is effective.*

In our setting, using adelic divisors over a projective variety will not be enough because the completions of \mathcal{M}_D have several components at infinity. The following lemma shows that Green functions of divisors do not behave well under maximum.

Lemma 2.3. *Let X be a projective variety, let $E, F \subset X$ be two prime divisors of X . Let $D_i = a_i E + b_i F, i = 1, 2$ be two \mathbf{R} -divisors of X^{an} with Green function g_i such that $0 < a_1 < a_2, b_1 > b_2 > 0$, then*

$$\max(g_1, g_2) \quad (39)$$

is not the Green function of an \mathbf{R} -divisor in X .

Proof. Let $g = \max(g_1, g_2)$ and suppose that there exists a divisor D such that g is a Green function of D . Then, D must be of the form $D = aE + bF$. Let $p \in (E \cap F)^{\text{an}}$ and let x, y be local equations of E and F respectively. Then, locally at p we have

$$g_i \asymp -a_i \log |x| - b_i \log |y| \text{ and } g \asymp -a \log |x| - b \log |y| \quad (40)$$

where \asymp means equality up to a (bounded) continuous function. Pick $\varepsilon > 0$ such that $a_1 + b_1 \varepsilon < a_2$. In the region $U_\varepsilon = \{|\log |y|| \leq \varepsilon |\log |x||\}$, we have

$$g_1 \leq -(a_1 + \varepsilon b_1) \log |x| < -a_2 \log |x| \leq g_2 \quad (41)$$

where \leq means inequality up to a (bounded) continuous function. Thus, $g = g_2$ in that region and therefore $D = a_2 E + b_2 F$. But the same reasoning in the region $\{|\log |x|| \leq \varepsilon |\log |y||\}$ for $\varepsilon > 0$ small enough shows that $g = g_1$ in that region and therefore D should be equal to $D = a_1 E + b_1 F$ and this is a contradiction. \square

2.4. Over quasiprojective varieties. The main reference for this section is [YZ23]. Let U be a normal quasiprojective variety over a number field \mathbf{K} . A quasiprojective model \mathcal{U} of U is a quasiprojective variety \mathcal{U} over $\text{Spec } O_K$ with generic fiber isomorphic to U . A projective model of \mathcal{U} is a projective variety \mathcal{X} over $\text{Spec } O_K$ with an open embedding $\mathcal{U} \hookrightarrow \mathcal{X}$.

A model adelic divisor on \mathcal{U} is a model adelic divisor induced by a projective model of \mathcal{U} . If $\overline{\mathcal{D}}$ is a model adelic divisor on some projective model \mathcal{X} of \mathcal{U} . We say that $\overline{\mathcal{D}}$ is supported at infinity if $(\mathcal{D}_{\text{hor}})|_U = 0$. We write $\widehat{\text{Div}}(\mathcal{X}, \mathcal{U})$ for the set of model adelic divisor supported at

infinity induced by a fixed projective model \mathcal{X} of \mathcal{U} . Since the system of projective models of U is a projective system, we can define the inductive limit

$$\widehat{\text{Div}}(\mathcal{U})_{\text{mod}} := \varinjlim_{\mathcal{X}} \text{Div}(\mathcal{X}, \mathcal{U}). \quad (42)$$

Our definition differs from [YZ23] because we only take divisors supported outside U . This makes sense for our dynamical setting.

A *boundary divisor* on \mathcal{U} is a model adelic divisor $\overline{\mathcal{D}}_0 = (\mathcal{X}_0, g)$ such that $\text{Supp } \mathcal{D}_0 = \mathcal{X}_0 \setminus \mathcal{U}$. It defines a norm on $\widehat{\text{Div}}(\mathcal{U})$ given by

$$\| \overline{\mathcal{D}} \| = \inf \{ \varepsilon > 0 : -\varepsilon \overline{\mathcal{D}}_0 \leq \overline{\mathcal{D}} \leq \varepsilon \overline{\mathcal{D}}_0 \} \quad (43)$$

An adelic divisor \overline{D} on \mathcal{U} is an element of the completion of $\widehat{\text{Div}}(\mathcal{U})_{\text{mod}}$ with respect to this norm. More precisely, an adelic divisor on \mathcal{U} is a sequence of model adelic divisors $(\mathcal{X}_i, \overline{\mathcal{D}}_i)$ such that there exists a sequence $\varepsilon_i > 0, \varepsilon_i \rightarrow 0$ such that

$$\forall j \geq i, \quad -\varepsilon_i \overline{\mathcal{D}}_0 \leq \overline{\mathcal{D}}_j - \overline{\mathcal{D}}_i \leq \varepsilon_i \overline{\mathcal{D}}_0. \quad (44)$$

If we denote by g_i the Green function of the model adelic divisor $\overline{\mathcal{D}}_i$. Then, this is equivalent to asking that

$$-\varepsilon_i g_0 \leq g_j - g_i \leq \varepsilon_i g_0. \quad (45)$$

In particular, (g_i) converges uniformly locally to a continuous function g over U^{an} . We call it the *Green function* of \overline{D} . We write $\widehat{\text{Div}}(\mathcal{U})$ for the completion of $\widehat{\text{Div}}(\mathcal{U})_{\text{mod}}$ with respect to this norm. An adelic divisor on U is an element of

$$\widehat{\text{Div}}(U/O_{\mathbf{K}}) := \varinjlim_{\mathcal{U}} \widehat{\text{Div}}(\mathcal{U}) \quad (46)$$

Remark 2.4. In [YZ23] Yuan and Zhang use only \mathbf{Q} -model divisors for the definition of $\text{Div}(\mathcal{X}, \mathcal{U})$ and $\widehat{\text{Div}}(\mathcal{U})$ whereas here we also allow \mathbf{R} -model adelic divisors. If one wants to use adelic line bundles instead of adelic divisors and in particular the global section of line bundles then it makes sense to use only \mathbf{Q} -line bundles in the limit process defining adelic line bundles with the boundary topology but we do not need that here. Anyway, for adelic divisors, using \mathbf{R} -model divisors provides the same final space thanks to the following lemma.

Lemma 2.5. *Let $\overline{\mathcal{D}}$ be a model \mathbf{R} -adelic divisor supported at infinity and let $\overline{\mathcal{D}}_0$ be a boundary divisor. Then for any $\varepsilon > 0$, there exists a \mathbf{Q} -model divisor $\overline{\mathcal{D}}_\varepsilon$ such that*

$$-\varepsilon \overline{\mathcal{D}}_0 \leq \overline{\mathcal{D}} - \overline{\mathcal{D}}_\varepsilon \leq \varepsilon \overline{\mathcal{D}}_0 \quad (47)$$

Proof. Let \mathcal{X} be a model over $\text{Spec } O_{\mathbf{K}}$ such that $\mathcal{D}, \mathcal{D}_0$ are defined. It is easy to find a \mathbf{Q} -divisor over \mathcal{X} arbitrary close to \mathcal{D} so we just need to do some work for the archimedean places.

Fix an archimedean place v , and let $g : X(\mathbf{C}) \rightarrow \mathbf{R}$ be the Green function of $\overline{\mathcal{D}}$ over v . Let E_1, \dots, E_r be the irreducible components of the boundary of X in $\mathcal{X}_{\mathbf{C}}$ and write $D = \sum_i a_i E_i$. Let g_i be a Green

function E_i over $X(\mathbf{C})$ then $g - \sum_i a_i g_i = h$ is a continuous bounded function over $X(\mathbf{C})$. Suppose $a_1 \neq 0$ we replace g_1 by $g_1 + \frac{1}{a_1}h$ such that

$$g = \sum_i a_i g_i. \quad (48)$$

Now, let $A > 0$ such that for all i $\sup_{X(\mathbf{C})} \left| \frac{g_i}{g_0} \right| \leq A$ and let $a_{i,n}$ be a sequence of rational numbers converging towards a_i , then

$$\left| \frac{g - \sum_i a_{i,n} g_i}{g_0} \right| \leq rA \max_i (|a_i - a_{i,n}|) \quad (49)$$

and we have the result. \square

Definition 2.6. An adelic divisor \overline{D} over U is

- *strongly nef* if for the Cauchy sequence $(\overline{\mathcal{D}}_i)$ defining it we can take for every $\overline{\mathcal{D}}_i$ a semipositive model adelic divisor.
- *nef* if there exists a strongly nef adelic divisor \overline{A} such that for all $m \geq 1$, $\overline{D} + m\overline{A}$ is strongly nef.
- *integrable* if it is the difference of two strongly nef divisors.

If \overline{D} is an adelic divisor over U , then \overline{D} has an associated *height function*

$$h_{\overline{D}} : U(\overline{\mathbf{K}}) \rightarrow \mathbf{R} \quad (50)$$

which is computed as follows if $D|_U = 0$ which is our use case in this paper:

$$\forall p \in U(\overline{\mathbf{K}}), \quad h_{\overline{D}}(p) = \frac{1}{\deg p} \sum_{v \in M(\mathbf{K})} \sum_{q \in \text{Gal}(\overline{\mathbf{K}}/\mathbf{K}) \cdot p} n_v g_{\overline{D},v}(q) \quad (51)$$

where $n_v = [\mathbf{K}_v : \mathbf{Q}_v]$. Moreover, for any closed $\overline{\mathbf{K}}$ -subvariety Z of U , we define the height of Z to be

$$h_{\overline{D}}(Z) := \frac{\overline{D}|_Z^{\dim Z + 1}}{(1 + \dim Z) D|_Z^{\dim Z}} \quad (52)$$

where $\overline{D}|_Z^{\dim Z + 1}$ represents the intersection number of adelic divisors, see [YZ23] for more details.

Proposition 2.7 ([YZ23] §2.5.5). *If $f : X \rightarrow Y$ is a morphism between quasiprojective varieties over \mathbf{K} , then there is a pullback operator*

$$f^* : \widehat{\text{Div}}(Y/\mathcal{O}_{\mathbf{K}}) \rightarrow \widehat{\text{Div}}(X/\mathcal{O}_{\mathbf{K}}) \quad (53)$$

that preserves model, strongly nef, nef and integrable adelic divisors. If g is the Green function of $\overline{D} \in \widehat{\text{Div}}(Y/\mathcal{O}_{\mathbf{K}})$, then the Green function of $f^\overline{D}$ is $g \circ f^{\text{an}}$.*

3. PICARD-MANIN SPACE AT INFINITY

3.1. Completions. A completion of \mathcal{M}_D is a projective surface X with an open embedding $\iota_X : \mathcal{M}_D \hookrightarrow X$. We call $X \setminus \iota_X(\mathcal{M}_D)$ the *boundary* of \mathcal{M}_D in X . By [Goo69] Proposition 1, it is a curve. We will also refer to it at the part "at infinity" in X . For any completion X of \mathcal{M}_D we define $\text{Div}_\infty(X)_\mathbf{A} = \bigoplus \mathbf{A}E_i$ where $\mathbf{A} = \mathbf{Z}, \mathbf{Q}, \mathbf{R}$ and $X \setminus \mathcal{M}_D = \bigcup E_i$, the space of \mathbf{A} -divisors at infinity. For any two completions X, Y we have a birational map $\pi_{XY} = \iota_Y \circ \iota_X^{-1} : X \dashrightarrow Y$. If this map is regular, we say that π_{XY} is a *morphism of completions* and that X is *above* Y . For any completion X, Y there exists a completion Z above X and Y . Indeed, take Z to be a resolution of indeterminacies of $\pi_{XY} : X \dashrightarrow Y$. A morphism of completions defines a pullback and a pushforward operator $\pi_{XY}^*, (\pi_{XY})_*$ on divisors and Néron-Severi classes. We have the projection formula,

$$\forall \alpha \in \text{NS}(X), \beta \in \text{NS}(Y), \alpha \cdot \pi_{XY}^* \beta = (\pi_{XY})_* \alpha \cdot \beta. \quad (54)$$

Let $\overline{\mathcal{M}}_D \subset \mathbf{P}^3$ be the closure of \mathcal{M}_D in \mathbf{P}^3 . We have that $\overline{\mathcal{M}}_D \setminus \mathcal{M}_D$ is a triangle of lines all of self intersection -1 . The matrix of the intersection form on $\text{Div}_\infty(\overline{\mathcal{M}}_D)$ is

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \quad (55)$$

Therefore, the intersection form is non degenerate over $\text{Div}_\infty(\overline{\mathcal{M}}_D)_\mathbf{A}$ and we have the embedding

$$\text{Div}_\infty(\overline{\mathcal{M}}_D)_\mathbf{A} \hookrightarrow \text{NS}(\overline{\mathcal{M}}_D)_\mathbf{A}. \quad (56)$$

And this holds for every completion X of \mathcal{M}_D .

