# On the dynamics of endomorphisms of affine surfaces 

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## CHAPTER 1

## Introduction

In [FJ07], Favre and Jonsson developed tools from valuative theory to study the dynamics of a dominant endomorphism of the complex affine plane. We extend this theory to the case of any affine surface, over any field. We give a new method to construct an eigenvaluation of an endomorphism. We generalize the result of Favre and Jonsson and show that the first dynamical degree of a dominant endomorphism of a normal affine surface is an algebraic integer of degree $\leqslant 2$. Plus, we obtain a new result of rigidity. The set of first dynamical degrees of loxodromic automorphisms of a given affine surface must be contained in the set of integers or in the set of algebraic numbers of degree 2 .

### 1.1. Dynamical degrees

Let $X$ be a smooth projective variety over an algebraically closed field and let $d$ be its dimension. For $d$ Cartier divisors $D_{1}, \cdots, D_{d}$ of $X$ we can define the intersection product $D_{1} \cdots D_{d} \in \mathbf{Z}$ (see $\left.[\overline{\mathbf{L a z 0 4}}]\right)$. If $f: X \rightarrow X$ is a dominant rational transformation of $X$, we define for $0 \leqslant \ell \leqslant d$ the $\ell$-th dynamical degree of $f$ by

$$
\begin{equation*}
\lambda_{\ell}(f):=\lim _{n \rightarrow \infty}\left(\left(f^{n}\right)^{*} H^{\ell} \cdot H^{d-\ell}\right)^{1 / n} \tag{1}
\end{equation*}
$$

where $H$ is an ample divisor over $X$. One can show that these quantities are well defined and do not depend on the choice of $H$. Furthermore, the dynamical degrees are birational invariants: if $\varphi: X \rightarrow Y$ is a birational map, then

$$
\begin{equation*}
\lambda_{l}(f)=\lambda_{l}\left(\varphi \circ f \circ \varphi^{-1}\right), \quad \forall 0 \leqslant l \leqslant d . \tag{2}
\end{equation*}
$$

We have that $\lambda_{d}(f)$ is the topological degree of $f$ and $\lambda_{0}(f)=1$. The KhovanskiiTeissier inequalities (see [Gro90], [DN05]) imply that the sequence $\left(\lambda_{l}\right)_{0 \leqslant l \leqslant d}$ is logconcave; i.e

$$
\begin{equation*}
\frac{\log \lambda_{l-1}+\log \lambda_{l+1}}{2} \leqslant \log \lambda_{l}, \quad \forall 1 \leqslant l \leqslant d-1 . \tag{3}
\end{equation*}
$$

In particular, one has $\forall 1 \leqslant l \leqslant d, \lambda_{1}(f)^{l} \geqslant \lambda_{k}(f)$.
Let $X_{0}$ be a smooth affine variety of dimension $d$ and $f: X_{0} \rightarrow X_{0}$ a dominant endomorphism of $X_{0}$. We define the dynamical degrees of $f$ as follows. A completion of $X_{0}$ is a smooth projective variety $X$ equipped with an open immersion $1: X_{0} \hookrightarrow$
$X$ such that $\mathfrak{l}\left(X_{0}\right)$ is dense in $X$. The endomorphism $f$ induces a dominant rational transformation of $X$ via $\tilde{f}=\mathfrak{l} \circ f \circ \mathfrak{l}^{-1}$ and we define the dynamical degrees

$$
\begin{equation*}
\lambda_{l}(f):=\lambda_{l}(\widetilde{f}) \tag{4}
\end{equation*}
$$

As the dynamical degrees are birational invariants, these quantities do not depend on the choice of the completion $X$. The data of these dynamical degrees gives information on the dynamical system. For example over C, Dinh and Sibony showed in [DS03] that for all dominant rational transformation $f: X \rightarrow X$

$$
\begin{equation*}
h_{\text {top }}(f) \leqslant \max _{0 \leqslant l \leqslant d} \log \left(\lambda_{l}\right) \tag{5}
\end{equation*}
$$

where $h_{\text {top }}$ is the topological entropy of $f$, Gromov showed this result for endomorphisms of $\mathbf{P}^{N}$ in [Gro03]. Yomdin showed in Yom87] that we have an equality if $f$ is an endomorphism. The inequality is strict in general (see [Gue05]). Recently, Favre, Truong and Xie showed in [FTX22] that the inequality (5) still holds in the non archimedean case; however the equality does not hold even for endomorphisms.

### 1.2. Dynamical degrees on projective surfaces

A natural question is to ask what numbers can appear as the first dynamical degree of a rational transformation of a projective surface. For the topological degrees, it is easy to check that any integer $k$ is the topoological degree of a dominant rational transformation of $\mathbf{P}^{2}$ (consider $f(x, y)=\left(x^{k}, y^{k}\right)$ over $\mathbf{C}^{2}$ ).

In 2021, Bell, Diller and Jonsson showed in [BDJ20] that there exists a dominant rational transformation $\sigma: \mathbf{P}^{2} \rightarrow \mathbf{P}^{2}$ such that $\lambda_{1}(\sigma)$ is transcendental. The authors with Krieger showed in $[\mathbf{B D J 2 0}]$ this example can be generalised to give an example of a birational transformation of $\mathbf{P}^{N}, N \geqslant 3$ with a transcendental first dynamical degree. However in dimension 2 , there are strong constraints on $\lambda_{1}(f)$ for $f$ birational. In [DF01], Diller and Favre showed that the first dynamical degree of a birational transformation of a projective surface is an algebraic integer, but with arbitrary large degree. Indeed, Bedford, Kim and McMullen have given in [BK06] and [McM07] examples of birational transformations of projective surfaces with first dynamical degree an algebraic integer of arbitrary large degree. In particular, Theorem 1.1 of [McM07] states that for all $d \geqslant 10$ we can find a smooth complex projective surface with an automorphism with first dynamical degree an algebraic integer of degree $d$. This also holds in positive characteristic by the main theorem of [CD]. Blanc and Cantat showed in [BC13] that the set of all first dynamical degrees of elements of $\operatorname{Bir}\left(\mathbf{P}_{K}^{2}\right)$ is a well ordered set if $K$ is infinite.

### 1.3. Dynamical degrees of endomorphisms of affine surfaces and Perron numbers

The first example of an affine surface is the complex affine plane $\mathbf{C}^{2}$. An endomorphism is then a polynomial transformation. Even in that case, the first dynamical degree is not necessarily an integer. Indeed, let

$$
A=\left(\begin{array}{ll}
a & b  \tag{6}\\
c & d
\end{array}\right)
$$

be a matrix with nonnegative integer coefficients such that $a d-b c \neq 0$. Consider the following monomial transformation

$$
\begin{equation*}
f(x, y)=\left(x^{a} y^{b}, x^{c} y^{d}\right) \tag{7}
\end{equation*}
$$

then $f^{N}$ is the monomial transformation where the monomials are given by the coefficients of $A^{N}$ and $\lambda_{1}(f)$ is equal to the spectral radius of $A$. Hence, $\lambda_{1}(f)$ is an algebraic integer of degree 2 because it satisfies the equation

$$
\begin{equation*}
\lambda_{1}(f)^{2}-\operatorname{Tr}(A) \lambda_{1}(f)+\operatorname{det}(A)=0 \tag{8}
\end{equation*}
$$

It is in fact a Perron number. A ${ }^{1}$ (weak) Perron number is a real algebraic integer $\alpha \geqslant 1$ such that all its Galois conjugates have complex modulus $\leqslant|\alpha|$. Thus, there exist polynomial transformations $f$ of the affine plane with $\lambda_{1}(f)$ an integer or a Perron number of degree 2. Favre and Jonsson showed that these are the only two possibilities.

THEOREM 1.1. FJ07] Let $f: \mathbf{C}^{2} \rightarrow \mathbf{C}^{2}$ be a dominant polynomial transformation, then $\lambda_{1}(f)$ is a Perron number of degree $\leqslant 2$.

The first result of this memoir is to extend this result to all normal affine surfaces, in any characteristic. Even if the semigroup of endomorphisms can change drastically when one changes the affine surface. For example, Blanc and Dubouloz, in [BD13], build smooth affine surfaces with a big group of automorphisms, much bigger than the one of the affine plane. Bot used this construction to show the existence of smooth complex rational affine surfaces with uncountably many real forms (see [Bot23]). The results in this paper show that, even though structure wise these groups are a lot more complicated, from the point of view of the dynamics of a single element, this is not the case.

THEOREM A. Let $X_{0}$ be a normal affine surface over a field $\mathbf{k}$. If $f: X_{0} \rightarrow X_{0}$ is a dominant endomorphism, then $\lambda_{1}(f)$ is a Perron number of degree $\leqslant 2$.

[^0]The proof uses valuative techniques which we describe in the next section. If char $\mathbf{k}=0$ or if $f$ is an automorphism, we also obtain results on the dynamics of $f$. For any completion $X$ of $X_{0}$, the endomorphism $f$ extends to a rational transformation of $X$, it has a finite number of indeterminacy points at infinity i.e on $X \backslash X_{0}$. One cannot hope in general to find a completion $X$ such that $f$ extends to a regular endomorphism of $X$. The strategy of proof consists of studying the dynamics of $f$ at infinity. More specificallly, we find good completions where $f$ has an attracting fixed point at infinity, i.e a point $p \in X \backslash X_{0}$ where the lift $f: X \rightarrow X$ of $f$ is defined at $p$ and $f(p)=p$ we can then study the local dynamics at $p$ to compute the first dynamical degree of $f$. Theorem Ebelow provides a precise statement in the case of automorphisms; the most general results will be described in Chapter 14.

### 1.4. The dynamical spectrum of the algebraic torus

If $V$ is an algebraic variety, let $\operatorname{End}(V)$ be the semigroup of dominant endomorphisms of $V$. We define the dynamical spectrum of $V$ by

$$
\begin{equation*}
\Lambda(V):=\left\{\lambda_{1}(f): f \in \operatorname{End}(V)\right\} . \tag{9}
\end{equation*}
$$

As every $2 \times 2$ matrix with integer coefficients induces a monomial endomorphism of the algebraic torus $\mathbb{G}_{m}^{2}$, we have that $\Lambda\left(\mathbb{G}_{m}^{2}\right)$ is the set of Perron numbers of degree $\leqslant 2$. By Theorem A, this shows that $\Lambda\left(\mathbb{G}_{m}^{2}\right)$ is maximal among the dynamical spectra of normal affine surfaces. One might wonder if this is a characterization of the algebraic torus but we show that this is not the case.

## Theorem B. For any field $\mathbf{k}$,

$$
\begin{equation*}
\Lambda\left(\mathbf{A}_{\mathbf{k}}^{2}\right)=\Lambda\left(\mathbb{G}_{m, \mathbf{k}}^{2}\right) . \tag{10}
\end{equation*}
$$

### 1.5. Existence of an eigenvaluation

Let $A$ be the ring of regular functions of a normal affine surface $X_{0}$ over an algebraically closed field $\mathbf{k}$. A valuation is a map $v: A \rightarrow \mathbf{R} \cup\{\infty\}$ such that
(1) $v(P Q)=v(P)+v(Q)$;
(2) $v(P+Q) \geqslant \min (v(P), v(Q))$;
(3) $v(0)=\infty$;
(4) $v_{\mid \mathbf{k}}=0$

Two valuations $v$ and $\mu$ are equivalent if there exists $t>0$ such that $v=t \mu$. For example, if $X$ is a completion of $X_{0}$, for each irreducible curve $E \subset X$, the map $\operatorname{ord}_{E}$ defined by $\operatorname{ord}_{E}(P)$ being the order of vanishing of $P$ along $E$ is a valuation. Any valuation of the form $\lambda \operatorname{ord}_{E}$ with $\lambda>0$ is called divisorial. If $f$ is an endomorphism of $X_{0}$, then $f$
induces a ring homomorphism $f^{*}: A \rightarrow A$. We can then define the pushforward $f_{*} v$ of a valuation $v$ by

$$
\begin{equation*}
f_{*} v(P)=v\left(f^{*} P\right) . \tag{11}
\end{equation*}
$$

We say that a valuation is centered at infinity if there exists $P \in A$ such that $v(P)<0$. If $X$ is a completion of $X_{0}$, the divisorial valuations centered at infinity are exactly the one corresponding to the irreducible components of $X \backslash X_{0}$. Let $V_{\infty}$ the set of valuations centered at infinity and $\widehat{V}_{\infty}$ the set of valuations centered at infinity modulo equivalence. Suppose for the sake of simplicity that $f$ is an automorphism of $X_{0}$, then $f_{*}$ induces a bijection of $\mathcal{V}_{\infty}$ and of $\hat{\mathcal{V}}_{\infty}$ which will in fact be a homeomorphism for a topology that will be described in Chapter 4 .

If $X_{0}$ is the complex affine plane, then Favre and Jonsson proved the existence of a valuation $\nu_{*} \in \mathcal{V}_{\infty}$ such that $f_{*} \nu_{*}=\lambda_{1}(f) \nu_{*}$. Such a valuation is called an eigenvaluation of $f$. To do so, they show in [ [FJ04] that $\hat{V}_{\infty}$ has a real tree structure and $f_{*}$ is compatible with this structure. The existence of $\nu_{*}$ follows from a fixed point theorem on trees. The existence of this eigenvaluation has a big impact on the dynamics of $f$. In particular, it allows one to find a good completion $X$ of $\mathbf{C}^{2}$ and a point $q \in X \backslash \mathbf{C}^{2}$ (a point at infinity) which is an attracting fixed point for the dynamics of $f$ (extended to $X$ as a rational map). Xie uses this construction to prove the Zariski dense orbits conjecture and the dynamical Mordell-Lang conjecture for polynomial endomorphisms of the complex affine plane ([Xie17]). Jonsson and Wulcan use these techniques to build canonical heights for polynomial endomorphisms of the complex affine plane with small topological degree in [JW12].

THEOREM C. Let $X_{0}$ be a normal affine surface over an algebraically closed field $\mathbf{k}$ (of any characteristic) and let $f$ be a dominant endomorphism of $X_{0}$. Suppose that
(1) $\mathbf{k}\left[X_{0}\right]^{\times}=\mathbf{k}^{\times}$.
(2) For any completion $X$ of $X_{0}, \operatorname{Pic}^{0}(X)=0$.
(3) $\lambda_{1}(f)^{2}>\lambda_{2}(f)$.

Then, there exists a valuation $v_{*}$, unique up to equivalence, such that

$$
\begin{equation*}
f_{*}(v)=\lambda_{1}(f) v_{*} . \tag{12}
\end{equation*}
$$

The techniques we use do not use the global geometry of $\hat{\mathcal{V}}_{\infty}$ because it is not necessarily a tree anymore. If $X$ is a completion of $X_{0}$ and $v$ is a valuation centered at infinity, we associate in a canonical way a divisor $Z_{v, X}$ of $X$ supported outside of $X_{0}$. If $\pi: Y \rightarrow X$ is another completion of $X_{0}$ obtained from blowing up points of $X$ at infinity, and $\pi_{*} Z_{v, Y}=Z_{v, X}$ (see Proposition 7.6). This construction involves the Picard-Manin space of $X_{0}$. We give a brief description of this space. Consider the direct limit

$$
\begin{equation*}
\operatorname{Cartier}-\mathrm{NS}\left(\mathrm{X}_{0}\right)=\underset{X}{\lim } \mathrm{NS}(X)_{\mathbf{R}} \tag{13}
\end{equation*}
$$

indexed by all the completions of $X_{0}$. This is an infinite dimension real vector space. The intersection form can be extended in a natural way to Cartier-NS $\left(\mathrm{X}_{0}\right)$ and the Hodge Index theorem states that it is a non-degenerate form of Minkowski type, i.e its signature is $(1, \infty)$. We can use the intersection form to define a norm on Cartier-NS $\left(\mathrm{X}_{0}\right)$ and the Picard Manin space of $X_{0}$ will be the completion of Cartier-NS $\left(\mathrm{X}_{0}\right)$ with respect to this norm. It has a structure of a Hilbert space and any dominant endomorphism $f$ of $X_{0}$ induces two bounded operators $f^{*}, f_{*}$ on it. The spectral analysis of the operators $f_{*}, f^{*}$ (see [BFJ08, Can11]) allows one to construct the eigenvaluation $v_{*}$ and show its uniqueness. Namely, $\lambda_{1}(f)$ is the spectral radius of $f^{*}$ and $f_{*}$ and when $\lambda_{1}^{2}>\lambda_{2}$, there is a spectral gap property. The eigenvalue $\lambda_{1}$ is simple for $f^{*}$ and $f_{*}$. This process is similar to the techniques of [DF21] §6. These techniques were used by Gignac and Ruggiero in [GR21] to study the local dynamics of non invertible germs near a normal singularity in dimension 2 . This memoir can be considered to be the global counterpart to the local techniques developed by these two authors. Our construction of the valuation $v_{*}$ is however different.

### 1.6. Discussion of the assumptions of Theorem $\mathbf{C}$

The assumptions of Theorem C may seem arbitrary but they are not restrictive. Indeed, if assumption (1) or (2) is not satisfied, then one can show that $f$ preserves a fibration over a ${ }^{2}$ quasi-abelian variety. We can decompose the dynamics of $f$ with this fibration and it becomes easier to study. This is done in Chapter 10, we show that the only case with interesting dynamics is when $X_{0}$ is the algebraic torus $\mathbb{G}_{m}^{2}$.

THEOREM D. Let $X_{0}$ be a normal affine surface over an algebraically closed field. Suppose that $X_{0}$ does not satisfy Conditions (1) or (2) of Theorem C. then either
(1) $X_{0}$ is of log general type. Every dominant endomorphism of $X_{0}$ is an automorphism and $\operatorname{Aut}\left(X_{0}\right)$ is a finite group.
(2) There exists a curve $C$ and a regular map $\pi: X_{0} \rightarrow C$ and for every endomorphism $f$ of $X_{0}$, there exists an endomorphism $g: C \rightarrow C$ such that $\pi \circ f=g \circ \pi$. In that case $\lambda_{1}(f)$ is always an integer.
(3) $X_{0} \simeq \mathbb{G}_{m}^{2}$.

If Assumption (3) is not satisfied, then we have $\lambda_{1}(f)^{2}=\lambda_{2}(f)$. In that case, $\lambda_{1}(f)$ is automatically a Perron number of degree $\leqslant 2$ because $\lambda_{2}(f)$ is the topological degree of $f$, hence an integer. In the case of the complex affine plane, Favre and Jonsson manage to classify all polynomial transformations of the complex affine plane for which $\lambda_{1}^{2}=\lambda_{2}$ : either they preserve a rational fibration, or there exists a completion $X$ of $\mathbf{A}_{\mathbf{C}}^{2}$ with at most quotient singularities at infinity such that $f$ extends to an endomorphism

[^1]of $X$. We expect that such a classification should exist in general, with one exceptional counterexample: The monomial transformations of $\left(\mathbb{G}_{m}^{2}\right)$ that cannot be made algebraically stable (see [Fav03]). We conjecture that the only counterexamples of this classification should come from equivariant quotient of these monomial maps. In the local case of dyamics near a normal singularity Gignac and Ruggiero ([GR21]) showed such a classification. One can notice that in the invertible case, a classification exists: By [Giz69] and [Can01b], every birational transformation $\sigma: X \rightarrow X$ of a smooth projective surface such that $\lambda_{1}(\sigma)=1$ lifts to an automorphism or preserves a rational or elliptic fibration.

### 1.7. Statement of the theorem in the case of automorphisms

In the case of loxodromic automorphism (i.e with $\lambda_{1}>1$ ), we obtain informations on the dynamics. For this introduction we state the result in the complex case.

THEOREM E. Let $X_{0}$ be a normal affine surface over $\mathbf{C}$ such that $\mathbf{C}\left[X_{0}\right]^{\times}=\mathbf{C}^{\times}$. If $f$ is an automorphism of $X_{0}$ such that $\lambda_{1}(f)>1$, then there exists a completion $X$ of $X_{0}$ such that
(1) $f$ admits a unique attracting fixed point $p \in X(\mathbf{C}) \backslash X_{0}(\mathbf{C})$ at infinity.
(2) An iterate of $f$ contracts $X \backslash X_{0}$ to $p$.
(3) There exists local analytic coordinates centered at $p$ such that $f$ is locally of the form
(a)

$$
\begin{equation*}
f(z, w)=\left(z^{a} w^{b}, z^{c} w^{d}\right) \tag{14}
\end{equation*}
$$

with $a, b, c, d$ integers $\geqslant 1$, in that case $\lambda_{1}(f)$ is the spectral radius of $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$. In particular, $\lambda_{1}(f) \in \mathbf{R} \backslash \mathbf{Q}$, it is a Perron number of degree 2. (b) or

$$
\begin{equation*}
f(z, w)=\left(z^{a}, \lambda z^{c} w+P(z)\right) \tag{15}
\end{equation*}
$$

with $a \geqslant 2, c \geqslant 1$ and $P \not \equiv 0$ a polynomial, in that case $\lambda_{1}(f)=a$ is an integer.
(4) The attracting fixed points of $f$ and $f^{-1}$ are distinct.
(5) The local normal form of $f^{-1}$ at its attracting fixed point is the same as $f$.

TheoremEholds in fact for any complete algebraically closed field (in any characteristic) but we cannot be as precise with local normal forms in general, see Theorem 12.1 and 14.4

The cases (3)(a) et (3)(b) are mutually exclusive in the following way
THEOREM F. Let $X_{0}$ be a normal affine surface over a field $\mathbf{k}$ such that $\mathbf{k}\left[X_{0}\right]^{\times}=\mathbf{k}^{\times}$ and $f \in \operatorname{Aut}\left(X_{0}\right)$ a loxodromic automorphism. We have the following dichotomy

- If $\lambda_{1}(f) \in \mathbf{Z}_{\geqslant 0}$, then for any loxodromic automorphism $g$ of $X_{0}$, we have $\lambda_{1}(g) \in \mathbf{Z}_{\geqslant 0}$ and the local normal form of $g$ at its attracting fixed point is given by (15).
- If $\lambda_{1}(f) \notin \mathbf{Z}_{\geqslant 0}$ then it is a Perron number of degree 2 and this holds for any loxodromic automorphism $g$ of $X_{0}$. In particular, the local normal form of $g$ at its attracting fixed point is given by (14).

Plan of the memoir. This memoir is divided into two parts. In the first part we establish the main definitions and results needed for the proofs of the theorems stated in this introduction. In Chapter 3, we define completions of an affine surface and introduce the Picard Manin space of an affine surface. In Chapter 46, we define valuations and explain the geometry of the space of valuations centered at infinity of an affine surface. The main result of this part is that a valuation induces a linear form with special properties on the space of divisors at infinity and that this process is bijective. This is the goal of Chapters 7.9 .

The second part is dedicated to the proofs of the theorems of this introduction using the results established in the first part. We construct the eigenvaluation and prove Theorems $A, B$ and Cin Chapters 10 and 11. In Chapter 12, we show that the eigenvaluation constructed is an attracting fixed point in the space of valuations and derive results on the dynamics at infinity of our endomorphisms. We study examples in Chapter 13. For examples over the complex affine plane, we refer to [FJ07] and [FJ11]. For examples of affine surfaces with interesting automorphisms group, we refer to [BD13]. Theorems E and Fare proven in Chapter 14, where we apply the techniques of Chapter 12 ,

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## Part 1

## Valuations and Algebraic geometry

## CHAPTER 2

## Results from Algebraic Geometry

In this chapter, we recall several results from Algebraic Geometry that will be used throughout this memoir. Let $\mathbf{k}$ be an algebraically closed field. A variety is an integral scheme of finite type over $\mathbf{k}$. A surface is a variety of dimension 2. An affine variety over $\mathbf{k}$ is a variety $X_{0}=\operatorname{Spec} A$ with $A$ a finitely generated $\mathbf{k}$-algebra. We will denote by $\mathbf{k}\left[X_{0}\right]$ the ring of regular functions of the affine variety $X_{0}$.

### 2.1. Bertini

THEOREM 2.1 (Bertini's Theorem, [Har77]). Let $X \subset \mathbf{P}^{N}$ be a smooth quasiprojective variety over an algebraically closed field $\mathbf{k}$. The set of hyperplanes $H$ of $\mathbf{P}^{N}$ such that the intersection $H \cap X$ is a smooth irreducible subvariety of $X$ is a dense open subset of $\mathbf{P} \Gamma\left(\mathbf{P}^{N}, O(1)\right)$.

### 2.2. Local power series and local coordinates

Let $X$ be a variety and $x \in X$ a closed point. We will write $O_{X, x}$ for the ring of germs of regular functions at $x$. A regular sequence of $O_{X, x}$ is a sequence $t_{1}, \cdots, t_{r} \in O_{X, x}$ such that $t_{1}$ is not a zero divisor in $O_{X, x}$ and for all $i \geqslant 2, t_{i}$ is not a zero divisor in $O_{X, x} /\left(t_{1}, \cdots, t_{i-1}\right)$ (see [Har77] p.184). The point $x$ is regular if the local ring $O_{X, x}$ is regular, i.e there exists a regular sequence of length $\operatorname{dim} O_{X, x}$.

THEOREM 2.2 ([Har77], Theorem 5.5A). Let $R$ be a regular local k-algebra of dimension $n$ with maximal ideal $\mathfrak{m}$, then the completion of $R$ with respect to the $\mathfrak{m}$-adic topology is isomorphic to $\mathbf{k}\left[\left[t_{1}, \cdots, t_{n}\right]\right]$ where $\left(t_{1}, \cdots, t_{n}\right)$ is a regular sequence of $R$.

Let $X$ be a surface and $x$ a regular point of $X$. Then, we will say that $(z, w)$ are local coordinates at $x$ if $(z, w)$ is a regular sequence of $O_{X, x}$. If $(z, w)$ is a regular sequence of the completion $\widehat{O_{X, x}}$ we will say that they are local formal coordinates. By Theorem 2.2, $\widehat{O_{X, x}}$ is isomorphic to $\mathbf{k}[[z, w]]$. Finally, If $\mathbf{k}=\mathbf{C}_{v}$, is a complete algebraically closed field of any characteristic, we consider the local ring of germs of holomorphic functions at $x$, this is the subring of $\widehat{O_{X, x}}$ of power series with a positive radius of convergence. We denote it by $O_{X, x}^{h o l}$ it is also a local ring of dimension 2, if $(z, w)$ is a regular sequence of $O_{X, x}^{h o l}$, we say that $(z, w)$ are local analytic coordinates. If $E, F$ are two germs of
reduced irreducible curves at $x$ (algebraic, analytic of formal) we will say that $(z, w)$ are associated to $(E, F)$ if $z=0$ is a local equation of $E$ and $w=0$ is a local equation of $F$.

### 2.3. Boundary

Proposition 2.3 ([G0069], Proposition 1 and 2). Let $X_{0}$ be an affine variety and let $\mathrm{t}: X_{0} \hookrightarrow X$ be an open embedding into a projective variety, then the subvariety $X \backslash X_{0}$ is connected and of pure codimension 1 .

Set

$$
\begin{equation*}
\partial_{X} X_{0}:=X \backslash X_{0} \tag{16}
\end{equation*}
$$

we call it the boundary of $X_{0}$ in $X$; by Proposition 2.3 it is a curve when $X_{0}$ is a surface.
THEOREM 2.4 ([G0069]). Let $X$ be a normal proper surface and $U$ an open dense affine subset of $X$ (that is an open dense subset of $X$ that is also an affine variety) such that $V:=X \backslash U$ is locally factorial (each local ring is a unique factorization domain), then there exists an ample divisor $H$ on $X$ such that $\operatorname{Supp} H=V$.

In fact, Goodman shows that Theorem 2.4 holds in higher dimension with the only difference that you may need to do some blow-ups at infinity to find an ample divisor.

### 2.4. Surfaces

THEOREM 2.5 ([山ムar77] Proposition 5.3). Let $g: S_{1} \rightarrow S_{2}$ be a birational morphism between smooth projective surfaces. Then, $g$ is a composition of blow-ups of points and of an automorphism of $S_{2}$. Furthermore, if $h: S_{1} \rightarrow S_{2}$ is a birational map, then there exists a sequence of blow-ups $\pi: S_{3} \rightarrow S_{1}$ such that $h \pi: S_{3} \rightarrow S_{2}$ is regular and $S_{3}$ can be chosen minimal for this property.

Proposition 2.6. Let $g: S_{1} \rightarrow S_{2}$ be a birational map. Let $\pi: S_{3} \rightarrow S_{1}$ be a minimal resolution of indeterminacies of $g$ such that the lift $h: S_{3} \rightarrow S_{2}$ of $g$ is regular. Then, the first curve contracted by $h$ must be the strict transform of a curve in $S_{1}$.

Recall the Castelnuovo criterion
THEOREM 2.7 ([Har77] Theorem V.5.7). Let C be a curve in a projective surface $S$ such that $C \simeq \mathbf{P}^{1}$ and $C^{2}=-1$, then there exists a projective surface $S^{\prime}$, a birational morphism $\pi: S \rightarrow S^{\prime}$ and a point $p \in S^{\prime}$ such that $S$ is isomorphic via $\pi$ to the blow up of $p$ and $C$ is the exceptional divisor under this isomorphism.

We will use these results for the study of automorphisms of affine surfaces as they induce birational maps. Understanding the combinatorics of the blow ups and contractions induced by the automorphism will allow us to understand their dynamics.

Our work relies heavily on the elimination of indeterminacies for rational morphism. Since we are in dimension 2, it exists in any characteristic.

THEOREM 2.8. Let $f: S_{1} \rightarrow S_{2}$ be a dominant rational morphism between projective varieties over an algebraically closed field of any characteristic, then there exists a sequence of blow-ups $\pi: S \rightarrow S_{1}$ such that $f \circ \pi: S \rightarrow S_{2}$ is regular.

THEOREM 2.9 ([|Cut02], $[\mathbf{C P 0 0}])$. Suppose char $\mathbf{k}=0$. Let $f: S \rightarrow S^{\prime}$ be a dominant rational map between normal projective surfaces over $\mathbf{k}$. There exists blow ups $S_{1} \rightarrow S$ and $S_{1}^{\prime} \rightarrow S^{\prime}$ such that the lift $\widehat{f}: S_{1} \rightarrow S_{1}^{\prime}$ is monomial at every point. Meaning that for every closed point $p \in S_{1}$ there exists local coordinates $(x, y)$ at $p$ and local coordinates $(u, v)$ at $f(p)$ such that $f(x, y)=\left(x^{a} y^{b}, x^{c} y^{d}\right)$.

If char $\mathbf{k}>0$ the result remains true if $f$ is separable and tamely ramified (see $\$ 4.5$ ) in the sense of $[\mathbf{C P 0 0}]$. In particular, it holds if $f$ is birational.

### 2.5. Rigid contracting germs in dimension 2 and local normal forms

Let $\mathbf{k}$ be a complete algebraically closed field (of any characteristic). Let $f$ : $\left(\mathbf{k}^{2}, 0\right) \rightarrow\left(\mathbf{k}^{2}, 0\right)$ be the germ of a regular function fixing the origin. The critical set $\operatorname{Crit}(f)$ of $f$ is the set where the Jacobian of $f$ vanishes. A germ is said to be rigid if the generalized critical set $\cup_{n \geqslant 0} f^{-n}(\operatorname{Crit}(f))=\cup_{n \geqslant 1} \operatorname{Crit}\left(f^{n}\right)$ is a divisor with simple normal crossings (see [Fav00]).

A germ is contracting if there exists an open (euclidian) neighbourhood $U$ of 0 such that $f(U) \Subset U$. In [Fav00], Favre classified all the complex rigid contracting germs in dimension 2 up to holomorphic conjugacy. Ruggiero extended the classification to any dimension in [Rug13] and showed that it holds over any algebraically closed complete metrized field of characteristic zero. For this section, $\mathbf{C}$ will denote any algebraically closed complete field of characteristic zero. In dimension 2, there are 7 possible possibilities which we call local normal forms. We are interested in 3 of them that will appear in this text. However since we do not only work in characteristic zero, we start by more general local forms that works over any field and show their complex counterpart.

First normal form.- Suppose that there are local coordinates $(z, w)$ at the origin such that $f$ contracts $\{z=0\}$ with an index of ramification $a \geqslant 2, f$ admits no invariant curves and no other curves is contracted to the origin, then $f$ is of the form

$$
\begin{equation*}
f(z, w)=\left(z^{a} \varphi(z, w), z^{c} w \psi_{2}(z, w)+\psi_{1}(z)\right) \tag{17}
\end{equation*}
$$

with $\varphi$ invertible, $\psi_{1}(z) \neq 0$ and $\psi_{2}(0, w) \neq 0$. If $\mathbf{k}=\mathbf{C}$, then in the classification of Favre that this local normal form corresponds to Class 2 of Table II in [Fav00] and it is analytically conjugated to

$$
\begin{equation*}
f(x, y)=\left(x^{a}, \lambda x^{c} y+P(x)\right) \tag{18}
\end{equation*}
$$

with $a \geqslant 2, c \geqslant 1, \lambda \in \mathbf{k}^{\times}$and $P$ is a polynomial such that $P(0)=0$. This is the local normal form of a Hénon map at its attracting fixed point in $\mathbf{P}^{2}$ (see [Fav00] §2).

Second normal form.- If $f$ is a germ of a regular function such that there exists local coordinates $(z, w)$ at the origin with both axis $\{z=0\}$ and $\{w=0\}$ contracted and they are the only two germs of curves contracted. Then, $f$ is of the following pseudomonomial form

$$
\begin{equation*}
f(z, w)=\left(z^{a_{11}} w^{a_{12}} \varphi(z, w), z^{a_{21}} w^{a_{22}} \psi(z, w)\right) \tag{19}
\end{equation*}
$$

with $\varphi, \psi$ invertible and $a_{i j} \in \mathbf{Z}_{\geqslant 0}$. Suppose furthermore that $a_{11} a_{22}-a_{12} a_{21}$ is not divisible by chark. In particular, $a d-b c \neq 0$, then (19) is analytically conjugated to the monomial normal form

$$
\begin{equation*}
f(x, y)=\left(x^{a_{11}} y^{a_{12}}, x^{a_{21}} y^{a_{22}}\right) \tag{20}
\end{equation*}
$$

The germ of curves $\{x=0\},\{y=0\}$ are contracted to the origin. We have $\operatorname{Crit}\left(f^{n}\right)=$ $\{x y=0\}$. If $\mathbf{k}=\mathbf{C}$, we can characterize the matrix $A$ given by $\left(a_{i j}\right)$ in the following way. The local fundamental group of $\left(\mathbf{C}^{2}, 0\right) \backslash\{x y=0\}$ is isomorphic to $\mathbf{Z}^{2}$. The action of $f_{*}$ on $\mathbf{Z}^{2}$ is given by the matrix $A$ and we have that $|\operatorname{det} A|$ is equal to the topological degree of $f$. This corresponds to Class 6 of Table II of [Fav00].

Third normal form.- The third one is

$$
\begin{equation*}
f(x, y)=\left(x^{a} y^{b} \varphi, y \psi\right) \tag{21}
\end{equation*}
$$

with $a \geqslant 2, b \geqslant 1$ and $\varphi, \psi$ are germs of invertible regular functions vanishing at the origin. We have that $\{y=0\}$ is contracted to the origin. The germ $\{x=0\}$ is $f$-invariant with a ramification index equal to $a$. We have $\operatorname{Crit}\left(f^{n}\right)=\{x y=0\}$ and the origin is a noncritical fixed point of $f_{\mid\{x=0\}}$. Notice that this germ is rigid but not necessarily contracting. It is contracting if and only if $|\psi(0)|<1$. If the germ is contracting and $\mathbf{k}=\mathbf{C}$, then the germ is analytically conjugated to this normal form

$$
\begin{equation*}
f(z, w)=\left(z^{a} w^{b}, \psi(0) w\right) \tag{22}
\end{equation*}
$$

with the same numbers $a, b$ as in Equation 21. This corresponds to Class 5 of Table II in [Fav00].

## CHAPTER 3

## Divisors at infinity and Picard-Manin space

In this chapter, we introduce the notion of completions of an affine surface $X_{0}$. They are essentialy projective compatifications of $X_{0}$ and form a projective set. The PicardManin space of $X_{0}$ will be a completion of the direct limit of the Néron Sévéri groups of the completions of $X_{0}$. It is a Hilbert space on which every endomorphism of $X_{0}$ acts in a natural way. Let $\mathbf{k}$ be an algebraically closed field of any characteristic and let $X_{0}$ be a normal affine surface over $\mathbf{k}$. We will denote by $\mathbf{k}\left[X_{0}\right]$ the ring of regular functions on $X_{0}$.

### 3.1. Completions and divisors at infinity

A completion of $X_{0}$ is the data of a projective surface $X$ with an open embedding $\mathfrak{\imath}: X_{0} \hookrightarrow X$ such that $\mathfrak{l}\left(X_{0}\right)$ is an open dense subset of $X$ and such that there exists an open smooth neighbourhood of $\partial_{X} X_{0}$ in $X$. We will say that a completion is good if $\partial_{X} X_{0}$ is an effective divisor with simple normal crossings. From any completion of $X$, one obtains a good one by a finite number of blow ups at infinity (i.e on $\partial_{X} X_{0}$ ) see for example [Har77] Theorem 3.9 p. 391.

Let $X$ be a completion of $X_{0}$ with the embedding $\mathfrak{l}_{X}: X_{0} \rightarrow X$, we will still denote $\mathbf{1}_{X}\left(X_{0}\right)$ by $X_{0}$ and we will denote by $O_{X}\left(X_{0}\right)$ the subring of $\mathbf{k}(X)$ of functions $f \in \mathbf{k}(X)$ which are regular on $X_{0}$. By Proposition 2.3, the boundary $\partial_{X} X_{0}$ is a possibly reducible connected curve. We denote by $\operatorname{Div}(X)$ the group of divisors of $X$ and by $\operatorname{Div}_{\infty}(X)$ the subgroup of divisors of $X$ supported on $\partial_{X} X_{0}$. For $\mathbf{A}=\mathbf{Z}, \mathbf{Q}, \mathbf{R}$, we set $\operatorname{Div}(X)_{\mathbf{A}}:=\operatorname{Div}(X) \otimes \mathbf{A}$ and $\operatorname{Div}_{\infty}(X)_{\mathbf{A}}=\operatorname{Div}_{\infty}(X) \otimes \mathbf{A}$. Let $E_{1}, \cdots, E_{m}$ be the irreducible components of $\partial_{X} X_{0}$ (we will call them the prime divisors at infinity). Any element of $\operatorname{Div}_{\infty}(X)_{\mathbf{A}}$ is of the form $D=\sum_{i} a_{i}(D) E_{i}$ with $a_{i}(D) \in \mathbf{A}$. We will write $\operatorname{ord}_{E_{i}}(D)$ for $a_{i}(D)$ of $D$ at $E_{i}$. For a family $\left(D_{j}\right)_{j \in J}$ of elements of $\operatorname{Div}_{\infty}(X)$ the coefficients $a_{i}(D)$ are integers; so, using the natural order on $\mathbf{Z}$, we define the supremum $\bigvee_{j \in J} D_{j}$ and the infimum $\bigwedge_{j \in J} D_{j}$ by

$$
\begin{equation*}
\bigvee_{j} D_{j}=\sum_{i} \sup \left(\operatorname{ord}_{E_{i}}\left(D_{j}\right)\right) E_{i} \quad \text { and } \quad \bigwedge_{j} D_{j}=\sum_{i} \inf \left(\operatorname{ord}_{E_{i}}\left(D_{j}\right)\right) E_{i} \tag{23}
\end{equation*}
$$

It only exists if each $\left(\operatorname{ord}_{E_{i}}\left(D_{j}\right)\right)_{j \in J}$ is bounded respectively from above or from below. If $\bigwedge_{j} D_{j}$ (respectively $\bigvee_{j} D_{j}$ ) is well defined we say that the family $\left(D_{j}\right)$ is bounded from below (from above). Notice that we only define supremum and infimum for family of divisors with coefficients in $\mathbf{Z}$.

### 3.2. Morphisms between completions, Weil, Cartier divisors

Some notations. If $\pi: Y \rightarrow X$ is a projective birational morphism between smooth projective surfaces and $D_{X}$ is a divisor on $X$, we will denote by $\pi^{*} D_{X}$ the pull-back of $D_{X}$ under $\pi$ and if $D_{X}$ is effective, then $\pi^{\prime}\left(D_{X}\right)$ will be the strict transform of $D_{X}$ under $\pi$. For any projective surface $Z$, if $D_{Z}$ is a divisor on $Z$, we will denote by $O_{Z}\left(D_{Z}\right)$ the invertible sheaf on $Z$ associated to $D_{Z}$.

Let $X_{1}, X_{2}$ be two completions of $X_{0}$ with their embeddings $\mathrm{l}_{1}, \mathrm{l}_{2}$. There exists a unique birational map $\pi: X_{1} \rightarrow X_{2}$ such that the diagram

commutes. If $\pi$ is a morphism, we call it a morphism of completions. In that case we say that $X_{1}$ is above $X_{2}$. By Theorem 2.5, $\pi^{-1}$ is a composition of blow-ups; since $\pi$ is an isomorphism over $X_{0}$, the centers of these blowups are above $\partial_{X_{2}} X_{0}$. Conversely, let $X$ be a completion of $X_{0}$ with an embedding $1: X_{0} \hookrightarrow X$, let $\pi: Y \rightarrow X$ be the blowup of $X$ at a point $p \in \partial_{X} X_{0}$, then $Y$ with the embedding $\pi^{-1} \circ \mathfrak{\imath}: X_{0} \rightarrow Y$ is a completion of $X_{0}$ and $\pi$ is a morphism of completions. For a morphism of completions $\pi: Y \rightarrow X$, we will write $\operatorname{Exc}(\pi) \subset Y$ for the exceptional locus of $\pi$.

Lemma 3.1. The system of completions of $X_{0}$ is a projective system: For any two completions $X_{1}, X_{2}$ of $X_{0}$ there exists a completion $X_{3}$ above $X_{1}$ and $X_{2}$.

Proof. Let $X_{1}, X_{2}$ be two completions of $X_{0}$, let $\pi: X_{1} \rightarrow X_{2}$ be the birational map from Diagram 24, By Theorem 2.5, there exists a sequence of blow-ups $\pi_{1}: X_{3} \rightarrow X_{1}$ such that $g=\pi_{1} \circ \pi: X_{3} \rightarrow X_{2}$ is regular. It is clear that $\pi_{1}$ is a morphism of completions since by definition $\mathfrak{l}_{X_{3}}=: \mathfrak{l}_{3}=\mathfrak{l}_{1} \circ \pi_{1}{ }^{-}$. The map $g$ is also a morphism of completion because by construction $g=\pi \circ \pi_{1}$ and $\mathfrak{l}_{2}=\pi \circ \mathfrak{l}_{1}$, therefore $\mathfrak{l}_{3}=\pi_{1}^{-1} \circ \mathfrak{l}_{1}=g^{-1} \circ \pi \circ$ $\mathfrak{l}_{1}=g^{-1} \circ \boldsymbol{l}_{2}$

If $\pi: X_{1} \rightarrow X_{2}$ is a morphism of completions. We can define (see [Ful98], Section 1.4) the pushforward $\pi_{*}: \operatorname{Div}\left(X_{1}\right)_{\mathbf{A}} \rightarrow \operatorname{Div}\left(X_{2}\right)_{\mathbf{A}}$ and pullback $\pi^{*}: \operatorname{Div}\left(X_{2}\right)_{\mathbf{A}} \rightarrow$ $\operatorname{Div}\left(X_{1}\right)_{\mathbf{A}}$ of divisors. They define group homomorphisms

$$
\begin{equation*}
\pi_{*}: \operatorname{Div}_{\infty}\left(X_{1}\right)_{\mathbf{A}} \rightarrow \operatorname{Div}_{\infty}\left(X_{2}\right)_{\mathbf{A}} \quad \text { and } \quad \pi^{*}: \operatorname{Div}_{\infty}\left(X_{2}\right)_{\mathbf{A}} \hookrightarrow \operatorname{Div}_{\infty}\left(X_{1}\right)_{\mathbf{A}} \tag{25}
\end{equation*}
$$

the map $\pi^{*}$ is often called the total transform. Recall that ([Har77] Proposition 3.2 p.386)

$$
\begin{equation*}
\pi_{*} \pi^{*}=\mathrm{id}_{\operatorname{Div}\left(X_{2}\right)_{\mathbf{A}}} \tag{26}
\end{equation*}
$$

Let $X$ be a completion of $X_{0}$ and $P \in \mathbf{k}\left[X_{0}\right]$, then $\left(\mathbf{l}_{X}{ }^{-1}\right)^{*}(P) \in \mathbf{k}(X)$. We set $\left(\mathfrak{l}_{X}\right)_{*}:=\left(\mathfrak{l}_{X}{ }^{-1}\right)^{*}$ and we denote by $\operatorname{div}_{X}(P):=\operatorname{div}\left(\left(\mathfrak{l}_{X}\right)_{*} P\right)$ the divisor of the rational function $P$ in $X$. In particular, if $\pi: Y \rightarrow X$ is a morphism of completions above $X_{0}$, then by Diagram (24), one has $\mathfrak{l}_{Y}=\pi^{-1} \circ \mathfrak{l}_{X}$. Therefore $\operatorname{div}_{Y}(P)=\operatorname{div}\left(\left(\pi^{-1} \circ \mathfrak{l}_{X}\right)_{*}(P)\right)=$ $\operatorname{div}\left(\pi^{*}\left(\left(\mathfrak{l}_{X}\right)_{*}(P)\right)\right)=\pi^{*} \operatorname{div}_{X}(P)$. We will write $\operatorname{div}_{\infty, X}(P) \in \operatorname{Div}_{\infty}(X)$ the divisor on $X$ supported at infinity such that

$$
\operatorname{div}_{X}(P)=D+\operatorname{div}_{\infty, X}(P)
$$

where $D$ is an effective divisor and no components of its support is in $\partial_{X} X_{0}$.
Example 3.2. Let $X_{0}=\mathbf{A}^{2}=\operatorname{Spec} \mathbf{k}[x, y]$ and let $P=x y$. Take the completion $\mathbf{P}^{2}$ of $\mathbf{A}^{2}$ with homogeneous coordinates $X, Y, Z$ such that $x=X / Y$ and $y=Y / Z$. Then,

$$
\begin{equation*}
\operatorname{div}_{\mathbf{P}^{2}}(P)=\{X=0\}+\{Y=0\}-2\{Z=0\} \tag{27}
\end{equation*}
$$

and $\operatorname{div}_{\infty, \mathbf{P}^{2}}(P)=-2\{Z=0\}$. Let $\pi: X \rightarrow \mathbf{P}^{2}$ be the blow-up of $[1: 0: 0]$, we can take $W$ to be the subscheme of $\mathbf{P}^{2} \times \mathbf{P}^{1}$ given by the equation

$$
\begin{equation*}
U Z=V Y \tag{28}
\end{equation*}
$$

where $U, V$ are the homogeneous coordinates of $\mathbf{P}^{1}$. Then $\pi$ is the projection onto the first factor. We take the affine chart $X=1$ in $\mathbf{P}^{2}$ with affine coordinates $y^{\prime}=Y / X$ and $z^{\prime}=Z / X$. Take the chart $U=1$ with affine coordinate $v$ in $\mathbf{P}^{1}$, then $W \cap\{X=1\} \times$ $\{U=1\}$ is an affine chart of $W$ with coordinates $v, y^{\prime}$ and we have the relation $z^{\prime}=v y^{\prime}$; $y^{\prime}=0$ is a local equation of the exceptional divisor and $v=0$ is a local equation of the strict transform of $z^{\prime}=0$.

$$
\begin{equation*}
\pi^{*}(P)=\pi^{*}\left(\frac{y^{\prime}}{\left(z^{\prime}\right)^{2}}\right)=\frac{y^{\prime}}{v^{2}\left(y^{\prime}\right)^{2}}=\frac{1}{v^{2} y^{\prime}} \tag{29}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\operatorname{div}_{X}(P)=\pi^{\prime}\{X=0\}+\pi^{\prime}\{Y=0\}-2 \pi^{\prime}\{Z=0\}-\widetilde{E}=\pi^{*}\left(\operatorname{div}_{\mathbf{P}^{2}}(P)\right) \tag{30}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{div}_{\infty, X}(P)=-2 \pi^{\prime}\{Z=0\}-\widetilde{E} \tag{31}
\end{equation*}
$$

The system of completions of $X_{0}$ is a projective system by Lemma 3.1. Consider the system of groups $\left(\operatorname{Div}_{\infty}(X)\right)_{\mathbf{A}}$ for $X$ a completion of $X_{0}$ with compatibility morphisms

$$
\begin{equation*}
\pi_{*}: \operatorname{Div}_{\infty}(X) \rightarrow \operatorname{Div}_{\infty}(Y) \tag{32}
\end{equation*}
$$

for any morphism of completions $\pi: X \rightarrow Y$. This is a projective system of groups. Analogously, the same system of groups with $\pi^{*}$ as compatibility morphisms is an inductive system. We define the space of Cartier and Weil divisors at infinity of $X_{0}$ by

$$
\begin{equation*}
\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}=\underset{X}{\lim _{\overleftrightarrow{X}}} \operatorname{Div}_{\infty}(X)_{\mathbf{A}}, \text { and } \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}={\underset{X}{\lim }}_{\operatorname{liv}}^{\infty}(X)_{\mathbf{A}} . \tag{33}
\end{equation*}
$$

Concretely, an element $D \in \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}$ is a collection $D=\left(D_{X}\right)$ such that $D_{X}$ is an element of $\operatorname{Div}_{\infty}(X)_{\mathbf{A}}$ for every completion $X$ of $X_{0}$ and such that for any morphism of completions $\pi: X \rightarrow Y, \pi_{*} D_{X}=D_{Y} ; D_{X}$ is called the incarnation of $D$ in $X$. An element of $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}$ is the data of a completion $X$ and a divisor $D \in \operatorname{Div}_{\infty}(X)$ where two pairs $(X, D)$ and $\left(X^{\prime}, D^{\prime}\right)$ are equivalent if there exists a completion $Z$ above $X$ and $X^{\prime}$ with morphisms of completion $\pi: Z \rightarrow X, \pi^{\prime}: Z \rightarrow X^{\prime}$ such that $\pi^{*} D=\left(\pi^{\prime}\right)^{*} D^{\prime}$. We will say that $D \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}$ is defined over a completion $X$ if $D$ is the equivalence class of $\left(X, D_{X}\right)$ for some $D_{X} \in \operatorname{Div}_{\infty}(X)_{\mathbf{A}}$. We have a natural inclusion

$$
\begin{equation*}
\varphi: \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}} \hookrightarrow \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}} \tag{34}
\end{equation*}
$$

defined as follows. If $(X, D) \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}$, then we need to define the incarnation $\varphi(D)_{Y}$ for any completion $Y$. First of all, set $\varphi(D)_{X}=D$. Then, for any completion $Y$, by Lemma 3.1, there exists a completion $Z$ above $Y$ and $X$; denote by $\pi_{Y}: Z \rightarrow Y$ and $\pi_{Z}: Z \rightarrow X$ the respective morphism of completions. We define $\varphi(D)_{Y}:=\left(\pi_{Y}\right)_{*} \pi_{X}^{*} D$. This does not depend on the choice of $Z$ because of Equation (26). In the rest of the paper, we will drop the notation $\varphi(D)$ and denote by $D$ the image of $(X, D)$ in Weil $_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}$. We equip Weil ${ }_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}$ with the projective limit topology.

In the same manner we define $\operatorname{Cartier}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}:=\underset{\longrightarrow}{\lim } \operatorname{Div}(X)_{\mathbf{A}}$ and $\operatorname{Weil}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}:=$ $l_{\leftrightarrows} \operatorname{Div}(X)_{\mathbf{A}}$ and we have the following commutative diagram


REmARK 3.3. We have that $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}=\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right) \otimes \mathbf{A}$ but $\operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}$ is strictly larger than Weil $\infty_{\infty}\left(\mathrm{X}_{0}\right) \otimes \mathbf{A}$ when $\mathbf{A}=\mathbf{Q}, \mathbf{R}$. Indeed, let $W_{1}, \ldots, W_{r} \in \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)$, $\lambda_{1}, \ldots, \lambda_{r} \in \mathbf{A}$ and set $W:=\sum_{i} \lambda_{i} W_{i}$. Then, for every completion $X$ and for every prime divisor $E$ at infinity in $X$ we have

$$
\begin{equation*}
\operatorname{ord}_{E}\left(W_{X}\right)=\operatorname{ord}_{E}\left(\sum_{i} \lambda_{i} W_{i, X}\right)=\sum_{i} \lambda_{i} \operatorname{ord}_{E}\left(W_{i, X}\right) \in \mathbf{Z} \lambda_{1}+\cdots+\mathbf{Z} \lambda_{r} \tag{36}
\end{equation*}
$$

In particular, the group $G(W)$ generated by $\left(\operatorname{ord}_{E}\left(W_{X}\right)\right)_{(X, E)}$ for all completions $X$ and all prime divisor $E$ at infinity in $X$ is a finitely generated subgroup of $\mathbf{R}$. Now pick a completion $X_{1}$ and consider a sequence of blow ups $\pi_{n}: X_{n+1} \rightarrow X_{n}$ starting with $X_{1}$. Let
$E_{n}$ be the exceptional divisor of $\pi_{n}$. We still denote by $E_{n}$ the strict transform of $E_{n}$ in every $X_{m}, m \geqslant n+1$. Define the Weil divisor $W \in \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}$ such that its incarnation in $X_{n+1}$ is $W_{X_{n+1}}=\sum_{k=1}^{n} \frac{1}{k} E_{k}$. Then, $G(W)$ is not finitely generated, therefore $W \notin$ $\operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right) \otimes \mathbf{A}$.

An element $D$ of $\mathrm{Weil}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}$ with $\mathbf{A}=\mathbf{Z}, \mathbf{Q}, \mathbf{R}$ is called effective (denoted by $D \geqslant$ 0 ) if its incarnation in every completion $X$ is effective; if $D$ belongs to $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$ this is equivalent to $D_{X} \geqslant 0$ for one completion $X$ where $D$ is defined. If $D_{1}, D_{2} \in$ $\operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}$, we will write $W_{1} \geqslant W_{2}$ for $W_{1}-W_{2} \geqslant 0$.

### 3.3. A canonical basis

Let $X$ be a completion of $X_{0}$, we define $\mathcal{D}_{X}$ as follows. Elements of $\mathcal{D}_{X}$ are equivalence classes of prime divisors exceptional above $X$ at infinity in completions $\pi_{Y}: Y \rightarrow X$ above $X$ where two prime divisors $E$ and $E^{\prime}$ belonging respectively to $Y$ and $Y^{\prime}$ are equivalent if the birational map $\pi_{Y^{\prime}}{ }^{-1} \circ \pi_{Y}: Y \rightarrow Y^{\prime}$ induces an isomorphism $\pi_{Y^{\prime}}{ }^{-1} \circ \pi_{Y}: E \rightarrow E^{\prime}$. We call $\mathcal{D}_{X}$ the set of prime divisors above $X$. We also define $\mathcal{D}_{\infty}\left(X_{0}\right)$ as the set of equivalence classes of prime divisors at infinity modulo the same equivalence relation. We write $\mathbf{A}^{\mathcal{D}_{X}}$ for the set of functions $\mathcal{D}_{X} \rightarrow \mathbf{A}$ and $\mathbf{A}^{\left(\mathcal{D}_{X}\right)}$ for the subset of functions with finite support.

Proposition 3.4. If $X$ is a completion of $X_{0}$, then

$$
\begin{equation*}
\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}=\operatorname{Div}_{\infty}(X)_{\mathbf{A}} \oplus \mathbf{A}^{\left(\mathcal{D}_{X}\right)}, \quad \text { and } \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}=\operatorname{Div}_{\infty}(X)_{\mathbf{A}} \oplus \mathbf{A}^{\mathcal{D}_{X}} . \tag{37}
\end{equation*}
$$

This is a homeomorphism with respect to the product topology of $\mathbf{A}^{\mathcal{D}_{X}}$.
Proof. Following [BFJ08] Proposition 1.4, for any $E \in \mathcal{D}_{X}$ there exists a minimal completion $X_{E}$ above $X$ such that $E$ is a prime divisor in $X_{E}$. We denote by $\alpha_{E} \in$ $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$ the element $\alpha_{E}:=\left(X_{E}, E\right)$. Let $E_{1}, \ldots, E_{r}$ be the prime divisor at infinity in $X$, then

$$
\begin{equation*}
\left(E_{0}, \ldots, E_{r}\right) \cup\left\{\alpha_{E}: E \in \mathcal{D}_{X}\right\} \tag{38}
\end{equation*}
$$

is a $\mathbf{A}$-basis of $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}$. In the same fashion we obtain the second homeomorphism.

REMARK 3.5. Since for any completion $X$, one can find a good completion $Y$ above $X$ and the blow up of a good completion is still a good completion, the projective system of good completions is cofinal in the projective system of completions, so in the rest of the paper any completion that we take will be a good completion.

If $f: X_{0} \rightarrow X_{0}$ is a dominant endomorphism, then we can define

$$
\begin{equation*}
f^{*}: \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}} \rightarrow \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}} \text { and } f_{*}: \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}} \rightarrow \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}} \tag{39}
\end{equation*}
$$

as follows. Let $D=\left(X, D_{X}\right) \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}$. Let $Y$ be a completion of $X_{0}$ such that the lift $F: Y \rightarrow X$ of $f$ is regular, then we define

$$
\begin{equation*}
f^{*} D:=\left(Y, F^{*} D_{X}\right) \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}} \tag{40}
\end{equation*}
$$

This does not depend on the choice of $Y$. If $D \in \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)$, let $X, Y$ be completions of $X_{0}$ such that the lift $F: Y \rightarrow X$ is regular, then

$$
\begin{equation*}
\left(f_{*} D\right)_{X}:=F_{*} D_{Y} \tag{41}
\end{equation*}
$$

Again, this does not depend on the choice of $Y$.

### 3.4. Local version of the canonical basis

Let $X$ be a completion and let $p \in X$ be a closed point at infinity i.e on $\partial_{X} X_{0}$. We denote by $\operatorname{Weil}(X, p)_{\mathbf{A}}$ the subspace of $\operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}$ defined as follows: $D \in \operatorname{Weil}(X, p)_{\mathbf{A}}$ if and only if $D_{X}=0$ and for every completion $\pi: Y \rightarrow X$ above $X$ and every prime divisor $E$ at infinity in $Y$, one has $E \in \operatorname{Supp} D_{Y}$ if and only if $\pi(E)=p$. We define

$$
\begin{equation*}
\operatorname{Cartier}(X, p)_{\mathbf{A}}=\operatorname{Weil}(X, p)_{\mathbf{A}} \cap \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{A}} . \tag{42}
\end{equation*}
$$

We can define the set $\mathcal{D}_{X, p}$ of prime divisors above $p$ as follows. We will say that a completion $\pi: Y \rightarrow X$ is exceptional above $p$ if $\pi(\operatorname{Exc}(\pi))=p$. We will write $\pi$ : $(Y, \operatorname{Exc}(\pi)) \rightarrow(X, p)$ for such a completion. Elements of $\mathcal{D}_{X, p}$ are equivalence classes of prime divisors $E \in \operatorname{Exc}(\pi)$ for all completions $\pi:(Y, \operatorname{Exc}(\pi)) \rightarrow(X, p)$.

Proposition 3.6. If $X$ is a completion of $X_{0}$, then $\mathcal{D}_{X}=\bigsqcup_{p \in \partial_{X} X_{0}} \mathcal{D}_{X, p}$ and

$$
\begin{align*}
\operatorname{Cartier}(X, p)_{\mathbf{A}} & =(\mathbf{A})^{\left(\mathcal{D}_{X, p}\right)}  \tag{43}\\
\operatorname{Weil}(X, p)_{\mathbf{A}} & =(\mathbf{A})^{\mathcal{D}_{X, p}} \tag{44}
\end{align*}
$$

### 3.5. Supremum and infimum of divisors

Let $\left(D_{i}\right)_{i \in I}$ be a family of elements of $\operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)$ such that for all completions $X$, the family $\left(D_{i, X}\right)$ is bounded from below, we define $\bigwedge_{i \in I} D_{i}$ with its incarnation in $X$ being

$$
\begin{equation*}
\left(\bigwedge D_{i}\right)_{X}=\bigwedge_{i} D_{i, X} \tag{45}
\end{equation*}
$$

We have an analogous definition for $\bigvee_{i} D_{i}$ when each $\left(D_{i, X}\right)$ is bounded from above.
Lemma 3.7. If $D, D^{\prime} \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$, then $D \wedge D^{\prime}, D \vee D^{\prime} \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$.
Proof. It suffices to show that $D \wedge D^{\prime} \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$ because $D \vee D^{\prime}=-(-D \wedge$ $\left.-D^{\prime}\right)$. So take $D, D^{\prime} \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$, we have to show that $D \wedge D^{\prime}$ belongs to $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$.

Now, it suffices to show this for $D, D^{\prime}$ effective, indeed let $X$ be a completion such that $D$ and $D^{\prime}$ are defined over $X$. Then, there exists $D_{2} \in \operatorname{Div}_{\infty}(X)$ such that $D-D_{2}$
and $D^{\prime}-D_{2}$ are effective. Indeed, take $D_{2}$ as the Cartier class determined by $D \wedge D^{\prime}$ in $X$, Then

$$
\begin{equation*}
D \wedge D^{\prime}=\left(D-D_{2}\right) \wedge\left(D^{\prime}-D_{2}\right)+D_{2} \tag{46}
\end{equation*}
$$

Therefore, suppose $D, D^{\prime}$ are effective. Then $\mathfrak{a}=O_{X}(-D)+O_{X}\left(-D^{\prime}\right)$ is a coherent sheaf of ideals such that $\mathfrak{a}_{\mid X_{0}}=O_{X_{0}}$, let $\pi: Y \rightarrow X$ be the blow-up along $\mathfrak{a}$. Since $\mathfrak{a}_{\mid X_{0}}$ is trivial, $\pi$ is an isomorphism over $X_{0}$, therefore $Y$ is a completion of $X_{0}$ with respect to the embedding $\mathfrak{l}_{Y}:=\pi^{-1} \circ \mathfrak{l}_{X}$ and $\pi$ is a morphism of completions. Then, $\mathfrak{b}:=\pi^{*} \mathfrak{a} \cdot O_{Y}$ is an invertible sheaf over $Y$ trivial over $X_{0}$, so there exists a divisor $D_{Y} \in \operatorname{Div}_{\infty}(Y)$ such that $\mathfrak{b}=O_{Y}\left(-D_{Y}\right)$.

Claim 3.8. The Cartier class in $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$ induced by $D_{Y}$ is $D \wedge D^{\prime}$.
We postpone the proof of this claim to the end of Chapter 7 , page 70 .
Example 3.9. Let $X$ be a completion that contains two prime divisors $E, E^{\prime}$ at infinity in $X$ such that they intersect (transversely) at a point $p$. The sheaf of ideals $\mathfrak{a}=O_{X}(-E)+O_{X}\left(-E^{\prime}\right)$ is the ideal of regular functions vanishing at $p$. The blow up of $\mathfrak{a}$ is exactly the blow up $\pi: Y \rightarrow X$ at $p$ since by universal property of the blow-up $\pi^{*} \mathfrak{a}=O_{Y}(-\widetilde{E})$ where $\widetilde{E}$ is the exceptional divisor above $p$. If we still denote by $E, E^{\prime}, \widetilde{E}$ the elements they define in $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$, then $E \wedge E^{\prime}=\widetilde{E}$.

Let $X$ be a good completion of $X_{0}$. Let $D_{1}, D_{2} \in \operatorname{Div}_{\infty}(X)$. Let $E, F$ be two prime divisors at infinity that intersect. We say that $\left(D_{1}, D_{2}\right)$ is well ordered at $E \cap F$ if

$$
\begin{equation*}
\operatorname{ord}_{E}\left(D_{1}\right)<\operatorname{ord}_{E}\left(D_{2}\right) \Leftrightarrow \operatorname{ord}_{F}\left(D_{1}\right)<\operatorname{ord}_{F}\left(D_{2}\right) \tag{47}
\end{equation*}
$$

We say that $\left(D_{1}, D_{2}\right)$ is a well ordered pair if it is well ordered at $E \cap F$ for every prime divisor $E, F$ at infinity that intersect.

Lemma 3.10. If $D_{1} \wedge D_{2}$ or $D_{1} \vee D_{2}$ is defined in $X$, then $\left(D_{1}, D_{2}\right)$ is a well ordered pair.

Proof. Suppose for example that $D_{1} \vee D_{2}$ is defined in $X$ and that $D_{1}, D_{2}$ is not a well ordered pair and let $E, F$ be two prime divisors at infinity that intersect such that at $E \cap F, D_{i}=\alpha_{i} E+\beta_{i} F$ with $\alpha_{1}<\alpha_{2}$ and $\beta_{1}>\beta_{2}$. Then, $D_{1} \vee D_{2}=\alpha_{2} E+\beta_{1} F$. Let $\widetilde{E}$ be the exceptional divisor above $E \cap F$, then we have $\operatorname{ord}_{\widetilde{E}}\left(D_{1} \vee D_{2}\right)=\alpha_{2}+\beta_{1}$. But

$$
\begin{equation*}
\operatorname{ord}_{\widetilde{E}} D_{i}=\alpha_{i}+\beta_{i}<\alpha_{2}+\beta_{1}=\operatorname{ord}_{\widetilde{E}}\left(D_{1} \vee D_{2}\right) \tag{48}
\end{equation*}
$$

This is a contradiction.
REMARK 3.11. This is actually an equivalence, if $D_{1}, D_{2}$ is a well ordered pair, then $D_{1} \wedge D_{2}$ and $D_{1} \vee D_{2}$ is defined in $X$. This gives an algorithmic procedure by successive blow ups to find the minimum and maximum of two Cartier divisors.

DEFINITION 3.12. Let $S_{\infty}\left(X_{0}\right)$ be the semigroup of Weil ${ }_{\infty}\left(\mathrm{X}_{0}\right)$ of elements $D \in$ Weil $\infty_{\infty}\left(\mathrm{X}_{0}\right)$ such that there exists a (potentially uncountable) family $\left(D_{i}\right)_{i \in I} \subset \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$ such that

$$
\begin{equation*}
D=\bigvee_{I} D_{i} \tag{49}
\end{equation*}
$$

Proposition 3.13. (1) $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right) \subset \mathcal{S}_{\infty}\left(X_{0}\right)$.
(2) For $a, b \geqslant 0$ and $D, D^{\prime} \in \mathcal{S}_{\infty}\left(X_{0}\right)$, one has $a D+b D^{\prime} \in \mathcal{S}_{\infty}\left(X_{0}\right)$.
(3) If $D_{i} \in \mathcal{S}_{\infty}\left(X_{0}\right)$ for each $i \in I$ and $\left(D_{i}\right)$ is bounded from above then $\bigvee_{i \in I} D_{i} \in$ $S_{\infty}\left(X_{0}\right)$.
(4) If $D, D^{\prime} \in S_{\infty}\left(X_{0}\right)$, then $D \wedge D^{\prime} \in \mathcal{S}_{\infty}\left(X_{0}\right)$.

Proof. The first assertion is trivial as for $D \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), D=\bigvee D$. For Property (2), let $X$ be a completion of $X_{0}$ then $\bigvee_{i} a D_{i, X}+\bigvee_{j} b D_{j, X}^{\prime}=\bigvee_{i, j}\left(a D_{i}+b D_{j}^{\prime}\right)_{X}$. For Property (3), if $D_{i}=\bigvee_{j} D_{i, j}$, then $\bigvee_{i} D_{i}=\bigvee_{(i, j)} D_{i, j}$. Finally, the fourth assertion is a corollary of Lemma 3.7.

