HOLOMORPHIC ACTIONS, KUMMER EXAMPLES, AND ZIMMER PROGRAM

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ABSTRACT. We classify compact Kähler manifolds M of dimension $n \geq 3$ on which acts a lattice of an almost simple real Lie group of rank $\geq n-1$. This provides a new line in the so-called Zimmer program, and characterizes certain complex tori as compact Kähler manifolds with large automorphisms groups.

RÉSUMÉ. Nous classons les variétés compactes kählériennes M de dimension $n \geq 3$ munies d'une action d'un réseau Γ dans un groupe de Lie réel presque simple de rang $\geq n-1$. Ceci complète le programme de Zimmer dans ce cadre, et caractérise certains tores complexes compacts par des propriétés de leur groupe d'automorphismes.

AMS CLASSIFICATION. 22E40 (32J27)

KEY WORDS. Lattices, superrigidity, complex tori, automorphism groups, Hodge theory, invariant cones, holomorphic dynamics.

CONTENTS

1. INTRODUCTION

1.1. **Zimmer Program.** Let G be an almost simple real Lie group. The **real rank** $\mathsf{rk}_{\mathbf{R}}(G)$ of G is the dimension of a maximal abelian subgroup of G that acts by $\mathbf{R}$-diagonalizable endomorphisms in the adjoint representation of G

Date: 2010.

on its Lie algebra $\mathfrak{g}$; for example, the real Lie groups $\mathsf{SL}_n(\mathbf{R})$ and $\mathsf{SL}_n(\mathbf{C})$ have rank $n-1$. When $\mathsf{rk}_{\mathbf{R}}(G)$ is at least 2, we say that G is a **higher rank** almost simple Lie group. Let Γ be a **lattice** in such a higher rank Lie group G; by definition, Γ is a discrete subgroup of G such that G/Γ has finite Haar volume. Margulis superrigidity theorem implies that all finite dimensional linear representations of Γ are built from representations in unitary groups and representations of the Lie group G itself. In particular, there is no faithful linear representation of Γ in dimension $\leq \mathsf{rk}_{\mathbf{R}}(G)$ (see Remark 3.4 below).

Zimmer's program predicts a similar picture for actions of Γ by diffeomorphims on compact manifolds, at least when the dimension $\dim(V)$ of the manifold V is close to the minimal dimension of non trivial linear representations of G (see [28]). For instance, a central conjecture asserts that lattices in simple Lie groups of rank n do not act faithfully on compact manifolds of dimension less than n (see [64, 63, 65, 32]).

In this article, we pursue the study of Zimmer's program for holomorphic actions on compact Kähler manifolds, as initiated in [16] and [19, 20].

1.2. **Automorphisms.** Let M be a compact complex manifold of dimension n. By definition, diffeomorphisms of M which are holomorphic are called **automorphisms**. According to Bochner and Montgomery [11, 14], the group $\mathsf{Aut}(M)$ of all automorphisms of M is a complex Lie group, its Lie algebra is the algebra of holomorphic vector fields on M. Let $\mathsf{Aut}(M)^0$ be the connected component of the identity in $\mathsf{Aut}(M)$, and

$$\mathsf{Aut}(M)^\sharp = \mathsf{Aut}(M)/\mathsf{Aut}(M)^0$$

be the group of connected components. This group can be infinite, and is hard to describe: For example, it is not known whether there exists a compact complex manifold M for which $\mathsf{Aut}(M)^\sharp$ is not finitely generated.

When M is a Kähler manifold, Lieberman and Fujiki proved that $\mathsf{Aut}(M)^0$ has finite index in the kernel of the action of $\mathsf{Aut}(M)$ on the cohomology of M (see [29, 45]). Thus, if a subgroup Γ of $\mathsf{Aut}(M)$ embeds into $\mathsf{Aut}(M)^\sharp$, the action of Γ on the cohomology of M has finite kernel; in particular, the group $\mathsf{Aut}(M)^\sharp$ almost embeds in the group $\mathrm{Mod}(M)$ of isotopy classes of smooth diffeomorphisms of M. When M is simply connected or, more generally, has nilpotent fundamental group, $\mathrm{Mod}(M)$ is naturally described as the group of integer matrices in a linear algebraic group (see [54]). Thus, $\mathsf{Aut}(M)^\sharp$ sits naturally in an arithmetic lattice. Our main result goes in the other direction: It describe the largest possible lattices contained in $\mathsf{Aut}(M)^\sharp$.

1.3. **Rigidity and Kummer examples.** The main examples that provide large groups $\Gamma \subset \mathsf{Aut}(M)^\sharp$ come from linear actions on carefully chosen complex tori.

Example 1.1. Let $E = \mathbf{C}/\Lambda$ be an elliptic curve and n be a positive integer. Let T be the torus $E^n = \mathbf{C}^n/\Lambda^n$. The group $\mathsf{Aut}(T)$ is the semi-direct product of $\mathsf{SL}(n, \mathsf{End}(E))$ by T, acting by translations on itself. In particular, the connected component $\mathsf{Aut}(T)^0$ coincides with the group of translations, and $\mathsf{Aut}(T)$ contains all linear transformations $z \mapsto B(z)$ where B is in $\mathsf{SL}_n(\mathbf{Z})$. If Λ is the lattice of integers O_d in an imaginary quadratic number field $\mathbf{Q}(\sqrt{d})$, where d is a squarefree negative integer, then $\mathsf{Aut}(T)$ contains a copy of $\mathsf{SL}_n(O_d)$.

Example 1.2. Starting with the previous example, one can change Γ into a finite index subgroup Γ_0, and change T into a quotient T/F where F is a finite subgroup of $\mathsf{Aut}(T)$ which is normalized by Γ_0. In general, T/F is an orbifold (a compact manifold with quotient singularities), and one needs to resolve the singularities in order to get an action on a smooth manifold M. The second operation that can be done is blowing up finite orbits of Γ. This provides infinitely many compact Kähler manifolds of dimension n with actions of lattices $\Gamma \subset \mathsf{SL}_n(\mathbf{R})$ (resp. $\Gamma \subset \mathsf{SL}_n(\mathbf{C})$).

In these examples, the group Γ is a lattice in a real Lie group of rank $(n-1)$, namely $\mathsf{SL}_n(\mathbf{R})$ or $\mathsf{SL}_n(\mathbf{C})$, and Γ acts on a manifold M of dimension n. Moreover, the action of Γ on the cohomology of M has finite kernel and a finite index subgroup of Γ embeds in $\mathsf{Aut}(M)^\sharp$. Since this kind of construction is at the heart of the article, we introduce the following definition, which is taken from [17, 19].

Definition 1.3. *Let Γ be a group, and $\rho : \Gamma \to \mathsf{Aut}(M)$ a morphism into the group of automorphisms of a compact complex manifold M. This morphism is a* **Kummer example** *(or, equivalently, is of* **Kummer type***) if there exists*

- *a birational morphism $\pi : M \to M_0$ onto an orbifold M_0,*
- *a finite orbifold cover $\varepsilon : T \to M_0$ of M_0 by a torus T, and*
- *morphisms $\eta : \Gamma \to \mathsf{Aut}(T)$ and $\eta_0 : \Gamma \to \mathsf{Aut}(M_0)$*

such that $\varepsilon \circ \eta(\gamma) = \eta_0(\gamma) \circ \varepsilon$ and $\eta_0(\gamma) \circ \pi = \pi \circ \rho(\gamma)$ for all γ in Γ.

The notion of **orbifold** used in this text refers to compact complex analytic spaces with a finite number of singularities of quotient type; in other words, M_0 is locally the quotient of $(\mathbf{C}^n, 0)$ by a finite group of linear tranformations (see Section 2.3).

Since automorphisms of a torus $\mathbf{C}^n/\Lambda$ are covered by affine transformations of $\mathbf{C}^n$, all Kummer examples come from actions of affine transformations on affine spaces.

1.4. **Results.** The following statement is our main theorem. It confirms Zimmer's program, in its strongest versions, for holomorphic actions on compact Kähler manifolds: We get a precise description of all possible actions of lattices $\Gamma \subset G$ for $\mathsf{rk}_{\mathbf{R}}(G) \geq \dim_{\mathbf{C}}(M) - 1$.

Main Theorem. *Let G be an almost simple real Lie group and Γ be a lattice in G. Let M be a compact Kähler manifold of dimension $n \geq 3$. Let $\rho : \Gamma \to \mathsf{Aut}(M)$ be a morphism with infinite image.*

(0) *The real rank $\mathsf{rk}_{\mathbf{R}}(G)$ is at most equal to the complex dimension of M.*

(1) *If $\mathsf{rk}_{\mathbf{R}}(G) = \dim(M)$, the group G is locally isomorphic to $\mathsf{SL}_{n+1}(\mathbf{R})$ or $\mathsf{SL}_{n+1}(\mathbf{C})$ and M is biholomorphic to the projective space $\mathbb{P}^n(\mathbf{C})$.*

(2) *If $\mathsf{rk}_{\mathbf{R}}(G) = \dim(M) - 1$, there exists a finite index subgroup Γ_0 in Γ such that either*

(2-a) *$\rho(\Gamma_0)$ is contained in $\mathsf{Aut}(M)^0$, and $\mathsf{Aut}(M)^0$ contains a subgroup which is locally isomorphic to G, or*

(2-b) *G is locally isomorphic to $\mathsf{SL}_n(\mathbf{R})$ or $\mathsf{SL}_n(\mathbf{C})$, and the morphism $\rho : \Gamma_0 \to \mathsf{Aut}(M)$ is a Kummer example.*

Moreover, all examples corresponding to assertion (2-a) are described in Section 4.6 and all Kummer examples of assertion (2-b) are described in Section 7. In particular, for these Kummer examples, the complex torus T associated to M and the lattice Γ fall in one of the following three possible examples:

• $\Gamma \subset \mathsf{SL}_n(\mathbf{R})$ is commensurable to $\mathsf{SL}_n(\mathbf{Z})$ and T is isogenous to the product of n copies of an elliptic curve $\mathbf{C}/\Lambda$;

• $\Gamma \subset \mathsf{SL}_n(\mathbf{C})$ is commensurable to $\mathsf{SL}_n(O_d)$ where O_d is the ring of integers in $\mathbf{Q}(\sqrt{d})$ for some negative integer d, and T is isogenous to the product of n copies of the elliptic curve $\mathbf{C}/O_d$;

• In the third example, $n = 2k$ is even. There are positive integers a and b such that the quaternion algebra $\mathbf{H}_{a,b}$ over the rational numbers $\mathbf{Q}$ defined by the basis $(1, \mathsf{i}, \mathsf{j}, \mathsf{k})$, with

$$\mathsf{i}^2 = a, \ \mathsf{j}^2 = b, \ \mathsf{ij} = \mathsf{k} = -\mathsf{ji}$$

is an indefinite quaternion algebra and the lattice Γ is commensurable to the lattice $\mathsf{SL}_k(H_{a,b}(\mathbf{Z}))$. The torus T is isogenous to the product of k copies of an abelian surface Y which contains $\mathbf{H}_{a,b}(\mathbf{Q})$ in its endomorphism algebra $\mathsf{End}_{\mathbf{Q}}(Y)$. Once a and b are fixed, those surfaces Y depend on one complex parameter, hence T depends also on one parameter; for some parameters Y is isogenous to the product of 2 copies of the elliptic curve $\mathbf{C}/\mathcal{O}_d$, and T is isogenous to $(\mathbf{C}/\mathcal{O}_d)^n$ with $d = -ab$ (see §7 for precise definitions and details).

As a consequence, Γ *is not cocompact, T is an abelian variety and M is projective.*

Remark 1.4. In dimension 2, [18] shows that all faithful actions of infinite discrete groups with Kazhdan property (T) by birational transformations on projective surfaces are birationally conjugate to actions by automorphisms on the projective plane $\mathbb{P}^2(\mathbf{C})$; thus, part (1) of the Main Theorem holds in the more general setting of birational actions and groups with Kazhdan property (T). Part (2) does not hold in dimension 2 for lattices in the rank 1 Lie group $\mathsf{SO}_{1,n}(\mathbf{R})$ (see [18, 27] for examples).

1.5. **Strategy of the proof and complements.** Sections 2 and 3 contain important preliminary facts, as well as a side result which shows how representation theory and Hodge theory can be used together in our setting (see §3.4).

The proof of the Main Theorem starts in §4: Assertion (1) is proved, and a complete list of all possible pairs (M,G) that appear in assertion (2-a) is obtained. This makes use of a previous result on Zimmer conjectures in the holomorphic setting (see [16], in which assertion (0) is proved), and classification of homogeneous or quasi-homogeneous spaces (see [2, 33, 39]). On our way, we describe Γ-invariant analytic subsets $Y \subset M$ and show that these subsets can be blown down to quotient singularities.

The core of the paper is to prove assertion (2-b) when the image $\rho(\Gamma_0)$ is not contained in $\mathsf{Aut}(M)^0$ (for all finite index subgroups of Γ) and $\mathsf{rk}_{\mathbf{R}}(G)$ is equal to $\dim(M) - 1$.

In that case, Γ acts almost faithfully on the cohomology of M, and this linear representation extends to a continuous representation of G on $H^*(M,\mathbf{R})$ (see §3). Section 5 shows that G preserves a non-trivial cone contained in the closure of the Kähler cone $\mathcal{K}(M) \subset H^{1,1}(M,\mathbf{R})$; this general fact holds for all linear representations of semi-simple Lie groups G for which a lattice $\Gamma \subset G$ preserves a salient cone (here Γ preserves the Kähler cone in $H^{1,1}(M)$). Section 5 can be skipped in a first reading.

Then, in §6, we apply ideas of Dinh and Sibony, of Zhang, and of our previous manuscripts together with representation theory. We fix a maximal torus A in G and study the eigenvectors of A in the G-invariant cone: Hodge index Theorem constrains the set of weights and eigenvectors; since the Chern classes are invariant under the action of G, this provides strong constraints on them. When there is no Γ-invariant analytic subset of positive dimension, Yau's Theorem can then be used to prove that M is a torus. To conclude the proof, we blow down all invariant analytic subsets to quotient singularities (see §4), and apply Hodge and Yau's Theorems in the orbifold setting.

Section 7 lists all tori of dimension n with an action of a lattice in a simple Lie group of rank $n-1$; it provides also a few consequences that follow easily from this classification. Since Sections 4.6 and 7 provide complements to the Main Theorem, we recommend to skip them in a first reading.

1.6. **Aknowledgment.** Thanks to Sébastien Boucksom, Michel Brion, Antoine Chambert-Loir, Jean-Pierre Demailly, Igor Dolgachev, Stéphane Druel, Ludovic Marquis, François Maucourant and Jean-François Quint for nice discussions, comments, and ideas. Demailly provided the proof of Theorem 2.2, Brion and Dolgachev helped us clarify Section 4.1, and Chambert-Loir made several important suggestions, including a better use of complex multiplication theory in Section 7. The anonymous referee did a wonderful job, which greatly improved the quality of the exposition; thanks to his careful reading, several inaccuracies have been corrected.

This paper contains the main results of our manuscripts [19] and [20]; we thank the editorial board of the Annales Scientifiques for its proposition to merge these two texts in a new one.

2. Cohomology, Hodge theory, orbifolds

Let M be a connected, compact, Kähler manifold of complex dimension n.

2.1. **Hodge Theory and cohomological automorphisms.**

2.1.1. *Hodge decomposition.* Hodge theory implies that the cohomology groups $H^k(M,\mathbf{C})$ decompose into direct sums

$$H^k(M,\mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(M,\mathbf{C}),$$

where cohomology classes in $H^{p,q}(M,\mathbf{C})$ are represented by closed forms of type (p,q). This bigraded structure is compatible with the cup product. Complex conjugation permutes $H^{p,q}(M,\mathbf{C})$ with $H^{q,p}(M,\mathbf{C})$. In particular, the cohomology groups $H^{p,p}(M,\mathbf{C})$ admit a real structure, with real part

$$H^{p,p}(M,\mathbf{R}) = H^{p,p}(M,\mathbf{C}) \cap H^{2p}(M,\mathbf{R}).$$

If $[\kappa]$ is a **Kähler class** (i.e. the cohomology class of a Kähler form), then $[\kappa]^p \in H^{p,p}(M,\mathbf{R})$ for all p.

2.1.2. *Notation.* In what follows, the vector space $H^{1,1}(M,\mathbf{R})$ is denoted W.

2.1.3. *Cohomological automorphisms.*

Definition 2.1. A **cohomological automorphism** of M is a linear isomorphism of the real vector space $H^*(M,\mathbf{R})$ that preserves the Hodge decomposition, the cup product, and the Poincaré duality.

Note that cohomological automorphisms are not assumed to preserve the set of Kähler classes or the lattice $H^*(M,\mathbf{Z})$, as automorphisms f^* with $f \in \mathsf{Aut}(M)$ do.

2.1.4. *Primitive classes and Hodge index Theorem.* Let $[\kappa] \in W$ be a Kähler class (alternatively, Kähler classes are also called **ample** classes). The set of primitive classes with respect to $[\kappa]$ is the vector space of classes $[u]$ in W such that

$$\int_M [\kappa]^{n-1} \wedge [u] = 0.$$

Hodge index theorem implies that the quadratic form

$$([u],[v]) \mapsto \int_M [\kappa]^{n-2} \wedge [u] \wedge [v]$$

is negative definite on the space of primitive forms (see [58], §6.3.2). We refer the reader to [15], [26], [25] and [62] for stronger results and for consequences of Hodge index Theorem on groups of automorphisms of M.