3.2. Weil and Cartier classes. If $\pi_{YX} : Y \rightarrow X$ are two completions of \mathcal{M}_D then we have the embedding defined by the pullback operator

$$\pi_{YX}^* : \text{Div}_\infty(X)_\mathbf{A} \hookrightarrow \text{Div}_\infty(Y)_\mathbf{A}. \quad (57)$$

We define the space of Cartier divisors at infinity of \mathcal{M}_D to be the direct limit

$$\mathcal{C}(\mathcal{M}_D) := \varinjlim_X \text{Div}_\infty(X)_\mathbf{R}. \quad (58)$$

In the same way we define the space of Cartier classes of \mathcal{M}_D

$$\mathcal{Z}(\mathcal{M}_D) := \varinjlim_X \text{NS}(X)_\mathbf{R}. \quad (59)$$

An element of $\mathcal{Z}(\mathcal{M}_D)$ is an equivalence class of pairs (X, α) where X is a completion of \mathcal{M}_D and $\alpha \in \text{NS}(X)_\mathbf{R}$ such that $(X, \alpha) \simeq (Y, \beta)$ if and only if there exists a completion Z above X, Y such that $\pi_{ZX}^* \alpha = \pi_{ZY}^* \beta$. We say that $\alpha \in \mathcal{Z}(\mathcal{M}_D)$ is *defined* in X if it is represented by (X, α) . We have a

natural embedding $\mathcal{C}(\mathcal{M}_D) \hookrightarrow \mathcal{Z}(\mathcal{M}_D)$, we still write $\mathcal{C}(\mathcal{M}_D)$ for its image in $\mathcal{Z}(\mathcal{M}_D)$. We also define the space of Weil classes

$$\widehat{\mathcal{Z}}(\mathcal{M}_D) := \varprojlim_X \mathrm{NS}(X)_{\mathbf{R}} \quad (60)$$

where the compatibility morphisms are given by the pushforward morphisms $(\pi_{YX})_* : \mathrm{NS}(Y) \rightarrow \mathrm{NS}(X)$ for a morphism of completions $\pi_{YX} : Y \rightarrow X$. An element of this inverse limit is a family $\alpha = (\alpha_X)_X$ such that if X, Y are two completions of \mathcal{M}_D with Y above X , then $(\pi_{YX})_* \alpha_Y = \alpha_X$. We call α_X the *incarnation* of α in X . We have a natural embedding $\mathcal{Z}(\mathcal{M}_D) \hookrightarrow \widehat{\mathcal{Z}}(\mathcal{M}_D)$. We also define the space of Weil divisors at infinity

$$\mathcal{W}(\mathcal{M}_D) := \varprojlim_X \mathrm{Div}_{\infty}(X)_{\mathbf{R}} \quad (61)$$

and we have the commuting diagram

$$\begin{array}{ccc} \mathcal{C}(\mathcal{M}_D) & \hookrightarrow & \mathcal{Z}(\mathcal{M}_D) \\ \downarrow & & \downarrow \\ \mathcal{W}(\mathcal{M}_D) & \hookrightarrow & \widehat{\mathcal{Z}}(\mathcal{M}_D) \end{array} \quad (62)$$

Thanks to the projection formula, the intersection form defines a perfect pairing

$$\mathcal{Z}(\mathcal{M}_D) \times \widehat{\mathcal{Z}}(\mathcal{M}_D) \rightarrow \mathbf{R} \quad (63)$$

defined as follows. If $\alpha \in \mathcal{Z}(\mathcal{M}_D)$ is defined in X and $\beta \in \widehat{\mathcal{Z}}(\mathcal{M}_D)$, then

$$\alpha \cdot \beta = \alpha_X \cdot \beta_X \quad (64)$$

An element $\alpha \in \mathcal{W}(\mathcal{M}_D)$ is *effective* if for every completion X , α_X is an effective divisor. We write $\alpha \geq \beta$ if $\alpha - \beta$ is effective. An element $\beta \in \widehat{\mathcal{Z}}(\mathcal{M}_D)$ is *nef* if for every completion X , β_X is nef.

3.3. The Picard-Manin space of \mathcal{M}_D . We provide $\widehat{\mathcal{Z}}(\mathcal{M}_D)$ with the topology of the inverse limit, we call it the weak topology, $\mathcal{Z}(\mathcal{M}_D)$ is dense in $\widehat{\mathcal{Z}}(\mathcal{M}_D)$ for this topology. Analogously, $\mathcal{C}(\mathcal{M}_D)$ is dense in $\mathcal{W}(\mathcal{M}_D)$.

We define \mathcal{D}_{∞} for the set of prime divisors at infinity. An element of \mathcal{D}_{∞} is an equivalence class of pairs (X, E) where X is a completion of \mathcal{M}_D and E is a prime divisor at infinity. Two pairs $(X, E), (Y, E')$ are equivalent if the birational map π_{XY} sends E to E' . We will just write $E \in \mathcal{D}_{\infty}$ instead of (X, E) . We define the function $\mathrm{ord}_E : \mathcal{W}(\mathcal{M}_D) \rightarrow \mathbf{R}$ as follows. Let $\alpha \in \mathcal{M}_D$, if X is any completion where E is defined (in particular (X, E) represents $E \in \mathcal{D}_{\infty}$), then α_X is of the form

$$\alpha_X = a_E E + \sum_{F \neq E} a_F F \quad (65)$$

and we set $\mathrm{ord}_E(\alpha_X) = a_E$. This does not depend on the choice of (X, E) .

Lemma 3.1 ([BFJ08] Lemma 1.5). *The map*

$$\alpha \in \mathcal{W}(\mathcal{M}_D) \mapsto (\text{ord}_E(\alpha))_{E \in \mathcal{D}_\infty} \in \mathbf{R}^{\mathcal{D}_\infty} \quad (66)$$

is a homeomorphism for the product topology.

We can define a stronger topology on $\mathcal{Z}(\mathcal{M}_D)$ as follows. The intersection product defines a non-degenerate bilinear form

$$\mathcal{Z}(\mathcal{M}_D) \times \mathcal{Z}(\mathcal{M}_D) \rightarrow \mathbf{R} \quad (67)$$

of signature $(1, \infty)$ by the Hodge Index Theorem. Take an ample class ω on some completion X of \mathcal{M}_D such that $\omega^2 = 1$. Every Cartier class $\alpha \in \mathcal{Z}(\mathcal{M}_D)$ can be decomposed with respect to ω and ω^\perp

$$\alpha = (\alpha \cdot \omega)\omega + (\alpha - (\alpha \cdot \omega)\omega). \quad (68)$$

By the Hodge index theorem, the intersection form is negative definite on ω^\perp and we define the norm

$$\|\alpha\|_\omega^2 = (\alpha \cdot \omega)^2 - (\alpha - (\alpha \cdot \omega)\omega)^2. \quad (69)$$

This defines a topology on $\mathcal{Z}(\mathcal{M}_D)$ which is independent of the choice of ω . We call it the strong topology. We define $\overline{\mathcal{Z}}(\mathcal{M}_D)$ to be the completion of $\mathcal{Z}(\mathcal{M}_D)$ with respect to this topology. As this topology is stronger than the weak topology, $\overline{\mathcal{Z}}(\mathcal{M}_D)$ is a subspace of $\widehat{\mathcal{Z}}(\mathcal{M}_D)$. This is a Hilbert space and the intersection product extends to a continuous non-degenerate bilinear form

$$\overline{\mathcal{Z}}(\mathcal{M}_D) \times \overline{\mathcal{Z}}(\mathcal{M}_D) \rightarrow \mathbf{R}. \quad (70)$$

We call $\overline{\mathcal{Z}}(\mathcal{M}_D)$ the *Picard-Manin* space of \mathcal{M}_D . We also write $\overline{\mathcal{C}}(\mathcal{M}_D)$ for the closure of $\mathcal{C}(\mathcal{M}_D)$ for the strong topology. We have in particular that every nef class in $\widehat{\mathcal{Z}}(\mathcal{M}_D)$ belongs to $\overline{\mathcal{Z}}(\mathcal{M}_D)$ (see [BFJ08] Proposition 1.4).

Remark 3.2. In [BFJ08] or [CLC13], the Picard-Manin space is defined by allowing blow up with arbitrary centers not only at infinity. Since we study dynamics of automorphism of \mathcal{M}_D the indeterminacy points are only at infinity. This justifies our restricted definition of the Picard-Manin space. A similar construction is used in [FJ11] for the affine plane.

3.4. Spectral property of the dynamical degree. If $f \in \text{Aut}(\mathcal{M}_D)$ we define the operator f^* on $\mathcal{Z}(\mathcal{M}_D)$ as follows. Let $\alpha \in \mathcal{Z}(\mathcal{M}_D)$ defined in a completion X . Let Y be a completion of \mathcal{M}_D such that the lift $F : Y \rightarrow X$ of f is regular. We define $f^*\alpha$ as the Cartier class defined by $F^*\alpha$. This does not depend on the choice of X or Y . We write f_* for $(f^{-1})^*$. If X is a completion of \mathcal{M}_D , we write $f_X^* : \text{Div}_\infty(\mathcal{M}_D) \rightarrow \text{Div}_\infty(\mathcal{M}_D)$ for the following operator:

$$f_X^*(D) = (f^*D)_X \quad (71)$$

where we consider the class of D and f^*D in $\mathcal{C}(\mathcal{M}_D)$. We also define the operator $f_X^* : \text{NS}(X) \rightarrow \text{NS}(X)$ in a similar way.

Proposition 3.3 (Proposition 2.3 and Theorem 3.2 of [BFJ08]). *The operator f^* extends to a continuous bounded operator $f^* : \overline{\mathcal{Z}}(\mathcal{M}_D) \rightarrow \overline{\mathcal{Z}}(\mathcal{M}_D)$ that satisfy*

- (1) $f^* \alpha \cdot \beta = \alpha \cdot f_* \beta$.
- (2) $f^* \alpha \cdot f^* \beta = \alpha \cdot \beta$
- (3) $\lambda_1(f)$ is the spectral radius and an eigenvalue of f^* .

If $\lambda_1(f) > 1$, then λ_1 is simple and there is a spectral gap property.

Theorem 3.4 (Theorem 3.5 of [BFJ08]). *Let f be a loxodromic automorphism of \mathcal{M}_D , there exist nef elements $\theta^+, \theta^- \in \overline{\mathcal{C}}(\mathcal{M}_D)$ unique up to renormalisation such that*

- (1) θ^+ and θ^- are effective.
- (2) $(\theta^+)^2 = (\theta^-)^2 = 0, \theta^+ \cdot \theta^- = 1$.
- (3) $f^* \theta^+ = \lambda_1 \theta^+, (f^{-1})^* \theta^- = \lambda_1 \theta^-$
- (4) For any $\alpha \in \overline{\mathcal{Z}}(\mathcal{M}_D)$,

$$\frac{1}{\lambda_1^N} (f^{\pm N})^* \alpha = (\theta^\mp \cdot \alpha) \theta^\pm + O_\alpha\left(\frac{1}{\lambda_1^N}\right) \quad (72)$$

Proof. The only assertion not following from [BFJ08] is that θ^+ and θ^- are effective and belong to $\overline{\mathcal{C}}$. We show the result for θ^+ . Following the proof of [BFJ08] Theorem 3.2, we have that θ^+ is obtained as a cluster value of a sequence of Cartier divisors (X_n, θ_n) on completions of \mathcal{M}_D such that $f_{X_n}^* \theta_n = \rho_n \theta_n$ where ρ_n is the spectral radius of $f_{X_n}^*$. The existence of θ^+ follows from the fact that $\rho_n \rightarrow \lambda_1$. We show that θ_n can be chosen nef and effective. Fix a completion X of \mathcal{M}_D , f_X^* preserves the nef cone $\text{Nef}(X)$ of $\text{NS}(X)$. It also preserves the subcone of $\text{Nef}(X)$ consisting of effective divisors supported at infinity. This subcone is nonempty because for example in the completion $\overline{\mathcal{M}}_D \subset \mathbf{P}^3$, the divisor

$$H = \{X = T = 0\} + \{Y = T = 0\} + \{Z = T = 0\} \quad (73)$$

is very ample as it is equal to $\overline{\mathcal{M}}_D \cap \{T = 0\}$. By Perron-Frobenius theorem, there exists θ_X in this subcone such that $f_X^* \theta_X = \rho_X \theta_X$ with ρ_X the spectral radius of f_X^* . \square

3.5. Compatibility with adelic divisors. We have a forgetful group homomorphism

$$c : \widehat{\text{Div}}(\mathcal{M}_D/O_{\mathbf{K}}) \rightarrow \mathcal{W}(\mathcal{M}_D) \quad (74)$$

defined as follows. Let \mathcal{U} be a quasiprojective model of \mathcal{M}_D over $O_{\mathbf{K}}$ and let $\overline{\mathcal{D}}$ be a model adelic divisor on \mathcal{U} . Then, $c(\overline{\mathcal{D}}) = \mathcal{D}_{\mathbf{K}}$ is the horizontal part of \mathcal{D} , this is an element of $\mathcal{C}(\mathcal{M}_D)$ because $\overline{\mathcal{D}}$ is supported at infinity.

Proposition 3.5. *The group homomorphism c extends to a continuous group homomorphism*

$$c : \widehat{\text{Div}}(\mathcal{M}_D/O_{\mathbf{K}}) \rightarrow \mathcal{W}(\mathcal{M}_D). \quad (75)$$

Furthermore, if \overline{D} is integrable then $c(\overline{D}) \in \overline{\mathcal{C}}(\mathcal{M}_D)$.

Proof. Let $\overline{D} \in \widehat{\text{Div}}(\mathcal{M}_D/\mathcal{O}_{\mathbf{K}})$ be given by a Cauchy sequence of model adelic divisors $(\overline{\mathcal{D}}_i)$. Let X be a completion of \mathcal{M}_D . There exists a sequence ε_i converging to zero such that

$$-\varepsilon_i \overline{\mathcal{D}}_0 \leq \overline{\mathcal{D}}_j - \overline{\mathcal{D}}_i \leq \varepsilon_i \overline{\mathcal{D}}_0 \quad (76)$$

Applying c , we get (write $D_j = c(\overline{\mathcal{D}}_j)$)

$$-\varepsilon_i D_0 \leq D_j - D_i \leq \varepsilon_i D_0 \quad (77)$$

Thus, for every $E \in \mathcal{D}_\infty$, $\text{ord}_E(D_i)$ is a Cauchy sequence and converges to a number $\text{ord}_E(D)$. By Lemma 3.1 this defines a Weil divisor $c(D) \in \mathcal{W}(\mathcal{M}_D)$. It is clear that c is continuous, again using Lemma 3.1.

If \overline{D} is integrable, then it is the difference of two strongly nef adelic divisors and nef classes in $\mathcal{W}(\mathcal{M}_D)$ belong to $\mathcal{C}(\mathcal{M}_D)$. \square

We will drop the notation $c(\overline{D})$ and just write $D = c(\overline{D})$.