Example 3.14. We have $\mathcal{S}_{\infty}\left(X_{0}\right) \nsubseteq \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)$. Let $X_{0}=\mathbf{A}^{2}$ and $X=\mathbf{P}^{2}$. Let $E_{0}$ denote the line at infinity, a canonical divisor in $\mathbf{P}^{2}$ is given by $K_{\mathbf{P}^{2}}=-3 E_{0}$. We can define an element $K \in \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)$ by taking for any completion $Y$ of $\mathbf{A}^{2}$ the canonical divisor supported at infinity. More precisely, let $Y$ is any completion of $\mathbf{A}^{2}$ above $\mathbf{P}^{2}$. We still denote by $E_{0}$ the strict transform of $E_{0}$ in $Y$. Then, $K_{Y}$ is of the form

$$
\begin{equation*}
K_{Y}=-3 E_{0}+\sum_{E \subset \partial_{X} X_{0}, E \neq E_{0}} E . \tag{50}
\end{equation*}
$$

Suppose that $K=\sup _{i}\left(D_{i}\right)$ for some $D_{i} \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$. Let $D \in\left(D_{i}\right)$ such that $D$ is defined over some completion $Y$ and for some prime divisor $E \neq E_{0}$ at infinity, $\operatorname{ord}_{E}(D)=1$. Then, we must have $K \geqslant D$ meaning that for any completion $Z, K_{Z} \geqslant D_{Z}$. Consider the following blow ups. Let $\pi_{1}: Y_{1} \rightarrow Y$ be the blow-up of a point $p$ of $E$ that does not belong to any other divisor at infinity. Let $\widetilde{E}$ be the exceptional divisor of $\pi$. Now let $\pi_{2}: Y_{2} \rightarrow Y_{1}$ be the blow-up at $\pi_{1}^{\prime} E \cap \widetilde{E}$ and let $\widetilde{F}$ be the exceptional divisor of $\pi_{2}$. Then, $\operatorname{ord}_{\widetilde{F}}\left(K_{Y_{2}}\right)=1$ but $\operatorname{ord}_{\widetilde{F}}\left(D_{Y_{2}}\right)=\operatorname{ord}_{\widetilde{F}}\left(\left(\pi_{2} \circ \pi_{1}\right)^{*} D\right)=2$ and this is a contradiction.

### 3.6. Picard-Manin Space at infinity

3.6.1. Cartier and Weil classes of $X_{0}$. Let $X$ be a completion of $X_{0}$ and let $\operatorname{NS}(X)$ be the Néron-Severi group of $X$. We have a perfect pairing given by the intersection form

$$
\begin{equation*}
\mathrm{NS}(X)_{\mathbf{R}} \times \mathrm{NS}(X)_{\mathbf{R}} \rightarrow \mathbf{R} \tag{51}
\end{equation*}
$$

Recall the Hodge index theorem

Theorem 3.15 (Hodge Index Theorem, [Har77] Theorem 1.9 p.364). Let $X$ be a projective surface over a smooth projective surface over an algebraically closed field. Let $\alpha \in \operatorname{NS}(X)$ and let $H$ be an ample divisor on $X$. If $\alpha \cdot H=0$, then

$$
\begin{equation*}
\alpha^{2}<0 \tag{52}
\end{equation*}
$$

In particular, the signature of the quadratic form induced by the intersection form is $(1, \rho-1)$ where $\rho$ is the rank of $\mathrm{NS}(X)$.

A class $\alpha \in \operatorname{NS}(X)$ is nef if for all irreducible curve $C \subset X, \alpha \cdot[C] \geqslant 0$. If $\pi: Y \rightarrow X$ is a morphism of completions we have two group homomorphisms

$$
\begin{equation*}
\pi_{*}: \mathrm{NS}(Y)_{\mathbf{A}} \rightarrow \mathrm{NS}(X)_{\mathbf{A}}, \pi^{*}: \mathrm{NS}(X)_{\mathbf{A}} \rightarrow \mathrm{NS}(Y)_{\mathbf{A}} \tag{53}
\end{equation*}
$$

with the following properties
(1) $\pi_{*} \circ \pi^{*}=\mathrm{id}_{\mathrm{NS}(X)_{\mathrm{A}}}$
(2) $\pi^{*} \alpha \cdot \pi^{*} \beta=\alpha \cdot \beta$
(3) $\pi^{*} \alpha \cdot \beta=\alpha \cdot \pi_{*} \beta$ (Projection Formula)

Furthermore, if $\pi: Y \rightarrow X$ is the blow up of one point, let $\widetilde{E}$ be the exceptional divisor, then

$$
\begin{equation*}
[\widetilde{E}]^{2}=-1, \text { and } \operatorname{NS}(Y)_{\mathbf{A}}=\pi^{*} \mathrm{NS}(X)_{\mathbf{A}} \oplus \mathbf{A} \cdot[\widetilde{E}] \tag{54}
\end{equation*}
$$

Therefore, the system of groups $(\mathrm{NS}(X))$ with compatibility morphisms $\pi_{*}$ is a projective system of groups and ( $\mathrm{NS}(X)$ ) with compatibility morphisms $\pi^{*}$ is an inductive system of groups.

Definition 3.16. The spaces of Cartier and Weil classes of $X_{0}$ are defined as

We equip Weil-NS $\left(\mathrm{X}_{0}\right)_{\mathbf{A}}$ with the topology of the projective limit. An element of Weil-NS $\left(\mathrm{X}_{0}\right)$ is a family $\alpha=\left(\alpha_{X}\right)_{X}$ where $\alpha_{X} \in \operatorname{NS}(X)$ such that for all $\pi: Y \rightarrow X$, we have

$$
\pi_{*} \alpha_{Y}=\alpha_{X}
$$

We call $\alpha_{X}$ the incarnation of $\alpha$ in $X$.
An element of Cartier-NS $\left(\mathrm{X}_{0}\right)$ is the data of a completion $X$ of $X_{0}$ and a class $\alpha \in \mathrm{NS}(X)$ with the following equivalence relation: $(X, \alpha) \simeq(Y, \beta)$ if there exists a completion $Z$ with a morphism of completion

$$
\pi_{Y}: Z \rightarrow Y, \quad \pi_{X}: Z \rightarrow X
$$

such that $\pi_{Y}^{*} \beta=\pi_{X}^{*} \alpha$. We say that the Cartier class is defined (by $\alpha$ ) in $X$. We have a natural embedding

$$
\begin{equation*}
\text { Cartier-NS }\left(\mathrm{X}_{0}\right) \hookrightarrow \text { Weil-NS }\left(\mathrm{X}_{0}\right) \tag{56}
\end{equation*}
$$

We have a pairing

$$
\begin{equation*}
\text { Weil-NS }\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \times \text { Cartier-NS }\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \rightarrow \mathbf{R} \tag{57}
\end{equation*}
$$

given by the following: let $\alpha \in$ Weil-NS $\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$ and $\beta \in \operatorname{Cartier-NS}\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$; let $X$ be a completion where $\beta$ is defined i.e $\beta=\left(X, \beta_{X}\right)$; then

$$
\begin{equation*}
\alpha \cdot \beta:=\alpha_{X} \cdot \beta_{X} . \tag{58}
\end{equation*}
$$

This is well defined because if $\pi: Y \rightarrow X$ then

$$
\begin{equation*}
\alpha_{Y} \cdot \beta_{Y}=\alpha_{Y} \cdot \pi^{*} \beta_{X}=\pi_{*} \alpha_{Y} \cdot \beta_{X}=\alpha_{X} \cdot \beta_{X} \tag{59}
\end{equation*}
$$

by the projection formula.
An element $\alpha \in$ Weil-NS $\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$ is nef if for all completion $X, \alpha_{X}$ is nef.
Proposition 3.17 ([BFJ08] Proposition 1.7). The intersection pairing

$$
\begin{equation*}
\text { Weil-NS }\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \times \text { Cartier-NS }\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \rightarrow \mathbf{R} \tag{60}
\end{equation*}
$$

is a perfect pairing and it induces a homeomorphism Weil-NS $\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \simeq \operatorname{Cartier-NS}\left(\mathrm{X}_{0}\right)_{\mathbf{R}}^{*}$ endowed with the weak-* topology.

Using the canonical basis of divisors introduced in $\$ 3.3$ we have a more explicit description of the space of Cartier and Weil classes of $X_{0}$.

Proposition 3.18. Let $X$ be a completion of $X_{0}$, then

$$
\begin{equation*}
\text { Cartier-NS }\left(\mathrm{X}_{0}\right)_{\mathbf{A}}=\operatorname{NS}(X)_{\mathbf{A}} \oplus \mathbf{A}^{\left(\mathcal{D}_{X}\right)}, \quad \text { Weil-NS }\left(\mathrm{X}_{0}\right)_{\mathbf{A}}=\mathrm{NS}(X) \oplus \mathbf{A}^{\mathcal{D}_{X}} \tag{61}
\end{equation*}
$$

Moreover, the intersection product is negative definite over $\mathbf{A}^{\left(\mathcal{D}_{X}\right)}$ and $\left\{\alpha_{E}: E \in \mathcal{D}_{X}\right\}$ is an orthonormal basis for the quadratic form $\alpha \in \mathbf{A}^{\left(\mathcal{D}_{X}\right)} \mapsto-\alpha^{2}$.

Proof. The decomposition follows from Equation (54). The fact that the intersection form is negative definite follows from the existence of an ample divisor on $X$, the Hodge Index theorem and the projection formula. The fact that $\left\{\alpha_{E}: E \in \mathcal{D}_{X}\right\}$ is an orthonormal basis is again a consequence of the projection formula and Equation (54).
3.6.2. Local Cartier and Weil classes. Let $X$ be a completion of $X_{0}$ and let $p$ be a point at infinity. Then, by Proposition 3.18 we have the canonical embeddings

$$
\begin{equation*}
\operatorname{Cartier}(X, p)_{\mathbf{A}} \hookrightarrow \operatorname{Cartier-NS}\left(\mathrm{X}_{0}\right)_{\mathbf{A}}, \quad \operatorname{Weil}(X, p)_{\mathbf{A}} \hookrightarrow \operatorname{Weil-NS}\left(\mathrm{X}_{0}\right)_{\mathbf{A}} \tag{62}
\end{equation*}
$$

Proposition 3.19. The space $\operatorname{Cartier}(X, p)_{\mathbf{R}}$ is an infinite dimensional $\mathbf{R}$-vector space and the intersection product defines a negative definite quadratic form over it. The set $\left\{\alpha_{E}: E \in \mathcal{D}_{X, p}\right\}$ is an orthonormal basis for the scalar product $\alpha \mapsto-\alpha^{2}$. Furthermore, the pairing

$$
\begin{equation*}
\operatorname{Weil}(X, p)_{\mathbf{R}} \times \operatorname{Cartier}(X, p)_{\mathbf{R}} \rightarrow \mathbf{R} \tag{63}
\end{equation*}
$$

is perfect.
3.6.3. The divisors supported at infinity. Fix a completion $X$ of $X_{0}$, we have a natural linear map $\tau: \operatorname{Div}_{\infty}(X)_{\mathbf{R}} \rightarrow \mathrm{NS}(X)_{\mathbf{R}}$.

Proposition 3.20. The intersection pairing restricted to $\tau\left(\operatorname{Div}_{\infty}(X)_{\mathbf{R}}\right)$ is non degenerate.

Proof. Let $D \in \tau\left(\operatorname{Div}_{\infty}(X)_{\mathbf{R}}\right)$, suppose that $D \cdot D^{\prime}=0$ for all $D^{\prime} \in \tau\left(\operatorname{Div}_{\infty}(X)_{\mathbf{R}}\right)$. Then, by Theorem 2.4, there exists $H \in \operatorname{Div}_{\infty}(X)$ ample. We have $D \cdot H=0$. By the Hodge index theorem, if $D$ is not numerically equivalent to zero, then $D^{2}<0$ and this is a contradiction.

Let $V \subset \mathrm{NS}(X)$ be the orthogonal subspace of $\tau\left(\operatorname{Div}_{\infty}(X)_{\mathbf{R}}\right)$. Then,

$$
\begin{equation*}
\mathrm{NS}(X)_{\mathbf{R}}=V \oplus \tau\left(\operatorname{Div}_{\infty}(X)_{\mathbf{R}}\right) . \tag{64}
\end{equation*}
$$

For example if $X_{0}=\mathbf{A}^{2}$ and $X=\mathbf{P}^{2}$, then $V=0$. Since we only blow up at infinity we get

Proposition 3.21. Let $X_{0}$ be an affine surface, then
Cartier-NS $\left(\mathrm{X}_{0}\right)_{\mathbf{R}}=V \oplus \tau\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}}\right), \quad$ Weil-NS $\left(\mathrm{X}_{0}\right)_{\mathbf{R}}=V \oplus \tau\left(\operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}}\right)$
3.6.4. Functoriality. Let $f: X_{0} \rightarrow X_{0}$ be a dominant endomorphism of $X_{0}$. We define $f^{*}, f_{*}$ on the spaces of Cartier and Weil classes as follows. We first define

$$
\begin{equation*}
f^{*}: \text { Cartier-NS }\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \rightarrow \text { Cartier-NS }\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \tag{66}
\end{equation*}
$$

Let $\beta \in \operatorname{Cartier-NS}\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$ and let $X$ be a completion where $\beta$ is defined. Let $Y$ be a completion of $X_{0}$ such that the lift $F: Y \rightarrow X$ is regular, then we define $f^{*} \beta$ as the Cartier class defined in $Y$ by

$$
\begin{equation*}
f^{*} \beta:=\left(Y, F^{*} \beta_{X}\right) \tag{67}
\end{equation*}
$$

this does not depend on the choice of $Y$. Indeed, if $Y^{\prime}$ is another completion such that $F^{\prime}: Y^{\prime} \rightarrow X$ is well defined, then there exists a completion $Z$ such that we have the following diagram.


Then, the lift of $f: Z \rightarrow X$ is $F \circ \pi_{Y}=F^{\prime} \circ \pi_{Y^{\prime}}$, hence we get

$$
\begin{equation*}
\pi_{Y}^{*} \circ F^{*}=\pi_{Y^{\prime}}^{*} \circ\left(F^{\prime}\right)^{*} \tag{69}
\end{equation*}
$$

and the pull back of Cartier classes is well defined.
Next, we define $f_{*}:$ Weil-NS $\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \rightarrow$ Weil-NS $\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$. Let $\alpha \in$ Weil-NS $\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$. Let $X, Y$ be completions of $X_{0}$ such that the lift $F: Y \rightarrow X$ is regular, then the incarnation of $f_{*} \alpha$ in $X$ is

$$
\begin{equation*}
\left(f_{*} \alpha\right)_{X}:=F_{*} \alpha_{Y} \tag{70}
\end{equation*}
$$

Again, this does not depend on the choice of $Y$ by a similar argument as for the pullback. We have the following proposition

Proposition 3.22 ([][BFJ08] Section 2). We have the following properties.

- The operator $f^{*}$ extends to an operator

$$
\begin{equation*}
f^{*}: \text { Weil-NS }\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \rightarrow \text { Weil-NS }\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \tag{71}
\end{equation*}
$$

- the operator $f_{*}$ restricts to an operator

$$
\begin{equation*}
f_{*}: \text { Cartier-NS }\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \rightarrow \text { Cartier-NS }\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \tag{72}
\end{equation*}
$$

- Let $\alpha \in$ Weil-NS $\left(\mathrm{X}_{0}\right)$, let $X, Y$ be completions of $X_{0}$ such that the lift $f: X \rightarrow$ $Y$ does not contract any curves, then

$$
\begin{equation*}
\left(f^{*} \alpha\right)_{X}=\left(f^{*} \alpha_{Y}\right)_{X} \tag{73}
\end{equation*}
$$

REMARK 3.23. For a completion $X$, we can also define the restriction of $f^{*}$ and $f_{*}$ to $\mathrm{NS}(X)$. We denote them respectively by $f_{X}^{*}$ and $\left(f_{X}\right)_{*}$. They are defined by

$$
\begin{equation*}
\forall \beta \in \operatorname{NS}(X), \quad f_{X}^{*} \beta=\left(f^{*} \beta\right)_{X}, \quad\left(f_{X}\right)_{*} \beta=\left(f_{*} \beta\right)_{X} \tag{74}
\end{equation*}
$$

3.6.5. The Picard-Manin space and spectral property of the first dynamical degree. Consider a completion $X$ of $X_{0}$ and $\omega \in \mathrm{NS}(X)$ an ample class. By the Hodge index theorem, the intersection form on Cartier-NS $\left(\mathrm{X}_{0}\right) \times \operatorname{Cartier}-\mathrm{NS}\left(\mathrm{X}_{0}\right)$ is negative definite on $\omega^{\perp}$. If $\alpha \in \operatorname{Cartier-NS}\left(\mathrm{X}_{0}\right)$, the projection of $\alpha$ on $\omega^{\perp}$ is $\alpha-(\alpha \cdot \omega) \omega$. Consider the quadratic form on Cartier-NS $\left(\mathrm{X}_{0}\right)$ given by

$$
\begin{equation*}
\forall \alpha \in \operatorname{Cartier}-\mathrm{NS}\left(\mathrm{X}_{0}\right),\left\|\alpha^{2}\right\|:=(\omega \cdot \alpha)^{2}-\frac{1}{\omega^{2}}(\alpha-(\alpha \cdot \omega) \omega)^{2} \tag{75}
\end{equation*}
$$

This defines a norm on Cartier-NS $\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$ and Cartier-NS $\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$ is not complete for this norm. We define the Picard-Manin space of $X_{0}$ as the completion of Cartier-NS $\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$ with respect to this norm and we denote it by $\mathrm{L}^{2}\left(\mathrm{X}_{0}\right)$; Had we chosen a different ample class, we would have gotten an equivalent norm so the space $\mathrm{L}^{2}\left(\mathrm{X}_{0}\right)$ is independent of the choice of $\omega$. This is a Hilbert space and we have

Proposition 3.24 ([BFJ08] Proposition 1.10). There is a continuous injection

$$
\begin{equation*}
\mathrm{L}^{2}\left(\mathrm{X}_{0}\right) \hookrightarrow \text { Weil-NS }\left(\mathrm{X}_{0}\right) \tag{76}
\end{equation*}
$$

and the topology on $\mathrm{L}^{2}\left(\mathrm{X}_{0}\right)$ induced by Weil- $\mathrm{NS}\left(\mathrm{X}_{0}\right)$ coincides with its weak topology as a Hilbert space. If $\alpha \in$ Weil- $\mathrm{NS}\left(\mathrm{X}_{0}\right)$ then $\alpha$ belongs to $\mathrm{L}^{2}\left(\mathrm{X}_{0}\right)$ if and only if
$\inf _{X}\left(\alpha_{X}^{2}\right)>-\infty$, in which case $\alpha^{2}=\inf _{X}\left(\alpha_{X}^{2}\right)$. Furthermore, the intersection product. defines a continuous bilinear form on $\mathrm{L}^{2}\left(\mathrm{X}_{0}\right)$.

REMARK 3.25. In particular, any nef class belongs to $L^{2}\left(X_{0}\right)$. Recall that $\alpha \in$ Weil-NS $\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$ is nef if for every completion $X, \alpha_{X}$ is nef. The cone theorem ( $[\mathbf{L a z 0 4}]$ Theorem 1.4.23) states that $\alpha_{X}$ is a limit of ample classes in $\operatorname{NS}(X)_{\mathbf{R}}$, therefore $\left(\alpha_{X}\right)^{2} \geqslant$ 0 and $\alpha \in \mathrm{L}^{2}\left(\mathrm{X}_{0}\right)$.

Using the canonical basis of exceptional divisors we can have an explicit description of $\mathrm{L}^{2}\left(\mathrm{X}_{0}\right)$. Let $\alpha \in$ Cartier- $\mathrm{NS}\left(\mathrm{X}_{0}\right)$ and let $\alpha_{X}$ be the incarnation of $\alpha$ in $X$. Then, since $\alpha$ is a Cartier class, we have for all but finitely many $E \in \mathcal{D}_{X}$ that $\alpha \cdot \alpha_{E}=0$ and

$$
\begin{equation*}
\alpha=\alpha_{X}+\sum_{E \in \mathcal{D}_{X}}\left(\alpha \cdot \alpha_{E}\right) \alpha_{E} \tag{77}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\left\|\alpha^{2}\right\|=\left\|\alpha_{X}\right\|^{2}+\sum_{E \in \mathcal{D}_{X}}\left(\alpha \cdot \alpha_{E}\right)^{2}, \tag{78}
\end{equation*}
$$

and

$$
\begin{equation*}
\alpha^{2}=\alpha_{X}^{2}-\sum_{E \in \mathcal{D}_{X}}\left(\alpha \cdot \alpha_{E}\right)^{2} \tag{79}
\end{equation*}
$$

Therefore, $L^{2}\left(X_{0}\right)$ is isomorphic to the Hilbert space

$$
\begin{equation*}
\mathrm{L}^{2}\left(\mathrm{X}_{0}\right)=\mathrm{NS}(X) \oplus \ell^{2}\left(\mathcal{D}_{X}\right) \tag{80}
\end{equation*}
$$

We also have the local version of this statement
Proposition 3.26. Let $X$ be a completion of $X_{0}$ and $p \in X$ be a point at infinity. Then,

$$
\begin{equation*}
\mathrm{L}^{2}\left(\mathrm{X}_{0}\right) \cap \operatorname{Weil}(X, p)=\ell^{2}\left(\mathcal{D}_{X, p}\right) \tag{81}
\end{equation*}
$$

and $\left\{\alpha_{E}: E \in \mathcal{D}_{X, p}\right\}$ is a Hilbert basis of this space.
Proposition 3.27 ([|[BFJ08]). Let $f$ be a dominant endomorphism of $X_{0}$. The linear maps

$$
\begin{equation*}
f^{*}, f_{*}: \text { Weil-NS }\left(\mathrm{X}_{0}\right) \rightarrow \text { Weil-NS }\left(\mathrm{X}_{0}\right) \tag{82}
\end{equation*}
$$

induce continuous operators

$$
\begin{equation*}
f^{*}, f_{*}: \mathrm{L}^{2}\left(\mathrm{X}_{0}\right) \rightarrow \mathrm{L}^{2}\left(\mathrm{X}_{0}\right) \tag{83}
\end{equation*}
$$

Furthermore, we have the following properties in $\mathrm{L}^{2}\left(\mathrm{X}_{0}\right)$.
(1) $\left(f^{n}\right)^{*}=\left(f^{*}\right)^{n}$;
(2) $\forall \alpha, \beta \in \mathrm{L}^{2}\left(\mathrm{X}_{0}\right), f^{*} \alpha \cdot \beta=\alpha \cdot f_{*} \beta$.
(3) $\forall \alpha \in \mathrm{L}^{2}\left(\mathrm{X}_{0}\right), f^{*} \alpha \cdot f^{*} \alpha=e(f) \alpha \cdot \alpha$ where $e(f)$ is the topological degree of $f$.

In particular, if $f \in \operatorname{Aut}\left(X_{0}\right)$ then $f^{*}$ is an isometry of $\mathrm{L}^{2}\left(\mathrm{X}_{0}\right)$ viewed as an infinite dimensional hyperbolic space (see [CLC13]).

THEOREM 3.28 ([[BFJ08, [DF21] $)$. Suppose that $\lambda_{1}(f)^{2}>\lambda_{2}(f)$, then there exist nef classes $\theta^{*}, \theta_{*} \in \mathrm{~L}^{2}\left(\mathrm{X}_{0}\right)$ unique up to multiplication by a positive constant such that
(1) $f^{*} \theta^{*}=\lambda_{1} \theta^{*}$.
(2) $f_{*} \theta_{*}=\lambda_{1} \theta_{*}$.
(3) For all $\alpha \in \mathrm{L}^{2}\left(\mathrm{X}_{0}\right)$,

$$
\begin{align*}
\frac{1}{\lambda_{1}^{n}}\left(f^{n}\right)^{*} \alpha & =\left(\alpha \cdot \theta_{*}\right) \theta^{*}+1 O_{\alpha}\left(\left(\frac{\lambda_{2}}{\lambda_{1}^{2}}\right)^{n / 2}\right)  \tag{84}\\
\frac{1}{\lambda_{1}^{n}}\left(f^{n}\right)_{*} \alpha & =\left(\alpha \cdot \theta^{*}\right) \theta_{*}+O_{\alpha}\left(\left(\frac{\lambda_{2}}{\lambda_{1}^{2}}\right)^{n / 2}\right) \tag{85}
\end{align*}
$$

In particular, for all $\alpha, \beta \in \mathrm{L}^{2}\left(\mathrm{X}_{0}\right)$,

$$
\begin{equation*}
\lim _{n} \frac{1}{\lambda_{1}^{n}}\left(f^{n}\right)^{*} \alpha \cdot \beta=\left(\alpha \cdot \theta_{*}\right)\left(\beta \cdot \theta^{*}\right) \tag{86}
\end{equation*}
$$

Furthermore, $\theta^{*}$ and $\theta_{*}$ satisfy

$$
\begin{equation*}
\left(\theta^{*}\right)^{2}=0, \quad \theta_{*} \cdot \theta^{*}>0 \tag{87}
\end{equation*}
$$

We call $\theta^{*}$ and $\theta_{*}$ the eigenclasses of $f$.
SKETCH OF PROOF. We sketch here the proof for $\theta^{*}$. Let $X$ be a completion of $X_{0}$. The pull back $f^{*}$ induces a linear map $f_{X}^{*}: \mathrm{NS}(X) \rightarrow \mathrm{NS}(X)$. Let $\rho_{X}$ be the spectral radius of this map. We have for any ample class $w \in \operatorname{NS}(X)$ that $\rho_{X}=$ $\lim _{n \rightarrow \infty}\left(\left(f_{X}^{*}\right)^{n} w \cdot w\right)^{1 / n}$. Now, $f_{X}^{*}$ preserves the cone $C_{X}$ of nef classes in $\operatorname{NS}(X)_{\mathbf{R}}$. This is a closed convex cone with compact basis and non-empty interior. By a PerronFrobenius type argument, there exists $\theta_{X} \in C_{X}$ such that $f_{X}^{*} \theta_{X}=\rho_{X} \theta_{X}$.

Now, Let $\left(X_{N}\right)$ be a sequence of completions of $X_{0}$ such that $X_{1}=X$ and $X_{N+1}$ is a composition of blowups of $X_{N}$ at infinity such that the lift of $f$ to $F_{N}: X_{N+1} \rightarrow X_{N}$ is regular, we denote by $\pi_{N}: X_{N+1} \rightarrow X_{N}$ the induced morphism of completions. Let $\rho_{N}:=\rho_{X_{N}}$ and $\theta_{N}:=\theta_{X_{N}}$. One can show that $\lim _{N} \rho_{N}=\lambda_{1}$. By construction, we have that for all $N \geqslant 1$, the element $f^{*} \theta_{N}-\rho_{N} \theta_{N} \in$ Weil-NS $\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$ has incarnation zero in $X_{N}$, hence it tends to zero in Weil-NS $\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$. We can normalize all $\theta_{N}$ such that $\theta_{N} \cdot w=$ 1 where $w$ is an ample class of $\operatorname{NS}(X)$. Now, the set $\left\{W \in \operatorname{Weil-NS}\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \mid W \cdot w=1\right\}$ is a compact subset of Weil-NS $\left(\mathrm{X}_{0}\right)$ so the sequence $\left(\theta_{N}\right)$ has an accumulation point $\theta^{*} \in$ Weil-NS $\left(\mathrm{X}_{0}\right)$ that is nef, effective and we get $f^{*} \theta^{*}=\lambda_{1} \theta^{*}$.

[^2]
## CHAPTER 4

## Valuations

We introduce the notion of valuations and describe some properties. We will especially focus on valuations over the ring of power series in two variables $\mathbf{k}[[x, y]]$ as they allow one to describe every valuation over $\mathbf{k}\left[X_{0}\right]$ for $X_{0}$ a normal affine surface.

### 4.1. Valuations and completions

Our general reference for the theory of valuations is [Vaq00]. Let $R$ be a commutative $\mathbf{k}$-algebra that is also an integral domain, a valuation on $R$ is a function $v: R \rightarrow \mathbf{R} \cup\{\infty\}$ such that
(i) $v\left(\mathbf{k}^{*}\right)=0$;
(ii) For all $P, Q \in R, v(P Q)=v(P)+v(Q)$;
(iii) For all $P, Q \in R, v(P+Q) \geqslant \min (v(P), v(Q))$;
(iv) $\mathrm{v}(0)=+\infty$.

If $I$ is an ideal of $R$, we set $v(I):=\min _{i \in I} v(i)$. If $S \subset I$ is a set of generators, then

$$
\begin{equation*}
v(I)=\min _{s \in S} v(s) . \tag{88}
\end{equation*}
$$

REMARK 4.1. In Abh56 A valuation can take the value $+\infty$ only at 0 but we do not require such a property. Let $\mathfrak{p}_{v}=\{a \in R: v(a)=\infty\}$ then $\mathfrak{p}_{v}$ is a prime ideal of $R$ that we call the bad ideal of $v$. If $v$ is a valuation on $R$, it defines naturally a valuation in the sense of [Abh56] on the quotient field $R / \mathfrak{p}_{\mathrm{v}}$. Furthermore $v$ can be naturally extended to a valuation on the ring $R_{\mathfrak{p}_{\mathrm{v}}}$ via the formula $v(p / q)=v(p)-v(q)$. In particular, if $\mathfrak{p}_{v}=\{0\}$, then $v$ defines a valuation over Frac $R$.

Let $X$ be a completion of $X_{0}$ and let $v$ be a valuation over $B:=O_{X}\left(X_{0}\right)$. Let $\mathfrak{p}_{v}$ be the bad ideal of $v$. Consider $B_{\mathfrak{p}_{v}}$ the localization of $B$ at $\mathfrak{p}_{v}$. Set

$$
\begin{equation*}
\mathcal{O}_{v}:=\left\{x \in B_{\mathfrak{p}_{v}}: v(x) \geqslant 0\right\} . \tag{89}
\end{equation*}
$$

This is a subring of $B_{\mathfrak{p}_{v}}$. If $\mathfrak{p}_{v}=\{0\}$, then this is the classical valuation ring of $v$.
Lemma 4.2. The subring $O_{v}$ is a local ring, its maximal ideal is

$$
\begin{equation*}
\mathfrak{m}_{v}:=\left\{x \in O_{v}: v(x)>0\right\} \tag{90}
\end{equation*}
$$

Proof. It suffices to show that if $v(x)=0$, then $x$ is invertible in $O_{v}$ but this is obvious since $v\left(x^{-1}\right)=-v(x)=0$.

One defines naturally a valuation $v$ on $C:=B / \mathfrak{p}_{v}$, let $L$ be the fraction field of $C$ and $O$ be the valuation ring of $L$ with respect to $v$. Then, we have the natural isomorphisms

$$
\begin{equation*}
L \simeq B_{\mathfrak{p}_{v}} / \mathfrak{p}_{v} \text { and } O_{v} / \mathfrak{p}_{v} \simeq O \tag{91}
\end{equation*}
$$

Geometrically, the Zariski closure of $\mathfrak{p}_{v}$ inside $X$ defines an irreducible closed subscheme $Y$ of $X$ and $L$ is isomorphic to the field of rational functions on $Y$.

Two valuations $v_{1}, \nu_{2}$ are equivalent if there exists a real number $\lambda>0$ such that $\nu_{1}=\lambda \nu_{2}$. Let $R, R^{\prime}$ be two integral domains with a homomorphism of schemes $\varphi$ : $\operatorname{Spec} R^{\prime} \rightarrow \operatorname{Spec} R$; it induces a ring homomorphism $\varphi^{*}: R \rightarrow R^{\prime}$. If $v$ is a valuation on $R^{\prime}$ we define $\varphi_{*} \vee$ the pushforward by $\varphi$ of $\vee$ by

$$
\begin{equation*}
\forall P \in R, \varphi_{*} v(P)=v\left(\varphi^{*}(P)\right) \tag{92}
\end{equation*}
$$

Let $X_{0}$ be an affine surface. Denote by $\mathcal{V}$ the set of valuations on $\mathbf{k}\left[X_{0}\right]$. We equip this space with the topology of weak convergence, that is the coarsest topology such that the evaluation map $v \in \mathcal{V} \mapsto v(P)$ is continuous for all $P \in \mathbf{k}\left[X_{0}\right]$. If $f$ is an endomorphism of $X_{0}$, then $f$ induces a continuous map $f_{*}: \mathcal{V} \rightarrow \mathcal{V}$.

Via the natural isomorphism $\mathrm{t}_{X}^{*}: O_{X}\left(X_{0}\right) \rightarrow \mathbf{k}\left[X_{0}\right]$, every $v \in \mathcal{V}$ induces a valuation $\left(\mathfrak{l}_{X}\right)_{*} V$ on $O_{X}\left(X_{0}\right)$, namely

$$
\begin{equation*}
\forall P \in O_{X}\left(X_{0}\right), \quad\left(\mathrm{v}_{X}\right)_{*} v(P):=\mathrm{v}\left(\mathrm{v}_{X}^{*} P\right) \tag{93}
\end{equation*}
$$

We will denote $\left(\mathbf{l}_{X}\right)_{*} \vee$ by $v_{X}$ for every valuation $v$ on $\mathbf{k}\left[X_{0}\right]$.
REMARK 4.3. Take a morphism of completions $\pi: X_{1} \rightarrow X_{2}$ and $v$ a valuation on $\mathbf{k}\left[X_{0}\right]$. Then, $\left(\boldsymbol{1}_{X_{2}}\right)_{*} v=\left(\pi^{-1} \circ \mathbf{1}_{X_{1}}\right)_{*} v$. In particular $\pi_{*} v_{X_{2}}=v_{X_{1}}$.

REMARK 4.4. In the language of Berkovich theory, the set $\mathcal{V}$ is the Berkovich analytification of $X_{0}$ over $\mathbf{k}$ where we have endowed $\mathbf{k}$ with the trivial valuation (see [Ber12]).

Example 4.5 (Divisorial valuations). Let $X$ be a completion of $X_{0}$ and $E$ be a prime divisor of $X$. Let $\operatorname{ord}_{E}$ be the valuation on $\mathbf{k}(X)$ such that for any $f \in \mathbf{k}(X), \operatorname{ord}_{E}(f)$ is the order of vanishing of $f$ along $E$. Any valuation $v$ on $\mathbf{k}\left[X_{0}\right]$ such that $v_{X}$ is equivalent to $\operatorname{ord}_{E}$ for some prime divisor $E$ in some completion $X$ is called a divisorial valuation. In that case $\mathfrak{p}_{v}=\{0\}$ and $v$ extends uniquely to a valuation on Frack $\left[X_{0}\right]$. For example if $X_{0}=\mathbf{A}^{2}$ and $X=\mathbf{P}^{2}$, then let $L_{\infty}$ be the line at infinity, we have $\forall P \in \mathbf{k}[x, y], \operatorname{ord}_{L_{\infty}}(P)=-\operatorname{deg}(P)$. If instead we take the completion $P^{1} \times \mathbf{P}^{1}$, decompose $\mathbf{A}^{2}=\mathbf{A}^{1} \times \mathbf{A}^{1}$ and let $x, y$ be the affine coordinate of $\mathbf{A}^{2}$ each being an affine coordinate of $\mathbf{A}^{1}$. Let $L_{x}=\{\infty\} \times \mathbf{P}^{1}$ and $L_{y}=\mathbf{P}^{1} \times\{\infty\}$, then

$$
\begin{equation*}
\forall P \in \mathbf{k}[x, y], \operatorname{ord}_{L_{x}}(P)=-\operatorname{deg}_{x}(P), \quad \operatorname{ord}_{L_{y}}(P)=-\operatorname{deg}_{y}(P) \tag{94}
\end{equation*}
$$

where $\operatorname{deg}_{x}\left(\right.$ respectively $\left.\operatorname{deg}_{y}\right)$ is the degree with respect to the variable $x$ (respectively $y)$.

Example 4.6 (Curve valuations). Let $X$ be a completion of $X_{0}$, let $p \in \partial_{X} X_{0} C$ be the germ of a (formal) curve at $p$. This means that $C$ is defined as $\varphi=0$ for $\varphi$ in the completion $\widehat{O}_{X, p}$ of the local ring $O_{X, p}$ at $p$. If $\psi \in \widehat{O}_{X, p}$ is another germ of a formal curve at $p$, we define the intersection number at $p$ by

$$
\begin{equation*}
\{\varphi=0\} \cdot p\{\psi=0\}:=\operatorname{dim}_{\mathbf{k}} \widehat{O}_{X, p} /\langle\varphi, \psi\rangle \tag{95}
\end{equation*}
$$

This number is equal to $\infty$ exactly when one of the germs divides the other. We first define a valuation $v_{C, p}$ on $\widehat{O}_{X, p}$ by

$$
\begin{equation*}
v_{C, p}(\psi)=\left\{\psi=0 \cdot{ }_{p} C\right\} \tag{96}
\end{equation*}
$$

Suppose $\varphi$ is not divisible by the local equation of any component of $\partial_{X} X_{0}$. For any $P \in O_{X}\left(X_{0}\right), P$ can be written as $P=\psi_{1}^{\alpha_{1}} \cdots \psi_{r}^{\alpha_{r}}$ with $\psi_{i} \in \widehat{O}_{X, p}$ irreducible and $\alpha_{i} \in \mathbf{Z}$. We define

$$
\begin{equation*}
v_{C, p}(P):=\sum_{i} \alpha_{i} v_{C, p}\left(\psi_{i}\right) \in \mathbf{R} \cup\{\infty\} \tag{97}
\end{equation*}
$$

Then $v_{C, p}$ is a valuation on $O_{X}\left(X_{0}\right)$. Any valuation on $\mathbf{k}\left[X_{0}\right]$ such that $v_{X}$ is equivalent to $v_{C, p}$ is called a curve valuation. If $v$ is a valuation such that $\mathfrak{p}_{v} \neq\{0\}$, then $v$ is a curve valuation (see [FJ04] and Proposition 4.9 below). We will make the following distinction, if $C$ is defined by $\varphi \in O_{X, p}$ we will say that $v_{C, p}$ is an algebraic curve valuation. Otherwise, we will say that it is a formal curve valuation.

If $\varphi$ was divisible by the local equation of a component of $\partial_{X} X_{0}$, then $v_{C, p}$ would not define a valuation on $\mathbf{k}\left[X_{0}\right]$ as some regular functions $P \in \mathbf{k}\left[X_{0}\right]$ would have a pole along $C$ and $v(P)$ would be equal to $-\infty$.

### 4.2. Valuations over $\mathbf{k}[[x, y]]$

We recall some results about valuations from [FJ04] and [FJ07]. Let $R$ be a regular local ring with maximal ideal $\mathfrak{m}$. We say that a valuation on $R$ is centered if $v \geqslant 0$ and $v_{\mid \mathfrak{m}}>0$. Here we set $R:=\mathbf{k}[[x, y]]$ for our local ring. Its maximal ideal is $\mathfrak{m}:=(x, y)$ we will study the set of centered valuations on $R$.

Proposition 4.7 (Proposition 2.10 [ $\mathbf{F J 0 4}$ ], [Spi90]). Any valuation on $\mathbf{k}[x, y]$ centered at the origin extends uniquely to a centered valuation on $R$ as follows. Let $\varphi \in R$ and let $\varphi_{n}$ be the polynomial of degree $n$ such that $\varphi=\lim \varphi_{n}$. Then,

$$
\begin{equation*}
v(\varphi)=\lim _{n \rightarrow \infty} \min \left(v\left(\varphi_{n}\right), n\right) \tag{98}
\end{equation*}
$$

COROLLARY 4.8. Let $R^{\prime}$ be regular local ring of dimension 2 over $\mathbf{k}$, then the $\mathfrak{m}_{R^{\prime}}$ adic completion $\widehat{R^{\prime}}$ of $R^{\prime}$ is isomorphic to $R$. Any centered valuation on $R^{\prime}$ extends uniquely to a centered valuation on $\widehat{R^{\prime}}$.

Proof. Let $(x, y)$ be a regular sequence of $R^{\prime}$, that is $\mathfrak{m}_{R^{\prime}}=(x, y)$. It exists because $R^{\prime}$ is a regular local ring of dimension 2. Then, $\hat{R}^{\prime}$ is isomorphic to $\mathbf{k}[[x, y]]$. Let $v$ be a centered valuation on $R^{\prime}$. We have that $\mathbf{k}[x, y] \subset R^{\prime}$, so $v$ induces a valuation on $\mathbf{k}[x, y]$ that is centered at the origin and we can apply the previous proposition to conclude.

Let $p$ be a regular point on a surface $X$ and let $R=\widehat{O_{X, p}}$ we define 4 types of valuations over $R$.
4.2.1. Divisorial valuations. A valuation $v$ over $R$ is divisorial if there exists a sequence of blow-up $\pi:(Y, \operatorname{Exc}(\pi)) \rightarrow(X, x)$ such that $v$ is equivalent to $\pi_{*} \operatorname{ord}_{E}$ for some prime divisor $E \subset \operatorname{Exc}(\pi)$.
4.2.2. Quasimonomial valuations. Let $\pi:(Y, \operatorname{Exc}(\pi)) \rightarrow(X, x)$ be a sequence of blow-ups and let $q \in \operatorname{Exc}(\pi)$. A monomial valuation at $q$ is a valuation $v$ on $\widehat{O_{Y, q}}$ such that there exists $s, t>0$,

$$
\begin{equation*}
\nu\left(\sum_{i, j} a_{i j} x^{i} y^{j}\right)=\min \left\{s i+t j: a_{i j} \neq 0\right\} \tag{99}
\end{equation*}
$$

for some local coordinates at $q$. We write $v=v_{s, t}$.
A valuation over $\widehat{O_{X, p}}$ is called quasimonomial if there exists a sequence of blowups $\pi:(Y, \operatorname{Exc}(\pi)) \rightarrow(X, p)$ such that $\mathrm{v}=\pi_{*} \mathrm{v}_{s, t}$. Quasimonomial valuations split into two categories: if $s / t \in \mathbf{Q}$, one can show actually that V is divisorial. Otherwise $s / t \in$ $\mathbf{R} \backslash \mathbf{Q}, v$ is not divisorial and we say that it is irrational.
4.2.3. Curve valuations. Let $\varphi \in \widehat{\mathfrak{m}_{p}}$ be irreducible, we define $v_{\varphi}$ by

$$
\begin{equation*}
\forall \psi \in \widehat{O_{X, p}}, \quad v_{\varphi}(\psi)=\frac{\{\varphi=0\} \cdot\{\psi=0\}}{m(\varphi)} \tag{100}
\end{equation*}
$$

where $m(\varphi)$ is the order of vanishing of $\varphi$ at the origin. A curve valuation is a valuation equivalent to $v_{\varphi}$ for some $\varphi \in \widehat{\mathfrak{m}_{p}}$ irreducible.
4.2.4. Infinitely singular valuations. These are all the remaining valuations. They have a nice description in term of Puiseux series (see [FJ04] Section 4.1 for more details). Briefly, to any valuation $\nu$ of $\mathbf{k}[[x, y]]$, one can associated a generalized power series

$$
\begin{equation*}
\widehat{\varphi}=\sum_{j} a_{j} x^{\beta_{j}} \tag{101}
\end{equation*}
$$

with $a_{j} \in \mathbf{k}$ and $\beta_{j} \in \mathbf{Q}$. The infinitely singular valuations are exactly the valuations such that $\lim _{j} \beta_{j} \neq+\infty$.

Proposition 4.9 ([|[504]). There are four types of centered valuations on R: divisorial, irrational, curve valuations and infinitely singular valuations. The only type of valuation $v$ such that $\mathfrak{p}_{v}=\{v=+\infty\} \neq 0$ are curve valuations

REMARK 4.10. Instead of looking at valuations over $R$ with values in $\mathbf{R}$, we can look at valuations with values in a totally ordered abelian group $\Gamma$, these are called Krull valuations (see [FJ04], section 1.3) and they have the advantage to always extend to Frac $R$. We can make any curve valuation into a Krull valuation by the following procedure (see [FJ04], section 1.5.5): Let $\varphi \in \mathfrak{m}$ and consider the curve valuation $v_{\varphi}$. Let $\Gamma=\mathbf{Z} \times \mathbf{Q}$ with the lexicographical order, we define $\widehat{v}_{\varphi}: R \rightarrow \Gamma$ as follows. For any $\psi \in R$, there exists an integer $k \in \mathbf{N}$ such that

$$
\begin{equation*}
\psi=\varphi^{k} \widehat{\psi} \tag{102}
\end{equation*}
$$

where $\hat{\psi}$ is not divisible by $\varphi$. Set

$$
\begin{equation*}
\widehat{v}(\Psi):=\left(k, v_{\varphi}(\widehat{\psi})\right) \tag{103}
\end{equation*}
$$

Notice that $v_{\varphi}(\psi)=\infty \Leftrightarrow p_{1}\left(\widehat{v}_{\varphi}(\psi)\right)>0$ where $p_{1}: \Gamma \rightarrow \mathbf{Z}$ is the projection to the first coordinate and if $v_{\varphi}(\psi)<+\infty$, then $\widehat{v}_{\varphi}(\psi)=\left(0, v_{\varphi}(\psi)\right)$. We will not need Krull valuations in the rest of the text. But this argument comes in handy for the proof of Proposition 4.18 so we state it here.

### 4.3. The center of a valuation

Let $X$ be a completion of $X_{0}$ and let $v$ be a valuation on $O_{X}\left(X_{0}\right)$. A center of $v$ on $X$ is a scheme-theoretic point $p \in X$ such that $O_{V}$ dominates the local ring $O_{X, p}$ (i.e $O_{X, p} \subset O_{v}$ and $\mathfrak{m}_{p} \subset \mathfrak{m}_{v}$ ). If such a $p$ exists then $v$ induces a centered valuation on $O_{X, p}$ (cf4.2) and in particular for any open affine subset $U \subset X$ that contains $p, \mathrm{v}$ induces a valuation on $O_{X}(U)$ via the inclusion $O_{X}(U) \subset O_{X, p}$.

Lemma 4.11. The center of $v$ on $X$ always exists and is unique.
Proof. Let $O_{v}$ be the subring of $\mathbf{k}(X)$ where $v$ is $\geqslant 0$; it contains $\mathbf{k}^{*}$. Let $L=$ $B_{\mathfrak{p}_{v}} / \mathfrak{p}_{v}$ and $O=O_{v} / \mathfrak{p}_{v}$. If $p$ is a center of $v$ on $X$ then we have the following commutative diagram of ring homomorphism

$$
\begin{equation*}
O_{X, p} \longleftrightarrow O_{v} \longrightarrow O \longleftrightarrow L \longleftrightarrow B_{\mathfrak{p}_{v}} \tag{104}
\end{equation*}
$$

inducing the following commutative diagram of scheme morphisms


Since $X$ is proper over $\mathbf{k}$ (it's a projective variety), the valuative criterion of properness ([Har77]) shows that if the center exists, then it is unique. For the existence, Let $x \in X$ be the image of the maximal ideal of $O$, then $x$ is the center of $v$ on $X$. Indeed, the image of $\operatorname{Spec} L$ is the prime ideal $\mathfrak{p}_{v}$ of $O_{X}\left(X_{0}\right)$ and $x$ belongs to its closure, therefore $O_{X, x} \subset B_{\mathfrak{p}_{v}}$ and the morphism of local rings $O_{X, x} \rightarrow O$ shows that $O_{v}$ dominates $O_{X, x}$.

The center of $v$ on $X$ is the center of $v_{X}$ we will denote it by $c_{X}(v)$.
EXAMPLE 4.12. Let $v$ be a divisorial valuation over $\mathbf{k}\left[X_{0}\right]$ and let $X$ be a completion of $X_{0}$ such that $v_{X} \simeq \operatorname{ord}_{E}$ for some prime divisor $E$ of $X$, then the center of $v$ on $X$ is the generic point $x_{E}$ of $E$. Indeed, the ring of regular function at the generic point of $E$ is a discrete valuation ring since $E$ is of codimension 1 . In that case, we will identify the center with its closure and say that the center of $v$ on $X$ is the prime divisor $E$. In fact a valuation is divisorial if and only if its center on some completion of $X_{0}$ is a prime divisor because if $c_{X}(v)=x_{E}$, then $v$ and $\operatorname{ord}_{E}$ defines the same valuation ring which is a discrete valuation ring, therefore they are equivalent.

Example 4.13. If $v$ is a curve valuation and $X$ is a completion of $X_{0}$ such that $\left(v_{X}\right)_{*} \nu \simeq v_{C, p}$, then the center of $v$ on $X$ is the closed point $p$.

A valuation over $\mathbf{k}\left[X_{0}\right]$ is centered at infinity if there exists a completion $X$ such that $c_{X}(v) \notin X_{0}$.

COROLLARY 4.14. Let $X_{0}$ be a smooth affine surface, there are exactly four types of valuations centered at infinity over $\mathbf{k}\left[X_{0}\right]$ : divisorial valuations, irrational valuations, curve valuations and infinitely singular valuations. If $\vee$ is a valuation such that $\mathfrak{p}_{v} \neq$ $\{0\}$, then $v$ is a curve valuation.

Proof. let $v$ be a valuation over $\mathbf{k}\left[X_{0}\right]$ and let $c_{X}(v)$ be its center on some completion $X$. If $c_{X}(v)$ is a prime divisor at infinity then $v$ is divisorial. Otherwise, $c_{X}(v)$ is a regular point at infinity and $v$ induces a centered valuation over $\widehat{O_{X, p}}$. The result follows from the classification of centered valuations over $\mathbf{k}[[x, y]]$ from Proposition 4.9

DEFINITION 4.15. - Let $X$ be a good completion of $X_{0}$ and $p \in \partial_{X} X_{0}$ a point at infinity. Following [FJ04], we say that $p$ is a free point if it belongs to a unique prime divisor at infinity and we say that it is a satellite point otherwise, i.e it is the intersection point of two prime divisors at infinity.

- Let $v$ be a valuation over $\mathbf{k}\left[X_{0}\right]$ centered at infinity. Let $p_{1}=c_{X}(v)$ be its center on $X$ and $X_{1}:=X$. We define the following sequence: If $p_{n}$ is a prime divisor, then the sequence stops, else $p_{n}$ is a closed point of $X_{n}$ and we define $X_{n+1}$ as the blow up of $p_{n}$, then define $p_{n+1}:=c_{X_{n+1}}(v)$. This is the sequence of centers of $v$ with respect to $X$.

We adopt the following convention: When we write "let $p \in E$ be a free point (at infinity)" this means that $E$ is the unique prime divisor at infinity on which $p$ lies. If we write "let $p=E \cap F$ be a satellite point", this means that $E$ and $F$ are the two prime divisors at infinity such that $p=E \cap F$ (Recall that we only work with good completions).

Proposition 4.16 ([|[FJ04], Section 6.2 ). Let v be a valuation centered at infinity. Let $X$ be a completion of $X_{0}$ and $\left(p_{n}\right)$ the sequence of centers (above $X$ ) associated to $v$. Then,
(1) v is divisorial if and only if the sequence $\left(p_{n}\right)$ is finite.
(2) If v is irrational, then $\left(p_{n}\right)$ contains finitely many free points.
(3) if v is a curve valuation, then $\left(p_{n}\right)$ contains finitely many satellite points.
(4) If $v$ is infinitely singular, then $\left(p_{n}\right)$ contains infinitely many free points.

Proof. Assertion 1 is clear since the sequence $\left(p_{n}\right)$ stops if and only if $p_{n}$ is a prime divisor at infinity. Assertion 2 and 4 follows from [FJ04] Theorem 6.10 and Assertion 3 follows from [ $\mathbf{F J 0 4}]$ Proposition 6.12.

### 4.4. Image of a valuation via an endomorphism

Let $f: X_{0} \rightarrow X_{0}$ be a endomorphism of $X_{0}$, it induces a map $f_{*}$ on the space of valuation $f_{*}: \mathcal{V} \rightarrow \mathcal{V}$ via the formula

$$
\begin{equation*}
\forall P \in \mathbf{k}\left[X_{0}\right], \forall v \in \mathcal{V}, \quad f_{*} v(\varphi) \tag{106}
\end{equation*}
$$

We will denote by $f_{\bullet}$ the induced map $f_{\bullet}: \widehat{\mathcal{V}} \rightarrow \hat{\mathcal{V}}$.
Proposition 4.17 (Proposition 2.4 of [FJ07]). Suppose that $f$ is dominant, the map $f_{*}$ preserves the sets of divisorial, of irrational and of infinitely singular valuations. If $v_{C}$ is a curve valuation such that $f$ does not contract $C$, then $f_{*} v_{C}$ is a curve valuation. If $f$ contracts $C$, then $f_{*} v_{C}$ is a divisorial valuation.

We will use this proposition in the following context. Let $X, Y$ be two completions of $X_{0}$ such that the lift $F: X \rightarrow Y$ of $f$ is regular. For any point $p \in X \backslash X_{0}$, we have a map $F_{*}: \mathcal{V}_{X}(p) \rightarrow \mathcal{V}_{Y}(F(p))$ that preserves the type of the valuations. The only curve that might be contracted by $F$ to $q$ are the divisors at infinity; but the curve valuation that they define do not define valuations on $\mathbf{k}\left[X_{0}\right]$.

Proposition 4.18. Let $f: X_{0} \rightarrow X_{0}$ be a dominant endomorphism of topological degree $\lambda_{2}$. Then, every valuation $v$ on $\mathbf{k}\left[X_{0}\right]$ has at most $\lambda_{2}$ preimages under $f_{*}$.

Proof. Suppose first that the valuation $v$ takes the value $+\infty$ only for 0 . Therefore, it extends to a valuation on $K=\operatorname{Frac} \mathbf{k}\left[X_{0}\right]$. The extension $f^{*} K \hookrightarrow K$ is a finite extension of degree $\lambda_{2}$. The valuation $v$ induces a valuation on $f^{*} K$ and every valuation $w$ such that $f_{*} w=v$ is an extension of $v_{\mid f^{*} K}$ to $K$. By [ZS60] Theorem 19 p.55, there are at most $\lambda_{2}$ extension of $v_{\mid f * K}$.

If now $\mathfrak{p}_{v}=\{v=+\infty\} \neq 0$, then we know that $v$ is a curve valuation. By Remark 4.10, $v$ can be made into a Krull valuation $\widehat{v}$. Since $\hat{v}$ is a Krull valuation, it extends to a Krull valuation over $K$ and $f_{*} v$ extends to a Krull valuation over $f^{*} K$. The same argument as above still works as [ZS60] deals with Krull valuations.

### 4.5. Tamely ramified endomorphisms

Let $K \hookrightarrow L$ be a field extension, let $v$ be a valuation over $K$ and let $w$ be a valuation over $L$ such that $w_{\mid K}=v$. If $\Gamma_{v}$ and $\Gamma_{w}$ are the value group of $v$ and $w$ respectively, we have $\Gamma_{v} \subset \Gamma_{w}$ and we define the ramification index $e(w \mid v)=\Gamma_{v}: \Gamma_{w}$.

If ${ }_{v}$ is the valuation ring of $v$ and ${ }_{w}$ the valuation ring of $w$. Let $\kappa_{v}$ be the residue field of $v$, then we have a field extension $\kappa_{v} \hookrightarrow \kappa_{w}$, the inertia degree is defined as $f(w \mid v):=\left[\kappa_{w}: \kappa_{v}\right]$. If $L / K$ is finite of degree $n$, then

$$
\begin{equation*}
e(w \mid v) f(w \mid v) \leqslant n \tag{107}
\end{equation*}
$$

Now consider a dominant endomorphism $f: X_{0} \rightarrow X_{0}$, let $L=\mathbf{k}\left(X_{0}\right)$ and $K=f^{*} L$. Following [ $\mathbf{C P 0 0}]$, we say that $f$ is tamely ramified if $f$ is separable and for every divisorial valuation $v$ of $X_{0}, e\left(v \mid f_{*} v\right)$ is not divisible by chark and the residue field extension $\kappa_{v} / \kappa_{f_{*} v}$ is separable.

In particular, if char $\mathbf{k}=0$ or $f$ is an automorphism, $f$ is automatically tamely ramified.

## CHAPTER 5

## Tree structure on the space of valuations

We show that the space of valuation centered at infinity of a normal affine surface $X_{0}$ has a local tree structure. Namely, the set of (normalized) valuations centered at a closed point is isomorphic to the valuative tree constructed in [FJ04]. We recall some of its properties.

### 5.1. Trees

For this section, we refer to [FJ04] Section 3.1. Let $(\mathcal{T}, \leqslant)$ be a partially ordered set, a subset $\mathcal{S} \subset \mathcal{T}$ is full if for every $\sigma, \sigma^{\prime} \in \mathcal{S}, \tau \in \mathcal{T}, \sigma \leqslant \tau \leqslant \sigma^{\prime} \Rightarrow \tau \in \mathcal{S}$.

Definition 5.1. Let $\Lambda=\mathbf{N}, \mathbf{Q}, \mathbf{R}$. An interval in $\Lambda$ is a subset $I \subset \Lambda$ such that for all $x, y, z \in \Lambda$, if $x \leqslant y \leqslant z$ and $x, z \in I$, then $y \in I$. If $(\mathcal{T}, \leqslant)$ be a partially ordered set, then $(\mathcal{T}, \leqslant)$ is a rooted $\Lambda$-tree if
(i) $\mathcal{T}$ has a unique minimal element $\tau_{0}$ called the root of $\mathcal{T}$.
(ii) If $\tau \in \mathcal{T}$, the set $\{\sigma \in \mathcal{T}: \sigma \leqslant \tau\}$ is ${ }^{1}{ }^{1}$ somorphic to an interval in $\Lambda$.
(iii) Every full, totally ordered subset of $\mathcal{T}$ is isomorphic to an interval in $\Lambda$.

A parametrized- $\Lambda$ tree is a rooted $\Lambda$-tree $\mathcal{T}$ with a map $\alpha: \mathcal{T} \rightarrow \Lambda \cup\{\infty\}$ such that the restriction of $\alpha$ to any full totally ordered subset of $\mathcal{T}$ induces a bijection with an interval in $\Lambda$. The map $\alpha$ is called the parametrisation.

A rooted $\mathbf{R}$-tree is called complete if every increasing sequence has an upper bound.
A subtree $S$ of a $\Lambda$-tree $\mathcal{T}$ is a subset such that $\left(S, \leqslant_{\mid S}\right)$ is a $\Lambda$-tree. An inclusion of trees is an order preserving injection $1: \mathcal{S} \rightarrow \mathcal{T}$. Where $\mathcal{S}$ is a $\Lambda$-tree, and $\mathcal{T}$ is a $\Lambda^{\prime}$-tree, we do not require $\Lambda=\Lambda^{\prime}$. For example $\mathbf{N} \hookrightarrow \mathbf{R}$ is an inclusion of trees. In particular, if $\Lambda=\Lambda^{\prime}$, then $1(S)$ is a subtree of $\mathcal{T}$.

If $\mathcal{T}$ is an $\mathbf{R}$-tree and $\tau_{1}, \tau_{2} \in \mathcal{T}$, then the minimum $\tau_{1} \wedge \tau_{2} \in \mathcal{T}$ exists by completeness of $\mathbf{R}$. We define the set

$$
\begin{equation*}
\left[\tau_{1}, \tau_{2}\right]:=\left\{\tau \in \mathcal{T}: \tau_{1} \wedge \tau_{2} \leqslant \tau \leqslant \tau_{1} \text { or } \tau_{1} \wedge \tau_{2} \leqslant \tau \leqslant \tau_{2}\right\} \tag{108}
\end{equation*}
$$

and we call it a segment. The segments $\left[\tau_{1}, \tau_{2}\right),\left(\tau_{1}, \tau_{2}\right]$ and $\left(\tau_{1}, \tau_{2}\right)$ are defined similarly. A finite subtree of $\mathcal{T}$ is a subtree that consists of a finite union of segments in $\mathcal{T}$.

[^3]If $\mathcal{T}$ is an $\mathbf{R}$-tree, a tangent vector $\vec{v}$ at $\tau \in \mathcal{T}$ is an equivalence class of elements $\tau^{\prime} \in \mathcal{T} \backslash\{\tau\}$ where

$$
\begin{equation*}
\tau^{\prime} \sim \tau^{\prime \prime} \Leftrightarrow\left(\tau, \tau^{\prime}\right] \cap\left(\tau, \tau^{\prime \prime}\right] \neq \varnothing . \tag{109}
\end{equation*}
$$

We define the weak topology on $\mathcal{T}$ by the topology generated by the sets

$$
\begin{equation*}
U(\vec{v}):=\left\{\tau^{\prime} \in \mathcal{T} \backslash\{\tau\}: \tau^{\prime} \text { represents } \vec{v}\right\} . \tag{110}
\end{equation*}
$$

THEOREM 5.2 ([|[J04] Proposition 3.12). We have the following

- Every rooted $\mathbf{R}$-tree $\mathcal{T}$ admits a completion $\overline{\mathcal{T}}$ that is a complete rooted $\mathbf{R}$-tree.
- Every rooted $\mathbf{Q}$-tree $\mathcal{T}_{\mathbf{Q}}$ admits a completion $\mathcal{T}_{\mathbf{R}}$ into a complete rooted $\mathbf{R}$-tree, i.e there exists an order preserving injection $\mathfrak{1}: \mathcal{T}_{\mathbf{Q}} \hookrightarrow \mathcal{T}_{\mathbf{R}}$ such that
(1) If $\tau_{0}$ is the root of $\mathcal{T}_{\mathbf{Q}}, \mathbf{l}\left(\tau_{0}\right)$ is the root of $\mathcal{T}_{\mathbf{R}}$.
(2) $\mathbf{l}\left(\mathcal{T}_{\mathbf{Q}}\right)$ is weakly dense in $\mathcal{T}_{\mathbf{R}}$
(3) $\mathcal{T}_{\mathbf{R}}$ is minimal for this property.
- If $\alpha_{\mathbf{Q}}: \mathcal{T}_{\mathbf{Q}} \rightarrow \mathbf{Q}_{+}$is a parametrisation of $\mathcal{T}_{\mathbf{Q}}$, then there exists a unique parametrisation $\alpha_{\mathbf{R}}$ of $\mathcal{T}_{\mathbf{R}}$ such that $\alpha_{\mathbf{Q}}=\alpha_{\mathbf{R}} \circ \mathbf{v}$.


### 5.2. The local tree structure of the space of valuations

We denote by $\mathcal{V}_{0}$ the set of centered valuations on $R$ where $R=\mathbf{k}[[x, y]]$. Define the multiplicity valuation $\nu_{\mathfrak{m}}$ by $\nu_{\mathfrak{m}}(\varphi)=\max \left\{n \geqslant 0: \varphi \in \mathfrak{m}^{n}\right\}$. We will sometimes write $m(\varphi)$ instead of $v_{\mathfrak{m}}(\varphi)$. Let $\mathcal{V}_{\mathfrak{m}} \subset \mathcal{V}_{0}$ be the set of centered valuations on $R$ such that $v(\mathfrak{m})=1$ and consider the following order relation on $\mathcal{V}_{\mathfrak{m}}$ :

$$
\begin{equation*}
v \leqslant w \Longleftrightarrow \forall \varphi \in R, v(\varphi) \leqslant w(\varphi) . \tag{111}
\end{equation*}
$$

With this order relation $\mathcal{V}$ becomes a complete rooted $\mathbf{R}$-tree called the valuative tree ([FJ04] Theorem 3.14) rooted in $v_{\mathfrak{m}}$. The ends of $\mathcal{V}_{\mathfrak{m}}$ consist of the curve valuations and the infinitely singular ones. The interior points are all quasimonomial valuations, all divisorial valuations are branching points whereas all the irrational valuations are regular points (i.e admit only two tangent vectors). Define on $\mathcal{V}_{\mathfrak{m}}$ the following function

$$
\begin{equation*}
\alpha(v):=\sup \left\{\frac{v(\varphi)}{m(\varphi)}: \varphi \in \mathfrak{m}\right\} \tag{112}
\end{equation*}
$$

It is called the skewness function (see [FJ04] §3.3)
Proposition 5.3 (Proposition 3.25 of [FJ04]). The skewness function $\alpha: \mathcal{V}_{\mathfrak{m}} \rightarrow$ $[1,+\infty]$ defines a parametrisation of $\mathcal{V}_{\mathfrak{m}}$. We have the following properties.

- $\alpha(v)=1 \Leftrightarrow v=v_{\mathfrak{m}}$.
- Let $\varphi \in \mathfrak{m}$ be irreducible and let $v \in \mathcal{V}_{\mathfrak{m}}$, then

$$
\begin{equation*}
\forall \varphi \in \mathfrak{m}, v(\varphi)=\alpha\left(v \wedge v_{\varphi}\right) m(\varphi) \tag{113}
\end{equation*}
$$

- If $v$ is divisorial, then $\alpha(v) \in \mathbf{Q}$.
- If v is irrational, then $\alpha(\mathrm{v}) \in \mathbf{R} \backslash \mathbf{Q}$.
- If $\vee$ is a curve valuation, then $\alpha(v)=+\infty$.
- If $\vee$ is infinitely singular, then $\alpha(v) \in(1,+\infty]$ and every value is realised.
- If $\mathcal{V}_{\mathfrak{m} \text {,div }}$ is the subset of $\mathcal{V}_{\mathfrak{m}}$ consisting of the divisorial valuations, then $\left(\mathcal{V}_{\mathfrak{m}, \mathrm{div}}, \alpha\right)$ is a parametrized $\mathbf{Q}$-tree.

We can define two topologies over $\mathcal{V}_{\mathfrak{m}}$. The first one is the weak topology being the coarsest topology such that for all $\varphi \in R$, the evaluations map $v \in \mathcal{V}_{\mathfrak{m}} \mapsto \nu(\varphi)$ is continuous. The second is the weak topology given by the $\mathbf{R}$-tree structure on $\mathcal{V}_{\mathfrak{m}}$.

Proposition $5.4\left([\mathbf{F J 0 4}]\right.$, Theorem 5.1). The weak topology over $\mathcal{V}_{\mathfrak{m}}$ given by the evaluation maps $v \in \mathcal{V}_{\mathfrak{m}} \mapsto \mathrm{v}(\varphi)$ and the weak topology induced by the tree structure of $\mathcal{V}_{\mathfrak{m}}$ are the same.

Let $X$ be a good completion of $X_{0}$ and let $p$ be a smooth point of $X$. Take local coordinates $z, w$ at $p$, then the completion of the local ring $O_{X, p}$ with respect the maximal ideal $\mathfrak{m}_{p}$ is isomorphic to $\mathbf{k}[[z, w]]$. Let $\mathcal{V}_{X}(p)$ be the set of valuations $v$ on $\mathbf{k}\left[X_{0}\right]$ centered at $p$. We will denote by $\mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$ the subset of $\mathcal{V}_{X}(p)$ of valuations $v$ such that $v\left(\mathfrak{m}_{p}\right)=1$. The space $\mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$ is an $\mathbf{R}$-tree isomorphic rooted in $v_{\mathfrak{m}_{p}}$. We make its structure precise.