2.2. **Nef cone and big classes.** Recall that a convex cone in a real vector space is **salient** when its closure does not contain any line. In particular, a salient cone is strictly contained in a half space.

The **Kähler cone** of M is the subset $\mathcal{K}(M) \subset W$ of Kähler classes. This set is an open convex cone; its closure $\overline{\mathcal{K}}(M)$ is a salient and closed convex cone, the interior of which coincides with $\mathcal{K}(M)$. We shall say that $\overline{\mathcal{K}}(M)$ is the cone of **nef** cohomology classes of type $(1,1)$ ("nef" stands for "numerically effective" and also for "numerically eventually free"). All these cones are invariant under the action of $\mathsf{Aut}(M)$.

A class w in $H^{1,1}(M,\mathbf{R})$ is **big and nef** if it is nef and $\int_M w^n > 0$. The cone of big and nef classes plays an important role in this paper. The following result extends a theorem proved by Nakamaye in the case of projective varieties (see [44], volume II, chapter 10.3, and [48]).

Theorem 2.2 (Demailly and Paun)**.** *Let M be a compact Kähler manifold, and $w \in H^{1,1}(M,\mathbf{R})$ be a big and nef class which is not a Kähler class. Then*

(0) *for all irreducible analytic subsets $X \subset M$, $\int_X w^{\dim(X)} \geq 0$;*
(1) *there exists an irreducible analytic subset $Y \subset M$ of positive dimension such that*
$$\int_Y w^{\dim(Y)} = 0;$$
(2) *the union of all these analytic subsets Y is (contained in) a proper Zariski closed subset $Z \subset M$.*

The proof follows from two important results proved by Demailly and Paun, using closed positive currents with logarithmic singularities. By definition, a $(1,1)$-current is positive if

$$\int_M T \wedge (\mathrm{i}u_1 \wedge \overline{u_1}) \wedge \ldots \wedge (\mathrm{i}u_{n-1} \wedge \overline{u_{n-1}}) \geq 0$$

for all smooth $(1,0)$-forms u_1, ..., u_{n-1}; it is closed if $\int_M T \wedge (d\beta) = 0$ for all smooth $(n-2)$-forms. Let κ be a Kähler form on M. According to [24], a **Kähler current** is a closed $(1,1)$-current such that $T - \delta\kappa$ is a positive current for some positive real number δ. The cohomology class of a Kähler current needs not be a Kähler class; for example, on a surface X, the sum $T = \varepsilon\kappa + \{E\}$ of a Kähler form $\varepsilon\kappa$, $\varepsilon > 0$, and the current of integration on a curve E is a Kähler current, but if E has negative self-intersection and ε is small, then the cohomology class of T has negative self-intersection.

Theorem 2.3 (see [23], [24], and [7])**.** *Let M be a compact Kähler manifold.*

(1) *If $w \in H^{1,1}(M,\mathbf{R})$ is a nef and big class (as defined above), then w contains a Kähler current.*
(2) *Let T_0 be a Kähler current. There exists a Kähler current T_1 which satisfies the following two properties: (i) T_1 is cohomologous to T_0 and (ii) there is an open cover $\{U_j\}$ of M such that, on each U_j, T_1 is equal to $dd^c(\phi_j + u_j)$ where u_j is a smooth function and*
$$\phi_j = c_j \log\left(\sum_{i=1}^{m} |g_{j,i}|^2\right)$$
for some positive constant c_j and holomorphic functions $g_{j,i} \in O(U_j)$.

Let w be a big and nef class. According to Theorem 2.3, the class w is represented by a Kähler current T which is locally of the form $dd^c(\phi+u)$ with u smooth and $\phi = c\log(\sum_i |g_i|^2)$ on elements U of an open cover of M (we drop the index j for simplicity). Let Z_T be the analytic subset of M which is locally defined as the intersection of the zero-locus of the functions g_i.

Lemma 2.4. *Let Y be an irreducible analytic subset of M which is not contained in Z_T. If $\dim_{\mathbf{C}}(Y) > 0$ then*

$$\int_Y w^{\dim_{\mathbf{C}}(Y)} > 0.$$

Since Y may be singular, the proof makes use of closed positive currents and plurisubharmonic functions on singular analytic subsets $Y \subset M$; the required definitions and properties are described in [22], Chapter 1, and [24].

Proof Lemma 2.4. Let κ be a Kähler form on M. Volumes and distances are defined with respect to the hermitian metric associated to κ.

Since Y is not contained in Z_T, the restriction $\phi_{|Y}$ of ϕ to Y is not everywhere $-\infty$, and defines a plurisubharmonic function on Y. Thus, the restriction $S = T_{|Y}$ of T to Y can be defined locally as

$$S = dd^c(\phi_{|Y} + u_{|Y}), \tag{2.1}$$

where u is smooth and $\phi = c\log(\sum_i |g_i|^2)$, as above.

By assumption, $T \geq \delta\kappa$ for some positive number δ. Let us show, first, that $S - (\delta/2)\kappa \geq 0$, so that S is a Kähler current on Y. Let $\gamma = (\mathrm{i}u_1 \wedge \overline{u_1}) \wedge \ldots \wedge (\mathrm{i}u_{p-1} \wedge \overline{u_{p-1}})$ where the u_i are $(1,0)$-forms. We need to prove that

$$\int_Y S \wedge \gamma \geq \int_Y (\delta/2)\kappa \wedge \gamma;$$

for this, we can restrict the study to an open set U on which S is defined by Formula 2.1. Let O_ε be the intersection of the ε-tubular neighborhood of Z_T with Y. Let $\{f, (1-f)\}$ be a partition of 1 on the open set U, with $f = 1$ near Z_T and $f = 0$ in the complement of O_ε. On U, we have

$$S - (\delta/2)\kappa = dd^c(\phi + w), \quad \text{with} \quad w = u + v,$$

where $(-2/\delta)v$ is a smooth local potential of κ. One has

$$\left| \int_{O_\varepsilon} dd^c(w) \wedge (f\gamma) \right| \leq c^{ste} \parallel \gamma \parallel \mathrm{vol}_\kappa(O_\varepsilon),$$

and this quantity goes to 0 with ε (the constant c^{ste} depends on w but does not depend on ε). Since ϕ is plurisubharmonic, $\int_{O_\varepsilon} dd^c(\phi) \wedge (f\gamma) \geq 0$ and thus

$$\lim_{\varepsilon \to 0} \int_{O_\varepsilon} (S - (\delta/2)\kappa) \wedge (f\gamma) \geq 0.$$

On the support of $(1-f)$ both functions ϕ and w are smooth, and $dd^c(\phi+w)\geq(\delta/2)\kappa$, by definition of δ. Thus

$$\begin{aligned}\int_{U\cap Y}(S-(\delta/2)\kappa)\wedge\gamma &= \int_{O_\varepsilon}(S-(\delta/2)\kappa)\wedge(f\gamma)+\int_U(S-(\delta/2)\kappa)\wedge(1-f)\gamma\\ &\geq \int_{O_\varepsilon}S\wedge(f\gamma)+\int_U(1-f)(\delta/2)\kappa\wedge\gamma.\end{aligned}$$

Since the first term of the right-hand side has a positive limit with ε going to 0, we obtain

$$\int_U(S-(\delta/2)\kappa)\wedge\gamma\geq\int_U(\delta/2)\kappa\wedge\gamma.$$

We can now prove that $\int_Y w^p>0$, where p denotes the dimension of Y. Since w is nef, $T+\varepsilon\kappa$ is cohomologous to a Kähler form κ_ε for all $\varepsilon>0$. We then have

$$\begin{aligned}\int_Y(w+\varepsilon\kappa)^p &= \int_Y(w+\varepsilon\kappa)\wedge(\kappa_\varepsilon)^{p-1}\\ &= \int_Y(S+\varepsilon\kappa)\wedge(\kappa_\varepsilon)^{p-1}\\ &\geq \int_Y S\wedge(\kappa_\varepsilon)^{p-1}\\ &\geq \int_Y(\delta/2)\kappa\wedge(\kappa_\varepsilon)^{p-1}\\ &\geq \int_Y(\delta/2)\kappa\wedge(w+\varepsilon\kappa)\wedge(\kappa_\varepsilon)^{p-2}.\end{aligned}$$

We can then replace $(w+\varepsilon\kappa)$ by $S+\varepsilon\kappa$ and apply the inequality $S\geq(\delta/2)\kappa$ again; in $p-1$ steps we get

$$\int_Y(w+\varepsilon\kappa)^p\geq(\delta/2)^p\int_Y\kappa^p=(\delta/2)^p\mathrm{vol}_\kappa(Y).$$

The Lemma is proved. □

Proof of Theorem 2.2. Property (0) is a direct consequence of the fact that w is in the closure of the Kähler cone. The existence of Y in property (1) follows from Theorem 0.1 in [24]: According to that result, the Kähler cone is a connected component of the set of classes $w\in H^{1,1}(M,\mathbf{R})$ such that $\int_Y w^{\dim_{\mathbf{C}}Y}>0$ for all irreducible analytic subsets of M.

To prove Property (2) define Z as the intersection of the analytic subsets $Z_T\subset M$, where T describes the set of Kähler currents with logarithmic singularities and cohomology class equal to w. By Theorem 2.3, this is a proper analytic subset of M. Lemma 2.4 implies that $\int_Y w^{\dim(Y)}>0$ as soon as Y is not contained in Z, is irreducible, and has positive dimension. Thus, all analytic subsets Y that occur in Property (1) are contained in Z. □

2.3. **Orbifolds.** In this paper, an orbifold M_0 of dimension n is a compact complex analytic space with a finite number of quotient singularities q_i; in a neighborhood of each q_i, M_0 is locally isomorphic to the quotient of $\mathbf{C}^n$ near the origin by a finite group of linear transformations. All examples of orbifolds considered in this paper are locally isomorphic to $\mathbf{C}^n/\eta_i$ where η_i is a scalar multiplication of finite order k_i. Thus, the singularity q_i can be resolved by one blow-up: The point q_i is then replaced by a hypersurface Z_i which is isomorphic to $\mathbb{P}^{n-1}(\mathbf{C})$ with normal bundle $O(-k_i)$. On our way to the proof of the Main Theorem, this type of singularities occurs when we contract hypersurfaces of the smooth manifold M which are invariant under the action of the lattice Γ (see Theorem 4.5, where we show that these hypersurfaces are copies of $\mathbb{P}^{n-1}(\mathbf{C})$, with a negative normal bundle).

All classical objects from complex differential geometry are defined on M_0 as follows. Usual definitions are applied on the smooth part $M_0 \setminus \{q_1, ..., q_k\}$ and, around each singularity q_i, one requires that the objects come locally from η_i-invariant objects on $\mathbf{C}^n$. Classical facts, like Hodge decomposition, Hodge index Theorem, Yau's Theorem, remain valid in the context of orbifolds. The reader will find more details in [51], [53], [13] and [61].

3. Lattices, rigidity, and action on cohomology

In Sections 3.1 and 3.2, we list important facts concerning lattices in Lie groups. The reader may consult [56] and [4, 6] for two nice introductions to this subject. Those properties are applied to actions of automorphisms on cohomology groups in Sections 3.3 and 3.4.

3.1. **Classical properties of lattices.** One feature of the theory of semi-simple Lie groups is that we can switch viewpoint from Lie groups to linear algebraic groups. We shall use this almost permanently.

3.1.1. *Borel density theorem (see* [56]*, page 37, or* [4]*).* Let G be a linear algebraic semi-simple Lie group with no compact normal subgroup of positive dimension. If Γ is a lattice in G, then Γ is Zariski-dense in G.

3.1.2. *Kazhdan property* (T) *(see* [21, 3]*).* We shall say that a topological group F has **Kazhdan property (T)** if F is locally compact and every continuous action of F by affine unitary motions on a Hilbert space has a fixed point (see [3] for equivalent definitions). If F has Kazhdan property (T) and Λ is a lattice in F, then Λ inherits property (T). If G is a simple real Lie group with rank $\mathsf{rk}_{\mathbf{R}}(G) \geq 2$, then G and all its lattices satisfy Kazhdan property (T). If F is a discrete group with property (T), then

- F is finitely generated;
- every morphism from F to $\mathsf{GL}_2(k)$, k any field, has finite image (see [37] and [63]);
- every morphism from Γ to a solvable group has finite image.

Lemma 3.1. *Every morphism from a discrete Kazhdan group F to the group of automorphisms of a compact Riemann surface has finite image.*

Proof. The automorphisms group of a connected Riemann surface X is either finite (when the genus $g(X)$ is > 1), virtually abelian (when $g(X) = 1$), or isomorphic to $\mathsf{PGL}_2(\mathbf{C})$. □

3.2. Margulis superrigidity.

3.2.1. *Superrigidity.* Let H be a group. A property is said to hold **virtually** for H if a finite index subgroup of H satisfies this property. Similarly, a morphism $h : \Gamma \to L$ from a subgroup Γ of H to a group L virtually extends to H if there is a finite index subgroup Γ_0 in Γ and a morphism $\hat{h} : H \to L$ such that $\hat{h}$ coincides with h on the subgroup Γ_0.

The following theorem is one version of the superrigidity phenomenon for linear representations of lattices (see [46] or [56]).

Theorem 3.2 (Margulis)**.** *Let G be a semi-simple connected Lie group with finite center, with rank at least* 2, *and without non trivial compact factor. Let $\Gamma \subset G$ be an irreducible lattice. Let $h : \Gamma \to \mathsf{GL}_k(\mathbf{R})$ be a linear representation of Γ.*

1. *The Zariski closure H of $h(\Gamma)$ is a semi-simple Lie group.*
2. *Assume H does not have any infinite compact factor; if H is adjoint or G is simply connected there exists a continuous representation $\hat{h} : G \to \mathsf{GL}_k(\mathbf{R})$ which coincides with h on a finite index subgroup of Γ.*

When H does not have any infinite compact factor but is not adjoint, one can always find a finite cover $\pi\colon \hat{G} \to G$ and a morphism $\hat{h}$ from $\hat{G}$ to H such that $\hat{h}$ extends virtually the morphism $\rho \circ \pi$ from $\pi^{-1}(\Gamma)$ to $\mathsf{GL}_k(\mathbf{R})$. We then say that h virtually extends to a continuous representation of a finite cover $\hat{G}$ of G.

In particular, if G is almost simple, its Lie algebra embeds into the Lie algebra of H.

Corollary 3.3. *If the representation h takes values into the group $\mathsf{GL}_k(\mathbf{Z})$ and has an infinite image, then h extends virtually to a continuous representation of a finite cover of G with finite kernel.*

If G is a semi-simple linear algebraic group, any continuous linear representation of G on a finite dimensional vector space is algebraic. As a consequence, up to finite indices and finite covers, representations of Γ into $\mathsf{GL}_k(\mathbf{Z})$ with infinite image are restrictions of algebraic linear representations of G.

Remark 3.4. Let G and Γ be as in Theorem 3.2. Assume, moreover, that G is almost simple. Let $\rho_0 : \Gamma \to \mathsf{GL}_k(\mathbf{R})$ be an injective morphism. If the image of Γ is relatively compact, one can find another morphism $\rho_1 : \Gamma \to \mathsf{GL}_k(\mathbf{C})$ with unbounded image (see [46], Theorem 6.6, and [16], Section 3.3). We can therefore assume that Γ embeds into $\mathsf{GL}_k(\mathbf{C})$ with unbounded image. Thus, Margulis theorem provides an injective morphism from the Lie algebra of G to the Lie algebra $\mathfrak{gl}_k(\mathbf{C})$; this implies that the rank of G is at most $k-1$. In particular, there is no faithful linear representation of Γ in dimension $\leq \mathsf{rk}_{\mathbf{R}}(G)$.

3.2.2. *Normal subgroups (see* [46] *or* [56]*).* According to another result of Margulis, if Γ is an irreducible higher rank lattice, then Γ is almost simple: All normal subgroups of Γ are finite or have finite index in Γ.

In particular, if $\alpha : \Gamma \to H$ is a morphism of groups, either α has finite image, or α is virtually injective, which means that we can change the lattice Γ in a sublattice Γ_0 and assume that α is injective.

3.3. **Action on cohomology.**

Proposition 3.5. *Let G and Γ be as in Theorem 3.2. Let $\rho : \Gamma \to \mathsf{Aut}(M)$ be a representation into the group of automorphisms of a compact Kähler manifold M. Let $\rho^* : \Gamma \to \mathsf{GL}(H^*(M,\mathbf{Z}))$ be the induced action on the cohomology ring of M.*

- a.- *If the image of ρ^* is finite, the image of ρ is virtually contained in $\mathsf{Aut}(M)^0$.*
- b.- *If the image of ρ^* is infinite, then ρ^* virtually extends to a representation $\hat{\rho^*} : \hat{G} \to \mathsf{GL}(H^*(M,\mathbf{R}))$ by cohomological automorphisms on a finite cover of G.*

Hence, to prove our Main Theorem, we can assume that either (a) the image of ρ is contained in $\mathsf{Aut}(M)^0$ or (b) the action of the lattice Γ on the cohomology of M extends to a linear representation of G (changing G into an appropriate finite cover).