4. REPRESENTATION THEORY

4.1. Fuchsian and Quasi-Fuchsian representation. A *Fuchsian* group is a discrete subgroup Γ of $\text{PSL}_2(\mathbf{R})$. A *Quasi-Fuchsian* group is a discrete subgroup Γ of $\text{PSL}_2(\mathbf{C})$ such that its limit set in $\widehat{\mathbf{C}} := \mathbf{P}^1(\mathbf{C})$ is a Jordan curve. Let S be an oriented compact (real) surface of negative Euler characteristic. We say that a representation $\rho : \pi_1(S) \rightarrow \text{SL}_2(\mathbf{C})$ is Fuchsian (resp, Quasi-Fuchsian) if $\overline{\rho}(S) \subset \text{PSL}_2(\mathbf{C})$ is Fuchsian (resp. Quasi-Fuchsian).

Let $\text{Teich}(S)$ be the Teichmüller space of S , that is the set of complete finite hyperbolic metrics over S . Every point of $\text{Teich}(S)$ induces a Fuchsian representation of S . We can actually parametrize the set of Quasi-Fuchsian representations of S using $\text{Teich}(S)$ by the simultaneous uniformization theorem of Bers.

Theorem 4.1 ([Ber60]). *There is a biholomorphic map*

$$\text{Bers} : \text{Teich}(S) \times \text{Teich}(\overline{S}) \rightarrow \text{QF}(S) \quad (78)$$

where \overline{S} is the surface S with its reversed orientation.

Using this theorem, one can apply an iterative process to find a fixed point in the character variety of S .

Theorem 4.2 ([McM96]). *Let S be an oriented compact surface of negative Euler characteristic. Let $(X, Y) \in \text{Teich}(S) \times \text{Teich}(\overline{S})$, let $\Phi \in \text{Mod}(S)$ be pseudo-Anosov, then the sequence*

$$\text{Bers}(\Phi^n(X), \Phi^{-n}(Y)) \quad (79)$$

has an accumulation point $\rho_\infty : \pi_1(S) \rightarrow \text{PSL}_2(\mathbf{C})$. Furthermore,

- (1) ρ_∞ is discrete and faithful.
- (2) The limit set of $\rho_\infty(\pi_1(S))$ is the whole boundary \mathbb{S}^2 of \mathbb{H}^3 .
- (3) ρ_∞ is a fixed point of Φ and Φ is conjugated to an isometry α of $\tilde{M}_\Phi = \mathbb{H}^3/\rho_\infty(\pi_1(S))$.

- (4) *The group of isometries of M_∞ is discrete and α is of infinite order.*
- (5) *The mapping torus M_Φ is isomorphic as an hyperbolic manifold to $\tilde{M}_\Phi / \langle \alpha \rangle$.*
- (6) *The subgroup generated by α of the group of isometries of \tilde{M}_Φ is of finite index.*

4.2. The surface \mathcal{M}_0 and a Theorem of Minsky. We are interested in this section with the Markov surface \mathcal{M}_0 that is when $\text{Tr}(K) = -2$, therefore $\rho(K)$ is a parabolic Möbius transformation. The real points $\mathcal{M}_0(\mathbf{R})$ consist of an isolated point $(0,0,0)$ and four diffeomorphic connected components that are given by the signs of x and y . We will denote by $\mathcal{M}_0(\mathbf{R})^+$ the connected component such that $x, y > 0$, of area 2π . This is equivalent to asking that there is a cusp at the puncture. It is known that $\text{Teich}(\mathbb{T}_1)$ (\mathbb{T}_1 the punctured torus) is isomorphic to the upper half plane \mathbb{H}^+ and we make this identification from now on. The action of $\text{Mod}(\mathbb{T}_1)$ on $\text{Teich}(\mathbb{T}_1)$ is conjugated to the usual action of $\text{PSL}_2(\mathbf{Z})$ by isometries on \mathbb{D} . The point $(0,0,0)$ is the only singular point of \mathcal{M}_0 and it corresponds to the conjugacy class of the representation

$$\rho(a) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (80)$$

Its image is the quaternionic group of order eight. The automorphism group $\text{Aut}(\mathcal{M}_0)$ fixes $(0,0,0)$, preserves $\mathcal{M}_0(\mathbf{R})^=$ and permutes the three remaining connected components of $\mathcal{M}_0(\mathbf{R}) \setminus (0,0,0)$.

Any point in $\text{Teich}(\mathbb{T}_1)$ gives rise to a representation $\bar{\rho} : \pi_1(\mathbb{T}_1) \rightarrow \text{PSL}_2(\mathbf{R})$ which can be lifted to four distinct representations $\rho : \pi_1(\mathbb{T}_1) \rightarrow \text{SL}_2(\mathbf{R})$. The cusp condition gives the condition $\text{Tr}(\rho(a, b)) = -2$ (because $\text{Tr} = 2$ corresponds to reducible representations). Therefore, we get an embedding of $\text{Teich}(\mathbb{T}_1)$ into the 4 different connected component of $\mathcal{M}_0(\mathbf{R}) \setminus (0,0,0)$. We will restrict our attention to the embedding $\text{Teich}(\mathbb{T}_1) \hookrightarrow \mathcal{M}_0(\mathbf{R})^+$.

The set $\mathcal{M}_0(\mathbf{R})^+$ is made of (conjugacy class of) Fuchsian representations. Let $\text{DF}_0 \subset \mathcal{M}_0(\mathbf{C})$ be the subset of discrete and faithful representation of $\pi_1(\mathbb{T}_1)$. Then DF_0 has four different connected components, one of them contains $\mathcal{M}_0(\mathbf{R})^+$. We denote it by DF_0^+ and we denote by QF_0^+ the set of Quasi-Fuchsian representation inside DF_0^+ . In fact, QF_0^+ is the interior of DF_0^+ (see [Min02]). We can identify $\text{Teich}(\mathbb{T}_1)$ with the upper half plane \mathbb{H}^+ and $\text{Teich}(\overline{\mathbb{T}}_1)$ with the lower half plane \mathbb{H}^- . The group $\text{PSL}_2(\mathbf{Z})$ acts on $\mathbf{P}^1(\mathbf{C})$ via Möbius transformation. It preserves $\mathbb{H}^+, \mathbb{H}^-$ and $\mathbf{P}^1(\mathbf{R})$. In particular, the mapping class group $\text{MCG}(\mathbb{T}_1) = \text{SL}_2(\mathbf{Z})$ acts on $\mathbf{P}^1(\mathbf{C})$ and we can conjugate this action to the action on $\mathcal{M}_0(\mathbf{R})^+$ via the Bers mapping. Namely, let $\Phi \in \text{MCG}(\mathbb{T}_1)$ and let $f_\Phi \in \text{Aut}(\mathcal{M}_0)$ induced by the map from Equation (11). We have for every $(s, t) \in \mathbb{H}^+ \times \mathbb{H}^-$,

$$\text{Bers}(\Phi(s, t)) = f_\Phi(\text{Bers}(s, t)) \quad (81)$$

Theorem 4.2 is not applicable directly as \mathbb{T}_1 is not compact. However, Minsky showed that the Bers mapping can be extended to almost all the boundary of $\text{Teich}(\mathbb{T}_1) \times \text{Teich}(\overline{\mathbb{T}}_1)$. The boundary of \mathbb{H}^+ is $\mathbf{P}^1(\mathbf{R})$. We denote by Δ the diagonal in $\partial \text{Teich}(\mathbb{T}_1) \times \partial \text{Teich}(\overline{\mathbb{T}}_1)$.

Theorem 4.3 ([Min99]). *The Bers mapping extend to a continuous bijection*

$$\text{Bers} : \overline{\text{Teich}(\mathbb{T}_1) \times \text{Teich}(\overline{\mathbb{T}}_1)} \setminus \Delta \rightarrow \text{DF}^+ \quad (82)$$

In particular, let $\Phi \in \mathrm{SL}_2(\mathbf{Z}) = \mathrm{MCG}(\mathbb{T}_1)$ be a loxodromic element and let f_Φ be its associated automorphism over \mathcal{M}_0 . The isometry Φ has a repulsive fixed point $\alpha(\Phi)$ on $\mathbf{P}^1(\mathbf{R})$ and an attractive one $\omega(\Phi)$. By Minsky's theorem, this gives two unique fixed point $p(\Phi) = \mathrm{Bers}((\alpha(\Phi), \omega(\Phi)))$ and $q(\Phi) = \mathrm{Bers}((\omega(\Phi), \alpha(\Phi)))$ of f_Φ in $\mathrm{DF}^+ \setminus \mathrm{QF}^+$.

5. DYNAMICS OF LOXODROMIC AUTOMORPHISMS OF THE MARKOV SURFACE

5.1. Cyclic completions and circle at infinity. Let $\overline{\mathcal{M}}_D \subset \mathbf{P}^3$ be the closure of \mathcal{M}_D in \mathbf{P}^3 . We have that $\overline{\mathcal{M}}_D \setminus \mathcal{M}_D$ is a triangle of lines. We use the following result.

Proposition 5.1 ([EH74, CdC19]). *Let X be projective surface and U an open subset of X such that $X \setminus U$ is a cycle of rational curves. Assume that $X \setminus U$ is not an irreducible curve with one nodal singularity. Let g be an automorphism of U , then the indeterminacy points of g can only be intersection points of two components of the cycle.*

This shows that to understand the dynamics of a loxodromic automorphism at infinity it suffices to blow up the intersection points of the divisor at infinity. Therefore we can remain with completions X of \mathcal{M}_D such that $X \setminus \mathcal{M}_D$ is a cycle of rational curves. We call them cyclic completions.

Start with $\overline{\mathcal{M}}_D \subset \mathbf{P}^3$. If we blow up the three intersection points of the triangle at infinity we get a new completion of \mathcal{M}_D with a hexagon of rational curves at infinity. If we repeat the process we get a sequence of completions with an increasing polygon of rational curves at infinity, let X_n be this sequence of completions. Let G_n be the dual graph of $X_n \setminus \mathcal{M}_D$, i.e the vertices of G_n are the irreducible components E_i of $X_n \setminus \mathcal{M}_D$ and we have an edge between E_i and E_j if and only if $E_i \cap E_j \neq \emptyset$. We have a natural embedding of graphs $G_n \hookrightarrow G_{n+1}$. We define the direct limit $G = \varinjlim_n G_n$ and $\mathrm{Aut}(\mathcal{M}_D)$ acts naturally on G . There is a parametrization of the vertices of G called the *Farey parametrization* (see [CdC19] §8.2), and G is isomorphic to the set of rational points of the circle \mathbb{S}^1 with this parametrization. If X is a cyclic completion, the irreducible components E_i (enumerated cyclically) of the boundary corresponds to rational points $v_i \in \mathbb{S}^1$. The rational points of $]v_i, v_{i+1}[$ are obtained by blowing up above the point $E_i \cap E_{i+1}$.

Following §2.6 of [Can09], we identify $\mathbb{S}^1 \simeq \mathbf{R} \cup \{\infty\}$ with the boundary of the upper half plane of \mathbb{H}^+ . If v_x, v_y, v_z are the three vertices of G representing the three curves $\{X = T = 0\}$, $\{Y = T = 0\}$ and $\{Z = T = 0\}$ in $\overline{\mathcal{M}}_D$, then this identification sends v_x, v_y, v_z to $0, -1, \infty \in \partial\mathbb{H}^+$ which we write j_x, j_y, j_z respectively. Recall the notations of §1.2, The generator σ_x (resp. σ_y, σ_z) of Γ^* acts on $\partial\mathbb{H}^+$ as the reflection with respect to the geodesic (j_y, j_z) (resp. $(j_x, j_z), (j_x, j_y)$) under this identification. The isometries of \mathbb{H}^+ induced by $\sigma_x, \sigma_y, \sigma_z$ with this identification are exactly given by (14). Hence, the action of Γ^* on G is given by the action of G on \mathbb{H}^+ via isometries. Thus every loxodromic automorphism $f \in \Gamma^*$ admits two irrational fixed points $\alpha(f), \omega(f) \in \partial\mathbb{H}^+$, $\alpha(f)$ is repulsive and $\omega(f)$ is attracting.

The circle \mathbb{S}^1 has an interpretation as a special subset of the set of valuations of the ring of regular functions of \mathcal{M}_D . The two fixed point $\alpha(f), \omega(f)$ correspond to a repulsive and attracting fixed point in this space of valuation. See §14.5 of [Abb23] for more details.

5.2. Dynamics of loxodromic automorphism at infinity. From the previous discussion, we get

Proposition 5.2. *Let $D \in \mathbf{C}$, $\mathbf{K} = \mathbf{Q}(D)$ and $f \in \text{Aut}(\mathcal{M}_D)$. For any cyclic completion Y of \mathcal{M}_D , there exists a cyclic completion X above Y such that there exists two closed points $p_+, p_- \in X \setminus \mathcal{M}_D(\mathbf{K})$ satisfying the following properties*

- (1) $p_+ \neq p_-$
- (2) *There exists a unique attracting fixed point $p \in X \setminus \mathcal{M}_D(\mathbf{K})$ at infinity.*
- (3) *some positive iterate of $f^{\pm 1}$ contracts $X \setminus \mathcal{M}_D$ to p_{\pm} .*
- (4) $f^{\pm 1}$ is defined at p_{\pm} , $f^{\pm 1}(p_{\pm}) = p_{\pm}$ and p_{\mp} is the unique indeterminacy point of $f^{\pm N}$ for N large enough.
- (5) *There exists local algebraic coordinates (u, v) at p_{\pm} such that $uv = 0$ is a local equation of the boundary and $f^{\pm 1}$ is of the form*

$$f^{\pm 1}(u, v) = (u^a v^b \phi, u^c v^d \psi) \quad (83)$$

with $ad - bc = \pm 1$ and ϕ, ψ invertible. In particular, for any place v of \mathbf{K} there exists an open neighbourhood U_v^{\pm} of p_{\pm} in X_v^{an} such that $f^{\pm}(U_v^{\pm}) \subseteq U_v^{\pm}$.