Proposition 5.5. The $\mathbf{R}$-tree $\mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$ is not complete.
(1) If $p \in E$ is a free point then $\mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$ is isomorphic to $\mathcal{V}_{\mathfrak{m}} \backslash\left\{v_{z}\right\}$ where $z$ is a local equation of $E$.
(2) If $p=E \cap F$ is a satellite point, then $\mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$ is isomorphic to $\mathcal{V}_{\mathfrak{m}} \backslash\left\{\mathbf{v}_{z}, v_{w}\right\}$ where $z, w$ are local coordinates at $p$ with $z$ a local equation of $E$ and $w$ a local equation of $F$.
Proof. If $p \in E$ is a free point, let $z, w$ be local coordinates at $p$ such that $z$ is a local equation of $E$. Then, the completion of the local ring at $p$ is isomorphic to $\mathbf{k}[[z, w]]$ by Theorem 2.2. Every $P \in \mathbf{k}\left[X_{0}\right]$ is of the form $P=\frac{\varphi}{z^{a}}$ with $a \geqslant 0$ and $\varphi \in O_{X, p}$. Hence, a centered valuation on $\mathbf{k}[[z, w]]$ defines a valuation over $\mathbf{k}\left[X_{0}\right]$ if and only if it is not the curve valuation $v_{z}$. Hence we have an isomorphism $\mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right) \simeq \mathcal{V}_{\mathfrak{m}} \backslash\left\{\mathrm{v}_{z}\right\}$.

If $p=E \cap F$ is a satellite point, then let $z, w$ be local coordinates at $p$ such that $z$ is a local equation of $E$ and $w$ is a local equation of $F$. Every $P \in \mathbf{k}\left[X_{0}\right]$ is of the form $P=\frac{\varphi}{z^{a} w^{b}}$ where $a, b \geqslant 0$ and $\varphi \in O_{X, p}$. Therefore a centered valuation on $\mathbf{k}[[z, w]]$ defines a valuation over $\mathbf{k}\left[X_{0}\right]$ if and only if it is not the curve valuation $v_{z}$ or $\nu_{w}$. Hence we have an isomorphism $\mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right) \rightarrow \mathcal{V}_{\mathfrak{m}} \backslash\left\{\mathrm{v}_{z}, \mathrm{v}_{w}\right\}$.

### 5.3. The relative tree with respect to a curve $z=0$

Let $R=\mathbf{k}[[x, y]]$ and let $\mathfrak{m}$ be the maximal ideal of $R$. Let $z \in \mathfrak{m}$ be irreducible such that $v_{\mathfrak{m}}(z)=1$. One can consider the set $V_{z}$ of centered valuations on $R$ such that
$v(z)=1$; we also add the valuation $\operatorname{ord}_{z}$ to $\mathcal{V}_{z}$ defined by $\operatorname{ord}_{z}(\varphi)=\max \left\{n \geqslant 0: z^{n} \mid \varphi\right\}$. (notice that $\operatorname{ord}_{z}$ is not centered, because for example if $x \neq z, \operatorname{ord}_{z}(x)=0$ ). This is also a tree rooted in $\operatorname{ord}_{z}$ called the relative tree (see [FJ04] Proposition 3.61) with the order relation $v \leqslant_{z} \mu \Leftrightarrow \forall \varphi \in R, v(\varphi) \leqslant \mu(\varphi)$. We can define the weak topology on $\mathcal{V}_{z}$ being the coarsest topology such that the for all $\varphi \in R$, the evaluation map $v \in \mathcal{V}_{z} \mapsto v(\varphi)$ is continuous. There is also the weak topology given by the tree structure of $\mathcal{V}_{z}$.

Proposition 5.6 (Relative version of 5.4). The weak topology over $\mathcal{V}_{z}$ given by the evaluation maps $v \in \mathcal{V}_{z} \mapsto v(\varphi)$ and the weak topology induced by the tree structure of $\mathcal{V}_{z}$ are the same.

Proposition 5.7 ([[FJ04] Lemma 3.59). We have an onto map $N_{z}: \mathcal{V}_{0} \rightarrow \mathcal{V}_{z}$ defined by

$$
\begin{aligned}
N_{z}(\mathrm{v}) & =\mathrm{v} / \mathrm{v}(z) \text { if } \mathrm{v} \neq \mathrm{v}_{z} \\
N_{z}\left(\mathrm{v}_{z}\right) & =\operatorname{ord}_{z} .
\end{aligned}
$$

This map restricts to a homeomorphism $N_{z}: \mathcal{V}_{\mathfrak{m}} \rightarrow \mathcal{V}_{z}$ with respect to the weak topology. If $w \in \mathfrak{m}$ is irreducible, then the map $N_{z, w}:=N_{w} \circ N_{w}{ }^{-1}: \mathcal{V}_{z} \rightarrow \mathcal{V}_{w}$ is a homeomorphism for the weak topology.

The tree $\mathcal{V}_{z}$ comes with a skewness function $\alpha_{z}: \mathcal{V}_{z} \rightarrow[0,+\infty]$ and a multiplicity function $m_{z}(\varphi)=\nu_{z}(\varphi)$. The skewness is defined by

$$
\begin{equation*}
\alpha_{z}(v):=\sup \left\{\left.\frac{v(\psi)}{m_{z}(\psi)} \right\rvert\, \psi \in \mathfrak{m}\right\} \tag{114}
\end{equation*}
$$

Proposition 5.8 (Relative version of Proposition 5.3). The function $\alpha_{z}: V_{z} \rightarrow$ $[0,+\infty]$ defines a parametrisation of the tree $\mathcal{V}_{z}$. We have the following properties.

- $\alpha_{z}(v)=0 \Leftrightarrow v=\operatorname{ord}_{z}$.
- Let $\varphi \in \mathfrak{m}$ be irreducible and let $v \in \mathcal{V}_{z}$, then

$$
\begin{equation*}
v(\varphi)=\alpha_{z}\left(v \wedge N\left(v_{\varphi}\right)\right) m_{z}(\varphi) \tag{115}
\end{equation*}
$$

- If v is divisorial or $\mathrm{v}=\operatorname{ord}_{z}$, then $\alpha_{z}(v) \in \mathbf{Q}$
- If $\vee$ is irrational, then $\alpha_{z}(v) \in \mathbf{R} \backslash \mathbf{Q}$.
- If v is a curve valuation, then $\alpha_{z}(\mathrm{v})=+\infty$.
- If $v$ is infinitely singular, then $\alpha_{z}(v) \in(0,+\infty]$ and every value is realised.
- If $\mathcal{V}_{z, \text { div }}$ is the subset of $\mathcal{V}_{z}$ consisting of $\operatorname{ord}_{z}$ and divisorial valuations, then $\left(\mathcal{V}_{z, \text { div }}, \alpha_{z}\right)$ is a parametrised $\mathbf{Q}$-tree.
Proposition 5.9 ([[]J04], Proposition 3.65). We have the following relation

$$
\begin{equation*}
\forall v \in \mathcal{V}_{0}, \quad v(z)^{2} \alpha_{z}\left(\frac{v}{v(z)}\right)=\min (v(x), v(y))^{2} \alpha\left(\frac{v}{\min (v(x), v(y))}\right) \tag{116}
\end{equation*}
$$

If $w \in \mathfrak{m}$ is another irreducible element with $m(w)=1$, then

$$
\begin{equation*}
\forall v \in \mathcal{V}_{0}, v(z)^{2} \alpha_{z}\left(\frac{v}{v(z)}\right)=v(w)^{2} \alpha_{w}\left(\frac{v}{v(w)}\right) \tag{117}
\end{equation*}
$$

Proposition 5.10 ([[FJ04], Lemma 3.60 and 6.47). The map $N: \mathcal{V}_{\mathfrak{m}} \rightarrow \mathcal{V}_{z}$ is not an isomorphism of trees. The two orders on $\mathcal{V}_{\mathfrak{m}}$ and $\mathcal{V}_{z}$ are compatible except on the segments $\left[v_{\mathfrak{m}}, v_{z}\right]$ and $\left[\operatorname{ord}_{z}, N\left(v_{\mathfrak{m}}\right)\right]$ where they are reversed. More precisely,
(1) $\forall v, \mu \in\left[v_{\mathfrak{m}}, v_{z}\right] \subset \mathcal{V}_{\mathfrak{m}}, v \leqslant_{\mathfrak{m}} \mu \Leftrightarrow N(v) \geqslant_{z} N(\mu)$.
(2) $\forall v_{1}, v_{2} \in \mathcal{V}_{z} \backslash\left\{\operatorname{ord}_{z}\right\}, v_{1} \leqslant_{z} v_{2} \Leftrightarrow\left[N^{-1}\left(v_{1}\right), v_{z}\right] \subset\left[N^{-1}\left(\mathrm{v}_{2}\right), v_{x}\right]$.

The situation is summed up in Figure 1 where we have put arrows on the branches of the tree to indicate the order.


Figure 1. The homeomorphism between $\mathcal{V}_{\mathfrak{m}}$ and $\mathcal{V}_{z}$
We will use the relative tree in the following context. Let $E$ be a prime divisor at infinity of some good completion $X$, let $p$ be a point of $E$ and let $z, w$ be local coordinates at $p$ such that $E=\{z=0\}$. The completion of the local ring at $p$ is isomorphic to $\mathbf{k}[[z, w]]$. We define $\mathcal{V}_{X}(p ; E)$ as follows; an element of $\mathcal{V}_{X}(p ; E)$ is either a valuation $v$ on $\mathbf{k}\left[X_{0}\right]$ centered at $p$ such that $v(z)=1$ or the divisorial valuation $\operatorname{ord}_{E}$. Notice that the definition of $\mathcal{V}_{X}(p ; E)$ does not depend on the local equation $z=0$ of $E$ because the quotient of two local equations is a regular invertible function.

Proposition 5.11. Let $X$ be a completion and let $p \in X$ be a closed point at infinity.
(1) If $p \in E$ is a free point, then $\mathcal{V}_{X}(p ; E)$ is isomorphic to $\mathcal{V}_{z}$.
(2) If $p=E \cap F$ is a satellite point. Let $z, w$ be local coordinates at $p$ such that $z$ is a local equation of $E$ and $w$ a local equation of $F$ then $\mathcal{V}_{X}(p ; E)$ is isomorphic to $\mathcal{V}_{z} \backslash\left\{\mathrm{v}_{w}\right\}$ and $\mathcal{V}_{X}(p ; F)$ is isomorphic to $\mathcal{V}_{w} \backslash\left\{\mathrm{v}_{z}\right\}$.
The map $N_{z}: \mathcal{V}_{\mathfrak{m}} \rightarrow \mathcal{V}_{z}$ induces a homeomorphism

$$
\begin{equation*}
N_{p, E}: \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right) \rightarrow \mathcal{V}_{X}(p ; E) \backslash\left\{\operatorname{ord}_{E}\right\} \tag{118}
\end{equation*}
$$

Furthermore, if $p=E \cap F$, then the map

$$
\begin{equation*}
N_{p, F} \circ N_{p, E}^{-1}: \mathcal{V}_{X}(p ; E) \backslash\left\{\operatorname{ord}_{E}\right\} \rightarrow \mathcal{V}_{X}(p ; F) \backslash\left\{\operatorname{ord}_{F}\right\} \tag{119}
\end{equation*}
$$

is a homeomorphism.
Proof. If $p \in E$ is a free point. Let $z, w$ be local coordinates at $p$ such that $z$ is a local equation of $E$. The completion of the local ring at $p$ is isomorphic to $\mathbf{k}[[z, w]]$ by Theorem 2.2. For every $P \in \mathbf{k}\left[X_{0}\right], P$ is of the form $P=\frac{\varphi}{z^{a}}$ where $a \geqslant 0$ and $\varphi \in O_{X, p}$. Therefore, a centered valuation on $\mathbf{k}[[z, w]]$ defines a valuation over $\mathbf{k}\left[X_{0}\right]$ if and only if it is not the curve valuation $v_{z}$. Since $\mathcal{v}_{z} \notin \mathcal{V}_{z}$ we have that $\mathcal{V}_{X}(p ; E) \simeq \mathcal{V}_{z}$. Call $\sigma: V_{X}(p ; E) \rightarrow \mathcal{V}_{z}$ the isomorphism. We define $N_{p, E}$ as follows. Recall by Proposition 5.7 that there is a homeomorphism $N: \mathcal{V}_{\mathfrak{m}} \rightarrow \mathcal{V}_{z}$ where in particular $N\left(\mathcal{V}_{z}\right)=\operatorname{ord}_{z}$. Here we have that $\operatorname{ord}_{z}$ is canonically identified with $\operatorname{ord}_{E}$ and $\mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$ is isomorphic to $\mathcal{V}_{\mathfrak{m}} \backslash\left\{\mathrm{v}_{z}\right\}$, call $\mathfrak{i}: \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right) \rightarrow \mathcal{V}_{\mathfrak{m}} \backslash\left\{\mathrm{v}_{z}\right\}$ the isomorphism. Define

$$
\begin{equation*}
N_{p, E}:=\sigma^{-1} \circ N \circ \tau: \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right) \rightarrow \mathcal{V}_{X}(p ; E) \backslash\left\{\operatorname{ord}_{E}\right\} \tag{120}
\end{equation*}
$$

it is a homeomorphism.
If $p=E \cap F$ is a satellite point. Let $(z, w)$ be local coordinates at $p$ such that $z$ is a local equation of $E$ and $w$ is a local equation of $F$. The completion of the local ring at $p$ is isomorphic to $\mathbf{k}[[z, w]]$ by Theorem 2.2. Every $P \in \mathbf{k}\left[X_{0}\right]$ is of the form $P=\frac{\varphi}{z^{a} w^{b}}$ where $a, b \geqslant 0$ and $\varphi \in O_{X, p}$. Therefore a centered valuation on $\mathbf{k}[[z, w]]$ defines a valuation over $\mathbf{k}\left[X_{0}\right]$ if and only if it is not the curve valuation associated to $z$ or $w$. Or $v_{z}$ does not belong to $V_{z}$ but $v_{w}$ does. Therefore, $V_{X}(p ; E)$ is isomorphic to $V_{z} \backslash\left\{v_{w}\right\}$. If $N_{z}: \mathcal{V}_{\mathfrak{m}} \rightarrow \mathcal{V}_{z}$ is the map from Proposition 5.7, then $N\left(v_{z}\right)=\operatorname{ord}_{z}$ and $N\left(v_{w}\right)=v_{w}$. Therefore, $N_{w} \circ N_{z}^{-1}: \mathcal{V}_{z} \rightarrow \mathcal{V}_{w}$ is a homeomorphism that sends ord ${ }_{z}$ to $v_{z}$ and $\nu_{w}$ to $\operatorname{ord}_{w}$. Fix an isomorphism $\tau_{E}: \mathcal{V}_{X}(p ; E) \rightarrow \mathcal{V}_{z}\left\{v_{w}\right\}$ and $\tau_{F}: \mathcal{V}_{X}(p ; F) \rightarrow \mathcal{V}_{w} \backslash \mathcal{V}_{z}$. We have that the map

$$
\begin{equation*}
N_{p, F} \circ N_{p, E}^{-1}=\tau_{F}^{-1} \circ N_{w} \circ N_{z}^{-1} \circ \tau_{E}: \mathcal{V}_{X}(p ; E) \backslash\left\{\operatorname{ord}_{E}\right\} \rightarrow \mathcal{V}_{X}(p ; F) \backslash\left\{\operatorname{ord}_{F}\right\} \tag{121}
\end{equation*}
$$

is a homeomorphism.

Proposition 5.12. Let $X$ be a completion of $X_{0}$ and let $E$ be a prime divisor at infinity. If $p_{1}, p_{2} \in E$ are closed points with $p_{1} \neq p_{2}$, then $\mathcal{V}_{X}\left(p_{1} ; E\right) \cap \mathcal{V}_{X}\left(p_{2} ; E\right)=$ $\left\{\operatorname{ord}_{E}\right\}$. Define the set $\mathcal{V}_{X}(E ; E)$ of valuations $v$ such that $c_{X}(v) \in E$ and $v(z)=1$ where $z$ is a local equation of $E$ at $c_{X}(v)$. Then

$$
\begin{equation*}
\mathcal{V}_{X}(E ; E)=\bigcup_{p \in E} \mathcal{V}_{X}(p ; E) \tag{122}
\end{equation*}
$$

and it has a natural structure of a rooted $\mathbf{R}$-tree rooted in $\operatorname{ord}_{E}$. The skewness functions $\alpha_{E}$ glue together to give $\mathcal{V}_{X}(E ; E)$ the structure of a parametrized rooted tree. Every point $p \in E$ defines a tangent vector at $\operatorname{ord}_{E}$ given by $\mathcal{V}_{X}(p ; E) \backslash\left\{\operatorname{ord}_{E}\right\}$.

Furthermore, Let $Y$ be a completion of $X_{0}$ and $q \in Y$ a closed point at infinity. Let $\pi: Z \rightarrow Y$ be the blow up of $q$ and let $\widetilde{E}$ be the exceptional divisor of $\pi$. Then, for every $\widetilde{q} \in \widetilde{E}$, the map $\pi_{\bullet}: \mathcal{V}_{Z}(\widetilde{q} ; \widetilde{E}) \rightarrow \mathcal{V}_{Y}\left(q ; \mathfrak{m}_{q}\right)$ is actually equal to $\pi_{*}$ and they glue together to give a map

$$
\begin{equation*}
\pi_{*}: \mathcal{V}_{Z}(\widetilde{E} ; \widetilde{E}) \rightarrow \mathcal{V}_{Y}\left(q ; \mathfrak{m}_{q}\right) \tag{123}
\end{equation*}
$$

which is an isomorphism of trees. We have the relation $\alpha_{\mathfrak{m}_{q}} \circ \pi_{*}=1+\alpha_{E}$ and $b_{\mathfrak{m}_{q}} \circ \pi_{*}=$ $b_{E}$.

We postpone the proof to $\$ 5.5$. If $E \simeq \mathbf{P}^{1}$, this tree is isomorphic to the tree of normalized valuations centered at infinity over $\mathbf{A}^{2}$ constructed in (FJ07], Appendix.

### 5.4. The monomial valuations centered at an intersection point at infinity

Let $X$ be a good completion of $X_{0}$ and let $E, F$ be two divisors at infinity that intersect at a point $p$. Let $(x, y)$ be local coordinates at $p$ such that $E=\{x=0\}$ and $F=\{y=0\}$. There are three spaces to consider: $\mathcal{V}_{X}\left(p, \mathfrak{m}_{p}\right), \mathcal{V}_{X}(p ; E)$ and $\mathcal{V}_{X}(p ; F)$. We explain here how they are related. For $(s, t) \in[0,+\infty]^{2} \backslash\{(0,0),(\infty, \infty)\}$, we denote by $v_{s, t}$ the monomial valuation defined by

$$
\begin{equation*}
v_{s, t}\left(\sum a_{i j} x^{i} y^{j}\right)=\min \left\{s i+t j \mid a_{i j} \neq 0\right\} \tag{124}
\end{equation*}
$$

Notice that $v_{0,1}=\operatorname{ord}_{F}, v_{1,0}=\operatorname{ord}_{E}, \nu_{1, \infty}=v_{y}, \nu_{\infty, 1}=v_{x}$. We will denote the set of such valuation by $\left[\operatorname{ord}_{E}, \operatorname{ord}_{F}\right]$. We use this notation because of the following: $\left[\operatorname{ord}_{E}, \operatorname{ord}_{F}\right] \cap$ $\mathcal{V}_{X}(p ; E)$ consists of the valuations $v_{1, t}$ for $t \in[0,+\infty)$ and $\left[\operatorname{ord}_{E}, \operatorname{ord}_{F}\right] \cap \mathcal{V}_{X}(p ; F)$ consists of the valuations $v_{s, 1}$ for $s \in[0,+\infty)$. So they define segments in the respective trees. In particular we have

$$
\begin{equation*}
N_{p, F} \circ N_{p, E}^{-1}\left(v_{1, t}\right)=v_{1 / t, 1}, \quad \forall t \in(0,+\infty) \tag{125}
\end{equation*}
$$

One can show with the definition of the skewness function $\alpha$ that $\alpha_{E}\left(v_{1, t}\right)=t$. Therefore we show

Lemma 5.13. Let $\vee$ be a monomial valuation centered at $p=E \cap F$. One has

$$
\begin{aligned}
& \alpha_{E}\left(\frac{v}{v(x)}\right)=\frac{v(y)}{v(x)}=\frac{s}{t} \text { if } v=v_{s, t} \\
& \alpha_{F}\left(\frac{v}{v(y)}\right)=\frac{v(x)}{v(y)}=\frac{t}{s} \text { if } v=v_{s, t}
\end{aligned}
$$

In particular we have that $\alpha_{E}\left(\frac{v}{v(x)}\right)=\alpha_{F}\left(\frac{v}{v(y)}\right)^{-1}$ on $] \operatorname{ord}_{E}, \operatorname{ord}_{F}[$.

### 5.5. Geometric interpretations of the valuative tree

Let $X$ be a completion of $X_{0}$ and let $p \in X$ be a closed point at infinity. We consider in this section only completions above $X$ that are exceptional above $p$. If $\pi:(Y, \operatorname{Exc}(\pi)) \rightarrow(X, p)$ is such a completion, then we call $\Gamma_{\pi}$ the dual graph which vertices consist of the exceptional divisors of $\pi$. Two exceptional divisors are linked by an edge if they intersect. The graph $\Gamma_{\pi}$ is connected without cycles, it is therefore an $\mathbf{N}$-tree. We set the root of $\Gamma_{\pi}$ to be the exceptional divisor $\widetilde{E}$ that appears after blowing up $p$.

If $E$ is a prime divisor at infinity of $X$ such that $p \in E$. We define the dual graph

$$
\begin{equation*}
\Gamma_{\pi, E}:=\Gamma_{\pi} \cup\{E\} . \tag{126}
\end{equation*}
$$

It is also a $\mathbf{N}$-tree. We set the root of $\Gamma_{\pi, E}$ to be $E$.
Lemma 5.14 ([|[J04], Proposition 6.2). Let $\pi: Y \rightarrow(X, p)$ be a completion exceptional above $p$. if $\tau: Z \rightarrow Y$ is the blow up of a point in the exceptional locus of $\pi$, then there are natural inclusions of $\mathbf{N}$-trees

$$
\begin{equation*}
\Gamma_{\pi} \hookrightarrow \Gamma_{\pi \circ \tau}, \quad \Gamma_{\pi, E} \hookrightarrow \Gamma_{\pi \circ \tau, E} . \tag{127}
\end{equation*}
$$

Therefore, the direct limits $\Gamma:=\xrightarrow[\longrightarrow]{\lim _{\|}} \Gamma_{\pi}, \Gamma_{E}:=\underline{\lim }_{\rightarrow} \Gamma_{\pi, E}$ are well defined. The points of $\Gamma$ are in bijection with $\mathcal{D}_{X, p}$ and $\Gamma_{E}=\Gamma \cup\{E\}$ and they have a structure of $\mathbf{Q}$-trees.

Lemma 5.15 ([FJ04] Theorem 6.9). We have a map $\pi_{\bullet}: \Gamma_{\pi} \hookrightarrow \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)_{\text {div }} d e$ fined by

$$
\begin{equation*}
\pi_{\bullet}(F)=v_{F} \tag{128}
\end{equation*}
$$

where $v_{F}$ is the valuation equivalent to $\pi_{*} \operatorname{ord}_{F}$ that belongs to $\mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$. These maps are compatible with the direct limit and give a map $\Gamma \hookrightarrow \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$.

Lemma 5.16. We have a map $\pi_{\bullet}: \Gamma_{\pi, E} \hookrightarrow \mathcal{V}_{E, \text { div }}$ defined by

$$
\begin{equation*}
\pi_{\bullet}(F)=v_{F} \tag{129}
\end{equation*}
$$

where $v_{F}$ is the valuation equivalent to $\pi_{*} \operatorname{ord}_{F}$ that belongs to $V_{X}(p ; E)$. These maps are compatible with the direct limit and give a map $\Gamma_{E} \hookrightarrow \mathcal{V}_{X}(p ; E)$.

Proposition 5.17 ([|[J04], Lemma 6.28). Let $\pi:(Y, \operatorname{Exc}(\pi)) \rightarrow(X, p)$ be a completion exceptional above $p$. Let $q \in Y$ be a closed point that belongs to the exceptional component of $\pi$. Let $\widetilde{F}$ be the exceptional divisor above $q$.
(1) If $q \in F$ with $F \in \Gamma_{\pi}$, then $v_{\widetilde{F}}>v_{F}$.
(2) If $q=F_{1} \cap F_{2}$ with $F_{1}, F_{2} \in \Gamma_{\pi}$, suppose that $\nu_{F_{1}}<\nu_{F_{2}}$, then $\nu_{F_{1}}<\nu_{\widetilde{F}}<\nu_{F_{2}}$.

Proposition 5.18 (Relative version of Proposition 5.17). Let $\pi:(Y, \operatorname{Exc}(\pi)) \rightarrow$ $(X, p)$ be a completion exceptional above p. Let $q \in \operatorname{Exc}(\pi)$. Let $\widetilde{F}$ be the exceptional divisor above $q$.
(1) If $q \in F$ is a free point with $F \in \Gamma_{\pi, E}$, then $\vee_{\widetilde{F}}>\mathrm{v}_{F}$.
(2) If $q=F_{1} \cap F_{2}$ is a satellite point with $F_{1}, F_{2} \in \Gamma_{\pi, E}$, if $\vee_{F_{1}}<\mathrm{v}_{F_{2}}$, then $\mathrm{v}_{F_{1}}<$ $\mathrm{V}_{\tilde{F}}<\mathrm{V}_{F_{2}}$.
(3) In particular, if $q=E \cap F$, then $\operatorname{ord}_{E}<\nu_{\tilde{F}}<\nu_{F}$.

THEOREM 5.19 ([[FJ04], Theorem 6.22). We have an isomorphism of $\mathbf{Q}$-trees

$$
\begin{equation*}
\Gamma \simeq \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)_{\operatorname{div}}, \quad \Gamma_{E} \simeq \mathcal{V}_{X}(p ; E)_{\operatorname{div}} \tag{130}
\end{equation*}
$$

given by $F \simeq \mathrm{v}_{F}$. We can take the completion of the $\mathbf{Q}$-trees to get the isomorphism

$$
\begin{equation*}
\bar{\Gamma} \simeq \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right), \quad \bar{\Gamma}_{E} \simeq \mathcal{V}_{X}(p ; E) \tag{131}
\end{equation*}
$$

Proposition 5.20. Let $X$ be a completion of $X_{0}$ and let $p \in X$ be a closed point at infinity. Let $\mathcal{V}_{*}$ be either $\mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$ or $\mathcal{V}_{X}(p ; E)$ for some prime divisor $E$ at infinity such that $p \in E$. Let $\Gamma_{*}$ be either $\Gamma$ or $\Gamma_{E}$. Let $\pi:(Y, \operatorname{Exc}(\pi)) \rightarrow(X, p)$ be a completion exceptional above $p$. Let $q \in \operatorname{Exc}(\pi)$ be a closed point. The map $\pi$ induces a map $\pi_{*}: \mathcal{V}_{Y}(q) \rightarrow \mathcal{V}_{X}(p)$.
(1) If $q \in E_{q}$ is a free point with $E_{q} \in \Gamma_{*}$, then we have an inclusion map $\pi_{\bullet}$ : $\mathcal{V}_{Y}\left(q ; E_{q}\right) \hookrightarrow \mathcal{V}_{*}$. The order relation in $\mathcal{V}_{Y}\left(q ; E_{q}\right)$ and $\mathcal{V}_{*}$ are compatible and $\pi_{0}$ is an inclusion of trees.
 relations on $\mathcal{V}_{*}$ and $\mathcal{V}_{Y}\left(q ; E_{q}\right)$ are compatible and $\pi_{\bullet}: V_{Y}\left(q ; E_{q}\right) \hookrightarrow \mathcal{V}_{*}$ is an inclusion of trees.

Proof. We only need to show that the orders are compatible on the divisorial valuations of $\mathcal{V}_{Y}\left(q ; E_{q}\right)$. Therefore we show the following,

CLAIM 5.21. For every completion $\tau:(Z, \operatorname{Exc}(\tau)) \rightarrow(Y, q)$ exceptional above $q$, we have the following
(1) For all $F_{1}, F_{2} \in \Gamma_{\tau, E_{q}}$,

$$
\begin{equation*}
v_{F_{1}}<_{*} v_{F_{2}} \Leftrightarrow v_{F_{1}}<_{E_{q}} v_{F_{2}} \tag{132}
\end{equation*}
$$

(2) If $F \in \Gamma_{\tau, E_{q}}$ satisfies $F \cap F_{q} \neq \varnothing$, then

$$
\begin{equation*}
\mathrm{v}_{F}<\mathrm{v}_{F_{q}} \tag{133}
\end{equation*}
$$

Here there is a slight abuse of notation as we denote by $v_{F_{i}}$ the image of $F_{i}$ both in $\mathcal{V}_{Y}\left(q ; E_{q}\right)$ and $\mathcal{V}_{*}$. This is done to lighten notations, we believe that it does not provide any confusion.

We prove this by induction on the number of blow ups above $q$. If $\tau=\mathrm{id}$, then $\operatorname{ord}_{E_{q}}$ is the root of $\mathcal{V}_{Y}\left(q ; E_{q}\right)$ and $\nu_{E_{q}}<\nu_{F_{q}}$ by assumption so there is nothing to do.

Let $\tau:(Z, \operatorname{Exc}(\tau)) \rightarrow(Y, q)$ be a completion exceptional above $q$ such that Claim (5.21) is true. Let $q^{\prime} \in \operatorname{Exc}(\tau)$ be a closed point, let $\tau^{\prime}: Z^{\prime} \rightarrow Z$ be the blow up of $q^{\prime}$ and let $\widetilde{F}$ be the exceptional divisor above $q^{\prime}$.

- If $q^{\prime} \in F$ is a free point with $F \in \Gamma_{\tau, E_{q}}$, then by Proposition 5.18 we have

$$
\begin{equation*}
v_{F}<E_{q} v_{\widetilde{F}} \tag{134}
\end{equation*}
$$

Now we have two possibilities.

- If $q^{\prime}$ is also a free point with respect to $\Gamma_{*}$, then by Proposition 5.17 and 5.18 we also get

$$
\begin{equation*}
v_{F}<_{*} v_{\widetilde{F}} . \tag{135}
\end{equation*}
$$

Since $\widetilde{F} \cap F_{q}=\varnothing$, Claim 5.21 is shown for $\Gamma_{\tau \circ \tau^{\prime}, E_{q}}$.

- If $q^{\prime}$ is the satellite point $F \cap F_{q}$, then by induction hypothesis we have $\nu_{F}<_{*} v_{F_{q}}$ and therefore $\widetilde{F} \cap F_{q} \neq \varnothing$ and by Proposition 5.17 and 5.18 we get

$$
\begin{equation*}
v_{F}<_{*} v_{\widetilde{F}}<_{*} v_{F_{q}} \tag{136}
\end{equation*}
$$

So Claim5.21 is shown for $\Gamma_{\tau \circ \tau^{\prime}, E_{q}}$.

- If $q^{\prime}$ is a satellite point. Let $F_{1}, F_{2} \in \Gamma_{\tau, E_{q}}$ such that $q=F_{1} \cap F_{2}$. Suppose without loss of generality that $v_{F_{1}}<_{E_{q}} \nu_{F_{2}}$, then by the induction hypothesis we have $\nu_{F_{1}}<_{*} \nu_{F_{2}}$ and by Proposition 5.17 and 5.18, we get

$$
\begin{equation*}
v_{F_{1}}<E_{E_{q}} v_{\widetilde{F}}<_{E_{q}} v_{F_{2}} \text { and } v_{F_{1}}<_{*} v_{\widetilde{F}}<_{*} v_{F_{2}} \tag{137}
\end{equation*}
$$

Since $\widetilde{F} \cap F_{q}=\varnothing$ we have proven Claim 5.21 for $\Gamma_{\tau \circ \tau^{\prime}, E_{q}}$.

Proof of Proposition 5.12, Let $Y$ be a completion of $X_{0}$ and let $q \in Y$ be a closed point at infinity. Let $\pi: Z \rightarrow Y$ be the blow up of $q$. Let $\widetilde{E}$ be the exceptional divisor and let $\widetilde{q} \in \widetilde{E}$ be a closed point. Apply Proposition 5.20 with $\mathcal{V}_{*}=\mathcal{V}_{Y}\left(q ; \mathfrak{m}_{q}\right)$. The map $\pi_{\bullet}: V_{Z}(\widetilde{q} ; \widetilde{E}) \rightarrow \mathcal{V}_{Y}\left(q ; \mathfrak{m}_{q}\right)$ is an inclusion of trees. There exists local coordinates $z, w$ at $q$ and $x, y$ at $p$ such that $\pi(z, w)=(z, z w)$ where $z$ is a local equation of $\widetilde{E}$. We therefore get

$$
\begin{equation*}
v(z)=1 \Leftrightarrow \min \left(\pi_{*} v(x), \pi_{*} v(y)\right)=1 \tag{138}
\end{equation*}
$$

Hence, $\pi_{\bullet}=\pi_{*}$ and $\pi_{*}\left(\operatorname{ord}_{\tilde{E}}\right)=\nu_{\mathfrak{m}_{q}}$. Therefore we can glue these maps to obtain an isomorphism of trees

$$
\begin{equation*}
\pi_{*}: \mathcal{V}_{Z}(\widetilde{E} ; \widetilde{E}) \rightarrow \mathcal{V}_{Y}\left(q ; \mathfrak{m}_{q}\right) \tag{139}
\end{equation*}
$$

We get the relation on the skewness functions by Proposition 5.28 which will be proven in the next section.

### 5.6. Properties of skewness

We have two valuative tree structures. We describe some properties of the skewness function for these two structures and how they behave after blowing up. Fix a completion $X$, let $p \in X$ be a closed point at infinity and let $E$ be a prime divisor at infinity in $X$ such that $p \in E$. In accordance with the notations of the previous section, set $\Gamma=\mathcal{D}_{X, p}$ and $\Gamma_{E}=\mathcal{D}_{X, p} \cup\{E\}$.

DEFINITION 5.22. If $F \in \Gamma$ is a prime divisor above $p$, we define the generic multiplicity $b(F)$ inductively as follows.

- $b(\widetilde{E})=1$ where $\widetilde{E}$ is the exceptional divisor above $p$.
- If $q \in F$ is a free point with $F \in \Gamma$, then $b(\widetilde{F})=b(F)$ where $\widetilde{F}$ is the exceptional divisor above $q$.
- If $q=F_{1} \cap F_{2}$ is a satellite point with $F_{1}, F_{2} \in \Gamma$, then $b(\widetilde{F})=b\left(F_{1}\right)+b\left(F_{2}\right)$.

If $v \in V_{X}\left(p ; \mathfrak{m}_{p}\right)$ is divisorial then we define $b(v):=b(E)$ where $E$ is the center of $v$ in some completion above $X$.

DEFINITION 5.23. If $F \in \Gamma_{E}$, we define the relative generic multiplicity $b_{E}(F)$ inductively as follows.

- $b_{E}(E)=1$.
- If $q \in F$ is a free point with $F \in \Gamma_{E}$, then $b_{E}(\widetilde{F})=b_{E}(F)$.
- If $q=F_{1} \cap F_{2}$ is a satellite point with $F_{1}, F_{2} \in \Gamma_{E}$, then $b_{E}(\widetilde{F})=b_{E}\left(F_{1}\right)+$ $b_{E}\left(F_{2}\right)$.
If $v \in \mathcal{V}_{X}\left(p ; E_{z}\right)$ is divisorial, then we set $b_{E}(v):=b_{E}(F)$ where $F$ is the center of $v$ in some completion above $X$.

Figure 2 sums up the definition of the generic multiplicity.
The term generic multiplicity is justified by the following proposition.
Proposition 5.24 ([|[5J04] Proposition 6.26). Let $\mathfrak{v} \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$ be divisorial, let $E \in \Gamma$ be the center of $\vee$ over some completion $\pi: Y \rightarrow X$ above $X$. Then,

$$
\begin{equation*}
\pi_{*} \operatorname{ord}_{E}\left(\mathfrak{m}_{p}\right)=b(v) \tag{140}
\end{equation*}
$$



Figure 2. Algorithm for computing the generic multiplicity
Proposition 5.25 (Relative version of Proposition 5.24. If $v \in \mathcal{V}_{X}(p ; E)$ is divisorial, let $F$ be the center of $v$ over some completion $\pi: Y \rightarrow X$ above $X$. Then,

$$
\begin{equation*}
\pi_{*} \operatorname{ord}_{F}(z)=b_{E}(F) \tag{141}
\end{equation*}
$$

where $z \in O_{X, p}$ is a local equation of $E$. This means that $\operatorname{ord}_{F}\left(\pi^{*} E\right)=b_{E}(F)$.
From now on we write $\mathcal{V}_{*}$ for either $\mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$ and $\mathcal{V}_{X}(p ; E)$ and we write $\alpha_{*}, b_{*}$ for the skewness function and the generic multiplicity function associated to the tree structure.

For a valuation $v \in \mathcal{V}_{*}$, we define the infinitely near sequence of $v$ as follows, set $v_{0}=v_{*}$ the root of $\mathcal{V}_{*}$ and let $p_{n}$ be the sequence of centers above $X$ associated to $v$. Let $E_{n}$ be the exceptional divisor above $p_{n}$. Set $v_{n}=\frac{1}{b_{*}\left(E_{n}\right)} \operatorname{ord}_{E_{n}}$, if $v$ is quasimonomial $\left(v_{n}\right)$ is the infinitely near sequence of $v$, in particular it is finite if and only if $v$ is divisorial. If $v$ is a curve valuation or infinitely singular we define the infinitely near sequence of $v$ as the subsequence of $v_{n}$ where $c_{X_{n}}(v)$ is a free point (at infinity).

Proposition 5.26. Let $v \in \mathcal{V}_{*}$ and let $\mathrm{v}_{n}$ be its infinitely near sequence

- the sequence $\mathrm{v}_{n}:=\frac{1}{b_{n}} \operatorname{ord}_{E_{n}}$ converges weakly towards v .
- $\alpha_{*}(v)=\lim _{n} \alpha_{*}\left(v_{n}\right)$.

Proof. The infinitely near sequence is constructed in Section 6.2.2 of [FJ04] (this sequence does not have a name in [FJ04]). The fact $v_{n}$ converges weakly towards $v$ is shown there. To show the statement for skewness, we split the proof with respect to the type of $v$.

If $v$ is a curve valuation or an infinitely singular one, then $v_{n}<v$ and $v_{n}$ increases towards $v$. Since $\alpha$ induces an order preserving bijection of the segment $\left[v_{*}, v\right]$. We have that $\alpha\left(v_{n}\right) \leqslant \alpha(v)$ and is increasing. So it converges towards a real number $\alpha_{0} \in$ $\left[\alpha_{*}\left(v_{*}\right), \alpha_{*}(v)\right]$. If $\alpha_{0}<\alpha_{*}(v)$, then $v_{n} \leqslant v_{0}<v$ where $\alpha_{*}\left(v_{0}\right)=\alpha_{0}$ and this is absurd.

If $v$ is irrational, then there exists $N_{0}$ such that for all $n \geqslant N_{0}, p_{n}$ is a satellite point. We can split the sequence $\left(v_{n}\right)_{n \geqslant N_{0}}$ into two subsequences $\left(v_{k}^{+}\right),\left(v_{k}^{-}\right)$such that $v_{k}^{+}$is increasing and converges towards $v$ and $\left(v_{k}^{-}\right)$is decreasing and converges towards $v$. We therefore get

$$
\begin{equation*}
\alpha\left(v_{k}^{+}\right)<\alpha(v)<\alpha\left(v_{k}^{-}\right) \tag{142}
\end{equation*}
$$

and it is clear that $\lim _{k} \alpha\left(v_{k}^{+}\right)=\lim _{k} \alpha\left(v_{k}^{-}\right)=\alpha(v)$.
We will say that two divisorial valuations $v, v^{\prime}$ are adjacent if there exists a completion $Y$ above $X$ such that the centers of $v$ and $v^{\prime}$ are both prime divisors and they intersect.

Proposition 5.27 ([[FJ04], Corollary 6.39). Let $\mathrm{v}, \mathrm{v}^{\prime} \in \mathcal{V}_{*}$. Assume $\mathrm{v}<\mathrm{v}^{\prime}$ and that they are adjacent, then

$$
\begin{equation*}
\alpha_{*}\left(v^{\prime}\right)-\alpha_{*}(v)=\frac{1}{b_{*}(v) b_{*}\left(v^{\prime}\right)} \tag{143}
\end{equation*}
$$

Proposition 5.28 ([|[J04], Theorem 6.51). Let $\pi: Y \rightarrow X$ be a completion above $X$ and let $q \in E_{q}$ be a free point of $Y$ such that $\pi\left(E_{q}\right)=p$. By Proposition 5.20 $\pi_{\mathbf{0}}$ : $\mathcal{V}_{Y}\left(q ; E_{q}\right) \rightarrow \mathcal{V}_{*}$ is an inclusion of trees.
(1) The normalization of $\pi_{*} \operatorname{ord}_{E_{q}}$ (to get a valuation in $\mathcal{V}_{*}$ ) is

$$
\begin{equation*}
\pi_{\bullet} \operatorname{ord}_{E_{q}}=: v_{E_{q}}=\frac{1}{b_{*}\left(E_{q}\right)} \pi_{*} \operatorname{ord}_{E_{q}} \tag{144}
\end{equation*}
$$

(2)

$$
\begin{array}{ll}
\forall v \in \mathcal{V}_{Y}(p ; E), & \alpha_{*}\left(\pi_{\bullet} v\right)=\alpha_{*}\left(v_{E_{q}}\right)+\frac{1}{b_{*}\left(E_{q}\right)^{2}} \alpha_{E_{q}}(v) \\
& b_{*}\left(\pi_{\bullet} v\right)=b_{*}\left(E_{q}\right) b_{E_{1}}(v) \tag{146}
\end{array}
$$

Proof. It suffices to show this formula for every divisorial valuation $v \in \mathcal{V}_{Y}\left(q ; E_{q}\right)$ and then use infinitely near sequences by Proposition 5.26. We prove the result by induction on the number of blow-ups above $q$. Namely we show the following

Claim 5.29. For every completion $\tau:(Z, \operatorname{Exc}(\tau)) \rightarrow(Y, q)$ exceptional above $q$, for every $F \in \Gamma_{\tau, E_{q}}$,

$$
\begin{align*}
b_{*}(F) & =b_{E_{q}}(F) b_{*}\left(E_{q}\right)  \tag{147}\\
\alpha_{*}\left(v_{F}\right) & =\alpha_{*}\left(v_{E_{q}}\right)+\frac{1}{b_{*}\left(E_{q}\right)} \alpha_{E_{q}}\left(v_{F}\right) \tag{148}
\end{align*}
$$

If $\tau=\mathrm{id}: Y \rightarrow Y$, then $\Gamma_{\tau, E_{q}}=\left\{E_{q}\right\}$. We have by definition that $b_{E_{q}}\left(E_{q}\right)=1, \alpha_{E_{q}}\left(\operatorname{ord}_{E_{q}}\right)=$ 0 . Therefore Equations (147) and (148) holds.

Suppose the claim to be true for a completion $\tau:(Z, \operatorname{Exc}(\tau)) \rightarrow(Y, q)$ exceptional above $q$. Let $\tau^{\prime}: Z^{\prime} \rightarrow Z$ be the blow up of a closed point $q^{\prime} \in \operatorname{Exc}(\tau)$. Let $\widetilde{E}$ be the exceptional divisor above $q^{\prime}$.

If $q^{\prime} \in F$ is a free point with $F \in \Gamma_{\tau, E}$, then $q^{\prime}$ is also a free point with respect to $\Gamma_{*, \pi \circ \tau}$ because $q \in Y$ is a free point. Therefore by definition

$$
\begin{equation*}
b_{*}(\widetilde{E})=b_{*}(F), \quad b_{E_{q}}(\widetilde{E})=b_{E_{q}}(F) \tag{149}
\end{equation*}
$$

So Equation (147) is true for $\widetilde{E}$ by induction. Now, by Proposition 5.27

$$
\begin{equation*}
\alpha_{*}\left(v_{\widetilde{E}}\right)=\alpha_{*}\left(v_{F}\right)+\frac{1}{b_{*}(F) b_{*}\left(E_{q}\right)}, \quad \alpha_{E_{q}}\left(v_{\widetilde{E}}\right)=\alpha_{E_{q}}\left(v_{F}\right)+\frac{1}{b_{E_{q}}(\widetilde{E}) b_{E_{q}}(F)} \tag{150}
\end{equation*}
$$

By induction, Equation (148) is true for $\widetilde{E}$.
If $q^{\prime}=F_{1} \cap F_{2}$ is a satellite point with $F_{1}, F_{2} \in \Gamma_{\tau, E_{q}}$, then

$$
\begin{equation*}
b_{*}(\widetilde{E})=b_{*}\left(F_{1}\right)+b_{*}\left(F_{2}\right), \quad b_{E_{q}}(\widetilde{E})=b_{E_{q}}\left(F_{1}\right)+b_{E_{q}}\left(F_{2}\right) \tag{151}
\end{equation*}
$$

So by induction Equation (147) holds for $\widetilde{E}$. Suppose without loss of generality that $v_{F_{1}}<v_{F_{2}}$ both in $\mathcal{V}_{*}$ and $\mathcal{V}_{Y}\left(q ; E_{q}\right)$. This is possible by Proposition 5.20. By Proposition 5.27

$$
\begin{equation*}
\alpha_{*}\left(v_{\widetilde{E}}\right)=\alpha_{*}\left(v_{F_{1}}\right)+\frac{1}{b_{*}\left(F_{1}\right) b_{*}(\widetilde{E})}, \quad \alpha_{E_{q}}\left(v_{\widetilde{E}}\right)=\alpha_{E_{q}}\left(v_{F_{1}}\right)+\frac{1}{b_{E_{q}}\left(F_{1}\right) b_{E_{q}}(\widetilde{E})} . \tag{152}
\end{equation*}
$$

Therefore, Equation (148) holds for $\widetilde{E}$. And the claim is shown by induction.
Proposition 5.30. Let $v$ be a valuation over $\mathbf{k}\left[X_{0}\right]$ centered at infinity. Let $X$ be a completion of $X_{0}$ and let $E$ be a prime divisor of $X$ at infinity such that $\widetilde{v} \in \mathcal{V}_{X}(E ; E)$ for some valuation $\widetilde{v}$ equivalent to $v$. If $\alpha_{E}(\widetilde{v})<+\infty$, then for every completion $Y$ of $X_{0}$ if $\widetilde{\mathcal{v}} \in \mathcal{V}_{Y}(F, F)$ for some prime divisor $F$ at infinity in $Y$, then $\alpha_{F}(\widetilde{v})<+\infty$.

Proof. If $v$ is quasimonomial, this is immediate as for any prime divisor $E$ at infinity and any closed point $p \in E$, we have that $\alpha_{E}(v)<+\infty$ for $v=\operatorname{ord}_{E}$ or $v$ quasimonomial centered at $p$. If $v$ is a curve valuation, then $\alpha_{E}(v)=+\infty$ for any prime divisor $E$ of any completion $X$ such that $c_{X}(v) \in E$. So it remains to show the result for $v$ an infinitely singular valuation.

We show that if $\pi: Y \rightarrow X$ is a completion above $X$, then $\alpha_{E^{\prime}}(v)<+\infty \Leftrightarrow \alpha_{E}(v)<$ $+\infty$ where $E^{\prime}$ is a prime divisor of $Y$ at infinity such that some multiple of $v$ belongs to $\mathcal{V}_{Y}\left(E^{\prime}, E^{\prime}\right)$. Let $p=c_{X}(v)$ and $q=c_{Y}(v)$. Since $v$ is infinitely singular, by Proposition 4.16 there exists a completion $\tau:(Z, \operatorname{Exc}(\tau)) \rightarrow(Y, q)$ exceptional above $q$ such that $c_{Z}(v)$ is a free point $q^{\prime}$ lying over a unique prime divisor $F$ at infinity. We apply Proposition 5.28. We have that

$$
\begin{align*}
& \alpha_{E}(v)=\alpha_{E}\left(v_{F}\right)+\frac{1}{b_{E}(F)^{2}} \alpha_{F}(v)  \tag{153}\\
& \alpha_{E^{\prime}}(v)=\alpha_{E^{\prime}}\left(v_{F}\right)+\frac{1}{b_{E^{\prime}}(F)^{2}} \alpha_{F}(v) \tag{154}
\end{align*}
$$

Thus $\alpha_{E}(v)<+\infty \Leftrightarrow \alpha_{F}(v)<+\infty \Leftrightarrow \alpha_{E^{\prime}}(v)<+\infty$.
Proposition 5.31 ([|FJ04] Proposition 6.35). Let $\pi:(Y, \operatorname{Exc}(\pi)) \rightarrow(X, p)$ be a completion exceptional above $p$. Let $q=E \cap F \in \operatorname{Exc}(\pi)$ be a satellite point with $E, F \in$ $\Gamma_{*, \pi}$. Define $v_{E}=\frac{1}{b_{*}(E)} \pi_{*} \operatorname{ord}_{E}$ and $v_{F}=\frac{1}{b_{*}(F)} \pi_{*} \operatorname{ord}_{F}$. Let $z, w$ be local coordinates at $q$ associated to $(E, F)$. Let $v_{s, t}$ be the monomial valuation centered at $q$ such that $v(z)=s$ and $v(w)=t$. Then, the map $\pi_{*}$ induces a homeomorphism from the set $\left\{v_{s, t} \mid s, t \geqslant 0, s b_{*}(E)+t b_{*}(F)=1\right\}$ and $\left[v_{E}, v_{F}\right] \subset \mathcal{V}_{*}$ for the weak topology.

Furthermore, the skewness function is given by

$$
\begin{equation*}
\alpha_{*}\left(\pi_{*} v_{s, t}\right)=\alpha\left(v_{E}\right)+\frac{t}{b_{*}(E)} \tag{155}
\end{equation*}
$$

Proof. The first part of the proposition is exactly the content of [FJ04] Proposition 6.35. We compute the skewness. It suffices to show (155) for the divisorial valuations in $\left[v_{E}, v_{F}\right]$ and then use infinitely near sequences. We show (155) by induction on the number of blowups. The result holds for $v_{E}=v_{1 / b_{*}(E), 0}$ and $v_{F}=v_{0,1 / b_{*}(F)}$. Let $v_{s_{1}, t_{1}}, v_{s_{2}, t_{2}}$ be adjacent divisorial valuations such that (155) holds. Let $E_{1}, E_{2}$ be the associated prime divisor and let $\tau:(Z, \operatorname{Exc}(\tau) \rightarrow(Y, E \cap F)$ be a completion exceptional above $E \cap F$ such that $E_{1}, E_{2}$ intersect in $Z$ and let $q=E_{1} \cap E_{2}$. Let $(x, y)$ be local coordinates at $q$ associated to $\left(E_{1}, E_{2}\right)$ and let $\widetilde{E}$ be the exceptional divisor above $q$ and $\omega$ be the blow up of $q$. We want to compute $s, t \geqslant 0$ such that

$$
\begin{equation*}
(\tau \circ \omega)_{*} \operatorname{ord}_{\widetilde{E}}=v_{s, t} \tag{156}
\end{equation*}
$$

To do so, we need to compute $\operatorname{ord}_{\widetilde{E}}\left((\tau \circ \omega)^{*} z\right)$ and $\left((\tau \circ \omega)^{*} \operatorname{ord}_{\tilde{E}} w\right)$ Let $b_{1}=b_{*}\left(E_{1}\right)$ and $b_{2}=b_{*}\left(E_{2}\right)$. We have by the first part of the proposition that

$$
\begin{equation*}
v_{E_{i}}=(\pi \circ \tau)_{*} \frac{1}{b_{i}} \operatorname{ord}_{E_{i}}=\pi_{*}\left(v_{s_{i}, t_{i}}\right) . \tag{157}
\end{equation*}
$$

Thus, $\tau_{*} \frac{1}{b_{i}} \operatorname{ord}_{E_{i}}=v_{s_{i}, t_{i}}$. In local coordinates $(u, v), \omega$ is given by

$$
\begin{equation*}
\omega(u, v)=(u, u v) \tag{158}
\end{equation*}
$$

where $u=0$ is a local equation of $\widetilde{E}$ and $v=0$ is a local equation of the strict transform of $E_{2}$. By (157), we get up to multiplication by invertible germs of functions that

$$
\begin{equation*}
\omega^{*}\left(\tau^{*} z\right)=\omega^{*}\left(x^{s_{1} b_{1}} y^{s_{2} b_{2}}\right)=u^{s_{1} b_{1}+s_{2} b_{2}} v^{s_{2} b_{2}} . \tag{159}
\end{equation*}
$$

and

$$
\begin{equation*}
\omega^{*}\left(\tau^{*} w\right)=u^{t_{1} b_{1}+t_{2} b_{2}} v^{t_{2} b_{2}} \tag{160}
\end{equation*}
$$

Thus, $s=s_{1} b_{1}+s_{2} b_{2}$ and $t=t_{1} b_{1}+t_{2} b_{2}$. This implies that

$$
\begin{equation*}
\pi_{*} \mathbf{v}_{\frac{s_{1} b_{1}+s_{2} b_{2}}{b_{1}+b_{2}}, \frac{t_{1} b_{1}+t_{2} b_{2}}{b_{1}+b_{2}}}=v_{\widetilde{E}} \tag{161}
\end{equation*}
$$

We compute the skewness, by Proposition 5.27 we have that

$$
\begin{equation*}
\alpha_{*}\left(v_{\tilde{E}}\right)=\frac{b_{1} \alpha_{*}\left(v_{E_{1}}\right)+b_{2} \alpha_{*}\left(v_{E_{2}}\right)}{b_{1}+b_{2}} \tag{162}
\end{equation*}
$$

and by induction, we get

$$
\begin{equation*}
\alpha_{*}\left(v_{\widetilde{E}}\right)=\frac{b_{1}\left(\alpha\left(v_{E}\right)+\frac{t_{1}}{b(E)}\right)+b_{2}\left(\alpha\left(v_{E}\right)+\frac{t_{2}}{b(E)}\right)}{b_{1}+b_{2}}=\alpha\left(v_{E}\right)+\frac{t_{1} b_{1}+t_{2} b_{2}}{b_{E}\left(b_{1}+b_{2}\right)} \tag{163}
\end{equation*}
$$

and the result is shown by induction.

## CHAPTER 6

## Different topologies over the space of valuations

We define two topologies on the space of valuations centered at infinity. We saw in the previous chapter that the space of valuations centered at infinity can be viewed as a space with an atlas of open subsets given by valuation trees. The valuation tree comes with a weak and a strong topology and they glue together to define the weak and the strong topology on the whole space of valuations centered at infinity.

### 6.1. The weak topology

Let $X_{0}$ be an affine surface and let $\mathcal{V}_{\infty}$ be the space of valuations centered at infinity. We define $\widehat{V}_{\infty}$ to be the space of valuations centered at infinity modulo equivalence and $\eta: \mathcal{V}_{\infty} \rightarrow \widehat{\mathcal{V}_{\infty}}$ the quotient map. We define the weak topology over $\mathcal{V}_{\infty}$ as follows. A basis for the topology is given by

$$
\begin{equation*}
\left\{v \in \mathcal{V}_{\infty}: t<v(P)<t^{\prime}\right\} \tag{164}
\end{equation*}
$$

for some $t, t^{\prime} \in \mathbf{R}, P \in \mathbf{k}\left[X_{0}\right]$. A sequence $v_{n}$ of $\mathcal{V}_{\infty}$ converges towards $v$ if and only if for every $P \in \mathbf{k}\left[X_{0}\right]$, the sequence $v_{n}(P)$ converges towards $v(P)$. We define the weak topology over $\widehat{\mathcal{V}_{\infty}}$ to be the thinnest topology such that $\eta: \mathcal{V}_{\infty} \rightarrow \widehat{\mathcal{V}_{\infty}}$ is continuous with respect to the weak topology.

Proposition 6.1. Let $X$ be a completion of $X_{0}$. Let $v \in \mathcal{V}_{\infty}$ and $\left(v_{n}\right)$ a sequence of elements of $\mathcal{V}_{\infty}$. Suppose that $v_{n} \rightarrow v$ with respect to the weak topology. Then,

- If $c_{X}(v)=p$ is a closed point at infinity, then for all $n$ large enough $c_{X}\left(v_{n}\right)=p$.
- If $c_{X}(v)=E$ is a prime divisor at infinity, then for all $n$ large enough $c_{X}\left(v_{n}\right) \in$ E.

Proof. Suppose first that $c_{X}(v)=p$ is a closed point at infinity. Let $(x, y)$ be local coordinates at $p$. By definition of the center we have $v(x), v(y)>0$. We can find $P_{1}, P_{2}, Q_{1}, Q_{2} \in O_{X}\left(X_{0}\right)$ such that $x=P_{1} / Q_{1}, y=P_{2} / Q_{2}$ and such that $v\left(Q_{1}\right), v\left(Q_{2}\right) \neq \infty$. Indeed by Lemma 4.11, $O_{X, p}$ is a subring of $O_{X}\left(X_{0}\right)_{\mathfrak{p}_{v}}$ where $\mathfrak{p}_{v}=\{v=+\infty\}$. Now, we have that $v_{n}\left(P_{i}\right) \rightarrow v\left(P_{i}\right)$ and $v_{n}\left(Q_{i}\right) \rightarrow v\left(Q_{i}\right)$ as $n \rightarrow \infty$, therefore for all n large enough

$$
\begin{equation*}
v_{n}(x), v_{n}(y)>0 \tag{165}
\end{equation*}
$$

Thus, for all $n$ large enough $c_{X}\left(v_{n}\right)=p$.

If $c_{X}(v)=E$, then $v=\lambda \operatorname{ord}_{E}$ for some $\lambda>0$. Let $U$ be an open affine subset of $X$ such that $U \cap E \neq \varnothing$. Let $z$ be a local equation of $E$ over $U$. Similarly, we can write $z=P / Q$ with $v(Q) \neq \infty$. Since $v_{n}(P) \rightarrow v(Q)$ and $v_{n}(Q) \rightarrow v(Q)$, we get that $v_{n}(z) \rightarrow v(z)>0$. Therefore for $n$ large enough, $v_{n}(z)>0$ and therefore $c_{X}\left(v_{n}\right) \in E$.

Proposition 6.2. Let $X$ be a completion and let $p \in X$ be a closed point at infinity. Let $v \in \mathcal{V}_{X}(p)$ and $v_{n} \in \mathcal{V}_{X}(p)$. Then, $v_{n} \rightarrow v$ weakly if and only if for every $\varphi \in$ $O_{X, p}, v_{n}(\varphi) \rightarrow v(\varphi)$.

Proof. Indeed, every $\varphi \in O_{X, p}$ can be written as $\varphi=\frac{P}{Q}$ with $v(Q) \neq \infty$. This shows one implication. Conversely, every $P \in \mathbf{k}\left[X_{0}\right]$ is of the form $\frac{\varphi}{\psi}$ where $\varphi, \psi \in O_{X, p}$. Furthermore, if $p \in E$ is a free point then $\psi=u^{a}$ where $a \in \mathbf{Z}_{\geqslant 0}$ and $u$ is a local equation of $E$. If $p=E \cap F$ is a satellite point, then $\psi=u^{a} v^{b}$ where $u v$ is a local equation of $E \cup F$. Now since $v_{n}$ and $v$ are valuations over $\mathbf{k}\left[X_{0}\right]$, they cannot be the curve valuations associated to a prime divisor at infinity. Therefore, for all $n, v_{n}(\psi) \neq \infty$ and $v(\psi) \neq \infty$. This shows the other implication.

Proposition 6.3. Let $X$ be a completion of $X_{0}$ and let $p \in X$ be a closed point. Let $E$ be a prime divisor at infinity in $X$ such that $p \in X$. Let $\eta_{p}: \mathcal{V}_{X}(p) \rightarrow \mathcal{V}_{X}(p ; E)$ be the natural map defined by $\eta_{p}(v)=\frac{v}{v(z)}$ where $z \in O_{X, p}$ is a local equation of $E$. Let $\left(v_{n}\right)$ be a sequence of $\mathcal{V}_{X}(p)$ and let $v \in \mathcal{V}_{X}(p)$. If $v_{n} \rightarrow v$ for the weak topology of $\mathcal{V}_{\infty}$, then $\eta_{p}\left(v_{n}\right) \rightarrow \eta_{p}(v)$ for the weak topology of $\mathcal{V}_{X}(p ; E)$.

Proof. If $v_{n} \rightarrow v$ for the weak topology, then, $v_{n}(z) \rightarrow v(z)$ by Proposition 6.2. Therefore $\eta_{p}\left(v_{n}\right) \rightarrow \eta_{p}(v)$, again by Proposition 6.2. This shows the first implication.

THEOREM 6.4. Let $X$ be a completion of $X_{0}$. The weak topology on $\widehat{\mathcal{V}_{\infty}}$ is the topology induced by the open subsets $\mathcal{V}_{X}(E ; E)$ for all prime divisor $E$ at infinity.

Proof. Let $X$ be a completion at infinity and let $E$ be a prime divisor at infinity. Let $V_{X}(E)$ be the set of valuations $v$ over $\mathbf{k}\left[X_{0}\right]$ such that $c_{X}(v) \in E$ (this includes $c_{X}(v)=E$, i.e $v=\operatorname{ord}_{E}$ ). We have that

$$
\begin{equation*}
\mathcal{V}_{X}(E)=\left\{\operatorname{ord}_{E}\right\} \cup \bigcup_{p \in E} \mathcal{V}_{X}(p) \tag{166}
\end{equation*}
$$

Let $U_{1}, \cdots, U_{r}$ be a finite open affine cover of $E$ such that for every $i=1, \cdots, r$ there exists $z_{i} \in O_{X}\left(U_{i}\right)$ a local equation of $E$. Then, every $z_{i}$ is of the form $z_{i}=P_{i} / Q_{i}$ with $P_{i}, Q_{i} \in \mathbf{k}\left[X_{0}\right]$. Then,

$$
\begin{equation*}
\mathcal{V}_{X}(E)=\bigcup_{i}\left\{v\left(Q_{i}\right)<+\infty, v\left(P_{i}\right)-v\left(Q_{i}\right)>0\right\} \tag{167}
\end{equation*}
$$

and, it follows that $\mathcal{V}_{X}(E)$ is an open subset of $\mathcal{V}_{\infty}$. Set $\widehat{\mathcal{V}_{\infty}}(p):=\eta\left(\mathcal{V}_{X}(p)\right)$. Define a $\operatorname{map} \sigma_{p}: \widehat{\mathcal{V}_{\infty}}(p) \rightarrow \mathcal{V}_{X}(p ; E) \backslash\left\{\operatorname{ord}_{E}\right\} \subset \mathcal{V}_{X}(p)$ by

$$
\begin{equation*}
\sigma_{p}([v])=\eta_{p}(v) \tag{168}
\end{equation*}
$$

where $\eta_{p}$ is the map from Proposition 6.3 and $[v]$ is the class of $v$ in $\widehat{V_{\infty}}$. By Proposition 6.3. $\sigma_{p}$ is a continuous section of $\eta_{\mid V_{X}(p)}: \mathcal{V}_{X}(p) \rightarrow \widehat{\mathcal{V}}_{\infty}(p)$. Still by Proposition 6.3., the map $\sigma_{p}:\left[\operatorname{ord}_{E}\right] \cup \widehat{\mathcal{V}_{\infty}}(p) \rightarrow \mathcal{V}_{X}(p ; E)$ extended by $\sigma_{p}\left(\left[\operatorname{ord}_{E}\right]\right)=\operatorname{ord}_{E}$ is also a continuous section of $\eta:\left\{\lambda \operatorname{ord}_{E}: \lambda>0\right\} \cup \mathcal{V}_{X}(p) \rightarrow\left\{\left[\operatorname{ord}_{E}\right]\right\} \cup \widehat{\mathcal{V}_{\infty}}(p)$. These maps $\sigma_{p}$ glue together to give a continuous section $\sigma_{E}: \widehat{\mathcal{V}_{\infty}}(E) \rightarrow \mathcal{V}_{X}(E ; E) \subset \mathcal{V}_{X}(E)$ of $\eta: \mathcal{V}_{X}(E) \rightarrow \widehat{\mathcal{V}_{\infty}}(E)$.

To finish the proof we need to understand the behaviour of $\sigma_{F}, \sigma_{E}$ on

$$
\begin{equation*}
\widehat{\mathcal{V}_{\infty}}(E) \cap \widehat{\mathcal{V}_{\infty}}(F)=\widehat{\mathcal{V}_{\infty}}(p) \tag{169}
\end{equation*}
$$

for $p=E \cap F$ where $E, F$ are two prime divisors at infinity. By Proposition 5.11, we have that the map $N_{p, F} \circ N_{p, E}^{-1}: \mathcal{V}_{X}(p ; E) \backslash\left\{\operatorname{ord}_{E}\right\} \rightarrow \mathcal{V}_{X}(p ; F) \backslash\left\{\operatorname{ord}_{F}\right\}$ is a homeomorphism and we have

$$
\begin{equation*}
\left(\sigma_{F}\right)_{\mid \widehat{\mathcal{V}_{\infty}}(p)}=\left(N_{p, F} \circ N_{p, E}-1\right) \circ\left(\sigma_{E}\right)_{\mid \widehat{\mathcal{V}_{\infty}}(p)} \tag{170}
\end{equation*}
$$

### 6.2. The strong topology

Let $R=\mathbf{k}[[x, y]]$ and let $\mathfrak{m}=(x, y)$. Let $\mathcal{V}_{*}$ be the valuative tree with either the normalization by $\mathfrak{m}$ or with respect to a curve $z$. We will write $\alpha_{*}$ for the skewness function over $\mathcal{V}_{*}$. We consider a stronger topology on $\mathcal{V}_{*}$. Let $\mathcal{V}_{*}^{q m}$ be the subset of quasimonomial valuations. We define the following distance

$$
\begin{equation*}
d\left(v_{1}, v_{2}\right)=\alpha\left(v_{1}\right)-\alpha\left(v_{1} \wedge v_{2}\right)+\alpha\left(v_{2}\right)-\alpha\left(v_{1} \wedge v_{2}\right) \tag{171}
\end{equation*}
$$

The topology induced by this distance is called the strong topology.
Proposition 6.5 ([ [FJ04] Proposition 5.12). We have the following

- The strong topology is stronger than the weak topology.
- The closure of $\mathcal{V}_{*}^{q m}$ with respect to the strong topology is the subspace of $\mathcal{V}_{*}$ consisting of valuations of finite skewness.
Proposition 6.6. Let $R=\mathbf{k}[[z, w]]$ and let $\mathcal{V}_{\mathfrak{m}}, \mathcal{V}_{z}, \mathcal{V}_{w}$ be the three valuation trees. Let $\mathcal{V}_{\mathfrak{m}}^{\prime}, V_{z}^{\prime}, \mathcal{V}_{w}^{\prime}$ be the three subtrees of valuations of finite skewness. Then, the maps

$$
\begin{equation*}
N_{z}: \mathcal{V}_{\mathfrak{m}}^{\prime} \rightarrow \mathcal{V}_{z}^{\prime} \backslash\left\{\operatorname{ord}_{z}\right\}, \quad N_{w} \circ N_{z}^{-1}: \mathcal{V}_{z}^{\prime} \rightarrow \mathcal{V}_{w}^{\prime} \tag{172}
\end{equation*}
$$

are homeomorphisms with respect to the strong topology.

This follows from Proposition 5.9
Let $\mathcal{V}_{\infty}^{\prime}$ be the subset of $\mathcal{V}_{\infty}$ of valuations of finite skewness, this set is well defined thanks to Proposition 5.30. We define the strong topology on $\mathcal{V}_{\infty}^{\prime}$ as follows. First define the strong topology on $\widehat{\mathcal{V}}_{\infty}^{\prime}:=\eta\left(\mathcal{V}_{\infty}^{\prime}\right)$ using the notations from the proof of Theorem 6.4. Consider the map $\sigma_{E}: \widehat{\mathcal{V}}_{\infty}^{\prime} \cap \widehat{\mathcal{V}_{\infty}}(E) \rightarrow \mathcal{V}_{X}(E ; E)^{\prime}$. We define the strong topology on $\widehat{\mathcal{V}}_{\infty}^{\prime} \cap \widehat{\mathcal{V}}_{\infty}(E)$ as the coarsest topology such that $\sigma_{E}$ is continuous for the strong topology on $\mathcal{V}_{X}(E ; E)^{\prime}$. This defines a topology on ${\widehat{V_{\infty}}}^{\prime}$ thanks to Proposition 6.6.

