A uniform Tits alternative for endomorphisms of the projective line

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Abstract

The recent article [BHPT24] establishes an analog of the Tits alternative for semigroups of endomorphisms of the projective line. The proof involves a ping-pong argument on arithmetic height functions. Extending this method, we obtain a uniform version of the same alternative. In particular, we show that semigroups of $\mathrm{End}(\mathbb{P}^1)$ of exponential growth are of uniform exponential growth.

1 Introduction

Let $\mathbb{K}$ be an algebraically closed field and let V be a projective variety over $\mathbb{K}$. We denote by $\mathrm{End}(V)$ the set of endomorphisms of V defined over $\mathbb{K}$, which has the structure of a semigroup when endowed with the composition operation. Which types of growth rate can occur for finitely generated subsemigroups of $\mathrm{End}(V)$, and how to characterize semigroups of a given growth rate? This is a natural generalization of the study of finitely generated linear semigroups, an overview of which can be found in [Okn98].

For any $f \in \mathrm{End}(V)$, let $\mathrm{PrePer}(f)$ be its set of *preperiodic points*, that is, the set of points $z \in V(\mathbb{K})$ whose orbits under the iteration of f are finite. An endomorphism $f \in \mathrm{End}(V)$ is said to be *polarized* by an ample line bundle $\mathcal{L}$ if $f^*\mathcal{L} \cong \mathcal{L}^{\otimes d}$ for some $d \in \mathbb{Z}_{\geq 2}$. The integer d is called the *algebraic degree* of f. The notion of polarized endomorphisms was introduced in [Zha95]. Such an endomorphism is also finite (see [Ser60], and [Fak03, §5] for the case of positive characteristic).

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The purpose of this note is to prove the following result:

Theorem 1.1. *Let* $f_1, f_2 \in \mathrm{End}(V)$ *be polarized by the same line bundle and suppose that* $\mathrm{PrePer}(f_1) \neq \mathrm{PrePer}(f_2)$. *Then* f_1 *and* f_2 *generate a free semigroup of rank* 2.

This theorem is proven using a ping-pong argument on *height functions*, which will be introduced in Section 3. The argument used is essentially a refinement of the methods in [BHPT24, §3], and answers Question 5.1 of the same article.

Theorem 1.1 allows us to obtain results concerning uniform independence and uniform exponential growth for certain subsemigroups of $\mathrm{End}(V)$. These results are best stated using the following definitions:

Let S be any semigroup. Two elements of S are said to be *independent* if they generate a free semigroup of rank 2. For any non-empty finite subset $F \subset S$, we define the *diameter of independence* of F as

$$\Delta(F) := \inf \{n \in \mathbb{Z}_{\geq 1} \mid F \cup F^2 \cup \cdots \cup F^n \text{ contains two independent elements}\}$$

where $F^n = \{f_1 \cdots f_n;\ f_i \in F\}$. The *algebraic entropy* of F is defined as

$$\Sigma(F) := \lim_{n \to \infty} \frac{1}{n} \log \#(F \cup F^2 \cup \cdots \cup F^n).$$

Note that $\Delta(F)$ may be infinite, whereas $\Sigma(F)$ is always finite. Moreover, we have the following inequality:

$$\Sigma(F) \geq \log(2)/\Delta(F). \tag{1}$$

If S is finitely generated, its diameter of independence and its algebraic entropy are respectively defined as

$$\Delta(S) := \sup_F \Delta(F) \quad \text{and} \quad \Sigma(S) := \inf_F \Sigma(F)$$

where F ranges over all finite generating sets of S. The semigroup S is of *exponential growth* if $\Sigma(F) > 0$ for some (equivalently, for any) finite generating set F of S; it is of *uniform exponential growth* if $\Sigma(S) > 0$.

In order to accomodate for automorphisms, we introduce the following notion: $f \in \mathrm{End}(V)$ is *semi-polarized* by $\mathcal{L}$ if $f^*\mathcal{L} \cong \mathcal{L}^{\otimes d}$ for some $d \in \mathbb{Z}_{\geq 1}$. In particular, we allow $d = 1$.