Proof. Assertion (a) is a direct consequence of Lieberman-Fujiki Theorem (see §1.2).

In case (b), Margulis superrigidity implies that the morphism ρ^* extends virtually to a linear representation $\hat{\rho^*}$ of a finite cover $\hat{G}$. Since Γ acts by holomorphic diffeomorphisms on M, Γ preserves the Hodge decomposition, the cup product, and Poincaré duality. Since lattices of semi-simple Lie groups are Zariski dense (see §3.1.1), the same is true for $\hat{\rho^*}(\hat{G})$. □

Let us focus on case (b) in the previous Proposition. Changing G in $\hat{G}$, we assume that h extends virtually to G. Let A be a **maximal torus** of G; A is a connected subgroup of G, its adjoint representation on $\mathfrak{g}$ is diagonalizable, and A is maximal for these properties. Since ρ^* is not trivial, there exist a vector $u \neq 0$ in $H^{1,1}(M, \mathbf{R})$ and a surjective morphism $\chi : A \to \mathbf{R}^*$ such that

$$\rho^*(a)(u) = \chi(a)u$$

for all a in A. By [49], a conjugate $b^{-1}\Gamma b$ of Γ intersects A on a lattice $A_\Gamma \subset A$; since A_Γ is a lattice, it is not contained in a hyperplane of $A \simeq \mathbf{R}^n$, and there are pairs (γ, λ) with γ in Γ and λ in the interval $(1, \infty)$ such that

$$\rho^*(\gamma)(\rho^*(b)(u)) = \lambda \rho^*(b)(u).$$

Let us now reformulate this remark in terms of spectral radii and topological entropy. For $f \in \mathsf{Aut}(M)$, define $\lambda(f)$ as the spectral radius of the linear transformation f^* induced by f on $H^*(M, \mathbf{R})$. Then, by results due to Yomdin and Gromov (see [36]), we know that

(1) the topological entropy of f is the logarithm of $\lambda(f)$:

$$\mathsf{h}_{top}(f) = \log(\lambda(f));$$

(2) $\lambda(f) > 1$ if and only if f^* has an eigenvalue > 1 on $W = H^{1,1}(M, \mathbf{R})$.

Example 3.6. The entropy of a projective linear transformation of a projective space is equal to 0. This can be seen from Property (1), or can be proved directly from the definition of topological entropy (all orbits converge or are contained in tori on which the map is conjugate to a translation).

Examples 1.1 and 1.2 give rise to automorphisms with positive entropy: The automorphism associated to an element B in $\mathsf{SL}_n(\mathbf{Z})$ has positive entropy if and only if the spectral radius of B is positive.

From the properties of entropy and Proposition 3.5, we get the following statement.

Corollary 3.7. *If the image of* ρ^* *is infinite, the group* Γ *contains elements* γ *such that* $\rho(\gamma)$ *is an automorphism with positive entropy and* γ *has an eigenvalue* >1 *on* W.

3.4. **An example in dimension three.** In this paragraph, we show how Hodge index Theorem and representation theory can be used together to provide constraints in the spirit of our Main Theorem.

Proposition 3.8. *Let* M *be a compact Kähler manifold of dimension* 3. *Let* G *be a semi-simple connected Lie group with finite center, rank at least* 2, *and no non-trivial compact factor. Let* Γ *be an irreducible lattice in* G, *and* $\rho : \Gamma \to \mathsf{Aut}(M)$ *be an injective morphism. If* G *is not isogenous to a simple group, then* $\rho^* : \Gamma \to \mathsf{GL}\,(H^*(M,\mathbf{Z}))$ *has a finite image.*

This proposition shows that our Main Theorem can be generalized to semi-simple Lie groups when $\dim(M) = 3$. We sketch the proof of this proposition below, and refer to [19] for details and more advanced consequences of this method, including the following statement, which extends our Main Theorem to semi-simple Lie groups G when $\dim(M) = 3$

Theorem 3.9. *Let* G *be a connected semi-simple real Lie group. Let* K *be the maximal compact, connected, and normal subgroup of* G. *Let* Γ *be an irreducible lattice in* G. *Let* M *be a connected compact Kähler manifold of dimension* 3, *and* $\rho : \Gamma \to \mathsf{Aut}(M)$ *be a morphism. If the real rank of* G *is at least* 2, *then one of the following holds*

- *the image of* ρ *is virtually contained in the connected component of the identity* $\mathsf{Aut}(M)^0$, *or*
- *the morphism* ρ *is virtually a Kummer example.*

In the second case, G/K *is locally isomorphic to* $\mathsf{SL}_3(\mathbf{R})$ *or* $\mathsf{SL}_3(\mathbf{C})$ *and* Γ *is commensurable to* $\mathsf{SL}_3(\mathbf{Z})$ *or* $\mathsf{SL}_3(O_d)$, *where* O_d *is the ring of integers in an imaginary quadratic number field* $\mathbf{Q}(\sqrt{d})$ *for some negative integer* d.

3.4.1. *Action on* W. Suppose that the action of Γ on the cohomology of M does not factor through a finite group. Then, changing G in a finite cover, ρ^* virtually extends to a non-trivial representation of G; by Corollary 3.7, the representation $\hat{\rho^*}$ of G on $W = H^{1,1}(M,\mathbf{R})$ is also non-trivial. The representation on $H^{2,2}(M,\mathbf{R})$ is the dual representation W^*, because M has dimension 3. Since this action is by cohomological automorphisms, it preserves the symmetric bilinear mapping

$$\wedge : W \times W \to W^*$$

given by the cup product. By Hodge index Theorem, (HIT) the symmetric mapping $\wedge$ *does not vanish identically on any subspace of dimension* 2 *in* W,

because it does not vanish on hyperplanes made of primitive classes (with respect to any Kähler class, see §2.1.4). In what follows, we describe the induced action of the Lie algebra $\mathfrak{g}$ on W.

3.4.2. *Representations of* $\mathsf{SL}_2(\mathbf{R})$. For all positive integers n, the Lie algebra $\mathfrak{sl}_2(\mathbf{R})$ acts linearly on the space of degree n homogeneous polynomials in two variables. Up to isomorphism, this representation is the unique irreducible linear representation of $\mathfrak{sl}_2(\mathbf{R})$ in dimension $n+1$. The weights of this representation with respect to the Cartan subalgebra of diagonal matrices are

$$-n, -n+2, -n+4, \ldots, n-4, n-2, n,$$

and the highest weight n characterizes this irreducible representation. These representations are isomorphic to their own dual representations.

Lemma 3.10. *Let* $\mu : \mathfrak{sl}_2(\mathbf{R}) \to \mathfrak{g}$ *be an injective morphism of Lie algebras. Then the highest weights of the representation*

$$\mathfrak{sl}_2(\mathbf{R}) \to \mathfrak{g} \to \mathsf{End}(W)$$

are bounded from above by 4, *and the weight* 4 *appears at most once.*

Proof. Let $V \subset W$ be an irreducible subrepresentation of $\mathfrak{sl}_2(\mathbf{R})$ such that its highest weight m is the highest possible weight that appears in W. Let u_m and u_{m-2} be non-zero elements in V with respective weights m and $m-2$. By Property (HIT), one of the products

$$u_m \wedge u_m,\ u_m \wedge u_{m-2},\ u_{m-2} \wedge u_{m-2}$$

is different from 0. The weight of this vector is at least $2(m-2)$, and is bounded from above by the highest weight of W^*, that is by m; thus, $2(m-2) \leq m$, and $m \leq 4$. If the weight $m = 4$ appears twice, there are 2 linearly independant vectors u_4 and v_4 of weight 4. Since the highest weight of W^* is also 4, all products $u_4 \wedge u_4$, $u_4 \wedge v_4$, and $v_4 \wedge v_4$ vanish, contradicting (HIT). $\square$

3.4.3. *Conclusion.* If G is not almost simple, its Lie algebra $\mathfrak{g}$ contains a copy of $\mathfrak{sl}_2(\mathbf{R}) \oplus \mathfrak{sl}_2(\mathbf{R})$, because $\mathsf{rk}_R(G) \geq 2$ and G does not have compact factors. Hence, Proposition 3.8 follows from the following Lemma.

Lemma 3.11. *The Lie algebra* $\mathfrak{g}$ *does not contain any copy of* $\mathfrak{sl}_2(\mathbf{R}) \oplus \mathfrak{sl}_2(\mathbf{R})$.

Proof. Suppose that $\mathfrak{g}$ contains $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_1$ and $\mathfrak{g}_2$ isomorphic to $\mathfrak{sl}_2(\mathbf{R})$. Let n_1 be the highest weight of $\mathfrak{g}_1$ and n_2 be the highest weight of $\mathfrak{g}_2$ in W. Since the representation of $\mathfrak{g}$ is faithful, both n_1 and n_2 are positive integers. After permutation of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, we assume that $n_1 = \max(n_1, n_2)$.

Let $\mathfrak{h} \leq \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be the diagonal copy of $\mathfrak{sl}_2(\mathbf{R})$; the highest weight of $\mathfrak{h}$ is n_1. Let $u_i \in W$ be a vector of weight n_i for $\mathfrak{g}_i$. Since $\mathfrak{g}_2$ commutes to $\mathfrak{g}_1$, the orbit of u_1 by $\mathfrak{g}_2$ is made of highest weight vectors for $\mathfrak{g}_1$. Thus, we can assume that u_1 is not colinear to u_2. If $u_i \wedge u_j$ is not zero, its weight for $\mathfrak{h}$ is $n_i + n_j$. This implies that $u_1 \wedge u_1 = 0$ and $u_1 \wedge u_2 = 0$. Since n_2 is a highest weight for $\mathfrak{g}_2$, we also know that $u_2 \wedge u_2 = 0$ because $2n_2$ does not appear as a weight for $\mathfrak{g}_2$ on W^*. Hence, $\wedge$ should vanish identically on the vector space spanned by u_1 and u_2, contradicting Property (HIT). This concludes the proof. $\square$

4. Lie group actions and invariant analytic subsets

4.1. **Homogeneous manifolds.** The following theorem is a simple consequence of the classification of maximal subgroups in simple Lie groups (see [55], chapter 6, or Section 4.6 below).

Theorem 4.1. *Let H be a connected almost simple complex Lie group of rank $\mathsf{rk}_{\mathbf{C}}(H) = n$. If H acts faithfully and holomorphically on a connected compact complex manifold M of dimension $\leq n$ then, up to holomorphic diffeomorphism, M is the projective space $\mathbb{P}^n(\mathbf{C})$, H is locally isomorphic to $\mathsf{PGL}_{n+1}(\mathbf{C})$, and the action of H on M is the standard action by linear projective transformations.*

Following a suggestion by Brion and Dolgachev, we sketch a proof that does not use the classification of maximal subgroups of Lie groups. In contrast, we use algebraic geometry: Toric varieties and results of Kushnirenko, Borel and Remmert, and Tits concerning homogeneous spaces.

Proof: First Step. [1] In this first step, we assume that M is a complex projective variety, H is an almost simple complex algebraic group, and the action $H \times M \to M$ is algebraic.

[1] As published, the proof contains a mistake: we do as if every closed connected subgroup of $(\mathbf{C}^*)^n$ is given by an algebraic morphism $(\mathbf{C}^*) \to (\mathbf{C}^*)^n$. For instance, if t has a positive imaginary part, the image of $z \in \mathbf{C} \to (\exp(z), \exp(tz)) \in (\mathbf{C}^*)^2$ is a closed subgroup such that the quotient group is the elliptic curve $\mathbf{C}/(\mathbf{Z} + \mathbf{Z}t)$. The proof is correct when M is a projective variety, H is an almost simple complex algebraic group, and the action $H \times M \to M$ is algebraic. We thank Sorin Dumitrescu for pointing out the mistake.

Let A be a maximal torus in H. Since H has rank n this group is isomorphic to the multiplicative group $\mathbb{G}_m^n(\mathbf{C}) = (\mathbf{C}^*)^n$. Since H acts faithfully on M, the action of A is also faithful. If x is a point of M, its stabilizer in A is an algebraic subgroup A_x of A; denote by A_x^0 the irreducible component of the identity in A_x. The set of irreducible algebraic subgroups in a complex multiplicative group is discrete (characters form a discrete group). There exists an open subset U of M, on which the stabilizer A_x depends continuously on $x \in U$ and A_x^0 is therefore a constant subgroup of A on U. Since the action of A is faithful, we deduce that A_x^0 is generically trivial, A has an open orbit, and $\dim(M) = n$. Thus M is a toric variety of dimension n with respect to the action of the multiplicative group A. In particular, there is no faithful action of H on compact complex manifolds of dimension less than n. Since H is almost simple and connected, all actions of H in dimension $< n$ are trivial.

Let us show, as a corollary, that H acts transitively on M. If not, H has a proper Zariski closed orbit. This orbit has dimension $< n$ and, as such, must be a point $m \in M$. By Kushnirenko's Theorem (see [42], Theorem 2), the action of H at m can be linearized. Since the action is analytic, non-trivial, and M is connected, this implies that the first jet of the action at m gives a non-trivial morphism from H to $\mathsf{GL}\,(T_mM) \simeq \mathsf{GL}_n(\mathbf{C})$, in contradiction with $\mathsf{rk}_{\mathbf{C}}(H) = n$. Thus M is homogeneous under the action of H, and $M = H/L$ for some closed analytic subgroup L.

Let L^0 be the connected component of the identity of L, and $N(L^0)$ be the normalizer of L^0 in H. Then $N(L^0)$ is a parabolic subgroup of H, and is the smallest parabolic subgroup of H containing L; the quotient space $H/N(L^0)$ is a flag manifold, and the fibration $\pi : M = H/L \to H/N(L^0)$ is the Tits fibration of M (see [2], §3.5). Since the dimension of M is the smallest positive dimension of a H-homogeneous space, $N(L^0) = L$ and $N(L^0)$ is a maximal parabolic subgroup. Since $N(L^0)$ is maximal, the Picard number ([2]) of M is equal to 1 (see [2], §4.2).

As a consequence, M is a smooth toric variety with Picard number 1 and, as such, is isomorphic to $\mathbb{P}^n(\mathbf{C})$ (see [30]). Since the group of automorphisms of $\mathbb{P}^n(\mathbf{C})$ is the rank n group $\mathsf{PGL}_{n+1}(\mathbf{C})$, the conclusion follows. □

Proof: Second Step. Now, we prove Theorem 4.1 (without assuming the action $H \times M \to M$ to be algebraic).

[2]The Picard number is the rank of the abelian subgroup of $H^2(M, \mathbf{Z})$ generated by the Chern classes of all holomorphic line bundles on M.

Let $\mathfrak{h}$ be the Lie algebra of H; the adjoint representation provides a homomorphism $\mathrm{Ad}: H \to \mathsf{GL}(\mathfrak{h})$ with finite kernel. Denote by $S_x \subset H$ the stabilizer of $x \in M$ and by $\mathfrak{s}(x) \subset \mathfrak{h}$ its Lie algebra. The dimension of $\mathfrak{s}(x)$ varies semicontinuously on M. Let d be the maximum of $\dim(\mathfrak{s}(x))$; the subset

$$F = \{x \in M \,;\, \dim(\mathfrak{s}(x)) = d\} \subset M$$

is closed and H-invariant. For x in F, denote by $[\mathfrak{s}(x)]$ the point of the Grassman variety $\mathsf{Grass}(\mathfrak{h}; d)$ of d-dimensional subspaces of $\mathfrak{h}$. The map $x \in F \mapsto [\mathfrak{s}(x)] \in \mathsf{Grass}(\mathfrak{h}; d)$ is continuous, and equivariant with respect to (1) the action of H on M and (2) the action of H on $\mathsf{Grass}(\mathfrak{h}; d)$ given by the adjoint representation of H. Thus, the image of this map is a compact, H-invariant subset V in $\mathsf{Grass}(\mathfrak{h}; d)$.

Note that $[\mathfrak{s}(y)] \in V$ is a fixed point of H if and only if S_y is normalized by H, if and only if $\mathfrak{s}(y) = \{0\}$ or $\mathfrak{h}$ (because H is almost simple). If $\mathfrak{s}(y) = \{0\}$, the orbit of y has dimension $\dim(H) > n$; so $\mathfrak{s}(y) = \mathfrak{h}$ and y is fixed by the connected group H. As in the first step, Kushnirenko's theorem provides a contradiction. So V contains no fixed point.

The action on $\mathsf{Grass}(\mathfrak{h}; d)$ factors through the algebraic action of $\mathrm{Ad}(H)$, so

(a) by the first step of the proof: every invariant irreducible subvariety $W \subset \mathsf{Grass}(\mathfrak{h}; d)$ has dimension 0 or $\geq n$;
(b) the closure of every H-orbit in $\mathsf{Grass}(\mathfrak{h}; d)$ is an algebraic subvariety;
(c) every H-invariant subvariety W of $\mathsf{Grass}(\mathfrak{h}; d)$ contains an orbit which is smooth, algebraic, and of minimal dimension among orbits in W.

(the second and third properties are consequences of the theorems of Chevalley on the constructibility of $\mathrm{Ad}(H)$-orbits.)