Corollary 6.7. Let $v$ be a valuation centered at infinity, let $X$ be a completion of $X_{0}$ and let $\left(v_{n}\right)$ be the infinitely near sequence of $v$ from Proposition 5.26. If $v \in \mathcal{V}_{\infty}^{\prime}$, then $\eta\left(v_{n}\right)$ converges towards $\eta(v)$ with respect to the strong topology.

Proof. Let $p=c_{X}(v)$ and we can suppose that $v_{n}, \nu \in \mathcal{V}_{X}(p ; E)$ for some prime divisor $E$ at infinity with $p \in E$. Then, we have $v_{n} \leqslant v$ for all $n$ and $\alpha\left(v_{n}\right) \rightarrow \alpha(v)$. Therefore

$$
\begin{equation*}
d\left(v_{n}, v\right)=\alpha(v)-\alpha\left(v_{n}\right) \xrightarrow[n \rightarrow \infty]{ } 0 \tag{173}
\end{equation*}
$$

## CHAPTER 7

## Valuations as Linear forms

As done in [JM12], we can view valuations on $X_{0}$ as

- linear forms with values in $\mathbf{R}$ over the space of integral Cartier Divisors over $X$ supported at infinity
- as real-valued functions over the set of coherent fractional ideal sheaves of $X$ co-supported at infinity.
We recall how to do so. For a divisor $D$, we denote by $H^{0}\left(X, O_{X}(D)\right)$ the set of global sections of the line bundle $O_{X}(D)$ and

$$
\begin{equation*}
\Gamma\left(X, O_{X}(D)\right)=\left\{h \in \mathbf{k}(X)^{\times}: D+\operatorname{div}(h) \geqslant 0\right\} . \tag{174}
\end{equation*}
$$

### 7.1. Valuations as linear forms over $\operatorname{Div}_{\infty}(X)$

Lemma 7.1. Let $D \in \operatorname{Div}(X)$ such that the negative part (if any) of $D$ is supported in $\partial_{X} X_{0}$. For any point $p \in X$, there exists an open neighbourhood $U$ of $p$ such that a local equation of $D$ on $U$ is of the form $\varphi=P \cdot \psi$ with $P \in O_{X}\left(X_{0}\right)$ and $\psi \in O_{X}(U)$.

Proof. Let $\varphi \in \mathbf{k}\left(U^{\prime}\right)^{*}=\mathbf{k}(X)^{*}$ be a local equation of $D$ where $U^{\prime}$ is an open subset of $X$ containing $p$.

Let $H$ be an ample effective divisor such that $\operatorname{Supp}(H)=\partial_{X} X_{0}$. There exists an integer $n \geqslant 1$ such that $D+n H \geqslant 0$. Pick $P$ general in $\Gamma\left(X, O_{X}(n H)\right) \subset O_{X}\left(X_{0}\right)$, then $\operatorname{div} P=Z_{P}-n H$ with $Z_{P} \geqslant 0$ and $p \notin \operatorname{Supp} Z_{P}$ because we chose $P$ general and $|n H|$ is basepoint free, in particular $P$ restricts to a regular function over $X_{0}$. Set $\psi:=\varphi / P$, one has

$$
\begin{equation*}
\operatorname{div}\left(\psi_{\mid U}\right)=D_{\mid U}+n H_{\mid U}-Z_{P \mid U} \tag{175}
\end{equation*}
$$

Set $U=U^{\prime} \backslash \operatorname{Supp} Z_{P}$, then $\operatorname{div}(\psi)_{\mid U^{\prime}} \geqslant 0$, i.e $\psi \in O_{X}(U)$ and we are done.
Corollary 7.2. If $D$ is a divisor such that the negative part (if any) of $D$ is at infinity and $v$ is a valuation on $\mathbf{k}\left[X_{0}\right]$, then for all small enough affine open subsets $U \subset X$ containing $c_{X}(v)$,

$$
\begin{equation*}
\Gamma\left(U, O_{X}(-D)\right) \subset O_{X}\left(X_{0}\right)_{\mathfrak{p}_{v_{X}}} \tag{176}
\end{equation*}
$$

and $v_{X}$ can be extended to $\Gamma\left(U, O_{X}(-D)\right)$.

Proof. If $U$ is small enough, then $\Gamma\left(U, O_{X}(-D)\right)$ is the $O_{X}(U)$-module generated by $\varphi$ where $\varphi$ is a local equation of $D$. Now, by Lemma 7.1, $\varphi$ is of the form $\varphi=P \cdot \psi$ where $P \in O_{X}\left(X_{0}\right)$ and $\psi \in O_{X}(U)$. By definition we have $O_{X}\left(X_{0}\right) \subset O_{X}\left(X_{0}\right)_{\mathfrak{p}_{v_{X}}}$ and for all affine open neighbourhood $U$ of $c_{X}(v), O_{X}(U) \subset O_{X}\left(X_{0}\right)_{\mathfrak{p}_{w_{X}}}$ by the proof of Lemma 4.11.

Let $D$ be divisor of $X$ supported at infinity and let $\varphi \in \mathbf{k}(X)$ be a local equation of $D$ at $c_{X}(v)$. Then we set

$$
\begin{equation*}
L_{v, X}(D):=v_{X}(\varphi) \tag{177}
\end{equation*}
$$

This is well defined because by Corollary 7.2 because by definition there exists an open affine neighbourhood $U$ of $c_{X}(v)$ such that $\varphi \in \Gamma\left(U, O_{X}(-D)\right)$. This does not depend on the choice of the local equation because if $\psi$ is another local equation of $D$, then $\frac{\varphi}{\psi}$ is a regular invertible function near $c_{X}(v)$ and $v_{X}(\varphi / \psi)=0$.

Lemma 7.3. Let $\vee$ be a valuation over $\mathbf{k}\left[X_{0}\right]$ and let $X$ be a completion of $X_{0}$, then for all $D \in \operatorname{Div}_{\infty}(X)_{\mathbf{R}}, L_{v, X}(D)<\infty$.

Proof. It suffices to show Lemma 7.3 for $D$ an integral divisor supported at infinity in $X$. We can apply corollary 7.2 to $D$ and $-D$, therefore if $\varphi$ is a local equation of $D$, we have that both $\imath_{X}^{*}(\varphi)$ and $\imath_{X}^{*}(1 / \varphi)$ belong to $A_{\mathfrak{p}_{v}}$ and this means that $v_{X}(\varphi)<\infty$.

Remark 7.4. We can in fact define $L_{v, X}$ at any divisor $D$ on $X$ such that the negative part of $D$ is supported at infinity but it could happen that $L_{v, X}(D)$ is infinite. For example, let $X_{0}=\mathbf{A}^{2}, X=\mathbf{P}^{2}$. Let $v$ be the curve valuation centered at $[1: 0: 0]$ associated to the curve $y=0$, then

$$
\begin{equation*}
L_{v, \mathbf{P}^{2}}(\{Y=0\}-\{Z=0\})=v(Y / Z)=+\infty . \tag{178}
\end{equation*}
$$

Example 7.5. If $X$ is a completion of $X_{0}$, let $E$ be a prime divisor at infinity. Let $D \in \operatorname{Div}_{\infty}(X)$. Recall that we have defined in Section 3.1 that $\operatorname{ord}_{E}(D)$ is the weight of $D$ along $E$, then

$$
\begin{equation*}
L_{\text {ord }_{E}}(D)=\operatorname{ord}_{E}(D) \tag{179}
\end{equation*}
$$

Indeed, at the generic point of $E$, a local equation of $D$ is $z^{\operatorname{ord}_{E}(D)} \varphi$ where $z$ is a local equation of $E$ and $\varphi$ is regular not divisible by $z$.

Proposition 7.6. If $v$ is a valuation over $\mathbf{k}\left[X_{0}\right]$, and $X$ is a completion of $X_{0}$ then
(1) $L_{v, X}\left(0_{\operatorname{Div}_{\infty}(X)}\right)=0$.
(2) For any $D, D^{\prime} \in \operatorname{Div}_{\infty}(X), L_{v, X}\left(D+D^{\prime}\right)=L_{v, X}(D)+L_{v, X}\left(D^{\prime}\right)$, and $L_{v, X}(m D)=$ $m L_{v, X}(D)$ for all $m \in \mathbf{Z}$.
(3) If $D \geqslant 0$, then $L_{v, X}(D) \geqslant 0$ and $L_{v, X}(D)>0 \Leftrightarrow c_{X}(v) \in \operatorname{Supp} D$. In particular, if $v$ is not centered at infinity then $L_{v}=0$.
(4) If $P \in O_{X}\left(X_{0}\right)$, then $v_{X}(P)=L_{v, X}(\operatorname{div} P)$.
(5) If $Y$ is another completion of $X_{0}$ above $X$, and $\pi: Y \rightarrow X$ is the morphism of completions over $X_{0}$, then $L_{v, X}(D)=L_{v, Y}\left(\pi^{*} D\right)$.

Thus, we can extend $L_{v, X}$ to $\operatorname{Div}_{\infty}(X)_{\mathbf{R}}$ by linearity:

$$
\begin{equation*}
L_{v, X}: \operatorname{Div}_{\infty}(X)_{\mathbf{R}} \rightarrow \mathbf{R} . \tag{180}
\end{equation*}
$$

Proof. The first assertion is trivial as 1 is a local equation of the trivial divisor. The second assertion follows from the fact that if $\varphi, \psi$ are local equations of $D$ and $D^{\prime}$ respectively, then $\varphi \psi$ is a local equation of $D+D^{\prime}$ and $1 / \varphi$ is a local equation of $-D$. For the third one, suppose $D$ is an integral divisor. If $D$ is effective and $f$ is a local equation at $c_{X}(v)$, then $f$ is regular at $p$ and by definition of the center $v(f) \geqslant 0$, now if $c_{X}(v)$ belongs to $\operatorname{Supp} D$, then $f$ vanishes at $c_{X}(v)$; thus, $v(f)>0$. If on the other hand $c_{X}(v) \notin \operatorname{Supp} D$, then $f$ is invertible at the center of $v_{X}$ and $v_{X}(f)=0$. The fourth assertion follows from $f$ being a local equation of $\operatorname{div}(f)$ and the fact that $f$ has no pole over $X_{0}$. Finally, if $f \in \mathbf{k}(X)$ is a local equation of $D$ at $c_{X}(v)$, then $\pi^{*} f$ is a local equation of $\pi^{*} D$ at $c_{Y}(v)$ and by Remark 4.3, one has $v_{X}(f)=v_{Y}\left(\pi^{*} f\right)$.

Proposition 7.7. Let $f: X_{0} \rightarrow X_{0}$ be a dominant endomorphism of $X_{0}$. Let $Y, X$ be two completions of $X_{0}$ such that the lift $F: Y \rightarrow X$ of $f$ is regular. Then,

$$
\begin{equation*}
F\left(c_{Y}(v)\right)=c_{X}\left(f_{*} v\right) \text { and } \forall D \in \operatorname{Div}_{\infty}(X), L_{f_{*} v, X}(D)=L_{v, Y}\left(F^{*} D\right) \tag{181}
\end{equation*}
$$

Proof. Let $p=c_{Y}(\mathrm{v})$ and $q=c_{X}\left(f_{*} \mathrm{v}\right)$. Then, $F$ induces a local ring homomorphism

$$
F^{*}: O_{X, q} \rightarrow O_{Y, p}
$$

Now, for any $\varphi \in O_{X, q}$, there exists $P, Q \in \mathbf{k}\left[X_{0}\right]$ such that $\varphi=\frac{P}{Q}$. Therefore,

$$
F^{*} \varphi=\frac{f^{*} P}{f^{*} Q}
$$

and therefore $f_{*} v(\varphi)=v\left(F^{*} \varphi\right)>0$. Therefore, $q=c_{X}\left(f_{*} v\right)$.
Now, to show the second result. If $g$ is a local equation of $D$ at the center of $v_{X}$, then $F^{*} g$ is a local equation of $F^{*} D$ at the center of $v_{Y}$. Since $\pi_{*} v_{Y}=v_{X}$, one has

$$
\begin{equation*}
v_{Y}\left(F^{*} g\right)=v_{X}\left(\left(F \circ \pi^{-1}\right)^{*} g\right)=v_{X}\left(f^{*} g\right)=\left(f_{*} v\right)_{X}(g) \tag{182}
\end{equation*}
$$

and this shows the result.

### 7.2. Valuations as real-valued functions over the set of fractional ideals co-supported at infinity in $X$

An ideal of $X$ is a sheaf of ideals of $O_{X}$ and a fractional ideal is a coherent sub- $O_{X^{-}}$ module of the constant sheaf $\mathbf{k}(X)$. Let $\mathfrak{a}$ be a fractional ideal of $X$, we say that $\mathfrak{a}$ is co-supported at infinity if $\mathfrak{a}_{X_{0}}=O_{X_{0}}$. For example, for any divisor $D \in \operatorname{Div}(X), O_{X}(D)$ is a fractional ideal of $X$ and if $D \in \operatorname{Div}_{\infty}(X)$ then $O_{X}(D)$ is co-supported at infinity.
7.2. VALUATIONS AS REAL-VALUED FUNCTIONS OVER THE SET OF FRACTIONAL IDEALS CO-SUPPORTED AT INFINITY IS

Proposition 7.8. Let $\mathfrak{a}$ be a fractional ideal of $X$ co-supported at infinity and let $p \in X$, for any finite system $\left(f_{1}, \cdots, f_{r}\right)$ of local generators of $\mathfrak{a}$ at $p$ there exists an open neighbourhood $U$ of $p$ such that $f_{i \mid U}$ is of the form

$$
\begin{equation*}
f_{i}=F_{i} g_{i} \tag{183}
\end{equation*}
$$

with $F_{i} \in O_{X}\left(X_{0}\right)$ and $g_{i} \in O_{X}(U)$.
Proof. Pick $U^{\prime}$ an open neighbourhood containing $p$. Since $f_{i}$ is regular over $X_{0}$, we have div $f_{i}=D^{+}-D_{1}^{-}-D_{2}^{-}$where $D^{+}, D_{1}^{-}$and $D_{2}^{-}$are effective divisors such that $\operatorname{Supp} D_{1}^{-} \subset \partial_{X} X_{0}$ and $D_{2}^{-}{ }_{\mid U^{\prime}}=0$. By Lemma 7.1 there exists an open neighbourhood $U_{i} \subset U^{\prime}$ of $p$ such that $\left(D^{+}-D_{1}^{-}\right)_{\mid U_{i}}=\operatorname{div} F_{i} g_{i}^{\prime}$ with $F_{i} \in O_{X}\left(X_{0}\right)$ and $g_{i}^{\prime} \in O_{X}\left(U_{i}\right)$. Therefore, there exists $g_{i}^{\prime \prime} \in O_{X}\left(U_{i}\right)$ such that $f_{i}=F_{i} g_{i}^{\prime} g_{i}^{\prime \prime}$. Set $U=\cap U_{i}$ and $g_{i}=g_{i}^{\prime} g_{i}^{\prime \prime}$.

COROLLARY 7.9. Let $\mathfrak{a}$ be a fractional ideal co-supported at infinity and let $v$ be a valuation over $\mathbf{k}\left[X_{0}\right]$, then for all affine open neighbourhood of $c_{X}(v), \Gamma(U, \mathfrak{a}) \subset$ $O_{X}\left(X_{0}\right)_{\mathfrak{p}_{v_{X}}}$ and $v_{X}$ is defined over $\Gamma(U, \mathfrak{a})$.

If $v$ is a valuation over $\mathbf{k}\left[X_{0}\right]$, then we define $L_{v, X}(\mathfrak{a})$ as

$$
\begin{equation*}
L_{v, X}(\mathfrak{a}):=\min _{f} v_{X}(f) \tag{184}
\end{equation*}
$$

where the $f$ runs over the germs of sections of $\mathfrak{a}$ at $c_{X}(v)$. This makes sense by Corollary 7.9 .

Proposition 7.10. If $v$ is a valuation over $\mathbf{k}\left[X_{0}\right]$, then
(1) $L_{v, X}\left(O_{X}\right)=0$.
(2) If $\mathfrak{a}, \mathfrak{b}$ are two fractional ideals of $X$ co-supported at infinity, then
$L_{v, X}(\mathfrak{a} \cdot \mathfrak{b})=L_{v, X}(\mathfrak{a})+L_{v, X}(\mathfrak{b})$ and $L_{v, X}(\mathfrak{a}+\mathfrak{b})=\min \left(L_{v, X}(\mathfrak{a}), L_{v, X}(\mathfrak{b})\right)$
(3) If $f_{1}, \cdots, f_{r} \in \mathbf{k}(X)$ is a set of local generators of $\mathfrak{a}$ at $c_{X}(v)$, then

$$
\begin{equation*}
L_{v, X}(\mathfrak{a})=\min \left(v_{X}\left(f_{1}\right), \cdots, v_{X}\left(f_{r}\right)\right) \tag{186}
\end{equation*}
$$

(4) If $D \in \operatorname{Div}(X)$ is a divisor, then $L_{v, X}(D)=L_{v, X}\left(O_{X}(-D)\right)$.
(5) If $Y$ is another completion of $X_{0}$ above $X$, and $\pi: Y \rightarrow X$ is the morphism of completions over $X_{0}$, then $\tilde{\mathfrak{a}}:=\pi^{*} \mathfrak{a} \cdot O_{Y}$ is a fractional ideal over $Y$ and $L_{v, X}(\mathfrak{a})=L_{v, Y}(\widetilde{\mathfrak{a}})$.

Proof. The first assertion is trivial since 1 is a local generator of the trivial sheaf. For Assertion (2), notice that if $\left(f_{1}, \ldots, f_{r}\right)$ are local generators of $\mathfrak{a}$ at $c_{X}(v)$ and $\left(g_{1}, \ldots, g_{s}\right)$ local generators of $\mathfrak{b}$ at $c_{X}(v)$ then $\left(f_{i} g_{j}\right)_{i, j}$ is a set of local generators of $\mathfrak{a} \cdot \mathfrak{b}$ at $c_{X}(v)$ and $\left(f_{1}, \ldots, f_{r}, g_{1}, \ldots, g_{s}\right)$ is a set of local generators of $\mathfrak{a}+\mathfrak{b}$ at $c_{X}(v)$, so

Assertion (2) follows from Assertion (3). To show Assertion (3), let $f_{1}, \cdots, f_{r}$ be local generators of $\mathfrak{a}$ at $c_{X}(v)$. This implies that $\mathfrak{a}_{c_{X}(v)}=f_{1} O_{c_{X}(v)}+f_{2} O_{c_{X}(v)}+\cdots+f_{r} O_{c_{X}(v)}$. Since $v$ is nonnegative on $O_{c_{X}(v)}$ by definition of the center, the assertion follows. For assertion 5 , if $f_{1}, \cdots, f_{r}$ are local generators of $\mathfrak{a}$, then $\pi^{*} f_{1}, \cdots, \pi^{*} f_{r}$ are local generators of $\tilde{\mathfrak{a}}$ at $c_{Y}(v)$ and the result follows since $\pi_{*} \nu_{Y}=\nu_{X}$. Assertion (4) follows from the fact $O_{X}(-D)$ is locally generated by an equation of $D$ and Assertion (5) follows from the fact that if $\left(f_{1}, \cdots, f_{r}\right)$ are local generators of $\mathfrak{a}$ at $c_{X}(v)$ then $\left(\pi^{*} f_{1}, \cdots, \pi^{*} f_{r}\right)$ are local generators of $\tilde{\mathfrak{a}}$ at $c_{Y}(\mathrm{v})$.

Proposition 7.11. If $v$ is a valuation over $\mathbf{k}\left[X_{0}\right]$ and $\mathfrak{a}$ is a fractional ideal cosupported at infinity, then $L_{v, X}(\mathfrak{a})<\infty$.

Proof. Take $f_{1}, \cdots, f_{r}$ local generators of $\mathfrak{a}$ at $p$ the center of $v$ on $X$. The proof of Lemma 7.1 shows that there exists an affine open neighbourhood $U$ of $p$ such that $f_{i \mid U}=h_{i} g_{i}$ with $h_{i} \in \mathbf{k}\left[X_{0}\right]$ and $g_{i} \in O_{X}(U)$ and such that $f_{i}^{-1}$ can be put into the same form. This shows that for all $i, v\left(f_{i}\right)<\infty$.

REMARK 7.12. The same definition would allow one to define $L_{v, X}(\mathfrak{a})$ for any fractional ideal such that $\mathfrak{a}$ is a sheaf of ideals of $X_{0}$ but we have to allow infinite values. In particular, $L_{v, X}(\mathfrak{a})$ is defined for any sheaf of ideals over $X$.

### 7.3. Valuations centered at infinity

Recall that a valuation $v$ over $\mathbf{k}\left[X_{0}\right]$ is centered at infinity, if $v$ does not admit a center on $X_{0}$. We denote by $V_{\infty}$ the set of valuations over $\mathbf{k}\left[X_{0}\right]$ centered at infinity.

Lemma 7.13. Let $\vee$ be valuation over $\mathbf{k}\left[X_{0}\right]$. The following assertions are equivalent.
(1) $v$ is centered at infinity.
(2) There exists $P \in \mathbf{k}\left[X_{0}\right]$ such that $v(P)<0$.
(3) For any completion $X$ of $X_{0}$ and any effective divisor $H$ in $X$ such that $\operatorname{Supp} H=$ $\partial_{X} X_{0}$, one has $L_{v, X}(H)>0$.
(4) There exists a completion $X$ of $X_{0}$ and an effective divisor $H \in X$ with $\operatorname{Supp} H=$ $\partial_{X} X_{0}$ such that $L_{v, X}(H)>0$.

Proof. We will show the following implications $2 \Rightarrow 1 \Rightarrow 3 \Rightarrow 4$. Then, we will show that $1 \Rightarrow 2$ and finally that $4 \Rightarrow 2$.
$2 \Rightarrow 1 \Rightarrow 3 \Rightarrow 4$. If there exists a regular function $P$ over $X_{0}$ such that $v(P)<0$ then the center of $v$ cannot be a point of $X_{0}$ because $P$ is regular at every point of $X_{0}$. This shows $2 \Rightarrow 1$, then if $v$ is centered at infinity, take a completion $X$ of $X_{0}$, let $E$ be a prime divisor at infinity in $X$ such that $c_{X}(v) \in E$. Then, since $H$ is effective and $E \in \operatorname{Supp} H$, $L_{v, X}(H) \geqslant v(E)>0$ by Proposition 7.6(1). This shows $1 \Rightarrow 3$ and $3 \Rightarrow 4$ is clear.
$1 \Rightarrow 2$. Conversely, suppose $v$ is centered at infinity and fix a closed embedding $X_{0} \hookrightarrow \mathbf{A}^{N}$ for some integer $N$. Let $X$ be the Zariski closure of $X_{0}$ in $\mathbf{P}^{N}$ with homogeneous coordinates $x_{0}, \cdots, x_{N}$ such that $\left\{x_{0}=0\right\}$ is the hyperplane at infinity. The surface $X$ might not be smooth so it is not necessarily a completion of $X_{0}$ but it still is proper and the center $p$ of $v$ on $X$ belongs to $\left\{x_{0}=0\right\} \cap X$. Let $1 \leqslant i \leqslant N$ be an integer such that $p$ belongs to the open subset $\left\{x_{i} \neq 0\right\}$. Then, the rational function $P:=\frac{x_{i}}{x_{0}}$ is a regular function on $X_{0}$ and $1 / P$ vanishes at $p$. Therefore, $v(P)<0$.
$4 \Rightarrow 1$. Suppose that $v$ is not centered at infinity, i.e the center of $v$ belongs to $X_{0}$. Then, for any completion $X$ and for any divisor $D \in \operatorname{Div}_{\infty}(X)$, one has $L_{v, X}(D)=0$ by Proposition 7.6(1) since $c_{X}(v) \notin \operatorname{Supp} D$.

This lemma shows that being centered at infinity is a property that can be tested on only one completion $X_{0}$.

Corollary 7.14. The space $\mathcal{V}_{\infty}$ is an open subset of $\mathcal{V}$.
Proof. We have by Lemma 7.13 that

$$
\begin{equation*}
\mathcal{V}_{\infty}=\bigcup_{P \in \mathbf{k}\left[X_{0}\right]}\{v(P)<0\} . \tag{187}
\end{equation*}
$$

Therefore, it is a union of open subsets.
7.3.1. Topologies over the set of valuations centered at infinity. Let $X$ be a completion of $X_{0}$. Call $\sigma$ the coarsest topology such that the evaluation maps $\varphi_{f}: v \in \mathcal{V}_{\infty} \mapsto$ $\mathrm{v}(f)$ are continuous for all $f \in \mathbf{k}\left[X_{0}\right]$ and $\tau$ the coarsest topology such that the evaluation maps $\psi_{\mathfrak{a}}: v \in \mathcal{V}_{\infty} \mapsto L_{v}(\mathfrak{a})$ are continuous for all fractional ideals $\mathfrak{a}$ of $X$ such that $\mathfrak{a}_{\mid X_{0}}$ is a sheaf of ideals over $X_{0}$. Recall that we allow in both cases infinite values.

Proposition 7.15. [JM12] These two topologies on $\mathcal{V}$ are the same.
Proof. First if $f \in \mathbf{k}\left[X_{0}\right]$, then $v(f)=L_{\mathbf{v}}((f))$ where $(f)$ is the fractional ideal generated by $f$. So $\sigma$ is finer than $\tau$. For the other way, Let $H$ be an ample divisor supported at infinity and let $\mathfrak{a}$ be a fractional ideal co-supported at infinity. There exists an integer $n>0$ such that $\mathfrak{a} \otimes O_{X}(n H)$ and $O_{X}(n H)$ are generated by global sections $\left(f_{1}, \cdots, f_{r}\right)$ and $\left(g_{1}, \cdots, g_{s}\right)$ respectively. Notice that for all $i, j$, the rational functions $f_{i}, g_{j}$ belong to $O_{X}\left(X_{0}\right)$. Now, we have that $L_{v}(\mathfrak{a})=L_{v}\left(\mathfrak{a} \otimes O_{X}(n H) \otimes O_{X}(-n H)\right)$, therefore

$$
L_{v}(\mathfrak{a})=\min _{i, j}\left(v\left(\frac{f_{i}}{g_{j}}\right)\right)=\min _{i, j}\left(v\left(f_{i}\right)-v\left(g_{j}\right)\right)
$$

Therefore, $\tau$ is finer than $\sigma$ and the result is shown.

### 7.3.2. Valuations centered at infinity as linear forms over $\operatorname{Cartier}_{\infty}\left(X_{0}\right)$.

Definition 7.16. Let $v$ be a valuation over $\mathbf{k}\left[X_{0}\right]$. Let $D \in \operatorname{Cartier}_{\infty}\left(X_{0}\right)$ and $X$ be a completion of $X_{0}$ such that $D$ is defined by $D_{X}$. We define

$$
\begin{equation*}
L_{v}(D):=L_{v, X}\left(D_{X}\right) \tag{188}
\end{equation*}
$$

This does not depend on the choice $X$ and defines a linear map on $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$ by Proposition 7.6 and $L_{v}(D)<+\infty$ by Lemma 7.3. Notice that $L_{v}=0$ if and only if $v$ is not centered at infinity.

Proposition 7.17. If $v$ is a valuation on $\mathbf{k}\left[X_{0}\right]$ centered at infinity then $L_{v}$ is a linear form $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right) \rightarrow \mathbf{R}$ and satisfies
(1) If $D \geqslant 0$, then $L_{v}(D) \geqslant 0$.
(2) For $D, D^{\prime} \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), L_{v}\left(D \wedge D^{\prime}\right)=\min \left(L_{v}(D), L_{v}\left(D^{\prime}\right)\right)$.

We will say that an element of hom $\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \mathbf{R}\right)$ that satisfies these 2 properties satisfies property (+).

Proof. Assertion 1 follows from Proposition 7.6(3). We show the second assertion. Take $D, D^{\prime} \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$ and $X$ a completion of $X_{0}$ such that $D, D^{\prime}$ are defined by their incarnation $D_{X}, D_{X}^{\prime}$. By Claim 3.8 (that we prove in the next section), we know that there exists a completion $Y$ along with a morphism of completions $\pi: Y \rightarrow X$ such that $D \wedge D^{\prime}$ is the Cartier class determined by some divisor $D_{Y}$ in $Y$ such that $\pi^{*}\left(O_{X}\left(-D_{X}\right)+O_{X}\left(-D_{X}^{\prime}\right)\right) \cdot O_{Y}=O_{Y}\left(-D_{Y}\right)$. Using Proposition 7.10, it follows that

$$
\begin{aligned}
L_{v}\left(D \wedge D^{\prime}\right) & =L_{v, Y}\left(D_{Y}\right) \\
& =L_{v, Y}\left(O_{Y}\left(-D_{Y}\right)\right) 7.10(4) \\
& =L_{v, X}\left(O_{X}\left(-D_{X}\right)+O_{X}\left(-D_{X}^{\prime}\right)\right) 3.8 \\
& =\min \left(L_{v, X}\left(O_{X}\left(-D_{X}\right)\right), L_{v, X}\left(O_{X}\left(-D_{X}^{\prime}\right)\right)\right) \quad 7.10(2) \\
& =\min \left(L_{v}(D), L_{v}\left(D^{\prime}\right)\right) \quad 7.10(4)
\end{aligned}
$$

Proposition 7.18. Let $\vee$ be a valuation over $\mathbf{k}\left[X_{0}\right]$ and $f: X_{0} \rightarrow X_{0}$ a dominant endomorphism, then for all $D \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$,

$$
\begin{equation*}
L_{f_{*} v}(D)=L_{v}\left(f^{*} D\right)=\left(f_{*} L_{v}\right)(D) \tag{189}
\end{equation*}
$$

Proof. Let $X$ be a completion of $X_{0}$ where $D$ is defined, then $f$ induces a dominant rational map $f: X \rightarrow X$. Let $\pi: Y \rightarrow X$ be a projective birational morphism such that the lift $F: Y \rightarrow X$ is regular. Since $f$ is an endomorphism of $X_{0}$ we can suppose that $\pi$ is the identity over $X_{0}$, hence $Y$ is a completion of $X_{0}$ and $\pi$ is a morphism of completions.

Now, if $\varphi$ is a local equation of $D$ near the center of $v_{X}$, then $F^{*} \varphi$ is a local equation of $F^{*} D$ near the center of $v_{Y}$. Since $\pi_{*} v_{Y}=v_{X}$, one has

$$
\begin{equation*}
v_{Y}\left(F^{*} g\right)=v_{X}\left(\left(F \circ \pi^{-1}\right)^{*} g\right)=v_{X}\left(f^{*} g\right)=\left(f_{*} v\right)_{X}(g) \tag{190}
\end{equation*}
$$

We equip hom $\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \mathbf{R}\right)$ with the weak- $\star$ topology, that is the coarsest topology such that the map $L \in \operatorname{hom}\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \mathbf{R}\right) \mapsto L(D)$ is continuous for all $D \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$. We extend $L_{v}$ to $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$ by linearity.

Proposition 7.19. The map $v \in \mathcal{V}_{\infty} \mapsto L_{v} \in \operatorname{hom}\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \mathbf{R}\right)$ is a continuous embedding.

Proof. For the injectivity, let $v, w \in \mathcal{V}_{\infty}$ such that $v \neq w$. First, if $w=t v$ with $t>0$, then since $L_{v} \neq 0$, we have $L_{v} \neq L_{w}$. Otherwise, there exists a completion $X$ such that $c_{X}(v) \neq c_{X}(w)$. If the centers are both prime divisors at infinity then it is clear that $L_{v} \neq L_{w}$. If $c_{X}(v)$ is a point, let $\widetilde{E}$ be the exceptional divisor above it. Then, by Proposition 7.6, $L_{v}(\widetilde{E})>0$, but $L_{w}(\widetilde{E})=0$.

By definition, to show continuity we have to show that for all $D \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$, the map $v \in \mathcal{V}_{\infty} \mapsto L_{v}(D)$ is continuous. Let $X$ be a completion where $D$ is defined, then by Proposition $7.6 L_{V}(D)=L_{v}\left(O_{X}(-D)\right)$ and by Proposition 7.15 the map $v \in \mathcal{V}_{\infty} \mapsto$ $L_{V}\left(O_{X}(-D)\right)$ is continuous.

Proposition 7.20. Let $X$ be a completion of $X_{0}$ and $p \in X$ a closed point at infinity. Let $v \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$. If $E$ is a prime divisor of $X$ at infinity such that $p \in E$, then

$$
\begin{equation*}
1 \leqslant L_{v}(E) \leqslant \alpha(v) \tag{191}
\end{equation*}
$$

Proof. Let $z \in O_{X, p}$ be a local equation of $E, z$ is irreducible and we have $L_{v}(E)=$ $\mathrm{v}(z)$. We have that $z \in \mathfrak{m}_{p}$, therefore $\mathrm{v}(z) \geqslant \mathrm{v}\left(\mathfrak{m}_{p}\right)=1$. This shows the first inequality. For the second one, let $v_{z}$ be the curve valuation associated to $z$. It does not define a valuation over $\mathbf{k}\left[X_{0}\right]$ but it defines a valuation over $O_{X, p}$ by Proposition 5.3, we get

$$
\begin{equation*}
v(z)=\alpha\left(v_{z} \wedge v\right) \leqslant \alpha(v) \tag{192}
\end{equation*}
$$

### 7.3.3. Special look at divisorial valuations centered at infinity.

Lemma 7.21. Let $X$ be a completion of $X_{0}$ and let $E$ be a prime divisor at infinity. One has $L_{\mathrm{ord}_{E}}(E)=1$ and for any prime divisor $F \neq E$ in $X, L_{\mathrm{ord}_{E}}(F)=0$.

Furthermore, if $\pi: Y \rightarrow X$ is some blow-up of $X$, and $\pi^{\prime}(E)$ the strict transform of E by $\pi$, then

$$
\begin{equation*}
\pi_{*} \operatorname{ord}_{\pi^{\prime}(E)}=\operatorname{ord}_{E} \tag{193}
\end{equation*}
$$

Proof. The first assertion follows from Proposition 7.6(3). We show the second assertion. It suffices to show it when $\pi$ is the blow-up of one point of $X$. Let $D=$ $a E+\sum_{F \neq E} \operatorname{ord}_{F}(D) F$, then $\pi^{*} D$ is of the form

$$
\begin{equation*}
\pi^{*} D=a \pi^{\prime}(E)+b \widetilde{E}+\sum_{F \neq E} a_{F}(D) \pi^{\prime}(F) \tag{194}
\end{equation*}
$$

where $\widetilde{E}$ is the exceptional divisor of $\pi$. Therefore $\operatorname{ord}_{\pi^{\prime}(E)}\left(\pi^{*}(D)\right)=a=\operatorname{ord}_{E}(D)$.
Proposition 7.22. Let $v$ be a divisorial valuation, then $L_{v}$ can be extended naturally to Weil $_{\infty}\left(\mathrm{X}_{0}\right)$ in a compatible way with the definition of $L_{V}$ over $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$.

Proof. Take $W \in \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)$. Since $v$ is divisorial, there exists a completion $X$ of $X_{0}$ that contains a prime divisor $E$ at infinity such that $\left(\mathfrak{l}_{X}\right)_{*} v=\lambda \operatorname{ord}_{E}$. We set

$$
\begin{equation*}
L_{v}(W):=L_{v, X}\left(W_{X}\right) \tag{195}
\end{equation*}
$$

This does not depend on the completion $X$. To show this, it suffices to show that we get the same result if we blow up one point of $X$. So, let $\pi: Y \rightarrow X$ be the blow up of one point of $X_{0}$ at infinity. Then, by Lemma 7.21, $v_{Y}=\lambda \operatorname{ord}_{\pi^{\prime}(E)}$ and $\operatorname{ord}_{\pi^{\prime}(E)}\left(W_{Y}\right)=\operatorname{ord}_{E}\left(\pi_{*} W_{Y}\right)=\operatorname{ord}_{E}\left(W_{X}\right)$. If $D \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$, then this is compatible with the previous definition of $L_{V}(D)$ because if $D$ is defined over $X$, there exists a completion $\pi: Y \rightarrow X$ such that the center of $v$ on $Y$ is a prime divisor at infinity and by Proposition 7.6 (5) $L_{v, Y}\left(\pi^{*} D\right)=L_{v, X}(D)$.

REMARK 7.23. Recall that we have defined in $\S 3.1$ the set $\mathcal{D}_{\infty}\left(X_{0}\right)$ as the set of equivalence classes of prime divisors at infinity modulo the following equivalence relations: $\left(X_{1}, E_{1}\right) \sim\left(X_{2}, E_{2}\right)$ if $\pi=\mathfrak{l}_{2} \circ \mathfrak{l}_{1}^{-1}: X_{1} \rightarrow X_{2}$ satisfies $\pi\left(E_{1}\right)=E_{2}$. Lemma 7.21 shows that it makes sense to define $\operatorname{ord}_{E}$ for $E \in \mathcal{D}_{\infty}\left(X_{0}\right)$ and that $\operatorname{ord}_{E}$ is defined over Weilo ${ }_{\infty}\left(\mathrm{X}_{0}\right)$.

Proposition 7.24. Let $W, W^{\prime} \in \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)$, then $W^{\prime \prime}=W \wedge W^{\prime}$ if and only if for any divisorial valuation $E \in \mathcal{D}_{\infty}\left(X_{0}\right)$,

$$
\begin{equation*}
\operatorname{ord}_{E}\left(W^{\prime \prime}\right)=\min \left(\operatorname{ord}_{E}(W), \operatorname{ord}_{E}\left(W^{\prime}\right)\right) \tag{196}
\end{equation*}
$$

Proof. This is immediate as for any completion $X$,

$$
\begin{equation*}
W_{X}=\sum_{E \in \partial_{X} X_{0}} \operatorname{ord}_{E}(W) \cdot E \tag{197}
\end{equation*}
$$

We can now show that the minimum of two Cartier divisors is still a Cartier divisor.

Proposition 7.25. Let $X$ be a completion of $X_{0}$, let $D, D^{\prime} \in \operatorname{Div}_{\infty}(X)$ be two effective divisor and let $\mathfrak{a}$ be the sheaf of ideals $\mathfrak{a}=O_{X}(-D)+O_{X}\left(-D^{\prime}\right)$. Then, $D \wedge D^{\prime}$ is the Cartier divisor defined by $\pi^{*} \mathfrak{a}$ where $\pi$ is the blow up of $\mathfrak{a}$.

Notice that $\mathfrak{a}$ is not locally principle only at satellite points, so $\pi$ is a sequence of blow-ups of satellite points. This shows the Claim 3.8 .

Proof of Claim 3.8. Define the sheaf of ideals $\mathfrak{a}=O_{X}(-D)+O_{X}\left(-D^{\prime}\right)$ and let $\pi: Y \rightarrow X$ be the blow up of $\mathfrak{a}$. There exists a Cartier divisor $D_{Y}$ on $Y$ such that $\mathfrak{b}=O_{Y}\left(-D_{Y}\right)=\pi^{*} \mathfrak{a} \cdot O_{Y}$. We show that $D_{Y}=D \wedge D^{\prime}$ in $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$. By Proposition 7.24, we only need to show that for any divisorial valuation $\mathrm{v}, L_{v, Y}\left(D_{Y}\right)=$ $\min \left(L_{v, X}(D), L_{v, X}\left(D^{\prime}\right)\right)$, but by Proposition 7.10 we have the following equalities

$$
\begin{equation*}
L_{v, Y}\left(D_{Y}\right)=L_{v, Y}(\mathfrak{b})=L_{v, X}(\mathfrak{a})=\min \left(L_{v, X}(D), L_{v, X}\left(D^{\prime}\right)\right) \tag{198}
\end{equation*}
$$

### 7.4. Local divisor associated to a valuation

Let $X$ be a completion of $X_{0}$ and let $p \in X$ be a closed point at infinity. Let $v$ be a valuation centered at $p$. We know by Section 7.3.2 that $v$ induces a linear form $L_{v}$ on $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$. By restriction, it induces a linear form $L_{v, X, p}$ on $\operatorname{Cartier}(X, p)_{\mathbf{R}}$. Now by Proposition 3.19, the pairing

$$
\begin{equation*}
\operatorname{Weil}(X, p)_{\mathbf{R}} \times \operatorname{Cartier}(X, p)_{\mathbf{R}} \rightarrow \mathbf{R} \tag{199}
\end{equation*}
$$

induced by the intersection product is perfect. Thus, there is a unique $Z_{v, X, p} \in \operatorname{Weil}(X, p)_{\mathbf{R}}$ such that

$$
\begin{equation*}
\forall D \in \operatorname{Cartier}(X, p)_{\mathbf{R}}, \quad Z_{v, X, p} \cdot D=L_{v, X, p}(D) \tag{200}
\end{equation*}
$$

EXAMPLE 7.26. If $\widetilde{E}$ is the exceptional divisor above $p$, then $Z_{\text {ord }_{\tilde{E}}, X, p}=-\widetilde{E}$.
Proposition 7.27. For any valuation $v \in \mathcal{V}_{X}(p)$, we have $Z_{v, X, p} \in \operatorname{Cartier}(X, p)$ if and only if v is divisorial. Furthermore, $Z_{v, X, p}$ is defined over any completion such that the center of v is a prime divisor at infinity. Furthermore, for any $E \in$ $\mathcal{D}(X, p), Z_{\text {ord }_{E}, X, p} \in \operatorname{Cartier}(X, p)_{\mathbf{Q}}$.

Proof. Let $E \in \mathcal{D}_{X, p}$, for every $W \in \operatorname{Weil}(X, p), \operatorname{ord}_{E}(W)=\operatorname{ord}_{E}\left(W_{Y}\right)$ where $Y$ is a completion exceptional above $p$ by Proposition 7.22, Let $E, E_{1}, \cdots, E_{r}$ be the component of $\partial_{Y} X_{0}$ that are exceptional above $p$. The intersection form is non degenerate on

$$
\begin{equation*}
V:=\mathbf{Q} E \oplus\left(\bigoplus_{i} \mathbf{Q} E_{i}\right) \tag{201}
\end{equation*}
$$

Let $L$ be the restriction of $\operatorname{ord}_{E}$ to $V$, by duality there exists a unique $Z \in V$ such that for all $W \in V, W \cdot Z=L(W)=\operatorname{ord}_{E}(W)$. This implies that $Z=Z_{\operatorname{ord}_{E}, X, p}$. Conversely, if $v$
is a valuation such that $Z_{v, X, p} \in \operatorname{Cartier}(X, p)$ then let $Y$ be a completion where $Z_{v, X, p}$ is defined. If $c_{Y}(v)$ is a point at infinity, then let $\widetilde{E}$ be the exceptional divisor above $c_{Y}(v)$. Then, we must have $Z_{v, X, p} \cdot \widetilde{E}>0$ but it is equal to 0 , this is a contradiction.

Proposition 7.28. The embedding $\mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right) \hookrightarrow \operatorname{Weil}(X, p)_{\mathbf{R}}$ is continuous with respect to the weak topology.

Proof. This is a direct consequence of Proposition 7.19 and Proposition 6.2 .
Thus, For all completion $\pi: Y \rightarrow X$, for all $E \in \Gamma_{\pi}$, we can consider $Z_{\text {ord }_{E}, X, p}$ as an element of $\operatorname{Div}_{\infty}(Y)_{\mathbf{R}}$.

Proposition 7.29. Let $\pi:(Y, \operatorname{Exc}(\pi)) \rightarrow(X, p)$ be a completion exceptional above p. Let $v$ be a valuation such that $c_{X}(v)=p$. Suppose that $c_{Y}(v)$ is a point at infinity. Consider $\mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$ with its generic multiplicity function $b$.
(1) If $c_{Y}(v) \in E$ is a free point with $E \in \Gamma_{\pi}$, then the incarnation of $Z_{v, X, p}$ in $Y$ is

$$
\begin{equation*}
\left(Z_{V, X, p}\right)_{Y}=L_{V}(E) Z_{\operatorname{ord}_{E}, X, p} \tag{202}
\end{equation*}
$$

Moreover if $v \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$, then $L_{v}(E)=\frac{1}{b(E)}$.
(2) If $c_{Y}(v)=E \cap F$ is a satellite point with $E, F \in \Gamma_{\pi}$, then

$$
\begin{equation*}
\left(Z_{v, X, p}\right)_{Y}=L_{V}(E) Z_{\operatorname{ord}_{E}, v, p}+L_{V}(F) Z_{\operatorname{ord}_{F}, X, p} \tag{203}
\end{equation*}
$$

Moreover if $v \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$, then $L_{v}(E) b(E)+L_{v}(F) b(F)=1$.
Furthermore, if $q \neq c_{Y}(v)$ and $\tau: Z \rightarrow Y$ is the blow up of $q$ then

$$
\begin{equation*}
\left(Z_{V, X, p}\right)_{Z}=\tau^{*}\left(Z_{v, X, p}\right)_{Y} \tag{204}
\end{equation*}
$$

Proof. For any prime divisor $E$ at infinity of $Y, L_{v}(E)>0 \Leftrightarrow c_{Y}(v) \in E$ by item (3) ofProposition 7.6. Therefore, if $c_{Y}(\mathrm{v}) \in E$ is a free point with $E \in \Gamma_{\pi}$, then for $F \in \Gamma_{\pi}, L_{v}(F) \neq 0 \Leftrightarrow F=E$, hence

$$
\begin{equation*}
\left(L_{V}\right)_{\mid \operatorname{Div}_{\infty}(Y)_{\mathbf{R}}}=\left(L_{V}(E)\right)\left(L_{\operatorname{ord}_{E}}\right)_{\mid \operatorname{Div}_{\infty}(Y)_{\mathbf{R}}} \tag{205}
\end{equation*}
$$

by definition (see Equation (179)). This shows the result if $c_{Y}(v)$ is a free point. Now, if $c_{Y}(v)=E \cap F$ is a satellite point with $E, F \in \Gamma_{\pi}$, then for all prime divisors $F^{\prime}$ of $Y$ at infinity $L_{v}\left(F^{\prime}\right)>0 \Leftrightarrow F^{\prime}=E$ or $F^{\prime}=F$. We therefore have

$$
\begin{equation*}
\left.\left(L_{V}\right)_{\mid \operatorname{Div}_{\infty}(Y)_{\mathbf{R}}}=\left(L_{V} \cdot E\right)\left(L_{\operatorname{ord}_{E}}\right)\right)_{\operatorname{Div}_{\infty}(Y)_{\mathbf{R}}}+\left(L_{V} \cdot F\right)\left(L_{\operatorname{ord}_{F}}\right)_{\mid \operatorname{Div}_{\infty}(Y)_{\mathbf{R}}} \tag{206}
\end{equation*}
$$

This shows the result in the satellite case.
If $v \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$. Let $\tau: Z \rightarrow X$ be the blow up of $p$. We know then that $L_{\mathcal{V}}(\widetilde{E})=1$ where $\widetilde{E}$ is the exceptional divisor above $p$ by Proposition 5.12 . Let $b_{\widetilde{E}}$ be the generic multiplicity function of the tree $\mathcal{V}_{Z}(\widetilde{E} ; \widetilde{E})$. We have for every prime divisor $F$ exceptional above $p$ that $\operatorname{ord}_{F}(\widetilde{E})=b_{\widetilde{E}}(F)$ again by Proposition 5.12. In the free point case,
we get $1=L_{v}(\widetilde{E})=L_{v}\left(b_{\widetilde{E}}(E) E\right)$ by Proposition 7.6 (3) and (5). In the satellite point case, we get

$$
\begin{equation*}
1=L_{v}(\widetilde{E})=L_{v}\left(b_{\widetilde{E}}(E) E+b_{\widetilde{E}}(F) F\right) \tag{207}
\end{equation*}
$$

again by Proposition 7.6 (3) and (5).
For the last assertion, if $\widetilde{F}$ is the exceptional divisor above $q$, we have

$$
\begin{equation*}
\left(Z_{v, X, p}\right)_{Z}=\tau^{*}\left(Z_{v, X, p}\right)_{Y}-\left(Z_{v, X, p} \cdot \widetilde{F}\right) \widetilde{F} . \tag{208}
\end{equation*}
$$

Since $c_{Z}(v) \notin \widetilde{F}$, we have $L_{V}(\widetilde{F})=0$ by Proposition 7.6 (3).
From now on let $b$ be the generic multiplicity function of $\mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$ and for any prime divisor $E \in \mathcal{D}_{X, p}=\Gamma$, set $\nu_{E}=\frac{1}{b(E)} \operatorname{ord}_{E}$.

Proposition 7.30. Let $\pi:(Y, \operatorname{Exc}(\pi)) \rightarrow(X, p)$ be a completion exceptional above p. Let $q \in \operatorname{Exc}(\pi)$ be a closed point. Let $\tau: Z \rightarrow Y$ be the blow up of $q$ and let $\widetilde{E}$ be the exceptional divisor above $q$.
(1) If $q \in E$ is a free point with $E \in \Gamma_{\pi}$, then

$$
\begin{equation*}
Z_{v_{\tilde{E}}, X, p}=\tau^{*}\left(Z_{v_{E}, X, p}\right)-\frac{1}{b(\widetilde{E})} \widetilde{E} \in \operatorname{Div}_{\infty}(Z)_{\mathbf{Q}} \tag{209}
\end{equation*}
$$

(2) If $q=E \cap F$ is a satellite point with $E, F \in \Gamma_{\pi}$, then

$$
\begin{equation*}
Z_{v_{\tilde{E}}, X, p}=\frac{b(E)}{b(E)+b(F)} \tau^{*} Z_{v_{E}, X, p}+\frac{b(F)}{b(E)+b(F)} \tau^{*} Z_{v_{F}, X, p}-\frac{1}{b(\widetilde{E})} \widetilde{E} \in \operatorname{Div}_{\infty}(Z)_{\mathbf{Q}} \tag{210}
\end{equation*}
$$

Proof. If $q \in E$ is a free point with $E \in \Gamma_{\pi}$, we have by Proposition 7.29 that the incarnation of $Z_{\text {ord }_{\tilde{E}}, X, p}$ in $Y$ is

$$
\begin{equation*}
\tau_{*}\left(Z_{\operatorname{ord}_{\tilde{E}}, X, p}\right)=Z_{\operatorname{ord}_{E}, X, p} \tag{211}
\end{equation*}
$$

because $\operatorname{ord}_{\tilde{E}}(E)=1$. Therefore

$$
\begin{equation*}
Z_{\operatorname{ord}_{\tilde{E}}, X, p} \tau^{*} Z_{\operatorname{ord}_{E}, X, p}+\lambda \widetilde{E} \tag{212}
\end{equation*}
$$

with $\lambda \in \mathbf{R}$. Since $Z_{\text {ord }_{\tilde{E}}, X, p} \cdot \widetilde{E}=1$, we get $\lambda=-1$. Now, by the definition of the generic multiplicity, we have $b(\widetilde{E})=b(E)$. Therefore,

$$
\begin{equation*}
Z_{\widetilde{\tilde{E}}_{\tilde{E}}, X, p}=\tau^{*} Z_{v_{E}, X, p}-\frac{1}{b(\widetilde{E})} \widetilde{E} \tag{213}
\end{equation*}
$$

If $q=E \cap F$ is a satellite point with $E, F \in \Gamma_{\pi}$, then $b(\widetilde{E})=b(E)+b(F)$. Note that $\operatorname{ord}_{\widetilde{E}}(E)=\operatorname{ord}_{\widetilde{E}}(F)=1$. We have by Proposition 7.29

$$
\begin{equation*}
\tau_{*} Z_{\operatorname{ord}_{\tilde{E}}, X, p}=Z_{\operatorname{ord}_{E}, X, p}+Z_{\operatorname{ord}_{F}, X, p} \tag{214}
\end{equation*}
$$

and since $\operatorname{ord}_{\tilde{E}}(\widetilde{E})=1$, we get

$$
\begin{equation*}
Z_{\operatorname{ord}_{\tilde{E}}, X, p}=\tau^{*} Z_{\operatorname{ord}_{E}, X, p}+\tau^{*} Z_{\operatorname{ord}_{F}, X, p}-\widetilde{E} . \tag{215}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
Z_{\tilde{V}_{\tilde{E}}, X, p}=\frac{b(E)}{b(E)+b(F)} \tau^{*} Z_{\operatorname{ord}_{E}, X, p}+\frac{b(F)}{b(E)+b(F)} \tau^{*} Z_{\operatorname{ord}_{F}, X, p}-\frac{1}{b(\widetilde{E}))} \widetilde{E} . \tag{216}
\end{equation*}
$$

THEOREM 7.31. Let $\mathrm{v}, \mathrm{v}^{\prime} \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$, then

$$
\begin{equation*}
Z_{v, X, p} \cdot Z_{v^{\prime}, X, p}=-\alpha\left(v \wedge v^{\prime}\right) \tag{217}
\end{equation*}
$$

Proof. We show by induction the
CLAIM 7.32. For every completion $\pi:(Y, \operatorname{Exc}(\pi)) \rightarrow(X, p)$ exceptional above $p$, for all $E \in \Gamma_{\pi}$, for all $v \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$,

$$
\begin{equation*}
Z_{v_{E}, X, p} \cdot Z_{v, X, p}=-\alpha\left(v_{E} \wedge v\right) \tag{218}
\end{equation*}
$$

First if $\pi: Y \rightarrow X$ is the blow up of $p$ with exceptional divisor $\widetilde{E}$. Recall that $\pi_{*} \operatorname{ord}_{\widetilde{E}}=v_{\mathfrak{m}_{p}}$ then $Z_{\operatorname{ord}_{\tilde{E}}, X, p}=-E$ and

$$
\begin{equation*}
Z_{\text {ord }_{\widetilde{E}}, X, p} \cdot Z_{v, X, p}=Z_{v, X, p} \cdot(-\widetilde{E})=\cdot L_{v}(-\widetilde{E}) \tag{219}
\end{equation*}
$$

By definition, $v\left(\mathfrak{m}_{p}\right)=1$ and $\pi^{*} \mathfrak{m}_{p}=O_{Y}(-\widetilde{E})$. Therefore, by Proposition 7.10, we get $Z_{\text {ord }_{\tilde{E}}, X, p} \cdot Z_{v, X, p}=-1=-\alpha\left(v_{\mathfrak{m}_{p}} \wedge v\right)$.

Suppose that $\pi:(Y, \operatorname{Exc}(\pi)) \rightarrow(X, p)$ is a completion exceptional above $p$ for which the claim holds. Let $q \in Y$ be a closed point at infinity, let $\tau: Z \rightarrow Y$ be the blow up of $q$ and let $\widetilde{E}$ be the exceptional divisor. Let $v \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$, we show that $Z_{v, X, p} \cdot Z_{v_{\tilde{E}}, X, p}=$ $-\alpha\left(\nu \wedge v_{\tilde{E}}\right)$. We divide the proof in 2 different cases.

Case 1: $q \in E$ is a free point with $E \in \Gamma_{\pi}$. In that case $v_{\widetilde{E}}>v_{E}$ by Proposition 5.17. We also have $b(\widetilde{E})=b(E)$ and $Z_{v_{\tilde{E}}, X, p}=Z_{v_{E}, X, p}-\frac{1}{b(\widetilde{E})} \widetilde{E}$ by Proposition 7.30 . If $c_{Y}(v) \neq(q)$ (this includes the case where $c_{Y}(v)$ is a prime divisor at infinity. Then, $v \wedge v_{\widetilde{E}}=v \wedge v_{E}$. We have by Proposition 7.30 that $Z_{v_{\tilde{E}}, X, p}=\tau^{*}\left(Z_{v_{E}, X, p}\right)-\frac{1}{b(\widetilde{E})} \widetilde{E}$. Since $Z_{\vee, X, p} \cdot \widetilde{E}=0$, we get

$$
\begin{equation*}
Z_{v, X, p} \cdot Z_{v_{\tilde{E}}, X, p}=Z_{v, X, p} \cdot Z_{v_{E}, X, p} \tag{220}
\end{equation*}
$$

This is equal to $-\alpha\left(v \wedge v_{E}\right)$ by induction and therefore it is equal to $-\alpha\left(v \wedge v_{\tilde{E}}\right)$.
If $c_{Y}(v)=q$, then $c_{Z}(v) \in \widetilde{E}$. We either have $v_{\widetilde{E}} \leqslant v$ or $v_{E}<\nu \wedge v_{\widetilde{E}}<v_{\widetilde{E}}$.
(1) If $v \geqslant v_{\widetilde{E}}$, then $v \wedge v_{\widetilde{E}}=v_{\widetilde{E}}$ and $c_{Z}(v)$ is either $\widetilde{E}$ or a free point on $\widetilde{E}$. In both cases by Proposition 7.29 , the incarnation of $Z_{v, X, p}$ in $Z$ is $Z_{v_{\tilde{E}}, X, p}$. Therefore

$$
\begin{equation*}
Z_{\vee, X, p} \cdot Z_{v_{\tilde{E}, X, p}}=\left(Z_{v_{\tilde{E}}, X, p}\right)^{2}=\left(Z_{v_{E}, X, p}\right)^{2}-\frac{1}{b(\widetilde{E})^{2}} . \tag{221}
\end{equation*}
$$

By induction $\left(Z_{v_{E}, X, p}\right)^{2}=-\alpha\left(v_{E}\right)$ and $\alpha\left(v_{\tilde{E}}\right)=\alpha\left(v_{E}\right)+\frac{1}{b(\tilde{E})^{2}}$ by Proposition 5.27, so the claim is shown in that case.
(2) If $v_{E}<v \wedge v_{\tilde{E}}<v_{\tilde{E}}$. Then, $v \wedge v_{E}$ is a monomial valuation centered at $E \cap \widetilde{E}$ (we still denote by $E$ the strict transform of $E$ in $Z$ ). Therefore, by Proposition 5.31 there exists $s, t>0$ such that $s b(E)+t b(\widetilde{E})=1$ and $v \wedge v_{\widetilde{E}}=v_{s, t}$ is the monomial valuation with weight $s, t$ with respect to local coordinates associated to $E$ and $\widetilde{E}$ respectively. By Proposition 7.29 , we have

$$
\begin{equation*}
\left(Z_{V, X, p}\right)_{Z}=s Z_{\operatorname{ord}_{E}, X, p}+t Z_{\operatorname{ord}_{\tilde{E}}, X, p}=s b_{E} Z_{v_{E}, X, p}+t b_{\widetilde{E}} Z_{v_{\tilde{E}}, X, p} \tag{222}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
Z_{v, X, p} \cdot Z_{v_{\widetilde{E}}, X, p}=s b(E) Z_{v_{E}, X, p} \cdot Z_{v_{\tilde{E}}, X, p}+t b(\widetilde{E})\left(Z_{v_{\tilde{E}}, X, p}\right)^{2} . \tag{223}
\end{equation*}
$$

By induction and the previous case this is equal to $-b(E)\left(s \alpha\left(v_{E}\right)+t \alpha\left(v_{\tilde{E}}\right)\right)$. By Proposition 5.27, we have $\alpha\left(v_{\widetilde{E}}\right)=\alpha\left(v_{E}\right)+\frac{1}{b(E)^{2}}$. Therefore, we get

$$
\begin{equation*}
-b(E)\left(s \alpha\left(v_{E}\right)+t \alpha\left(v_{\widetilde{E}}\right)\right)=-\alpha\left(v_{E}\right)-\frac{t}{b(E)} \tag{224}
\end{equation*}
$$

and this is equal to $-\alpha\left(\pi_{*} v_{s, t}\right)$ by Proposition 5.31 .
Case 2: $q=E_{1} \cap E_{2}$ is a satellite point. We can suppose without loss of generality that $v_{E_{1}}<v_{E_{2}}$. In that case we get $v_{E_{1}}<v_{\widetilde{E}}<v_{E_{2}}, b(\widetilde{E})=b\left(E_{1}\right)+b\left(E_{2}\right)$ and

$$
\begin{equation*}
Z_{v_{\widetilde{E}}, X, p}=\frac{b\left(E_{1}\right)}{b\left(E_{1}\right)+b\left(E_{2}\right)} Z_{v_{E_{1}}, X, p}+\frac{b\left(E_{2}\right)}{b\left(E_{1}\right)+b\left(E_{2}\right)} Z_{v_{E_{2}}, X, p}-\frac{1}{b(\widetilde{E})} \widetilde{E} \tag{225}
\end{equation*}
$$

by Proposition 7.30 .
If $c_{Y}(v) \neq q$, then $v \wedge v_{E_{2}} \leqslant v_{E_{1}}$ or $v \geqslant v_{E_{2}}$ and we get

$$
\begin{equation*}
Z_{v, X, p} \cdot Z_{v_{\tilde{E}}, X, p}=\frac{b\left(E_{1}\right)}{b\left(E_{1}\right)+b\left(E_{2}\right)}\left(Z_{v, X, p} \cdot Z_{v_{E_{1}}, X, p}\right)+\frac{b\left(E_{2}\right)}{b\left(E_{1}\right)+b\left(E_{2}\right)}\left(Z_{v, X, p} \cdot Z_{v_{E_{2}}, X, p}\right) \tag{226}
\end{equation*}
$$

By induction, this is equal to $-\frac{b\left(E_{1}\right)}{b\left(E_{1}\right)+b\left(E_{2}\right)} \alpha\left(v_{E_{1}} \wedge v\right)-\frac{b\left(E_{2}\right)}{b\left(E_{1}\right)+b\left(E_{2}\right)} \alpha\left(v_{E_{2}} \wedge v\right)$.
If $v \wedge v_{E_{2}} \leqslant v_{E_{1}}$, then $v \wedge v_{E_{2}}=v \wedge \nu_{\widetilde{E}}=v \wedge v_{E_{1}}$ and the quantity in Equation (226) is equal to $-\alpha\left(v \wedge v_{\tilde{E}}\right)$.

If $v \geqslant v_{E_{2}}$, then $v>v_{\tilde{E}}$ and $v \wedge v_{\tilde{E}}=v_{\tilde{E}}$. In that case $v \wedge v_{E_{1}}=v_{E_{1}}$ and $v \wedge \nu_{E_{2}}=$ $v_{E_{2}}$. Therefore, the quantity in Equation (226) is equal to

$$
\begin{equation*}
-\frac{b\left(E_{1}\right)}{b\left(E_{1}\right)+b\left(E_{2}\right)} \alpha\left(v_{E_{1}}\right)-\frac{b\left(E_{2}\right)}{b\left(E_{1}\right)+b\left(E_{2}\right)} \alpha\left(v_{E_{2}}\right) . \tag{227}
\end{equation*}
$$

By Proposition 5.27, $\alpha\left(v_{E_{2}}\right)=\alpha\left(v_{E_{1}}\right)+\frac{1}{b\left(E_{1}\right) b\left(E_{2}\right)}$, so we get

$$
\begin{equation*}
Z_{v, X, p} \cdot Z_{v_{\tilde{E}}, X, p}=-\alpha\left(v_{E_{1}}\right)-\frac{1}{b\left(E_{1}\right)\left(b\left(E_{1}\right)+b\left(E_{2}\right)\right)}=-\alpha\left(v_{E_{1}}\right)-\frac{1}{b\left(E_{1}\right) b(\widetilde{E})} \tag{228}
\end{equation*}
$$

and this is equal to $-\alpha\left(v_{\tilde{E}}\right)$ again by Proposition 5.27.
If $c_{Y}(v)=q$, then $c_{Z}(v) \in \widetilde{E}$. We have that $v_{E_{1}}<v \wedge v_{\widetilde{E}}<v_{E_{2}}$. Therefore either $\mathrm{v}=\mathrm{v}_{\widetilde{E}}$ or $c_{Z}(\mathrm{v}) \in \widetilde{E}$ is a point and $\nu \wedge \nu_{\widetilde{E}}$ is a monomial valuation centered at $E_{1} \cap \widetilde{E}$ or $E_{2} \cap E$. We show again the claim by induction in an analogous way as in Case 1 . We have thus shown the claim by induction.