- (6) f is algebraically stable over X and $f_X^* \theta_X^+ = \lambda_1 \theta_X^+, (f_X^{-1})^* \theta_X^- = \lambda_1 \theta_X^-$.

Proof. From the previous section, write E_1, \dots, E_r for the irreducible components of Y enumerated in cyclic order. We write p_+ for the unique point $p_+ = E_i \cap E_{i+1}$ such that $\omega(f) \in]v_i, v_{i+1}[$ and we define p_- similarly with respect to $\alpha(f)$. It is clear that for N large enough every E_i is contracted to p_{\pm} by $f^{\pm N}$ because of the attractingness of $\omega(f)$ and $\alpha(f) = \omega(f^{-1})$. Thus, to get a completion above Y that satisfy Properties (1)-(5) we just need that $p_+ \neq p_-$ in that completion. Since f is loxodromic, we have $\alpha(f) \neq \omega(f)$, thus after enough blow-ups this will be the case. \square

Corollary 5.3 (Corollary 3.4 from [Can09]). *Any loxodromic automorphism of \mathcal{M}_D does not admit any invariant algebraic curves.*

Proof. Let $f \in \text{Aut}(\mathcal{M}_D)$ be loxodromic. Let X be a completion of \mathcal{M}_D given by Proposition 5.2. If $C \subset \mathcal{M}_D$ was an invariant algebraic curve, then its closure \overline{C} in X should intersect $X \setminus \mathcal{M}_D$. Since the boundary is contracted by f , we must have $p_+ \in \overline{C}$ and $f : \overline{C} \rightarrow C$ is an automorphism with a superattractive fixed point. This is a contradiction. \square

Proposition 5.4. *Let X be a completion of \mathcal{M}_D given by Proposition 5.2, replace f by one of its iterates such that $f^{\pm 1}$ contracts $X \setminus \mathcal{M}_D$ to p_{\pm} . Then,*

- (1) *There exists $D^- \in \text{Div}_{\infty}(X)_{\mathbf{R}}$ such that $f_X^* D^- = \frac{1}{\lambda_1} D^-$.*
- (2) *If E_+, F_+ are the two divisors at infinity such that $p_+ = E_+ \cap F_+$, then for all $R \in \text{Div}_{\infty}(X)_{\mathbf{R}}$ such that $E_+, F_+ \notin \text{Supp } R$, then $f_X^* R = 0$ and $\theta^- \cdot R = 0$.*
- (3) $D^- \cdot \theta^- = 0$.
- (4) $\{\theta_X^+, D^-\} \cup \{E : E \notin \{E_+, F_+\}\}$ is a basis of $\text{Div}_{\infty}(X)$.

Proof. If E is a prime divisor at infinity distinct from E_+ and F_+ , then $f_X^*E = 0$ because every prime divisor at infinity is contracted to p_+ by f and $p_+ \notin E$. Now if R satisfies $f_X^*R = 0$, then

$$0 = f_X^*R \cdot \theta_X^- = R \cdot (f_X^{-1})^* \theta_X^- = \lambda_1 R \cdot \theta_X^-. \quad (84)$$

Thus $R \cdot \theta_X^- = 0$. This shows (2).

Since $f_X^+ \theta_X^+ = \lambda_1 \theta_X^+$, we have that

$$\theta_X^+ = \alpha E_+ + \beta F_+ + \cdots \quad (85)$$

where (α, β) is an eigenvector of $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ of eigenvalue λ_1 . Now, the other eigenvalue of A is $\frac{1}{\lambda_1}$ by Proposition 5.2 (5) (up to replacing f by f^2), let (γ, δ) be an associated eigenvector, then

$$f_X^*(\gamma E_+ + \delta F_+) = \frac{1}{\lambda_1}(\gamma E_+ + \delta F_+) + R \quad (86)$$

where R is a divisor at infinity which support does not contain E_+ or F_+ . Set $D^- = \gamma E_+ + \delta F_+ + \lambda_1 R$, then by (2), D^- satisfies $f_X^*D^- = \frac{1}{\lambda_1}D^-$. This shows (1).

Now,

$$\frac{1}{\lambda_1}D^- \cdot \theta^- = \frac{1}{\lambda_1}D^- \cdot \theta_X^- = (f_X^*D^-) \cdot \theta_X^- = D^- \cdot (f_X)_* \theta_X^- = \lambda_1 D^- \cdot \theta^-. \quad (87)$$

Thus $D^- \cdot \theta^- = 0$. This shows (3).

Finally, we just have to show that the family $\{\theta_X^+, D^-\} \cup \{E; E \notin \{E_+, F_+\}\}$ is free. Suppose that

$$\alpha \theta_X^+ + \beta D^- + R = 0 \quad (88)$$

with $\alpha, \beta \in \mathbf{R}$ and $E_+, F_+ \notin \text{Supp } R$. Intersecting with θ^- in (88) and using (2) and (3), we get $\alpha = 0$. Then, applying f_X^* to (88) we get $\beta = 0$. Thus $R = 0$ and we have shown (4). \square

6. AN INVARIANT ADELIC DIVISOR

We now suppose that $D \in \mathbf{C}$ is algebraic and write $\mathbf{K} = \mathbf{Q}(D)$ which is a number field.

Theorem 6.1. *Let f be a loxodromic automorphism of \mathcal{M}_D , then there exists a unique adelic divisor $\overline{\theta}^+ \in \widehat{\text{Div}}(\mathcal{M}_D/\mathbf{O}_{\mathbf{K}})$ of θ^+ such that*

$$f^* \overline{\theta}^+ = \lambda_1 \overline{\theta}^+. \quad (89)$$

Furthermore, $\overline{\theta}^+$ is a strongly nef adelic divisor in the sense of Yuan and Zhang.

Remark 6.2. Of course, the same result holds for f^{-1} with the existence of a unique effective strongly nef invariant adelic divisor $\overline{\theta}^-$, the proof is symmetric. With the notations of §3.5, we must have $c(\overline{\theta}^\pm) = \theta^\pm$ (up to multiplication by a positive constant) because of Theorem 3.4. So our notation of $\overline{\theta}^\pm$ is compatible with Theorem 3.4.

The rest of this section is dedicated to the proof of this theorem.

6.1. Two lemmas. Start with the following lemma

Lemma 6.3. *If $\overline{\mathcal{D}}$ is a model vertical divisor, then*

$$\frac{1}{\lambda_1^n} (f^n)^* \overline{\mathcal{D}} \rightarrow 0 \quad (90)$$

in $\widehat{\text{Div}}(\mathcal{M}_D/O_K)$.

Proof. Let g be the Green function induced by $\overline{\mathcal{D}}$ over $\mathcal{M}_D^{\text{an}}$. It is a continuous bounded function over $\mathcal{M}_D^{\text{an}}$. Thus

$$\frac{1}{\lambda_1^n} g \circ (f^{\text{an}})^n \quad (91)$$

converges uniformly to zero over $\mathcal{M}_D^{\text{an}}$. Thus it also converges to zero for the boundary topology. \square

From now on, we fix a completion X of \mathcal{M}_D that satisfies Proposition 5.2. We also replace f by one of its iterate f^{N_0} such that $f^{\pm N_0}$ contracts $X \setminus \mathcal{M}_D$ to p_{\pm} .

Lemma 6.4. *Let $D \in \text{Div}_{\infty}(X)_{\mathbf{R}}$ be a \mathbf{R} -divisor at infinity. Let $(\mathcal{X}, \mathcal{D})$ be a model of (X, D) over O_K . Let $V \subset \text{Spec } O_K$ be an open subset such that*

(1) *The indeterminacy locus I^{\pm} of the rational map $f^{\pm 1} : \mathcal{X} \dashrightarrow \mathcal{X}$ satisfies*

$$I|_V = (I \cap \overline{\{p_{\pm}\}})|_V. \quad (92)$$

(2) *No vertical components of the support of \mathcal{D} lies above V .*

Then, for every finite place v above V , let U_v^{\pm} be the open subset of X_v^{an} such that $r_{\mathcal{X}_v}(x) = r_{\mathcal{X}_v}(p_{\pm})$. Then, $f^{\pm 1}$ is defined over U_v^{\pm} , U_v^{\pm} is $f^{\pm 1}$ -invariant and if $W_v^{\pm} = X_v^{\text{an}} \setminus U_v^{\pm}$, then

$$(g_{(\mathcal{X}_v, \mathcal{D}_v)} \circ f^{\text{an}})|_{W_v^-} = g_{(\mathcal{X}_v, f_X^* D)}|_{W_v^-} \quad (93)$$

$$(g_{(\mathcal{X}_v, \mathcal{D}_v)} \circ (f^{-1})^{\text{an}})|_{W_v^+} = g_{(\mathcal{X}_v, (f_X^{-1})^* D)}|_{W_v^+} \quad (94)$$

Proof. let $\pi : \mathcal{Y} \rightarrow \mathcal{X}$ be a minimal sequence of blow ups such that the lift $F : \mathcal{Y} \rightarrow \mathcal{X}$ of f is regular. We have that

$$g_{(\mathcal{Y}, \mathcal{D})} \circ F^{\text{an}} = g_{(\mathcal{Y}, F^* \mathcal{D})}. \quad (95)$$

Now, over V the birational morphism π consists only of horizontal blow ups that is blow ups of \overline{p} where p is a closed point of the generic fiber. By hypothesis, π induces an isomorphism between $\mathcal{X}_V \setminus \overline{\{p_{-}\}}$ and $\pi^{-1}(\mathcal{X}_V \setminus \overline{\{p_{-}\}})$ because all the indeterminacy points of $f : X \dashrightarrow X$ are above p_{-} . Let v be a finite place above V , π_v^{an} induces an isomorphism between W_v^- and $\pi^{-1} W_v^-$ since $r_{\mathcal{X}_v} \circ \pi_v^{\text{an}} = \pi_v^{\text{an}} \circ r_{\mathcal{Y}_v}$. Finally, since \mathcal{D} has no vertical component above v , we get that $(F^* \mathcal{D})_v = (F^* D)_v$. Now if $x \in W_v^-$,

$$g_{(\mathcal{X}_v, \mathcal{D}_v)}(f^{\text{an}}(x)) = g_{(\mathcal{X}_v, \mathcal{D}_v)}\left((F \circ \pi^{-1})^{\text{an}}(x)\right) = g_{(\mathcal{Y}_v, F^* \mathcal{D})}\left((\pi^{-1})^{\text{an}}(x)\right). \quad (96)$$

Now it is clear that

$$g(\mathcal{D}_v, F^* \mathcal{D}) \left((\pi^{-1})^{\text{an}}(x) \right) = g(\mathcal{D}_v, f_X^* D)(x) \quad (97)$$

and thus the result is shown. \square

6.2. An iterative process.

Proposition 6.5. *Let $D \in \text{Div}_\infty(X)$ be such that $f_X^* D = \mu D$ for some $\mu \in \mathbf{R}$. Let $(\mathcal{X}, \mathcal{D})$ be a model of (X, D) , then*

- *If $\mu = \lambda_1$, then $D = \theta_X^+$ up to renormalisation and $\frac{1}{\lambda_1^n} (f^n)^* \mathcal{D}$ converges towards an element $\bar{\theta}^+(X)$ such that $f^* \bar{\theta}^+(X) = \lambda_1 \bar{\theta}^+(X)$.*
- *Else $|\mu| < |\lambda_1|$ and $\frac{1}{\lambda_1^n} (f^n)^* \mathcal{D}$ converges towards 0.*

To prove the proposition, we study the sequence of adelic divisors

$$\frac{1}{\lambda_1^n} (f^n)^* \mathcal{D} - \left(\frac{\mu}{\lambda_1} \right)^n \mathcal{D}. \quad (98)$$

We show that if $|\mu| < \lambda_1$, then this sequence tends to 0 and if $\mu = \lambda_1$, then this sequence converges towards an adelic divisor. Looking at Green functions, we need to show that the sequence

$$u_n := \frac{1}{\lambda_1^n} g(\mathcal{X}, \mathcal{D}) \circ (f^n)^{\text{an}} - \left(\frac{\mu}{\lambda_1} \right)^n g(\mathcal{X}, \mathcal{D}). \quad (99)$$

converges over $\mathcal{M}_D^{\text{an}}$ with respect to the boundary topology. We split the proof into two parts. First we show that away from the indeterminacy point p_- , the convergence is actually uniform. Then, we will study in more details what happens near p_- and show the convergence for the boundary topology there.

6.3. Convergence away from p_- . There exists an open neighbourhood U^- of p_- in X^{an} such that f^{-1} is defined over U^- and U^- is f^{-1} -invariant. Indeed, let $V \subset \text{Spec } O_{\mathbf{K}}$ be an open subset that satisfies the hypothesis of Lemma 6.4 for every finite place v above V we set U_v^- as the open subset of X_v^{an} defined in Lemma 6.4. For the finite number of remaining places v (including also all the infinite ones), we know by Proposition 5.2 that p_v^- is an attracting fixed point of $(f_v^{-1})^{\text{an}}$ (here we write p_v^- for the point defined by p_- in X_v^{an}). Let U_v^- be an $(f_v^{-1})^{\text{an}}$ invariant open neighbourhood of p_v^- such that $(f_v^{-1})^{\text{an}}(U_v^-) \subseteq U_v^-$. Define

$$U^- := \bigcup_{v \in M(\mathbf{K})} U_v^-. \quad (100)$$

It is an $(f^{-1})^{\text{an}}$ invariant open subset. Let W^- be its complement, it is f^{an} -invariant.