Theorem 1.2. *Let* V *be a projective variety over an algebraically closed field* $\mathbb{K}$. *Let* S *be a finitely generated subsemigroup of endomorphsims of* V, *all semi-polarized by the same line bundle. Suppose there are* $f_1, f_2 \in S$ *of algebraic degree at least* 2 *such that* $\mathrm{PrePer}(f_1) \neq \mathrm{PrePer}(f_2)$. *Then* $\Delta(S) \leq 2$. *In particular,* S *is of uniform exponential growth:* $\Sigma(S) \geq \log(2)/2$.

In the case where $\mathrm{char}(\mathbb{K}) = 0$ and $V = \mathbb{P}^1$, we may combine Theorem 1.2 with [BHPT24, Proposition 4.10], along with results of E. Breuillard and T. Gelander [BG05] in the linear case, to obtain the following

Theorem 1.3. *Let S be a finitely generated subsemigroup of $\mathrm{End}(\mathbb{P}^1)$ over a field of characteristic 0. Then either S is of polynomial growth, or $\Delta(S) < +\infty$. In particular, semigroups of exponential growth in $\mathrm{End}(\mathbb{P}^1)$ are of uniform exponential growth.*

2 A ping-pong lemma for contractions

The notions introduced in this section are presented in greater detail in the book [Fal85]. The setting of the book is Euclidean space, but the results still hold in the generality in which we use them.

Let (X, d) be a non-empty complete metric space, in which we fix a base point x_0. A map $\alpha : X \to X$ is called a *contraction with ratio* $c \in]0, 1[$ if

$$\forall x, y \in X,\ d(\alpha(x), \alpha(y)) \leq c \cdot d(x, y).$$

By Banach's fixed point theorem, α has a unique fixed point in X. We denote by $\mathrm{Con}(X)$ the set of contractions on X, a semigroup when endowed with the composition operation. For any subset $F \subset \mathrm{Con}(X)$, we denote by $\langle F \rangle$ the subsemigroup generated by F.

The attractor. Let α_1, α_2 be two contractions with ratios c_1, c_2, and let $C = \{\alpha_1, \alpha_2\}^{\mathbb{N}}$. For any sequence $u = (\alpha_{n_i})_{i\geq 0} \in C$, the sequence $((\alpha_{n_0} \cdots \alpha_{n_i})(x_0))_{i\geq 0}$ is Cauchy, and therefore converges to an element x_u in X. If we endow C with the product topology, the map

$$\pi : C \to X,\ u \mapsto x_u \tag{2}$$

is continuous: its image A is therefore a compact subset of X. The set A is called the *attractor* associated with the system $\{\alpha_1, \alpha_2\}$. For $i \in \{1, 2\}$, we have a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\sigma_i} & C \\ \downarrow \pi & & \downarrow \pi \\ A & \xrightarrow{\alpha_i} & A \end{array}$$

where $\sigma_i : (\alpha_{n_0}, \alpha_{n_1}, \cdots) \mapsto (\alpha_i, \alpha_{n_0}, \alpha_{n_1}, \cdots)$. Since $C = \sigma_1(C) \cup \sigma_2(C)$, A satisfies the self-similarity relation $A = \alpha_1(A) \cup \alpha_2(A)$. It is in fact the unique non-empty compact set that satisfies this relation (see [Fal85, §8.3]).

For any subset $Y \subset X$, we denote its diameter by $\mathrm{diam}(Y)$. The *one-dimensional Hausdorff measure* of Y is defined as follows (see [Fal85, §1.2]):

$$H^1(Y) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_i \mathrm{diam}(U_i) \;\middle|\; Y \subset \bigcup_i U_i \ , \ 0 < \mathrm{diam}(U_i) \leq \varepsilon \right\}$$

Proposition 2.1. *Let $\alpha_1, \alpha_2 : X \to X$ be two injective contractions with ratios $c_1, c_2 \in]0,1[$. Suppose that α_1 and α_2 have distinct fixed points and $c_1 + c_2 \leq 1$. Then $\langle \alpha_1, \alpha_2 \rangle$ is a free semigroup of rank 2.*

Proof. Let A be the attractor associated with $\{\alpha_1, \alpha_2\}$ and $\pi : C \to A$ be the map introduced in equation (2). The first step of the proof is adapted from [BH85].