To show the Proposition, let $v, v^{\prime} \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$. If $v \neq v^{\prime}$, then there exists a completion $\pi:(Y, \operatorname{Exc}(\pi)) \rightarrow(X, p)$ exceptional above $p$ such that $c_{Y}(v) \neq c_{Y}\left(v^{\prime}\right)$. Then, we have that

$$
\begin{equation*}
Z_{v, X, p} \cdot Z_{\boldsymbol{v}^{\prime}, X, p}=\left(Z_{\vee, X, p}\right)_{Y} \cdot\left(Z_{\boldsymbol{v}^{\prime}, X, p}\right)_{Y} \tag{229}
\end{equation*}
$$

If $v^{\prime}$ is infinitely singular or a curve valuation, we can suppose that $c_{Y}\left(v^{\prime}\right)$ is a free point lying over a unique prime divisor $E$ at infinity. Then, $\mathrm{v}^{\prime}>\mathrm{v}_{E}$ and $\mathrm{v}^{\prime} \wedge \mathrm{v}=$ $v^{\prime} \wedge v_{E}$. Furthermore, the incarnation of $Z_{v, X, p}$ in $Y$ is exactly $Z_{v_{E}, X, p}$ by Proposition 7.29. Therefore,

$$
\begin{equation*}
Z_{v, X, p} \cdot Z_{v^{\prime}, X, p}=Z_{v, X, p} \cdot Z_{v_{E}, X, p} \tag{230}
\end{equation*}
$$

This is equal to $-\alpha\left(v \wedge v_{E}\right)=-\alpha\left(v \wedge v^{\prime}\right)$ by the Claim.
If $v^{\prime}$ is irrational, then we can suppose that $c_{Y}\left(v^{\prime}\right)=E_{1} \cap E_{2}$ for $E_{1}, E_{2}$ two prime divisors at infinity. Suppose without loss of generality that $v_{E_{1}}<v_{E_{2}}$. By Proposition 5.31, we have that $\mathrm{v}^{\prime}=\pi_{*} \mathrm{v}_{s, t}$ for some $s, t>0$ such that $\operatorname{sb}\left(E_{1}\right)+t b\left(E_{2}\right)=1$ and $\alpha\left(v^{\prime}\right)=\alpha\left(v_{E_{1}}\right)+\frac{t}{b\left(E_{1}\right)}$. Furthermore, by Proposition 7.29, the incarnation of $Z_{v^{\prime}, X, p}$ in $Y$ is

$$
\begin{equation*}
\left(Z_{v^{\prime}, X, p}\right)_{Y}=s b\left(E_{1}\right) Z_{v_{E_{1}}, X, p}+t b\left(E_{2}\right) Z_{v_{E_{2}}, X, p} \tag{231}
\end{equation*}
$$

And we have

$$
\begin{equation*}
Z_{v, X, p} \cdot Z_{V^{\prime}, X, p}=s b\left(E_{1}\right)\left(Z_{v, X, p} \cdot Z_{v_{E_{1}}, X, p}\right)+t b\left(E_{2}\right)\left(Z_{v, X, p} \cdot Z_{v_{E_{2}}, X, p}\right) \tag{232}
\end{equation*}
$$

Either $v \wedge v^{\prime}=v \wedge v_{E_{1}}$ or $v \wedge v^{\prime}=v^{\prime}$. If $v \wedge v^{\prime}=v \wedge v_{E_{1}}$, then we also have $v \wedge v_{E_{2}}=$ $v \wedge v_{E_{1}}$. The quantity in Equation (232) is then equal to

$$
\begin{equation*}
-s b\left(E_{1}\right) \alpha\left(v \wedge v_{E_{1}}\right)-t b\left(E_{2}\right) \alpha\left(v \wedge v_{E_{2}}\right)=\alpha\left(v \wedge v_{E_{1}}\right)=-\alpha\left(v \wedge v^{\prime}\right) \tag{233}
\end{equation*}
$$

If $v \wedge v^{\prime}=v^{\prime}$, then $v \wedge v_{E_{1}}=v_{E_{1}}$ and $v \wedge v_{E_{2}}=v_{E_{2}}$. The quantity in Equation (232) is then equal to

$$
\begin{equation*}
-s b\left(E_{1}\right) \alpha\left(v_{E_{1}}\right)-t b\left(E_{2}\right) \alpha\left(v_{E_{2}}\right)=-\alpha\left(v_{E_{1}}\right)-\frac{t}{b\left(E_{1}\right)}=-\alpha\left(v^{\prime}\right) \tag{234}
\end{equation*}
$$

To get the last two equalities we use Proposition 5.27 and 5.31 .
Finally, if $v=v^{\prime}$, we need to show that $\left(Z_{v, X, p}\right)^{2}=-\alpha(v)$. We know the result if $v$ is divisorial. We use infinitely near sequence to conclude in general. If $v$ is infinitely singular or a curve valuation. Let $\left(X_{n}, p_{n}\right)$ be the sequence of infinitely near points associated to $v$. The infinitely near sequence of $v$ (Proposition 5.26) is the subsequence $v_{n}=\frac{1}{b\left(E_{n}\right)} \operatorname{ord}_{E_{n}}$ where $p_{n}$ is a free point lying over a unique prime divisor $E_{n}$ at infinity. We have that $\alpha\left(v_{n}\right) \rightarrow \alpha(v)$ and the incarnation of $Z_{v, X, p}$ in $X_{n}$ is $Z_{v_{n}, X, p}$. Therefore,

$$
\begin{equation*}
\left(Z_{v, X, p}\right)^{2}=\lim _{n}\left(Z_{v_{n}, X, p}\right)^{2}=-\lim _{n} \alpha\left(v_{n}\right)=-\alpha(v) \tag{235}
\end{equation*}
$$

If $v$ is irrational, then let $\left(X_{n}, p_{n}\right)$ be the sequence of infinitely near points associated to $v$. For every $n$ large enough, $p_{n}=E_{n} \cap F_{n}$ for $E_{n}, F_{n}$ two prime divisors at infinity. Suppose that for all $n, \mathrm{v}_{E_{n}}<\mathrm{v}_{F_{n}}$. Then, we have $\mathrm{v}_{E_{n}}<\mathrm{V}<\mathrm{v}_{F_{n}}$, $\alpha\left(v_{E_{n}}\right) \rightarrow \alpha(v), \alpha\left(v_{F_{n}}\right) \rightarrow \alpha(v)$ and $b\left(E_{n}\right) \rightarrow+\infty, b\left(F_{n}\right) \rightarrow+\infty$. We have by Proposition 7.29 that the incarnation of $Z_{v, X, p}$ in $X_{n}$ is

$$
\begin{equation*}
s_{n} b\left(E_{n}\right) Z_{v_{E_{n}}, X, p}+t_{n} b\left(F_{n}\right) Z_{v_{F_{n}}, X, p} \tag{236}
\end{equation*}
$$

for some $s_{n}, t_{n}>0$ such that $s_{n} b\left(E_{n}\right)+t_{n} b\left(F_{n}\right)=1$. We have

$$
\begin{align*}
\left(Z_{v, X, p}\right)^{2} & =\lim _{n}\left(s_{n} b\left(E_{n}\right) Z_{v_{n}, X, p}+t_{n} b\left(F_{n}\right) Z_{v_{F_{n}, X, p}}\right)^{2}  \tag{237}\\
& =\lim _{n}-s_{n}^{2} b\left(E_{n}\right)^{2} \alpha\left(v_{E_{n}}\right)-2 s_{n} t_{n} b\left(E_{n}\right) b\left(F_{n}\right) \alpha\left(v_{E_{n}}\right)-t_{n}^{2} b\left(F_{n}\right)^{2} \alpha\left(v_{F_{n}}\right) \tag{238}
\end{align*}
$$

Therefore we get

$$
\begin{equation*}
\lim _{n}-\alpha\left(v_{E_{n}}\right) \leqslant\left(Z_{v, X, p}\right)^{2} \leqslant \lim _{n}-\alpha\left(v_{F_{n}}\right) \tag{239}
\end{equation*}
$$

Hence $\left(Z_{v, X, p}\right)^{2}=-\alpha(v)$.
Corollary 7.33. If $v \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$, then $Z_{v, X, p} \notin \operatorname{Weil}(X, p)_{\mathbf{Q}}$ if and only if $v$ is irrational.

Proof. If $v$ is divisorial, let $E \in \mathcal{D}_{X, p}$ such that $v$ is equivalent to $\operatorname{ord}_{E}$. Then,

$$
\begin{equation*}
Z_{V, X, p}=\frac{1}{b(E)} Z_{\mathrm{ord}_{E}, X, p} \in \operatorname{Weil}(X, p)_{\mathbf{Q}} \tag{240}
\end{equation*}
$$

by Proposition 7.27. If $v$ is infinitely singular or a curve valuation, let $\mu$ be any divisorial valuation. We have that $\mu \wedge \nu$ must be a divisorial valuation, therefore by Theorem 7.31 we have

$$
\begin{equation*}
Z_{\mu} \cdot Z_{v}=-\alpha(v \wedge \mu) \in \mathbf{Q} \tag{241}
\end{equation*}
$$

Hence $Z_{v, X, p} \in \operatorname{Weil}(X, p)_{\mathbf{Q}}$.
If $v$ is irrational, then for all $\mu \geqslant v$ divisorial we have $\alpha(\mu \wedge v)=\alpha(v) \in \mathbf{R} \backslash \mathbf{Q}$. Therefore, $Z_{v, X, p} \notin \operatorname{Weil}(X, p)_{\mathbf{Q}}$.

Proposition 7.34. Let $X$ be a completion, let $p \in X$ be a closed point at infinity. If $\left(v_{n}\right)$ is a sequence of $\mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$ such that $\alpha\left(v_{n}\right)<+\infty$ for all $n$ and $v \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$, then $v_{n} \rightarrow v$ for the strong topology if and only if $Z_{v_{n}, X, p} \rightarrow Z_{v, X, p}$ for the strong topology of $L^{2}\left(\mathrm{X}_{0}\right)$.

Proof. This all comes from Theorem 7.31 as

$$
\begin{align*}
\left|\left(Z_{v, X, p}-Z_{v_{n}, X, p}\right)^{2}\right| & =\left|-\alpha(v)+2 \alpha\left(v \wedge v_{n}\right)-\alpha\left(v_{n}\right)\right|  \tag{242}\\
& =\left|\alpha(v)-\alpha\left(v \wedge v_{n}\right)+\alpha\left(v_{n}\right)-\alpha\left(v \wedge v_{n}\right)\right| . \tag{243}
\end{align*}
$$

## CHAPTER 8

## From linear forms to valuations

Suppose now that we have an element $L$ of $\operatorname{hom}\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \mathbf{R}\right)$ satisfying property (+), we want to construct a valuation $\nu_{L}: \mathbf{k}\left[X_{0}\right] \rightarrow \mathbf{R} \cup\{\infty\}$ centered at infinity such that $v_{f_{*} L}=f_{*} v_{L}$.

### 8.1. Construction of $v_{L}$

First we extend $L$ to $S_{\infty}\left(X_{0}\right)$ (see Definition 3.12) by setting

$$
\begin{equation*}
\text { If } D=\bigvee_{i} D_{i} \text { with } D_{i} \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \quad L(D):=\sup _{i} L\left(D_{i}\right) . \tag{244}
\end{equation*}
$$

Proposition 8.1. This definition does not depend on the representation of $D$ as a supremum $D=\bigvee_{i} D_{i}$ with $D_{i} \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$.

Proof. If $D=\bigvee_{i \in I} D_{i}=\bigvee_{j \in J} D_{j}^{\prime}$. Let $j \in J$ be an index and $X$ a completion such that $D_{j}^{\prime}$ is defined on $X$. Let $\varepsilon>0$ and let $H$ be an effective divisor such that $\operatorname{Supp}(H)=\partial_{X} X_{0}$. There exists an index $i \in I$ such that $D_{i}+\varepsilon H \geqslant D_{j}^{\prime}$, since otherwise we would get $D+\varepsilon H \leqslant D_{j}^{\prime} \leqslant D$. Therefore we have by property ( + ) item (1)

$$
\begin{equation*}
L\left(D_{j}^{\prime}\right) \leqslant L\left(D_{i}\right)+\varepsilon L(H) \leqslant \sup _{k} L\left(D_{k}\right)+\varepsilon L(H) . \tag{245}
\end{equation*}
$$

Letting $\varepsilon$ go to 0 , we get $\sup _{j} L\left(D_{j}^{\prime}\right) \leqslant \sup _{k} L\left(D_{k}\right)$ and the result holds by symmetry.

PROPOSITION 8.2. We have the following properties: for $D, D^{\prime} \in S_{\infty}\left(X_{0}\right)$
(1) $L\left(D+D^{\prime}\right)=L(D)+L\left(D^{\prime}\right)$.
(2) $L\left(D \wedge D^{\prime}\right)=\min \left(L(D), L\left(D^{\prime}\right)\right)$.
(3) If $D \geqslant 0$, then $L(D) \geqslant 0$.

Proof. For (1), write

$$
\begin{aligned}
L\left(D+D^{\prime}\right) & =\sup _{(i, j) \in I \times J} L\left(D_{i}+D_{j}^{\prime}\right) \\
& =\sup _{i \in I} L\left(D_{i}\right)+\sup _{j \in J} L\left(D_{j}^{\prime}\right)=L(D)+L\left(D^{\prime}\right)
\end{aligned}
$$

For (2), let $D=\bigvee_{i} D_{i}$ and $D^{\prime}=\bigvee_{j} D_{j}^{\prime}$ be two elements of $S_{\infty}\left(X_{0}\right)$. Then,

$$
\begin{equation*}
D \wedge D^{\prime}=\bigvee_{i, j} D_{i} \wedge D_{j}^{\prime} \tag{246}
\end{equation*}
$$

and

$$
\begin{align*}
L\left(D \wedge D^{\prime}\right) & =\sup _{i, j} \min \left(L\left(D_{i}\right), L\left(D_{j}^{\prime}\right)\right)  \tag{247}\\
& =\min ^{\left(\sup _{i} L\left(D_{i}\right), \sup _{j} L\left(D_{j}^{\prime}\right)\right)}  \tag{248}\\
& =\min \left(L(D), L\left(D^{\prime}\right)\right) \tag{249}
\end{align*}
$$

For (3), if $D=0$, then $L(D)=0$. Otherwise, $D>0$ and there exists a Cartier divisor $D_{i}$ defined in some completion $X$ of $X_{0}$ such that $D_{X} \geqslant D_{i} \geqslant 0$ and therefore

$$
\begin{equation*}
L(D) \geqslant L\left(D_{i}\right) \geqslant 0 . \tag{250}
\end{equation*}
$$

Recall the notations of Section 3.2, Define

$$
\begin{equation*}
w(P):=\left(\operatorname{div}_{\infty, X}(P)\right)_{X} . \tag{251}
\end{equation*}
$$

Proposition 8.3. For $P \in \mathbf{k}\left[X_{0}\right], w(P)$ defines an element of Weil ${ }_{\infty}\left(X_{0}\right)$, moreover if one identifies for any completion $X$ the divisor $\operatorname{div}_{\infty_{, X}}(P) \in \operatorname{Div}_{\infty}(X)$ with its image in $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$, then

$$
\begin{equation*}
w(P)=\bigvee_{X} \operatorname{div}_{\infty, X}(P) \tag{252}
\end{equation*}
$$

Thus, $w(P)$ defines an element of $S_{\infty}\left(X_{0}\right)$.
Proof. To prove both assertions it suffices to show that if $X$ is a completion of $X_{0}$ and $Y$ is the blow up of some point at infinity, then $\pi_{*} \operatorname{div}_{\infty, Y}(P)=\operatorname{div}_{\infty, X}(P)$ and $\pi^{*} \operatorname{div}_{\infty, X}(P) \leqslant \operatorname{div}_{\infty, Y}(P)$. Let $\widetilde{E}$ be the exceptional divisor of $\pi$ and let $E_{1}, \ldots, E_{r}$ be the prime divisors in $\partial_{X} X_{0}$. Since $P$ is regular over $X_{0}, \operatorname{div}_{X}(P)$ is of the form

$$
\begin{equation*}
\operatorname{div}_{X}(P)=D+\sum_{i=1}^{r} a_{i} E_{i} \tag{253}
\end{equation*}
$$

where $D$ is an effective divisor such that no irreducible component of its support is one of the $E_{i}$ 's; by definition $\operatorname{div}_{\infty 0, X}(P)=\sum_{i=1}^{r} a_{i} E_{i}$. $\operatorname{Then}^{\operatorname{div}} \operatorname{div}_{Y}(P)$ is of the form

$$
\begin{equation*}
\operatorname{div}_{Y}(P)=\operatorname{div}_{Y}(P \circ \pi)=\pi^{*} \operatorname{div}_{X}(P)=\pi^{\prime}(D)+b \widetilde{E}+\sum_{i=1}^{r} a_{i} \pi^{\prime}\left(E_{i}\right) \tag{254}
\end{equation*}
$$

for some $b \in \mathbf{Z}$. So $\operatorname{div}_{\infty, Y}(P)=b \widetilde{E}+\sum_{i=1}^{r} a_{i} \pi^{\prime}\left(E_{i}\right)$ and we get $\pi_{*}\left(\operatorname{div}_{\infty, Y}(P)\right)=$ $\operatorname{div}_{\infty, X}(P)$ as $\pi_{*}(\widetilde{E})=0$, This shows that $w(P)$ is an element of Weil ${ }_{\infty}\left(\mathrm{X}_{0}\right)$.

To show that $\pi^{*} \operatorname{div}_{\infty, X}(P) \leqslant \operatorname{div}_{\infty, Y}(P)$ we have to be more precise about the coefficient $b$. We can write $b=c+d$, where $\pi^{*} D=\pi^{\prime}(D)+d \widetilde{E}$ and $\pi^{*} \operatorname{div}_{\infty, X}(P)=$ $c \widetilde{E}+\sum_{i} a_{i} \pi^{\prime}\left(E_{i}\right)$. Since, $D$ is effective, we have $d \geqslant 0$ and the result follows.

We define

$$
\begin{equation*}
v_{L}(P):=L(w(P)) \tag{255}
\end{equation*}
$$

Remark 8.4. The class $w(P)$ is not in general a Cartier class. Indeed, take $X_{0}=$ $\mathbf{A}^{2}, X=\mathbf{P}^{2}$ with homogeneous coordinates $[x: y: z]$ such that $\{z=0\}$ is the line at infinity. Consider $P=y / z \in \mathbf{k}\left(\mathbf{P}^{2}\right)$. Define a sequence of blow ups $X_{i}$ by $X_{0}=\mathbf{P}^{2}, E_{0}=$ $\{z=0\}$ and $\pi_{i+1}: X_{i+1} \rightarrow X_{i}$ the blow up of the intersection point of the strict transform of $\{y=0\}$ in $X_{i}$ and $E_{i}$, where $E_{i}$ is the exceptional divisor in $X_{i}$. Let $C_{y}$ be the strict transform of $\{y=0\}$ in any the $X_{i}$. We still denote by $E_{i}$ its strict transform in every $X_{j}, j \geqslant i$. Then,

$$
\begin{aligned}
\operatorname{div}_{\mathbf{P}^{2}}(P) & =C_{y}-E_{0} \\
\operatorname{div}_{X_{1}}(P) & =C_{y}-E_{0} \\
\operatorname{div}_{X_{2}}(P) & =C_{y}+E_{2}-E_{0} \\
\operatorname{div}_{X_{3}}(P) & =C_{y}+2 E_{3}+E_{2}-E_{0}
\end{aligned}
$$

and by induction, we get for all $k \geqslant 2$

$$
\begin{equation*}
\operatorname{div}_{X_{k}}(P)=C_{y}+\sum_{j=2}^{k}(j-1) E_{j}-E_{0} \tag{256}
\end{equation*}
$$

Therefore, for all $k \geqslant 2$

$$
\begin{aligned}
\pi_{k+1}^{*} \operatorname{div}_{\infty, X_{k}}(P) & =(k-1) E_{k+1}+\sum_{j=2}^{k}(j-1) E_{j}-E_{0} \\
& \neq k E_{k+1}+\sum_{j=2}^{k}(j-1) E_{j}-E_{0}=\operatorname{div}_{\infty, X_{k+1}}(P) .
\end{aligned}
$$

Thus, $w(P)$ is not a Cartier class.

### 8.2. Proofs

We show that $\nu_{L}$ is a valuation centered at infinity and satisfies $f_{*} \nu_{L}=v_{f_{*} L}$.

## Proposition 8.5. The function $v_{L}$ is a valuation on $\mathbf{k}\left[X_{0}\right]$ centered at infinity.

Proof. We first show that $v_{L}$ is in fact a valuation
(1) For any $\lambda \in \mathbf{k}^{*}$ and for any completion $X$ of $X_{0}, \operatorname{div}_{X}(\lambda)=0$ so $v_{L}(\lambda)=0$.
(2) If $f, g \in \mathbf{k}\left[X_{0}\right]$, then $\operatorname{div}_{X}(f g)=\operatorname{div}_{X}(f)+\operatorname{div}_{X}(g)$. So, $w(f g)=w(f)+w(g)$ and by Proposition $8.2 v_{L}(f g)=v_{L}(f)+v_{L}(g)$.
(3) Let $f, g \in \mathbf{k}\left[X_{0}\right], f \neq-g$, then $\operatorname{div}_{X}(f+g) \geqslant \operatorname{div}_{X}(f) \wedge \operatorname{div}_{X}(g)$, therefore

$$
\begin{equation*}
w(f+g) \geqslant w(f) \wedge w(g) \tag{257}
\end{equation*}
$$

and by Proposition $8.2 v_{L}(f+g) \geqslant \min \left(v_{L}(f), v_{L}(g)\right)$.
If $L \neq 0$, there exists a completion $X$ and a prime divisor $E$ at infinity such that $L(E)>0$. By Theorem 2.4, there exists $H \in \operatorname{Div}_{\infty}(X)$ ample such that $H \geqslant 0, \operatorname{Supp} H=$ $\partial_{X} X_{0}$. We have by item (1) of $(+)$ that $L(H) \geqslant L(E)>0$. To show that $v_{L}$ is centered at infinity, it suffices to show that $L_{v_{L}}(H)>0$. Up to replacing $H$ by one of its multiples (which does not change the hypothesis $L(H)>0$ ), we can suppose that $H$ is very ample and that it induces an embedding $\tau: X \hookrightarrow \mathbf{P}^{N}$ such that $\tau(H)$ is the intersection of $\tau(X)$ with the hyperplane $\left\{x_{0}=0\right\}$. By Bertini's theorem, we can find a hyperplane $M=\left\{\sum_{i} \lambda_{i} x_{i}=0\right\} \neq\left\{x_{0}=0\right\}$ such that $M \cap \tau(X)$ is a smooth irreducible subvariety $C$ in $X$ satisfying
(1) The intersection of $C$ with any divisor at infinity of $X$ is transverse.
(2) If $v_{L}$ is not divisorial, the center of $v_{L}$ is not contained in $C$.

Indeed, by Bertini theorem, the set $U_{X}$ of hyperplanes $H$ such that $H \cap X$ is a smooth irreducible curve is an open dense subset. Let $E_{1}, \cdots, E_{n}$ be the primes at infinity in $X$. Applying Bertini theorem to $E_{i}$ yields an open subset $U_{i}$ of hyperplanes that meet $E_{i}$ transversally. Finally, if the center of $v_{L}$ is a subvariety $Y$ of codimension $\geqslant 2$, then the set of hyperplanes that contain $Y$ is a closed nowhere dense subset of $\mathbf{P}\left(\Gamma\left(\mathbf{P}^{n}, O(1)\right)\right)$ because $|H|$ is base point free, so its complementary is a non-empty open subset $U_{Y}$. Now, $U_{1} \cap \cdots \cap U_{n} \cap U_{Y}$ is an open subset that intersects $U_{X}$ since it is dense, we then choose $M$ in the intersection. Define

$$
\begin{equation*}
P=\sum_{i=0}^{N} \lambda_{i} \frac{x_{i}}{x_{0}} \tag{258}
\end{equation*}
$$

Then, $P$ is a regular function over $X_{0}$ such that $\operatorname{div}_{X}(P)=C-H$ and $1 / P$ is a local equation of $H$ at the center of $v_{L}$ (even if $v_{L}$ is divisorial). Hence,

$$
\begin{equation*}
L_{v_{L}}(H)=v_{L}(1 / P)=\sup _{Y}\left(L\left(\operatorname{div}_{\infty, Y}(1 / P)\right) \geqslant L(H)>0 .\right. \tag{259}
\end{equation*}
$$

In Chapter 7, we have constructed a map

$$
\begin{equation*}
L: \mathcal{V}_{\infty} \rightarrow \operatorname{hom}\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \mathbf{R}\right)_{(+)} ; \tag{260}
\end{equation*}
$$

here, we have constructed a map

$$
\begin{equation*}
v: \operatorname{hom}\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \mathbf{R}\right)_{(+)} \rightarrow \mathcal{V}_{\infty} \tag{261}
\end{equation*}
$$

where hom $\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \mathbf{R}\right)_{(+)}$are the linear forms over $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$ that satisfy property (+). We shall prove that they are mutual inverse in Chapter 9 . Using this result we show

Proposition 8.6. Let $f$ be a dominant endomorphism of $X_{0}$. If $\left.L \in \operatorname{hom}\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \mathbf{R}\right)_{(+)}\right)$, then $f_{*} L \in \operatorname{hom}\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \mathbf{R}\right)_{(+)}$and $v_{f_{*} L}=f_{*} v_{L}$.

Proof. Let $\left.L \in \operatorname{hom}\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \mathbf{R}\right)_{(+)}\right)$, then there exists a unique valuation $v \in \mathcal{V}_{\infty}$ such that $L=L_{v}$. Then, we have by Proposition 7.7 that

$$
\begin{equation*}
f_{*} L=f_{*} L_{v}=L_{f_{*} v} \tag{262}
\end{equation*}
$$

Therefore, $f_{*} L \in \operatorname{hom}\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \mathbf{R}\right)_{(+)}$and if $w \in \mathcal{V}_{\infty}$ such that $f_{*} L=L_{w}$ it is clear that $w=f_{*} v$.

Remark 8.7. If $P \in\left(X_{0}\right)$, it is not true that $w\left(f^{*} P\right)=\bigvee \operatorname{div}_{\infty, X}\left(f^{*} P\right)$. Indeed, the problem is that $f$ might not be proper. For example, take $f(x, y)=(x, x y)$ with $X_{0}=\mathbf{A}^{2}$. In $\mathbf{P}^{2}$, blow up $[0: 1: 0]$, let $E$ be the exceptional divisor and blow up again the intersection point of $E$ and the strict transform of $\{X=0\}$. Let $V$ be the completion obtained after the two blow ups and call $E_{1}$ the exceptional divisor. The lift $f: V \rightarrow \mathbf{P}^{2}$ is regular and $f_{*} E_{1}=\{X=0\}$. Thus, we have that for all $D \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \operatorname{ord}_{E_{1}} f^{*} D=$ 0 . Now take $P=x$, then $f^{*} P=P$ and

$$
\begin{equation*}
\operatorname{div}_{V}(P)=\{X=0\}+E_{1}-\{Z=0\} \tag{263}
\end{equation*}
$$

Thus, $w\left(f^{*} P\right) \neq \bigvee f^{*} \operatorname{div}_{\infty, X}(P)$. However it is true in general that

$$
\begin{equation*}
w\left(f^{*} P\right) \geqslant \bigvee f^{*} \operatorname{div}_{\infty, X}(P) \tag{264}
\end{equation*}
$$

## CHAPTER 9

## Proof that $v$ and $L$ are mutual inverses

Set $\mathcal{M}:=\operatorname{hom}\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \mathbf{R}\right)_{(+)}$. In Chapters 7 and 8 , we have defined $L: v \in$ $\mathcal{V}_{\infty} \mapsto L_{v} \in \mathcal{M}$ and $v: L \in \mathcal{M} \mapsto v_{L} \in \mathcal{V}_{\infty}$. The goal is to show that these two maps are inverse of each other.

$$
\text { 9.1. First step, } \vee \circ L=\operatorname{id}_{V_{\infty}}
$$

Proposition 9.1. For all valuation $v \in \mathcal{V}_{\infty}$ and for all $P \in O_{X}\left(X_{0}\right), v(P)=L_{v}(w(P))$.
Proof. Let $X$ be a completion of $X_{0}$. We have seen that $\operatorname{div}_{\infty, X}(P)=\operatorname{div}_{X}(P)-D$ where $D$ is an effective divisor not supported in $\partial_{X} X_{0}$. Therefore,

$$
\begin{equation*}
L_{v, X}\left(\operatorname{div}_{\infty, X}(P)\right)=v(P)-L_{v, X}(D) \leqslant v(P) \tag{265}
\end{equation*}
$$

Taking the supremum over $X$, we get $L_{v}(w(P)) \leqslant v(P)$.
To show the other inequality, take a valuation $v$ centered at infinity and let $X$ be a completion of $X_{0}$. Up to further blow ups of point at infinity, we can suppose that $D:=\operatorname{div}_{X}(P)$ is a divisor in $X$ with simple normal crossing on $\partial_{X} X_{0}$. Let $E_{1}, \cdots, E_{r}$ be the prime divisors at infinity of $X$. Then, $D$ is of the form

$$
\begin{equation*}
D=\sum_{i=1}^{r} a_{i} E_{i}+\sum_{j \in J} b_{j} F_{j} \tag{266}
\end{equation*}
$$

for some prime divisors $F_{j}$ not supported at infinity. Let $p$ be the center of $v$ on $X$, there are two cases.
(1) For all $j \in J, p \notin F_{j}$, in that case for all $j \in J, L_{v, X}\left(F_{j}\right)=0$ and $v(P)=L_{v, X}\left(\operatorname{div}_{\infty, X}(P)\right)$. Therefore, $v(P) \leqslant L_{v}(w(P))$ and they are equal.
(2) There exist a unique $j \in J$ and a unique $i$ such that $p=E_{i} \cap F_{j}$. The uniqueness comes from the fact that $D$ is a divisor with simple normal crossing. We denote them respectively by $E$ and $F$. Then, we construct a sequence of blow up of points $\pi_{i}: \overline{X_{i+1}} \rightarrow \overline{X_{i}}$ such that $\pi_{i}$ is the blow-up of the center of $v$ in $X_{i}$ and $X_{0}=X$. We still denote by $F$ the strict transform of $F$ in any of these blow-ups. There are two possibilities:
(a) Either there exists a number $k$ such that the center of $v$ in $X_{k}$ does not belong to $F$ (This includes the case where $v$ is divisorial, in that case
the center becomes a prime divisor and there are no more blow-ups to be done). In that case, we are back in case 1 and $v(P)=v_{X_{k}}\left(\operatorname{div}_{\infty, X_{k}}(P)\right) \leqslant$ $L_{v}(w(P))$ and we get the desired equality.
(b) Or for all $k \geqslant 0$, the center of $v$ in $X_{k}$ belongs to $F$, in that case $v$ is the curve valuation associated to $F$ at $p$ and $v(P)=+\infty$. We show that $v_{X_{k}}\left(\operatorname{div}_{\infty, X_{k}}(P)\right) \rightarrow+\infty$ using the following result.
LEMMA 9.2. In case 2.(b), set $E_{0}=E$ and for $k \geqslant 1, \widetilde{E}_{k}$ the exceptional divisor in $X_{k}$ above $c_{X_{k-1}}(\mathrm{v})$, then $L_{v, X_{0}}(E)=L_{v, X_{k}}\left(E_{k}\right)$ for all $k$ and the divisor $\operatorname{div}_{X_{k}}(P)$ is of the form

$$
\begin{equation*}
\operatorname{div}_{X_{k}}(P)=(a+k b) \widetilde{E}_{k}+b F+D_{k}^{\prime} \tag{267}
\end{equation*}
$$

where $a=\operatorname{ord}_{E}(P)>0, b=\operatorname{ord}_{F}(P)>0$ and $c_{X_{k+1}}(v)$ does not belong to the support of $D_{k}^{\prime}$.

Proof. First, since we are in case 2 b and we have supposed that $\operatorname{Suppdiv}_{X}(P)$ is with simple normal crossings, we have that for all $k \geqslant 0$ the center of v in $X_{k}$ is the intersection point $p_{k}:=\widetilde{E}_{k} \cap F$.

We proceed by induction on $k$. If $k=0$ then the result is true as $X_{0}=X$ and $c_{X}(v)=E \cap F$. Suppose the result true for a given index $k \geqslant 0$, then when we blow up $p_{k}, p_{k+1}$ is the intersection point of $\widetilde{E}_{k+1}$ and $F$ so it does not belong to $\pi_{k}^{\prime}\left(\widetilde{E}_{k}\right)$ therefore $L_{v, X_{k+1}}\left(\pi_{k}^{\prime}\left(\widetilde{E}_{k}\right)\right)=0$. By induction we have $\nu_{X_{k}}\left(\widetilde{E}_{k}\right)=L_{v, X_{0}}(E)$, and we know that
$L_{v, X_{k}}\left(\widetilde{E}_{k}\right)=L_{v, X_{k+1}}\left(\pi_{k}^{*} \widetilde{E}_{k}\right)=L_{v, X_{k+1}}\left(\pi_{k}^{\prime}\left(\widetilde{E}_{k}\right)+\widetilde{E}_{k+1}\right)=L_{v, X_{k+1}}\left(\widetilde{E}_{k+1}\right)$
so this shows the first assertion. Now, by induction $\operatorname{div}_{X_{k}}(P)$ is of the form

$$
\begin{equation*}
\operatorname{div}_{X_{k}}(P)=(a+k b) \widetilde{E}_{k}+b F+D_{k}^{\prime} \tag{269}
\end{equation*}
$$

Now, since $p_{k}=\widetilde{E}_{k} \cap F$ and $p_{k} \notin \operatorname{Supp} D_{k}^{\prime}$, one has

$$
\begin{equation*}
\operatorname{div}_{X_{k+1}(P)}=\pi_{k}^{*} \operatorname{div}_{X_{k}}(P)=(a+(k+1) b) \widetilde{E}_{k+1}+b F+(a+k b) \pi_{k}^{\prime}\left(\widetilde{E}_{k}\right)+\pi_{k}^{\prime}\left(D_{k}^{\prime}\right) \tag{270}
\end{equation*}
$$

Since $p_{k+1} \notin \pi_{k}^{\prime}\left(\widetilde{E}_{k}\right)$, the support of the divisor $D_{k+1}^{\prime}:=\pi_{k}^{\prime}\left(D_{k}^{\prime}\right)+(a+$ $k b) \pi_{k}^{\prime}\left(\widetilde{E}_{k}\right)$ does not contain $p_{k+1}$ and we are done.

Using this lemma we see that

$$
\begin{equation*}
L_{v, X_{k}}\left(\operatorname{div}_{\infty, X_{k}}(P)\right)=(a+k b) L_{v, X_{0}}(E) \underset{k \rightarrow \infty}{ }+\infty \tag{271}
\end{equation*}
$$

Therefore $L_{v}(w(P))=+\infty$ and since $v(P) \geqslant L_{v}(w(P))$ we have that $v(P)=$ $+\infty$

### 9.2. Second step, $L \circ v=\mathrm{id}_{\mathcal{M}}$

To show that $L \circ v=\mathrm{id}_{\mathcal{M}}$ we need some technical lemmas.

### 9.2.1. The center of $L$.

Proposition 9.3. Let $L \in \mathcal{M}$ and $X$ be a completion of $X_{0}$. If there exists two divisors $E, E^{\prime}$ at infinity in $X$ such that $L(E), L\left(E^{\prime}\right)>0$, then $E$ and $E^{\prime}$ must intersect.

Proof. Suppose that $E$ and $E^{\prime}$ do not intersect, then the sheaf of ideals $\mathfrak{a}=O_{X}(-E) \oplus$ $O_{X}\left(-E^{\prime}\right)$ is trivial, $\mathfrak{a}=O_{X}$. From Proposition 7.25, we get $E \wedge E^{\prime}=0$. Thus $L(E \wedge$ $\left.E^{\prime}\right)=0$. But $L\left(E \wedge E^{\prime}\right)=\min \left(L(E), L\left(E^{\prime}\right)\right)>0$ and this is a contradiction.

Corollary 9.4. Let $X$ be a completion of $X_{0}$, suppose there exists two prime divisors at infinity $E, F$ such that $L(E), L(F)>0$. Then, let $\widetilde{E}$ be the exceptional divisor above $p=E \cap F$, one has $L(\widetilde{E})>0$.

Proof. Let $\pi: Y \rightarrow X$ be the blow up of $p$ and suppose that $L(\widetilde{E})=0$. Since $\pi^{*} E=\pi^{\prime}(E)+\widetilde{E}$ and $\pi^{*} F=\pi^{\prime}(F)+\widetilde{E}$, one has $L\left(\pi^{\prime}(E)\right)>0$ and $L\left(\pi^{\prime}(F)\right)>0$ but the two divisors no longer meet and this is a contradiction.

Proposition 9.5. Let $X$ be a completion of $X_{0}$, there are two possibilities
(1) There exist a unique closed point $p$ in $X$ at infinity such that if $\widetilde{E}$ is the exceptional divisor above p, one has $L(\widetilde{E})>0$. We call this point the center of $L$ in $X$.
(2) If no point satisfy this property, then there exists a unique divisor at infinity $E$ in $X$ such that $L(E)>0$. In that case we call $E$ the center of $L$ in $X$.
and we have the following properties
(a) Let $E$ be a prime divisor at infinity in $X$. If the center of $L$ on $X$ is a point $p$, then $p \in E \Leftrightarrow L(E)>0$.
(b) If $Y$ is a completion of $X_{0}$ above $X$, then the center of $L$ in $Y$ belongs to the inverse image of the center of $X$.

Proof. Suppose there are two points $p_{1}, p_{2}$ satisfying this property on $X$. Let $\pi_{i}$ be the blow up of $p_{i}$ in $X$, we have commutative diagram

where on the left side we first blow up $p_{1}$ then we blow up the strict transform of $p_{2}$ and the other way around on the right. Now let $\widetilde{E}_{1}, \widetilde{E}_{2}$ be the exceptional divisors above $p_{1}$ and $p_{2}$ respectively in $X_{1}$ and in $X_{2}$ and suppose that $L\left(\widetilde{E}_{1}\right), L\left(\widetilde{E}_{2}\right)>0$. Then, since $p_{1}$ does not belong to $\widetilde{E}_{2}$ and $p_{2}$ does not belong to $\widetilde{E}_{1}$, we have that $L\left(\widetilde{E}_{1}\right)=L\left(\tau_{1}^{*} \widetilde{E}_{1}\right)=$ $L\left(\tau_{1}^{\prime}\left(\widetilde{E}_{1}\right)\right)>0$ and $L\left(\tau_{2}^{\prime}\left(\widetilde{E}_{2}\right)\right)>0$. But in $Y$ the prime divisors $\tau_{1}^{\prime}\left(\widetilde{E}_{1}\right)$ and $\tau_{2}^{\prime}\left(\widetilde{E}_{2}\right)$ do not intersect and that contradicts Proposition 9.3 .

Now, if $E, F$ are two divisors at infinity such that $L(E), L(F)>0$, Lemma 9.4 shows that $E \cap F$ must be the center of $L$ on $X$. Hence if no point of $X$ is the center of $L$ there is only one prime divisor at infinity $E$ such that $L(E)>0$.

To show assertion (a), suppose that the center of $L$ on $X$ is a point $p$ and let $\pi$ be the blow up of $p$. If $p \in E$, then $\pi^{*}(E)=\pi^{\prime}(E)+\widetilde{E}$ and $L(E)=L\left(\pi^{*} E\right) \geqslant L(\widetilde{E})>0$. If $L(E)>0$ then $p$ must belong to $E$ otherwise $\widetilde{E}$ and $E$ would not intersect and this contradicts Proposition 9.3 .

We now assertion (b), we only need to show it for the blow up of a point $\pi: Y \rightarrow X$. Suppose first that the center of $L$ on $X$ is a (closed) point $p$. If we blow up another point than $p$, then it is clear that the center of $L$ on $Y$ is the point $\pi^{-1} p$ as the order of the blow ups does not matter in that case.

Suppose now that we blow up $p$, then the exceptional divisor $\widetilde{E}$ verifies $L(\widetilde{E})>0$, if the center of $L$ on $Y$ is a prime divisor then it must be $\widetilde{E}$. If it is a point then it must belong to $\widetilde{E}$ by assertion (a).

If the center of $L$ on $X$ is a prime divisor $E$, then for any blow up $\pi: Y \rightarrow X$ of a point of $X$, we show that the center of $L$ on $Y$ is $\pi^{\prime}(E)$. The exceptional divisor $\widetilde{E}$ verifies $L(\widetilde{E})=0$ and $\pi^{\prime}(E)$ is the only prime divisor of $Y$ such that $L\left(\pi^{\prime}(E)\right)>0$. Thus, if the center of $L$ on $Y$ is not a point, it must be $\pi^{\prime}(E)$. If the center of $L$ on $Y$ is a point $q$, then it must belong to $\pi^{\prime}(E)$ by assertion (a). If $q$ is not the intersection point $\pi^{\prime}(E) \cap \widetilde{E}$, then it is the strict transform of a point $p \in E$ and in that case $p$ was the center of $L$ in $X$ this is a contradiction. If $q=\widetilde{E} \cap \pi^{\prime}(E)$, then $L(\widetilde{E})>0$ by assertion (a) and this is also a contradiction. Therefore, the center of $L$ on $Y$ cannot be a point, it is $\pi^{\prime}(E)$.
9.2.2. End of the proof. We say that $L$ is divisorial if there exists a completion $X$ of $X_{0}$ such that the center of $L$ on $X$ is a prime divisor at infinity.

Proposition 9.6. The map v sends divisorial valuations to divisorial elements of $\mathcal{M}$ and the map $L$ sends divisorial functions to divisorial valuations.

Proof. The fact that divisorial valuations induce divisorial functions on Cartier divisors is clear. Suppose that $L$ is a divisorial function and let $X$ be a completion such that the center of $L$ in $X$ is a prime divisor $E$ at infinity. Then, for all completion $\pi: Y \rightarrow X$ above $X$, the center of $L$ on $Y$ is the strict transform of $E$ by Proposition 9.5 and $L(E)=L\left(\pi^{\prime}(E)\right)$. Therefore, let $v$ be the divisorial valuation on $\mathbf{k}\left[X_{0}\right]$ such
that $v_{X}=\operatorname{ord}_{E}$ and let $P \in O_{X_{0}}\left(X_{0}\right)$, then for all completion $Y$ above $X$, we have by Proposition 9.5

$$
\begin{equation*}
L\left(\operatorname{div}_{\infty, Y}(P)\right)=L\left(\pi^{\prime}(E)\right) \operatorname{ord}_{E}\left(\operatorname{div}_{Y}(P)\right)=L(E) v(P) \tag{272}
\end{equation*}
$$

Therefore $v_{L}(P)=L(E) v(P)$ and it is a divisorial valuation.
PROPOSITION 9.7. One has $L \circ v=\mathrm{id}_{M_{M}}$.
Proof. We can assume that $L$ and $v_{L}$ are not divisorial. Let $X$ be a completion of $X_{0}$, we will show first that if $H \in \operatorname{Div}_{\infty}(X)$ is an effective divisor such that $|H|$ is base point free and $\operatorname{Supp} H=\partial_{X} X_{0}$, then $\nu_{L}(H)=L(H)$. Pick $f$ generic in $H^{0}\left(X, O_{X}(H)\right)$. We have that $\operatorname{div} f=Z_{f}-H$ with $Z_{f}$ effective, $\operatorname{Supp} Z_{f}$ does not contain any divisor at infinity and the center of $v_{L}$ and the center of $L$ do not belong to $\operatorname{Supp} Z_{f}$. Thus, $f$ defines a regular function over $X_{0}, 1 / f$ is a local equation of $H$ at the center of $v_{L}$ and we have

$$
\begin{equation*}
v_{L}(f)=\sup _{Y} L\left(\operatorname{div}_{\infty, Y}(f)\right) \tag{273}
\end{equation*}
$$

Now, by our assumptions on $f$ we have
Lemma 9.8. For all $Y$ above $X, \operatorname{div}_{Y}(f)$ is of the form $Z_{f, Y}+\operatorname{div}_{\infty, Y}(f)$ where $Z_{f, Y}$ is effective, supported on $X_{0}$ and $\operatorname{Supp} Z_{f, Y}$ does not contain the center of L. Furthermore, we have $L\left(\operatorname{div}_{\infty, Y}(f)\right)=L\left(\operatorname{div}_{\infty, X}(f)\right)$.

Proof. This is true for $Y=X$. We proceed by induction. Let $Y$ be a completion above $Y$ where the lemma is true and let $\pi: Y_{1} \rightarrow Y$ be a blow up of $Y$ at a point $p$. If $p$ is not the center of $L$ then the lemma is clearly true over $Y_{1}$, if $p$ is the center of $L$ over $Y$ then since $p$ does not belong to $\operatorname{Supp} Z_{f, Y}$ we have

$$
\begin{equation*}
\operatorname{div}_{f, Y_{1}}=\pi^{\prime}\left(Z_{f, Y}\right)+\pi^{*}\left(\operatorname{div}_{\infty, Y}(f)\right) \tag{274}
\end{equation*}
$$

and the lemma is true since $Z_{f, Y_{1}}=\pi^{\prime}\left(Z_{f, Y}\right)$ and $\operatorname{div}_{\infty, Y_{1}}(f)=\pi^{*}\left(\operatorname{div}_{\infty, Y}(f)\right)$.

Using this lemma we conclude that $v_{L}(f)=L\left(\operatorname{div}_{\infty, X}(f)\right)=-L(H)$. Therefore,

$$
\begin{equation*}
v_{L}(H)=v_{L}(1 / f)=L(H) \tag{275}
\end{equation*}
$$

Now take any divisor $D \in \operatorname{Div}_{\infty}(X)$. There exists an integer $n \geqslant 1$ such that $D+n H$ is effective and $|D+n H|$ is base-point free. Therefore,

$$
\begin{equation*}
v_{L}(D)=v_{L}(D+n H)-v_{L}(n H)=L(D+n H)-L(n H)=L(D) \tag{276}
\end{equation*}
$$

## Part 2

## Eigenvaluations and dynamics at infinity

## CHAPTER 10

## General case

In this chapter, we show Theorem A when either the condition $\mathbf{k}\left[X_{0}\right]^{\times}=\mathbf{k}^{\times}$or $\operatorname{Pic}^{0}\left(X_{0}\right)=0$ is not satisfied. We rely on the universal property of the quasi-Albanese variety (see [ $\mathbf{S e r 0 1 ]}$ ), as well as on the geometric properties of subvarieties of quasiabelian varieties (see [Abr94]).

### 10.1. Quasi-Albanese variety and morphism

Let $G$ be an algebraic group over $\mathbf{k}$ with $\mathbf{k}$ algebraically closed. We say that $G$ is a quasi-abelian variety if there exists an algebraic torus $T=\mathbb{G}_{m}^{r}$, an abelian variety $A$, and an exact sequence of $\mathbf{k}$-algebraic groups

$$
\begin{equation*}
0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0 \tag{277}
\end{equation*}
$$

ThEOREM 10.1 (see [Ser01], Théorème 7). Let $X$ be a variety over $\mathbf{k}$, then there exists a quasi-abelian variety $G$ and a morphism $q: X \rightarrow G$ such that for any quasiabelian variety $G^{\prime}$ and any morphism $\varphi: X \rightarrow G^{\prime}$ there exists a unique morphism $g$ : $G \rightarrow G^{\prime}$ and a unique $b \in G^{\prime}$ such that

$$
\varphi=g \circ q .
$$

Moreover, $g$ is the composition of a homomorphism $L_{g}: G \rightarrow G^{\prime}$ of algebraic groups and a translation $T_{g}: G^{\prime} \rightarrow G^{\prime}$ by some element $b \in G^{\prime}$.

Such a $G$ is unique up to (a unique) isomorphism. It is called the quasi-Albanese variety of $X$ and it will be denoted by $\mathrm{QAlb}(X)$; the universal morphism $q: X \rightarrow \mathrm{QAlb}(X)$ is "the" quasi-Albanese morphism (it is unique up to post-composition with an isomorphism of $G$ ). Of course if $X$ is projective, then $\operatorname{QAlb}(X)$ is the classical Albanese variety of $X$.

Proposition 10.2. Let $X_{0}$ be an affine variety. Then $\mathbf{k}\left[X_{0}\right]^{\times}=\mathbf{k}^{\times}$and $\operatorname{Pic}^{0}\left(X_{0}\right)=$ 0 if and only if $\mathrm{QAlb}\left(X_{0}\right)=0$.

Proof. Let $G=\operatorname{QAlb}\left(X_{0}\right)$ and $q: X_{0} \rightarrow G$ be a quasi-Albanese morphism. Let

$$
\begin{equation*}
0 \rightarrow T \rightarrow G \xrightarrow{\pi} A \rightarrow 0 \tag{278}
\end{equation*}
$$

be an exact sequence, as in Equation (277). Let $X$ be a completion of $X_{0}$ such that $\pi \circ q$ extends to a regular map $\pi \circ q: X \rightarrow A$.

Assume $\mathbf{k}\left[X_{0}\right]^{\times}=\mathbf{k}^{\times}$and $\operatorname{Pic}^{0}\left(X_{0}\right)=0$. Then, $\pi \circ q\left(X_{0}\right)$ is a point in $A$, and compos$\operatorname{ing} q$ with a translation of $G$, we can assume that this point is the neutral element of $A$. Then, $q\left(X_{0}\right) \subset T$, so $q$ is a regular map from $X_{0}$ to an algebraic torus, and $\mathbf{k}\left[X_{0}\right]^{\times}=\mathbf{k}^{\times}$ implies that $q\left(X_{0}\right)$ is a point. This shows that $\mathrm{QAlb}\left(X_{0}\right)$ is a point.

Now, suppose that $\mathbf{k}\left[X_{0}\right]^{\times} \neq \mathbf{k}^{\times}$, then any non-constant invertible function $X_{0} \rightarrow \mathbf{k}^{\times}$ provides a dominant morphism to a 1-dimensional torus, so $\operatorname{dim}\left(\mathrm{QAlb}\left(X_{0}\right)\right) \geqslant 1$ by the universal property. And if $\operatorname{Pic}^{0}\left(X_{0}\right) \neq 0$, the Albanese morphism also shows that $\operatorname{dim}\left(\operatorname{QAlb}\left(X_{0}\right)\right) \geqslant 1$. This concludes the proof.

In the following, we show that if $X_{0}$ is an irreducible normal affine surface with nontrivial quasi-Albanese variety and $f$ is a dominant endomorphism of $X_{0}$, then $\lambda_{1}(f)$ is a quadratic integer. See Proposition 10.9 below. We will rely on the following result.

THEOREM 10.3 (Theorem 3 of [Abr94]). Let Q be a quasi-abelian variety and let $V$ be a closed subvariety of $Q$. Let $K$ be the maximal closed subgroup of $Q$ such that $V+K=V$. Then, the variety $V / K$ is of general type.

### 10.2. Logarithmic Kodaira dimension

Let $V$ be an algebraic variety, let $\bar{V}$ be a good completion of $V$ and $\bar{D}=\bar{V} \backslash V$, it is a simple normal crossing divisor. For $q=1, \cdots, \operatorname{dim} V$, let $\Omega^{q}(\log D)$ be the sheaf of logarithmic $q$-forms along $D$, i.e the subsheaf of rational $q$-forms $\alpha$ on $\bar{V}$ such that locally at every point of $D, \alpha$ is of the form

$$
\begin{equation*}
\alpha=\sum_{\substack{r+s=q \\ I=(i(1), \cdots, i(r)) \\ J=(j(1), \cdots, j(s))}} \alpha_{I J}(z, w) \frac{d z_{i(1)}}{z_{i(1)}} \wedge \cdots \wedge \frac{d z_{i(r)}}{z_{i(r)}} \wedge w_{j(1)} \wedge \cdots \wedge w_{j(s)} \tag{279}
\end{equation*}
$$

where $(z, w)=\left(z_{1}, \cdots, z_{m}, w_{1}, \cdots, w_{n-m}\right)$ is a local system of coordinates such that $z_{1} \cdots z_{m}=0$ is a local equation of $D$ and $\alpha_{I J}(z, w)$ is a local germ of regular function.

In particular, $H^{0}\left(\bar{V}, \Omega^{\operatorname{dim} V}(\log D)\right)=H^{0}(\bar{V}, \bar{K}+\bar{D})$ where $\bar{K}$ is a canonical divisor over $\bar{V}$. Following [Iit77] we have that $\operatorname{dim} H^{0}(\bar{V}, \bar{K}+\bar{D})$ does not depend on the completion $\bar{V}$. Define the following invariant

$$
\begin{align*}
\bar{q}(V) & =\operatorname{dim} H^{0}\left(\bar{V}, \Omega^{1}(\log \bar{D})\right)  \tag{280}\\
\bar{\kappa}(V) & =\kappa(\bar{V}, \bar{K}+\bar{D}) \tag{281}
\end{align*}
$$

Where for a line bundle $L$ over $\bar{V}$,

$$
\begin{equation*}
\kappa(\bar{V}, L)=\limsup _{k \rightarrow+\infty} \frac{n!}{k^{n}} \operatorname{dim} H^{0}(\bar{V}, L) . \tag{282}
\end{equation*}
$$

The invariant $\bar{q}(V)$ is actually the dimension of the quasi-Albanese variety of $V$ (see [Fuj15]) and $\bar{\kappa}(V)$ is the logarithmic Kodaira dimension of $V$ (see [Iit77]). We have
the following characterization of the algebraic torus of dimension 2．If $V$ is projective， then the log kodaira dimension is nothing but the classical Kodaira dimension of $V$ ．

THEOREM 10.4 （Theorem 2 of［【it79］）．Let $V$ be a normal affine surface，then $V \simeq \mathbb{G}_{m}^{2}$ if and only if $\bar{\kappa}(V)=0$ and $\bar{q}(V)=2$ ．

Lemma 10.5 （［【it77］Proposition 1 and 2）．If $V$ is an affine variety and $f: V \rightarrow V$ is a dominant endomorphism，then $f$ induces an isomorphism

$$
\begin{equation*}
f^{*}: H^{0}(\bar{V}, m(\bar{K}+\bar{D})) \rightarrow H^{0}(\bar{V}, m(\bar{K}+\bar{D})) \tag{283}
\end{equation*}
$$

for all $m \geqslant 1$ ．
This lemma allows one to define the log Kodaira Iitaka fibration

$$
\begin{equation*}
\Phi_{m}: V \rightarrow \mathbf{P}\left(H^{0}, \bar{V}, m(\bar{K}+\bar{D})\right) . \tag{284}
\end{equation*}
$$

By the lemma，every dominant endomorphism of $V$ must preserve the log Kodaira Iitaka fibration for $m \gg 1$ ．

We say that $V$ is of $\log$ general type if $\bar{\kappa} V=\operatorname{dim} V$ ．
Corollary 10.6 （［【IT77］Proposition 2 and Corollary p．5）．If V is an affine variety of $\log$ general type，then End $V=\operatorname{Aut} V$ and this is a finite group．

10．3．Dynamical degree in presence of an invariant fibration
Proposition 10.7 （Stein Factorization）．Let $X$ ，$S$ be projective varieties and let $f: X \rightarrow X$ be a rational transformation．Suppose that there exists $\varphi: X \rightarrow S$ and $g: S \rightarrow S$ such that the following diagram commutes，


Then there exists a variety $\widetilde{S}$ and morphisms $\psi: X \rightarrow \widetilde{S}, \pi: \widetilde{S} \rightarrow S$ such that
－$\varphi=\pi \circ \psi$ ，
－$\pi$ is finite and $\psi$ has connected fibers
－there exists a rational transformation $\widetilde{g}: \widetilde{S} \rightarrow \widetilde{S}$ such that the diagram


## commutes.

Proof. The existence of $\widetilde{S}$ along with $\pi$ and $\psi$ is due to Stein Factorization theorem: It is known that one can take $\widetilde{S}=\operatorname{Spec}_{S} \varphi_{*} O_{X}$ where $\operatorname{Spec}_{S}$ is the relative Spec; that is for every affine open subset $U$ of $S$, one has

$$
\begin{equation*}
\pi^{-1}(U) \simeq \operatorname{Spec} O_{X}\left(\varphi^{-1}(U)\right) \tag{285}
\end{equation*}
$$

Now to construct $\widetilde{g}$, take affine open subsets $U$ and $V$ of $S$ such that $U \subset g^{-1}(V)$. Suppose also that $\varphi^{-1}(U)$ and $\varphi^{-1}(V)$ do not contain any indeterminacy of $f$. To construct

$$
\begin{equation*}
\tilde{g}_{\mid \pi^{-1}(U)}: \pi^{-1}(U) \rightarrow \pi^{-1}(V) \tag{286}
\end{equation*}
$$

we use the map $f^{*}: O_{X}\left(\varphi^{-1}(V)\right) \rightarrow O_{X}\left(\varphi^{-1}(U)\right)$ induced by $f$; this is well defined since $\varphi^{-1}(U) \subset f^{-1}\left(\varphi^{-1}(V)\right)$. It is clear that $\psi \circ f=\widetilde{g} \circ \psi$.

Proposition 10.8. Let $S$ be a quasiprojective surface and $f$ be a dominant endomorphism of $S$. Suppose there exists a quasiprojective curve $C$ with a dominant morphism $\pi: S \rightarrow C$ and an endomorphism $g: C \rightarrow C$ such that $\pi \circ f=g \circ \pi$. Then, the first dynamical degree of $f$ is an integer.

Proof. Let $X$ be a completion of $S ; f$ extends to a rational transformation of $X$. We can also suppose that $C$ is a projective curve, and then we apply Theorem 10.7 to suppose also that $\pi$ has connected fibers.

Let $P$ be a general point of $C$ and $H$ an ample divisor of $X$. We have by [DN11, Tru15] that

$$
\begin{equation*}
\left.\lambda_{1}(f)=\max \left(\lambda_{1}(g), \lambda_{1}\left(f_{\mid \pi}\right)\right)\right) \tag{287}
\end{equation*}
$$

where $\lambda_{1}(g)$ is the integer given by the topological degree of $g$ and

$$
\begin{equation*}
\lambda_{1}\left(f_{\mid \pi}\right):=\lim _{n}\left(H \cdot\left(f^{n}\right)_{*} \pi^{-1}(P)\right)^{1 / n} \tag{288}
\end{equation*}
$$

Since $C$ is a curve and $\pi$ is dominant we have that $\pi$ is flat ([Har77] Proposition III.9.7) so for any point $P \in C$,

- $\pi^{-1}(P)$ is an irreducible curve $C_{P}$ and the topological degree of $f: C_{P} \rightarrow C_{g(P)}$ is an integer $d$ that does not depend on $P$
- $d \cdot d_{\mathrm{top}}(g)=\lambda_{2}(f)$.

Indeed, consider the following 0 -cycle in $S \times S$ :

$$
\begin{equation*}
\alpha(P)=\left(\pi_{1}^{*} C_{P}\right) \cdot\left(\pi_{2}^{*} H\right) \cdot \Gamma_{f} \tag{289}
\end{equation*}
$$

where $\pi_{1}, \pi_{2}: S \times S \rightarrow S$ are the two projections and $\Gamma_{f}$ is the graph of $f$. The degree of $\alpha(P)$ is

$$
\begin{equation*}
\operatorname{deg} \alpha(P)=\left(H \cdot C_{g(P)}\right) \cdot \operatorname{deg}\left(f: C_{P} \rightarrow C_{g(p)}\right) . \tag{290}
\end{equation*}
$$

Now, since $C$ is a curve the morphism $\pi \circ \pi_{1}: S \times S \rightarrow C$ is flat, therefore $\operatorname{deg}(\alpha(P))$ does not depend on $P$ ([Ful98] §20.3) and since $\pi$ is flat, the intersection number $\left(H \cdot C_{P}\right)$ does not depend on $P$ either. Therefore, $\operatorname{deg}\left(f: C_{P} \rightarrow C_{g(P)}\right)$ is an integer $d$ independent of $P$. Hence, we infer

$$
\begin{equation*}
\lambda_{1}\left(f_{\mid \pi}\right)=\lim _{n}\left(H \cdot\left(f^{n}\right)_{*} \pi^{-1} P\right)=d \cdot \lim _{n}\left(H \cdot \pi^{-1} P\right)^{1 / n}=d \tag{291}
\end{equation*}
$$

and we get that $\lambda_{1}(f)$ is the integer $\max \left(d, \lambda_{1}(g)\right)$.

### 10.4. Dynamical degree when the quasi-Albanese variety is non-trivial

The goal of this section is to show the following proposition.
Proposition 10.9. Let $X_{0}$ be an irreducible normal affine surface and $f$ a dominant endomorphism of $X_{0}$. Suppose that $\operatorname{QAlb}\left(X_{0}\right)$ is non-trivial, then $\lambda_{1}(f)$ is an algebraic integer of degree $\leqslant 2$. Furthermore, if $\lambda_{1}(f)$ is not an integer, then $X_{0} \simeq \mathbb{G}_{m}^{2}$.

Set $Q_{0}=\operatorname{QAlb}\left(X_{0}\right)$ and let $q: X_{0} \rightarrow Q_{0}$ be a quasi-Albanese morphism. Let $V=$ $\overline{q\left(X_{0}\right)}$ be the closure of the image of $X_{0}$ in $Q_{0}$. By the universal property, there exists an endomorphism $g$ of $Q_{0}$ such that

$$
\begin{align*}
q \circ f & =g \circ q  \tag{292}\\
g(z) & =L_{g}(z)+b_{g} \tag{293}
\end{align*}
$$

for some algebraic homomorphism $L_{g}: Q_{0} \rightarrow Q_{0}$ and some translation $z \mapsto z+b_{g}$ (here, we denote the group law by addition). In particular $g_{\mid V}$ defines a regular endomorphism of $q\left(X_{0}\right)$ and since $f$ is dominant, so is $g_{\mid V}$. As in Theorem 10.3, set $K=\left\{x \in Q_{0} ; x+\right.$ $V=V\}$. Then, denote by $\pi_{V}: V \rightarrow V / K$ the canonical projection onto the quotient.

Proposition 10.10. There exists an endomorphism $g^{\prime}: V / K \rightarrow V / K$ such that $g^{\prime} \circ \pi_{V}=\pi_{V} \circ g_{\mid V}$.

Proof. We have to show that $g_{\mid V}$ is compatible with the quotient map. Take $v \in V$ and $k \in K$. Since $v+k \in V, g(v+k) \in V$. Now,

$$
\begin{equation*}
g(v+k)=L_{g}(v+k)+b_{g}=L_{g}(v)+L_{g}(k)+b_{g}=g(v)+L_{g}(k) . \tag{294}
\end{equation*}
$$

Thus, $L_{g}(k)+g(V) \subset V$. Taking the closure and knowing that $g_{\mid V}$ is dominant, we have $L_{g}(k)+V=V$. Therefore, $L_{g}(k) \in K$ and $g_{\mid V}$ is compatible with the quotient modulo $K$.

Case $\operatorname{dim} V / K=2$.- In that case, the map $\pi_{V} \circ q: X \rightarrow V / K$ is generically finite. Since $V / K$ is of general type, $g^{\prime}$ has finite order: there is some positive integer $n$ such that $\left(g^{\prime}\right)^{n}=I d_{V / K}$. Thus, $f$ is also a finite order automorphism, and $\lambda_{1}(f)=1$.

Case $\operatorname{dim} V / K=1$. - In that case $\pi_{V} \circ q$ induces a fibration of $X_{0}$ over a curve of general type and we conclude that $\lambda_{1}(f)$ is an integer by Proposition 10.8.

Case $\operatorname{dim} V / K=0$.- This means that $V$ is equal to $K$ up to translation. Therefore, by the universal property of the quasi-Albanese variety, $K=V=Q_{0}$ and $q: X_{0} \rightarrow Q_{0}$ is dominant.

If $\operatorname{dim} Q_{0}=1$, then $f$ preserves a fibration over a curve and Proposition 10.8 implies again that $\lambda_{1}(f)$ is an integer.

Suppose now that $\operatorname{dim} Q_{0}=2$. Then $q$ is generically finite, so that $\lambda_{1}(f)=\lambda_{1}(g)$. Since $\bar{\kappa}\left(Q_{0}\right)=0$, we have $\bar{\kappa}\left(X_{0}\right) \in\{0,1,2\}$.

If $\bar{\kappa}\left(X_{0}\right)=2$, then $X_{0}$ is of log general type. In that case, by Corollary 10.6, every endomorphism $f$ of $X_{0}$ is an automorphism and satisfy $\lambda_{1}(f)=1$ because it is of finite order.

If $\bar{\kappa}\left(X_{0}\right)=1$, then every endomorphism of $X_{0}$ preserves the $\log$ Kodaira Iitaka fibration and by Proposition 10.8, $\lambda_{1}(f)$ is an integer.

Finally, if $\bar{\kappa}\left(X_{0}\right)=0$, then by Theorem 10.4, $X_{0} \simeq \mathbb{G}_{m}^{2}$ and $\lambda_{1}(f)$ is an algebraic integer of degree $\leqslant 2$ because it is the spectral radius of a $2 \times 2$ matrix with integer entries. We see that this is the only case where we might have $\lambda_{1}(f) \notin \mathbf{Z}_{\geqslant 0}$.

Corollary 10.11. If $X_{0}$ is a normal affine surface with a loxodromic automorphism and $\mathrm{QAlb}\left(X_{0}\right)$ is not trivial, then $X_{0} \simeq \mathbb{G}_{m}^{2}$.

Proof. A loxodromic automorphism of $X_{0}$ satisfies $\lambda_{1}>\lambda_{2}=1$ and thus cannot preserve a fibration over a curve or be of finite order. Looking at the proof of Proposition 10.9, we see that this only happens when $X_{0} \simeq \mathbb{G}_{m}^{2}$.

## CHAPTER 11

$$
\text { Dynamics when } \mathbf{k}\left[X_{0}\right]^{\times}=\mathbf{k}^{\times} \text {and } \operatorname{Pic}^{0}\left(X_{0}\right)=0
$$

In this chapter, we will prove Theorem $C$ and derive Theorems $A$ and $B$. The two hypothesis allows one to describe the Picard-Manin space of $X_{0}$ more precisely. In particular, we show that $\mathcal{V}_{\infty}$ embeds into $\operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$ and $\mathcal{V}_{\infty}^{\prime}$ embeds into $\mathrm{L}^{2}\left(\mathrm{X}_{0}\right)$.

### 11.1. The structure of the Picard-Manin space of $X_{0}$

From $\$ 3.6$ we have linear maps

$$
\begin{equation*}
\tau: \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \rightarrow \text { Cartier-NS }\left(\mathrm{X}_{0}\right)_{\mathbf{R}}, \quad \tau: \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \rightarrow \text { Weil-NS }\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \tag{295}
\end{equation*}
$$

For this section we suppose that $X_{0}$ is a normal affine surface over an algebraically closed field $\mathbf{k}$ such that
(1) $\mathbf{k}\left[X_{0}\right]^{\times}=\mathbf{k}^{\times}$;
(2) For all completion $X$ of $X_{0}, \operatorname{Pic}^{0}(X)=0$.

It suffices to test the second condition on one completion of $X_{0}$ as the Albanese variety of a projective variety is a birational invariant. We will make an abuse of notations and write $\operatorname{Pic}^{0}\left(X_{0}\right)=0$ for the second hypothesis.

If these two conditions are satisfied, the finite dimensional subspace $\operatorname{Div}_{\infty}(X) \mathrm{em}$ beds into $\mathrm{NS}(X)$. Indeed, consider the composition

$$
\begin{equation*}
\operatorname{Div}_{\infty}(X) \rightarrow \operatorname{Pic}(X) \rightarrow \mathrm{NS}(X) \tag{296}
\end{equation*}
$$

the first map is injective since $\mathbf{k}\left[X_{0}\right]^{\times}=\mathbf{k}^{\times}$and the second is an isomorphism because $\operatorname{Pic}^{0}(X)=0$. Therefore the maps $\tau$ are injective and we have the orthogonal decomposition

$$
\begin{equation*}
\text { Weil-NS }\left(\mathrm{X}_{0}\right)_{\mathbf{R}}=\text { Weil }_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \oplus V \tag{297}
\end{equation*}
$$

where $V$ is a finite-dimensional vector space(this decomposition also holds over $\mathbf{Q}$ ); in fact let $X$ be a completion of $X_{0}$, then $V$ is the orthogonal of $\operatorname{Div}_{\infty}(X)$ in $\operatorname{NS}(X)$.

### 11.1.1. The intersection form at infinity.

Proposition 11.1. Let $X$ be a completion of $X_{0}$, then

- $\operatorname{Div}_{\infty}(X)_{\mathbf{A}}$ embeds into $\mathrm{NS}(X)_{\mathbf{A}}$ and the intersection form is non degenerate on $\operatorname{Div}_{\infty}(X)_{\mathbf{A}}$.
- The perfect pairing Cartier-NS $\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \times \mathrm{Weil-} \mathrm{NS}\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \rightarrow \mathbf{R}$ induces a pairing

$$
\begin{equation*}
\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \times \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \rightarrow \mathbf{R} \tag{298}
\end{equation*}
$$

that is also perfect.

- Weil $\infty_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$ is isomorphic, as a topological vector space, to $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}}^{*}$ endowed with the weak-* topology.

Proof. Everything follows from Propositions 3.20 and 3.17 and that $\tau: \operatorname{Div}_{\infty}(X) \hookrightarrow$ $\operatorname{NS}(X)$ is injective.

COROLLARY 11.2. The subspace $\operatorname{hom}\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \mathbf{R}\right)_{(+)}$is a closed subspace of $\mathrm{Weil}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$ with the weak-ぇ topology.

Proof. All the conditions that elements of hom $\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \mathbf{R}\right)_{(+)}$have to satisfy are closed conditions. Indeed, we have

$$
\begin{equation*}
\operatorname{hom}\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \mathbf{R}\right)_{(+)}=C_{1} \cap C_{2} \tag{299}
\end{equation*}
$$

where

$$
\begin{align*}
& C_{1}=\bigcap_{D \geqslant 0}\{L(D) \geqslant 0\}  \tag{300}\\
& C_{2}=\bigcap_{D, D^{\prime} \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)}\left\{L\left(D \wedge D^{\prime}\right)=\min \left(L(D), L\left(D^{\prime}\right)\right)\right\} \tag{301}
\end{align*}
$$

11.1.2. A continuous embedding of $\mathcal{V}_{\infty}$ into $\mathrm{Weil}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$. From Proposition 11.1 , we get the immediate corollary.

Corollary 11.3. For any valuation $v$ centered at infinity, there exists a unique $Z_{v} \in$ Weil $_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$ such that for all $D \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}}, L_{v}(D)=Z_{v} \cdot D$.