Set

$$h = u_1 = \frac{1}{\lambda_1} g(\mathcal{X}, \mathcal{D}) \circ f^{\text{an}} - \frac{\mu}{\lambda_1} g(\mathcal{X}, \mathcal{D}). \quad (101)$$

Lemma 6.6. *The function h extends to a continuous bounded function over W^- and*

$$h_{W_V^-} \equiv 0 \quad (102)$$

Proof. First, by Lemma 6.4 we have $h \equiv 0$ over W_v^- for every finite place $v \in V$. If $v \notin V[f]$, then since $f_X^* D = \mu D$, we have that h extends to a continuous function over W_v^- because $p_- \notin W_v^-$. Since $W^- \setminus W_V^-$ is compact we have that h is a bounded continuous function over W^- . \square

Proposition 6.7. *If $|\mu| < \lambda_1$, then u_n converges uniformly to 0 over W^- .*

If $\mu = \lambda_1$, then u_n converges uniformly towards a continuous function h^+ over W^- such that

$$(1) \ h_{W_V^-}^+ \equiv 0.$$

$$(2) \text{ If } G^+ = h^+ + g(\mathcal{X}, \mathcal{D}), \text{ then } G^+ \circ f^{\text{an}} = \lambda_1 G^+.$$

Proof.

$$u_n = \frac{1}{\lambda_1^{n-1}} \left(\frac{1}{\lambda_1} g(\mathcal{X}, \mathcal{D}) \circ f \right) \circ f^{n-1} - \left(\frac{\mu}{\lambda_1} \right)^n g(\mathcal{X}, \mathcal{D}) \quad (103)$$

$$= \frac{1}{\lambda_1^{n-1}} \left(h + \frac{\mu}{\lambda_1} g(\mathcal{X}, \mathcal{D}) \right) \circ f^{n-1} - \left(\frac{\mu}{\lambda_1} \right)^n g(\mathcal{X}, \mathcal{D}) \quad (104)$$

$$= \frac{1}{\lambda_1^{n-1}} h \circ f^{n-1} + \frac{\mu}{\lambda_1} u_{n-1}. \quad (105)$$

Therefore,

$$u_n = \sum_{\ell=0}^{n-1} \frac{\mu^\ell}{\lambda_1^{n-1}} h \circ f^{n-1-\ell}. \quad (106)$$

If $|\mu| < \lambda_1$, let $M = \max_{W^-} |h|$, then

$$\sup_{W^-} |u_n(x)| \leq \frac{M}{\lambda_1^{n-1}} \frac{|\mu|^n - 1}{|\mu| - 1} \xrightarrow{n \rightarrow +\infty} 0. \quad (107)$$

If $\mu = \pm 1$, then

$$\sup_{W^-} |u_n(x)| \leq \frac{Mn}{\lambda_1^{n-1}} \xrightarrow{n \rightarrow +\infty} 0. \quad (108)$$

If $\mu = \lambda_1$, then write \mathcal{D}^+ for \mathcal{D} , Equation (106) becomes

$$u_n = \sum_{k=0}^{n-1} \frac{1}{\lambda_1^k} h \circ f^k. \quad (109)$$

Since h is bounded over W^- , u_n converges uniformly over W^- towards a continuous function h^+ . By Lemma 6.4, it is clear that $u_n|_{W_V^-} \equiv 0$, thus $h_{W_V^-}^+ \equiv 0$. If $G^+ = h^+ + g(\mathcal{X}, \mathcal{D}^+)$, then it is defined on $W^- \cap \mathcal{M}_D^{\text{an}}$ and satisfies $G^+ \circ f^{\text{an}} = \lambda_1 G^+$. \square

If \mathcal{D}^- is a model of θ_X^- , we construct in the same fashion an open f^{an} -invariant neighbourhood U^+ of $(p_+^v)_{v \in M(O_K)}$ with $W^+ := X^{\text{an}} \setminus U^+$ and G^- which is defined over $W^+ \cap \mathcal{M}_D^{\text{an}}$ and satisfies $G^- \circ (f^{-1})^{\text{an}} = \lambda_1 G^-$ and such that $G^- - g_{(\mathcal{X}, \mathcal{D}^-)}$ extends to a continuous function h^- over W^+ that satisfies

$$h^-|_{W_V^+} \equiv 0 \quad (110)$$

Furthermore, we can choose U^- and U^+ such that $U^- \subseteq W^+$ and $U^+ \subseteq W^-$. For the places outside $V[f]$ we shrink U_v^+, U_v^- such that $G_{|U_v^+}^+ \geq 1, G_{|U_v^-}^- \geq 1$, this is always possible because $G^\pm - g_{(\mathcal{X}, \mathcal{D}^\pm)}$ extends to a continuous function at p_\pm .

6.4. Convergence everywhere. Define

$$\mathcal{U} = \mathcal{X} \setminus \left(\overline{X \setminus \mathcal{M}_D} \cup \bigcup_{m \notin V} \mathcal{X}_m \right) \quad (111)$$

where $\mathcal{X}_m = \mathcal{X} \times \text{Spec } O_K/m$. This is a quasiprojective model of \mathcal{M}_D over O_K . Let $\overline{\mathcal{D}_0}$ be a model of θ_X^- and a boundary divisor of \mathcal{U} in \mathcal{X} . If g_0 is the Green function of $\overline{\mathcal{D}_0}$ over $\mathcal{M}_D^{\text{an}}$ then we can suppose without loss of generality that for all $v \notin V[f], g_{0,v} \geq 1$. We have already constructed the Green functions G^\pm away from p_\mp .

Lemma 6.8. *For every place $v \notin V[f]$, over $U_v^- \cap \mathcal{M}_D^{\text{an}}$, the functions*

$$\frac{G^-}{g_0}, \quad \frac{g_0}{G^-} \quad (112)$$

are continuous and bounded.

Proof. With the notations of §6.3, there exists a constant $A > 0$ such that

$$-A\mathcal{D}_0 \leq \mathcal{D}^- \leq A\mathcal{D}_0 \quad (113)$$

and over U_v^- we have by Proposition 6.7 that

$$G^- = h^- + g_{(\mathcal{X}, \mathcal{D}^-)} \quad (114)$$

where h^- is continuous. This shows the result. \square

Let $\pi : \mathcal{Y} \rightarrow \mathcal{X}$ be a resolution of indeterminacies of $f : \mathcal{X} \dashrightarrow \mathcal{X}$ and let $\mathcal{F} : \mathcal{Y} \rightarrow \mathcal{X}$ be a lift of f , write $F : Y \rightarrow X$ for the restriction to the generic fiber. Then, h is a green function of $\frac{1}{\lambda_1} F^* D - \frac{\mu}{\lambda_1} \pi^* D$. Therefore, there exists a constant A such that

$$-A\overline{\mathcal{D}_0} \leq \frac{1}{\lambda_1} F^* \overline{\mathcal{D}} - \frac{\mu}{\lambda} \pi^* \overline{\mathcal{D}} \leq A\overline{\mathcal{D}_0}. \quad (115)$$

Thus, $\frac{h}{g_0}$ is a continuous bounded function over $\mathcal{M}_D^{\text{an}}$. We show that the sequence u_n converges with respect to the boundary topology. Set the following constants

$$M_0 = \sup_{\mathcal{M}_D^{\text{an}}} \left| \frac{h}{g_0} \right|, \quad M_1 = \sup_{\mathcal{M}_{D,V[f]^c}^{\text{an}}} \frac{1}{|g_0|}, \quad M_2 = \sup_{W^-} |h| \quad (116)$$

$$M_3 = \sup_{U_{V[f]^c}^- \cap \mathcal{M}_D^{\text{an}}} \left| \frac{g_0}{G^-} \right|, \quad M_4 = \sup_{U_{V[f]^c}^- \cap \mathcal{M}_D^{\text{an}}} \left| \frac{G^-}{g_0} \right| \quad (117)$$

where $V[f]^c$ is the set of places of \mathbf{K} outside $V[f]$.

Claim 6.9. *Set $M := \max(M_2 M_1, M_0 M_3 M_4)$, then for every $k \geq 0$*

$$-M g_0 \leq h \leq M g_0 \quad (118)$$

over $\mathcal{M}_D^{\text{an}}$.

Proof. We will write f instead of f^{an} as avoid heavy notations. Let $k \geq 0$ and $x \in \mathcal{M}_D^{\text{an}}$. Suppose first that $f^k(x) \in W^-$. If x lies above $v \in V[f]$, then $h(f^k(x)) = 0$ by Lemma 6.6 and (118) is obvious. Otherwise we have

$$\left| \frac{h(f^k(x))}{g_0(x)} \right| \leq M_2 M_1 \quad (119)$$

and (118) is satisfied.

If $f^k(x) \notin W^-$ then $x, f^k(x) \in U^- \subset W^+$. If x lies above $V[f]$, then by Proposition 6.7

$$G_{|W_V^+}^- = g_{(\mathcal{X}_V, \mathcal{D}_V^-)} = g_0, \quad (120)$$

thus

$$\left| h(f^k(x)) \right| \leq M_0 g_0(f^k(x)) = \frac{M_0}{\lambda_1^k} g_0(x). \quad (121)$$

Suppose x does not lie above $V[f]$, let $y = f^k(x)$, then

$$\left| \frac{h(f^k(x))}{g_0(x)} \right| = \left| \frac{h(y)}{g_0(f^{-k}(y))} \right| \leq M_4 \left| \frac{h(y)}{G^-(f^{-k}(y))} \right| = M_4 \left| \frac{h(y)}{\lambda_1^k G^-(y)} \right|. \quad (122)$$

Thus,

$$\left| \frac{h(f^k(x))}{g_0} \right| \leq \frac{M_4}{\lambda_1^k} \left| \frac{g_0(y)}{G^-(y)} \right| \left| \frac{h(y)}{g_0(y)} \right| \leq \frac{M_0 M_3 M_4}{\lambda_1^k} \quad (123)$$

□

End of proof of Proposition 6.5. (1) If $\mu = \lambda_1$, then u_n converges with respect to the boundary topology because by the claim

$$\sup_{\mathcal{M}_D^{\text{an}}} \left| \frac{h \circ f^k}{g_0} \right| \quad (124)$$

is the term of a converging sum.

(2) If $|\mu| < |\lambda_1|$, then $\left|\frac{u_n}{g_0}\right|$ converges uniformly towards 0 over $\mathcal{M}_D^{\text{an}}$. Indeed,

$$\sup_{\mathcal{M}_D^{\text{an}}} \left|\frac{u_n}{g_0}\right| \leq M \sum_{\ell=0}^{n-1} \frac{|\mu|^\ell}{\lambda_1^{n-1}} \leq \frac{M}{\lambda_1^{n-1}} \frac{|\mu^n - 1|}{|\mu - 1|} \xrightarrow{n \rightarrow +\infty} 0 \quad (125)$$

if $\mu \neq \pm 1$ and otherwise

$$\sup_{\mathcal{M}_D^{\text{an}}} \left|\frac{u_n}{g_0}\right| \leq \frac{Mn}{\lambda_1^{n-1}} \xrightarrow{n \rightarrow +\infty} 0. \quad (126)$$

□

Proposition 6.10. *For any completion X of \mathcal{M}_D , for any divisor $D \in \text{Div}_\infty(X)_\mathbf{R}$ and any adelic extension $\overline{\mathcal{D}}$ of D , we have*

$$\frac{1}{\lambda_1^N} (f^N)^* \overline{\mathcal{D}} \rightarrow (D \cdot \theta^-) \overline{\theta}^+(X). \quad (127)$$

Furthermore, $\overline{\theta}^+ := \overline{\theta}^+(X)$ does not depend on the completion X and is strongly nef and effective.

Proof. Using Proposition 5.4 (4) we write

$$D = \alpha \theta_X^+ + \beta D^- + R \quad (128)$$

where $E_+, F_+ \notin \text{Supp } R$. Intersecting with θ^- and using Proposition 5.4 (1) and (3) we get $\alpha = D \cdot \theta^-$. For any adelic extension $\overline{\mathcal{D}}$ of D we get by Proposition 6.5 that

$$\frac{1}{\lambda_1^N} (f^N)^* \overline{\mathcal{D}} \rightarrow (D \cdot \theta^-) \overline{\theta}^+(X). \quad (129)$$

Now we show that $\overline{\theta}^+(X)$ does not depend on X . It suffices to show that $\overline{\theta}^+(X) = \overline{\theta}^+(Y)$ for any completion Y above X . Let $D = \theta_X^+ \in \text{Div}_\infty(X)$, then $\pi^* D \in \text{Div}_\infty(Y)$ and satisfies $\pi^* D \cdot \theta_Y^- = \theta_X^+ \cdot \theta_X^- = \theta^+ \cdot \theta^- = 1$. Thus we get by (127) that $\overline{\theta}^+(X) = \overline{\theta}^+(Y)$.