Now, pick a point v in V and consider its H-orbit $\mathrm{Ad}(H)(v)$. Its closure is contained in V, and by (b) and (c) it contains a closed, algebraic orbit $\mathrm{Ad}(H)(w)$. Since V contains no fixed point, Property (a) implies that the dimension of $\mathrm{Ad}(H)(w)$ is at least n. Thus, the dimension of any H-orbit in V is at least n. Since $x \in F \mapsto [\mathfrak{s}(x)]$ is continuous, this implies that H-orbit in M has dimension $\geq n$; thus $\dim(M) \geq n$ and every orbit is open. So M is H-homogeneous and we conlude as in the first step of the proof. □

One consequence of (the beginning of the proof of) Theorem 4.1 is the following: If L is a Zariski closed subgroup of $\mathsf{SL}_n(\mathbf{C})$ of positive codimension, then $\mathrm{codim}_{\mathbf{C}}(L) \geq n-1$. The following section provides a similar property for the unitary group U_n.

4.2. **Subgroups of the unitary group.** Before stating the result of this section, recall that the compact real form of the symplectic group $\mathsf{Sp}_4(\mathbf{C}) \subset \mathsf{SL}_4(\mathbf{C})$ is the group $\mathsf{U}_2(\mathbb{H})$; its Lie algebra $\mathfrak{u}_2(\mathbb{H})$ is isomorphic to $\mathfrak{so}_5(\mathbf{R})$ (see [31], Lecture 26, and Example 4.7.a. below).

Proposition 4.2. *Let K be a closed subgroup in the unitary group U_n, and let $\mathfrak{k}$ be its Lie algebra. Then*

- *K has codimension 0 and $\mathfrak{k} = \mathfrak{u}_n$, or*
- *K has codimension 1 and then $\mathfrak{k} = \mathfrak{su}_n$, or*
- *$codim_{\mathbf{R}}(K) \geq 2n-2$, or*
- *$n = 4$, the Lie algebra $\mathfrak{k}$ is equal to $\mathrm{i}\mathbf{R} \oplus \mathfrak{u}_2(\mathbb{H})$, where $\mathrm{i}\mathbf{R}$ is the center of $\mathfrak{u}_4$, and then $codim_{\mathbf{R}}(K) = 5 = 2n-3$.*

TABLE 1. Minimal dimensions of faithful representations

the Lie algebra	its dimension	the dimension of its minimal representation
$\mathfrak{sl}_k(\mathbf{C}),\, k \geq 2$	k^2-1	k
$\mathfrak{so}_k(\mathbf{C}),\, k \geq 7$	$k(k-1)/2$	k
$\mathfrak{sp}_{2k}(\mathbf{C}),\, k \geq 2$	$2k^2+k$	$2k$
$\mathfrak{e}_6(\mathbf{C})$	78	27
$\mathfrak{e}_7(\mathbf{C})$	133	56
$\mathfrak{e}_8(\mathbf{C})$	248	248
$\mathfrak{f}_4(\mathbf{C})$	52	26
$\mathfrak{g}_2(\mathbf{C})$	14	7

The proof follows again from Dynkin's classification of maximal subalgebras of simple complex Lie algebras. We provide a slightly different argument, which introduces a few basic facts that are used later on. The main ingredient is the list of the minimal dimensions of faithful representations of simple complex Lie algebras (see [55], Table 1 on page 224).

Proof. To prove the proposition, we can, and do, assume that $\operatorname{codim}(K) \leq 2n-3$, i.e. $\dim(K) \geq n^2-2n+3$. We denote by V the natural representation of K on $\mathbf{C}^n$.

Since K is compact, the representation V splits into the direct sum of irreducible representations $V_1 \oplus \ldots \oplus V_l$, and K embeds into the product $\mathsf{U}_{n_1} \times \ldots \times \mathsf{U}_{n_l}$, where n_j is the dimension of V_j, $1 \leq j \leq l$. In particular, the dimension of K is at most $\sum_j n_j^2$. Since it must be larger than or equal to $n^2 - 2n + 3$ and $n = \sum_j n_j$, we deduce that $l = 1$ and V is irreducible.

Let $\mathfrak{k}(\mathbf{C}) = \mathfrak{k} + \mathrm{i}\mathfrak{k}$ be the complexification of $\mathfrak{k}$; it is a complex Lie subalgebra of $\mathfrak{gl}_n(\mathbf{C})$ of complex dimension $\dim_{\mathbf{R}}(\mathfrak{k})$. Since the action of K on $\mathbf{C}^n$ is irreducible two cases can occur:

(1) $\mathfrak{k}(\mathbf{C})$ is a semi-simple algebra which acts irreducibly on V;
(2) $\mathfrak{k}(\mathbf{C})$ is the sum of the center $\mathbf{C}$ of $\mathfrak{gl}_n(\mathbf{C})$ and a semi-simple algebra $\mathfrak{h}$ which acts irreducibly on V.

In the first case, we decompose $\mathfrak{k}(\mathbf{C})$ as a direct sum of simple algebras $\mathfrak{k}_1(\mathbf{C}) \oplus \ldots \oplus \mathfrak{k}_m(\mathbf{C})$. The representation V is the tensor product of faithful irreducible representations W_j of the $\mathfrak{k}_j$, $1 \leq j \leq m$. Let w_j be the dimension of W_j. Assume that $m \geq 2$. Since $m \geq 2$, $w_j \geq 2$ for all indices j, and $n = \prod_j w_j$, we deduce that $\sum_{j=1}^m w_j^2 < n^2 - 2n$ (this can be proved by induction on $m \geq 2$, by studying the function which maps w_m to $(\prod w_j)^2 - 2\prod w_j - \sum w_j^2)$. As a consequence,

$$n^2 - 2n + 3 \leq \dim(\mathfrak{k}(\mathbf{C})) \leq \sum_{j=1}^{m} \dim(\mathfrak{k}_j(\mathbf{C})) \leq \sum_{j=1}^{m} w_j^2 < n^2 - 2n,$$

a contradiction which implies that $m = 1$ and $\mathfrak{k}(\mathbf{C})$ is simple. We can thus use Table 1 to compare $\dim(K)$ ($\geq n^2 - 2n + 3$ by assumption) to the minimal dimension of its non trivial representations ($\leq n$ since $K \subset \mathsf{U}_n$). For example, if $\mathfrak{k}(\mathbf{C})$ is isomorphic to $\mathfrak{sp}_{2d}(\mathbf{C})$ for some integer $d \geq 2$, then $\dim(\mathfrak{k}(\mathbf{C})) = 2d^2 + d$ and $n = \dim(V) \geq 2d$; thus, $\dim(\mathfrak{k}) \geq n^2 - 2n + 3$ implies

$$2d^2 + d \geq 4d^2 - 4d + 3, \quad \text{i.e.} \quad 5d \geq 2d^2 + 3,$$

which contradicts $d \geq 2$ (note that $2d^2 + 2 = 10 = 5d$ for $d = 2$). Similarly, all other possibilities for $\mathfrak{k}(\mathbf{C})$ are excluded, except $\mathfrak{k}(\mathbf{C}) = \mathfrak{sl}_n(\mathbf{C})$, which corresponds to $\mathfrak{k} = \mathfrak{su}_n$.

In case (2), $\mathfrak{k}(\mathbf{C}) = \mathbf{C} \oplus \mathfrak{h}$ and we can apply the same argument to $\mathfrak{h}$: This complex Lie algebra is simple, has dimension at least $n^2 - 2n + 2$, and is isomorphic to $\mathfrak{sl}_n(\mathbf{C})$ or to $\mathfrak{sp}_4(\mathbf{C})$ (with $n = 4$ in this latter case). Since $\mathfrak{sp}_4(\mathbf{C})$ has a unique representation in dimension 4, $\mathfrak{k}$ is conjugate to $\mathrm{i}\mathbf{R} \oplus \mathfrak{u}_2(\mathbb{H})$ when $\mathfrak{h} \simeq \mathfrak{sp}_4(\mathbf{C})$. This concludes the proof of Proposition 4.2. $\square$

4.3. **First part of the Main Theorem.** Let us apply Theorem 4.1. Let Γ be a lattice in an almost simple real Lie group G. Assume that Γ acts faithfully on a connected compact Kähler manifold M, with $\dim_{\mathbf{C}}(M) \leq \mathsf{rk}_{\mathbf{R}}(G)$. By [16], the dimension of M is equal to the rank of G and, by hypothesis, the image of Γ in $\mathsf{Aut}(M)$ is virtually contained in $\mathsf{Aut}(M)^0$. Hence, we can assume that the action of Γ on M is given by an injective morphism $\rho\colon \Gamma \to \mathsf{Aut}(M)^0$. As explained in [16], §3.2, the complex Lie group $\mathsf{Aut}(M)^0$ contains a copy of an almost simple complex Lie group H with $\mathsf{rk}_{\mathbf{C}}(H) \geq \mathsf{rk}_{\mathbf{R}}(G)$. More precisely, if $\rho(\Gamma)$ is not relatively compact in $\mathsf{Aut}(M)^0$, one applies Theorem 3.2, change G into a finite cover $\hat{G}$ and extend virtually the morphism ρ to a morphism $\hat{\rho}\colon \hat{G} \to \mathsf{Aut}(M)^0$; if the image of ρ is relatively compact, then another representation $\rho'\colon \Gamma \to \mathsf{Aut}(M)^0$ extends virtually to a finite cover of G (see Remark 3.4); in both cases, the Lie algebra of H is the smallest complex Lie subalgebra containing $d\hat{\rho}_{Id}(\mathfrak{g})$.

Theorem 4.1 shows that M is the projective space $\mathbb{P}^n(\mathbf{C})$ and $\mathsf{Aut}(M)$ coincides with $\mathsf{PGL}_{n+1}(\mathbf{C})$ (and thus with H). As a consequence, the group G itself is locally isomorphic to $\mathsf{SL}_{n+1}(\mathbf{R})$ or $\mathsf{SL}_{n+1}(\mathbf{C})$.

Summing up, the inequality $\dim_{\mathbf{C}}(M) \geq \mathsf{rk}_{\mathbf{R}}(G)$ as well as property (1) in the Main Theorem have been proved.

4.4. **Invariant analytic subsets.** Let us now study Γ-invariant analytic subsets $Z \subset M$ under the assumption of assertion (2) in the Main Theorem; in particular $\dim_{\mathbf{C}}(M) = \mathsf{rk}_{\mathbf{R}}(G) + 1$.

Lemma 4.3. *Let Γ be a lattice in an almost simple Lie group of rank $n-1 \geq 2$. If Γ acts faithfully by holomorphic transformations on a compact Kähler manifold M of dimension n, then:*

1. *All irreducible Γ-invariant analytic subsets of positive dimension are smooth;*
2. *the set of fixed points of Γ is finite;*
3. *all irreducible Γ-invariant analytic subsets of positive dimension and codimension are smooth hypersurfaces isomorphic to $\mathbb{P}^{n-1}(\mathbf{C})$, on which Γ acts as a Zariski dense subgroup of $\mathsf{PGL}_n(\mathbf{C})$.*

Proof. Let Z be a Γ-invariant irreducible analytic subset of positive dimension and codimension. If Z is not smooth, its singular locus $\mathsf{Sing}(Z)$ is Γ-invariant and has dimension $\leq n-2$. If $\mathsf{Sing}(Z)$ is not smooth, we replace it by its singular locus, and so on: After k steps with $k \leq \dim(Z)$, we get a smooth Γ-invariant analytic subset $Z' \subset Z$ with $\dim(Z') \leq n-2$. By the first part of the Main Theorem, the image of Γ in $\mathsf{Aut}(Z')$ is finite, and changing Γ into a finite index subgroup, we assume that Γ fixes Z' pointwise. Let q be a point of

Z'. The image of the morphism $\delta_{q,1}\colon \Gamma \to \mathsf{GL}\,(T_qM)$ defined by the differential at q, i.e. by

$$\delta_{q,1}(\gamma) = \mathrm{d}\gamma_q,$$

preserves the tangent cone of Z at q; in particular, the Zariski closure of $\delta_{q,1}(\Gamma)$ in $\mathsf{PGL}\,(T_qM)$ is a proper algebraic subgroup of $\mathsf{PGL}\,(T_qM)$. Since proper semi-simple algebraic subgroups of $\mathsf{PGL}_n(\mathbf{C})$ have rank less than $n-1 = \mathsf{rk}_{\mathbf{R}}(G)$, Margulis superrigidity theorem implies that the image of $\delta_{q,1}$ is finite. Thus, changing Γ into a finite index subgroup, Γ fixes q and is tangent to the identity at q. Let $k \geq 2$ be the first positive integer such that some element γ in Γ has a non trivial k-jet $\delta_{q,k}(\gamma)$ at q. Then, the k-jet defines a morphism $\delta_{q,k}$ with infinite image from Γ to the group of k-jets of diffeomorphisms which are tangent to the identity up to order $k-1$; this group is abelian because $k \geq 2$. This contradicts property (T) for Γ (see §3.1.2) and shows by contradiction that Z is smooth.

This argument also shows that the set of fixed points of Γ does not contain analytic subsets of positive dimension. Since the set of fixed points is analytic, it must be finite.

To prove the third assertion, let Z be a Γ-invariant irreducible analytic subset of positive dimension. By assertion (1) of the Lemma, Z is smooth. By the first part of the Main Theorem (see §4.3), either Z is isomorphic to $\mathbb{P}^{n-1}(\mathbf{C})$ and the image of Γ in $\mathsf{Aut}(\mathbb{P}^{n-1}(\mathbf{C}))$ is Zariski dense, or a finite index subgroup of Γ acts trivially on Z. Thus, the conclusion follows from assertion (2) applied to finite index subgroups of Γ. □

Proposition 4.4. *Let Γ be a lattice in an almost simple Lie group of rank $n-1 \geq 2$. If Γ acts faithfully by holomorphic transformations on a compact Kähler manifold M of dimension n, any Γ-invariant analytic subset $Z \subset M$ is a disjoint union of isolated points and smooth hypersurfaces isomorphic to $\mathbb{P}^{n-1}(\mathbf{C})$.*

Proof. Let Z be a Γ-invariant analytic subset. Lemma 4.3 asserts that all irreducible components of Z of positive dimension are copies of $\mathbb{P}^{n-1}(\mathbf{C})$, on which a finite index subgroup Γ_1 of Γ acts as a Zariski dense subgroup of $\mathsf{PGL}_n(\mathbf{C})$. The Zariski density implies that Γ_1 does not preserve any proper analytic subset in $\mathbb{P}^{n-1}(\mathbf{C})$; in particular, all distinct components are disjoint, and the conclusion follows. □

4.5. **Contraction of invariant analytic subsets.**

Theorem 4.5. *Let Γ be a lattice in an almost simple Lie group G. Assume that Γ acts faithfully on a connected compact Kähler manifold M,*

$$\mathsf{rk}_{\mathbf{R}}(G) = \dim_{\mathbf{C}}(M) - 1,$$

and the image of Γ in $\mathsf{Aut}(M)$ is not virtually contained in $\mathsf{Aut}(M)^0$.

Let Z be the union of all Γ-invariant analytic subsets $Y \subset M$ with positive dimension. Then Z is the union of a finite number of disjoint copies of projective spaces $Z_i = \mathbb{P}^{n-1}(\mathbf{C})$. Moreover there exists a birational morphism $\pi\colon M \to M_0$ onto a compact Kähler orbifold M_0 such that

1. *π contracts all Z_i to points $q_i \in M_0$;*
2. *around each point q_i, the orbifold M_0 is either smooth, or locally isomorphic to a quotient of $(\mathbf{C}^n, 0)$ by a finite order scalar multiplication;*
3. *π is an isomorphism from the complement of Z to the complement of the points q_i;*
4. *π is equivariant: The group Γ acts on M_0 in such a way that $\pi \circ \gamma = \gamma \circ \pi$ for all γ in Γ.*

Proof. By Proposition 3.5 and its Corollary 3.7, the group Γ contains automorphisms γ with positive entropy ; in other words, there are elements γ in Γ with eigenvalues > 1 on the cohomology of M. We shall use this fact to prove the Theorem.

We already know that Z is made of hypersurfaces isomorphic to $\mathbb{P}^{n-1}(\mathbf{C})$. Let S be one of these hypersurfaces; changing Γ in a finite index subgroup, we assume that S is Γ-invariant. Let N_S be its normal bundle, and let r be the integer such that $N_S \simeq O(r)$. Let L be the line bundle on M which is defined by S. The adjunction formula shows that N_S is the restriction $L_{|S}$ of L to S (see [35], §1.1, p. 146-147).