Corollary 11.4. A valuation $v$ is divisorial if and only if $Z_{v}$ belongs to $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$. In particular, for any prime divisor $E$ at infinity, $Z_{\text {ord }_{E}} \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{Q}}$. The embedding

$$
\begin{equation*}
v \in \mathcal{V}_{\infty} \mapsto Z_{v} \in \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \tag{302}
\end{equation*}
$$

is a continuous map for the weak topology.
Proof. If $v$ is divisorial, then there exists a completion $X$ such that the center of $v$ is a prime divisor $E$ at infinity. For every $W \in \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right), L_{\text {ord }_{E}}(W)=L_{\text {ord }_{E}, X}\left(W_{X}\right)$, by Proposition 7.22. By non-degeneracy of the intersection pairing on $\operatorname{Div}_{\infty}(X)_{\mathbf{Q}}$, there exists $Z \in \operatorname{Div}_{\infty}(X)_{\mathbf{Q}}$ such that for all $D \in \operatorname{Div}_{\infty}(X)_{\mathbf{Q}}, L_{\text {ord }_{E}, X}(D)=Z \cdot D$. It follows that $Z_{\text {ord }_{E}}$ is the Cartier class defined by $Z$, hence it is an element of $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{Q}}$.

Conversely, if $Z_{v} \in \operatorname{Cartier}_{\infty}\left(X_{0}\right)_{\mathbf{R}}$, let $X$ be a completion where $Z_{v}$ is defined. The center of $v$ over $X$ cannot be a closed point $p$; otherwise let $\widetilde{E}$ be the exceptional divisor above $p$, we would have $L_{v}(\widetilde{E})>0$, but $Z_{v} \cdot \widetilde{E}=0$.

Now to show the continuity of the map of the Corollary, it suffices by Proposition 11.1 to show that for any $D \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}}$, the map $v \in \mathcal{V}_{\infty} \mapsto Z_{v} \cdot D$ is continuous. It actually suffices to show this for $D \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$ and this follows immediately from $Z_{v} \cdot D=L_{v}(D)$ and Proposition 7.19.

Proposition 11.5. Let $v$ be a valuation centered at infinity and $X$ a completion of $X_{0}$ such that $c_{X}(v) \in E$ is a free point. Then, the incarnation of $Z_{v}$ in $X$ is

$$
\begin{equation*}
Z_{v, X}=\left(Z_{V} \cdot E\right) Z_{\text {ord }_{E}} \tag{303}
\end{equation*}
$$

If $c_{X}(v)=E \cap F$ is a satellite point, then

$$
\begin{equation*}
Z_{v, X}=\left(Z_{v} \cdot E\right) Z_{\operatorname{ord}_{E}}+\left(Z_{v} \cdot F\right) Z_{\operatorname{ord}_{F}} \tag{304}
\end{equation*}
$$

Furthermore, if $\pi: Y \rightarrow X$ is the blow up of a point at infinity $p \neq c_{X}(v)$, then

$$
\begin{equation*}
Z_{v, Y}=\pi^{*} Z_{v, X} \tag{305}
\end{equation*}
$$

Proof. If $c_{X}(v) \in E$ is a free point. For any $D \in \operatorname{Div}_{\infty}(X)_{\mathbf{R}}$, one has $D=\sum_{F} L_{\text {ord }_{F}}(D) F$, therefore by Proposition 7.6 (2) and (3) $L_{v}(D)=L_{\text {ord }_{E}}(D) L_{v}(E)$. Since $\left(Z_{v} \cdot E\right)=$ $L_{v}(E)$, we get the result. The proof is similar for the case $c_{X}(v)=E \cap F$.

For the last assertion, if $\widetilde{E}$ is the exceptional divisor of $\pi: Y \rightarrow X$, then by definition

$$
\begin{equation*}
Z_{v, Y}=\pi^{*} Z_{v, X}-\left(Z_{v} \cdot \widetilde{E}\right) \widetilde{E} \tag{306}
\end{equation*}
$$

However, since $c_{X}(v) \neq p$, we have that $c_{Y}(v) \notin \widetilde{E}$ and therefore $Z_{v} \cdot \widetilde{E}=0$ by Proposition 7.6 .

Recall that in $\$ 7.4$, we have defined for a point $p$ at infinity in a completion $X$ the local divisor $Z_{v, X, p}$ for every valuation $v$ centered at $p$. The divisor is defined by duality via the following property

$$
\begin{equation*}
\forall D \in \operatorname{Cartier}(X, p)_{\mathbf{R}}, \quad L_{V}(D)=Z_{v, p, X} \cdot D \tag{307}
\end{equation*}
$$

COROLLARY 11.6. Let $X$ be a completion of $X_{0}$ and let $v$ be a valuation centered at infinity.

- If $p:=c_{X}(v) \in E$, then

$$
\begin{equation*}
Z_{v}=\left(Z_{v} \cdot E\right) Z_{\operatorname{ord}_{E}}+Z_{v, X, p} \tag{308}
\end{equation*}
$$

- If $p:=c_{X}(v)=E \cap F$ is a satellite point, then

$$
\begin{equation*}
Z_{v}=\left(Z_{v} \cdot E\right) Z_{\operatorname{ord}_{E}}+\left(Z_{v} \cdot Z_{\operatorname{ord}_{F}}\right) Z_{\operatorname{ord}_{F}}+Z_{v, X, p} \tag{309}
\end{equation*}
$$

In particular, $Z_{v} \in \mathrm{~L}^{2}\left(\mathrm{X}_{0}\right)$ if and only if v is quasimonomial or there exists a completion $X$ and a closed point $p \in X$ at infinity such that $c_{X}(v)=p$ and $\alpha(\widetilde{v})<+\infty$ where $\widetilde{\mathrm{v}}$ is the valuation equivalent to $v$ such that $\widetilde{\mathrm{v}} \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$.

Proof. We have that

$$
\begin{equation*}
Z_{v}=Z_{v, X}+Z^{\prime} \tag{310}
\end{equation*}
$$

where $Z^{\prime} \in$ Weil $_{\infty}\left(\mathrm{X}_{0}\right)$ is exceptional above $X$. Now, for every divisor $D$ exceptional above $X$, we have

$$
\begin{equation*}
L_{v}(D)=Z_{v} \cdot D=Z^{\prime} \cdot D \tag{311}
\end{equation*}
$$

If $D$ is exceptional above a point $q \neq p$, then $L_{v}(D)=0$ by Proposition 7.6 as $q \neq c_{X}(v)$. Therefore, we get that $Z^{\prime}=Z_{v, X, p}$.

Now, we have $Z_{v} \in \mathrm{~L}^{2}\left(\mathrm{X}_{0}\right) \Leftrightarrow\left(Z_{v}\right)^{2}<-\infty$. Replace $v$ by the equivalent valuation such that $v \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$, then by Theorem $7.31\left(Z_{v, X, p}\right)^{2}=-\alpha(v)$ and therefore

$$
\begin{equation*}
\left(Z_{v}\right)^{2}=\left(Z_{v, X}\right)^{2}-\alpha(v) \tag{312}
\end{equation*}
$$

This shows the result.
Corollary 11.7. Let $v \in \mathcal{V}_{\infty}$, then up to normalisation $Z_{v} \in \operatorname{Weil}_{\infty}\left(X_{0}\right)_{\mathbf{Q}}$ if and only v is not irrational.

Proof. First, if $v$ is divisorial, the result follows from Corollary 11.4. Then, if $v$ is infinitely singular or a curve valuation. Then, there exists a completion $X$ such that $c_{X}(v)$ is a free point $p \in E$. Then, replace $v$ by its equivalent valuation such that $\boldsymbol{v} \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$. Let $(z, w)$ be local coordinates at $p$ such that $z=0$ is a local equation of $E$. Then, $Z_{v}(E)=v(z)=\alpha\left(v \wedge v_{z}\right) \in \mathbf{Q}$ because $v \wedge v_{z}$ has to be a divisorial valuation. Therefore, by Corollary 7.33 and Proposition 11.5 , we get that $Z_{v} \in$ Weil $_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{Q}}$.

Finally, if $v$ is irrational then let $X$ be a completion such that $c_{X}(v)=E \cap F$ is a satellite point. Then, $Z_{v, X}=s Z_{\text {ord }_{E}}+t Z_{\text {ord }_{F}}$ with $s / t \notin \mathbf{Q}$ by Proposition 11.5. It is clear that no multiple of $Z_{v, X}$ can be in $\operatorname{Div}_{\infty}(X)_{\mathbf{Q}}$.

Corollary 11.8. Let $\mathcal{V}_{\infty}^{\prime}$ be the subspace of $\mathcal{V}_{\infty}$ consisting of $\mathcal{v} \in \mathcal{V}_{\infty}$ such that $Z_{v} \in \mathrm{~L}^{2}\left(\mathrm{X}_{0}\right)$, then

$$
\begin{equation*}
\mathcal{V}_{\infty}^{\prime} \hookrightarrow \mathrm{L}^{2}\left(\mathrm{X}_{0}\right) \tag{313}
\end{equation*}
$$

is a continuous embedding for the strong topology. Furthermore, it is a homeomorphism onto its image.

Proof. Let $X$ be a completion of $X_{0}$. Let $v_{n}$ be a sequence of $V_{\infty}^{\prime}$ converging towards $v \in \mathcal{V}_{\infty}^{\prime}$ for the strong topology. We treat two cases, whether $v$ is associated to a prime divisor of $X$ or $v$ is centered at a closed point $p \in X$ at infinity.

If $v$ is centered at a closed point $p$ at infinity, then since $v_{n}$ converges strongly towards $v$ then it converges also weakly, therefore for $n$ big enough, $v_{n}$ is centered at
$p$ by Proposition 6.1. We can replace each $v_{n}$ and $v$ by their representative such that $v_{n}, v \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$. Then

- If $p \in E$ is a free point,

$$
\begin{equation*}
Z_{v_{n}}=\left(Z_{v_{n}} \cdot E\right) Z_{\operatorname{ord}_{E}}+Z_{V_{n}, X, p} \tag{314}
\end{equation*}
$$

- If $p=E \cap F$ is a satellite point, then

$$
\begin{equation*}
Z_{v_{n}}=\left(Z_{v_{n}} \cdot E\right) Z_{\operatorname{ord}_{E}}+\left(Z_{v_{n}} \cdot F\right) Z_{\operatorname{ord}_{F}}+Z_{v_{n}, X, p} \tag{315}
\end{equation*}
$$

and we have similar formulas for $Z_{v}$. Now the incarnation of $Z_{v_{n}}$ in $X$ converges towards the incarnation of $Z_{v}$ in $X$ in both the free and the satellite case by weak convergence. Let $\|\cdot\|$ be any norm over $\operatorname{NS}(X)_{\mathbf{R}}$, then

$$
\begin{equation*}
\left\|Z_{v}-Z_{v_{n}}\right\|_{\mathrm{L}^{2}\left(\mathrm{X}_{0}\right)}^{2}=\left\|Z_{v, X}-Z_{v_{n}, X}\right\|^{2}-\left(Z_{v, X, p}-Z_{v_{n}, X, p}\right)^{2} \tag{316}
\end{equation*}
$$

where $f=g$ means that there exists constants $A, B>0$ such that $A g \leqslant f \leqslant B g$. By Proposition 7.34, we have that $\left\|Z_{v}-Z_{v_{n}}\right\|_{\mathrm{L}^{2}\left(\mathrm{X}_{0}\right)}^{2} \rightarrow 0$.

If $v \simeq \operatorname{ord}_{E}$ for some prime divisor $E$ at infinity in $X$, then for all $n$ large enough, $c_{X}\left(v_{n}\right) \in E$. We can suppose that $\nu=\operatorname{ord}_{E}$ and for all $\mathrm{n} \nu_{n}(E)>0$, i.e $\vee, v_{n} \in \mathcal{V}_{X}(E)$ and $Z_{v_{n}} \cdot E \rightarrow 1$ as $n \rightarrow \infty$. We show that

$$
\begin{equation*}
\frac{Z_{v_{n}}}{Z_{v_{n}} \cdot E} \xrightarrow[n \rightarrow+\infty]{ } Z_{\text {ord }_{E}} \tag{317}
\end{equation*}
$$

in $\mathrm{L}^{2}\left(\mathrm{X}_{0}\right)$. We can replace $v_{n}$ by its equivalent valuation such that $v_{n} \in \mathcal{V}_{X}\left(p_{n}, \mathfrak{m}_{p_{n}}\right)$ where $p_{n}=c_{X}\left(v_{n}\right)$. Then, we have that $Z_{v_{n}, X} / Z_{v_{n}} \cdot E$ converges towards $Z_{\operatorname{ord}_{E}}$ in $\mathrm{NS}(X)_{\mathbf{R}}$ by weak convergence. It suffices to show

$$
\begin{equation*}
\frac{\left(Z_{v_{n}, X, p}\right)^{2}}{\left(Z_{v_{n}} \cdot E\right)^{2}} \rightarrow 0 \tag{318}
\end{equation*}
$$

but this is equal to

$$
\begin{equation*}
-\frac{\alpha_{\mathfrak{m}_{p_{n}}}\left(v_{n}\right)}{v_{n}(E)^{2}}=-\frac{\alpha_{E}\left(v_{n}\right)}{v(E)^{2}} \xrightarrow[n \rightarrow+\infty]{ } 0 \tag{319}
\end{equation*}
$$

by Theorem 7.31 and Proposition 5.9 so we are done.
Finally, to show the homeomorphism, we have to show that if $Z_{v_{n}} \rightarrow Z_{v}$ in $L^{2}\left(X_{0}\right)$, then $v_{n}$ converges strongly towards $v$. Let $X$ be a completion of $X_{0}$. Suppose first that $c_{X}(v)$ is a point at infinity. Let $\widetilde{E}$ be the exceptional divisor above $c_{X}(v)$, we have $Z_{v} \cdot \widetilde{E}>0$, therefore for all n large enough $Z_{v_{n}} \cdot \widetilde{E}>0$ and $c_{X}\left(v_{n}\right)=c_{X}(v)=: p$. Now, we can suppose that $v_{n}, v \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$, it suffices to show that $v_{n} \rightarrow v$ for the strong topology of $V_{X}\left(p ; \mathfrak{m}_{p}\right)$ and this is a direct consequence of Proposition 7.34 .

If $c_{X}(v)=E$ a prime divisor at infinity, then for all $n$ large enough, $Z_{V_{n}} \cdot E>0$. Suppose that $v=\operatorname{ord}_{E}$ and $v_{n} \in \mathcal{V}_{X}(E)$. We have that $Z_{v_{n}, X} / Z_{v} \cdot E \rightarrow Z_{\operatorname{ord}_{E}}$ in $\operatorname{NS}(X)_{\mathbf{R}}$.

We need to show that $\alpha_{E}\left(\frac{v_{n}}{v_{n}(E)}\right) \rightarrow 0$. We can suppose that $v_{n} \in \mathcal{V}_{X}\left(p_{n}, \mathfrak{m}_{p_{n}}\right)$ where $p_{n}=c_{X}\left(v_{n}\right)$, then by Proposition 5.9 .

$$
\begin{equation*}
\alpha_{E}\left(\frac{v_{n}}{v_{n}(E)}\right)=\frac{\alpha_{\mathfrak{m}_{p_{n}}}\left(v_{n}\right)}{v_{n}(E)^{2}} . \tag{320}
\end{equation*}
$$

Thus, by Proposition 5.9 and Theorem 7.31

$$
\begin{equation*}
\alpha_{E}\left(\frac{v_{n}}{v_{n}(E)}\right)=\left|\frac{Z_{v_{n}, X, p_{n}}^{2}}{\left(Z_{v_{n}} \cdot E\right)^{2}}\right| \xrightarrow[n \rightarrow+\infty]{ } 0 \tag{321}
\end{equation*}
$$

COROLLARY 11.9. If $v$ is a curve valuation, then $Z_{v}$ is a Weil class satisfying $Z_{v}^{2}=-\infty$.

Proof. Let $X$ be a completion of $X_{0}$, let $p=c_{X}(v)$ and replace $v$ by the valuation equivalent to $v$ such that $v \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$. We have by Corollary 11.6 that

$$
\begin{equation*}
Z_{v}=Z_{v, X}+Z_{v, X, p} \tag{322}
\end{equation*}
$$

Therefore, by Theorem 7.31

$$
\begin{equation*}
\left(Z_{v}\right)^{2}=Z_{v, X}^{2}+\left(Z_{v, X, p}\right)^{2}=Z_{v, X}^{2}-\alpha(v)=-\infty \tag{323}
\end{equation*}
$$

because $\alpha(v)=-\infty$ for any curve valuation $v$ (see [FJ04] Lemma 3.32).

### 11.2. Endomorphisms

Proposition 11.10. Let $f$ be an endomorphism of $X_{0}$ and let $X, Y$ be completions of $X_{0}$ such that the lift $F: X \rightarrow Y$ of $f$ is regular. Let $p \in X$ be a closed point and $q:=F(p) \in Y$. Then,

- $f_{*} \mathcal{V}_{X}(p) \subset \mathcal{V}_{Y}(q)$.
- $f_{*}$ preserves the set of divisorial, irrational and infinitely singular valuations.
- If $v_{C}$ is a curve valuation centered at infinity and such that $f_{*} v_{C}$ is still centered at infinity, then $f_{*} v_{C}$ is also a curve valuation.
Proof. The map $F$ induces a local ring homomorphism $F^{*}: \widehat{O_{Y}(q)} \rightarrow \widehat{O_{X}(p)}$. Let $v$ be a valuation centered at $p$. For $\varphi \in O_{Y}(q), f_{*} v(\varphi)=v\left(F^{*} \varphi\right) \geqslant 0$ and for $\psi \in \mathfrak{m}_{Y, q}, f_{*} v(\psi)=v\left(F^{*} \psi\right)>0$. Therefore $f_{*} v$ is centered at $q$. The fact that $f_{*}$ preserves the type of valuations is shown in Proposition 4.17. It only remains to show the statement for curve valuations. Let $p=c_{X}\left(v_{C}\right)$ and $q=c_{Y}\left(f_{*} v_{C}\right)$. We have that $F(p)=q$. By Proposition $4.17 f_{*} v_{C}$ is not a curve valuation only if it is contracted by $F$. But the only germ of holomorphic curve at $p$ that can be contracted by $F$ is the germ of a prime divisor $E$ at infinity on which $p$ lies, and the curve valuation associated to $E$ does not define a valuation on $\mathbf{k}\left[X_{0}\right]$. So, $f_{*} v_{C}$ is a curve valuation.

EXAMPLE 11.11. It might happen that $f_{*} v$ is not centered at infinity even though $v$ is; if this is the case then $f$ is not proper. For example, let $X_{0}=\mathbf{A}^{2}$ with affine coordinates $(x, y)$ and consider the completion $\mathbf{P}^{2}$ with homogeneous coordinates $[X$ : $Y: Z]$. We have the relation $x=X / Z, y=Y / Z$. Consider the chart $X \neq 0$ with affine coordinates $y^{\prime}=Y / X$ and $z^{\prime}=Z / X$. Define $v_{t}$ to be the monomial valuation centered at $[1: 0: 0]$ such that $v_{t}\left(y^{\prime}\right)=1$ and $v_{t}\left(z^{\prime}\right)=t$ with $t>0$. Let $P=\sum_{i, j} a_{i j} x^{i} y^{j} \in \mathbf{k}[x, y]$, we have that $\mathrm{v}_{t}(P)=\min \left\{j+(j-i) t \mid a_{i j} \neq 0\right\}$. Now take the map $f:(x, y) \in \mathbf{A}^{2} \mapsto(x y, y)$, $f$ contracts the curve $\{y=0\}$ to the point $(0,0)$ in $\mathbf{A}^{2}$, hence it is not proper. For any polynomial $P=\sum_{i, j} a_{i j} x^{i} y^{j}, f^{*} P=\sum_{i, j} a_{i j} x^{i} y^{i+j}$. We get

$$
\begin{equation*}
v_{1, t}\left(f^{*} P\right)=\min _{i, j}\left\{i+j(t+1) \mid a_{i j} \neq 0\right\} \tag{324}
\end{equation*}
$$

The center of $f_{*} v_{t}$ is $[0: 0: 1]$ and $f_{*} v_{t}$ is the monomial valuation centered at $[0: 0: 1]$ such that $\mathrm{v}_{t}(x)=1, v_{t}(y)=t+1$.

Lemma 11.12 (Proposition 3.2 of [FJ07]). Let $f: X_{0} \rightarrow X_{0}$ be a dominant endomorphism and let $X, Y$ be completions of $X_{0}$. Let $F: X \rightarrow Y$ be the lift of $f$, let $p$ be a closed point of $X$ at infinity and $\mathcal{V}_{X}(p)$ be the set of valuations on $\mathbf{k}\left[X_{0}\right]$ centered at $p$. Then, $F$ is defined at $p$ if and only if $f_{*} \mathcal{V}_{X}(p)$ does not contain any divisorial valuation associated to a prime divisor (not necessarily at infinity) of $Y$. If $F$ is defined at $p$, then $F(p)$ is the unique point $q$ such that $f_{*} \mathcal{V}_{X}(p) \subset \mathcal{V}_{Y}(q)$.

Proof. If $\hat{f}$ is defined at $p$, then let $q=\widehat{f}(p)$, we have that $f_{*} \mathcal{V}_{X}(p) \subset \mathcal{V}_{Y}(q)$ by Proposition 11.10 .

Conversely, If $p$ is an indeterminacy point of $\hat{f}$. Let $\pi: Z \rightarrow X$ be a completion above $X$ such that the lift $F: Z \rightarrow Y$ is regular. Then, $F\left(\pi^{-1}(p)\right)$ contains a prime divisor $E^{\prime}$ of $Y$. Let $E$ be a prime divisor at infinity in $Z$ above $p$ such that $F(E)=E^{\prime}$, then $F_{*} \operatorname{ord}_{E}=f_{*}\left(\pi_{*} \operatorname{ord}_{E}\right)=\lambda \operatorname{ord}_{E^{\prime}}$ for some constant $\lambda>0$ and $\operatorname{ord}_{E^{\prime}} \in f_{*} \mathcal{V}_{X}(p)$.

Proposition 11.13. Let $\vee$ be a valuation over $\mathbf{k}\left[X_{0}\right]$ and let $f: X_{0} \rightarrow X_{0}$ be a dominant endomorphism, then

- $f_{*} Z_{v}=Z_{f_{*} v} \bmod \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)^{\perp}$.
- If $f$ is proper then $f_{*}$ preserves $\operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)$ and $f_{*} Z_{v}=Z_{f_{*} v}$.

Proof. Indeed, let $D \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$, then

$$
\begin{equation*}
f_{*} Z_{v} \cdot D=Z_{v} \cdot f^{*} D=L_{v}\left(f^{*} D\right)=L_{f_{*}}(D)=Z_{f_{*} v} \cdot D \tag{325}
\end{equation*}
$$

Therefore, we get that $Z_{f_{*} v}-f_{*} Z_{v}$ belongs to $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)^{\perp}$. If $f$ is proper, then Weil $\infty_{\infty}\left(\mathrm{X}_{0}\right)$ is $f_{*}$-stable and $f_{*} Z_{v} \in \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)$, thus $Z_{f_{*} v}=f_{*} Z_{v}$.

Example 11.14. Suppose that $P(x)$ and $Q(x)$ are two rational fractions of degree two and $E$ in $\mathbf{P}^{1} \times \mathbf{P}^{1}$ defined by the equation

$$
\begin{equation*}
y^{2}-P(x) y+Q(x)=0 \tag{326}
\end{equation*}
$$

if $P, Q$ are general, then $E$ is smooth and irreducible and it is an elliptic curve. Let $X=\mathbf{P}^{1} \times \mathbf{P}^{1}$ and $X_{0}=X \backslash E$. We have $\operatorname{Pic}^{0}\left(X_{0}\right)=0$ because it is a rational surface and $\mathbf{k}\left[X_{0}\right]^{\times}=\mathbf{k}^{\times}$because $X \backslash X_{0}$ consists of a single irreducible curve. We have $Z_{\text {ord }_{E}}=$ $\frac{1}{8} E$. Consider the projection $\mathrm{pr}_{1}: X \rightarrow \mathbf{P}^{1}$ to the first coordinates. Each fiber of $\mathrm{pr}_{1}$ is isomorphic to $\mathbf{P}^{1}$ and generically it has two intersection points with $E$. Let $x_{0}, x_{1}, x_{2}, x_{3}$ be the four roots of the discriminant $\Delta=P(x)^{2}-4 Q(x)$. Then, $\mathrm{pr}_{1}^{-1}\left(x_{i}\right)$ has only one intersection point with $E$. Consider the following selfmap of $X_{0}$

$$
\begin{equation*}
f(x, y)=\left(x, \frac{y^{2}-Q(x)}{2 y-P(x)}\right) . \tag{327}
\end{equation*}
$$

It preserves the fibers of $\mathrm{pr}_{x}$ and it acts as $z \mapsto z^{2}$ in each fiber where the points 0 and $\infty$ of $\mathbf{P}^{1}$ are the intersection point of the fiber with $E$. See Figure 1 . There are exactly 4 indeterminacy points on $X$, they are the points $\left(x_{i}, y_{i}\right)$ where $x_{i}$ is one of the roots of $\Delta$ and $y_{i} \in \mathbf{P}^{1}$ is such that $\left(x_{i}, y_{i}\right) \in E$.


Figure 1. The endomorphism $f$ on $X_{0}$

Let $C_{0}=\left\{x_{0}\right\} \times \mathbf{P}^{1}$. Then, $\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)^{\perp}=\mathbf{R} \cdot\left(4 C_{0}-E\right)$ because $C_{0} \cdot E=2, E^{2}=8$ and $\operatorname{dimNS}(X)_{\mathbf{R}}=2$.

The endomorphism $f$ is not proper, indeed we have in $\mathrm{NS}(X), f_{*} E=E+4 C_{0}$. Since $f^{*} E$ is of the form $f^{*} E=2 E+\ldots$, we have $f_{*} \operatorname{ord}_{E}=2 \operatorname{ord}_{E}$. And we get

$$
\begin{align*}
f_{*} Z_{\text {ord }_{E}}=\frac{1}{8} E+\frac{1}{2} C_{0} &  \tag{328}\\
& =\frac{1}{8} E+\frac{1}{8}\left(4 C_{0}-E\right)+\frac{1}{8} E  \tag{329}\\
& =2 Z_{\text {ord }_{E}}+\frac{1}{8}\left(4 C_{0}-E\right) \tag{330}
\end{align*}
$$

### 11.3. Existence of Eigenvaluations

Recall from Theorem 3.28 that there exists unique nef classes $\theta^{*}, \theta_{*} \in L^{2}\left(\mathrm{X}_{0}\right)$ up to normalization such that $f^{*} \theta^{*}=\lambda_{1} \theta^{*}$ and $f_{*} \theta_{*}=\lambda_{1} \theta^{*}$.

Proposition 11.15. If $\mathbf{k}\left[X_{0}\right]^{\times}=\mathbf{k}^{\times}$and $\operatorname{Pic}^{0}\left(X_{0}\right)=0$, then $\theta^{*} \in \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right) \cap \mathrm{L}^{2}\left(\mathrm{X}_{0}\right)$ and is effective.

Proof. We have that $\operatorname{Weil-NS}\left(\mathrm{X}_{0}\right)=V \oplus \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right)$ where $V$ is a finite dimensional vector space. Furthermore, Weil $\infty_{\infty}\left(\mathrm{X}_{0}\right)$ is $f^{*}$-invariant as $f$ is an endomorphism of $X_{0}$. In the proof of Theorem 3.28 , for every completion $X$ we can consider the cone $C_{X}^{\prime} \subset \operatorname{Div}_{\infty}(X)_{\mathbf{R}}$ of nef, effective divisors supported at infinity. By Theorem 2.4, there exists an ample effective divisor $H \in \operatorname{Div}_{\infty}(X)$ such that $\operatorname{Supp} H=\partial_{X} X_{0}$. Therefore, $C_{X}^{\prime}$ is a closed convex cone with compact basis and non-empty interior, the PerronFrobenius type argument shows that there exists $\theta_{X} \in C_{X}^{\prime}$ such that $f_{X}^{*} \theta_{X}^{*}=\rho_{X} \theta_{X}$ and the rest of the proof is unchanged.

THEOREM 11.16. Let $X_{0}$ be an irreducible normal affine surface such that $\mathbf{k}\left[X_{0}\right]^{\times}=$ $\mathbf{k}^{\times}$and $\operatorname{Pic}^{0}\left(X_{0}\right)=0$. Let $f$ be a dominant endomorphism such that $\lambda_{1}(f)^{2}>\lambda_{2}(f)$, then there exists a unique valuation $\nu_{*}$ centered at infinity up to equivalence satisfying

$$
\begin{align*}
\forall P \in \mathbf{k}\left[X_{0}\right], v_{*}(P) & \leqslant 0  \tag{331}\\
f_{*} v_{*} & =\lambda_{1}(f) v_{*}  \tag{332}\\
Z_{v_{*}}^{2} & >-\infty \tag{333}
\end{align*}
$$

In particular, there exists $w \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)^{\perp}$ such that $\theta_{*}=w+Z_{v_{*}}$. Furthermore, $v_{*}$ is not a curve valuation.

We call $\nu_{*}$ the eigenvaluation of $f$.
Proof. By Theorem 3.28, there exists nef classes $\theta_{*}, \theta^{*} \in \mathrm{~L}^{2}\left(\mathrm{X}_{0}\right)$ that satisfy
(1) $f^{*} \theta^{*}=\lambda_{1} \theta^{*}$
(2) $f_{*} \theta_{*}=\lambda_{1} \theta_{*}$
(3) $\forall \alpha \in \mathrm{L}^{2}\left(\mathrm{X}_{0}\right), \frac{1}{\lambda_{1}^{n}}\left(f^{n}\right)^{*} \alpha \rightarrow\left(\theta_{*} \cdot \alpha\right) \theta^{*}$

Let $X$ be a completion of $X_{0}$. Write the decomposition $\theta_{*}=w+Z$ with $w \in \operatorname{Div}_{\infty}(X)^{\perp}$ and $Z \in$ Weil $_{\infty}\left(\mathrm{X}_{0}\right)_{\mathbf{R}} \cap \mathrm{L}^{2}\left(\mathrm{X}_{0}\right)$. Let $E$ be a prime divisor at infinity in $X$ such that $Z_{\text {ord }_{E}} \cdot \theta^{*}>0$, it exists because $\theta^{*}$ is effective and nef. Then, Item (3) and the continuity of the intersection product in $\mathrm{L}^{2}\left(\mathrm{X}_{0}\right)$ imply that for all $D \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$,

$$
\begin{equation*}
Z_{\mathrm{ord}_{E}} \cdot\left(\frac{1}{\lambda_{1}^{n}}\left(f^{n}\right)^{*} D\right) \rightarrow\left(Z_{\operatorname{ord}_{E}} \cdot \theta^{*}\right)\left(\theta_{*} \cdot D\right)=\left(Z_{\operatorname{ord}_{E}} \cdot \theta^{*}\right)(Z \cdot D) \tag{334}
\end{equation*}
$$

Now, set $v_{n}:=\frac{1}{\lambda_{1}^{n}}\left(f^{n}\right)_{*} \operatorname{ord}_{E}$. Equation (334) shows that $Z_{v_{n}}$ converges towards $Z$ in Weil $_{\infty}\left(\mathrm{X}_{0}\right)$. But, for all $n, Z_{v_{n}}$ belongs to hom $\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \mathbf{R}\right)_{(+)}$which is a closed set of Weil $\infty_{\infty}\left(\mathrm{X}_{0}\right)$ by Corollary 11.2. Therefore, $Z \in \operatorname{hom}\left(\operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right), \mathbf{R}\right)_{(+)}$and it defines a valuation $\nu_{*}$ by Proposition 8.5. From the relation $f_{*} \theta_{*}=\lambda_{1} \theta_{*}$ we get that $f_{*} \nu_{*}=\lambda_{1} \nu_{*}$.

Using the decomposition $\theta_{*}=w+Z_{v_{*}}$ we have

$$
\begin{equation*}
0 \leqslant \theta_{*}^{2}=\omega^{2}+Z_{v_{*}}^{2} \tag{335}
\end{equation*}
$$

Therefore we get $Z_{v_{*}}^{2} \neq-\infty$ and by Corollary 11.9, $v_{*}$ is not a curve valuation.
Now to show the uniqueness of $v_{*}$, if $v$ is another valuation satisfying Equations (331), (332), (333), then for all $D \in \operatorname{Cartier}_{\infty}\left(\mathrm{X}_{0}\right)$, Item (3) implies

$$
\begin{equation*}
Z_{v} \cdot D=\frac{1}{\lambda_{1}^{n}} Z_{v} \cdot\left(f^{n}\right)^{*} D \underset{n \rightarrow \infty}{\longrightarrow}\left(Z_{v} \cdot \theta^{*}\right)\left(\theta_{*} \cdot D\right) \tag{336}
\end{equation*}
$$

Since $v \neq 0$, we get $Z_{v} \cdot \theta^{*}>0$. And then $v=v_{*}$ up to a scalar factor.
REMARK 11.17. It can happen that $f$ admits a curve valuation $\mu$ such that $f_{*} \mu=$ $\lambda_{1} \mu$. For example take the dominant endomorphism of $\mathbf{C}^{2}$

$$
\begin{equation*}
f(x, y)=\left(x^{2}, y^{3}\right) . \tag{337}
\end{equation*}
$$

Then, $\lambda_{1}(f)=3, \lambda_{2}(f)=6$. The curves $x= \pm 1$ are invariant by $f$, so they defines curve valuations at infinity centered at $[0: 1: 0]$ in $\mathbf{P}^{2}$. The extension of $f$ to $\mathbf{P}^{2}$ is the rational map

$$
\begin{equation*}
f[X: Y: Z]=\left[X^{2} Z: Y^{3}: Z^{3}\right] \tag{338}
\end{equation*}
$$

We see that $p=[0: 1: 0]$ is a fixed point of $f$. Take the local coordinates $u=X / Y, v=$ $Z / Y$, then we have

$$
\begin{equation*}
f(u, v)=\left(u^{2} v, v^{3}\right) \tag{339}
\end{equation*}
$$

The curve $x= \pm 1$ becomes $u= \pm v$ in these coordinates. We can see that they are both invariant by $f$ and their curve valuations satisfy $f_{*} \mu=3 \mu$. Now, if $v_{1,1}$ is the
multiplicity valuation at $p$, then we get also that $f_{*} v_{1,1}=v_{3,3}=3 v_{1,1}$. Thus, this is the eigenvaluation of $f$ and it is divisorial.

Corollary 11.18. With the hypothesis of Theorem 11.16 The dynamical degree $\lambda_{1}(f)$ is a Perron number of degree $\leqslant 2$. More precisely,

- If $\mathrm{v}_{*}$ is divisorial or infinitely singular, then $\lambda_{1} \in \mathbf{Z}_{>1}$.
- If $\nu_{*}$ is irrational, then $\lambda_{1}$ is a Perron number of degree 2, in particular $\lambda_{1} \notin \mathbf{Z}$.

This finishes the proof of Theorem $A$
Proof. By Theorem $11.16 f$ admits an eigenvaluation $v_{*}$ satisfying Equations (331), (332), (333). We know that $v_{*}$ cannot be a curve valuation, so there are three cases. It can either be a divisorial valuation, an irrational one or an infinitely singular one. Hence, $v_{*}(P)=\infty \Leftrightarrow P=0$ and it defines a valuation over $K=\operatorname{Frack}\left[X_{0}\right]$. Let $G=v\left(K^{\times}\right)$be the value group of $v_{*}$. The value group of $f_{*} \nu_{*}$ is a subgroup of $G$ and $f_{*}$ induces a Z-linear map $f_{*}: G \rightarrow G$.
(1) If $\nu_{*}$ is divisorial, then $G$ is isomorphic to $\mathbf{Z}$. Since $f_{*} \nu_{*}=\lambda_{1} v_{*}$ we get that $\lambda_{1}$ is an integer.
(2) If $\nu_{*}$ is irrational, then $G$ is isomorphic to $\mathbf{Z}^{2}$. Since $f_{*} \nu_{*}=\lambda_{1} \nu_{*}, \lambda_{1}$ is the spectral radius of a $2 \times 2$ matrix with integer coefficients. Therefore, it is a Perron number of degree 2 by Proposition 11.20 which will be proven in $\$ 11.4$
(3) If $v_{*}$ is infinitely singular. We will show in Proposition 12.3 page 109 , the following.

Claim 11.19. There exists a completion $X$ of $X_{0}$ such that $p:=c_{X}(v) \in E$ is a free point at infinity, the lift $f: X \rightarrow X$ is defined at $p, f(p)=p$ and $f$ contracts $E$ to $p$.

Suppose the claim is true. Let $(z, w)$ be local coordinates at $p$ such that $z=0$ is a local equation of $E, f^{*} z$ is of the form $z^{a} \Phi(z, w)$ where $\Phi$ is a unit. Then,

$$
\begin{equation*}
\lambda_{1} L_{v_{*}}(E)=L_{f_{*} v_{*}}(E)=L_{v_{*}}\left(f^{*} E\right)=a L_{v_{*}}(E) . \tag{340}
\end{equation*}
$$

Since $L_{v_{*}}(E)>0$ we get $\lambda_{1}=a$ and it is an integer.

### 11.4. The dynamical spectrum of the complex algebraic torus

For an algebraic variety $V$, we have defined in the introduction the dynamical spectrum of $V$ by

$$
\begin{equation*}
\Lambda(V):=\left\{\lambda_{1}(f): f \in \operatorname{End}(V)\right\} \tag{341}
\end{equation*}
$$

Recall the definition of Perron numbers, given in the introduction.

Proposition 11.20. For any field $\mathbf{k}, \Lambda\left(\mathbb{G}_{m}^{2}\right)$ is the set of Perron numbers of degree $\leqslant 2$.

Proof. Any endomorphism $f$ of $\mathbb{G}_{m}^{2}$ is given by the composition of a monomial transformation and a translation. Let $A$ be the matrix associated to the monomial transformation of $f$. Then, $\lambda_{1}(f)$ is equal to the spectral radius $\rho$ of $A$. We show that $\rho$ is a Perron number of degree $\leqslant 1$. Let $P=T^{2}-(\operatorname{Tr} A) T+\operatorname{det} A$ be the characteristic polynomial of $A$. Set $\Delta=(\operatorname{Tr} A)^{2}-4 \operatorname{det} A$ the discriminant.

If $\Delta<0$, then $\operatorname{det} A>0$ and the two roots of $P$ are complex conjugate and their modulus is $\sqrt{\operatorname{det} A}$, so $\rho=\sqrt{\operatorname{det} A}$ which is a Perron number of $\operatorname{degree} 2$ if $\operatorname{det} A$ is not a square in $\mathbf{Z}$, otherwise it is a positive integer.

If $\Delta=0$, then $(\operatorname{Tr} A)^{2}=4 \operatorname{det} A$. Therefore $\operatorname{Tr} A$ is even and $P=\left(T-\frac{\operatorname{Tr} A}{2}\right)^{2}$, so $\rho=\left|\frac{\operatorname{Tr} A}{2}\right|$ which is a positive integer.

If $\Delta>0$, set $a:=\operatorname{Tr} A$. If $a \geqslant 0$, then $\rho=\frac{a+\sqrt{\Delta}}{2}$ which is the largest root of $P$ and so $\rho$ is a Perron number of degree 2. If $a<0$, then $\rho=\frac{-a+\sqrt{\Delta}}{2}$ which is a Perron number of degree 2 as it is the largest root of $T^{2}+a T+\operatorname{det} A$.

By Theorem A, any normal affine surface satisfies $\Lambda\left(X_{0}\right) \subset \Lambda\left(\mathbb{G}_{m}^{2}\right)$. Thus, $\Lambda\left(\mathbb{G}_{m}^{2}\right)$ is maximal and one might think that this is a characterisation of the algebraic torus but this is not the case. We now prove Theorem B which states

$$
\begin{equation*}
\Lambda\left(\mathbf{A}^{2}\right)=\Lambda\left(\mathbb{G}_{m}^{2}\right) \tag{342}
\end{equation*}
$$

Proof of Theorem B. By Theorem 12.1 and Proposition 11.20 we have $\Lambda\left(\mathbf{A}^{2}\right) \subset$ $\Lambda\left(\mathbb{G}_{m}^{2}\right)$. We show the equality using the following lemma.

LEMMA 11.21. Every Perron number of degree $\leqslant 2$ is the spectral radius of a $2 \times 2$ matrix with nonnegative integer entries.

Using the lemma, we have that every $\lambda \in \Lambda\left(\mathbb{G}_{m}^{2}\right)$ is the dynamical degree of a monomial transformation of $\mathbf{A}^{2}$, thus $\Lambda\left(\mathbf{A}^{2}\right)=\Lambda\left(\mathbb{G}_{m}^{2}\right)$.

Proof of the lemma. Let $\lambda$ be a Perron number of degree $\leqslant 2$.
If $\lambda$ is an integer then it is the spectral radius of $\left(\begin{array}{ll}\lambda & 0 \\ 0 & 1\end{array}\right)$.
If $\lambda=\sqrt{m}$ with $m$ a positive integer which is not a square, then $\lambda$ is the spectral radius of $\left(\begin{array}{cc}0 & 1 \\ m & 0\end{array}\right)$.

Finally, suppose $\lambda$ is the largest root of $T^{2}-a T+b$ with $a>0, b \neq 0$. If $b<0$, then $\lambda$ is the spectral radius of $\left(\begin{array}{cc}a & 1 \\ -b & 0\end{array}\right)$. If $b>0$, then the discriminant must satisfy

$$
\begin{equation*}
\Delta=a^{2}-4 b>0 \Rightarrow\left(\frac{a}{2}\right)^{2}>b \tag{343}
\end{equation*}
$$

If $a=2 k$ is even, then $\lambda$ is the spectral radius of

$$
\left(\begin{array}{cc}
k & 1  \tag{344}\\
k^{2}-b & k
\end{array}\right) .
$$

If $a=2 k+1$ is odd, then $(k+1 / 2)^{2}>b \Rightarrow k(k+1) \geqslant b$ and $\lambda$ is the spectral radius of

$$
\left(\begin{array}{cc}
k & 1  \tag{345}\\
k(k+1)-b & k+1
\end{array}\right) .
$$

## CHAPTER 12

## Local normal forms

We now suppose that we are in the conditions of Theorem 2.9, i.e either chark= 0 or chark $>0$ and $f$ is tamely ramified, e.g an automorphism. Since everything is defined over a finitely generated field over the prime subfield of $\mathbf{k}$, we can suppose that $\mathbf{k}$ is a subfield of $\mathbf{C}_{v}$, which is a complete algebraically closed field. We show that the existence of this eigenvaluation allows one to find an attracting fixed point at infinity and a local normal form at this fixed point.

THEOREM 12.1. Let $X_{0}$ be an irreducible normal affine surface over a complete algebraically closed field $\mathbf{C}_{v}$. Let $f$ be a dominant tamely ramified endomorphism of $X_{0}$ such that $\lambda_{1}^{2}>\lambda_{2}$. Suppose that $\operatorname{Pic}^{0}\left(X_{0}\right)=0$ and $\mathbf{k}\left[X_{0}\right]^{\times}=\mathbf{k}^{\times}$then
(1) If $v_{*}$ is infinitely singular or irrational, there exists a completion $X$ such that the lift $f: X \rightarrow X$ is defined at $c_{X}\left(\mathrm{v}_{*}\right), f\left(c_{X}\left(\mathrm{v}_{*}\right)\right)=c_{X}\left(\mathrm{v}_{*}\right)$ and $f$ defines a rigid contracting germ of holomorphic function at $c_{X}\left(v_{*}\right)$ with no $f$-invariant germ of curves at $c_{X}\left(v_{*}\right)$. Furthermore, there exists an open (euclidian) $f$ invariant neighbourhood $U^{*}$ of $c_{X}\left(v_{*}\right)$ such that $f\left(U^{*}\right) \Subset U^{*}$. We have the following local normal form:
(a) If $\mathrm{v}_{*}$ is infinitely singular, $c_{X}\left(\mathrm{v}_{*}\right) \in E$ is a free point and $f$ has the local normal form (17) and (18) if char $\mathbf{C}_{v}=0$ with $\{x=0\}$ a local equation of $E \lambda_{1}=a \in \mathbf{Z}_{\geqslant 2}$.
(b) If $\mathrm{v}_{*}$ is irrational, $c_{X}\left(\mathrm{v}_{*}\right)=E \cap F$ is a satellite point. The local normal form is monomial (359) with $(x, y)$ associated to $(E, F)$. The dynamical degree $\lambda_{1}$ is the spectral radius of the matrix $\left(a_{i j}\right)$. It is a Perron number of degree 2 ; in particular $\lambda_{1} \notin \mathbf{Z}$.
(2) If $\mathrm{v}_{*}$ is divisorial, then there exists a completion such that $c_{X}\left(v_{*}\right)$ is a prime divisor $E$ at infinity. In that case, $E$ is f-invariant and $\lambda_{1} \in \mathbf{Z}_{\geqslant 2}$ is such that $f_{X}^{*} E=\lambda_{1} E+D$ where $D \in \operatorname{Div}_{\infty}(X)$ and $E \notin \operatorname{Supp} D$.
(a) Up to replacing $f$ by some iterate, there exists a noncritical fixed point $p \in E$ of $f_{\mid E}, p=E \cap E_{0}$ is a satellite point, $f: X \rightarrow X$ is defined at $p$, $f(p)=p$ and $f$ is a rigid germ (not necessarily contracting) at $p$ with $E$ the only $f$-invariant germ of curves at $p$. The local normal form of $f$ at $p$ is (21) with $(x, y)$ associated to $\left(E, E_{0}\right)$ and $\lambda_{1}=a$.
(b) The curve $E$ is an elliptic curve and $f_{\mid E}$ is a translation by a non-torsion element.
In particular, the dynamical degree of $f$ is a Perron number of degree $\leqslant 2$, and if it is not an integer then the eigenvaluation $v_{*}$ of $f$ is irrational and the normal form is monomial.

We will call (2)b the elliptic case. The rest of this section is devoted to the proof of Theorem 12.1, we will prove the Theorem page 123 .

To prove the theorem we need to understand the dynamics of $f_{*}$ on the space of valuations.

Proposition 12.2. Let $v \in \mathcal{V}_{\infty}$ such that $Z_{v} \in \mathrm{~L}^{2}\left(\mathrm{X}_{0}\right)$. If $Z_{v} \cdot \theta^{*}>0$, then $\frac{1}{\lambda_{1}^{n}} f_{*}^{n} v$ strongly converges towards $\left(Z_{v} \cdot \theta^{*}\right) \nu_{*}$.

Proof. This is a direct consequence of Equation (85) and Corollary 11.8
We will use this to show that $f$ admits a fixed point at infinity on some completion and that $f$ contracts a divisor at infinity there.

For the rest of Chapter 12, we suppose that we are in the conditions of Theorem 11.16

### 12.1. Attractingness of $v_{*}$, the infinitely singular case

For the infinitely singular case we do not assume chark $=0$ or that $f$ is tamely ramified. We show the following

Proposition 12.3. Let $\mathbf{k}$ be an algebraically closed field (of any characteristic). If the eigenvaluation $\mathrm{v}_{*}$ is infinitely singular, then there exists a completion $X$ of $X_{0}$ such that
(1) $p:=c_{X}\left(v_{*}\right) \in E$ is a free point at infinity.
(2) $f_{*} \mathcal{V}_{X}(p) \subset \mathcal{V}_{X}(p)$;
(3) $f$ contracts $E$ to $p$.
(4) Let $f_{\bullet}: \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right) \rightarrow \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$, then for all $v \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right), f_{\bullet}^{n} v \rightarrow v_{*}$.

Furthermore, the set of completions $Y$ above $X$ that satisfy these 4 properties is cofinal in the set of all completions above $X$.

Let $X$ be a completion of $X_{0}$ such that $c_{X}\left(v_{*}\right)$ is a free point $p_{X} \in E_{X}$. Such a completion $X$ exists and there are infinitely many of them above $X$ by Proposition 4.16. Let $Y$ be a completion above $X$ such that $c_{Y}\left(v_{*}\right)$ on $Y$ is a free point $p_{Y} \in E_{Y}$ and such that the diagram

commutes, where $F$ is regular and $F\left(p_{Y}\right)=p_{X}$. Let $x, y$ be coordinates at $p_{X}$ such that $x=0$ is a local equation of $E_{X}$ and $z, w$ be coordinates at $p_{Y}$ such that $z=0$ is a local equation for $E_{Y}$. We use the notations of Chapter 5 . We have that $f_{*} \mathcal{V}_{Y}\left(p_{Y}\right) \subset \mathcal{V}_{X}\left(p_{X}\right)$ by Lemma 11.12. We define $F_{\bullet}: \mathcal{V}_{Y}\left(p_{Y} ; E_{Y}\right) \mapsto \mathcal{V}_{X}\left(p_{X}, \mathfrak{m}_{p_{X}}\right)$ as follows:

$$
\begin{equation*}
\forall v \in \mathcal{V}_{Y}\left(p_{Y} ; E_{Y}\right), \quad F_{\bullet}(v):=\frac{F_{*} v}{\min \left(v\left(F^{*} x\right), v\left(F^{*} y\right)\right)} \tag{346}
\end{equation*}
$$

Similarly, we define

$$
\begin{equation*}
\forall v \in \mathcal{V}_{Y}\left(p_{Y} ; E_{Y}\right), \quad \pi_{\bullet}(v):=\frac{\pi_{*} v}{\min \left(v\left(\pi^{*} x\right), v\left(\pi^{*} y\right)\right)} \tag{347}
\end{equation*}
$$

By Proposition 5.20 item (1), the map $\pi_{\bullet}: \mathcal{V}_{Y}\left(p_{Y} ; E_{Y}\right) \rightarrow \mathcal{V}_{X}\left(p_{X} ; \mathfrak{m}_{p_{X}}\right)$ is an inclusion of trees and allows one to view $\mathcal{V}_{Y}\left(p_{Y} ; E_{Y}\right)$ as a subtree of $\mathcal{V}_{X}\left(p_{X} ; \mathfrak{m}_{p_{X}}\right)$.

See Figure 11. The tree $\mathcal{V}_{X}\left(p_{X}, \mathfrak{m}_{p_{X}}\right)$ is in black with its root $v_{\mathfrak{m}_{p_{X}}}$ in blue, the tree $\mathcal{V}_{Y}\left(p_{Y} ; E_{Y}\right)$ is in orange with its root ord $E_{E_{Y}}$ in red. One can see how $\pi_{\bullet}$. maps homeomorphically $\mathcal{V}_{Y}\left(p_{Y} ; E_{Y}\right)$ to a subtree of $\mathcal{V}_{X}\left(p_{X}, \mathfrak{m}_{p_{X}}\right)$.


Figure 1. The embedding $\pi$ 。

REMARK 12.4. Since the orders $\leqslant_{\mathfrak{m}_{p_{X}}}$ and $\leqslant_{E_{Y}}$ are compatible on $\mathcal{V}_{Y}\left(p_{Y} ; E_{Y}\right)$ and $\pi_{\bullet} \mathscr{V}_{Y}\left(p_{Y} ; E_{Y}\right)$ we will not write $\pi_{\bullet}$ or $\leqslant_{E_{Y}}$ when no confusion is possible to avoid heavy notations.

By Proposition 5.28, we have the following relation

$$
\begin{equation*}
\alpha_{\mathfrak{m}_{p_{X}}}\left(\pi_{\bullet} \mu\right)=\alpha_{\mathfrak{m}_{p_{X}}}\left(\pi_{\bullet} \operatorname{ord}_{E_{Y}}\right)+b\left(E_{Y}\right)^{-2} \alpha_{E_{Y}}(\mu) \tag{348}
\end{equation*}
$$

where $b$ is the generic multiplicity function of the tree $\mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$ and $\alpha_{\mathfrak{m}_{p_{X}}}, \alpha_{E_{Y}}$ are the skewness functions defined in Chapter [5. Indeed, with the notation of Proposition5.28, $v_{E_{Y}}=\pi_{\bullet} \operatorname{ord}_{E_{Y}}$.

Lemma 12.5. There exists $v \in \mathcal{V}_{Y}\left(p_{Y} ; E_{Y}\right)$ such that $v<\nu_{*}$ and for all $\mu \geqslant v$,

$$
\begin{equation*}
\min \left(\mu\left(F^{*} x\right), \mu\left(F^{*} y\right)\right)=b\left(E_{Y}\right) \lambda_{1} \tag{349}
\end{equation*}
$$

I.e set $U=\{\mu \geqslant v\}$, we have $F_{\bullet}=\frac{F_{*}}{b\left(E_{Y}\right) \lambda_{1}}$ over $U$. In particular, $F_{\bullet}$ is order preserving over $U$ and $F_{\bullet}\left(\left[v, v_{*}\right]\right) \subset\left[v_{\mathfrak{m}_{p_{X}}}, v_{*}\right]$.

Proof. Using Proposition5.3, we see that the map $v \mapsto \min \left(v\left(f^{*} x, f^{*} y\right)\right)$ is locally constant outside a finite subtree of $\mathcal{V}_{Y}\left(p_{Y} ; E_{p_{Y}}\right)$. Indeed, one has $f^{*} x=\prod_{i} \psi_{i}$ with $\psi_{i}$ irreducible and therefore

$$
\begin{align*}
v\left(f^{*} x\right) & =\sum_{i} v\left(\psi_{i}\right)  \tag{350}\\
& =\sum_{i} \alpha_{E_{Y}}\left(v \wedge v_{\psi_{i}}\right) m_{E_{Y}}\left(\psi_{i}\right) \quad \text { by Proposition 5.3. } \tag{351}
\end{align*}
$$

Let $S_{x}$ be the finite subtree consisting of the segments $\left[\operatorname{ord}_{E_{Y}}, \nu_{\psi_{i}}\right.$ ], then the map $\mu \mapsto$ $\left.\mu\left(f^{*} x\right)\right)$ is locally constant outside of $S_{x}$. Let $S$ be the maximal finite subtree of $\mathcal{V}_{Y}\left(p_{Y} ; E_{p_{Y}}\right)$ such that the evaluation maps on $f^{*} x, f^{*} y$ and $z$ are locally constant outside of $S$. Since $\nu_{*}$ is an infinitely singular valuation it does not belong to $S$ and these three evaluation maps are constant on the open connected component $V$ of $\mathcal{V}_{Y}\left(p_{Y} ; E_{p_{Y}}\right) \backslash S$ containing $\nu_{*}$. Since $f_{*} \nu_{*}=\lambda_{1} \nu_{*}$, this means that $F_{*} v_{*}=\lambda_{1} \pi_{*} v_{*}$. Since the ideal generated by $\pi^{*} x, \pi^{*} y$ is the ideal generated by $z^{b\left(E_{Y}\right)}$, we have $f_{\bullet} \left\lvert\, V=\frac{f_{*}}{b\left(E_{Y}\right) \lambda_{1}}\right.$ and the map $F_{\bullet}$ is order preserving on $V$. Following Remark 12.4 , the two orders $\leqslant_{\mathfrak{m}_{p_{X}}}$ and $\leqslant_{E_{Y}}$ agree on $V$. Let $v \in\left[\operatorname{ord}_{E_{Y}}, \nu_{*}\right] \cap V$ be a divisorial valuation, $F_{\bullet}$ sends the segment $\left[v, v_{*}\right] \subset \mathcal{V}_{Y}\left(p_{Y} ; E_{Y}\right)$ inside the segment $\left[v_{\mathfrak{m}_{p_{X}}}, v_{*}\right] \subset \mathcal{V}_{X}\left(p_{X} ; \mathfrak{m}_{p_{X}}\right)$. Notice that $U:=\{\mu \geqslant v\} \subset V$ so the valuation $v$ satisfies Lemma 12.5 ,

Proposition 12.6 ([FJ07], Theorem 3.1). Let $v$ be as in Lemma 12.5. For $t \in$ $\left[\alpha_{E_{Y}}(\mathrm{v}), \alpha_{E_{Y}}\left(v_{*}\right)\right]$, let $\mathrm{v}_{t}$ be the unique valuation in $\left[\mathrm{v}, \mathrm{v}_{*}\right]$ such that $\alpha_{E_{Y}}\left(\mathrm{v}_{t}\right)=t$. Then, there exists a divisorial valuation $\mathrm{v}^{\prime} \in\left[\mathrm{v}, \mathrm{v}_{*}\right]$ such that the map

$$
\begin{equation*}
t \in\left[\alpha_{E_{Y}}\left(v^{\prime}\right), \alpha_{E_{Y}}\left(v_{*}\right)\right] \mapsto \alpha_{m_{p_{X}}}\left(F_{\bullet} v_{t}\right) \tag{352}
\end{equation*}
$$

is an affine function of $t$ with nonnegative coefficients.
Proof. Let $v_{1}, v_{2} \in \mathcal{V}_{Y}\left(p_{Y} ; E_{Y}\right)$ be such that $v<v_{1}<v_{2}<v_{*}$. Since $F_{\bullet}$ is order preserving on $U=\{\mu \geqslant v\}$ one has that $F_{\bullet}$ maps [ $v_{1}, v_{2}$ ] homeomorphically to $\left[F_{\bullet} v_{1}, F_{\bullet} v_{2}\right]$. Let $\psi \in \widehat{O_{X, p_{X}}}$ be irreducible such that $v_{\psi}>F_{\bullet} v_{2}$, then by Proposition 5.3, for all $\mu \in\left[v_{1}, v_{2}\right]$ one has

$$
\begin{equation*}
\alpha_{\mathfrak{m}_{p_{X}}}\left(F_{\bullet} \mu\right)=\frac{F_{\bullet} \mu(\psi)}{m_{p_{X}}(\psi)}=\frac{\mu\left(f^{*} \psi\right)}{m_{p_{X}}(\psi) b\left(E_{Y}\right) \lambda_{1}} \tag{353}
\end{equation*}
$$

Now let $\psi_{1}, \cdots, \psi_{r} \in \widehat{O_{Y, p_{Y}}}$ be irreducible (not necessarily distinct) such that $f^{*} \psi=$ $\psi_{1} \cdots \psi_{r}$. One has,

$$
\begin{equation*}
\mu\left(f^{*} \psi\right)=\sum_{i} \mu\left(\psi_{i}\right)=\sum_{i} \alpha_{E_{Y}}\left(\mu \wedge v_{\psi_{i}}\right) m_{E_{Y}}\left(\psi_{i}\right) \tag{354}
\end{equation*}
$$

Take one of the $\psi_{i}$ and call it $\psi_{0}$, we shall study the map $\mu \in\left[v_{1}, v_{2}\right] \mapsto \alpha_{E_{Y}}\left(\mu \wedge v_{\psi_{0}}\right)$. Let $\mu_{0}=v_{2} \wedge v_{\psi_{0}}$, this map is equal to $\alpha_{E_{Y}}$ on $\left[v_{1}, \mu_{0}\right]$ and constant equal to $\alpha_{E_{Y}}\left(\mu_{0}\right)$ on [ $\mu_{0}, \nu_{2}$ ]. Therefore, the map $\mu \in\left[v_{1}, \nu_{2}\right] \mapsto \mu\left(f^{*} \psi\right)$ is a piecewise affine function with nonnegative coefficients of $\alpha_{E_{Y}}(\mu)$. The points on $\left[v_{1}, v_{2}\right]$ where this map is not smooth are exactly the valuations $\nu_{*} \wedge \nu_{\psi_{i}}$ and there are at most $\lambda_{2}$ of them by Proposition 4.18. Therefore the map $\mu \mapsto v\left(f^{*} \psi\right)$ is an affine function of $\alpha_{E_{Y}}$ with nonnegative coefficients on the segment $\left[\mu^{\prime}, \nu_{*}\right]$ for any $\mu^{\prime}<\nu_{*}$ close enough to $\nu_{*}$.

As a corollary of the proof, we get the following proposition.
Proposition 12.7. Let $v \in \mathcal{V}_{Y}\left(p_{Y} ; E_{Y}\right)$ be as in Proposition 12.6. let $v_{0} \in\left[v, v_{*}\right]$ and let $\psi \in \widehat{O}_{X, p}$ be irreducible such that $v_{\psi}>f_{\bullet} v_{0}$. Then, for all $\varphi \in \widehat{O_{Y, p_{Y}}}$ such that $f_{\bullet} v_{\varphi}=v_{\psi}$, one has two possibilities:
(1) Either $v_{\varphi}>v_{0}$.
(2) or $\nu_{0} \wedge \nu_{\varphi}=\nu_{*} \wedge \nu_{\varphi} \leqslant \nu$.
I.e the configuration of Figure 2 cannot occur.

Proof. The map $\mu \in\left[v, v_{0}\right] \mapsto \alpha_{m_{p_{X}}}\left(F_{\bullet} \mu\right)$ is a smooth affine function of $\alpha_{E_{Y}}(\mu)$. If (1) and (2) were not satisfied, then we would get $\nu_{\varphi} \wedge \nu_{*} \in\left[v, \nu_{*}\right]$ and this would contradict the smoothness of the map $\mu \in\left[v, \nu_{*}\right] \mapsto \alpha_{m_{p_{X}}}\left(F_{\bullet} \mu\right)$

Lemma 12.8. Let $\vee$ be as in Proposition 12.6. If $\mu \in\left[v, v_{*}\right]$ is sufficiently close to $v_{*}$, then $F_{\bullet} \mu>\mu$ and $F_{\bullet}\left(\left\{\mu^{\prime} \geqslant \mu\right\}\right) \Subset U(\vec{v})$ where $\vec{v}$ is the tangent vector at $\mu$ defined by $\nu_{*}$ and $U(\vec{v})$ is its associated open subset.

We sum up Lemma 12.8 in Figure 3


Figure 2. Configuration which is not possible
Proof. Let $U=\{\mu \geqslant v\}$. Recall that $F_{\mathbf{\bullet}}$ is order preserving over $U$. We first notice that if every $\mu \in\left[v, v_{*}\right]$ close enough to $v_{*}$ satisfies $F_{\bullet} \mu>\mu$, it is clear that $F_{\bullet}\left\{\mu^{\prime} \geqslant \mu\right\} \Subset$ $U(\vec{v})$. Indeed, let $\mu^{\prime} \geqslant \mu$ and set $\mu_{0}:=\mu^{\prime} \wedge v_{*} \geqslant \mu$. Then, $F_{\bullet} \mu^{\prime} \geqslant F_{\bullet} \mu_{0}>\mu_{0}$. In particular, $F_{\bullet} \mu^{\prime} \wedge \nu_{*}>\mu^{\prime} \wedge \nu_{*} \geqslant \mu$.

Secondly, by Proposition 12.6, the map $t \in\left[\alpha_{E_{Y}}(v), \alpha_{E_{Y}}\left(v_{*}\right)\right] \mapsto \alpha_{m_{p_{X}}}\left(v_{t}\right)$ is affine and we know that it is non decreasing.

Lemma 12.9. Let $a: \mathbf{R} \rightarrow \mathbf{R}$ be a non-decreasing non constant affine function that admits a fixed point $t_{0}$. If there exists $s<t_{0}, a(s)>s$ then the slope of $a$ is $<1$ and for all $t<t_{0}, a(t)>t$.

Proof of Lemma 12.9. We can suppose that $t_{0}=0$ by a linear change of coordinate. Then, $a(t)$ is of the form

$$
\begin{equation*}
a(t)=\alpha t \tag{355}
\end{equation*}
$$

with $\alpha>0$. Now, if $s<0$ satisfies $a(s)>s$, this means that $0<\alpha<1$ and therefore for all $t<0, a(t)>t$.

We show that there exists $\mu \in\left[v, v_{*}\right]$ such that $F_{\bullet} \mu>\mu$. If not, then for all $\mu \in$ $\left[v, v_{*}\left[, F_{\bullet} \mu \leqslant \mu\right.\right.$. Under such an assumption, we show the following

Claim. For all $\mu^{\prime} \geqslant v$ we have $F_{\bullet} \mu^{\prime} \wedge \nu_{*} \leqslant \mu^{\prime} \wedge \nu_{*}$.
Suppose that the claim is false and let $\mu^{\prime}$ be a valuation that contradicts this statement. It is clear that $\mu^{\prime}$ does not belong to $\left[v, v_{*}\right]$. Pick $v_{0} \in\left[v, \nu_{*}\right]$ such that $v \leqslant$ $\mu^{\prime} \wedge \nu_{*}<v_{0}<F_{\bullet} \mu^{\prime} \wedge \nu_{*}$. Let $\varphi \in \widehat{O}_{Y, p_{Y}}$ be such that $v_{\varphi}>\mu^{\prime}$ and let $\psi \in \widehat{O}_{X, p}$ be such that $f_{\bullet} v_{\varphi}=\nu_{\psi}$. Since $f$ is order preserving we get that $v_{\psi}>F_{\bullet} \mu^{\prime} \geqslant F_{\bullet} \mu^{\prime} \wedge \nu_{*}>v_{0}$, therefore $v_{\psi}>F_{\bullet} v_{0}$. But then $\varphi$ contradicts Proposition 12.7 since $v_{\varphi} \wedge v_{0}=\mu^{\prime} \wedge v_{0} \in\left[v, v_{0}\right]$. So the claim is shown.

Now, pick $\omega$ divisorial such that $Z_{\omega} \cdot \theta^{*}>0$ by Proposition 12.2 the sequence $\frac{1}{\lambda_{1}^{n}} f_{*}^{n} \omega$ converges towards $\left(Z_{\omega} \cdot \theta^{*}\right) \nu_{*}$. Hence, there exists an integer $N_{0}>0$ such that for all


Figure 3. An $f_{\bullet}$-invariant open subset of $\mathcal{V}_{\infty}$, infinitely singular case
$N \geqslant N_{0}, f_{*}^{N} v \in \mathcal{V}_{Y}\left(p_{Y}\right)$, replace $\omega$ by $f_{*}^{N_{0}} \omega$ and normalize it such that $\omega \in \mathcal{V}_{Y}\left(p_{Y}, E_{Y}\right)$. We can suppose up to choosing a larger $N_{0}$ that $\omega>\nu$. In that case $F_{\bullet}^{N} \omega$ converges towards $v_{*}$ but by the claim, $\forall N \geqslant 0, F_{\bullet}^{N} \omega \wedge \nu_{*} \leqslant \omega \wedge \nu_{*}$ which is a contradiction.