Let H be the ample divisor on $\overline{\mathcal{M}}_D \subset \mathbf{P}^3$ defined by $H = \{X = T = 0\} + \{Y = T = 0\} + \{Z = T = 0\}$. Let \overline{H} be a strongly nef extension of H . We can suppose that \overline{H} is also effective. Since H is ample we have $H \cdot \theta^- > 0$, and (127) shows that $\overline{\theta}^+$ is strongly nef and effective. □

Corollary 6.11. *For any $\overline{D} \in \widehat{\text{Div}}(\mathcal{M}_D/O_\mathbf{K})$, we have that*

$$\frac{1}{\lambda_1^N} (f^N)^* \overline{D} \rightarrow (D \cdot \theta^-) \overline{\theta}^+ \quad (130)$$

Proof. If \overline{D} is a model adelic divisor then we have already shown the result. Otherwise \overline{D} is defined by a sequence of model adelic divisors (\overline{D}_j) and we have rational numbers $\epsilon_j \rightarrow 0$ such that

$$-\epsilon_j \overline{D}_0 \leq \overline{D} - \overline{D}_j \leq \epsilon_j \overline{D}_0. \quad (131)$$

The operator f^* preserves effectiveness, therefore we get for all $N \geq 0$.

$$-\varepsilon_j \frac{1}{\lambda_1^N} (f^N)^* \bar{D}_0 \leq \frac{1}{\lambda_1^N} (f^N)^* \bar{D} - \frac{1}{\lambda_1^N} (f^N)^* \bar{D}_j \leq \varepsilon_j \frac{1}{\lambda_1^N} (f^N)^* \bar{D}_0 \quad (132)$$

Since $\frac{1}{\lambda_1^N} (f^N)^* \bar{D}_0$ converges to some multiple of $\bar{\theta}^+$ there exists a constant $C > 0$ such that for all N $\frac{1}{\lambda_1^N} (f^N)^* \bar{D}_0 \leq C \bar{D}_0$. Now, by the model case, for N large enough

$$-\varepsilon_j \bar{D}_0 \leq \frac{1}{\lambda_1^N} (f^N)^* \bar{D}_j - (D_j \cdot \theta^-) \bar{\theta}^+ \leq \varepsilon_j \bar{D}_0 \quad (133)$$

Thus, for N large enough

$$-(1+C)\varepsilon_j \bar{D}_0 \leq \frac{1}{\lambda_1^N} (f^N)^* \bar{D} - (D_j \cdot \theta^-) \bar{\theta}^+ \leq (1+C)\varepsilon_j \bar{D}_0 \quad (134)$$

letting $j \rightarrow +\infty$ finishes the proof. \square

Remark 6.12. This corollary shows that the fact that we replaced f by one of its iterate, Theorem 6.1 is proven for f . Indeed, Suppose we have shown the Corollary for f^{N_0} , then Corollary 6.11 holds for f as well: take $\bar{D} \in \widehat{\text{Div}}(\mathcal{M}_D/O_K)$, it suffices to show that for $\ell = 0, \dots, N_0 - 1$

$$\frac{1}{\lambda_1^{\ell+N_0k}} (f^{\ell+N_0k})^* \bar{D} \xrightarrow{k \rightarrow +\infty} (D \cdot \theta^-) \bar{\theta}^+. \quad (135)$$

And this holds by applying Corollary 6.11 with f^{N_0} and $\frac{1}{\lambda_1^\ell} (f^\ell)^* \bar{D}$.

Remark 6.13. Let X be the completion $\bar{\mathcal{M}}_D \subset \mathbf{P}^3$ and H the ample divisor $H = \{X = T = 0\} + \{Y = T = 0\} + \{Z = T = 0\}$. Write $D = A/B$ where $A, B \in O_K$, we have a natural model $(\mathcal{X}, \mathcal{H})$ of (X, H) given by

$$\mathcal{X} = \text{Proj } O_K[X, Y, Z, T] / (BT(X^2 + Y^2 + Z^2) - BXYZ - AT^3) \quad (136)$$

$$\mathcal{H} = \{X = T = 0\}_{\mathcal{X}} + \{Y = T = 0\}_{\mathcal{X}} + \{Z = T = 0\}_{\mathcal{X}} \quad (137)$$

Applying Corollary 6.11 with \mathcal{H} yields the definitions of the Green functions from the introduction.

Proposition 6.14. Let G^+ be the Green function of $\bar{\theta}^+$, then

- (1) $G^+ \geq 0$.
- (2) $G^+ \circ f^{\text{an}} = \lambda_1 G^+$.
- (3) $G^+(x) = 0$ if and only if the forward f^{an} orbit of x is bounded.
- (4) If v is archimedean, then G_v^+ is plurisubharmonic and pluriharmonic over the set $\{G_v^+ > 0\}$.
- (5) If X is a completion of \mathcal{M}_D , then for any Green function g of θ_X^+ , $G^+ - g$ extends to a continuous function over $X_v^{\text{an}} \setminus \{p_-\}$ for every archimedean place v .

Proof. (1) follows from $\bar{\theta}^+$ being effective. (2) follows from $f^*\bar{\theta}^+ = \lambda_1\theta^+$.

If X is any completion of \mathcal{M}_D , we can suppose that it satisfies (1)-(5) of Proposition 5.2 up to blowing up. Let $(\mathcal{X}, \mathcal{D})$ be a model of (X, θ_X^+) for a completion X that satisfies Proposition 5.2. Fix a place v of \mathbf{K} , let U^+ be the open neighborhood of p_+ in X^{an} constructed in this §6.3. Since $G^+ - g(\mathcal{X}, \mathcal{D})$ extends to a continuous function over U^+ and θ_X^+ is effective, we can shrink U_v^+ such that $G_{U_v^+}^+ > 0$.

By (2), we have that $G^+ > 0$ over

$$U_v := \bigcup_{n \geq 0} (f^{-n})^{\text{an}}(U_v^+ \cap \mathcal{M}_D^{\text{an}}) \quad (138)$$

To show (3), it suffices to show that if the forward orbit of $x \in (\mathcal{M}_D)_v^{\text{an}}$ is unbounded, then there exists N_0 such that $(f^{N_0})^{\text{an}}(x) \in U_v^+$. Since X_v^{an} is compact, the sequence $((f^n)^{\text{an}}(x))$ must have an accumulation point $q \in X_v^{\text{an}} \setminus (\mathcal{M}_D)_v^{\text{an}}$. If $q \neq p_-$, then since $f : X \setminus \{p_-\} \rightarrow X \setminus \{p_-\}$ contracts the complement of \mathcal{M}_D to p_+ we must have $f^{\text{an}}(q) = p_+$, thus by continuity there exists N_0 such that $(f^{N_0})^{\text{an}}(x) \in U_v^+$. Otherwise $(f^n)^{\text{an}}(x) \rightarrow p_-$. Since p_- is an attracting fixed point for $(f^{-1})^{\text{an}}$, there exists a basis of neighbourhood U_v^k of p_- in X_v^{an} such that $(f^{-1})^{\text{an}}(U_v^k) \subseteq U_v^k$ and we would get that $x \in U_v^k$ for all $k \geq 0$, this is absurd.

To show (4), let H be the ample divisor on $\overline{\mathcal{M}}_D \subset \mathbf{P}^3$ defined by $\{X = T = 0\} + \{Y = T = 0\} + \{Z = T = 0\}$ and let \bar{H} be a strongly nef effective extension of H and g_H the associated Green function. Suppose v is archimedean. Since g_H is plurisubharmonic over $\mathcal{M}_D(\mathbf{C})$, $\frac{1}{\lambda_1^n} g_H \circ f^n$ also is. By local uniform convergence, we get that G^+ is plurisubharmonic over $\mathcal{M}_D(\mathbf{C})$. To show the pluriharmonicity, it suffices to show that $G_{|U_v^+ \cap \mathcal{M}_D(\mathbf{C})}^+$ is pluriharmonic by the proof of (3). We can suppose that g_H is pluriharmonic over $U_v^+ \cap \mathcal{M}_D(\mathbf{C})$ up to shrinking U_v^+ , then since U_v^+ is f^{an} -invariant, $\frac{1}{\lambda_1^n} g_H \circ (f^{\text{an}})^n$ is also pluriharmonic over $U_v^+ \cap \mathcal{M}_D(\mathbf{C})$ and $G_{|U_v^+ \cap \mathcal{M}_D(\mathbf{C})}^+$ is pluriharmonic as the local uniform limit of pluriharmonic functions.

To show (5), Let v be any place of \mathbf{K} , let $q \in X_v^{\text{an}} \setminus (\mathcal{M}_D)_v^{\text{an}}$ such that $q \neq p_-$. From Proposition 5.2, we can find an open neighbourhood U_v^- of p_- such that $q \notin U_v^-$ and $(f^{-1})^{\text{an}}(U_v^-) \subseteq U_v^-$. The proof of §6.3 shows that if g_+ is a model Green function of θ_X^+ , then

$$\frac{1}{\lambda_1^n} g_+ \circ (f^n)^{\text{an}} - g_+ \quad (139)$$

converges uniformly over $W_v^- = X_v^{\text{an}} \setminus U_v^-$ to a continuous function which is equal to $G^+ - g_+$. Since two Green functions of the same divisor differ by a continuous function we get the result for any Green function of θ_X^+ . \square

Remark 6.15. In [CD12], Chambert-Loir and Ducros developed a theory of plurisubharmonic functions, currents and differential forms on Berkovich spaces. Following their definitions, (4) also holds for non-archimedean places with the same proof.

7. PERIODIC POINTS AND EQUILIBRIUM MEASURE

7.1. Equidistribution. Let (x_n) be a sequence of $X(\overline{\mathbf{K}}) \subset X(\overline{\mathbf{C}}_v)$ and let μ_v be a measure on X_v^{an} . We say that the Galois orbit of (x_n) is equidistributed with respect to μ_v if the sequence of measures

$$\delta(x_n) := \frac{1}{\deg(x_n)} \sum_{x \in \text{Gal}(\overline{\mathbf{K}}/\mathbf{K}) \cdot x_n} \delta_x \quad (140)$$

weakly converges towards μ , where δ_x is the Dirac measure at x .

We say that a sequence of points (x_n) of $X(\overline{\mathbf{K}})$ is *generic* if no subsequence of (x_n) is contained in a strict subvariety of X . In particular, a generic sequence is Zariski dense.

Lemma 7.1. *Let X be a projective variety over a number field \mathbf{K} and let (x_n) be a Zariski dense sequence of $X(\overline{\mathbf{K}})$, then one can extract a generic subsequence of (x_n) .*

Proof. The set of strict irreducible subvarieties of X is countable because \mathbf{K} is a number field. Let $(Y_q)_{q \in \mathbf{N}}$ be the set of strict irreducible subvarieties of X . We construct a generic subsequence $(x'_q)_{q \in \mathbf{N}}$ as follows. Set $Y'_q = \bigcup_{k \leq q} Y_k$. This is a strict subvariety of X . Let $n(1)$ be such that $x_{n(1)} \notin Y'_1 = Y'_1$ and suppose we have constructed $n(1) < \dots < n(q)$ such that $x_{n(i)} \notin Y'_i$. Since (x_n) is Zariski dense, there exists an integer $n(q+1) > n(q)$ such that $x_{n(q+1)} \notin Y'_q$. This defines an increasing sequence $n(q)$ and we set $x'_q = x_{n(q)}$. The sequence (x'_q) is a subsequence of (x_n) which is clearly generic. \square

Theorem 7.2 ([YZ23] Theorem 5.4.3). *Let X be a quasiprojective variety over a number field \mathbf{K} and let \overline{D} be a nef adelic divisor over X such that $D^{\dim X} > 0$. Let $(x_n) \in X(\overline{\mathbf{K}})$ be a generic sequence such that $\lim_n h_{\overline{D}}(x_n) \rightarrow h_{\overline{D}}(X)$, then at every place v the Galois orbit of the sequence (x_n) is equidistributed with respect to the equilibrium measure $\mu_{\overline{D},v}$.*

7.2. Equidistribution of periodic points. Suppose $D \in \mathbf{C}$ is algebraic. Let $f \in \text{Aut}(\mathcal{M}_D)$ be loxodromic. Let $\overline{\theta}^+, \overline{\theta}^-$ be the two strongly nef adelic divisors provided by Theorem 6.1 for f and f^{-1} . Recall that $\theta^+ \cdot \theta^- = 1$. Set

$$\overline{\theta} = \frac{\overline{\theta}^+ + \overline{\theta}^-}{2}. \quad (141)$$

It is a strongly nef adelic divisor over \mathcal{M}_D and satisfies $\theta^2 = 1$. For every place v , we write $\mu_{f,v}$ for the equilibrium measure of $\overline{\theta}$. We also write $h_f := h_{\overline{\theta}}$.

Theorem 7.3. *If (p_n) is a generic sequence of $\mathcal{M}_D(\overline{\mathbf{K}})$ of periodic points of f , then for every place v of \mathbf{K} the Galois orbit of (p_n) is equidistributed with respect to the measure $\mu_{f,v}$.*

Proof. We apply Yuan-Zhang's equidistribution theorem to the adelic divisor $\overline{\theta}$. We need to show that the sequence $h_f(p_n)$ converges to $h_f(X)$. Since the points p_n are periodic, this bounds to show that $h_f(X) = 0$. To do that we apply Theorem 5.3.3 of [YZ23]. Namely, let

$$e(X, (D, G)) := \sup_{U \subset X} \inf_{p \in U} h_f(p) \quad (142)$$

where U runs through open subsets of X . This quantity is called the *essential minimum* of (D, G) . Since we have a generic sequence of periodic points, we get $e(X, (D, G)) = 0$. Theorem 5.3.3 of [YZ23] states that

$$e(X, (D, G)) \geq h_f(X). \quad (143)$$

Therefore we get $h_f(X) = 0$ and Yuan's equidistribution theorem gives the desired result. \square

Corollary 7.4. *If D is algebraic and f, g are two loxodromic automorphisms of \mathcal{M}_D such that $\text{Per}(f) \cap \text{Per}(g)$ is Zariski dense, then for every place v of $\mathbf{Q}(D)$,*

$$\mu_{f,v} = \mu_{g,v} \quad (144)$$

Proof. Let (x_n) be a Zariski dense sequence of $\text{Per}(f) \cap \text{Per}(g)$. By Lemma 7.1, we can suppose that (x_n) is generic. By Theorem 7.3, for every place v of \mathbf{K} , the Galois orbit of the sequence (x_n) equidistributes with respect to both the measures $\mu_{f,v}$ and $\mu_{g,v}$. Thus, they must be equal. \square

8. SADDLE PERIODIC POINTS ARE IN THE SUPPORT OF THE EQUILIBRIUM MEASURE

Let $D \in \mathbf{C}$ be algebraic. Fix an archimedean place v of $\mathbf{Q}(D)$. For this section we work only over this place so we will drop the index v and fix an embedding $\mathbf{Q}(D) \hookrightarrow \mathbf{C}$.