If $r > 0$, then S moves in a linear system $|S|$ of positive dimension (see [38], page 110, Theorem 4.2, and the proof of Theorem 5.1 in chapter I); more precisely, the space of sections $H^0(M, L^{\otimes l})$ has dimension ≥ 2 for large enough $l \geq 1$. Moreover the line bundle $L_{|S} \simeq O(r)$ is very ample, so that the base locus of the linear system $|lS|$ is empty. As a consequence, this linear system determines a well defined morphism

$$\Phi_{L^{\otimes l}} : M \to \mathbb{P}(H^0(M, L^{\otimes l})^*)$$

where $\Phi_{L^{\otimes l}}(x)$ is the linear form which maps a section s of $L^{\otimes l}$ to its value at x (this is well defined up to a scalar multiple). The self intersection $(L^{\otimes l})^n$ being positive, the dimension of $\Phi_L(M)$ is equal to n and $\Phi_{L^{\otimes l}}$ is generically finite. Since S is Γ-invariant, Γ permutes linearly the sections of $L^{\otimes l}$. This gives a morphism $\eta : \Gamma \to \mathsf{PGL}\,(H^0(M, L^{\otimes l})^*)$ such that

$$\Phi_{L^{\otimes l}} \circ \gamma = \eta(\gamma) \circ \Phi_{L^{\otimes l}}$$

for all γ in Γ. Let γ be an element of Γ. The action of $\eta(\gamma)$ on $\Phi_{L^{\otimes l}}(M)$ is induced by a linear mapping. This implies that the topological entropy of $\eta(\gamma)$ on $\Phi_{L^{\otimes l}}(M)$ vanishes (see Example 3.6). Since $\Phi_{L^{\otimes l}}$ is generically finite to one, the entropy of γ vanishes, contradicting our first remark. (equivalently, one can prove that $(\eta(\gamma))^*$ and γ^* have finite order, see [25]).

Assume now that $r = 0$. From [41], we know that S moves in a pencil of hypersurfaces. In other words, $H^0(M, L)$ has dimension 2 and defines a holomorphic fibration $\Phi_L : M \to \mathbb{P}^1(\mathbf{C})$. This fibration is Γ-invariant, and replacing Γ by a finite index subgroup, we can assume that the action on the base $\mathbb{P}^1(\mathbf{C})$ is trivial (Lemma 3.1). All fibers of Φ_L are Γ-invariant hypersurfaces and, as such, are smooth projective spaces $\mathbb{P}^{n-1}(\mathbf{C})$. This implies that Φ_L is a locally trivial fibration: There is a covering U_i of $\mathbb{P}^1(\mathbf{C})$ such that $\Phi_L^{-1}(U_i)$ is isomorphic to $U_i \times \mathbb{P}^{n-1}(\mathbf{C})$. In such coordinates, Γ acts by

$$\gamma(u, v) = (u, A_u(\gamma)(v))$$

where $(u, v) \in U_i \times \mathbb{P}^{n-1}(\mathbf{C})$ and $u \mapsto A_u$ is a one parameter representation of Γ into $\mathsf{Aut}(\mathbb{P}^{n-1}(\mathbf{C}))$. Once again, this implies that all elements of Γ have zero entropy, a contradiction.

As a consequence, the normal bundle N_S is isomorphic to $O(r)$ with $r < 0$. Grauert's theorem shows that S can be blown down to a quotient singularity of type $(\mathbf{C}^n, 0)/\xi$ where ξ is the multiplication by a root of unity of order r (see [34], Theorem 3.8). $\square$

4.6. **Lie group actions in case $\mathsf{rk}_{\mathbf{R}}(G) = \dim(M) - 1$.** In case (2-a) of the Main Theorem, the group Γ is a lattice in a rank $n-1$ almost simple Lie group, and Γ virtually embeds into $\mathsf{Aut}(M)^0$. This implies that $\mathsf{Aut}(M)^0$ contains an almost simple complex Lie group H, with rank $\geq n-1$. Since the case $\mathsf{rk}_{\mathbf{R}}(H) \geq n$ has already been dealt with in Sections 4.1 and 4.3, our new goal is to list all examples such that

(i) H is an almost simple complex Lie group, and its rank is equal to $n-1$;
(ii) M is a connected, compact, complex manifold and $\dim_{\mathbf{C}}(M) = n \geq 3$;
(iii) H is contained in $\mathsf{Aut}(M)^0$.

We now list all such possible pairs (M, H).

Example 4.6. The following examples work in all dimensions $n \geq 3$.

a.- The group $\mathsf{SL}_n(\mathbf{C})$ acts on $\mathbb{P}^{n-1}(\mathbf{C})$ by linear projective transformations. In particular, $\mathsf{SL}_n(\mathbf{C})$ acts on products of type $\mathbb{P}^{n-1}(\mathbf{C}) \times B$ where B is any Riemann surface.

b.- The action of $\mathsf{SL}_n(\mathbf{C})$ on $\mathbb{P}^{n-1}(\mathbf{C})$ lifts to an action on the total space of the line bundles $O(k)$ for every $k \geq 0$; sections of $O(k)$ are in one-to-one correspondence with homogeneous polynomials of degree k, and the action of $\mathsf{SL}_n(\mathbf{C})$ on the space of sections $H^0(\mathbb{P}^{n-1}(\mathbf{C}), O(k))$ is the usual action on homogeneous polynomials in n variables. Let p be a positive integer and E the vector bundle of rank 2 over $\mathbb{P}^{n-1}(\mathbf{C})$ defined by $E = O \oplus O(p)$. Then $\mathsf{SL}_n(\mathbf{C})$ acts on E by isomorphisms of vector bundles. From this we get an action on the projectivized bundle $\mathbb{P}(E)$, i.e. on a compact Kähler manifold M which fibers over $\mathbb{P}^{n-1}(\mathbf{C})$ with rational curves as fibers.

c.- When $p = 1$, one can blow down the section of $\mathbb{P}(E)$ given by the line bundle $O(1)$ (the normal bundle of this $\mathbb{P}^{n-1}(\mathbf{C})$ is isomorphic to $O(-1)$). This provides a new smooth manifold with an action of $\mathsf{SL}_n(\mathbf{C})$ (for other values of p, a singularity appears). In that case, $\mathsf{SL}_n(\mathbf{C})$ has an open orbit Ω, the complement of which is the union of a point and a smooth hypersurface $\mathbb{P}^{n-1}(\mathbf{C})$.

d.- A similar example is obtained from the $\mathbf{C}^*$-bundle associated to $O(k)$. Let λ be a complex number with modulus different from 0 and 1. The quotient of this $\mathbf{C}^*$-bundle by multiplication by λ along the fibers is a compact manifold, with the structure of a torus principal bundle over $\mathbb{P}^{n-1}(\mathbf{C})$. Since multiplication by λ commutes with the $\mathsf{SL}_n(\mathbf{C})$-action on $O(k)$, we obtain a (transitive) action of $\mathsf{SL}_n(\mathbf{C})$ on this manifold. In this case, M is not Kähler; if $k = 1$, M is the Hopf manifold, i.e. the quotient of $\mathbf{C}^n \setminus \{0\}$ by the multiplication by λ.

Example 4.7. In these examples, the dimension of M is equal to 3 or 4.

a.- Let H be the group $\mathsf{SO}_5(\mathbf{C})$; its rank is equal to 2. The projective quadric $Q_3 \subset \mathbb{P}^4(\mathbf{C})$ given by the equation $\sum x_i^2 = 0$ is H-invariant, and has dimension 3.

The space of isotropic lines contained in Q_3 is parametrized by $\mathbb{P}^3(\mathbf{C})$, so that $\mathbb{P}^3(\mathbf{C})$ is a $\mathsf{SO}_5(\mathbf{C})$-homogeneous space: This comes from the isogeny between $\mathsf{SO}_5(\mathbf{C})$ and $\mathsf{Sp}_4(\mathbf{C})$ (see [31], page 278), and provides another homogeneous space of dimension $3 = \mathsf{rk}(\mathsf{SO}_5(\mathbf{C})) + 1$.

b.- Let H be the group $\mathsf{SO}_6(\mathbf{C})$, a group of rank 3. The quadric $Q_4 \subset \mathbb{P}^5(\mathbf{C})$ is H-invariant, and has dimension 4. This provides a new example with $\dim(M) = \mathsf{rk}(G) + 1$.

Note that $\mathsf{SO}_6(\mathbf{C})$ is isogenous to $\mathsf{SL}_4(\mathbf{C})$, and $\mathsf{PSO}_6(\mathbf{C})$ acts transitively on $\mathbb{P}^3(\mathbf{C})$. However, in this case, the rank of the group is equal to the dimension of the space (as in Theorem 4.1).

The following result shows that this list of examples exhausts all cases.

Theorem 4.8. *Let M be a connected compact complex manifold of dimension $n \geq 3$. Let H be an almost simple complex Lie group with $\mathsf{rk}_{\mathbf{C}}(H) = n-1$. If there exists an injective morphism $H \to \mathsf{Aut}(M)^0$, then M is one of the following:*

(1) *a projective bundle $\mathbb{P}(E)$ for some rank 2 vector bundle E over $\mathbb{P}^{n-1}(\mathbf{C})$, and then H is isogenous to $\mathsf{PGL}_n(\mathbf{C})$;*
(2) *a principal torus bundle over $\mathbb{P}^{n-1}(\mathbf{C})$, and H is isogenous to $\mathsf{PGL}_n(\mathbf{C})$;*
(3) *a product of $\mathbb{P}^{n-1}(\mathbf{C})$ with a curve B of genus $g(B) \geq 2$, and then H is isogenous to $\mathsf{PGL}_n(\mathbf{C})$;*
(4) *the projective space $\mathbb{P}^n(\mathbf{C})$, and H is isogenous to $\mathsf{PGL}_n(\mathbf{C})$ or to $\mathsf{PSO}_5(\mathbf{C})$ when $n=3$;*
(5) *a smooth quadric of dimension 3 or 4 and H is isogenous to $\mathsf{SO}_5(\mathbf{C})$ or to $\mathsf{SO}_6(\mathbf{C})$ respectively.*

The proof splits into three cases, according to the size of orbits of H in M.

4.6.1. *Transitive actions.* Let us come back to the rank – dimension inequality obtained in Theorem 4.1. Let M be a connected compact complex manifold on which a complex semi-simple Lie group S acts holomorphically and faithfully. Let $K \subset S$ be a maximal compact subgroup and let m be a point of M. Denote by $K(m)$ the orbit of the point m under the action of K and by K_m its stabilizer. The group K_m is compact; as such, it preserves a riemannian metric and fixes the point m; since M is connected, this implies that the linear part defines an injective morphism from K_m to $\mathsf{GL}\,(T_mM)$. Thus

(1) $\dim_{\mathbf{C}}(S) = \dim_{\mathbf{R}}(K)$;
(2) $\dim_{\mathbf{R}}(K) = \dim_{\mathbf{R}}(K(m)) + \dim_{\mathbf{R}}(K_m)$;
(3) K_m embeds into a maximal compact subgroup of $\mathsf{GL}\,(T_mM)$; hence, up to conjugacy, K_m is a closed subgroup of the unitary group U_n, with $n = \dim(M)$.

The inequality

$$\dim_{\mathbf{C}}(S) \leq \dim_{\mathbf{R}}(M) + \dim_{\mathbf{R}} \mathsf{U}_n = 2n + n^2 \tag{4.1}$$

follows. Moreover, if the rank of S is less than n, then K_m has positive codimension in U_n; Proposition 4.2 implies that the codimension of K_m in U_n is large: $\mathrm{codim}_{\mathbf{R}}(K_m) \geq 2n-2$, or K_m is locally isomorphic to $e^{i\mathbf{R}}\mathsf{U}_2(\mathbb{H})$ and $n=4$. The inequality (4.1) can therefore be strengthened, and gives

$$\dim_{\mathbf{C}}(S) \leq n^2+2, \quad \text{or} \quad n=4 \quad \text{and} \quad \dim_{\mathbf{C}}(S) \leq 19.$$

We now apply this inequality to the proof of Theorem 4.8 in case H acts transitively. Thus, the group S is now replaced by the almost simple complex Lie group H, with rank $r=n-1 \geq 2$ (note that $r=3$ if the special case $n=4$).

If the Lie algebra of H is of type B_r or C_r, i.e. H is locally isomorphic to $\mathsf{SO}_{2r+1}(\mathbf{C})$ or $\mathsf{Sp}_{2r}(\mathbf{C})$, we have

$$\dim_{\mathbf{C}}(H) = 2r^2+r \leq (r+1)^2+2$$

(because for $r=3$ we have $2r^2+r=21>19$); thus $r^2 \leq r+3$. This implies $r \leq 2$, and thus $r=2$ because the rank of H is at least 2. Since $r=2$, the group H is locally isomorphic to $\mathsf{SO}_5(\mathbf{C})$ and $\mathsf{Sp}_4(\mathbf{C})$; there are two examples of compact quotients of dimension 3: The quadric $Q \subset \mathbb{P}^4(\mathbf{C})$, and the projective space $\mathbb{P}^3(\mathbf{C})$ parametrizing the set of lines contained in this quadric (see example 4.7, [55], page 169, [2], page 65).

Let us now assume that H is of type D_r, i.e. H is isogenous to $\mathsf{SO}_{2r}(\mathbf{C})$, with $r \geq 3$ ($\mathfrak{so}_4(\mathbf{C})$ is not simple). We get $r^2 \leq 3r+3$ or $n=4$ and $r=3$. In all cases, $r=3$ and H is isogenous to $\mathsf{SO}_6(\mathbf{C})$. There is a unique homogeneous space M of dimension 4 for this group, namely the quadric $Q \subset \mathbb{P}^5(\mathbf{C})$.

Similarly, the inequality excludes the five exceptional groups $\mathsf{E}_6(\mathbf{C})$, $\mathsf{E}_7(\mathbf{C})$, $\mathsf{E}_8(\mathbf{C})$, $\mathsf{F}_4(\mathbf{C})$, and $\mathsf{G}_2(\mathbf{C})$: None of them acts transitively on a compact complex manifold of dimension $\mathsf{rk}(H)+1$ (see Table 1).

The remaining case concerns the group $H=\mathsf{SL}_n(\mathbf{C})$, acting transitively on a compact complex manifold M of dimension $n \geq 3$. Write $M=H/L$ where L is a closed subgroup of H. Two cases may occur: Either L is parabolic, or not.

If L is parabolic, then M is a flag manifold of dimension n for $\mathsf{SL}_n(\mathbf{C})$. Flag manifolds for $\mathsf{SL}_n(\mathbf{C})$ are well known, and only two examples satisfy our constraints. The first one is given by the incidence variety $F \subset \mathbb{P}^2(\mathbf{C}) \times \mathbb{P}^2(\mathbf{C})^\vee$ of pairs (x,l) where x is a point contained in the line l, or equivalently the set of complete flags of $\mathbf{C}^3$: This is a homogeneous space under the natural action of $\mathsf{PGL}_3(\mathbf{C})$ and, at the same time, this is a $\mathbb{P}^1(\mathbf{C})$-bundle over $\mathbb{P}^2(\mathbf{C})$. The second example is given by the Grassmannian $\mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$: This space has dimension 4 and is homogeneous under the natural action of $\mathsf{PGL}_4(\mathbf{C})$. This example appears for the second time: By Plücker embedding, $\mathbb{G}(1,3)$ is a smooth quadric in $\mathbb{P}^5(\mathbf{C})$ and, as such, is a homogeneous space for $\mathsf{SO}_6(\mathbf{C})$ (the groups $\mathsf{SO}_6(\mathbf{C})$ and $\mathsf{SL}_4(\mathbf{C})$ are isogenous, see page 286 of [31]).

If the group L is not parabolic, we denote by $N(L^0)$ the normalizer of its connected component of the identity, as in the proof of Theorem 4.1. Then $L \subset N(L^0)$ and $\dim(N(L^0)) \geq \dim(L)+1$. This gives rise to a H-equivariant fibration, the Tits fibration

$$M \to H/N(L^0),$$

with $\dim(H/N(L^0)) < n$. By Theorem 4.1, $H/N(L^0)$ is the projective space $\mathbb{P}^{n-1}(\mathbf{C})$ and $\dim(N(L^0)) = \dim(L)+1$. The fibers of the projection $M \to H/N(L^0)$ are quotients of a one parameter group by a discrete subgroup and, as such, are elliptic curves. This implies that M is an elliptic fibre bundle over $\mathbb{P}^{n-1}(\mathbf{C})$, as in Example 4.6.a, with B of genus 1, and Example 4.6.d.

4.6.2. *Almost homogeneous examples.* Let us now assume that M is not homogeneous under the action of H, but that H has an open orbit $\Omega = H/L$; let $Z = M \setminus \Omega$ be its complement; this set is analytic and H-invariant. A theorem due to Borel ([12]) asserts that the number of connected components of Z is at most 2. By Proposition 4.4, each component of Z is either a point or a copy of $\mathbb{P}^{n-1}(\mathbf{C})$; if one component is isomorphic to $\mathbb{P}^{n-1}(\mathbf{C})$ then H is isogenous to $\mathsf{SL}_n(\mathbf{C})$ and acts transitively on this component. Assume now that Z contains an isolated point m. This point is fixed by the action of H, and this action can be linearized locally around m. Since H has rank $n-1$ and M has dimension n, the group H is isogenous to $\mathsf{SL}_n(\mathbf{C})$. Blowing up the point m, we replace m by a copy of $\mathbb{P}^{n-1}(\mathbf{C})$. Thus, H is isogenous to $\mathsf{SL}_n(\mathbf{C})$, and blowing up the isolated points of Z, we can assume that Z is the union of one or two disjoint copies of $\mathbb{P}^{n-1}(\mathbf{C})$ on which H acts transitively. This situation has been studied in details in [39] and [33]; we now describe the conclusions of these papers without proof.

Let P be a maximal parabolic subgroup with $H/P = \mathbb{P}^{n-1}(\mathbf{C})$ (P is unique up to conjugacy).