Therefore, there exists a valuation $\mu \in\left[v, \nu_{*}\left[\right.\right.$ such that $F_{\bullet} \mu>\mu$.
Proposition 12.10. With the notations from Lemma 12.8 we have $F_{\bullet}(U(\vec{v})) \Subset$ $U(\vec{v})$ and for all $\mu^{\prime} \in U(\vec{v})$,

$$
\begin{equation*}
F_{\bullet}^{n} \mu^{\prime} \xrightarrow[n \rightarrow+\infty]{\longrightarrow} v_{*} \tag{356}
\end{equation*}
$$

for the weak topology.
Proof. For every $\mu^{\prime}$ in $U(\vec{v})$, write $\tilde{\mu^{\prime}}=\mu^{\prime} \wedge v_{*}$. By the proof of Lemma 12.8, $F_{\bullet}^{n}\left(\mu^{\prime}\right) \rightarrow \nu_{*}$ for the strong topology. Therefore, $F_{\bullet}^{n} \mu^{\prime} \wedge v_{*} \geqslant F_{\bullet}^{n}\left(\tilde{\mu^{\prime}}\right) \rightarrow v_{*}$ and $F_{\bullet}^{n} \mu^{\prime}$
converges weakly towards $v_{*}$ because for all $\varphi \in O_{Y, p}$ irreducible, we have

$$
\begin{equation*}
F_{\bullet}^{n}\left(\mu^{\prime}\right)(\varphi)=\alpha_{E_{Y}}\left(F_{\bullet}^{n} \mu^{\prime} \wedge v_{\varphi}\right) m_{E_{Y}}(\varphi) . \tag{357}
\end{equation*}
$$

For $n$ large enough we have $F_{\bullet}^{n} \mu^{\prime} \wedge \nu_{*} \geqslant v_{*} \wedge v_{\varphi}$, hence $F_{\bullet}^{n} \mu^{\prime} \wedge v_{\varphi}=\nu_{*} \wedge v_{\varphi}$ and

$$
\begin{equation*}
F_{\bullet}^{n}\left(\mu^{\prime}\right)(\varphi)=\alpha_{E_{Y}}\left(v_{*} \wedge v_{\varphi}\right) m_{E_{Y}}(\varphi)=v_{*}(\varphi) \tag{358}
\end{equation*}
$$

Proof of Proposition 12.3, Let $v$ be as in Proposition 12.6. Let $v_{n}$ be the infinitely near sequence of $v_{*}$ (see Proposition 5.26). We have for $n$ large enough $v_{n} \in\left[v, v_{*}\right]$ and $v_{n}$ satisfies Lemma 12.8. Set $\mu=v_{n}$ for some $n$ large enough and let $Z$ be a completion such that $c_{Z}(\mu)=E$ and $c_{Z}\left(v_{*}\right)=: p \in E$ is a free point. The open subset $U(\vec{v})$ associated to the tangent vector at $\mu$ defined by $v_{*}$ is exactly the image of $\mathcal{V}_{Z}(p)$ in $\mathcal{V}_{Y}\left(p_{Y} ; E_{Y}\right)$. By Proposition $12.10, F_{\bullet} U(\vec{v}) \Subset U(\vec{v})$, this means that $f_{*} \mathcal{V}_{Y}(p) \subset \mathcal{V}_{Y}(p)$. By Lemma 11.12, $f$ is defined at $p, f(p)=p$ and since $F_{\bullet} \mu>\mu$, we get $f$ contracts $E$ to $p$. We have that for every $\mu \in \mathcal{V}_{Z}\left(p ; \mathfrak{m}_{p}\right), f_{\bullet}^{n} \mu \rightarrow \boldsymbol{v}_{*}$ also by Proposition 12.10 .

The statement about cofinalness follows from the fact that the sequence of infinitely near points associated to $v_{*}$ contains infinitely many free points, so for every completion $X$ of $X_{0}$, there exists a completion above it where the center of $v_{*}$ is a free point at infinity.

### 12.2. Attractingness of $v_{*}$, the irrational case

Suppose now that char $\mathbf{k}=0$ or that $f$ is tamely ramified, this is necessary as we will use Theorem 2.9 in this paragraph. Suppose now that $v_{*}$ is an irrational valuation. There exists a completion $X$ such that the center of $v_{*}$ on $X$ and on any completion above $X$ is the intersection of two divisors at infinity $E, F$. We still write $f: X \rightarrow X$ for the lift of $f$.

Let $X_{1}=X$ and for all $n \geqslant 1$, let $X_{n+1}$ be the blow up of $X_{n}$ at $c_{X_{n}}\left(v_{*}\right)$. (The center of $\mathrm{v}_{*}$ is always a point since $\mathrm{v}_{*}$ is not divisorial). Let $p_{n}=c_{X_{n}}\left(\nu_{*}\right)$ and $E_{n}, F_{n}$ be the divisors at infinity in $X_{n}$ such that $p_{n}=E_{n} \cap F_{n}$. A consequence of Theorem 2.9 is

Proposition 12.11. There exist integers $N \geqslant M$ such that the lift $\hat{f}: X_{N} \rightarrow X_{M}$ is regular at $p_{N}:=c_{X_{N}}\left(v_{*}\right)$ and such that $\widehat{f}$ is monomial at $p_{N}$ in the coordinates that have $E_{N}, F_{N}$ and $E_{M}, F_{M}$ for axis respectively.

Proof. Apply Theorem 2.9 to $f: X \rightarrow X$. There exist completions $Y, Z$ above $X$ such that the lift $F: Y \rightarrow Z$ of $f$ is regular and monomial at every point. Let $N_{Y}=$ $\max \left\{N: Y\right.$ is above $\left.X_{N}\right\}$ and define $N_{Z}$ in the same way. By construction, the morphism of completions $\pi: Y \rightarrow X_{N_{Y}}$ consists of blow up of points that are not $p_{N_{Y}}$. The same holds for $\tau: Z \rightarrow X_{N_{Z}}$. This shows that the lift $f: X_{N_{Y}} \rightarrow X_{N_{Z}}$ is defined at $p_{N_{Y}}$. We
therefore have that $f\left(p_{N_{Y}}\right)=p_{N_{Z}}$ because $f_{*}\left(v_{*}\right)=\lambda_{1} v_{*}$ and $f$ is monomial at $p_{N_{Y}}$ in the coordinates that have $E_{N_{Y}, F_{M_{Y}}}$ and $E_{N_{Z}}, F_{N_{Z}}$ for axis respectively by Theorem 2.9 . We set $M=N_{Z}$. If $N_{Y}<M$, we keep blowing up $p_{N_{Y}}$ until $N_{Y} \geqslant M$. This does not change the result because in local coordinates the blow up is given by a monomial map $\pi(u, v)=(u v, v)$ where $u$ and $v$ are local equation of the prime divisors at infinity to which the center of $v_{*}$ belong.

Using this we show

## PROPOSITION 12.12. There exists a completion $Y$ such that

(1) The lift $\hat{f}: Y \rightarrow Y$ is defined at $p=c_{Y}\left(v_{*}\right)$;
(2) $\widehat{f}(p)=p$;
(3) If $E, F$ are the two divisors at infinity such that $p=E \cap F$, then $\hat{f}$ contracts at least one of the two divisors and $\widehat{f}^{2}$ contracts both of them.
(4) Define $f_{\bullet}: \mathcal{V}_{Y}\left(p ; \mathfrak{m}_{p}\right) \rightarrow \mathcal{V}_{Y}\left(p ; \mathfrak{m}_{p}\right)$. For all $\boldsymbol{\mu} \in \mathcal{V}_{Y}\left(p ; \mathfrak{m}_{p}\right), f_{\bullet}^{n} m u \rightarrow \boldsymbol{v}_{*}$ for the weak topology of $\mathcal{V}_{Y}\left(p ; \mathfrak{m}_{p}\right)$.
Furthermore, If $Z$ is a completion above $Y$, then (1)-(4) remain true.
Proof. Let $N \geqslant M$ given by Proposition 12.11. We still write $f: X_{N} \rightarrow X_{M}$ for the lift of $f$ and $\pi: X_{N} \rightarrow X_{M}$ for the composition of blow ups. Let $x, y$ be local coordinates at $p_{N}$ such that $E_{N}=\{x=0\}$ and $F_{N}=\{y=0\}$ and let $z, w$ be local coordinates at $p_{M}$ such that $E_{M}=\{z=0\}$ and $F_{M}=\{w=0\}$. Both maps $f$ and $\pi$ are monomial at $p_{N}$ with respect to these coordinates. Write

$$
\begin{equation*}
f(x, y)=\left(x^{a} y^{b}, x^{c} y^{d}\right) . \tag{359}
\end{equation*}
$$

Consider the tree $V_{X_{M}}\left(p_{M} ; E_{M}\right)$ with its order $<_{M}$, its skewness function $\alpha_{M}$ and the generic multiplicity function $b_{M}$. This tree is rooted in $\operatorname{ord}_{E_{M}}$ and $F_{M}$ defines the end $v_{w}$ that we denote by $v_{F_{M}}$. Let $v_{E_{N}}=\frac{1}{b_{M}\left(E_{N}\right)} \operatorname{ord}_{E_{N}}, v_{F_{N}}=\frac{1}{b_{M}\left(F_{N}\right)} \operatorname{ord}_{F_{N}}$. Suppose without loss of generality that $\nu_{E_{N}}<_{M} \nu_{F_{N}}$. Consider the tree $V_{X_{N}}\left(p_{N} ; E_{N}\right)$ with its order $<_{N}$ and skewness function $\alpha_{N}$. We have by Proposition 5.20 item (2) that the map $\pi_{\bullet}: V_{X_{N}}\left(p_{N} ; E_{N}\right) \rightarrow \mathcal{V}_{X_{M}}\left(p_{M} ; E_{M}\right)$ is an inclusion of trees. Hence, the orders $<_{M},<_{N}$ are compatible and $\mathcal{V}_{X_{N}}\left(p_{N} ; E_{N}\right)$ is naturally a subtree of $\mathcal{V}_{X_{M}}\left(p_{M} ; E_{M}\right)$ via the map $\pi_{\bullet}$. We also have the map $f_{\bullet}: \mathcal{V}_{X_{N}}\left(p_{N} ; E_{N}\right) \rightarrow \mathcal{V}_{X_{M}}\left(p_{M} ; E_{M}\right)$. The root of $\mathcal{V}_{X_{N}}\left(p_{N} ; E_{N}\right)$ is $\operatorname{ord}_{E_{N}}$ and $F_{N}$ defines the end $v_{y}$ in $\mathcal{V}_{X_{N}}\left(p_{N} ; E_{N}\right)$ that we also denote by $\nu_{F_{N}}$. We have that $\operatorname{ord}_{E_{N}}<_{N} v_{*}<_{N} v_{F_{N}}$. Using Equation (359), we can write

$$
\begin{equation*}
\forall v \in \mathcal{V}_{X_{N}}\left(p_{N} ; E_{N}\right), \quad f_{\bullet}(v)=\frac{f_{*} v}{a+b v(y)} \tag{360}
\end{equation*}
$$

Now, both maps $f_{\bullet}$ and $\pi_{\bullet}$ send the segment $\left[\operatorname{ord} E_{N}, \nu_{F_{N}}\right]$ into the segment $\left[\operatorname{ord}_{E_{M}}, \nu_{F_{M}}\right]$ via a Möbius transformation. Indeed, if $v_{1, t} \in \mathcal{V}_{X_{N}}\left(p_{N} ; E_{N}\right)$ is a monomial valuation at
$p_{N}$, then $f_{*} v_{1, t}=v_{a+b t, c+t d}$ and one has by Lemma 5.13 and Equation (360)

$$
\begin{equation*}
\alpha_{M}\left(f_{\bullet} v_{1, t}\right)=\alpha_{M}\left(v_{1, \frac{c+t d}{a+t b}}\right)=\frac{c+\alpha_{N}\left(v_{1, t}\right) d}{a+\alpha_{N}\left(v_{1, t}\right) b}=M_{f}\left(\alpha_{N}\left(v_{1, t}\right)\right) \tag{361}
\end{equation*}
$$

Where $M_{f}$ is the Möbius transformation associated to the matrix $\left(\begin{array}{ll}d & c \\ b & a\end{array}\right)$. We can do the same process with the map $\pi$. to get a Möbius transformation represented by a matrix $M_{\pi}$. Set $M$ to be the Möbius transformation $M_{f} \circ M_{\pi}^{-1}$.

Lemma 12.13. The Möbius map $M$ is loxodromic with an attracting fixed point $t_{*}=\alpha_{M}\left(\pi_{\bullet} \nu_{*}\right)$ and the multiplier of $M$ at $t_{*}$ is $\leqslant \sqrt{\frac{\lambda_{2}}{\lambda_{1}^{2}}}<1$.

In particular, for every $\mathrm{v}_{1}, \mathrm{v}_{2} \in \mathcal{V}_{X_{N}}\left(p_{N} ; E_{N}\right)$ close enough to $\mathrm{v}_{*}$ such that $\mathrm{v}_{1}<\mathrm{v}_{*}<$ $\nu_{2}, f_{\bullet}\left(\left[\nu_{1}, \nu_{2}\right]\right) \Subset\left[\pi_{\bullet} \nu_{1}, \pi_{\bullet} \nu_{2}\right]$.

Proof of Lemma 12.13. First of all, $M$ cannot be of finite order. Indeed, for every $v \in\left[v_{E_{N}}, \nu_{E_{M}}\right]$ sufficiently close to $v_{*}$, we have $Z_{v} \cdot \theta^{*}>0$ since $\theta^{*} \cdot \theta_{*}=1$. So $f_{\bullet}^{n} v \rightarrow v_{*}$ by Proposition 12.2 .

We know that $M\left(t_{*}\right)=t_{*}$ and we want to show that $\left|M^{\prime}\left(t_{*}\right)\right|<1$. The only way that the proposition is not true is if $t_{*}$ is a parabolic fixed point of $M$. This means up to reversing the orientation that $t_{*}$ is attracting for $t<t_{*}$ sufficiently close to $t_{*}$ and $t_{*}$ is repelling for $t>t_{*}$ sufficiently close to $t$. In particular, there exists $t^{\prime}$ such that the segment $\left[t^{\prime}, t_{*}\right]$ is sent strictly into itself, so we can iterate $M$ on it, and there exist two constant $c_{1}, c_{2}>0$ such that $\frac{c_{1}}{n} \leqslant\left|M^{n}(s)-t_{*}\right| \leqslant \frac{c_{2}}{n}$. We will show that we have actually an exponential speed of convergence and this leads to a contradiction. Let $v$ be the valuation centered at $p_{N}$ such that $\alpha_{M}\left(\pi_{\bullet} v\right)=t^{\prime}$, we can suppose that $v$ is divisorial up to shrinking $\left[t^{\prime}, t_{*}\right]$. Since $f_{\bullet}^{n} v \rightarrow v_{*}$, we have $Z_{v} \cdot \theta^{*}>0$. We have by Equation (84)

$$
\begin{align*}
& \frac{1}{\lambda_{1}^{k}}\left(f_{*}^{k} Z_{v}\right) \cdot E_{M}=\left(\theta_{*} \cdot E_{M}\right)\left(Z_{v} \cdot \theta^{*}\right)+O\left(\left(\frac{\lambda_{2}}{\lambda_{1}^{2}}\right)^{k / 2}\right)  \tag{362}\\
& \frac{1}{\lambda_{1}^{k}}\left(f_{*}^{k} Z_{v}\right) \cdot F_{M}=\left(\theta_{*} \cdot F_{M}\right)\left(Z_{v} \cdot \theta^{*}\right)+O\left(\left(\frac{\lambda_{2}}{\lambda_{1}^{2}}\right)^{k / 2}\right) . \tag{363}
\end{align*}
$$

Using Lemma 5.13 we get that

$$
\begin{equation*}
\left|M^{k}\left(\alpha_{M}\left(\pi_{\bullet} v\right)\right)-t_{*}\right|=\left|\frac{f_{*}^{k} Z_{v} \cdot F_{M}}{f_{*}^{k} Z_{v} \cdot E_{M}}-\frac{\theta_{*} \cdot F_{M}}{\theta_{*} \cdot E_{M}}\right|=O\left(\left(\frac{\lambda_{2}}{\lambda_{1}^{2}}\right)^{k / 2}\right) \tag{364}
\end{equation*}
$$

Therefore the speed of convergence is exponential and this shows that $\left|M^{\prime}\left(t_{*}\right)\right|<$ 1.

End of Proof of Proposition 12.12., By Lemma 12.13, pick $v_{1}, v_{2} \in \mathcal{V}_{X_{N}}\left(p_{N} ; E_{N}\right)$ divisorial sufficiently close to $v_{*}$ such that

$$
\begin{equation*}
\operatorname{ord}_{E_{N}}<_{N} v_{1}<_{N} v_{*}<_{N} v_{2}<_{N} v_{F_{N}} \tag{365}
\end{equation*}
$$

and

$$
\begin{equation*}
f_{\bullet}\left(\left[v_{1}, v_{2}\right]\right) \subset\left[\pi_{\bullet} v_{1}, \pi_{\bullet} v_{2}\right] . \tag{366}
\end{equation*}
$$

If $f_{\bullet}$ is order preserving, then we must have $\left.f_{\bullet}\left[v_{1}, v_{2}\right] \subset\right] \pi_{\bullet} v_{1}, \pi_{\bullet} v_{2}\left[\right.$. If $f_{\bullet}$ is not orderpreserving, it is possible to have $f_{\bullet}\left(v_{2}\right)=\pi_{\bullet} v_{1}$ and $\left.f_{\bullet}\left(v_{1}\right) \in\right] \pi_{\bullet} v_{1}, \pi_{\bullet} v_{2}[$. In that case, $f_{\bullet}^{2}$ is order-preserving and we have $\left.f_{\bullet}^{2}\left[v_{1}, v_{2}\right] \subset\right] \pi_{\bullet} v_{1}, \pi_{\bullet} \nu_{2}[$.

Let $U_{N}=\left\{v: v_{1}<v \wedge v_{F_{N}}<v_{2}\right\} \subset \mathcal{V}_{X_{N}}\left(p_{N} ; E_{N}\right)$. It is clear that $v_{F_{N}} \notin U_{N}$. Let $\psi \in \widehat{O_{X_{M}, p_{M}}}$ be such that $v_{\psi}>_{M} f_{\bullet}\left(\left[v_{1}, v_{2}\right]\right)$. Let $\psi_{1}, \cdots, \psi_{r} \in \widehat{O_{X_{N}, p_{N}}}$ be irreducible such that $f^{*} \psi=\psi_{1} \cdots \psi_{r}$. We can shrink the segment $\left[\nu_{1}, v_{2}\right]$ to make sure that none of the $\psi_{i}$ belong to $U_{N}$ (see Figure 4). If this is the case, then for all $\mu \in U_{N}$, set $\widetilde{\mu}=\mu \wedge \nu_{2}$, then for all i

$$
\begin{equation*}
\mu \wedge v_{\Psi_{i}}=\widetilde{\mu} \wedge v_{\Psi_{i}} \tag{367}
\end{equation*}
$$

and

$$
\begin{equation*}
\mu \wedge v_{F_{N}}=\tilde{\mu} \wedge v_{F_{N}} \tag{368}
\end{equation*}
$$

Now, for all $\mu \in U_{N}$, by Equation (360) and Proposition 5.3

$$
\begin{equation*}
\left(f_{\bullet} \mu\right)(\psi)=\frac{\mu\left(f^{*} \psi\right)}{a+b \mu(y)}=\frac{\sum_{k} \alpha_{N}\left(\mu \wedge v_{\psi_{k}} m\left(\psi_{k}\right)\right)}{a+b \mu(y)} \tag{369}
\end{equation*}
$$

By Equations (367) and (368), we get

$$
\begin{equation*}
\left(f_{\bullet} \mu\right)(\psi)=\left(f_{\bullet} \widetilde{\mu}\right)(\psi) \tag{370}
\end{equation*}
$$

This means that

$$
\begin{equation*}
\forall \mu \in U_{N}, \quad \alpha_{M}\left(\left(f_{\bullet} \mu\right) \wedge v_{\psi}\right)=\alpha_{M}\left(\left(f_{\bullet} \widetilde{\mu}\right) \wedge v_{\psi}\right) \tag{371}
\end{equation*}
$$

In particular, $f_{\bullet}\left(U_{N}\right) \Subset \pi_{\bullet}\left(U_{N}\right)$. So we can iterate $f_{\bullet}$ on $U_{N}$.
Proposition 12.14. For every $\mu \in U_{N}, f_{\bullet}^{n} \mu \rightarrow v_{*}$ for the weak topology.
Proof. Let $\mu \in U_{N}$ and let $\tilde{\mu}:=\mu \wedge \nu_{2}$. We have $f_{\bullet}^{n} \widetilde{\mu} \rightarrow v_{*}$ for the strong topology by Lemma 12.13. By equation (368), we have $f_{\bullet}^{n} \mu \wedge \nu_{2}=f_{\bullet}^{n} \widetilde{\mu} \wedge v_{2}$. Therefore for $\varphi \in O_{X_{N}, p_{N}}$ irreducible and for $n$ large enough, $f_{\bullet}^{n} \mu \wedge v_{\varphi}=f_{\bullet}^{n} \widetilde{\mu} \wedge v_{\varphi}$. Therefore,

$$
\begin{align*}
f_{\bullet}^{n} \mu(\varphi) & =\alpha_{N}\left(f_{\bullet}^{n} \mu \wedge v_{\varphi}\right) m_{N}(\varphi)  \tag{372}\\
& =\alpha_{N}\left(f_{\bullet}^{n} \widetilde{\mu} \wedge v_{\varphi}\right) m_{N}(\varphi)  \tag{373}\\
& =f_{\bullet}^{n} \widetilde{\mu}(\varphi) \xrightarrow[n \rightarrow+\infty]{ } v_{*}(\varphi) . \tag{374}
\end{align*}
$$



Figure 4. An $f_{\bullet}$-invariant open subset of $\mathcal{V}_{\infty}$, irrational case

Now pick a completion $X$ above $X_{N}$ such that for $i=1,2$, the center of $\mathrm{v}_{i}$ is a prime divisor $E_{i}$ at infinity such that $E_{1}$ and $E_{2}$ intersect at a unique point $p$. We have $c_{X}\left(v_{*}\right)=p$. The open set $U_{N} \subset \mathcal{V}_{X_{N}}\left(p_{N} ; E_{N}\right)$ is the image of $\mathcal{V}_{X}\left(p ; E_{1}\right)$ via the inclusion $\mathcal{V}_{X}\left(p ; E_{1}\right) \hookrightarrow \mathcal{V}_{X_{N}}\left(p_{N} ; E_{N}\right)$. Since $f_{\bullet} U_{N} \subset \pi_{\bullet}\left(U_{N}\right)$, this shows that $f_{*} \mathcal{V}_{X}(p) \subset \mathcal{V}_{X}(p)$. Therefore by Lemma 11.12 the lift $f: X \rightarrow X$ is defined at $p, f(p)=p$ and since $f_{\bullet}$ (or $f_{\bullet}^{2}$ ) contracts the segment $\left[v_{1}, v_{2}\right]$ we have that $f$ contracts $E_{1}$ and $E_{2}$ to $p$. We have for every $\mu \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right), f_{\bullet}^{n} \mu \rightarrow v_{*}$ by Proposition 12.14 .

If $Y$ is a completion above $X$, then $c_{Y}\left(v_{*}\right)=F_{1} \cap F_{2}$ where $F_{i}$ is a prime divisor at infinity because $v_{*}$ is irrational. The segment $\left[v_{F_{1}}, v_{F_{2}}\right.$ ] is a subsegment of $\left[v_{E_{1}}, v_{E_{2}}\right.$ ] and the same proof applies. This shows that $Y$ satisfies also Proposition 12.12.

### 12.3. Attractingness of $v_{*}$, the divisorial case

Here we also suppose that $f$ is tamely ramified. Suppose that $v_{*}$ is divisorial and let $X$ be a completion such that the center of $v_{*}$ on $X$ is a prime divisor $E$ at infinity. Since $f_{*} \operatorname{ord}_{E}=\lambda_{1} \operatorname{ord}_{E}$ we have that $f$ induces a dominant rational selfmap of $E$.

LEMMA 12.15. Either there exists an integer $N>0$ such that $f_{\mid E}^{N}$ admits a noncritical fixed point on $E$, or $E$ is an elliptic curve and $f_{\mid E}$ is a translation by a non-torsion element of $E$.

Proof. The rational transformation $f$ induces a rational selfmap on $E$. If $E$ is rational, then $E \simeq \mathbf{P}^{1}$ and $f_{\mid E}$ is given by a rational fraction $\frac{P(x)}{Q(x)}$ and therefore admits fixed points. It admits a noncritical fixed point if and only if $f_{\mid E}$ is not a rational fraction of the form $\frac{P_{1}\left(x^{p}\right)}{Q_{1}\left(x^{p}\right)}$ where $p=\operatorname{chark}$ that is if and only if $f_{\mid E}$ is separable. If $E$ is of general type, then some iterate of $f$ induces the identity on $E$. Finally, if $E$ is an elliptic curve, then $f_{\mid E}=t_{-b} \circ g$ where $g: E \rightarrow E$ is a homomorphism of elliptic curves and $t_{-b}$ is the translation by $-b$. We have $f_{\mid E}(p)=p \Leftrightarrow g(p)-p=b$. Thus $f_{\mid E}$ admits a fixed point if and only if $g-$ id is not the trivial homomorphism, i.e $g \neq \mathrm{id}$. Now, by [Sil09] III.5, there exists an invariant 1-form $\omega$ with no poles or zeros on $E$ such that $g^{*} \omega=a(g) \omega$ where $a(g) \in \mathbf{k}$. If $a(g) \neq 0$ then every fixed point of $f$ is non-critical, if $a(g)=0$, then every fixed point of $f_{\mid E}$ is critical and $f_{\mid E}$ is inseparable.

Suppose that $E \simeq \mathbf{P}^{1}$ or $E$ is an elliptic curve with $f_{\mid E}$ inseparable. We show that this is not possible if $f$ is tamely ramified. Pick a general free point $p$ on $E$, then $f$ is defined at $p$ and we can find local coordinates $(x, y)$ at $p$ and $(z, w)$ at $f(p)$ such that $x=0$ and $z=0$ is a local equation of $E$ at $p$ and $f(p)$ respectively and $f^{*} w$ is divisible by $y$. Thus $f$ is of the form

$$
\begin{equation*}
f(x, y)=\left(x^{\lambda_{1}} \psi(x, y), y^{m} \varphi(x, y)\right) \tag{375}
\end{equation*}
$$

with $\psi, \varphi$ invertible regular functions and $m$ is an integer divisible by chark. Indeed, $E$ is $f$-invariant and $y=0$ is the local equation of an algebraic curve $C$ such that $C \cap X_{0} \neq$ $\varnothing$ so it can't be sent to $E$ by $f$ because $f$ is an endomorphism. Let $C_{1}$ be the curve $f(C)$ with its reduced structure, then we have $f_{*} \operatorname{ord}_{C}=m \operatorname{ord}_{C_{1}}$ and this contradicts the fact that $f$ is tamely ramified because chark|m.

In the case where $f_{\mid E}$ is not a translation by a non-torsion element on an elliptic curve, $f$ defines a regular fixed point germ at $p$ and we can proceed as in [FJ07] §5.2 to show that there exists a completion $X$ that contains a prime divisor $E_{0}$ at infinity such that $p=E \cap E_{0}$ and $f_{\bullet}$ maps the segment of monomial valuations [ $v_{E}, \nu_{E_{0}}$ ] strictly into itself. Here is how to proceed.

Set $X_{0}=X, p_{0}=p$. Define the sequence of completions $\left(X_{n}\right)$ as follows: $\pi_{n}$ : $X_{n+1} \rightarrow X_{n}$ is the blow up of $X_{n}$ at $p_{n}$ and $p_{n+1}$ is the intersection point of the strict transform of $E$ with the exceptional divisor of $\pi_{n+1}$. We still denote by $E$ its strict transform in every $X_{n}$. For every $n$, we have $f_{\mid E}\left(p_{n}\right)=p_{n}$ and if $f: X_{n} \rightarrow X$ is defined at $p_{n}$, we have $f\left(p_{n}\right)=p$. We apply Theorem 2.9 to get

Proposition 12.16. There exists integers $N \geqslant M$ such that the lift $f: X_{N} \rightarrow X_{M}$ is defined at $p_{N}, f\left(p_{N}\right)=p_{M}$. Furthermore, there exists local coordinates $(x, y),(z, w)$ respectively at $p_{N}, p_{M}$ such that $x=0$ and $z=0$ are local equations of the strict transform of $E$ in $X_{N}$ and $X_{M}$ respectively and $f$ is monomial in these coordinates.

The proof is the same as in Proposition 12.11.

Proposition 12.17. If $v_{*}$ is divisorial, there exists a completion $X$ such that
(1) $c_{X}\left(\nu_{*}\right)$ is a prime divisor $E$ at infinity.
(2) $E$ intersects another prime divisor $E_{0}$ at infinity and we set $p=E \cap E_{0}$.
(3) Up to replacing $f$ by an iterate, $f: X \rightarrow X$ is defined at $p, f(p)=p$.
(4) $p$ is a noncritical fixed point of $f_{\mid E}$.
(5) $f$ leaves $E$ invariant and contracts $E_{0}$ to $p$.
(6) Define $f_{\bullet}: V_{X}(p ; E) \rightarrow \mathcal{V}_{X}(p ; E)$, then for all $\mu \in c V_{X}(p ; E), f_{\bullet}^{n} \mu \rightarrow \operatorname{ord}_{E}$ for the weak topology.
If $\pi:(Y, \operatorname{Exc}(\pi)) \rightarrow(X, p)$ is a completion exceptional above $p$, then all the item above remain true in $Y$.

Proof. Let $N \geqslant M$ be as in Proposition 12.16. Let $F: X_{N} \rightarrow X_{M}$ be the lift of $f$. We can suppose that $N \geqslant M$ and denote by $\pi: X_{N} \rightarrow X_{M}$ the morphism of completions. We therefore have a map $f_{\bullet}: \mathcal{V}_{Y}\left(p_{N}, E\right) \rightarrow \mathcal{V}_{X}\left(p_{M}, E\right)$. Again, the tree $\mathcal{V}_{Y}\left(p_{N}, E\right)$ is a subtree via the map $\pi_{\bullet}$ and they are both rooted at the divisorial valuation $\operatorname{ord}_{E}$.

Let $(x, y),(z, w)$ be the local coordinates at $p_{N}$ and $p_{M}$ respectively given by Proposition 12.16. We have that $x=0$ is a local equation of $E$ in $X_{N}$ and $z=0$ is a local equation of $E$ in $X_{M}$.

$$
\begin{equation*}
f(x, y)=\left(x^{a} y^{b}, x^{c} y^{d}\right) \tag{376}
\end{equation*}
$$

Since we know that $E$ is not contracted by $f$ we actually have $c=0$. We can therefore write

$$
\begin{equation*}
\forall v \in \mathcal{V}_{X_{N}}\left(p_{N} ; E\right), \quad f_{\bullet}(v)=\frac{f_{*} v}{a+b v(y)} \tag{377}
\end{equation*}
$$

(Recall from Chapter5 that $\mathcal{V}_{X_{N}}\left(p_{n} ; E\right)$ is defined by the normalization $v(E)=1$ ). We have

$$
\begin{equation*}
f_{\bullet}\left[\operatorname{ord}_{E}, v_{y}\right] \subset\left[\operatorname{ord}_{E}, v_{w}\right] \tag{378}
\end{equation*}
$$

and the map is given by the following formula

$$
\begin{equation*}
f_{\bullet} v_{1, s}=v_{1, \frac{s d}{a+s b}} . \tag{379}
\end{equation*}
$$

As in the irrational case, we can consider the matrix $M_{f}$ and $M_{\pi}$ and study the type of the Möbius transformation induced by $M_{\pi}^{-1} \circ M_{f}$. Since $\operatorname{ord}_{E}$ is a fixed point, we show that it is not repelling on the segment $\left[\operatorname{ord}_{E}, \vee_{y}\right]$.

Let $v_{0} \in\left[\operatorname{ord}_{E}, \mathrm{v}_{w}\right]$ be a divisorial valuation. We have $f_{\bullet}\left(\left[\operatorname{ord}_{E}, v_{0}\right]\right) \subset\left[\operatorname{ord}_{E}, v_{w}\right]$. Let $U_{N}=\left\{\mu: \operatorname{ord}_{E} \leqslant \mu \wedge v_{y}<v_{0}\right\} \subset \mathcal{V}_{X_{N}}\left(p_{N} ; E\right)$. It is clear that $v_{y} \notin U_{N}$. Let $\psi \in$ $\widehat{O_{X_{M}, p_{M}}}$ be irreducible such that $v_{\psi}>f_{\bullet}\left(\left[\operatorname{ord}_{E}, v_{0}\right]\right)$. Let $\psi_{1}, \cdots, \psi_{r}, \in \widehat{O}_{X_{N}, p_{N}}$ be irreducible such that $f^{*} \psi=\psi_{1} \cdots \psi_{r}$. Up to shrinking the segment $\left[\operatorname{ord}_{E}, \nu_{0}\right.$ ] we can suppose that none of the $v_{\psi_{i}}$ belong to $U_{N}$ (See Figure 5). If this is the case, then for all $\mu \in U_{N}$, set $\widetilde{\mu}=\mu \wedge v_{0}$, then for all i

$$
\begin{equation*}
\mu \wedge v_{\psi_{i}}=\tilde{\mu} \wedge v_{\psi_{i}}, \quad \mu \wedge v_{y}=\tilde{\mu} \wedge v_{y} . \tag{380}
\end{equation*}
$$

Now, for all $\mu \in U_{N}$, by Equation (377) and Proposition 5.3

$$
\begin{equation*}
\left(f_{\bullet} \mu\right)(\psi)=\frac{\mu\left(f^{*} \psi\right)}{a+b \mu(y)}=\frac{\sum_{k} \alpha_{N}\left(\mu \wedge v_{\psi_{k}}\right) m\left(\psi_{k}\right)}{a+b \mu(y)} . \tag{381}
\end{equation*}
$$

By Equation (380), we get

$$
\begin{equation*}
\left(f_{\bullet} \mu\right)(\psi)=\left(f_{\bullet} \widetilde{\mu}\right)(\psi) \tag{382}
\end{equation*}
$$

This means that

$$
\begin{equation*}
\forall \mu \in U_{N}, \quad \alpha_{M}\left(\left(f_{\bullet} \mu\right) \wedge v_{\psi}\right)=\alpha_{M}\left(\left(f_{\bullet} \widetilde{\mu}\right) \wedge v_{\psi}\right) \tag{383}
\end{equation*}
$$

If $v \in \mathcal{V}_{\infty}$ is divisorial such that $Z_{v} \cdot \theta^{*}>0$, then $\frac{1}{\lambda_{1}^{n}} f_{*}^{n} v \rightarrow v_{*}$ by Proposition 12.2. Then, there exists $N_{0} \geqslant 1$ such that for $n \geqslant N_{0}, \frac{1}{\lambda_{1}^{n}} f_{*}^{n} v \in U_{N}$. Replace $v$ by $\frac{1}{\lambda_{1}^{N_{0}}} f_{*}^{N_{0}}(v)$. If $\operatorname{ord}_{E}$ was a repelling fixed point, then we could not have $f_{\bullet}^{n} v \rightarrow v_{*}$ by Equation (380) and (383). Therefore, we can pick $v_{0}$ such that $f_{\bullet}\left[\operatorname{ord}_{E}, v_{0}\right] \Subset \pi_{\bullet}\left[\operatorname{ord}_{E}, v_{0}\right]$. In that case $f_{\bullet}\left(U_{N}\right) \Subset \pi_{\bullet}\left(U_{N}\right)$. So we can iterate $f_{\bullet}$ on $U_{N}$.


Figure 5. An $f_{\bullet}$-invariant open subset of $\mathcal{V}_{\infty}$, divisorial case

PROPOSITION 12.18. For all $\mu \in U_{N}, f_{\bullet}^{n} \mu \rightarrow \operatorname{ord}_{E}$ for the weak topology.
Proof. The proof is similar to the proof of Proposition 12.14. Let $\mu \in U_{N}$ and set $\tilde{\mu}=\mu \wedge v_{0}$. Since $\operatorname{ord}_{E}$ is an attracting fixed point for $f_{\bullet}$ and $f_{\bullet}\left[\operatorname{ord}_{E}, v_{0}\right] \Subset\left[\operatorname{ord}_{E}, v_{0}\right]$, we have $f_{\bullet}{ }^{n} \widetilde{\mu} \rightarrow \operatorname{ord}_{E}$ for the strong topology. Then, by Equation (383), $f_{\bullet}^{n} \mu \wedge v_{0}=f_{\bullet}^{n} \widetilde{\mu}$. Let $\varphi \in O_{X_{N}, p_{N}}$ be irreducible such that $\varphi$ is not a local equation of $E$, then for $n$ large
enough

$$
\begin{align*}
f_{\bullet}^{n} \mu(\varphi) & =\alpha_{E}\left(f_{\bullet}^{n} \mu \wedge \nu_{\varphi}\right) m_{E}(\varphi)  \tag{384}\\
& =\alpha_{E}\left(f_{\bullet}^{n} \widetilde{\mu} \wedge v_{\varphi}\right) m_{E}(\varphi)  \tag{385}\\
& =\alpha_{E}\left(f_{\bullet}^{n} \widetilde{\mu}\right) m_{E}(\varphi) \xrightarrow[n \rightarrow+\infty]{ } 0 \tag{386}
\end{align*}
$$

Let $E_{0}$ be the divisor associated to the divisorial valuation $v_{0}$ and let $Z$ be a completion such that $c_{Z}\left(v_{0}\right)$ is the divisor $E_{0}$ and such that $E_{0} \cap E$ is a point $p$. Then, the open subset $U_{N}$ corresponds to $\mathcal{V}_{Z}(p)$ and we have $f_{*} \mathcal{V}_{Z}(p) \subset \mathcal{V}_{Z}(p)$. By Lemma 11.12, we have that the lift $\widehat{f}: Z \rightarrow Z$ is regular at $p, \widehat{f}(p)=p$ and since we know that $f_{\bullet} v_{0}<v_{0}$ and $f_{*} \operatorname{ord}_{E}=\lambda_{1}(f) \operatorname{ord}_{E}$ we have that $\widehat{f}$ contracts $E_{0}$ at $p, E$ is $f$-invariant and for all $\mu \in \mathcal{V}_{Z}(p ; E), f_{\bullet}^{n} \mu \rightarrow v_{*}$ by Proposition 12.18 .

If $\pi:\left(Z^{\prime}, \operatorname{Exc}(\pi)\right) \rightarrow(Z, p)$ is a completion exceptional above $p$, then $\operatorname{Exc}(\pi)$ is a tree of rational curves, let $E_{0}^{\prime}$ be the irreducible component of $\operatorname{Exc}(\pi)$ that intersect the strict transform of $E$. Then $E_{0}^{\prime}$ corresponds to a divisorial valuation $v_{0}^{\prime}$ such that $\operatorname{ord}_{E}=v_{*}<v_{0}^{\prime}<v_{0}$ and all the proofs above apply so Proposition 12.17 holds also for $Z^{\prime}$.

Lemma 12.19. When $v_{*}$ is divisorial, $\lambda_{1} \leqslant \lambda_{2}$, with equality if and only if $f_{\mid E}$ : $E \rightarrow E$ has degree 1 .

Proof. Let $X$ be a completion such that the center of $v_{*}$ is a prime divisor $E$ at infinity. Since $f_{*} \nu_{*}=\lambda_{1} \nu_{*}$, we have that $f^{*} E=\lambda_{1} E+R$ where $R$ denotes an effective divisor supported at infinity. Now, we also have $f_{*} E=d E+R^{\prime}$. From the equality $f_{*} \circ f^{*}=\lambda_{2}$ id, we get that $\lambda_{1} d \leqslant \lambda_{2}$. In particular, $\lambda_{1} \leqslant \lambda_{2}$.

### 12.4. Local normal form of $f$

We are now ready to proof Theorem 12.1 .
Proof of Theorem 12.1. Suppose $v_{*}$ is infinitely singular. From Proposition 12.3, there exists a completion $X$ such that $c_{X}\left(v_{*}\right)=: p \in E$ is a free point, $f: X \rightarrow X$ is defined at $p$ and $f_{*}\left(\mathcal{V}_{X}(p)\right) \Subset \mathcal{V}_{X}(p)$. We need to show that the germ of holomorphic functions induced by $f$ at $p$ is contracting and rigid. It is clear that $E \subset \operatorname{Crit}(f)$ (Recall the notations from $\S(2.5)$. If $\operatorname{Crit}(f)$ admits another irreducible component, it induces a curve valuation in $\mathcal{V}_{X}(p)$, we can blow up $p$ to get another completion above $X$ satisfying Proposition 12.3 such that $\mathcal{V}_{X}(p)$ does not contain any curve valuation associated an irreducible component of $\operatorname{Crit}(f)$. Thus, $f$ is rigid at $p$ it remains to show that it is contracting. Let $(x, y)$ be local coordinates at $p$ such that $x=0$ is a local equation of $E$. We must have that $f^{*} x=x^{a} \varphi$ with $a \geqslant 1$ and $\varphi \in O_{X, p}^{\times}$because no other germs of curve
is contracted to $p$ or sent to $E$ since $f$ is an endomorphism of $X_{0}$. Since $v_{*}(E)>0$ and $f_{*} \nu_{*}=\lambda_{1} \nu_{*}$ we get that

$$
\begin{equation*}
\lambda_{1} v_{*}(x)=f_{*} v_{*}(x)=v_{*}\left(x^{a} \varphi\right)=a v_{*}(x) . \tag{387}
\end{equation*}
$$

Thus, $\lambda_{1}=a$ is an integer. Now, since $E$ is contracted by $f$, we get that $f^{*} y=x \psi$ with $\psi \in O_{X, p}$ but we must have $\psi \in \mathfrak{m}_{p}$ because the image of the curve $y=0$ is a curve that contains $p$. Hence, we get that

$$
\begin{equation*}
f(x, y)=\left(x^{\lambda_{1}} \varphi, x \psi\right) \tag{388}
\end{equation*}
$$

with $\varphi \in O_{X, p}^{\times}$and $\psi \in \mathfrak{m}_{p}$. Consider the norm $\|(x, y)| |=\max (|x|,|y|)$ associated to the coordinates $x, y$ and let $U^{*}$ be the ball of center $p$ and radius $\varepsilon>0$. If $\varepsilon>0$ is small enough, then $U^{*}$ is $f$-invariant and $f\left(U^{*}\right) \Subset U^{*}$, so $f$ is contracting at $p$. Finally, there are no $f$-invariant germ of curves at $p$. Indeed, if $\varphi \in \widehat{O_{X, p}}$ is $f$-invariant, then $f_{\bullet} v_{\varphi}=v_{\varphi}$. But we have by Proposition 12.3 that $f_{\bullet}^{n} v_{\varphi} \rightarrow v_{*}$ and this is a contradiction. Thus, we get that $f$ has the local normal form of (17) with $a=\lambda_{1}$. If $\mathbf{k}=\mathbf{C}$, Looking at the classification of the rigid contracting germs in dimension 2, we see that $f$ is in Class 4 of Table 1 in [Fav00] hence of type (18) thus there exists local analytic coordinates $(z, w)$ at $p$

$$
\begin{equation*}
\widehat{f}(z, w)=\left(z^{a}, \lambda z^{c} w+P(z)\right) \tag{389}
\end{equation*}
$$

where $a \geqslant 2, c \geqslant 1, \lambda \in \mathbf{C}^{\times}$and $P$ is a polynomial such that $P(0)=0$. Since $E$ is the only germ of curve contracted by $f$ (all the other germs of analytic curves are contained in $X_{0}$ they cannot be contracted to $p$ by $f$ since $f$ is an endomorphism of $X_{0}$ ), we have that $z=0$ is a local equation of $E$. Furthermore, since $f$ does not have any invariant germ of analytic curve, we get that $P \not \equiv 0$.

Suppose now that $v_{*}$ is irrational, by Proposition 12.12, there exists a completion $X$ of $X_{0}$ such that the lift $f: X \rightarrow X$ is defined at $p=c_{X}\left(v_{*}\right), X$ contains two divisors at infinity $E, F$ such that $p=E \cap F$ and $\hat{f}$ contracts both $E$ and $F$ at $p$. It remains to show that $f$ is contracting and rigid at $p$. First we can suppose up to further blow ups that $\operatorname{Crit}(f) \cap X_{0}=\varnothing$. Therefore $f$ is rigid, now since both $E, F$ are contracted to $p, f$ is contracting. Finally, there are no $f$-invariant germs of curves at $p$ since for all $\mu \in \mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right), f_{\bullet}^{n} \mu \rightarrow v_{*}$ by Proposition 12.12 . Let $(z, w)$ be local coordinates at $p$ associated to $(E, F)$. We have that $f$ is of the pseudomonomial form

$$
\begin{equation*}
f(z, w)=\left(z^{a} w^{b} \varphi, z^{c} w^{d} \psi\right) \tag{390}
\end{equation*}
$$

with $\varphi, \psi \in O_{X, p}^{\times}$and $a, b, c, d \geqslant 0$. Notice that $f_{*} \operatorname{ord}_{E}=v_{a, b}$ and $f_{*} \operatorname{ord}_{F}=v_{c, d}$. Consider the segment of monomial valuations $I$ centered at $p$ inside $\mathcal{V}_{X}\left(p ; \mathfrak{m}_{p}\right)$ we have that $f_{\bullet}: I \rightarrow I$ is injective, therefore $(a, b)$ is not proportional to $(c, d)$ and $a d-b c \neq 0$. We show that in fact $a d-b c$ is not divisible by chark. Otherwise, there would be positive integers $s, t$ such that $f_{*} v_{s, t}=p v_{s^{\prime}, t^{\prime}}$ and this contradicts the fact that $f$ is tamely
ramified because the value group of $f_{*} v_{s, t}$ in the value group of $v_{s^{\prime}, t^{\prime}}$ has index divisible by $p$. Thus, the normal form of $f$ at $p$ is analytically conjugated to the monomial form

$$
\begin{equation*}
f(z, w)=\left(z^{a} w^{b}, z^{c} w^{d}\right) . \tag{391}
\end{equation*}
$$

Furthermore the open subset $U^{*}$ corresponding to the ball of radius $\varepsilon>0$ is $f$-invariant for $\varepsilon>0$ small enough and $f\left(U^{*}\right) \Subset U^{*}$. In that case, we show that $\lambda_{1}(f)$ is the spectral radius of the invertible matrix $A=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$, hence a Perron number of degree 2. Indeed, $\nu_{*}=v_{s, t}$ where $(s, t)$ is an eigenvector of $A$ for the eigenvalue $\lambda_{1}$. Since $v_{*}$ is irrational, we have $s / t \notin \mathbf{Q}$ and therefore $\lambda_{1} \notin \mathbf{Q}$. Now, when we iterate $f$, we get that $f^{n}$ has a monomial form at $p$ given by the matrix $A^{n}$, hence we get

$$
\begin{equation*}
\lambda_{1}^{n}\binom{\mathrm{v}_{*}(z)}{\mathrm{v}_{*}(w)}=A^{n}\binom{s}{t} \tag{392}
\end{equation*}
$$

Now finally, suppose that $\nu_{*}$ is divisorial. Take a completion $X$ as in Proposition 12.17. Let $p=E \cap E_{0}$ with $v_{*}=\operatorname{ord}_{E}$. The lift $f: X \rightarrow X$ is defined at $p$. Up to further blow-ups we can suppose that $\operatorname{Crit}(f) \cap X_{0}=\varnothing$. Therefore, $\operatorname{Crit}(f) \subset E \cup E_{0}$ which is totally invariant as $f_{*} \mathcal{V}_{X}(p) \Subset \mathcal{V}_{X}(p)$ so $f$ is rigid at $p$. There are no $f$-invariant germs of curves apart from $E$ at $p$ since for all $\mu \in \mathcal{V}_{X}(p ; E), f_{\bullet}^{n} \mu \rightarrow \operatorname{ord}_{E}$ by Proposition 12.17. Let $(x, y)$ be local coordinates at $p$ associated to $\left(E, E_{0}\right)$. Since $f_{*} \operatorname{ord}_{E}=\lambda_{1} \operatorname{ord}_{E}$ with $\lambda_{1} \geqslant 2$ we have $f^{*} x=x^{\lambda_{1}} \varphi$ with $\varphi \in O_{X, p}$. Since no germ of curve is sent to $E$ apart from $E_{0}$, we have that up to multiplying $x$ by a constant that $f^{*} x=x^{\lambda_{1}} y^{b}(1+\varphi)$ with $\varphi \in O_{X, p}$. Then, $E_{0}$ is contracted to $p$ so $f^{*} y=y^{c} \psi$ with $\psi \in O_{X, p}^{\times}$and $c=1$ since $p$ is a noncritical fixed point of $f_{\mid E}$. Hence, in these coordinates the local normal form of $f$ is (21):

$$
\begin{equation*}
\widehat{f}(x, y)=\left(x^{a} y^{b}(1+\varphi), \lambda y(1+\psi)\right) \tag{393}
\end{equation*}
$$

with $a=\lambda_{1} \geqslant 2, b \geqslant 1, \lambda \in \mathbf{C}^{\times}$and $\varphi(0)=\psi(0)=0$.

## CHAPTER 13

## Examples

### 13.1. An affine surface with a lot of nonproper endomorphisms

### 13.1.1. A family of rational affine surface with no loxodromic automorphisms.

 In [Dub04] Example 2.23, Dubouloz gives an infinite family of examples of rational complex affine surfaces that admit a minimal completion for which the dual graph of the curve at infinity is neither a zigzag nor a cycle. This means by Theorem 14.4 that these surfaces do not admit loxodromic automorphism. The result is the following: Consider the affine surface $S_{0} \subset \mathbf{A}_{\mathbf{C}}^{3}$ given by the equation$$
\begin{equation*}
x^{n} y=P(z) \tag{394}
\end{equation*}
$$

where $n \geqslant 2$ and $P$ is a degree $r$ polynomial with $r \geqslant 2$ distinct roots. Then, $S_{0}$ admits a minimial completion for which the dual graph at infinity is given by

where $\square$ is a zigzag of $(-2)$-curves of length $n-3$ if $n \geqslant 3$ and $\square=\varnothing$ otherwise.
13.1.2. A subfamily with a lot of endomorphisms. In [DP18] §5A, Dubouloz and Palka study the following family of surfaces

$$
\begin{equation*}
S(n):=\left\{x^{n} y=z^{n}-1\right\} \quad(n \geqslant 2) . \tag{396}
\end{equation*}
$$

They fall inside the previous category of affine surfaces; $S(n)$ admits a $\mathbf{Z} / n \mathbf{Z}$ action given by

$$
\begin{equation*}
\forall a \in \mathbf{Z} / n \mathbf{Z}, \quad a \cdot(x, y, z)=\left(\varepsilon^{a} x, y, \varepsilon^{-a} z\right) \tag{397}
\end{equation*}
$$

where $\varepsilon$ is a primitive $n$-th root of unity. The quotient $S(n) /(\mathbf{Z} / n \mathbf{Z})$ is an affine surface $S^{\prime}(n)$ of equation

$$
\begin{equation*}
S^{\prime}(n)=\left\{u(1+u v)=w^{n}\right\} \tag{398}
\end{equation*}
$$

and the quotient map $\pi: S(n) \rightarrow S^{\prime}(n)$ is given by

$$
\begin{equation*}
\pi(x, y, z)=\left(x^{n}, y, x z\right) \tag{399}
\end{equation*}
$$

We have the surprising result

Proposition 13.1. For every $n \geqslant 2$ the affine surface $S^{\prime}(n)$ admits a strict open embedding into $S(n)$ given by the following formula

$$
\begin{equation*}
j(u, v, w)=\left(w, v R_{0}(-u v), R_{1}(-u v)\right) \tag{400}
\end{equation*}
$$

where $R_{1}(t)=(\varepsilon-1) t+1$ and $R_{0}(t)=\frac{R_{1}(t)^{n}-1}{t(t-1)} \in \mathbf{C}[t]$ where $\varepsilon \neq 1$ is an $n$-th root of unity. Different choices of $\varepsilon$ lead to different embeddings that are not conjugated by the $\mathbf{Z} / n \mathbf{Z}$ action over $S(n)$.

Hence we can define the endomorphism $f: S(n) \rightarrow S(n)$ defined by $f=j \circ \pi$. This yields a nonproper endomorphism of $S(n)$ of topological degree $n$. We can twist this example using the following result

Proposition 13.2. Let $n \geqslant 2$ be an integer. Every polynomial $P \in \mathbf{C}[x]$ yields an automorphism $g_{P}$ of $S(n)$ defined by

$$
\begin{equation*}
g_{P}(x, y, z):=\left(x, y+\frac{\left(z+P(x) x^{n}\right)^{n}-z^{n}}{x^{n}}, z+P(x) x^{n}\right) \tag{401}
\end{equation*}
$$

13.1.3. The surface $S(2)$. We treat in details the example of $S(2)=\left\{x^{2} y=z^{2}-1\right\}$. The $\mathbf{Z} / 2 / \mathbf{Z}$ action is given by $(-1) \cdot(x, y, z)=(-x, y,-z)$. To find a minimal completion of $S(2)$ we follow the computations of [Dub04] Example 2.23. Consider the birational morphism

$$
\begin{equation*}
\varphi:(x, y, z) \in S(2) \mapsto(x, z) \in \mathbf{A}^{2} \subset \mathbf{P}^{1} \times \mathbf{P}^{1} \tag{402}
\end{equation*}
$$

Define the following curves in $S(2), C_{\varepsilon}=\{x=0, z=\varepsilon\}$ where $\varepsilon= \pm 1, F_{0}:=\{0\} \times$ $\mathbf{P}^{1}, F_{\infty}=\{\infty\} \times \mathbf{P}^{1}$ and $L=\mathbf{P}^{1} \times\{\infty\}$, then

$$
\begin{equation*}
\varphi: S(2) \backslash\left(C_{1} \cup C_{-1}\right) \rightarrow \mathbf{P}^{1} \times \mathbf{P}^{1} \backslash\left(F_{0} \cup F_{\infty} \cup L\right) \tag{403}
\end{equation*}
$$

is an isomorphism with inverse given by

$$
\begin{equation*}
\varphi^{-1}(u, v)=\left(u, \frac{v^{2}-1}{u^{2}}, v\right) . \tag{404}
\end{equation*}
$$

The curve $C_{\varepsilon}$ is contracted by $\varphi$ to $(0, \varepsilon) \in F_{0}$. Let $F_{\varepsilon}$ be the exceptional divisor above $(0, \varepsilon)$. The lift of $\varphi$ contract $C_{\varepsilon}$ to a free point on $F_{\varepsilon}$ that we call $p_{\varepsilon}$. Let $X$ be the blow up of $p_{1}$ and $p_{-1}$, then $\varphi$ induces an open embedding $\varphi: S(2) \hookrightarrow X$ as $C_{\varepsilon}$ is sent by $\varphi$ to the exceptional divisor above $p_{\varepsilon}$. Hence, $X$ is a completion of $S(2)$ and the dual graph of the boundary is


Here, we stil denote by $F_{\infty}, F_{0}, L$ their strict transform in $X$. In particular, $F_{\infty}$ is not linearly equivalent to $F_{0}$ but to $F_{0}+F_{1}+F_{-1}+\varphi\left(C_{1}\right)+\varphi\left(C_{-1}\right)$ which is the strict transform of the "original" $F_{0}=\{0\} \times \mathbf{P}^{1} \subset \mathbf{P}^{1} \times \mathbf{P}^{1}$.

Proposition 13.3. The surface $S(2)$ satisfies $\mathrm{QAlb}(S(2))=0$. For every endomorphism $f$ of $S(2)$ such that $\lambda_{1}(f)^{2}>\lambda_{2}(f)$, the eigenvaluation $v_{*}$ of Theorem 11.16 satisfies $c_{X}\left(v_{*}\right) \in L$.

Proof. Since $S(2)$ is birational to $\mathbf{A}^{2}$ we have that it is a rational surface, hence $\operatorname{Pic}^{0}(S(2))=0$. It suffices to show that $S(2)$ does not admit any nonconstant invertible regular function. To do so, we consider the intersection form on $\operatorname{Div}_{\infty}(X)=\mathbf{Z} F_{\infty} \oplus$ $\mathbf{Z} L \oplus \mathbf{Z} F_{0} \oplus \mathbf{Z} F_{1} \oplus \mathbf{Z} F_{-1}$. It suffices to show that it is non degenerate. The matrix of the intersection form in the basis ( $F_{\infty}, L, F_{0}, F_{1}, F_{-1}$ ) is given by

$$
M=\left(\begin{array}{ccccc}
0 & 1 & 0 & 0 & 0  \tag{406}\\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & -2 & 1 & 1 \\
0 & 0 & 1 & -2 & 0 \\
0 & 0 & 1 & 0 & -2
\end{array}\right)
$$

It is inversible with inverse given by

$$
M^{-1}=\frac{1}{4}\left(\begin{array}{ccccc}
-4 & 4 & 4 & 2 & 2  \tag{407}\\
4 & 0 & 0 & 0 & 0 \\
4 & 0 & -4 & -2 & -2 \\
2 & 0 & -2 & -3 & -1 \\
2 & 0 & -2 & -1 & -3
\end{array}\right)
$$

Hence the intersection form in nondegenerate on $\operatorname{Div}_{\infty}(X)$ which shows that $\mathrm{QAlb}(S(2))=$ 0.

Therefore, we are in the condition of Theorem 11.16. Let $f$ be a dominant endomorphism with $\lambda_{1}(f)^{2}>\lambda_{2}(f)$. Let $v_{*}$ be its eigenvaluation. Then, the invariant class $\theta_{*} \in \mathbb{L}^{2}(S(2))$ is of the form

$$
\begin{equation*}
\theta_{*}=w+\mu Z_{v_{*}} . \tag{408}
\end{equation*}
$$

with $w \in \operatorname{Pic}(X) \cap \operatorname{Div}_{\infty}(X)^{\perp}$. Therefore, we must have $\left(Z_{v_{*}}\right)^{2} \geqslant 0$ as $w^{2} \leqslant 0$ and since $\theta_{*}$ is nef, we have $\left(\theta_{*}\right)^{2} \leqslant\left(\theta_{*, Y}\right)^{2}$ for every completion $Y$. This implies that $\left(Z_{v_{*}, X}\right)^{2} \geqslant 0$.

Now, for all prime divisor $E$ of $X$ at infinity, $Z_{\text {ord }}$ is given by a column of $M^{-1}$. Indeed, $M$ is the matrix of the intersection form in the basis ( $F_{\infty}, L, F_{0}, F_{1}, F_{-1}$ ) and therefore $M^{-1}$ is the matrix of the intersection form in the dual basis $\left(Z_{\operatorname{ord}_{F_{\infty}}}, Z_{\text {ord }_{L}}, Z_{\operatorname{ord}_{F_{0}}}, Z_{\operatorname{ord}_{F_{1}}}, Z_{\operatorname{ord}_{F_{-1}}}\right)$. For example $Z_{\text {ord }_{L}}=F_{\infty}$ and $Z_{\text {ord }_{F_{\infty}}}=-F_{\infty}+L+F_{0}+\frac{1}{2} F_{1}+\frac{1}{2} F_{-1}$. In particular, we have that $L$ is the unique prime divisor at infinity of $X$ such that $Z_{\text {ord }_{L}}^{2} \geqslant 0$. This implies that $c_{X}\left(\mathrm{v}_{*}\right)$ cannot be a free point on a prime divisor $E \neq L$ otherwise we would get $\left(Z_{v_{*}, X}\right)^{2}<0$. If $c_{X}\left(v_{*}\right)$ is a satellite point, then it cannot be $F_{0} \cap F_{\varepsilon}$ because in that case

$$
\begin{equation*}
Z_{v_{*}, X}=\lambda Z_{\operatorname{ord}_{F_{0}}}+\mu Z_{\operatorname{ord}_{F_{\varepsilon}}} \tag{409}
\end{equation*}
$$

with $\lambda, \mu>0$ and looking at the last three rows and colums of $M^{-1}$ we would get $\left(Z_{v_{*}, X}\right)^{2}<0$. Hence $c_{X}\left(v_{*}\right) \in L$ or $c_{X}\left(v_{*}\right)=L$.

First example of endomorphism.- The endomorphism $f=j \circ \pi$ is equal to

$$
\begin{equation*}
f(x, y, z)=\left(x z, 4 y, 2 z^{2}-1\right) . \tag{410}
\end{equation*}
$$

Using the map $\varphi: S(2) \rightarrow \mathbf{A}^{2}$ from Equation (402), we have that $f$ is conjugated to

$$
\begin{equation*}
\eta:(u, v) \in \mathbf{A}^{2} \mapsto\left(u v, 2 v^{2}-1\right) \in \mathbf{A}^{2} \tag{411}
\end{equation*}
$$

Hence we get that $\lambda_{1}(f)=\lambda_{2}(f)=2$. Consider the completion $X$ of $S(2)$ defined above with dual graph given by Equation (405). We know that the eigenvaluation $\boldsymbol{v}_{*}$ of $f$ must be centered on $L=\mathbf{P}^{1} \times\{\infty\}$. Therefore, we can study the local dynamics of $f$ on $L$ using $\eta$. Let $[U: T],[V: W]$ be the homogenous coordinates of $\mathbf{P}^{1} \times \mathbf{P}^{1}$ such that $u=\frac{U}{T}$ and $v=\frac{V}{W}$. In homogenous coordinates we get

$$
\begin{equation*}
\eta([U: T],[V: W])=\left([U V: T W],\left[2 V^{2}-W^{2}: W^{2}\right]\right) . \tag{412}
\end{equation*}
$$

Consider the affine coordinates $t=\frac{T}{U}$ and $w=\frac{W}{V}$. In particular, $t=0$ is a local equation of $F_{\infty}$ and $w=0$ is a local equation of $L$. Then, in these coordinates we have

$$
\begin{equation*}
\eta(t, w)=\left(t w, \frac{w^{2}}{2-w^{2}}\right) . \tag{413}
\end{equation*}
$$

Hence, we get that $(0,0)=F_{\infty} \cap L$ is a fixed point. From $\eta^{*} t=t w$ we infer $f_{*} \operatorname{ord}_{F_{\infty}}=\operatorname{ord}_{F_{\infty}}$. Hence, $\operatorname{ord}_{F_{\infty}}$ is not the eigenvaluation of $f$ because $\lambda_{1}(f)=2$. We have that $L$ is contracted to $(0,0)$ so $v_{*}$ must be centered at $(0,0)$. We blow up $(0,0)=F_{\infty} \cap L$. Let $\widetilde{E}$ be the exceptional divisor and let $s, s^{\prime}$ be local coordinates at $\widetilde{E} \cap L$ (we still denote by $L$ its strict transform), the blow up map is given by

$$
\begin{equation*}
\pi\left(s, s^{\prime}\right)=\left(s, s s^{\prime}\right) \tag{414}
\end{equation*}
$$

$s^{\prime}=0$ is a local equation of $L$ and $s=0$ a local equation of $\widetilde{E}$. At $\widetilde{E} \cap L$, we get

$$
\begin{equation*}
f\left(s, s^{\prime}\right)=\left(s^{2} s^{\prime}, \frac{s^{\prime}}{2-s^{2}\left(s^{\prime}\right)^{2}}\right) \tag{415}
\end{equation*}
$$

Thus, $f^{*} s=s^{2} s^{\prime}$ and therefore $f_{*} \operatorname{ord}_{\widetilde{E}}=2 \operatorname{ord}_{\tilde{E}}$. This implies that $v_{*}=\operatorname{ord}_{\tilde{E}}$.
Second example.- Consider Proposition 13.2 with $P=1$ and $f$ the endomorphism of $S(2)$ from the previous paragraph. Define $g=g_{1} \circ f$, then

$$
\begin{equation*}
g(x, y, z)=\left(x z, x^{2} z^{2}+4 z^{2}+4 y-2, x^{2} z^{2}+2 z^{2}-1\right) \tag{416}
\end{equation*}
$$

Let $\mathbf{A}^{2} \subset \mathbf{P}^{1} \times \mathbf{P}^{1}$ with affine coordinates $(u, v)$ and the birational morphism $\varphi: S(2) \rightarrow$ $\mathbf{A}^{2}$. Then, $g$ is conjugated by $\varphi$ to

$$
\begin{equation*}
\eta(u, v)=\left(u v, u^{2} v^{2}+2 v^{2}-1\right) . \tag{417}
\end{equation*}
$$

It is an endomorphism of $S(2)$ of topological degree 2. By Proposition 13.3, if $v_{*}$ is the eigenvaluation of $g$, then its center must belong to $L_{\infty}$. Consider the affine coordinates $t=1 / u, w=1 / v$ centered at $F_{\infty} \cap L_{\infty}$. In these coordinates we have

$$
\begin{equation*}
\eta(t, w)=\left(t w, \frac{t^{2} w^{2}}{1+2 t^{2}-t^{2} w^{2}}\right) \tag{418}
\end{equation*}
$$

Hence, $F_{\infty}$ and $L_{\infty}$ are both contracted to $L_{\infty} \cap F_{\infty}=p$ so it must be equal to $c_{X}\left(v_{*}\right)$. Blow up $p$ and let $E_{1}$ be the exceptional divisor. Blow up again the intersection point of $E_{1}$ and the strict transform of $L_{\infty}$ and let $E_{2}$ be the exceptional divisor. Then there exists local algebraic coordinates $(u, v)$ at $E_{2} \cap L_{\infty}$ associated to $\left(E_{2}, L_{\infty}\right)$ such that

$$
\begin{equation*}
g(u, v)=\left(u^{3} v, \frac{1}{1+u^{2}\left(2-u^{4} v^{2}\right)}\right) \tag{419}
\end{equation*}
$$

we see that the point $(0,1)$ in these coordinates is fixed. Consider the local analytic coordinates at this point given by $(u, w)=(u, v-1)$. Then,

$$
\begin{equation*}
g(u, w)=\left(u^{3}(1+w),-2 u^{2}+u^{2} \psi(u, w)\right) \tag{420}
\end{equation*}
$$

where $\psi(u, w)$ is a holomorphic function with $\psi(0,0)=0$. We have $g^{*} u=u^{3}(1+w)$ this implies that $\lambda_{1}(g)=3$ and since $\lambda_{1}(g)=3>\lambda_{2}(g)=2$ we have by Lemma 12.19 that $v_{*}$ is not divisorial. Therefore, we get that $\nu_{*}$ is infinitely singular and the center of $v_{*}$ is a free point on this completion.

### 13.2. An affine surface with an elliptic curve at infinity with an action by translation

We show that the Elliptic case in Theorem 12.1 can happen. Start with a generic $(2,2,2)$ divisor $V$ in $\mathbf{P}^{1} \times \mathbf{P}^{1} \times \mathbf{P}^{1}$. This is a K3 surface. It is given by one equation in the variables $(x, y, z)$ which is of degree 2 with respect to each $x, y, z$. The projection on the
plane $\mathbf{P}^{1} \times \mathbf{P}^{1}$ induced by forgetting the coordinate $x$ yields a 2:1 cover $V \rightarrow \mathbf{P}^{1} \times \mathbf{P}^{1}$. Indeed, we can rewrite the equation of $V$ as

$$
\begin{equation*}
a(y, z) x^{2}+b(y, z) x+c(y, z)=0 \tag{421}
\end{equation*}
$$

where $b, c$ are polynomials in $y, z$ of degree 2 with respect to each variable. Let $\sigma_{x}$ be the involution of $V$ that switches the folds of the cover. We can define similarly the involutions $\sigma_{y}, \sigma_{z}$. The group generated by $\sigma_{x}, \sigma_{y}, \sigma_{z}$ is $\mathbf{Z} / 2 \mathbf{Z} * \mathbf{Z} / 2 \mathbf{Z} * \mathbf{Z} / 2 \mathbf{Z}$ (see [Can01a]). Now, we restict our attention to the subgroup generated by $\sigma_{x}, \sigma_{y}$. Consider the family of curves defined by the hyperplane sections $E_{\alpha}:=V \cap\{z=\alpha\}$. The involutions $\sigma_{x}, \sigma_{y}$ preserve $E_{\alpha}$ for every $\alpha$. Thus, for a very general parameter $\alpha$ the subgroup of $\operatorname{Aut}\left(E_{\alpha}\right)$ generated by $\sigma_{x \mid E_{\alpha}}, \sigma_{y \mid E_{\alpha}}$ is isomorphic to $\mathbf{Z} / 2 \mathbf{Z} * \mathbf{Z} / 2 \mathbf{Z}$.

Proposition 13.4. For $\alpha \in \mathbf{C}$ very general, set $E:=E_{\alpha}, X_{0}=\mathbf{P}^{1} \times \mathbf{P}^{1} \backslash E$ where we have identified $\mathbf{P}^{1} \times \mathbf{P}^{1} \simeq \mathbf{P}^{1} \times \mathbf{P}^{1} \times \mathbf{P}^{1} \bigcap\{z=\alpha\}$. Then, $X_{0}$ is a smooth affine surface and there exists two endomorphism $f, g$ of $X_{0}$ such that
(1) $\lambda_{1}(f)=\lambda_{1}(g)=\lambda_{2}(f)=\lambda_{2}(g)=2$.
(2) $f_{\mid E}=\sigma_{y}$
(3) $g_{\mid E}=\sigma_{x}$
(4) let $h=g \circ f$, then $\lambda_{1}(h)=\lambda_{2}(h)=4$ and $h_{\mid E}$ is a translation by a non-torsion element.