Theorem 8.1. *Let f be a loxodromic automorphism of \mathcal{M}_D . Every periodic saddle point of f is in the support of the measure μ_f .*

This theorem, stated in [Can01] and [Can09], follows directly from the work of Dinh and Sibony in [DS13], which extends [BS91b], and an argument of [BLS93] for Hénon type automorphisms of the complex affine plane. A sketch of proof is given in [Can09] §3.1 and §3.2. We provide a more detailed proof here.

8.1. Green functions and bounded orbits. First, let us summarize some of the properties of the function $G_f^+ : \mathcal{M}_D(\mathbf{C}) \rightarrow \mathbf{R}^+$ of $\bar{\theta}^+$ from Proposition 6.14. A *saddle* point is a point where the differential of f does not have any eigenvalue of modulus 1. The *stable manifold* of a point $q \in \mathcal{M}_D(\mathbf{C})$ is the set of point p such that

$$\text{dist}(f^n(q), f^n(p)) \xrightarrow{n \rightarrow +\infty} 0. \quad (145)$$

Fix a completion X of \mathcal{M}_D that satisfies Proposition 5.2.

- (a) $\{G_f^+ = 0\}$ coincides with the set $K^+(f)$ of points with a bounded forward orbit;
- (b) G_f^+ is plurisubharmonic, and is pluriharmonic on the set $\{G_f^+ > 0\}$;
- (c) the set $K^+(f)$ is closed in $\mathcal{M}_D(\mathbf{C})$, its closure in $X(\mathbf{C})$ coincides with $K^+(f) \cup p_-$;
- (d) locally, near every point $q \neq p_-$ of $X(\mathbf{C}) \setminus \mathcal{M}_D(\mathbf{C})$,

$$G_f^+(x) = - \sum_i a_i \log(|z_i|) + u(x) \quad (146)$$

where the functions z_i are holomorphic equations of the boundary components containing q , the real numbers $a_i \geq 0$ are the weight of θ_X^+ , and $u(x)$ is a continuous (pluriharmonic) function.

- (e) there is an open neighborhood U^\pm of p_- in $X(\mathbf{C})$ such that $f^{-1}(U^-) \subseteq U^-$ and U^- is contained in the basin of attraction of p_- for the backward dynamics; there is an open neighborhood U^+ of p_+ with similar properties for f instead of f^{-1} ;
- (f) If q is a saddle periodic point, its stable manifold $W^s(q)$ is contained in $K^+(f)$; in fact, the proof of Proposition 5.1 in [BS91a] shows that $W^s(q)$ is contained in the boundary of $K^+(f)$;
- (g) f does not preserve any algebraic curve $C_0 \subset \mathcal{M}_D(\mathbf{C})$.

In particular, if S is a closed positive current supported by $\overline{K(f)}$, then its support does not intersect the open set U^- .

8.2. Rigidity of $\overline{K^+(f)}$ and equidistribution of stable manifolds. The properties (a) to (g) are sufficient to apply the arguments of Sections 4, 5, 6 of [DS13]. More precisely, one first obtains Theorem 6.6 of [DS13], because its proof relies only on the above properties and general results concerning closed positive currents (in particular Corollary 3.13 of [DS13]).¹ Then, one gets directly the following fact (which corresponds to a weak version of Theorem 6.5 of [DS13], with the same proof):

Theorem 8.2. *The set $\overline{K^+(f)}$ (resp. $K^-(f)$) supports a unique closed positive current, namely $T_f^+ = dd^c G_f^+$ (resp. T_f^-) up to multiplication by a positive constant.*

Consequently, we get the following result: *Given any algebraic curve $C_0 \subset \mathcal{M}_D$, the sequence of currents $\lambda_1(f)^{-n} \{(f^n)^* C_0\}$ converges towards a positive multiple of T_f^+ as n goes to $+\infty$ (see Corollary 6.7 of [DS13]). Thus, T_f^+ can be approximated by a sequence of currents of integration on algebraic curves of a fixed genus (properly renormalized); in this context, one can apply the theory of strongly approximable laminar currents, as developed by Dujardin (see [Can14, Duj04, Duj05] for an introduction and §8.3).*

This rigidity results provides automatic equidistribution theorems for $(1, 1)$ positive currents. We shall need the following specific application.

If q is a saddle periodic point of f , then its stable manifold $W^s(q)$ is biholomorphic to the complex line². Denote by $\xi: \mathbf{C} \rightarrow W^s(q) \subset \mathcal{M}_D(\mathbf{C})$ a one to one holomorphic parametrization of $W^s(q)$; ξ is an entire holomorphic curve. To such a curve, one can associate a family of currents of mass 1, constructed as follows. One fixes a Kähler form κ on $X(\mathbf{C})$ and one measures lengths, areas and volumes with respect to this form. For instance, if $\mathbb{D}_r \subset \mathbf{C}$ is the disk of radius r centered at the

¹The only changes in this proof are that (1) $\mathbf{P}^2(\mathbf{C})$ should be replaced by $X(\mathbf{C})$ and the line at infinity by $X \setminus \mathcal{M}_D$; and (2) the function $\log(1 + \|z\|^2)^{1/2}$ should be replaced by a smooth Green function associated to the \mathbf{R} -divisor θ_X^+ .

²Indeed, it is a Riemann surface, it is homeomorphic to \mathbf{R}^2 , and f acts on it as a contraction fixing q , so $W^s(q)$ cannot be a disk and Riemann uniformization theorem says that it is a copy of \mathbf{C}

origin, then

$$\text{Area}(\xi(\mathbb{D}_r)) = \int_{\xi(\mathbb{D}_r)} \kappa = \int_{\mathbb{D}_r} \xi^* \kappa \quad (147)$$

is the area of the image of \mathbb{D}_r by ξ . Averaging with respect to dr/r , one introduces the function

$$N(R) = \int_{t=0}^R \text{Area}(\xi(\mathbb{D}_t)) \frac{dt}{t}. \quad (148)$$

Now, for each disk \mathbb{D}_r , one can consider the current of integration over $\xi(\mathbb{D}_r)$: to a smooth form α of type $(1, 1)$, this current $\{\xi(\mathbb{D}_r)\}$ associates the number

$$\langle \{\xi(\mathbb{D}_r)\} | \alpha \rangle = \int_{\xi(\mathbb{D}_r)} \alpha = \int_{\mathbb{D}_r} \xi^* \alpha. \quad (149)$$

Taking averages with respect to the weight dr/r one obtains the following family of currents, parametrized by a radius $R > 0$:

$$\langle N_\xi(R) | \alpha \rangle = \frac{1}{N(R)} \int_{t=0}^R \langle \{\xi(\mathbb{D}_t)\} | \alpha \rangle \frac{dt}{t} \quad (150)$$

$$= \frac{1}{N(R)} \int_{t=0}^R \int_{\xi(\mathbb{D}_t)} \alpha \frac{dt}{t}. \quad (151)$$

The normalization by $1/N(R)$ assures that the mass $\langle N_\xi(R) | \kappa \rangle$ is equal to 1 for every $R > 0$. From an inequality of Ahlfors, and from the compactness of the space of positive currents of mass 1, there are sequences of radii (R_n) such that $N_\xi(R_n)$ converges to a closed positive current S . A priori, such a closed positive current S depends on the choice of the sequence R_n ; if there is a unique closed positive current S that can be obtained as such a limit, one says that there is a unique Ahlfors-Nevanlinna current (namely S) associated to ξ .

Corollary 8.3 (Proposition 4.10, Corollary 4.11 [DS13]). *Let q be a saddle periodic point of f . Let $\xi: \mathbb{C} \rightarrow \mathcal{M}_D(\mathbb{C})$ be a holomorphic parametrization of the stable manifold of f . Then, there is a unique Ahlfors-Nevanlinna current associated to ξ , and this current is equal to T_f^+ .*

8.3. Laminarity, Pesin theory and consequence. The measure $\mu_f = T_f^+ \wedge T_f^-$ is an ergodic measure of positive (and maximal) entropy for f , and tools from Pesin theory can be used to describe the dynamics of f with respect to this measure. In particular, in our setting, one can apply the work of Bedford, Lyubich, and Smillie in [BLS93] or the work of Dujardin in [Duj04].

Definition 8.4. (i) A family of disjoint horizontal graphs Γ in $\mathbb{D} \times \mathbb{D}$ is called a flow box. If it is equipped with a measure λ on $\{0\} \times \mathbb{D}$ we call it a *measured flow box*. It defines a closed positive current $T_{\Gamma, \lambda}$ in $\mathbb{D} \times \mathbb{D}$ defined by

$$\langle T_{\Gamma, \lambda}, \alpha \rangle = \int_{a \in \mathbb{D}} \int_{\Gamma_a} \alpha d\lambda(a). \quad (152)$$

- (ii) A current T is *uniformly laminar* if for every $x \in \text{Supp } T$, there exist an open subset $V \ni x$ such that V is biholomorphic to a bidisk and a measured flow box (Γ, λ) in V such that $T|_V = T_{\Gamma, \lambda}$.
- (iii) A current is *laminar* if there exists a family of disjoint measured flow boxes (Γ_i, λ_i) such that

$$T = \sum_i T_{\Gamma_i, \lambda_i}. \quad (153)$$

- (iv) A current is *strongly approximable* if it is the weak limit of a sequence of integration currents $\frac{1}{d_n}[C_n]$ such that

$$\text{genus}(C_n) + \sum_{p \in \text{Sing}(C_n)} n_p(C_n) = O(d_n). \quad (154)$$

- (v) A current is *diffuse* if it does not charge algebraic curves.

The main result of [Duj05] is that *if X is a projective rational surface and T is a strongly approximable diffuse current on X , then T is laminar and for every flow box Γ , $T|_\Gamma$ is uniformly laminar*. The discussion after Theorem 8.2 shows that T_f^+ and T_f^- are strongly approximable currents. They are also diffuse by Proposition 6.3 of [DF01].

Definition 8.5. If S_1, S_2 are two uniformly laminar diffuse currents with a representation

$$S_i = \int_{A_i} [D_{a,i}] d\mu_i(a) \quad (155)$$

then we define the *geometric intersection* of S_1, S_2 as

$$S_1 \mathbin{\dot{\wedge}} S_2 := \int_{A_1} \int_{A_2} [D_{a,1} \cap D_{b,2}] d\mu_1(a) \otimes d\mu_2(b) \quad (156)$$

where $[D_{a,1} \cap D_{b,2}]$ is the sum of Dirac masses at the intersection point if the intersection is finite and 0 otherwise. We extend the definition of geometric intersection to sums of uniformly laminar currents by taking geometric intersection with respect to each term of the sum. We say that a product is *geometric* if $S_1 \wedge S_2 = S_1 \mathbin{\dot{\wedge}} S_2$.

Definition 8.6. A Pesin box is a pair (U, K) where U is an open subset isomorphic to a bidisk $\mathbb{D} \times \mathbb{D}$ and a compact $K \subset U$ of positive μ_f -measure such that

- (i) Every point in K is a hyperbolic point of f .
- (ii) The local stable and unstable manifolds of the points of K are vertical and horizontal graphs in U .
- (iii) For all pair of distinct points $(x, y) \in K^2$, $W_{loc}^s(x) \cap W_{loc}^u(y)$ is a singleton contained in K .

In particular, the local stable and unstable manifolds define a lamination K^s and K^u in U . By the main theorem of [Duj05], $T_f^+|_{K^s}$ is uniformly laminar so there exists a transverse measure λ_K^+ such that

$$T_f^+|_{K^s} = T_{K^s, \lambda_K^+}. \quad (157)$$

$$T_f^+ = \sum_{(U,K)} T_{K^s, \lambda_K^+} \quad (158)$$

Proposition 8.8. *Every saddle periodic point of f is in the support of μ_f .*

$$\mu_{f|K} = T_{f|K^s}^+ \bigwedge T_{f|K^u}^- \quad (159)$$
$$\frac{1}{\lambda_f^n} (f^n)_* \left(T_{f|K^u}^- \right) |_W \leq T_f^-. \quad (160)$$
$$\mu_n := \frac{1}{\lambda_f^{2n}} \left((f^n)^* T_{f|K^s}^+ \right)_W \bigwedge \left((f^n)_* T_{f|K^u}^- \right)_W \leq \mu_f \quad (161)$$

Thus, Theorem 8.1 is proven. \square

9. PROOF OF THEOREM A

The proof of Theorem A relies on the following proposition.

Proposition 9.1. *Let $f \in \text{Aut}(\mathcal{M}_D)$ be a loxodromic automorphism with $D = 0$ or $D = -2 + 2\cos\left(\frac{2\pi}{q}\right)$ and let v be an archimedean place. Then, f admits a periodic saddle fixed point $q(f) \in \mathcal{M}_D(\mathbf{C})$ such that*

- (1) $q(f) \in \text{Supp}(\mu_{f,v})$
- (2) *If $g \in \text{Aut}(\mathcal{M}_D)$ is loxodromic such that f and g do not share a common iterate, then $(g^n(q(f)))$ is unbounded.*

Item (1) follows from Theorem 8.1.

Assuming the proposition, suppose that f, g share a Zariski dense subset of periodic points, we can suppose that they share a generic sequence of periodic points. Then by Theorem 7.3 we have equality of the equilibrium measures of f and g at every place so in particular at every archimedean place. Fix v one of them. Suppose that f and g do not share a common iterate, then $(g^n(q(f)))_n$ is unbounded. Let $\mu = \mu_{f,v} = \mu_{g,v}$. Since $\text{Supp}\mu = \text{Supp}\mu_{f,v} = \text{Supp}\mu_{g,v}$, we have that $\text{Supp}\mu$ is a compact subset of $\mathcal{M}_D(\mathbf{C})$ invariant by f and g . Since $q(f) \in \text{Supp}\mu_{f,v} = \text{Supp}\mu$ we get that $(g^n(q(f))) \subset \text{Supp}\mu$ which is a contradiction.