Step 1. Suppose that A is disconnected. We will prove that $\alpha_1(A) \cap \alpha_2(A) = \varnothing$ and conclude that $\langle \alpha_1, \alpha_2 \rangle$ is free of rank 2.

Since A is disconnected, there are non-empty compact subsets A_1, A_2 such that $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \varnothing$. By the compactness of A,

$$\inf \{ d(x_1, x_2);\ x_1 \in A_1,\ x_2 \in A_2 \} =: \delta > 0.$$

For $u_1, u_2 \in C$, let $\lambda(u_1, u_2)$ be their largest common prefix, and consider the set $\mathcal{P} = \{ \lambda(u_1, u_2);\ (u_1, u_2) \in \pi^{-1}(A_1) \times \pi^{-1}(A_2) \}$. If we choose an integer $n \geq 0$ such that $\delta \geq (\max(c_1, c_2))^n \cdot \mathrm{diam}(A)$, then the words in $\mathcal{P}$ are of length at most n. We may therefore choose a prefix $p \in \mathcal{P}$ of maximal length, as well as a pair $(pu_1, pu_2) \in \pi^{-1}(A_1) \times \pi^{-1}(A_2)$. If there existed an element $y \in \alpha_1(A) \cap \alpha_2(A)$, say

$$y = \pi(\alpha_1 v_1) = \pi(\alpha_2 v_2),$$

then without loss of generality, $p(y) = \pi(p\alpha_1 v_1) = \pi(p\alpha_2 v_2) \in A_1$, and pu_2 would have a common prefix with either $p\alpha_1 v_1$ or $p\alpha_2 v_2$ which would be longer than p, contradicting its maximality. We deduce that $\alpha_1(A) \cap \alpha_2(A) = \varnothing$.

This allows us to perform a ping-pong argument. Suppose there is a pair of distinct words w_1, w_2 in the alphabet $\{\alpha_1, \alpha_2\}$ such that $w_1 = w_2$ in $\langle \alpha_1, \alpha_2 \rangle$. Since α_1 and α_2 are injective, left cancellation implies that $\alpha_1 w_1' = \alpha_2 w_2'$ in $\langle \alpha_1, \alpha_2 \rangle$ for some pair of words w_1', w_2'. But then

$$\alpha_1 w_1'(A) = \alpha_1 w_1'(A) \cap \alpha_2 w_2'(A) \subset \alpha_1(A) \cap \alpha_2(A) = \varnothing$$

which is absurd. We conclude that $\langle \alpha_1, \alpha_2 \rangle$ is free of rank 2.

Step 2. Suppose now that A is connected. We will prove that $H^1(A) = \mathrm{diam}(A)$ and that $\alpha_1(A)$ and $\alpha_2(A)$ have an intersection of zero H^1-measure. This will allow us to perform a measurable version of the ping-pong argument above.

By the self-similarity relation that characterizes the attractor, we have for all $n \geq 0$,

$$\bigcup_{w\in\{\alpha_1,\alpha_2\}^n} w(A) = A.$$

Since α_1 and α_2 are contractions, we have

$$\mathrm{diam}(\alpha_1(A)) + \mathrm{diam}(\alpha_2(A)) \leq c_1 \cdot \mathrm{diam}(A) + c_2 \cdot \mathrm{diam}(A) = (c_1 + c_2) \cdot \mathrm{diam}(A)$$

and more generally, for all $n \geq 0$,

$$\sum_{w\in\{\alpha_1,\alpha_2\}^n} \mathrm{diam}(w(A)) \leq (c_1 + c_2)^n \cdot \mathrm{diam}(A).$$

This construction provides arbitrarily fine covers of A, so that $H^1(A) \leq \mathrm{diam}(A)$, and $H^1(A) = 0$ whenever $c_1 + c_2 < 1$.