Suppose, first, that Z is connected. Then $L \subset P$ (up to conjugacy), M is a projective rational manifold and it fibers equivariantly on $\mathbb{P}^{n-1}(\mathbf{C}) = H/P$; the fibers are isomorphic to $\mathbb{P}^1(\mathbf{C})$, each of them intersecting Z in one point [39]. The intersection of each fiber with the open orbit Ω is isomorphic to $\mathbf{C}$ and, at the same time, is isomorphic to P/L; this is not possible for $n > 2$ because all morphisms from the maximal parabolic group P to the group $\mathsf{Aff}(\mathbf{C})$ of holomorphic diffeomorphisms of $\mathbf{C}$ factor through the natural projection $P \to \mathbf{C}^*$, and there is no transitive action of $\mathbf{C}^*$ on $\mathbf{C}$.

Thus, Z has indeed two connected components (as in [33]). This case corresponds to $\mathbb{P}^1$-bundles over $\mathbb{P}^{n-1}(\mathbf{C})$, as in example 4.6: M fibers equivariantly on $\mathbb{P}^{n-1}(\mathbf{C})$ with fibers $F \simeq \mathbb{P}^1(\mathbf{C})$, each of them intersecting Z in two points; the two connected components of Z are two sections of the projection onto $\mathbb{P}^{n-1}(\mathbf{C})$, which correspond to the two line bundles O and $O(k)$ from example 4.6.

If $k = 1$, one of the sections can be blown down to a fixed point (this process inverses the blow up construction described at the beginning of this Section, i.e. of §4.6.2).

4.6.3. *No open orbit.* Let us now assume that H does not have any open orbit. Then, blowing up all fixed points of H, all orbits have dimension $n-1$. By Theorem 4.1, H is isogenous to $\mathsf{SL}_n(\mathbf{C})$ and its orbits are copies of $\mathbb{P}^{n-1}(\mathbf{C})$. In that case, the orbits define a locally trivial fibration of M over a curve B. Let A be the diagonal subgroup of $\mathsf{SL}_n(\mathbf{C})$. The set of fixed points of A defines n sections s_i of the fibration $M \to B$; for each point b in B the n-tuple $(s_1(b), ..., s_n(b))$ determines a projective basis of the fiber of M above b. This shows that the fibration is trivial: M is isomorphic to $\mathbb{P}^{n-1}(\mathbf{C}) \times B$. A posteriori, H had no fixed point on M.

5. Invariant cones for lattices and Lie groups

This paragraph contains preliminary results towards the proof of the Main Theorem in case (2-b). Under the assumption of assertion (2-b), Proposition 3.5 applies, and one can extend virtually the action of Γ on $W = H^{1,1}(M, \mathbf{R})$ to an action of G; a priori, the nef cone $\overline{\mathcal{K}}(M)$ is not G-invariant. In this section, we find a G-invariant subcone which is contained in $\overline{\mathcal{K}}(M)$. This is proved in the general context of a linear representation of a semi-simple Lie group G, for which a lattice $\Gamma \subset G$ preserves a salient cone. The reader may skip this section in a first reading.

5.1. **Proximal elements, proximal groups, and representations.**

5.1.1. *Proximal elements and proximal groups.* Let V be a real vector space of finite dimension k. Let g be an element of $\mathsf{GL}(V)$. Let $\lambda_1(g) \geq \lambda_2(g) \geq ... \geq \lambda_k(g)$ be the moduli of the eigenvalues of g, repeated according to their multiplicities. One says that g is **proximal** if $\lambda_1(g) > \lambda_2(g)$; in this case, g has a unique attracting fixed point x_g^+ in $\mathbb{P}(V)$. A subgroup of $\mathsf{GL}(V)$ is proximal if it contains a proximal element, and a representation $G \to \mathsf{GL}(V)$ is proximal if its image is a proximal subgroup. If Γ is a proximal subgroup of $\mathsf{GL}(V)$, the **limit set** $\Lambda_\Gamma^{\mathbb{P}}$ of the group Γ in $\mathbb{P}(V)$ is defined as the closure of the set $\{x_g^+ \mid g \in \Gamma, \text{ g is proximal}\}$; this set is the unique closed, minimal, invariant subset of $\mathbb{P}(V)$: $\Lambda_\Gamma^{\mathbb{P}}$ is invariant and closed, and is contained in every non-empty, closed, invariant subset.

5.1.2. *Proximal representations and highest weight vectors.* Let G be a real semi-simple Lie group and A be a maximal torus in G; let $\mathfrak{g}$ and $\mathfrak{a}$ be their respective Lie algebras, and Σ the system of restricted roots: By definition Σ is the set of non-zero weights for the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$. A scalar product $\langle\cdot|\cdot\rangle$ on $\mathfrak{a}$ is chosen, which is invariant under the action of the Weyl group. This scalar product provides an identification between $\mathfrak{a}$ and its dual.

A linear form $l: \mathfrak{a} \to \mathbf{R}$ is fixed, in such a way that its kernel does not contain any non-zero element of the lattice of roots; this form l determines an ordering of the root and a system of positive roots Σ^+, those with $l(\lambda) > 0$.

One denotes by Wt the set of **weights** of Σ; by definition

$$\mathsf{Wt} = \left\{ \lambda \in \mathfrak{a} \,|\, \forall \alpha \in \Sigma,\ 2\frac{\langle \lambda|\alpha\rangle}{\langle\alpha|\alpha\rangle} \in \mathbf{Z} \right\}.$$

The set of **dominant weights** is $\mathsf{Wt}^+ = \{\lambda \in \mathsf{Wt} \,|\, \forall \alpha \in \Sigma^+,\ \langle\lambda|\alpha\rangle \geq 0\}$.

Let $\rho\colon G \to \mathsf{GL}(V)$ be an irreducible representation of G. This provides a representation of the Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$. By definition, the weights of $\mathfrak{a}$ in V are the (restricted) weights of ρ. This finite set has a maximal element λ for the order defined on Wt; this **highest weight** λ is contained in Wt^+ and is unique.

The image of G in $\mathsf{GL}(V)$ is proximal if and only if the eigenspace of A corresponding to the highest weight λ has dimension 1 (see [1]); in that case, the highest weight determines the representation ρ up to isomorphism.

If one starts with a representation ρ which is not irreducible, one first splits it as a direct sum of irreducible factors, and then apply the previous description to each of them; this gives a list of highest weights, one for each irreducible factor. The maximal element in this list is the highest weight of ρ (see § 5.2.3).

5.2. **Invariant cones.** In this paragraph we prove the following proposition.

Proposition 5.1. *Let Γ be a lattice in a connected semi-simple Lie group G. Let $G \to \mathsf{GL}(V)$ be a real, finite dimensional, linear representation of G. If Γ preserves a salient cone $\Omega \subset V$ which is not reduced to $\{0\}$, its closure $\overline{\Omega}$ contains a subcone Ω_G such that:*

(1) *Ω_G is G-invariant, salient, and not reduced to $\{0\}$;*
(2) *If the action of Γ on the linear span of Ω is not trivial, the action of G on the subcone Ω_G is also not trivial.*

Remark 5.2. Note that we can find lattices Γ_0 in $\mathsf{SO}_{1,2}(\mathbf{R})$ and deformations Γ_t of Γ_0 in $\mathsf{SL}_3(\mathbf{R})$ such that Γ_t, $t \neq 0$, is Zariski dense in $\mathsf{SL}_3(\mathbf{R})$ and preserves an open, convex and salient cone $\Omega_t \subset \mathbf{R}^3$ (see [57]). This shows that Proposition 5.1 fails to be true for Zariski dense subgroups $\Gamma \subset G$.

Let G be a connected semi-simple Lie group and Γ be a Zariski dense subgroup of G. Let $\rho : G \to \mathsf{GL}(V)$ be a real, finite dimensional, linear representation of G. Assume that $\rho(\Gamma)$ preserves a salient cone Ω with $\Omega \neq \{0\}$. If the interior of Ω is empty, then Ω spans a proper Γ-invariant subspace of V; since Γ is Zariski dense in G this proper invariant subspace is G-invariant. We can therefore restrict the study to this invariant subspace and assume that the interior of Ω is not empty.

5.2.1. *Limit sets (see* [47], *lemma 8.5).* Let us recall a useful fact. Let G be a semi-simple Lie group having no compact normal subgroup of positive dimension. Let P be a parabolic subgroup of G. If Γ is a lattice in G, the closure of ΓP coincides with G: $\overline{\Gamma P} = G$. In other words, all orbits of Γ in G/P are dense. For example, if Γ is a lattice in $\mathsf{SL}_3(\mathbf{C})$, all orbits of Γ in $\mathbb{P}^2(\mathbf{C})$ are dense.

5.2.2. *Irreducible representations.* We first assume that ρ is irreducible (and non trivial). Proposition 3.1, page 164, of [5], implies that $\rho(\Gamma)$ is a proximal subgroup of $\mathsf{GL}(V)$, and the limit set $\Lambda^{\mathbb{P}}_{\rho(\Gamma)}$ of Γ is contained in $\mathbb{P}(\overline{\Omega})$. From §5.1, ρ is a proximal representation of the group G and the limit set $\Lambda^{\mathbb{P}}_{\rho(G)}$ of $\rho(G)$ coincides with the orbit of the highest weight line of its maximal torus: This orbit is the unique closed orbit of $\rho(G)$ in $\mathbb{P}(V)$. As such, $\Lambda^{\mathbb{P}}_{\rho(G)}$ is a homogeneous space G/P, where P is a parabolic subgroup of G.

Assume now that Γ is a lattice in G. By §5.2.1, all orbits $\Gamma \cdot x$ of Γ in G/P are dense, so that $\Lambda^{\mathbb{P}}_{\rho(G)} = G/P$ coincides with $\Lambda^{\mathbb{P}}_{\rho(\Gamma)}$. In particular, $\Lambda^{\mathbb{P}}_{\rho(G)}$ is a $\rho(G)$-invariant subset of $\mathbb{P}(\overline{\Omega})$. Let $\pi : V \setminus \{0\} \to \mathbb{P}(V)$ be the natural projection. The cone $\pi^{-1}(\Lambda^{\mathbb{P}}_{\rho(G)})$ is closed and G-invariant. This cone has two connected components and only one is contained in the closure of the salient cone $\overline{\Omega}$. Denote by $\Omega_G \subset \overline{\Omega}$ the convex hull of this component; since G is connected, both components are G-invariant, and thus Ω_G is a G-invariant convex cone contained in the closure of Ω. This proves Proposition 5.1 for irreducible representations.

5.2.3. *General case.* Let us now consider a linear representation $\rho : G \to \mathsf{GL}(V)$ which is not assumed to be irreducible. Prasad and Raghunathan proved in [49] that Γ intersects a conjugate A' of the maximal torus on a cocompact lattice $A'_\Gamma \subset A'$. Changing A into A', we assume that Γ intersects A on such a lattice A_Γ.

Since G is semi-simple, V splits into a direct sum of irreducible factors. Fix such a splitting, let λ be the highest weight of (ρ, V), let V_1, ..., V_m be the irreducible factors corresponding to this weight, and let V' be the direct sum of the V_i:

$$V' := \bigoplus_{1 \leq i \leq m} V_i.$$

By construction, all representations V_i, $1 \leq i \leq m$, are isomorphic, and we can assume that theses representations are not trivial.

Lemma 5.3. *Since Γ is a lattice, $\overline{\Omega}$ intersects the sum V' of the highest weight factors on a closed, salient cone Ω' which is not reduced to zero.*

Proof. If u is any element of Ω, one can decompose u as a sum $\sum_\chi u_\chi$ where each u_χ is an eigenvector of the maximal torus A corresponding to the weight χ. Since Ω has non empty interior, we can choose such an element u with a non zero component u_λ for the highest weight λ. Let $\| \cdot \|$ be a norm on the real vector space V. Since A_Γ is a lattice in A, there is a sequence of elements γ_n in A_Γ such that

$$\frac{\gamma_n(u)}{\|\gamma_n(u)\|} = \sum_\chi \frac{\chi(\gamma_n)}{\|\gamma_n(u)\|} u_\chi$$

converges to a non zero multiple of u_λ. Since Ω is Γ-invariant and all γ_n are in Γ, we deduce that $\overline{\Omega}$ intersects V'. $\square$

The subspace of V' which is spanned by Ω' is a direct sum of highest weight factors; for simplicity, we can therefore assume that V' is spanned by Ω'. In particular, the interior $\mathsf{Int}(\Omega')$ is a non-empty subset of V'.

Let π_i be the projection of $V' = \bigoplus V_i$ onto the factor V_i. The image of $\mathsf{Int}(\Omega')$ by π_1 is an open subcone in V_1.

If this cone is salient, the previous paragraph shows that the representation (ρ_1, V_1) is proximal. Thus, all V_i can be identified to a unique proximal representation R, with a given highest weight line $L = \mathbf{R}u^+$. We obtain m copies L_i of L, one in each copy V_i of R. Apply Lemma 5.3 and its proof: Since Ω' is Γ-invariant, Γ is a lattice, and Ω' has non empty interior, there is a point $v \in L_1 \oplus ... \oplus L_m$ which is contained in Ω'. Let $(a_1, ..., a_m)$ be the real numbers such that $v = (a_1u^+, ..., a_mu^+)$. The diagonal embedding $R \to V'$, $w \mapsto (a_1w, ..., a_mw)$ determines an irreducible sub-representation of G into V

that intersects Ω', and the previous paragraph shows that G preserves a salient subcone of Ω'.

If the (closed) cone $\pi_1(\Omega')$ is not salient, the fiber $\pi_1^{-1}(0)$ intersects Ω' on a non-zero Γ-invariant salient subcone; this reduces the number of irreducible factors from m to $m-1$, and enables us to prove Proposition 5.1 by induction on the number m of factors V_i.

6. Linear representations, ample classes and tori

We now prove the Main Theorem. Recall that G is a connected, almost simple, real Lie group with real rank $\mathsf{rk}_{\mathbf{R}}(G) \geq 2$, that A is a maximal torus of G, and that Γ is a lattice in G acting on a connected compact Kähler manifold M of dimension n.

From Section 4.3, we know that the rank of G is at most n and, in case $\mathsf{rk}_{\mathbf{R}}(G) = n$, the group G is isogenous to $\mathsf{SL}_{n+1}(\mathbf{R})$ or $\mathsf{SL}_{n+1}(\mathbf{C})$ and M is isomorphic to $\mathbb{P}^n(\mathbf{C})$. We now assume that the rank of G satisfies the next critical equality $\mathsf{rk}_{\mathbf{R}}(G) = n-1$. According to Proposition 3.5, two possibilities can occur.

- The image of Γ is virtually contained in $\mathsf{Aut}(M)^0$; Theorem 4.8 in Section 4.6 gives the list of possible pairs (M, G). This corresponds to assertion (2-a) in the Main Theorem.
- The action of Γ on the cohomology of M is almost faithful and virtually extends to a linear representation (a finite cover) of G on $H^*(M, \mathbf{R})$.

Thus, in order to prove the Main Theorem, we replace Γ by a finite index subgroup, G by a finite cover, and assume that the action of Γ on the cohomology of M is faithful and extends to a linear representation of G. Our aim is to prove that all such examples are Kummer examples when $\mathsf{rk}_{\mathbf{R}}(G) = \dim_{\mathbf{C}}(M) - 1$.

We denote by W the space $H^{1,1}(M, \mathbf{R})$. We fix a decomposition of W into irreducible factors. Then, we denote by λ_W the highest weight of the representation $G \to \mathsf{GL}(W)$ and by E the direct sum of the irreducible factors V_i of W corresponding to the weight λ_W (all V_i are isomorphic representations).

6.1. Invariant cones in $\overline{\mathcal{K}}(M)$.

Since the Kähler cone $\mathcal{K}(M)$ is a Γ-invariant, convex, and salient cone in W with non empty interior, Proposition 5.1 asserts that $\overline{\mathcal{K}}(M)$ contains a non-trivial G-invariant subcone. More precisely, $\overline{\mathcal{K}}(M) \cap E$ contains a G-invariant salient subcone $\overline{\mathcal{K}}_E$ which is not reduced to $\{0\}$, and the action of G on the linear span of $\overline{\mathcal{K}}_E$ is faithful (see §5.2).

From now on, we replace E by the linear span of the cone $\overline{\mathcal{K}}_E$. Doing this, the cone $\overline{\mathcal{K}}_E$ has non empty interior in E, and is a G-invariant subcone of

$\overline{\mathcal{K}}(M)$. Since G is almost simple, the representation $G \to \mathsf{GL}(E)$ is unimodular: Its image is contained in $\mathsf{SL}(E)$. Thus, *the action of the maximal torus A on E is unimodular, faithful and diagonalizable.*

6.2. **Actions of abelian groups.** We now focus on a slightly more general situation, and use ideas from [26]. Let E be a subspace of W of positive dimension and $\overline{\mathcal{K}_E}$ be a subcone of $\overline{\mathcal{K}}(M) \cap E$ with non empty interior. Let A be the additive abelian group $\mathbf{R}^m$, with $m \geq 1$; in the next paragraph, A will be a maximal torus of G, and thus $m = \mathsf{rk}_{\mathbf{R}}(G)$ will be equal to $\dim(M) - 1$. Let ρ be a continous representation of A into $\mathsf{GL}(H^*(M, \mathbf{R}))$ by cohomological automorphisms; thus, $\rho(A)$ preserves the Hodge structure and the Poincaré duality, and acts equivariantly with respect to the wedge product (see §2.1). Assume that

(i) $\rho(A)$ preserves E and $\overline{\mathcal{K}_E}$;
(ii) the restriction $\rho_E \colon A \to \mathsf{GL}(E)$ is diagonalizable, unimodular, and faithful.