9.1. Construction of the saddle fixed point $q(f)$. Suppose first that $D = 0$.

Up to taking an iterate of f we can suppose that there exists a loxodromic element $\Phi_f \in \text{SL}_2(\mathbf{Z})$ such that $f = f_{\Phi_f}$. Denote by $p(f) = p(f_{\Phi_f})$ and $q(f) = q(f_{\Phi_f})$ the fixed points constructed using Minsky theorem. These two fixed point are saddle fixed points by [McM96] Corollary 3.19. The fixed point $q(f)$ corresponds to a representation $\rho_\infty : F_2 \rightarrow \text{PSL}_2(\mathbf{C})$, one can show that ρ_∞ also satisfies Theorem 4.2 even though the punctured torus is not compact. One can show that for any automorphism g of \mathcal{M}_0 the differential of g at $(0,0,0)$ has order 1 or 2, thus $p(f), q(f) \neq (0,0,0)$ and it is a smooth point of \mathcal{M}_0 .

Suppose now that $D = 2 - 2\cos\frac{2\pi}{q}$. Following [McM96] §3.7, let S be the orbifold obtained from a genus 1 torus with a singular point of index q . The fundamental group of S is

$$\pi_1(S) = \langle a, b | [a, b]^q = 1 \rangle \quad (162)$$

The modular class group $\text{Mod}(S)$ of S is also $\text{SL}_2(\mathbf{Z})$. Let $\Phi_f \in \text{SL}_2(\mathbf{Z})$ be an element of $\text{Mod}(S)$ associated to f . There exists a smooth (real) surface \tilde{S} with a map $\tilde{S} \rightarrow S$ which is a finite characteristic covering. In particular, Φ_f lifts to \tilde{S} and defines an element of $\text{Mod}(\tilde{S})$ that we denote by $\tilde{\Phi}_f$. Apply Theorem 4.2 to $(\tilde{S}, \tilde{\Phi}_f)$, there exists a faithful and discrete representation $\tilde{\rho}_\infty : \pi_1(\tilde{S}) \rightarrow \text{PSL}_2(\mathbf{C})$. Let $\tilde{M}_\infty = \mathbb{H}^3 / \tilde{\rho}_\infty(\pi_1(\tilde{S}))$, the group of isometries of \tilde{M}_∞ contains the subgroup generated by $\tilde{\Phi}_f$. The quotient $\tilde{M}_\infty / \langle \tilde{\Phi}_f \rangle$ is the mapping torus $M_{\tilde{\Phi}_f}$ of $\tilde{\Phi}_f$ which is a finite cover of the mapping torus M_{Φ_f} . By Mostow rigidity theorem, the covering group can be realized by isometries, therefore the hyperbolic structure on $M_{\tilde{\Phi}_f}$ descends to a hyperbolic structure on the mapping torus M_{Φ_f} , which

yields a fixed point p_∞ of f in \mathcal{M}_D that we denote by $q(f)$. By [McM96] Corollary 3.19, $q(f)$ is a saddle fixed point.

9.2. The sequence $(g^n(q(f)))$ is unbounded. Suppose $D = 0$ we can consider S as the flat torus $T = \mathbf{R}^2/\mathbf{Z}^2$ with a puncture at the origin, i.e. $S = T \setminus \{o\}$, or as a complete hyperbolic surface X of finite area (we fix such a hyperbolic structure, it corresponds to some point X in the Teichmüller space $\text{Teich}(S) \simeq \mathbb{D}$).

An element f of $\text{Out}^+(F_2)$ is pseudo-Anosov if the corresponding matrix $A_f \in \text{SL}_2(\mathbf{Z})$ has $\text{Tr}(A_f)^2 \geq 4$. In that case, the matrix has two eigenvalues $\lambda(f) > 1$ and $1/\lambda(f) < 1$ and the mapping class is represented by a linear automorphism of the torus T (fixing the origin o) with stable and unstable linear foliations. In the hyperbolic surface X , these foliations give rise to two measured laminations F_- and F_+ (by geodesic lines). If $C \subset S$ is a closed curve (represented by some geodesic in X), one can define two intersection numbers $i(C, F_+)$ and $i(C, F_-)$; they depend only on the free homotopy class of C . The product $j(C) = i(C, F_+)i(C, F_-)$ is f -invariant, because f stretches F_+ by a dilatation factor $\lambda(f) > 1$, and contracts F_- by $1/\lambda(f)$; if C is not homotopic to a loop around the puncture $j(C)$ is strictly positive (any closed geodesic is transverse to F_+ and F_-).

If $D = 2 - 2\cos(2\pi/q)$, let S be the genus one torus with an orbifold singularity of order q . We have seen that there exists a characteristic finite covering $\tilde{S} \rightarrow S$ with \tilde{S} a compact surface of negative Euler characteristic. We let $X = \mathbb{H}^2/\Gamma$ be a hyperbolic surface homeomorphic to \tilde{S} (i.e. $X \in \text{Teich}(\tilde{S})$). If $f \in \text{Out}^+(F_2)$ is pseudo-Anosov then it lifts to a pseudo-Anosov $\tilde{f} \in \text{Mod}(X) = \text{Out}^+(F_2)$ pseudo-Anosov also. In that case, there exist two measured laminations F_+ and F_- over \tilde{S} (the stable and the unstable one) and by Proposition 1.5.1 of [Ota96]. We have that for any geodesic $\gamma \in \tilde{S}$,

$$\frac{(\tilde{f})_*^{\pm i} \gamma}{\ell((\tilde{f})_*^{\pm i} \gamma)} \xrightarrow{i \rightarrow +\infty} F_{\pm} \quad (163)$$

in the sense of measured laminations. (This also holds in the case $D = 0$). Here ℓ is the length induced by the hyperbolic structure from the quotient \mathbb{H}^2/Γ so $\ell((\tilde{f})_*^{\pm i} \gamma)$ grows like $\lambda(\tilde{f})^i$. We also have that $j(\gamma) = i(\gamma, F_+)i(\gamma, F_-)$ is f -invariant as $i(\tilde{f}_*(\gamma), F_{\pm}) = \lambda(\tilde{f})^{\mp 1} i(\gamma, F_{\pm})$ and if γ is a geodesic, then $j(\gamma) > 0$. To unify the notations we will still denote by f the lift \tilde{f} of f to X .

Lemma 9.2. *If f and g are two loxodromic elements of $\text{Out}^+(F_2) \simeq \text{SL}_2(\mathbf{Z})$ generating a non-elementary subgroup of $\text{SL}_2(\mathbf{Z})$, then given any geodesic $\gamma \subset X$, $j(g^n(\gamma))$ goes to $+\infty$ as n goes to $+\infty$.*

Proof. Let G_+ and G_- be the unstable and stable laminations associated to g in X . Since f and g generate a non-elementary subgroup of $\text{GL}_2(\mathbf{Z})$, G_+ is transverse to both F_+ and F_- (equivalently, the four fixed points of A_f and A_g on $\mathbf{P}^1(\mathbf{R})$ are distinct). Thus, by Equation (163) $j(g^n(C)) \simeq \lambda(g)^n i(G_+, F_+) i(G_-, F_-)$ by continuity of the intersection number (see [Ota96] p.151). \square

Lemma 9.3. *Let f and g be two loxodromic elements of $\text{Out}^+(F_2) \simeq \text{SL}_2(\mathbf{Z})$ generating a non-elementary subgroup of $\text{SL}_2(\mathbf{Z})$. Let $\gamma \subset X$ be a geodesic, and let $[\gamma]$ be its free homotopy class. Then the sequence $g^n[\gamma]$ intersects each orbit of f only finitely many times.*

Proof. This follows from the previous lemma and the fact that $j(\cdot)$ is f -invariant so it is constant in each orbit of f . \square

Recall the definition of M_{Φ_f} , \tilde{M}_{Φ_f} , ρ_∞ and α_f from Theorem 4.2 (here we consider $f \in \text{Mod}(\tilde{S})$ if we are in the orbifold case). In M_{Φ_f} , the number of simple closed geodesics of length $\leq L$ is finite (for every $L > 0$); thus, in \tilde{M}_{Φ_f} , given any upper bound L , there are only finitely many homotopy classes of simple closed curves *up to the action of $f^{\mathbb{Z}}$* (Note that, since α_f acts by isometry, each closed geodesic $C \subset \tilde{M}_f$ gives rise to infinitely many geodesics $\alpha_f^n(C)$ with the exact same length).

Proof of Proposition 9.1 (2). Fix a generator a in $\pi_1(S)$ where S is either the punctured torus or the genus 1 torus with an orbifold singularity of index q . Set k to be the degree of the finite cover $\tilde{S} \rightarrow S$ in the orbifold case and $k = 1$ otherwise. The element a^k gives rise to a closed geodesic A in \tilde{M}_{Φ_f} . From these preliminaries and the previous lemma, the sequence of homotopy classes $g^n(a^k)$ correspond to a sequence of closed geodesics in \tilde{M}_{Φ_f} , with length going to infinity because f acts by isometry on \tilde{M}_{Φ_f} .

Now, $g^n(a^k)$ corresponds to a (conjugacy class of a) matrix $\rho_\infty(g^n(a^k)) \in \text{SL}_2(\mathbb{C})$, and the trace of this matrix is related to the length of the geodesic by a simple formula; in particular, the fact that the length goes to infinity implies that the modulus of the trace goes to $+\infty$. Since for any matrix $A \in \text{SL}_2(\mathbb{C})$, $\text{Tr} A^k$ is a polynomial in $\text{Tr} A$ we get that $\text{Tr}(\rho_\infty(g^n(a)))$ goes to infinity. This implies that the orbit of $q(f)$ under the action of g on $\mathcal{M}_D(\mathbb{C})$ is discrete, going to infinity.

10. FOR A TRANSCENDENTAL PARAMETER D

We finish this paper by proving Theorem B which we restate.

Theorem 10.1. *Let $D \in \mathbb{C}$ be transcendental and let $f, g \in \text{Aut}(\mathcal{M}_D)$ be loxodromic automorphisms. The following assertions are equivalent:*

- (1) $\text{Per}(f) = \text{Per}(g)$.
- (2) $\exists N, M \in \mathbb{Z}, f^N = g^M$.

Proof. We can suppose that $f, g \in \text{SL}_2(\mathbb{Z})$, for any parameter $t \in \mathbb{C}$, we denote by f_t the automorphism induced by $f \in \text{SL}_2(\mathbb{Z})$ over S_t . For the parameter $t = 0$, we have constructed a hyperbolic fixed point $q(f) \in \mathcal{M}_0(\mathbb{C})$. Because $q(f) \neq (0, 0, 0)$, we can find local analytic coordinates u, v, w at $q(f) \in \mathbb{C}^3$ with $w = \text{Tr} \gamma$ such that f_t is locally of the form

$$f_w(u, v, w) = (\lambda u, \frac{1}{\lambda} v, w) \tag{164}$$

where $\lambda, \frac{1}{\lambda}$ are the two eigenvalues of the differential of f_0 at $q(f)$. By the analytic implicit function theorem, there exists $\varepsilon > 0$ a local analytic curve $c_\varepsilon : w \in \mathbb{D}(0, \varepsilon) \mapsto c_\varepsilon(w) \in S_w$ such that $c_\varepsilon(0) = q(f)$ and $f_w(c_\varepsilon(w)) = c_\varepsilon(w)$. Now, if f, g do not share a common iterate, then the orbit of $q(f)$ under g_0 is unbounded by Proposition 9.1. Thus, for all $k \in \mathbb{Z}$, we have $g_0^k(q(f)) \neq q(f)$. We show the following.

Lemma 10.2. *If $D \in \mathbf{C}$ is transcendental, then for all $k \in \mathbf{Z}_{\geq 0}$, there exists $p \in \mathcal{M}_D(\mathbf{C})$ such that*

$$f_D(p) = p \text{ and } g_D^\ell(p) \neq p, \forall 1 \leq \ell \leq k. \quad (165)$$

Using the lemma, we can conclude because f_D admits a finite number of fixed points since f_D does not admit an invariant curve, thus we must have $\text{Per}(f_D) \neq \text{Per}(g_D)$. \square

Proof of the lemma. Notice that this statement does not depend on the transcendental parameter D . Indeed, let D' be another transcendental parameter, then there exists a Galois automorphism $\sigma \in \text{Gal}(\mathbf{C}/\overline{\mathbf{Q}})$ that exchange D and D' . Since the family of surfaces \mathcal{M}_D gives a foliation of \mathbf{C}^3 we can view a point $p \in \mathcal{M}_D(\mathbf{C})$ as a point in \mathbf{C}^3 and apply σ to each coordinates, we denote by $p^\sigma \in \mathbf{C}^3$ the new obtained point. We apply σ to (165) to get

$$f_{\sigma(D)}(p^\sigma) = p^\sigma \text{ and } g_{\sigma(D)}^\ell(p^\sigma) \neq p^\sigma, \forall 1 \leq \ell \leq k. \quad (166)$$

Now, fix $k \geq 1$. For any transcendental parameter t small enough we have

$$f_t(c_\varepsilon(t)) = c_\varepsilon(t) \quad (167)$$

by construction of the curve c_ε and

$$\forall 1 \leq \ell \leq k, g_t^\ell(c_\varepsilon(t)) \neq c_\varepsilon(t) \quad (168)$$

by continuity since we have $g_0^m(q(f)) \neq q(f)$ for all $m \in \mathbf{Z}$. Thus, the lemma is shown. \square

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