Let $x, y \in A$ be such that $d(x, y) = \mathrm{diam}(A)$. The map

$$A \to [0, \mathrm{diam}(A)],\ z \mapsto d(x, z)$$

is surjective because it is continuous and A is connected. Moreover, this map is also 1-Lipschitz, so $\mathrm{diam}(A) = H^1([0, \mathrm{diam}(A)]) \leq H^1(A)$ ([Fal85, Lemma 1.8]). Note that $\mathrm{diam}(A) > 0$ because the fixed points of α_1 and α_2 are distinct. In view of the upper bounds on $H^1(A)$ obtained above, we must have $c_1 + c_2 = 1$ and $H^1(A) = \mathrm{diam}(A)$. Finally,

$$H^1(A) \leq H^1(\alpha_1(A)) + H^1(\alpha_2(A)) \leq c_1 H^1(A) + c_2 H^1(A) = H^1(A)$$

so $H^1(\alpha_1(A) \cap \alpha_2(A)) = 0$.

Suppose there is a distinct pair of words w_1, w_2 in the alphabet $\{\alpha_1, \alpha_2\}$ such that $w_1 = w_2$ in $\langle \alpha_1, \alpha_2 \rangle$. As in the first step, we may assume that $w_1 = \alpha_1 w_1'$ and $w_2 = \alpha_2 w_2'$. Let $B = w_1(A) = w_2(A)$, a connected compact subset of A. Since α_1 and α_2 are injective, B has non-zero diameter and thus $H^1(B) > 0$. But $H^1(B) = H^1(\alpha_1 w_1'(A) \cap \alpha_2 w_2'(A)) = 0$, which is absurd. We conclude that $\langle \alpha_1, \alpha_2 \rangle$ is free of rank 2. □

Remark 2.2. The bound $c_1 + c_2 \leq 1$ in Proposition 2.1 is sharp. Indeed, for $n \geq 2$, let c_n be the solution in $]\frac{1}{2}, 1[$ to the equation $x + x^2 + \cdots + x^n = 1$. Then $\alpha_1 : \mathbb{R} \to \mathbb{R},\ x \mapsto c_n x$ and $\alpha_2 : \mathbb{R} \to \mathbb{R},\ x \mapsto c_n x + 1$ are injective contractions (similitudes, even) with common ratio c_n, but $\alpha_1 \alpha_2^n = \alpha_2 \alpha_1^n$. By increasing n, we get values of c_n that are arbitrarily close to $\frac{1}{2}$.

3 Proof of the Theorems

Heights. Let K be a finitely generated field, V a projective variety over K and $\mathcal{L}$ an ample line bundle on V. We also fix an algebraic closure $\bar{K}$ of K. If K has positive characteristic, we may assume that it is infinite (otherwise every endomorphism over K has the same set of preperiodic points and our theorems are vacuous). We then fix a *Weil height function* $h_{\mathcal{L}} : V(\bar{K}) \to \mathbb{R}$, whose construction is detailed in [BG06, §2.4]. If K is of characteristic zero, we fix a *Moriwaki height function* $h_{\mathcal{L}} : V(\bar{K}) \to \mathbb{R}$, first defined in [Mor00]. Note that if K is a number field, the Moriwaki height coincides with the classical Weil height. In both cases, the function $h_{\mathcal{L}}$ satisfies two properties:

> **Northcott property.** The set $\{x \in V(\bar{K}) \mid h_{\mathcal{L}}(x) \leq a,\ [K(x) : K] \leq b\}$ is finite for any $a, b > 0$.
>
> **Functoriality.** If $f \in \mathrm{End}(V)$ is defined over K and is polarized by $\mathcal{L}$ with algebraic degree d, then $d \cdot h_{\mathcal{L}}$ and $f^* h_{\mathcal{L}}$ differ by a bounded function.

These constructions and their properties are also summarized in [BHPT24, §3.2]. The discussion above motivates the introduction of

$$\mathcal{H}_{\mathcal{L}} := \{h : V(\bar{K}) \to \mathbb{R} \mid \|h - h_{\mathcal{L}}\|_\infty < +\infty\},$$

a complete metric space when endowed with the metric $d(h_1, h_2) := \|h_1 - h_2\|_\infty$. We also define, for any endomorphism f polarized by $\mathcal{L}$ of algebraic degree d,

$$\alpha_f : \mathcal{H}_{\mathcal{L}} \to \mathcal{H}_{\mathcal{L}},\ h \mapsto \frac{1}{d} f^* h.$$