From (ii), there is a basis of E and morphisms $\lambda_i : A \to \mathbf{R}$, $1 \leq i \leq \dim(E)$, such that the matrix of $\rho_E(a)$ in this basis is diagonal, with diagonal coefficients $\exp(\lambda_i(a))$. The morphisms λ_i are the weights of ρ_E; the set of weights

$$\Lambda = \{\lambda_i, 1 \leq i \leq \dim(E)\}$$

is a finite subset of $A^\vee$ where $A^\vee$, the dual of A, is identified with the space of linear forms on the real vector space $A = \mathbf{R}^m$. The convex hull of Λ is a polytope $\mathcal{C}(\Lambda) \subset A^\vee$ and the set of its extremal vertices is a subset Λ^+ of Λ; equivalently, a weight λ is **extremal** if and only if there is an element $a \in A$ such that

$$\lambda(a) > \alpha(a), \quad \forall \alpha \in \Lambda \setminus \{\lambda\}.$$

Since any non-empty convex set is the convex hull of its extremal points, Λ^+ is not empty and $\mathcal{C}(\Lambda^+)$ coincides with $\mathcal{C}(\Lambda)$.

For all weights $\alpha \in \Lambda$, we denote by E_α the eigenspace of A of weight α:

$$E_\alpha = \left\{ u \in E \mid \quad \forall a \in A,\ \rho_E(a)(u) = e^{\alpha(a)} u \right\}.$$

We denote by E^+ the vector subspace of E which is spanned by the E_λ where λ describes Λ^+.

Lemma 6.1. *The following three properties are satisfied.*

(1) *The morphism $\rho_{E^+} \colon A \to \mathsf{GL}(E^+)$ is injective.*
(2) *The convex hull $\mathcal{C}(\Lambda)$ of Λ^+ contains the origin in its interior; in particular the cardinality of Λ^+ satisfies $|\Lambda^+| \geq \dim(A) + 1$.*

(3) *For all* $\lambda \in \Lambda^+$ *we have* $E_\lambda \cap \overline{\mathcal{K}}_E \neq \{0\}$.

Proof. The three properties are well known.

Property (1).— The kernel of ρ_{E^+} is defined by the set of linear equations $\lambda(a) = 0$ where λ describes Λ^+. Since all weights $\alpha \in \Lambda$ are barycentric combinations of the extremal weights, the kernel of ρ_{E^+} is contained in the kernel of ρ_E. Property (1) follows from the injectivity of ρ_E.

Property (2).— Since the sum of all weights $\lambda_i(a)$, repeated with multiplicities, is the logarithm of the determinant of $\rho_E(a)$ and the representation is unimodular, this sum is 0. If $\mathcal{C}(\Lambda)$ has empty interior, it is contained in a strict affine subspace of $A^\vee$; since it contains the origin, this subspace is a proper vector subspace of $A^\vee$, contradicting Property (1); this contradiction shows that the interior of $\mathcal{C}(\Lambda)$ is not empty. The origin of $A^\vee$ being a barycentric combination of all extremal weights with strictly positive masses, it is contained in the interior of $\mathcal{C}(\Lambda)$. As a consequence, the cardinality of Λ^+ satifies $|\Lambda^+| \geq \dim(A) + 1$.

Property (3).— The proof is similar to the proof of Lemma 5.3. Let λ be an extremal weight and let $a \in A$ satisfy $\lambda(a) > \alpha(a)$ for all $\alpha \in \Lambda \setminus \{\lambda\}$. Let u be any element of $\overline{\mathcal{K}}_E$; write u as a linear combination $u = \sum_{\alpha \in \Lambda} u_\alpha$ where $u_\alpha \in E_\alpha$ for all α in Λ. Since $\overline{\mathcal{K}}_E$ has non empty interior, we can choose u in such a way that $u_\lambda \neq 0$. Then the sequence

$$\frac{\rho_E(na)(u)}{\exp(n\lambda(a))}$$

is a sequence of elements of $\overline{\mathcal{K}}_E$ that converges towards u_λ when n goes to $+\infty$. Since $\overline{\mathcal{K}}_E$ is closed, property (3) is proved. □

Lemma 6.2. *Let k be an integer with* $1 \leq k \leq \dim(M)$. *Let* $\lambda_i \in \Lambda$, $1 \leq i \leq k$, *be distinct weights, and w_i be non zero elements in* $E_{\lambda_i} \cap \overline{\mathcal{K}}_E$. *Then, the wedge product*

$$w_1 \wedge \ldots \wedge w_k$$

is different from 0.

The proof makes use of the following proposition which is due to Dinh and Sibony (see [26], Corollary 3.3 and Lemma 4.4). Lemma 4.4 of [26] is stated for cohomological automorphisms that are induced by automorphisms of M, but the proof given in [26] extends to all cohomological automorphisms.

Proposition 6.3 (Dinh and Sibony)**.** *Let M be a connected compact Kähler manifold of dimension n. Let u and v be elements of the nef cone* $\overline{\mathcal{K}}(M)$.

(1) *If u and v are not colinear, then $u \wedge v \neq 0$.*
(2) *Let $v_1, \ldots, v_l$, $l \leq n-2$, be elements of $\overline{\mathcal{K}}(M)$. If $v_1 \wedge \ldots \wedge v_l \wedge u$ and $v_1 \wedge \ldots \wedge v_l \wedge v$ are non zero eigenvectors with distinct eigenvalues for a cohomological automorphism, then $(v_1 \wedge \ldots \wedge v_l) \wedge (u \wedge v) \neq 0$.*

Proof of Lemma 6.2. The proof is an induction on k. Since all w_i are assumed to be different from 0, the property is established for $k = 1$. Suppose that the property holds for all integers $l \leq k$. Assume $k+1 \leq \dim(M)$ and choose w_1, ..., w_{k+1} as in the Lemma. Let $v_1, \ldots, v_{k-1}$ denote the vectors $w_1, \ldots, w_{k-1}$, let u be equal to w_k and v be equal to w_{k+1}. Since the property is proved for length k, we apply it to the vectors v_j and get that

$$v_1 \wedge \ldots \wedge v_{k-1} \wedge u \quad \text{and} \quad v_1 \wedge \ldots \wedge v_{k-1} \wedge v$$

are two non zero eigenvectors of A with respective weights

$$\lambda_k + \sum_{j=1}^{j=k-1} \lambda_j \quad \text{and} \quad \lambda_{k+1} + \sum_{j=1}^{j=k-1} \lambda_j.$$

These two weights are different because $\lambda_k \neq \lambda_{k+1}$. Thus, property (2) of proposition 6.3 can be applied, and it implies that the wedge product $w_1 \wedge \ldots \wedge w_k \wedge w_{k+1}$ is different from zero. The Lemma follows by induction. □

Let us now assume that $\dim_{\mathbf{R}}(A) = \dim_{\mathbf{C}}(M) - 1$, i.e. $m = n-1$. According to property (2) in Lemma 6.1, we can find

$$(\textbf{weights}) \begin{cases} n = \dim(A) + 1 \text{ extremal weights } \lambda_i \text{ such that all linear maps} \\ a \mapsto (\lambda_1(a), \ldots, \lambda_{i-1}(a), \lambda_{i+1}(a), \ldots, \lambda_n(a)),\ 1 \leq i \leq n, \\ \text{are bijections from } A \text{ to } \mathbf{R}^{n-1}. \end{cases}$$

By property (3) in Lemma 6.1, there exist elements w_i in $E_{\lambda_i} \cap \overline{\mathcal{K}}_E \setminus \{0\}$ for all $1 \leq i \leq \dim(A) + 1$. Once such a choice of vectors w_i is made, we define w_A as the sum

$$(\mathbf{w_A}) \qquad w_A = w_1 + w_2 + \ldots + w_n.$$

This class is nef and

$$w_A^{\wedge n} = w_A \wedge w_A \wedge \ldots \wedge w_A > 0$$

because w_A^n is a sum of products of nef classes, and as such is a sum of non-negative (n,n)-classes, and Lemma 6.2 implies that at least one term in the sum is different from zero. According to the definitions in Section 2.2, this proves the following corollary.

Corollary 6.4. *If* $\dim_{\mathbf{R}}(A) = \dim_{\mathbf{C}}(M) - 1$, *the class w_A is nef and big.*

Note that the class w_A depends on the choice for w_i, $1 \leq i \leq n$; the statement holds for all choices that satisfy (**weights**) and ($\mathbf{w_A}$).

6.3. **A characterization of torus examples.** Let us apply the previous paragraph to the maximal torus A of G; by assumption, G has rank $n-1$ and thus

$$\dim_{\mathbf{R}}(A) = \dim_{\mathbf{C}}(M) - 1 = n - 1.$$

The groups G and A act on W and preserve E; we denote by $\rho_E(g)$ the endomorphism of E obtained by the action of $g \in G$ on E (thus, $\rho_E(g)$ is the restriction of g^* to E if g is in Γ).

We keep the notation used in Sections 6.1 and 6.2; in particular, we pick classes w_i in $E \cap \overline{\mathcal{K}}(M)$ that correspond to distinct extremal eigenweights λ_i as in Equation (**weights**), and we define w_A as in formula ($\mathbf{w_A}$). This construction involves choices for the w_i; according to Corollary 6.4, the class w_A is nef and big for all choices of w_i.

Proposition 6.5. *If, for some choice of the classes w_i as above, the class w_A is a Kähler class then, up to a finite cover, M is a torus.*

Remark 6.6. In Section 6.4, this result is applied in the slightly more general context where M is an orbifold with isolated singularities.

Proof. Let $c_1(M) \in H^{1,1}(M,\mathbf{R})$ and $c_2(M) \in H^{2,2}(M,\mathbf{R})$ be the first and second Chern classes of M. Both of them are invariant under the action of Γ, and therefore also under the action of G.

Let $u \in W$ be a G-invariant cohomology class. Let $I = (i_1, ..., i_{n-1})$ be a multi-index of length $n-1$, and w_I be the product $w_{i_1} \wedge \ldots \wedge w_{i_{n-1}}$. Let v be the class of type (n,n) defined by $v = w_I \wedge u$. Since u is A-invariant we have

$$\rho_E(a)(v) = \exp\left(\sum_{j=1}^{n-1} \lambda_{i_j}(a)\right) v.$$

Since v is an element of $H^{n,n}(M,\mathbf{R})$ and the action of G is trivial on $H^{n,n}(M,\mathbf{R})$, we get the alternative: Either $v = 0$ or $\sum_{j=1}^{j=n-1} \lambda_{i_j}(a) = 0$ for all $a \in A$. Thus, according to the choice of the extremal weights λ_i (see Equation (**weights**)), the class v is equal to 0 for all choices of multi-indices I of length $n-1$. As a consequence, u is a primitive element with respect to the Kähler class w_A:

$$\int_M w_A^{n-1} \wedge u = 0.$$

In the same way, one proves that $w_A^{n-2} \wedge u = 0$ for all G-invariant cohomology classes u in $H^{2,2}(M,\mathbf{R})$.

Let us apply this remark to the first Chern class $c_1(M)$. Since this class is invariant, it is primitive with respect to w_A. Since $c_1(M)^2$ is also G-invariant,

$$w_A^{n-2} \wedge c_1(M)^2 = 0;$$

From Hodge index theorem we deduce that $c_1(M) = 0$. Yau's Theorem (see [60]) provides a Ricci flat Kähler metric on M with Kähler form w_A, and Yau's Formula reads

$$\int_M w_A^{n-2} \wedge c_2(M) = \kappa \int_M \|\mathsf{Rm}\|^2 w_A^n$$

where Rm is the Riemannian tensor and κ is a universal positive constant (see [8], page 80, and [40], §IV.4 page 112–118). From the invariance of $c_2(M)$ we get $w_A^{n-2} \wedge c_2(M) = 0$ and then $\mathsf{Rm} = 0$. This means that M is flat and thus M is finitely covered by a torus T. □

Using this proposition, we now need to change the big and nef class w_A into an ample class by a modification of M; this modification will change M into an orbifold M_0. This is done in the following paragraph.

6.4. **Obstruction to ampleness, invariant subsets, and Kummer examples.**

6.4.1. Let us start with the following simple fact.

Proposition 6.7. *Let B be an irreducible real analytic subset of the vector space $H^{1,1}(M, \mathbf{R})$. Assume that*

(i) *all classes w in B are big and nef classes but*
(ii) *none of them is ample.*

Then there exists an integer d with $0 < d < n$ and a complex analytic subset $Y_0 \subset M$ of dimension d such that $\int_{Y_0} w^d = 0$ for all classes w in B.

Proof. The set of classes $[Y]$ of irreducible analytic subsets $Y \subset X$ is countable. For all such cohomology classes $[Y]$, let $Z_{[Y]}$ be the closed, analytic subset of B which is defined by

$$Z_{[Y]} = \left\{ w \in B \mid \int_Y w^{\dim(Y)} = 0 \right\}.$$

Apply Theorem 2.2 in Section 2.2. Since all elements of B are nef and big but none of them is ample, the family of closed subsets $Z_{[Y]}$ with $\dim(Y) \geq 1$ covers B. By Baire's theorem, one of the subsets $Z_{[Y]}$ has non empty interior. Let $Z_{[Y_0]}$ be such a subset, with $\dim(Y_0) \geq 1$. The map

$$w \mapsto \int_{Y_0} w^{\dim(Y_0)}$$

is algebraic and vanishes identically on an open subset of B. Since B is an irreducible analytic subset of $H^{1,1}(M,\mathbf{R})$, this map vanishes identically. $\square$

6.4.2. We now conclude the proof of assertion (2-b) in the Main Theorem; for this, we assume that w_A is not ample for all choices of the maximal torus A in G and the eigenvectors w_i of A as in §6.2, equations (**weights**) and ($\mathbf{w_A}$). Consider the orbit of the class w_A under the action of G. This orbit $B = G.w_A$ satisfies the following properties:

(1) B is made of big and nef classes, but none of them is ample;
(2) B is a connected Zariski open subset in an irreducible algebraic subset of E.

We can thus apply Proposition 6.7 to the set B. Let Z be the Zariski closure of the union of analytic subsets $Y \subset M$ such that $0 < \dim(Y) < \dim(M)$ and

$$\int_Y w^{\dim(Y)} = 0, \quad \forall w \in B.$$

Proposition 6.7 and Section 2.2 show that Z is a non empty proper analytic subset of M. Since B is the orbit of w_A under the action of G, this set is Γ-invariant.

6.4.3. Applying Theorem 4.5, all proper invariant analytic subsets, and in particular the subset $Z \subset M$, can be contracted. We get a birational morphism $\pi : M \to M_0$ and conclude that the image of w_A in M_0 is ample. Let us now explain how Section 6.3 and Proposition 6.5 can be applied in this orbifold context to deduce that M_0 is covered by a torus T.

Here, M_0 is a connected orbifold with trivial Chern classes $c_1(M_0)$ and $c_2(M_0)$. This implies that there is a flat Kähler metric on M_0 (see [40], and the end of the proof of Proposition 6.5). The universal cover of M_0 (in the orbifold sense) is then isomorphic to $\mathbf{C}^n$ and the (orbifold) fundamental group $\pi_1^{orb}(M_0)$ acts by affine isometries on $\mathbf{C}^n$ for the standard euclidean metric. In other words, $\pi_1^{orb}(M_0)$ is identified to a cristallographic group Δ of affine motions of $\mathbf{C}^n$. Let Δ^* be the group of translations contained in Δ. Bieberbach's theorem shows that (see [59], chapter 3, theorem 3.2.9).

a.- Δ^* is a lattice in $\mathbf{C}^n$;
b.- Δ^* is the unique maximal and normal free abelian subgroup of Δ of rank $2n$.

The torus T is the quotient of $\mathbf{C}^n$ by this group of translations. By construction, T covers M_0. Let F be the quotient group Δ/Δ^*; we identify it to the group of deck transformations of the covering $\varepsilon : T \to M_0$. To conlude the proof of the

Main Theorem, all we need to do is to lift virtually the action of Γ on M_0 to an action on T. This is done in the following lemma.

Lemma 6.8.

(1) *Some finite index subgroup of* Γ *lifts to* $\mathsf{Aut}(T)$.
(2) *Either* M_0 *is singular, or* M_0 *is a torus.*
(3) M_0 *is a quotient of the torus* T *by a homothety*

$$(x_1, x_2, \ldots x_n) \mapsto (\eta x_1, \eta x_2, \ldots, \eta x_n),$$

where η *is a root of* 1.