Proof. We look again at Example 11.14. We write the equation of $E$ in two different ways. There exists degree two rational fractions $P(x), Q(x) \in \mathbf{C}(x), R(y), S(y) \in \mathbf{C}(y)$ such that the equation of $E$ is of the form

$$
\begin{equation*}
y^{2}-P(x) y+Q(x)=0 \tag{422}
\end{equation*}
$$

and

$$
\begin{equation*}
x^{2}-R(y) x+S(y)=0 \tag{423}
\end{equation*}
$$

Let $X=\mathbf{P}^{1} \times \mathbf{P}^{1}$ and let $X_{0}=X \backslash E$. Let $k: X_{0} \rightarrow X_{0}$ be the endomorphism from Example 11.14. That is $k$ commutes with $\operatorname{pr}_{x}$ and acts as $z \mapsto z^{2}$ on each fiber with $0, \infty \in \mathbf{P}^{1}$ being the intersection points of $E$ with the fiber. We let $f$ be the endomorphism of $X_{0}$ that preserves the fibration $\mathrm{pr}_{x}$ and acts on each fiber $\left(\simeq \mathbf{P}^{1}\right)$ as $z \mapsto 1 / z^{2}$ where 0 and $\infty$ are the intersection points of the fiber and $E$. This defines an endomorphism with $\lambda_{1}=\lambda_{2}=2$ and $f_{\mid E}$ is an involution. Indeed, we have $f^{2}=k^{2}$, therefore the eigenvaluation of $f$ must be $\operatorname{ord}_{E}$ and $\lambda_{1}\left(k^{2}\right)=4$. The four points of $E$ where the discriminant with respect to $y$ vanishes are the four indeterminacy points of $f$ and they are fixed points of $f_{\mid E}$. In coordinates $(x, y), f$ is of the form

$$
\begin{equation*}
f(x, y)=\left(x, \frac{P(x) y^{2}-2\left(P(x)^{2}-2 Q(x)\right) y+P(x)\left(P(x)^{2}-Q(x)\right)}{y^{2}-2 P(x) y+\left(P(x)^{2}-2 Q(x)\right)}\right) \tag{424}
\end{equation*}
$$

It is clear that $f_{\mid E}=\sigma_{y \mid E}$.
Now, if we exchange the role of the coordinates $x, y$, we obtain an endomorphism $g$ with an expression similar to (424) with $(R(y), S(y))$ instead of $(P(x), Q(x))$. Set $h=g \circ f$, then $\lambda_{1}(h)=\lambda_{2}(h)=4$. Indeed, $\lambda_{2}(h)=\lambda_{2}(f) \lambda_{2}(g)$. And we have that $h_{*} Z_{\text {ord }_{E}}=g_{*} f_{*}\left(Z_{\text {ord }_{E}}\right)=4 Z_{\text {ord }_{E}}+w$ with $w \in \operatorname{Div}_{\infty}(X)^{\perp}$. Since $Z_{\text {ord }_{E}}=\frac{1}{8} E$ is nef, $\operatorname{ord}_{E}$ must be the eigenvaluation of $h$ and $\lambda_{1}(h)=4$.

Now, let $\Lambda \subset \mathbf{C}$ be a lattice such that $E \simeq \mathbf{C} / \Lambda$. An involution of $E$ lifts to a linear map over $\mathbf{C}$ of the form

$$
\begin{equation*}
z \mapsto-z+b \tag{425}
\end{equation*}
$$

with $b \in \mathbf{C}$ or

$$
\begin{equation*}
z \mapsto z+b \tag{426}
\end{equation*}
$$

with $2 b \in \Lambda$. However, (426) is impossible for $\sigma_{x \mid E}$ or $\sigma_{y \mid E}$ because they admit fixed points. So, they are both of the form (425) and we get that $\sigma_{x \mid E} \circ \sigma_{y \mid E}$ is a translation of infinite order because $\left\langle\sigma_{x \mid E}, \sigma_{y \mid E}\right\rangle \simeq \mathbf{Z} / 2 \mathbf{Z} * \mathbf{Z} / 2 \mathbf{Z}$.

Corollary 13.5. The endomorphism $g \circ f$ does not admit an invariant fibration over a curve.

Proof. First, $g \circ f$ or any of its iterate cannot admit an invariant curve in $X_{0}$ because $(g \circ f)_{\mid E}$ does not admit any fixed point. Now, suppose that there exists a curve $C$ and a rational transformation $\varphi: C \rightarrow C$ such that the diagram

commutes. Let $\bar{C}$ be the unique completion of $C$. We have the commutative diagram


We cannot have $g(\bar{C}) \geqslant 2$ or $g(\bar{C})=0$ because in both cases some iterate of $\varphi$ has a fixed point and its fiber would be an invariant curve in $X_{0}$. Now, $g(\bar{C})=1$ is also not possible because $X$ is rational. Thus, we have a contradiction.

## CHAPTER 14

## The automorphism case

Here we suppose that $X_{0}$ is an irreducible normal affine surface that admits a loxodromic automorphism. In this situation, we can actually deduce a lot more from the result of Chapter 11. In particular one can first check that $X_{0}$ has to be rational, see [DF01] Table 1 Class 5. So the condition $\operatorname{Pic}^{0}\left(X_{0}\right)$ is automatically satisfied. We change the notation for this section, we will denote $\theta^{*}$ and $\theta_{*}$ by $\theta^{+}$and $\theta^{-}$respectively. So that $\left(f^{ \pm 1}\right)^{*} \theta^{ \pm}=\lambda_{1} \theta^{ \pm}$. By Proposition 11.15 and Theorem 11.16, we get that

- $\theta^{+}, \theta^{-} \in \operatorname{Weil}_{\infty}\left(\mathrm{X}_{0}\right) \cap \mathrm{L}^{2}\left(\mathrm{X}_{0}\right)$ and they are both effective.
- $\theta^{+}=Z_{v_{-}}$and $\theta^{-}=Z_{v_{+}}$where $\nu_{+}$is the eigenvaluation of $f$ and $v_{-}$the eigenvaluation of $f^{-1}$.

Proposition 14.1. Let $X_{0}$ be a rational affine surface such that $\mathbf{k}\left[X_{0}\right]^{\times}=\mathbf{k}^{\times}$and let $f$ be a loxodromic automorphism of $X_{0}$, then
(1) The eigenvaluations $v_{+}, v_{-}$of $f$ and $f^{-1}$ respectively are of the same type.
(2) If $\lambda_{1} \in \mathbf{Z} \geqslant 0$, then $v_{+}$and $v_{-}$are infinitely singular.
(3) If $\lambda_{1} \in \mathbf{R} \backslash \mathbf{Z}_{\geqslant 0}$ then $v_{+}$and $v_{-}$are irrational.

Proof. If the eigenvaluation was divisorial, then we would get by Lemma 12.19 that $\lambda_{1} \leqslant \lambda_{2}$ and this is absurd because $\lambda_{1}>1, f$ being loxodromic. The dichotomy of the type of eigenvaluation follows from Theorem 12.1 and the fact that $\lambda_{1}(f)=$ $\lambda_{1}\left(f^{-1}\right)$.

Corollary 14.2. In that case, the nef eigenclasses $\theta^{-}$and $\theta^{+}$verify

$$
\left(\theta^{-}\right)^{2}=\left(\theta^{+}\right)^{2}=0
$$

and in any completion $X$ of $X_{0}$ one has $\left(\theta_{X}^{ \pm}\right)^{2}>0$.
Proof. The equalities $\left(\theta^{-}\right)^{2}=\left(\theta^{+}\right)^{2}=0$ come from Theorem 3.28 (87). Since the eigenvaluations are not divisorial, $\theta^{-}$and $\theta^{+}$are not Cartier divisors by Corollary 11.4 therefore for any completion $X$ of $X_{0},\left(\theta_{X}^{ \pm}\right)^{2}>0$. Indeed, if $\left(\theta_{X}^{ \pm}\right)^{2}=0$ then since $\theta^{ \pm}$is nef, we would get $\theta_{\bar{X}}^{ \pm}=\theta^{ \pm}$.

Let $X$ be a completion of $X_{0}$. We have a simple criterion to check whether a divisor at infinity is contracted thanks to Proposition 12.2 .

Proposition 14.3. Let $E$ be a prime divisor at infinity in a completion $X$ of $X_{0}$. If $Z_{\text {ord }_{E}} \cdot \theta^{-}>0$ then there exists $N>0$ such that $f^{N}$ contracts $E$ to the point $c_{X}\left(v_{+}\right)$.

### 14.1. Gizatullin's work on the boundary and applications

In [Giz71a], Gizatullin considers minimal completions of affine surface. That is a completion $X$ of $X_{0}$ minimal with respect to the following property:

- The boundary $\partial_{X} X_{0}$ does not have three prime divisors that intersect at the same point.
- If $\partial_{X} X_{0}$ has a singular irreducible component then $\partial_{X} X_{0}$ consists only of one irreducible curve with at most one nodal singularity.
For such a completion $\imath: X_{0} \hookrightarrow X$, Gizatullin defines the curve $E(\imath)$ as the union of the irreducible components $E$ of $\partial_{X} X_{0}$ that are contracted by an automorphism of $X_{0}$ (the automorphism depends on $E$ ).

We call a zigzag a chain of rational curves. That is a sequence $\left(E_{1}, \cdots, E_{r}\right)$ of rational curves such that $E_{i} \cdot E_{i+1}=1, i=1, \cdots, r-1$ and for all $i, j$ such that $|i-j| \geqslant$ $2, E_{i} \cdot E_{j}=0$. In particular the dual graph with respect to the $E_{i}$ 's is of the form


We will write $E_{1} \triangleright E_{2} \triangleright \cdots \triangleright E_{r}$ for the zigzag defined by $\left(E_{1}, \cdots, E_{r}\right)$.
A cycle of rational curves is a sequence $\left(E_{1}, \cdots, E_{r}\right)$ of rational curves such that $E_{i} \cdot E_{i+1}=1$ and $E_{1} \cdot E_{r}=1$. The dual graph with respect to the $E_{i}$ 's is of the form


THEOREM 14.4. Let $X_{0}=\operatorname{Spec} \mathbf{k}\left[X_{0}\right]$ be an irreducible normal affine surface such that $\mathbf{k}\left[X_{0}\right]^{\times}=\mathbf{k}^{\times}$and $\operatorname{Pic}^{0}\left(X_{0}\right)=0$. Suppose that $X_{0}$ admits an automorphism $f$ with $\lambda_{1}(f)>1$. If $X$ is a minimal completion of $X_{0}$, one has $E(\mathrm{l})=\partial_{X} X_{0}$. Furthermore we have two mutually excluding cases
(1) $\lambda_{1}(f)$ is an integer and in that case $E(\mathrm{l})$ is a zigzag.
(2) $\lambda_{1}(f)$ is irrational and $E(\mathfrak{l})$ is a cycle of rational curves.

Furthermore, there exists a completion $Y$ with two distinct points $p_{+}, p_{-} \in \partial_{Y} X_{0}$ and an integer $N>0$ such that

- $f^{ \pm 1}\left(p_{ \pm}\right)=p_{ \pm}$.
- $f^{ \pm N}$ contracts $\partial_{Y} X_{0}$ to $p_{ \pm}$.
- $f^{ \pm 1}$ has a normal form at $p_{ \pm}$given by Theorem 12.1 it is pseudomonomial or monomial in the cycle case and of type (18) or (17) in the zigzag case.
- In the cycle case, this set of properties remains true if we blow up $p_{+}$or $p_{-}$.
- In the zigzag case, the set of completions above Y that satisfy these properties is cofinal in the set of all completions above $Y$.
The normal form of $f$ at $p_{ \pm}$is monomial in the cycle case and of the form of Theorem 12.1 case (3) in the zigzag case.

This shows Theorem E We will prove Theorem 14.4 in $\$ 14.2$ and $\S 14.3$. We end this section with some technical result that will be useful in the proof of Theorem 14.4.

Lemma 14.5. Let $X$ be a completion of $X_{0}$ and let $E$ be a prime divisor at infinity such that $Z_{\text {ord }_{E}} \cdot \theta^{+}=0$ and $E$ intersects some prime divisor in the support of $\theta_{X}^{+}$, then $c_{X}\left(\mathrm{v}_{+}\right)$belongs to $E$.

Proof. Since $\theta^{+}$is effective and $\operatorname{ord}_{E}\left(\theta^{+}\right)=0$ we get $\theta^{+} \cdot E>0$ since $E$ intersects the support of $\theta^{+}$. This implies by Proposition 7.6 that $c_{X}\left(v_{+}\right)$belongs to $E$.

Lemma 14.6. Let $Y$ be a completion of $X_{0}$ and $E$ a prime divisor at infinity of $Y$ such that $Z_{\operatorname{ord}_{E}} \cdot \theta^{+}>0$. If $p \in E \backslash\left\{c_{X}\left(v_{+}\right)\right\}$, then for any divisorial valuation $v$ such that $c_{X}(v)=p$, one has $Z_{v} \cdot \theta^{+}>0$.

Proof. Let $Z$ be the blow up of $Y$ at $p$. Then, $\theta_{Z}^{+}=\left(\pi^{*} \theta_{Y}^{+}\right)+c \widetilde{E}$ for some $c \in \mathbf{R}$. Since the center of $v_{+}$is not on $\widetilde{E}$, one has $\theta_{Z}^{+} \cdot \widetilde{E}=0$, hence $c=0$. Now whether $p$ is a free point on $E$ or a satellite point, we have $Z_{\operatorname{ord}_{\tilde{E}}} \cdot \theta^{+} \geqslant Z_{\operatorname{ord}_{E}} \cdot \theta^{+}>0$.

LEMMA 14.7. Let $Y$ be a completion of $X_{0}$ such that the center of $v_{+}$is the intersection of two prime divisors at infinity $F_{1}, F_{2}$. Then, $Z_{\operatorname{ord}_{F_{1}}} \cdot \theta^{+}>0$ or $Z_{\operatorname{ord}_{F_{2}}} \cdot \theta^{+}>0$.

Proof. Recall that $\theta^{+}$is nef and effective. Suppose that $Z_{\text {ord }_{F_{i}}} \cdot \theta^{+}=0$ for $i=1,2$ and let $\widetilde{E}$ be the exceptional divisor above $p_{+}$. Let $\pi: Z \rightarrow Y$ be the blow-up at $p_{+}$. Then we have

$$
\theta_{Z}^{+}=\pi^{*}\left(\theta_{Y}^{+}\right)+c \widetilde{E}
$$

for some $c \in \mathbf{R}$. This implies $\theta^{+} \cdot \widetilde{E}=-c>0$ because $p_{+}$was the center of $\mathrm{v}_{+}$on $Y$, therefore $c<0$. But $Z_{\text {ord }_{\tilde{E}}} \cdot \theta_{Z}^{+}=\left(Z_{\operatorname{ord}_{F_{1}}}+Z_{\operatorname{ord}_{F_{2}}}\right) \theta_{Y}^{+}+c=c<0$ and this contradicts the fact that $\theta^{+}$is effective.

Proposition 14.8. For any completion $Y$ such that $c_{Y}\left(v_{+}\right)$is a free point, we have

$$
\begin{equation*}
\text { Supp } \theta_{Y}^{+}=\partial_{Y} X_{0} \tag{427}
\end{equation*}
$$

Hence, if $\mathrm{v}_{ \pm}$is an infinitely singular valuation, then for any completion $Z$, there exists an integer $N>0$ such that $f^{ \pm N}\left(\partial_{Z} X_{0}\right)=p_{ \pm}$.

Proof. Let $E$ be the unique prime divisor at infinity such that $c_{Y}\left(v_{+}\right) \in E$. If $\operatorname{Supp} \theta_{Y}^{+} \neq \partial_{Y} X_{0}$, there a prime divisor $F$ at infinity such that $Z_{\text {ord }_{F}} \cdot \theta^{+}=0$ and $F \cap$ $\operatorname{Supp} \theta_{Y}^{+} \neq \varnothing$. By Lemma 14.5 , we have $F=E$; therefore $Z_{\text {ord }_{E}} \cdot \theta^{+}=0$. But we have that $\theta_{Y}^{+}=\lambda Z_{\text {ord }_{E}}$ for some $\lambda>0$ by Proposition 11.5. So $\left(\theta_{Y}^{+}\right)^{2}=0$, but this is absurd by Corollary 14.2 .

For the second assertion, assume that $\nu_{ \pm}$is an infinitely singular valuation. Let $Z$ be a completion of $X_{0}$. Then, by Proposition 4.16, there exists a completion $Y$ above $Z$ such that $c_{Y}\left(v_{ \pm}\right)$is a free point. The first assertion shows that Supp $\theta_{Y}^{ \pm}=\partial_{Y} X_{0}$ and so the same is true for $\operatorname{Supp} \theta_{Z}^{ \pm}$. The fact that some iterate of $f^{ \pm 1}$ contracts the boundary on $p_{ \pm}$follows from Proposition 14.3 .

### 14.2. Proof of Theorem 14.4, the cycle case

In that case it was already proven by Gizatullin that $\partial_{X} X_{0}=E(\mathrm{l})$.
Proposition 14.9 ([Ѐట74, CdC19]). Let $X$ be projective surface and $U$ an open subset of $X$ such that $X \backslash U$ is a cycle of rational curves. Assume that $X \backslash U$ is not an irreducible curve with one nodal singularity. Let $g$ be an automorphism of $U$, then the indeterminacy points of $g$ can only be intersection points of two components of the cycle.

COROLLARY 14.10. In the cycle case, the eigenvaluation of a loxodromic automorphism must be irrational and therefore $\lambda_{1}$ is an algebraic integer of degree 2, in particular it is irrational.

Proof. Proposition 14.9 shows that for any completion $X$ of $X_{0}, p_{+}=c_{X}\left(v_{+}\right)$is a satellite point at infinity. Indeed, since $\theta^{+}$is nef, its incarnation in $X$ cannot be 0 . Therefore, there exists a prime divisor $E$ at infinity such that $Z_{\text {ord }_{E}} \cdot \theta^{+}>0$ because $\theta^{+}$ is effective. Therefore, by Proposition 14.3, $E$ must be contracted by $f^{N}$ to $p_{+}$so it must be an indeterminacy point of $f^{-N}$. Proposition 4.16 shows that the eigenvaluations $v_{ \pm}$are irrational.

Proof of Theorem 14.4. Corollary 14.10 shows the first part of the theorem. We get the normal form at $p_{ \pm}$by blowing up the center of $v_{ \pm}$enough times. Since these are always intersection points of two prime divisors at infinity we can suppose that $\partial_{Y} X_{0}$ is still a cycle.

It remains to show that $\partial_{Y} X_{0}$ is contracted by some iterate of $f$ and $f^{-1}$. Suppose that there exists a prime divisor $E$ that is not contracted to $p_{+}$by any iterate of $f$. In particular $Z_{\text {ord }_{E}} \cdot \theta^{+}=0$ by Proposition 14.3. By Lemma 14.5, we have that $E$ contains $c_{Y}\left(v_{-}\right)$and $f^{-1}$ contracts $E$ to $p_{-}$. And by Lemma 14.7 and Corollary 14.10 we have that $E$ is the unique prime divisor at infinity that satisfy this property. Either $f$ contracts $E$ to a satellite point $p \neq p_{+}$of the boundary or $f$ is sent to a prime divisor at infinity.

Indeed, we cannot have $f(E)=E$, otherwise $E$ is $f$-invariant but this contradicts that $f^{-1}$ contracts $E$. If $E$ is contracted, it cannot be contracted to $p_{-}$because it is not an indeterminacy point of $f^{-1}$. Therefore, we have that the center of $f_{*} \operatorname{ord}_{E}$ is either another prime divisor at infinity or a satellite point at infinity that is not the center of $v_{+}$. In both case, we get $f_{*} Z_{\text {ord }_{E}} \cdot \theta^{+}>0$ by Lemma 14.6 and this is a contradiction.

### 14.3. Proof of Theorem 14.4, the zigzag case

14.3.1. Some technical lemmas about zigzags. We will say following [GD75, BD11] that a zigzag $Z$ is standard if it is of the form

$$
\begin{equation*}
Z=F \triangleright E \triangleright Z^{\prime} \tag{428}
\end{equation*}
$$

where $F^{2}=0, E^{2} \leqslant-1$ and $Z$ is a negative zigzag meaning that every component of $Z^{\prime}$ has self-intersection $\leqslant-2$. Any zigzag can be put to a standard form via blow-up of points and contractions of (-1)-curves (see [GD75], §1.7)

Following [BD11], an almost standard zigzag is a zigzag $Z=B_{1} \triangleright B_{2} \triangleright \cdots \triangleright B_{r}$ such that
(1) There exists a unique irreducible component $B_{k}$ such that $\left(B_{k}\right)^{2} \geqslant 0$.
(2) There exists at most one component $B_{l}$ such that $\left(B_{l}\right)^{2}=-1$ and in that case we must have $l=k \pm 1$.
We need to state some technical results for the proof of Theorem 14.4, we will need to apply them to a quasiprojective surface which is not necessarily affine. If $U$ is a quasiprojective surface, a completion of $U$ is defined in the same way as the completion of an affine surface. All the results in this Section rely heavily on Proposition 2.6 and the Castelnuovo criterion.

LEMMA 14.11 (Proposition 3.1.3 of [BD11]). Let $U$ be a quasiprojective surface and $X$ a completion of $U$ such that $X \backslash U$ is an almost standard zigzag that has no component of self intersection -1 . Let $B_{k}$ be the unique irreducible component of nonnegative self-intersection of $X \backslash U$. Let $g$ be an automorphism of $U$, then
(1) $g$ has at most one indeterminacy point $q$ on $X$.
(2) $q$ has to be on $B_{k}$ (if it exists).
(3) If $B_{k}$ is not on the boundary of the zigzag then $q$ must be the intersection point of $B_{k}$ with $B_{k+1}$ or $B_{k-1}$.

Proof. Suppose that $g$ has an indeterminacy point, then $g^{-1}$ also has one and $g$ has to contract a curve of the zigzag. Let $\pi: Y \rightarrow X$ be the minimal resolution of indeterminacies of $g$ and let $\widetilde{g}$ be the lift of $g$. Then, the first curve contracted by $\widetilde{g}$ has to be the strict transform of $B_{k}$. So $g$ has at least one indeterminacy point on $B_{k}$.

There cannot be any indeterminacy point $q$ outside of $B_{k}$ because otherwise it belongs to components that have self-intersection $\leqslant-2$ and since the zigzag $X \backslash U$ contains no ( -1 )-curve any exceptional divisor above $q$ has to be contracted by $g$ so $q$ is not an indeterminacy point.

Suppose that $B_{k}$ is not on the boundary and that the indeterminacy point $p$ of $g$ is not an intersection point. Then, the map $\pi$ factorizes through the blow-up of $p$ and after contracting the strict transform of $B_{k}$, we get at infinity three prime divisors that intersect at the same point. But this is a contradiction because $\widetilde{g}$ consists only of blow ups of point at infinity and $X \backslash U$ does not have three divisors that intersects at the same point.

Finally, there cannot be more than one indeterminacy point on $X$. Suppose the contrary and let $p_{1}, p_{2}$ be two indeterminacy points, they both belong to $B_{k}$. Let $E_{1}, E_{2}$ be two exceptional divisor above $p_{1}$ and $p_{2}$ in $Y$ respectively. They cannot be contracted by $\widetilde{g}$ because $Y$ is the minimal resolution of singularities of $g$. Therefore, their strict transform is either a $(-1)$-curve or a curve with nonnegative self intersection. But this is absurd because $X \backslash U$ does not contain any ( -1 )-curve and has only one curve of nonnegative self-intersection.

Corollary 14.12. Let $X$ be a completion of $U$ such that $X \backslash U$ is an almost standard zigzag $Z$ and let $f$ be an automorphism of $U$. Suppose that $f$ has an indeterminacy point that is a free point on $B_{k}$, then one of the two sides of $Z$ can be contracted so that $B_{k}$ becomes a boundary component of the zigzag.

Proof. Suppose that $B_{k}$ is not a boundary component of the zigzag and that $f$ has an indeterminacy point that is a free point on $B_{k}$. Then, by Lemma 14.11, $B_{k-1}$ or $B_{k+1}$ has to be a $(-1)$-curve, suppose it is $B_{k+1}$. We contract it and we obtain an almost standard zigzag and $f$ still has an indeterminacy point that is a free point on $B_{k}$. If $B_{k}$ is on the boundary we are done, otherwise the only $(-1)$-curve is the strict transform of $B_{k+2}$ and we keep contracting until $B_{k}$ becomes a boundary component of the zigzag.

Lemma 14.13. Let $U$ be a quasiprojective variety and $X$ a completion of $U$ such that $X \backslash U$ is a zigzag of type $\left(-m_{1}, \cdots,-m_{k},-1,-1,-m_{k+1}, \cdots,-m_{r}\right)$ such that for all $i, m_{i} \geqslant 2$. Let $f$ be an automorphism of $U$. Then the intersection point of the two $(-1)$-curves cannot be an indeterminacy point of $f$.

If the zigzag is of type $(-1,-2, \ldots, \underbrace{-2}_{F}, \underbrace{-1}_{E},-m_{k+1}, \cdots,-m_{r})$ with $m_{i} \geqslant 2$, then $F \cap E$ cannot be an indeterminacy point of $f$.

Proof. Let $\pi: Z \rightarrow X$ be a minimal resolution of indeterminacy of $f: X \rightarrow X$ and let $\tilde{f}: Z \rightarrow X$ be the lift of $f$. The first curve contracted by $\tilde{f}$ must be the strict transform of one of the prime divisors at infinity of $X$. But if the intersection of the $(-1)$-curves
is an indeterminacy point of $f$, then all the strict transforms of the prime divisors at infinity of $X$ have self-intersections $\leqslant-2$ and this is a contradiction.

If $X \backslash U$ is a zigzag $Z$ of type $\left(-1,-2, \cdots,-2,-1,-m_{k+1}, \cdots,-m_{r}\right)$, suppose that $F \cap E$ is an indeterminacy point of $f$, then the first curve contracted by $\tilde{f}$ must be the strict transform of the $(-1)$-curve on the left of the zigzag. So we can start by contracting it and we get a zigzag $Z^{\prime}$ of type $\left(-1,-2, \cdots,-2,-1,-m_{k+1}, \cdots,-m_{r}\right)$

and of size $\# Z-1$. We can repeat this process until we get a zigzag of the form $\left(-1,-1,-m_{k+1}, \cdots,-m_{r}\right)$ and we have that $F \cap E$ cannot be an indeterminacy $\underbrace{-}_{F} \underbrace{-1}_{E}$
point of $f$ by the previous case, this is a contradiction.
LEMMA 14.14. Let $f$ be an automorphism of $X_{0}$ and let $X$ be a minimal completion of $X_{0}$ in the sense of Gizatullin. Then, $f$ defines an automorphism of $U=(E(\mathrm{t}))^{c} \subset X$, the complement of $E(\mathrm{l})$, i.e the birational map $f: X \rightarrow X$ does not have any indeterminacy point on $U$.

Proof. Suppose that $f$ admits an indeterminacy point $p$ on some component $E_{1}$ of $\partial_{X} X_{0}$ with $p \notin E(\mathbf{l})$. Let $\pi: Y \rightarrow X$ be a minimal resolution of indeterminacies for $f$ and let $F: Y \rightarrow X$ be the lift of $f$. The fiber $\pi^{-1}(p)$ contains at least one $(-1)$-curve and we claim that none of the irreducible components of $\pi^{-1}(p)$ can be contracted by $F$, indeed since $E_{1}$ is not contracted, one can only contract $(-1)$-curves of $\pi^{-1}(p)$ but that would contradict the minimality of $Y$. Therefore, the fiber $\pi^{-1}(p)$ is not affected by $F$ and neither are the self-intersections in the fiber. This would imply that $\partial_{X} X_{0}$ contains some $(-1)$-curves that can be contracted and this contradicts the minimality of $X$.

Corollary 14.15. Let $X_{\min }$ be a minimal completion of the affine surface $X_{0}$. The centers $c_{X_{\min }}\left(v_{ \pm}\right)$must belong to $E(\mathfrak{i})$.

We will apply all the results of this section with $U=(E(\mathrm{l}))^{c} \subset X_{\min }$ where $X_{\min }$ is a minimal completion of $X_{0}$.
14.3.2. Elementary links between almost standard zigzags. From now on $U=$ $(E(\mathrm{t}))^{c} \subset X_{\min }$ where $X_{\text {min }}$ is a minimal completion of the affine surface $X_{0}$. All the results of $\$ 14.3 .1$ will be applied to the following situation. If $X$ is a completion of $U$ (hence of $X_{0}$ ) and $f$ is a loxodromic automorphism of $X_{0}$, then some positive iterate of $f$ contracts a component of $X \backslash U$ to $c_{X}\left(v_{+}\right)$. Thus, $c_{X}\left(v_{+}\right)$is an indeterminacy point of some positive iterate of $f^{-1}$ on $X$.

Proposition 14.16. Let $X$ be a completion of $U$ such that $X \backslash U$ is an almost standard zigzag, then one can find a completion $Y$ of $U$ with a birational map $\varphi: X \rightarrow Y$ that is an isomorphism above $U$ such that
(1) $Y \backslash U$ is also an almost standard zigzag.
(2) Let $\widetilde{X}$ be the blow up of $X$ at $c_{X}\left(v_{+}\right)$, then the lift $\varphi: \widetilde{X} \rightarrow Y$ is defined at $c_{\tilde{X}}\left(v_{+}\right)$and is a local isomorphism there.

Proof. Let $B$ the unique irreducible component of $X \backslash U$ of nonnegative self intersection.

Case: $B$ is on the boundary. $X \backslash U$ is a zigzag of the form $B \triangleright E \triangleright Z$ where $B^{2} \geqslant$ $0, E^{2} \leqslant-1$ and $Z$ is a negative zigzag.

- $c_{X}\left(v_{+}\right)$is a free point on $B$ If $E^{2}=-1$, we blow up $c_{X}\left(v_{+}\right)$and then contract the strict transform of $E$. Let $Y$ be the new projective surface obtained, it satisfies the proposition.

Suppose $E^{2}<-1$, If $B^{2}>0$ we blow up $B \cap E$ to obtain a new zigzag $B \triangleright$ $E^{\prime} \triangleright Z^{\prime}$ which is still almost standard. We keep blowing up the strict transform of $B$ with the second component of the zigzag until $B^{2}=0$. After all these blowups, let $X^{\prime}$ be the newly obtained projective surface, we have that $X^{\prime} \backslash U$ is an almost standard zigzag of the form $B \triangleright E \triangleright Z$ where $B^{2}=0, E^{2}=-1$ and $Z$ is a negative zigzag. We blow up $c_{X^{\prime}}\left(v_{+}\right)$and let $\widetilde{E}$ be the exceptional divisor, by Lemma 14.13, the center of $v_{+}$cannot be the intersection point of $\widetilde{E}$ and the strict transform of $B$, therefore it is a free point of $\widetilde{E}$ and we can contract the strict transform of $B$. We call $Y$ the new obtained surface it satisfies the proposition.

- $c_{X}\left(\mathrm{v}_{+}\right)$is the satellite point $B \cap E$ We blow up $B \cap E$ and call $\widetilde{E}$ the exceptional divisor. If $B^{2}>0$ in $X$, then we still have an almost standard zigzag and we call $Y$ the new obtained surface. If $B^{2}=0$ in $X$, then by Lemma 14.13 is a free point of $\widetilde{E}$ and we can contract the strict transform of $B$, we call $Y$ the newly obtained surface.


## Case: $B$ is not on the boundary.

- $c_{X}\left(v_{+}\right)$is a free point of $B$ By Corollary 14.12, one of the two sides of $X \backslash U$ is contractible, so we contract it and call $X_{1}$ the newly obtained surface, we can now apply the proof of the boundary case to find $Y$.
- $c_{X}\left(v_{+}\right)$is the satellite point $B \cap E$ We can suppose up to contraction that if $X \backslash U$ contains a ( -1 )-component, it must be $E$. We start by blowing up $c_{X}\left(\mathrm{v}_{+}\right)$ and let $\widetilde{E}$ be the exceptional divisor.
- If $B^{2}>0$ in $X$, then we still have an almost standard zigzag and we call $Y$ the newly obtained surface.
- If $B^{2}=0$ in $X$, then by Lemma 14.13 the center of $\mathrm{v}_{+}$cannot be the intersection of $\widetilde{E}$ and the strict transform of $B$ where $\widetilde{E}$ is the exceptional divisor. So we can contract the strict transform of $B$ and we get an almost standard zigzag and we call $Y$ the newly obtained surface.

COROLLARY 14.17. If $\partial_{X} X_{0}$ is a zigzag, the eigenvaluation $\mathrm{v}_{+}$cannot be irrational, hence it is infinitely singular and $\lambda_{1}$ is an integer. Furthermore, $U=X_{0}$.

Proof. It suffices to show that the sequence of centers of $v_{+}$contains infinitely many free points. If not, we can apply Proposition 14.16 finitely many times so that we get a completion $X$ of $X_{0}$ such that $X \backslash U$ is an almost standard zigzag and the center of $\nu_{+}$is always a satellite point. We show that this leads to a contradiction.

Case 1: $c_{X}\left(\mathrm{v}_{+}\right)=B \cap E$ with $E$ a component of $X \backslash U$. We can suppose after contractions and blow ups that $B^{2}=0$. We will show that we can suppose that $B$ is a boundary component of the zigzag. The zigzag $X \backslash U$ is of the form $Z_{1} \triangleleft B \triangleright E \triangleright Z$. Denote by $\left(m_{1}, \cdots, m_{r}\right)$ the type of $Z_{1}$.

- Case $m_{1} \geqslant 2$ Blow up $B \cap E$ and call $\widetilde{E}$ the exceptional divisor. The center of $v_{+}$has to be $B \cap \widetilde{E}$ or $\widetilde{E} \cap E$, but it cannot be $B \cap \widetilde{E}$ by Lemma 14.13. So we can contract the strict transform of $B$. We get a new zigzag of the form $Z_{1}^{\prime} \triangleleft B^{\prime} \triangleright Z^{\prime}$ with $m_{1}^{\prime}=m_{1}-1$ and $\# Z_{1}^{\prime}=\# Z_{1}$.
- Case $m_{1}=1$ call $E_{1}$ the first component of $Z_{1}$. Blow up $B \cap E$. The center of $v_{+}$is either $B \cap \widetilde{E}$ or $\widetilde{E} \cap E$. Either way, we can contract the strict transform of $E_{1}$. We get a zigzag of the form $Z_{1}^{\prime} \triangleleft B \triangleright \widetilde{E} \triangleright E \triangleright Z$ where $\# Z_{1}^{\prime}=\# Z_{1}-1$.
We can apply this procedure recursively, it stops because the sequence $\left(\# Z_{1}, m_{1}\right)$ is strictly decreasing for the lexicographical order. And we never blow down a curve that contains the center of $v_{+}$nor do we blow down a curve to the center of $v_{+}$.

Now that we have that $B$ is a boundary component, we can suppose that $X \backslash U$ is a 1standard zigzag. Call $E$ the $(-1)$-component of $X \backslash U$, we will show that $Z_{v_{+}} \cdot E=+\infty$. Indeed, blow up $B \cap E$ and let $\widetilde{E}$ be the exceptional divisor. By Lemma 14.13, the center of $v_{+}$has to be $\widetilde{E} \cap E$. If we blow up the center of $v_{+}$again we can still apply Lemma 14.13, so the center of $v_{+}$is always the intersection point of the strict transform of $E$ with the exceptional divisor. This implies that $v_{+}$is the curve valuation associated to the curve $E$ and this is absurd.

Case 2: $c_{X}\left(\mathrm{v}_{+}\right)=B \cap C$ with $C$ a component of $\partial_{X} X_{0}$ but $C \cap U \neq \varnothing$. This means that $c_{X}\left(\mathrm{v}_{+}\right)$belongs to no other component of $X \backslash U$ than $B$. Using Lemma 14.11 we can contract one of the two sides of the zigzag so that $B$ is a boundary component of the zigzag $X \backslash U$, we can furthermore suppose that $X \backslash U$ has no ( -1 )-component. Call $m$ the self intersection of the component next to $B$ in the zigzag, we have by assumption $m \leqslant-2$.

- Case $B^{2}>0$ let $X^{\prime}$ be the blow up of $B \cap C$ and let $\tilde{E}$ be the exceptional divisor. Then, since the strict transform of $B$ has nonnegative self intersection $X^{\prime} \backslash U$ is an almost standard zigzag. We must have that $c_{X^{\prime}}\left(\mathrm{v}_{+}\right) \in \widetilde{E}$ and by Lemma $14.11 c_{X^{\prime}}\left(\mathrm{v}_{+}\right)$must be $B \cap \widetilde{E}$ and we are back in Case 1. This leads to a contradiction.
- Case $B^{2}=0$ Let $E$ be the component on $X \backslash U$ next to $B$ (if it exists). Let $X^{\prime}$ be the blow up of $B \cap C$ and let $\widetilde{E}$ be the exceptional divisor. By Lemma 14.13, $c_{X^{\prime}}\left(v_{+}\right)$cannot be $B \cap \widetilde{E}$ so it has to be $\widetilde{E} \cap C$. Let $X^{\prime \prime}$ be the blow down of the strict transform of $B$. The strict transform of $\widetilde{E}$ has nonnegative self-intersection and $X^{\prime \prime} \backslash U$ is an almost standard zigzag and $c_{X^{\prime \prime}}\left(v_{+}\right)=\widetilde{E} \cap C$. Rename $\widetilde{E}$ by $B$ in $X^{\prime \prime}$. If $E^{2}=m$ in $X$, then the strict transform of $E$ in $X^{\prime \prime}$ satisfies $E^{2}=m+1$. We repeat this procedure until $E^{2}=-1$. We then blow down $E$ and we end up back in the case $B^{2}>0$ and this leads to a contradiction.

The last case to treat is if $X \backslash U$ is a zigzag containing only $B$ with $B^{2}=0$. We will show in that case that $v_{+}(C)=+\infty$ which is a contradiction. Indeed, let $X^{\prime}$ be the blow up of $B \cap C$ and let $\widetilde{E}$ be the exceptional divisor. Then, by Lemma 14.13, $c_{X^{\prime}}$ cannot be $B \cap \widetilde{E}$ so it must be $\widetilde{E} \cap C$. Let $X^{\prime \prime}$ be the blow up of $\widetilde{E} \cap C$ and let $\widetilde{E}^{(2)}$ be the exceptional divisor. Again, by Lemma 14.13, $c_{X^{\prime \prime}}\left(\mathrm{v}_{+}\right)=\widetilde{E}^{(2)} \cap C$. By induction, we see that the centers of $\mathrm{v}_{+}$must always belong to the strict transform of $C$ in every blow up, this implies that $v_{+}$is the curve valuation associated to $C$ and this is absurd.
Thus, $v_{+}$is not irrational. Hence, by Proposition $14.1 v_{+}$is an infinitely singular valuation, so we get that $U=X_{0}$ by Proposition 14.8 .

### 14.4. A summary and applications

We sum up the content of Theorem 14.18 in Figure 1 and 2
THEOREM 14.18. Let $X_{0}$ be a normal affine surface defined over a field $\mathbf{k}$ such that $\mathbf{k}\left[X_{0}\right]^{\times}=\mathbf{k}^{\times}$and $\operatorname{Pic}^{0}\left(X_{0}\right)=0$. Let $f$ be a loxodromic automorphism of $X_{0}$. Then, there exists two unique (up to normalization) distinct valuations centered at $\mathrm{v}_{+}, v_{-}$such that $f_{*}^{ \pm 1}\left(v_{ \pm}\right)=\lambda_{1} v_{ \pm}$. Let $\theta^{-}=Z_{v_{+}}$and $\theta^{+}=Z_{v_{-}}$. We have that $\theta^{+}, \theta^{-}$are nef, effective and satisfy the following relations

$$
\begin{align*}
f^{*} \theta^{+}=\lambda_{1} \theta^{+}, & f^{*} \theta^{-}=\frac{1}{\lambda_{1}} \theta^{-}  \tag{429}\\
f_{*} \theta^{+}=\frac{1}{\lambda_{1}} \theta^{+}, & f_{*} \theta^{-}=\lambda_{1} \theta^{-} \tag{430}
\end{align*}
$$

Furthermore we have the following intersection relations: $\left(\theta^{+}\right)^{2}=\left(\theta^{-}\right)^{2}=0$ and $\theta^{+} \cdot \theta^{-}=1$.


Figure 1. Dynamics at infinity of $f$ when $\lambda_{1}(f) \in \mathbf{Z}_{\geqslant 0}$
We can find a completion $X$ of $X_{0}$ such that if $p_{+}:=c_{X}\left(v_{+}\right), p_{-}:=c_{X}\left(v_{-}\right)$, then
(1) $p_{+} \neq p_{-}$.
(2) some positive iterate of $f^{ \pm 1}$ contracts $\partial_{X} X_{0}$ to $p_{ \pm}$.
(3) $f^{ \pm 1}$ is defined at $p_{ \pm}, f^{ \pm 1}=p_{ \pm}$and $p_{\mp}$ is the unique indeterminacy point of $f^{ \pm}$.
(4) There exists an open neighbourhood $U^{ \pm}$of $p_{ \pm}$in $X\left(\mathbf{C}_{v}\right)$ and local coordinates at $p_{ \pm}$such that $f_{\mid U^{ \pm}}^{ \pm}$has a local normal form of (pseudo)monomial type (20) or ( (19)) if $\lambda_{1}(f) \notin \mathbf{Z}_{\geqslant 0}$ or of type (18) or (17) if $\lambda_{1}(f) \in \mathbf{Z}_{\geqslant 0}$.
Proof. Any completion provided by Theorem 14.4 satisfies item (1)-(4).
Proposition 14.19. Let $X_{0}$ be a normal affine surface defined over $\mathbf{k}$. If $f$ is a loxodromic automorphism of $X_{0}$, then, there are no $f$-invariant algebraic curves in $X_{0}$.

Proof. If $\mathrm{QAlb}\left(X_{0}\right) \neq 0$, then by Corollary $10.11 X_{0} \simeq \mathbb{G}_{m}^{2}$ and this is known.
If $\mathrm{QAlb}\left(X_{0}\right)=0$, let $X$ be a completion of $X_{0}$ given by Theorem 14.18 . Suppose that $C \subset X_{0}$ is an algebraic curve invariant by $f$. Let $\bar{C}$ be the closure of $C$ in $X$. We must have $\left\{p_{+}, p_{-}\right\} \cap\left(\bar{C} \cap \partial_{X} X_{0}\right) \neq \varnothing$. Indeed, $\bar{C} \cap \partial_{X} X_{0}$ is not empty so let $p$ be a point in it. If $p \notin\left\{p_{+}, p_{-}\right\}$, then $f$ is defined at $p$ and $f(p)=p_{+}$. Since $\bar{C}$ is $f$-invariant, we get $p_{+} \in \bar{C}$. This means that $C$ defines a germ of an analytic curve at $p_{+}$that is invariant by $f$ but this is not possible by Theorem 12.1 .


Figure 2. Dynamics at infinity of $f$ when $\lambda_{1}(f) \in \mathbf{R} \backslash \mathbf{Q}$
Corollary 14.20. If $X_{0}$ is a normal affine surface defined over a number field $K$ and $f$ is a loxodromic automorphism of $X_{0}$, then all periodic points of $f$ are defined over $\bar{K}$.

Proof. Suppose there exists $p \in X_{0}(\mathbf{C}) \backslash X_{0}(\bar{K})$ such that $f^{N}(p)=p$. Let $G:=$ $\operatorname{Gal}(\mathbf{C} / \overline{\mathbf{Q}})$, then for all $q \in G \cdot p$, we have $f^{N}(q)=q$. Since $p \notin X_{0}(\bar{K})$, the orbit $G \cdot p$ is infinite and its Zariski closure $\overline{G \cdot p} \subset X_{0} \times \operatorname{Spec} \mathbf{C}$ has dimension $>0$. If $\operatorname{dim} \overline{G \cdot p}=2$, then $f^{N}=$ id and this is impossible because $f$ is loxodromic. If $\operatorname{dim} \overline{G \cdot p}=1$, then $C=\overline{G \cdot p}$ is an $f^{N}$-invariant curve of $X_{0} \times \operatorname{Spec} \mathbf{C}$. This is impossible by Proposition 14.19

Corollary 14.21. Let $X_{0}$ be a normal affine surface defined over $\mathbf{C}_{v}$ such that $\mathrm{QAlb}\left(X_{0}\right)=0$. Let $f$ be a loxodromic automorphism of $X_{0}$ and let $X$ be a completion of $X_{0}$ from Theorem 14.18. If $p \in X_{0}\left(\mathbf{C}_{v}\right)$, we have two possibilities.
(1) The forward $f$-orbit of $p$ is bounded.
(2) $\left(f^{n}(p)\right)_{n \geqslant 0}$ converges towards $p_{+}$.

Proof. Suppose that $\left(f^{n}(p)\right)_{n}$ is not bounded. Since $X\left(\mathbf{C}_{v}\right)$ is compact, $\left(f^{n}(p)\right)$ has an accumulation point $q \in \partial_{X} X_{0}$. Let $U_{+}$be the open neighbourhood of $p_{+}$given by Theorem 14.18. We must have $q \in\left\{p_{+}, p_{-}\right\}$. Otherwise, since $f(q)=p_{+}$, if $f^{N_{0}}(p)$ is sufficiently close to $q$, then for all $N \geqslant N_{0}+1, f^{N}(p) \in U_{+}$and $q$ cannot be an accumulation point. Suppose that $q=p-$. Let $(x, y)$ be the local coordinates at $p_{-}$over $U^{-}$given by Theorem 14.18. Consider the norm $\max (|x|,|y|)$ over $U^{-}$. Looking at the normal form of $f$, for any $\varepsilon>0$ small enough, the ball $B\left(p_{-}, \varepsilon\right)$ of center $p_{-}$and radius $\varepsilon$, with respect to this norm, is $f^{-1}$-invariant and we have $f^{-1} B\left(p_{-}, \varepsilon\right) \Subset B\left(p_{-}, \varepsilon\right)$. Therefore if $f^{N_{0}}(p) \in B\left(p_{-}, \varepsilon\right)$, we have $p \in B\left(p_{-}, \varepsilon\right)$. Letting $\varepsilon \rightarrow 0$ we get $p=p_{-}$and this is a contradiction. Therefore, the only accumulation point of $\left(f^{N}(p)\right)_{N}$ is $p_{+}$and it is the limit of this sequence.

### 14.5. Affine surfaces with a cycle at infinity

Let $X_{0}$ be a normal affine surface and suppose that there exists a loxodromic automorphism $f$ of $X_{0}$ such that $\lambda_{1}(f) \notin \mathbf{Z}$. Then, by Theorem 14.4, for any minimal completion $X$ of $X_{0}, \partial_{X} X_{0}$ is a cycle of rational curves and to study the dynamics of a loxodromic automorphism it suffices to consider completions where the boundary remains a cycle of rational curves.
14.5.1. The circle at infinity. Let $X$ be such a completion and let $E_{1}, \cdots, E_{r}$ be the irreducible components of $\partial_{X} X_{0}$. Define $\mathcal{C}_{X} \subset \widehat{\mathcal{V}_{\infty}}$ by

$$
\begin{equation*}
\mathcal{C}_{X}=\bigcup_{i=1}^{r}\left[\operatorname{ord}_{E_{i}}, \operatorname{ord}_{E_{i+1}}\right] \tag{431}
\end{equation*}
$$

$\mathcal{C}_{X}$ consists only of quasimonomial valuations hence it is a subset of $\widehat{\mathcal{V}}_{\infty}^{\prime}$, the subset of valuations of finite skewness. It can therefore be equipped with the strong topology.

Proposition 14.22. For every completion $X$ such that $\partial_{X} X_{0}$ is a cycle of rational curves, one has
(1) $\mathcal{C}_{X}=: \mathcal{C}$ does not depend on $X$.
(2) $\mathcal{C}$ is homeomorphic to $\mathbb{S}^{1}$.
(3) C is characterized as follows: for every continuous embedding $c: \mathbb{S}^{1} \hookrightarrow \widehat{\mathcal{V}_{\infty}}$, $c\left(\mathbb{S}^{1}\right)=\mathcal{C}$.
Proof. For (1) we show that if $\pi: Y \rightarrow X$ is the blow up of a satellite point, then $\mathcal{C}_{Y}=\mathcal{C}_{X}$. Let $p=E \cap F$ be the center of the blow up and let $\widetilde{E}$ be the exceptional divisor. Then, $\left[\operatorname{ord}_{E}, \operatorname{ord}_{F}\right]=\left[\operatorname{ord}_{E}, \operatorname{ord}_{\tilde{E}}\right] \cup\left[\operatorname{ord}_{\tilde{E}}, \operatorname{ord}_{F}\right]$ and we see that $\mathcal{C}_{X}=\mathcal{C}_{Y}$.

For (2), recall that the segment $\left[\operatorname{ord}_{E}, \operatorname{ord}_{F}\right)$ is naturally a subsegment of $\mathcal{V}_{X}(p ; E)$ parametrized with the skewness function $\alpha_{E}$. By Proposition 5.31, we have that $\alpha_{E}=$ $\alpha_{F}{ }^{-1} \operatorname{over}\left(\operatorname{ord}_{E}, \operatorname{ord}_{F}\right)$.

For (3), let $c: \mathbb{S}^{1} \hookrightarrow \widehat{\mathcal{V}_{\infty}}$ be a continuous embedding. Suppose that $c\left(\mathbb{S}^{1}\right) \neq \mathcal{C}$ this means that there exists a completion $X$ and $t_{0} \in \mathbb{S}^{1}$ such that $c\left(t_{0}\right)$ is centered at a free point $p \in E$ at infinity. Let $\left.I_{0}=\right] a, b\left[\right.$ be the largest subsegment of $\mathbb{S}^{1}$ containing $t_{0}$ such that for all $s \in I_{0}, c(s) \in \mathcal{V}_{X}(p ; E)$. Because $\mathcal{V}_{X}(p ; E)$ is open we must have $a=b$ and $c(a)=c(b)=\operatorname{ord}_{E}$. Therefore $c$ is a continuous embedding of $\mathbb{S}^{1}$ into $\mathcal{V}_{X}(p ; E)$ but this is not possible since $V_{X}(p ; E)$ is a tree.
14.5.2. Farey parametrisation. Let $X$ be a completion of $X_{0}$ and let $E$ be a prime divisor at infinity and let $p \in E$. A Farey parametrisation of $\mathcal{D}_{X, p} \cup\{E\}$ is given by the following procedure. Pick positive integers $a_{0}, b_{0}$ such that $\operatorname{gcd}\left(a_{0}, b_{0}\right)=1$ and set $\operatorname{Far}_{\left(E, a_{0}, b_{0}\right)}(E)=\left(a_{0}, b_{0}\right)$, then do the following. Suppose that $\pi: Y \rightarrow X$ is a completion exceptional above $p$ such that $\operatorname{Far}_{\left(E, a_{0}, b_{0}\right)}(F)$ has been defined for every $F \in \Gamma_{\pi, E}$. Then, if $q \in F$ is a free point with respect to $\Gamma_{\pi, E}$, set

$$
\begin{equation*}
\operatorname{Far}_{\left(E, a_{0}, b_{0}\right)}(\widetilde{F})=\left(a_{F}+1, b_{F}\right) \tag{432}
\end{equation*}
$$

where $\operatorname{Far}_{\left(E, a_{0}, b_{0}\right)}(F)=\left(a_{F}, b_{F}\right)$. If $q=F \cap F^{\prime}$ is a satellite point with respect to $\Gamma_{\pi, E}$ then set

$$
\begin{equation*}
\operatorname{Far}(\widetilde{F})=\left(a+a^{\prime}, b+b^{\prime}\right) \tag{433}
\end{equation*}
$$

where $\operatorname{Far}_{\left(E, a_{0}, b_{0}\right)}(F)=(a, b)$ and $\operatorname{Far}_{\left(E, a_{0}, b_{0}\right)}\left(F^{\prime}\right)=\left(a^{\prime}, b^{\prime}\right)$.
Proposition 14.23. Let Far $_{\left(E, a_{0}, b_{0}\right)}$ be a Farey parametrisation of $\mathcal{D}_{X, p} \cup\{E\}$.
(1) $\operatorname{Set} A_{\left(E, a_{0}, b_{0}\right)}(F)=\frac{a}{b}$ where $\operatorname{Far}_{\left(E, a_{0}, b_{0}\right)}(F)=(a, b)$, then $A$ is a parametrisation of $\Gamma_{E}$.
(2) For any $F_{1}, F_{2} \in \Gamma_{E}$ that are adjacent such that $v_{F_{1}}<v_{F_{2}}$ we have

$$
\begin{equation*}
a_{2} b_{1}-a_{1} b_{2}=1 \tag{434}
\end{equation*}
$$

where $\operatorname{Far}_{\left(E, a_{0}, b_{0}\right)}\left(F_{i}\right)=\left(a_{i}, b_{i}\right)$.
(3) If $M=\left(\begin{array}{ll}\alpha & \beta \\ \gamma & \delta\end{array}\right) \in \operatorname{PSL}_{2}(\mathbf{Z})$, then $M \circ \operatorname{Far}(F):=(\alpha a+\beta, \gamma b+\delta)$ is another Farey parametrisation of $\mathcal{D}_{X, p}$.

Proposition 14.24. Let $X$ be a completion of $X_{0}$ and $p=E \cap F$ a satellite point at infinity. Then, the skewness function $\alpha_{E}$ is a Farey parametrisation of $\left[\operatorname{ord}_{E}, \nu_{F}\right)$.

Proof. This uses another parametrisation of the valuative tree defined in [FJ04] called the thinness function. The thinness function $A_{E}$ of the valuative tree $\mathcal{V}_{X}(p ; E)$
is defined by the Farey parametrisation starting with $\operatorname{Far}(E)=(1,1)$. The relation between $A_{E}$ and $\alpha_{E}$ is the following. Define the multiplicity function $m_{E}$ by

$$
\begin{equation*}
\forall \varphi \in \widehat{X, p}, \quad m_{E}(\varphi)=E \cdot{ }_{p}\{\varphi=0\} \tag{435}
\end{equation*}
$$

The multiplicity of a valuation is defined as

$$
\begin{equation*}
m_{E}(v):=\min \left\{m_{E}(\varphi): v_{\varphi} \geqslant v\right\} . \tag{436}
\end{equation*}
$$

and we have

$$
\begin{equation*}
A_{E}(v)=1+\int_{\operatorname{ord}_{E}}^{v} m_{E}(\mu) d \alpha_{E}(\mu) \tag{437}
\end{equation*}
$$

see [FJ04] Definition 3.64. It is clear that on the segment $\left[\operatorname{ord}_{E}, \nu_{F}\right.$ [ we get that $m_{E}$ is constant equal to 1 . Hence, over this segment $A_{E}=1+\alpha_{E}$ and $\alpha_{E}$ is a Farey parametrisation of the segment.

Proposition 14.25. Let $E, F$ be two prime divisors at infinity and let $p=E \cap$ $F$. Let $a_{E}, b_{E}, a_{F}, b_{F}$ be nonnegative integers such that $a_{F} b_{E}-a_{E} b_{F}=1$. If $M=$ $\left(\begin{array}{ll}a_{F} & a_{E} \\ b_{F} & b_{E}\end{array}\right) \in \operatorname{PSL}_{2}(\mathbf{Z})$, then

$$
\begin{equation*}
M \circ \alpha_{E} \tag{438}
\end{equation*}
$$

is the Farey parametrisation of $\left[\operatorname{ord}_{E}, \operatorname{ord}_{F}\right]$ such that $\operatorname{Far}(E)=\left(a_{E}, b_{E}\right)$ and $\operatorname{Far}(F)=$ $\left(a_{F}, b_{F}\right)$.
14.5.3. The Thompson group. The Thompson group is a subgroup of the group of homeomorphism of $\mathbb{S}^{1}$ defined as follows. A homeomorphism $g$ is in the Thompson group if there exists two subdivisions $\cup_{i=1}^{r} I_{i}, \cup_{i=1}^{r} J_{i}$ of $\mathbb{S}^{1}$ into Farey intervals such that $g$ sends $I_{i}$ to $J_{i}$ and $g_{i}: I_{i} \rightarrow J_{i}$ is given by a Mobius transformation with integer coefficients (i.e given by a matrix of $\mathrm{PGL}_{2}(\mathbf{Z})$ ). In particular, the group $\mathrm{PGL}_{2}(\mathbf{Z})$ acting on $\mathbb{S}^{1}$ via Mobius transformations is a subgroup of the Thompson group.

THEOREM 14.26. If $X_{0}$ is an affine surface such that $X \backslash X_{0}$ is a cycle of rational curves, then every automorphism of $X_{0}$ acts on $\mathcal{C} \simeq \mathbb{S}^{1}$ via an element of the Thompson group.

Proof. This is a consequence of Proposition 14.9. Let $f \in \operatorname{Aut}\left(X_{0}\right)$ be an automorphism. Suppose that $Y$ is a completion above $X$ such that the lift $F: Y \rightarrow X$ is regular. Then, satellite points of $Y$ must be sent to satellite points of $X$. Plus, by applying Theorem 2.9 there exists two completions $Y, Z$ above $X$ such that the lift $f: Y \rightarrow Z$ is regular and at every satellite point $f$ is monomial. Let $p=E \cap F$ be a satellite point of $Y$ and $q=f(p) \in Z$. Then, $f$ is of the form

$$
\begin{equation*}
f(x, y)=\left(x^{a} y^{b}, x^{c} y^{d}\right) \tag{439}
\end{equation*}
$$

with $a d-b c= \pm 1$ as the determinant of the matrix of a monomial map is equal to the topological degree and $f$ is invertible. We get $f_{*} \nu_{s, t}=v_{a s+b t, c s+d t}$. Hence, $f_{\bullet}$ sends the Farey interval determined by $\left[\operatorname{ord}_{E}, \operatorname{ord}_{F}\right]$ to another Farey interval of $\mathbb{S}^{1}$ via a Mobius transformation. It acts as an element of the Thompson group.

THEOREM 14.27. There is a group homomorphism $\operatorname{Aut}\left(X_{0}\right) \rightarrow G_{\text {Thompson. The }}$ kernel is up to finite index an algebraic torus of dimension $d \leqslant 2$. And we have the following
(1) If $d=2$, then $X_{0} \simeq \mathbb{G}_{m}^{2}$.
(2) If $X_{0} \not \not \mathbb{G}_{m}^{2}$ and $\operatorname{Aut}\left(X_{0}\right)$ contains a loxodromic element, then $d=0$. In particular, the kernel is finite and $\operatorname{Aut}\left(X_{0}\right)$ is countable.
(3) If $d=1$, then up to finite index

$$
\begin{equation*}
\operatorname{Aut}\left(X_{0}\right) \simeq \mathbb{G}_{m}, \text { or } \operatorname{Aut}\left(X_{0}\right) \simeq \mathbb{G}_{m} \times A \tag{440}
\end{equation*}
$$

where $A$ is a solvable group.
Proof. Let $X$ be a completion of $X_{0}$ such that $\partial_{X} X_{0}$ is a cycle. The kernel is the group $\mathfrak{U}_{1}(C)$ of [Giz71b] where $C=\partial_{X} X_{0}$. Gizatullin showed that the connected component of $\mathfrak{U}_{1}(C)$ must be an algebraic torus of dimension $d \leqslant 2$ and $d=2$ if and only if $X_{0} \simeq \mathbb{G}_{m}^{2}$. Let $K=\mathfrak{U}_{1}(C)$ be the kernel.

Now, Suppose $X_{0} \not \not \mathbb{G}_{m}^{2}$. By [Giz71b] Proposition 1, $X_{0}$ is rational, therefore we can suppose $\operatorname{Aut}\left(X_{0}\right) \subset \operatorname{Bir}\left(\mathbf{P}^{2}\right)$. If $\operatorname{Aut}\left(X_{0}\right)$ contains a loxodromic element, then $\S 7$ of [DP12] shows that if $K$ is infinite, then $K^{0}$ must have an open orbit in $X_{0}$ and therefore $\operatorname{dim} K^{0}=2$ which is a contradiction. Thus $\operatorname{dim} K^{0}=0$ and $K$ is finite, $\operatorname{Aut}\left(X_{0}\right)$ is countable as $G_{\text {Thompson }}$ is.

Finally, if $\operatorname{dim} K^{0}=1$, then $\operatorname{Aut}\left(X_{0}\right)$ does not contain loxodromic elements by the same argument as in the previous paragraph and $\operatorname{Aut}\left(X_{0}\right)$ preserves the fibration over the affine curve $X_{0} / / K^{0}=: C$. Plus $\operatorname{Aut}\left(X_{0}\right)$ acts by conjugation on $K^{0}$ by an action of algebraic groups. But the group of algebraic group automorphism of $K^{0}$ is $\{ \pm 1\}$ because $K^{0} \simeq \mathbb{G}_{m}$. Therefore, up to a finite index subgroup, every element of $\operatorname{Aut}\left(X_{0}\right)$ commutes with the element of $K^{0}$. We have a group homomorphism $\left.\operatorname{Aut}\left(X_{0}\right) \rightarrow \operatorname{Aut}(C)\right)$. Let $\bar{C}$ be the unique projective curve that is a completion of $C$.

If $g(\bar{C}) \geqslant 2$, then $\operatorname{Aut}(C)$ is finite because $\bar{C}$ is of general type and up to finite index $\operatorname{Aut}\left(X_{0}\right) \simeq \mathbb{G}_{m}$.

If $g(\bar{C})=1$, then $\operatorname{Aut}(C)$ is the subgroup of $\operatorname{Aut}(\bar{C})$ that preserves $\bar{C} \backslash C$. This is a finite subgroup, so up to finite index we also get $\operatorname{Aut}\left(X_{0}\right) \simeq \mathbb{G}_{m}$.

Finally, if $g(\bar{C})=0$, then $\bar{C} \simeq \mathbf{P}^{1}$ and $\operatorname{Aut}(C)$ is the subgroup of $\operatorname{Aut}\left(\mathbf{P}^{1}\right) \simeq \operatorname{PGL}_{2}(\mathbf{C})$ that preserves $\mathbf{P}^{1} \backslash C$. If $\#\left(\mathbf{P}^{1} \backslash C\right) \geqslant 3$, then $\operatorname{Aut}(C)$ is finite and we get $\operatorname{Aut}\left(X_{0}\right) \simeq \mathbb{G}_{m}$ up to finite index. Otherwise, up to finite index $\operatorname{Aut}(C)$ is solvable and $\operatorname{Aut}\left(X_{0}\right) \simeq \mathbb{G}_{m} \rtimes A$ where $A$ is a solvable group.

### 14.6. An example: The Markov surface

Let $\mathbf{k}$ be an algebraically closed field with $\operatorname{char} k \neq 2$, let $D \in \mathbf{k}$ and consider the affine surface $M_{D} \subset \mathbf{A}_{\mathbf{k}}^{3}$ of equation

$$
\begin{equation*}
x^{2}+y^{2}+z^{2}=x y z+D \tag{441}
\end{equation*}
$$

For any $D \in \mathbf{k}, M_{D}$ satisfies $\mathrm{QAlb}\left(M_{D}\right)=0$. This is because if we consider the Zariski closure $\bar{M}_{D}$ of $M_{D}$ in $\mathbf{P}^{3}$, it is defined by the equation

$$
\begin{equation*}
T\left(X^{2}+Y^{2}+Z^{2}\right)=X Y Z+D T^{3} \tag{442}
\end{equation*}
$$

Thus, $\bar{M}_{D} \backslash M_{D}$ is the triangle of lines defined by the equations $\{T=0, X Y Z=0\}$. One shows that each line has self intersection -1 , thus the matrix of the intersection form at infinity is given by

$$
\left(\begin{array}{ccc}
-1 & 1 & 1  \tag{443}\\
1 & -1 & 1 \\
1 & 1 & -1
\end{array}\right)
$$

which is nondegenerate. Therefore, $M_{D}$ does not admit nonconstant invertible regular functions and it admits loxodromic automorphisms, thus by Corollary $10.11 M_{D} \neq \mathbb{G}_{m}^{2}$ and $\operatorname{QAlb}\left(M_{D}\right)=0$.

If $D \neq 0,4$, this is a smooth affine surface. If $D=0, M_{0}$ is a normal affine surface with a singularity at $(0,0,0)$. If $D=4, M_{4}$ is a normal affine surface with 4 singularities at the points

$$
\begin{equation*}
( \pm 2, \pm 2, \pm 2) \tag{444}
\end{equation*}
$$

where two of the signs must be equal.
We see that each surface $M_{D}$ falls into the category of the surface with a cycle at infinity. Thus by Theorem 14.27 , there is a group homomorphism $\operatorname{Aut}\left(M_{D}\right) \rightarrow G_{\text {Thompson }}$ with finite kernel. In the case of the Markov surface there is a very explicit description of the automorphism group and its image in the Thompson group.

THEOREM 14.28 ([Can09]). Up to finite index $\operatorname{Aut}\left(M_{D}\right) \simeq \mathrm{GL}_{2}(\mathbf{Z})$, the kernel of $\operatorname{Aut}\left(M_{D}\right) \rightarrow G_{\text {Thompson }}$ is a finite group of size 24 given by the permutations of the coordinates and signs flip. The image of $\operatorname{Aut}\left(M_{D}\right)$ in $G_{\text {Thompson }}$ is exactly the group $\mathrm{PGL}_{2}(\mathbf{Z})$ acting on $\mathbb{S}^{1}$ by Mobius transformations.

For the parameter $D=4$, the action is very explicit. The surface $M_{4}$ is the quotient of $\mathbb{G}_{m}^{2}$ by the involution $\sigma:(u, v) \mapsto\left(u^{-1}, v^{-1}\right)$. The quotient map is given by

$$
\begin{equation*}
(u, v) \in \mathbb{G}_{m}^{2} \mapsto(u+1 / u, v+1 / v, u v+1 / u v) \in M_{4} . \tag{445}
\end{equation*}
$$

This involution has four fixed points: $( \pm 1, \pm 1)$ which gives the four singularities of $M_{4}$. The group $\mathrm{GL}_{2}(\mathbf{Z})$ acts by monomial automorphisms on $\mathbb{G}_{m}^{2}$ commuting with $\sigma$, this gives the embedding $\mathrm{GL}_{2}(\mathbf{Z}) /<\sigma>=\mathrm{PGL}_{2}(\mathbf{Z}) \hookrightarrow \operatorname{Aut}\left(M_{4}\right)$. For more results on the dynamics of $\operatorname{Aut}\left(M_{D}\right)$ and characterization of the case $D=4$, see [RR22].

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[^0]:    ${ }^{1}$ In the litterature, the Galois conjugates of a Perron number have a strictly smaller modulus but we want to include square roots of integers in our definition.

[^1]:    ${ }^{2}$ a quasi-abelian variety is an algebraic group such that there exists an algebraic torus $T$ and an abelian variety $A$ such that the sequence of algebraic groups $0 \rightarrow T \rightarrow X \rightarrow A \rightarrow 0$ is exact.

[^2]:    ${ }^{1} A=O_{\alpha}(B)$ means that there exists a constant $C(\alpha)>0$ such that $A \leqslant C(\alpha) B$.

[^3]:    ${ }^{1}$ isomorphic here means that there exists an order preserving bijection.