By the surjectivity of $f : V(\bar{K}) \to V(\bar{K})$, we have

$$\forall h_1, h_2 \in \mathcal{H}_{\mathcal{L}}, \quad \|\alpha_f(h_1) - \alpha_f(h_2)\|_\infty = \frac{1}{d}\|h_1 - h_2\|_\infty$$

so α_f is an injective contraction with ratio $\frac{1}{d}$. By Banach's fixed point theorem, α_f has a unique fixed point in $\mathcal{H}_{\mathcal{L}}$ which is called the *canonical height* h_f associated with f and $\mathcal{L}$. It therefore satisfies $f^* h_f = d \cdot h_f$. By the Northcott property, we have $\{x \in V(\bar{K}) \mid h_f(x) = 0\} = \mathrm{PrePer}(f)$.

Proof of Theorem 1.1. We are given two endomorphisms f_1, f_2 of a projective variety V, polarized by the same line bundle $\mathcal{L}$, say $f_i^* \mathcal{L} \cong \mathcal{L}^{\otimes d_i}$ with $d_i \geq 2$, and with distinct sets of preperiodic points. We may fix a finitely generated field over which f_1, f_2, V and $\mathcal{L}$ are defined. This allows us to define the space $\mathcal{H}_{\mathcal{L}}$, as well as the maps $\alpha_1, \alpha_2 : \mathcal{H}_{\mathcal{L}} \to \mathcal{H}_{\mathcal{L}}$ associated with f_1 and f_2. These are injective contractions with ratios $\frac{1}{d_1}, \frac{1}{d_2} \leq \frac{1}{2}$ and fixed points h_{f_1}, h_{f_2}.

Note that the preperiodic points of a polarized endomorphism are isolated, as a consequence of [Fak03, Corollary 2.2]. In particular, the points in $\mathrm{PrePer}(f_i)$ defined over our original field $\mathbb{K}$ coincide with the points in $\mathrm{PrePer}(f_i)$ over $\bar{K}$. Now, since we assume that $\mathrm{PrePer}(f_1) \neq \mathrm{PrePer}(f_2)$, we obtain $h_{f_1} \neq h_{f_2}$. We can therefore apply Proposition 2.1: the semigroup generated by α_1 and α_2 is free of rank 2. Finally, any relation of the form $f_{i_1} \cdots f_{i_n} = f_{j_1} \cdots f_{j_m}$ in $\mathrm{End}(V)$ would imply a relation $\alpha_{i_n} \cdots \alpha_{i_1} = \alpha_{j_m} \cdots \alpha_{j_1}$. From this we conclude that f_1 and f_2 generate a free semigroup of rank 2. □

In order to prove Theorem 1.2, we need to adapt [BHPT24, Lemma 4.5]. For any subset F of endomorphisms of V semi-polarized by the same line bundle, and any $d \geq 1$, we denote by F_d the set of elements of F of algebraic degree d, and by $F_{\geq d}$ the set of elements of F of algebraic degree at least d.

Lemma 3.1. *Let S be a subsemigroup of $\mathrm{End}(V)$, all of whose elements are semi-polarized by the same line bundle. Suppose there is a generating set F of S such that for any $\sigma, \tau \in F_1 \cup \{\mathrm{id}\}$ and any $f, g \in F_{\geq 2}$, we have $\mathrm{PrePer}(\sigma f) = \mathrm{PrePer}(\tau g)$. Then $\mathrm{PrePer}(w_1) = \mathrm{PrePer}(w_2)$ for any $w_1, w_2 \in S_{\geq 2}$.*

Proof. We may assume that $S_{\geq 2}$ is non-empty. Therefore $F_{\geq 2}$ is also non-empty: we fix $f \in F_{\geq 2}$ and define $P = \mathrm{PrePer}(f)$. For any $g \in F_{\geq 2}$, we may set $\sigma = \tau = \mathrm{id}$ to obtain $P = \mathrm{PrePer}(f) = \mathrm{PrePer}(g)$. Similarly, for any $\sigma = F_1$, we have $\mathrm{PrePer}(\sigma f) = \mathrm{PrePer}(f) = P$. In particular, P is both f and σf invariant, so $\sigma(P) = \sigma f(P) = P$. Let $w \in S_{\geq 2}$. Then w is a product of elements of F, and thus $w(P) = P$.

Fix a finitely generated field K over which S is defined, and consider the canonical height h_f of f. We have $P = \{x \in V(\bar{K}) \mid h_f(x) = 0\}$, so by the Northcott property, all w-orbits in P are finite. In other words, $P \subset \mathrm{PrePer}(w)$. As a result of [Fak03, Theorem 5.1], P is Zariski-dense in V. Hence, we can apply [YZ21, Theorem 1.3, (3) $\Rightarrow$ (1)], or [Car20, Theorem A, (3) $\Rightarrow$ (4)] for the case of positive characteristic, to conclude that $P = \mathrm{PrePer}(w)$. □

Proof of Theorem 1.2. Let S be a finitely generated subsemigroup of $\mathrm{End}(V)$, all of whose elements are semi-polarized by the same line bundle. Let F be any finite generating set of S. If $\mathrm{PrePer}(w_1) \neq \mathrm{PrePer}(w_2)$ for some $w_1, w_2 \in S_{\geq 2}$, then by Lemma 3.1, $\mathrm{PrePer}(\sigma f) \neq \mathrm{PrePer}(\tau g)$ for some $\sigma, \tau \in F_1 \cup \{\mathrm{id}\}$ and $f, g \in F_{\geq 2}$. Applying Theorem 1.1, we deduce that σf and τg are independent. Since $\sigma f, \tau g \in F \cup F^2$, we have $\Delta(F) \leq 2$ and thus $\Delta(S) \leq 2$. In particular, using inequality (1),

$$\Sigma(S) \geq \log(2)/\Delta(S) \geq \log(2)/2 > 0,$$

so S is of uniform exponential growth. □

In order to treat the linear case, we shall need the following result from E. Breuillard and T. Gelander.

Theorem 3.2 ([BG05, Theorem 2.3]). *Let K be a finitely generated field, and let $n \geq 1$. There exists a constant $c(n, K) < +\infty$ such that for any subset $F \subset \mathrm{GL}_n(K)$ that generates a non virtually nilpotent group, $\Delta(F) \leq c(n, K)$.*

Proof of Theorem 1.3. Let S be a finitely generated subsemigroup of $\mathrm{End}(\mathbb{P}^1)$ which is not of polynomial growth. Fix a finitely generated field K over which S is defined. Since constant maps in $\mathrm{End}(\mathbb{P}^1)$ are left-absorbing, they cannot be part of an independent pair. In other words, $\Delta(S) = \Delta(S_{\geq 1})$, so we may assume that S contains no non-constant maps.

If $S_{\geq 2}$ is non-emtpy, then by [BHPT24, Proposition 4.10], $\mathrm{PrePer}(f) \neq \mathrm{PrePer}(g)$ for some $f, g \in S_{\geq 2}$, and so by Theorem 1.2 we have $\Delta(S) \leq 2 < +\infty$. Suppose then that S only contains elements of degree 1. Let F be any finite generating set of S. The group generated by F in $\mathrm{PGL}_2(K)$ is not of polynomial growth, so is not virtually nilpotent ([Wol68]). By Theorem 3.2, there exists a constant $c < +\infty$ that only depends on K such that $\Delta(F) \leq c$. We conclude that $\Delta(S) \leq c < +\infty$.

In particular, if S is of exponential growth, then $\Sigma(S) \geq \log(2)/\Delta(S) > 0$ so S is of uniform exponential growth. □

Remark 3.3. Note that in the case $S_{\geq 2} \neq \varnothing$, the upper bound on the diameter of independence is absolute, whereas in the linear case, it depends on the choice of semigroup. Indeed, there is no uniform upper bound on $\Delta(S)$ for all semigroups $S \subset \mathrm{PGL}_2$ of exponential growth (see [Bre06, Theorem 1.7]; this phenomenon only occurs when S generates a virtually solvable, non virtually nilpotent group). On the other hand, the existence of a uniform lower bound on $\Sigma(S)$ for all semigroups $S \subset \mathrm{PGL}_2$ of exponential growth is still unknown; in fact it would imply Lehmer's conjecture (see [Bre06, Question 1.8]).

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