Proof. By property (b.) all automorphisms of M_0 lift to T. Let $\overline{\Gamma} \subset \mathsf{Aut}(T)$ be the group of automorphisms of T made of all possible lifts of elements of Γ. So, $\overline{\Gamma}$ is an extension of Γ by the group F:

$$1 \to F \to \overline{\Gamma} \to \Gamma \to 1.$$

Let $L : \mathsf{Aut}(T) \to \mathsf{GL}_n(\mathbf{C})$ be the morphism which maps each automorphism f of T to its linear part $L(f)$. Since T is obtained as the quotient of $\mathbf{C}^n$ by all translations contained in Δ, the restriction of L to F is injective. Let $N \subset \mathsf{GL}_n(\mathbf{C})$ be the normalizer of $L(F)$. The group $L(\overline{\Gamma})$ normalizes $L(F)$. Hence we have a well defined morphism $\overline{\Gamma} \to N$, and an induced morphism $\delta : \Gamma \to N/L(F)$. Changing Γ into a finite index subgroup, δ is injective. Since Γ is a lattice in an almost simple Lie group of rank $n-1$, the Lie algebra of $N/L(F)$ contains a simple subalgebra of rank $n-1$. Since $\mathfrak{sl}_n(\mathbf{C})$ is the unique complex simple subalgebra of rank $n-1$ in $\mathfrak{gl}_n(\mathbf{C})$, we conclude that N contains $\mathsf{SL}_n(\mathbf{C})$. It follows that $L(F)$ is contained in the center $\mathbf{C}^*\mathrm{Id}$ of $\mathsf{GL}_n(\mathbf{C})$.

Either F is trivial, and then M_0 coincides with the torus T, or F is a cyclic subgroup of $\mathbf{C}^*\mathrm{Id}$. In the first case, there is no need to lift Γ to $\mathsf{Aut}(T)$. In the second case, we fix a generator g of F, and denote by η the root of unity such that $L(g)$ is the multiplication by η. The automorphism g has at least one (isolated) fixed point x_0 in T. Changing Γ into a finite index subgroup Γ_1, we can assume that $\overline{\Gamma_1}$ fixes x_0. The linear part L embeds $\overline{\Gamma_1}$ into $\mathsf{GL}_n(\mathbf{C})$. Selberg's lemma assures that a finite index subgroup of $\overline{\Gamma_1}$ has no torsion. This subgroup does not intersect F, hence projects bijectively onto a finite index subgroup of Γ_1. This proves that a finite index subgroup Γ_1 of Γ lifts to $\mathsf{Aut}(T)$. $\square$

7. Classification of Kummer examples

In this section, we list all Kummer examples of dimension $n \geq 3$ with an action of a lattice Γ in a rank $n-1$ simple Lie group G, up to commensurability and isogenies.

Let G be an almost simple Lie group of rank $n-1$ and Γ be a lattice in G. Let $T = \mathbf{C}^n/\Lambda$ be a torus such that $\mathsf{Aut}(T)$ contains a copy of Γ. As seen in the proof of Lemma 6.8, a finite index subgroup of Γ lifts to a linear representation ρ_Γ into $\mathsf{SL}_n(\mathbf{C})$ that preserves the lattice Λ. Changing G in an appropriate finite index cover, Margulis theorem implies that ρ_Γ virtually extends to a representation ρ_G of G itself. Thus, we have to list triples (G, Γ, Λ) where G is a rank $n-1$, real, and almost simple Lie group represented in $\mathsf{SL}_n(\mathbf{C})$, Γ is a lattice in G, Λ is a lattice in $\mathbf{C}^n$, and Γ preserves Λ. This is done in paragraphs 7.1 to 7.3: The list is up to commensurability for Γ, and up to isogeny for the torus $\mathbf{C}^n/\Lambda$.

Since G has rank $n-1$ and almost embeds into $\mathsf{SL}_n(\mathbf{C})$, G is locally isomorphic to $\mathsf{SL}_n(\mathbf{R})$ or $\mathsf{SL}_n(\mathbf{C})$. This justifies why we restrict to these two possibilities in the following statement of the results.

Theorem 7.1. *Let T be a complex torus of complex dimension $n \geq 3$. If $\mathsf{Aut}(T)$ contains a lattice Γ of $\mathsf{SL}_n(\mathbf{C})$, then*

- *Γ is commensurable to $\mathsf{SL}_n(O_d)$ where O_d is the ring of integers in $\mathbf{Q}(\sqrt{d})$ for some negative integer d, and T is isogenous to the product of n copies of the elliptic curve $\mathbf{C}/O_d$.*

If $\mathsf{Aut}(T)$ contains a lattice Γ of $\mathsf{SL}_n(\mathbf{R})$. Then

- *either Γ is commensurable to $\mathsf{SL}_n(\mathbf{Z})$ and T is isogenous to the product of n copies of an elliptic curve $\mathbf{C}/\Lambda_0$;*
- *or $n = 2k$ is even and there exists an indefinite quaternion algebra $\mathbf{H}_{a,b}$ over $\mathbf{Q}$ such that Γ is commensurable to the group $\mathsf{SL}_k(\mathbf{H}_{a,b}(\mathbf{Z}))$ and T is isogenous to the product of k copies of an abelian surface Y such that $\mathsf{End}_{\mathbf{Q}}(Y)$ contains $\mathbf{H}_{a,b}(\mathbf{Q})$.*

In particular, Γ is not cocompact and T is an abelian variety

We discuss Kummer examples in paragraph 7.5; this amounts to list all possible quotients T/F where F is a finite group which is normalized by the action of the lattice Γ.

7.1. **Endomorphisms of complex tori.** Let us summarize the results we need concerning the endomorphism algebra of complex tori. These results are well know for abelian varieties; in the case of arbitrary complex tori, proofs can be found in [9].

Let T be a complex torus of dimension n. By definition, T is **simple** if it does not contain any proper subtorus of positive dimension; it is **indecomposable** if it is not isogenous to a product $T_1 \times T_2$ of nonzero complex tori. Thus, all simple tori are indecomposable but there are indecomposable complex tori which are not simple (a phenomenon which does not arise for abelian varieties, see [9], chapters I.5 and I.6). Given a complex torus $T \neq 0$, there is an isogeny

$$T \to T_1^{n_1} \times \ldots \times T_k^{n_k}$$

with indecomposable pairwise nonisogenous nonzero tori T_i and positive integers n_i; the complex tori T_i and integers n_i are uniquely determined up to isogenies and permutations.

Let $\mathsf{End}\,(T)$ denote the endomorphism algebra of T, and let $\mathsf{End}_{\mathbf{Q}}(T)$ be defined by

$$\mathsf{End}_{\mathbf{Q}}(T) = \mathsf{End}\,(T) \otimes_{\mathbf{Z}} \mathbf{Q};$$

it is a finite dimensional $\mathbf{Q}$-algebra. Its **radical** $\mathsf{Nil}_{\mathbf{Q}}(T)$ can be defined as the intersection of all maximal left ideals of $\mathsf{End}_{\mathbf{Q}}(T)$; this radical $\mathsf{Nil}_{\mathbf{Q}}(T)$ is two sided, contains all nilpotent two sided ideals, and is itself nilpotent: There exists a positive integer r such that $\mathsf{Nil}_{\mathbf{Q}}(T)^r = 0$ (cf. [43], XVII-6, Thm. 6.1). The quotient $\mathsf{End}_{\mathbf{Q}}(T)_{ss}$ of $\mathsf{End}_{\mathbf{Q}}(T)$ by $\mathsf{Nil}_{\mathbf{Q}}(T)$ is a semi-simple algebra; it is a division algebra if and only if T is indecomposable. Moreover

$$\mathsf{End}_{\mathbf{Q}}(T)_{ss} \simeq \mathsf{Mat}_{n_1}(\mathsf{End}_{\mathbf{Q}}(T_1)_{ss}) \times \ldots \times \mathsf{Mat}_{n_k}(\mathsf{End}_{\mathbf{Q}}(T_k)_{ss})$$

where the T_i and n_i are given by the decomposition of T as a product of indecomposable factors.

Let us now embed $\mathsf{End}_{\mathbf{Q}}(T)$ in $\mathsf{Mat}_n(\mathbf{C})$ as an algebra of linear transformations of $\mathbf{C}^n$ preserving $\Lambda \otimes_{\mathbf{Z}} \mathbf{Q}$. If T is indecomposable, $\mathsf{End}_{\mathbf{Q}}(T)_{ss}$ is a division algebra and thus $\det(u) \neq 0$ for all u in $\mathsf{End}_{\mathbf{Q}}(T)_{ss}$. Let $x \neq 0$ be an element of the lattice Λ, then

$$u \mapsto u(x)$$

is a morphism from the $\mathbf{Q}$-module $\mathsf{End}_{\mathbf{Q}}(T)_{ss}$ to $\Lambda \otimes_{\mathbf{Z}} \mathbf{Q}$ with trivial kernel if T is indecomposable. Thus, we get the fundamental simple fact: If T is indecomposable, then

$$\dim_{\mathbf{Q}}(\mathsf{End}_{\mathbf{Q}}(T)_{ss}) \leq 2\dim_{\mathbf{C}}(T). \tag{7.1}$$

7.2. **Lattices in endomorphism algebras of complex tori.** Let G be an almost simple, connected, real Lie group of rank $n-1$. Let Γ be a lattice in G, and assume that Γ embeds into the group of automorphisms of a complex torus $T = \mathbf{C}^n/\Lambda$. As explained in the introduction of this Section, up to finite index and finite cover, we can assume that Γ lifts to a group of linear transformations of $\mathbf{C}^n$ preserving Λ and that the linear representation ρ_Γ of Γ in $\mathbf{C}^n$ extends to a linear representation ρ_G of G. This implies that G is locally isomorphic to $\mathsf{SL}_n(\mathbf{R})$ or $\mathsf{SL}_n(\mathbf{C})$ and that $\rho_\Gamma(\Gamma)$ is Zariski dense in the complex algebraic group $\mathsf{SL}_n(\mathbf{C})$.

Lemma 7.2. *The radical* $\mathsf{Nil}_{\mathbf{Q}}(T)$ *is reduced to zero.*

Proof. Since $\mathsf{Nil}_{\mathbf{Q}}(T)$ is a nilpotent subalgebra of $\mathsf{Mat}_n(\mathbf{C})$, its kernel

$$\mathrm{Ker}(\mathsf{Nil}_{\mathbf{Q}}(T)) := \{x \in \mathbf{C}^n \mid u(x) = 0 \ \forall u \in \mathsf{Nil}_{\mathbf{Q}}(T)\}$$

has positive dimension. Since $\mathsf{Nil}_{\mathbf{Q}}(T)$ is a two-sided ideal and $\mathsf{End}_{\mathbf{Q}}(T)$ contains Γ, $\mathrm{Ker}(\mathsf{Nil}_{\mathbf{Q}}(T))$ is Γ-invariant. Since Γ is Zariski dense, $\mathrm{Ker}(\mathsf{Nil}_{\mathbf{Q}}(T))$ coincides with $\mathbf{C}^n$ and $\mathsf{Nil}_{\mathbf{Q}}(T) = \{0\}$. □

As a consequence, decomposing T as a product

$$T_1^{n_1} \times \ldots \times T_k^{n_k}$$

of indecomposable tori T_i, all endomorphism algebra $\mathsf{End}_{\mathbf{Q}}(T_i)$ are semi-simple.

Lemma 7.3. *The torus T has a unique indecomposable factor $T_1^{n_1}$ and the dimension of T_1 is equal to* 1 *or* 2*; moreover,* $\dim_{\mathbf{C}}(T_1) = 1$ *if the Lie group G is locally isomorphic to* $\mathsf{SL}_n(\mathbf{C})$.

Proof. If T has distinct indecomposable factors, its decomposition induces a $\mathsf{End}_{\mathbf{Q}}(T)$-invariant splitting of $\mathbf{C}^n$ as a direct sum of two non-trivial complex linear subspaces. In particular, $\mathsf{End}_{\mathbf{Q}}(T)$ cannot contain the Zariski dense subgroup Γ. Thus, there is an indecomposable torus T_1 and a positive integer n_1 such that T is (isogenous to) $T_1^{n_1}$. The endomorphism algebra of T_1 is a division algebra and $\mathsf{End}_{\mathbf{Q}}(T)$ is isomorphic to $\mathsf{Mat}_{n_1}(\mathsf{End}_{\mathbf{Q}}(T_1))$. We have $n = \dim_{\mathbf{C}}(T) = n_1 \times \dim_{\mathbf{C}}(T_1)$, and the fundamental inequality (7.1) gives

$$\dim_{\mathbf{Q}}(\mathsf{End}_{\mathbf{Q}}(T_1)) \leq 2\dim_{\mathbf{C}}(T_1).$$

Since the action of Γ on $\mathbf{C}^n$ extends to a linear representation of G, the real dimension of $\mathsf{End}_{\mathbf{Q}}(T) \otimes_{\mathbf{Q}} \mathbf{R}$ is at least $a_G n^2$ where $a_G = 1$ (resp. 2) if G is locally isomorphic to $\mathsf{SL}_n(\mathbf{R})$ (resp. $\mathsf{SL}_n(\mathbf{C})$). Thus, we have

$$a_G n_1^2 \dim_C(T_1)^2 = a_G n^2 \leq \dim_{\mathbf{R}}(\mathsf{Mat}_{n_1}(\mathsf{End}_{\mathbf{Q}}(T_1))) \otimes_{\mathbf{Q}} \mathbf{R}) \leq 2\dim_C(T_1) n_1^2.$$

As a consequence, $a_G \dim_C(T_1) \leq 2$ and the conclusion follows. □

7.3. **Complex tori of dimension 2.** Lemma 7.3 shows that T is isogenous to E^n where E is an elliptic curve, or that $n = 2m$ is even and T is isogenous to Y^m for some two-dimensional torus Y. We first study the case when T is isogenous to Y^m for some indecomposable complex torus of dimension 2; for this purpose we need the concept of quaternion algebra.

7.3.1. *Quaternion algebras and complex tori.* Let a and b be two integers. Let $\mathbf{H}_{a,b}$ (or $\mathbf{H}_{a,b}(\mathbf{Q})$) be the quaternion algebra over the rational numbers $\mathbf{Q}$ defined by its basis $(1, \mathsf{i}, \mathsf{j}, \mathsf{k})$, with

$$\mathsf{i}^2 = a,\ \mathsf{j}^2 = b,\ \mathsf{ij} = \mathsf{k} = -\mathsf{ji}.$$

This algebra embeds into the space of 2×2 matrices over $\mathbf{Q}(\sqrt{a})$ by mapping i and j to the matrices

$$\begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}.$$

In what follows, $\mathbf{H}_{a,b}(\mathbf{Z})$ denotes the set of quaternions with coefficients in $\mathbf{Z}$, and $\mathbf{H}_{a,b}(\mathbf{R})$ denotes the tensor product $\mathbf{H}_{a,b} \otimes_{\mathbf{Q}} \mathbf{R}$. The determinant of the matrix which is associated to a quaternion $q = x + y\mathsf{i} + z\mathsf{j} + t\mathsf{k}$ is equal to its reduced norm

$$\mathrm{Nrd}(q) = x^2 - ay^2 - bz^2 + abt^2.$$

The algebra $\mathbf{H}_{a,b}$ is a division algebra if and only if $\mathrm{Nrd}(q) \neq 0$ for all elements $q \neq 0$ in $\mathbf{H}_{a,b}(\mathbf{Q})$. One says that $\mathbf{H}_{a,b}$ is definite if $\mathbf{H}_{a,b} \otimes_{\mathbf{Q}} \mathbf{R}$ is isomorphic to the usual quaternion algebra $\mathbf{H} = \mathbf{H}_{-1,-1}$, and indefinite otherwise; indefinite quaternion algebras are isomorphic to $\mathsf{Mat}_2(\mathbf{R})$ over $\mathbf{R}$.

Note that $\mathbf{H}_{a,b}(\mathbf{Q})$ is isomorphic to $\mathbf{H}_{b,a}(\mathbf{Q})$ (permuting i and j), to $\mathbf{H}_{\mu^2 a,b}$ for μ in $\mathbf{Q}$ (changing i in $\mu\mathsf{i}$), and to $\mathbf{H}_{a,-ab}(\mathbf{Q})$ (permuting j and k). When $\mathbf{H}_{a,b}(\mathbf{Q})$ is indefinite, at least one of the parameters a and b is positive; thus, permuting a and b, and then changing b into $-ab$ if necessary, one can assume that a and b are positive.

Assume $\mathbf{H}_{a,b}(\mathbf{Q})$ is indefinite. The embedding of $\mathbf{H}_{a,b}(\mathbf{Q})$ into $\mathsf{Mat}_2(\mathbf{R})$ extends to an isomorphism between $\mathbf{H}_{a,b}(\mathbf{R})$ and $\mathsf{Mat}_2(\mathbf{R})$. The group

$$\mathsf{GL}_k(\mathbf{H}_{a,b}(\mathbf{Z}))$$

acts by left multiplication on $\mathbf{H}_{a,b}(\mathbf{R})^k$, preserving the lattice $\mathbf{H}_{a,b}(\mathbf{Z})^k$. This action commutes with right multiplication by elements of $\mathbf{H}_{a,b}(\mathbf{R})$, i.e. by elements of $\mathsf{Mat}_2(\mathbf{R})$; in particular, if J is an element of $\mathsf{SL}_2(\mathbf{R})$ with $J^2 = -\mathrm{Id}$, then J determines a $\mathsf{GL}_k(\mathbf{H}_{a,b}(\mathbf{Z}))$-invariant complex structure on $\mathbf{H}_{a,b}(\mathbf{R})^k$ and on the quotient $(\mathbf{H}_{a,b}(\mathbf{R})/\mathbf{H}_{a,b}(\mathbf{Z}))^k$.

Remark 7.4. These complex structures depend on one complex parameter, because the set of such operators $J \in \mathsf{SL}_2(\mathbf{R})$ depends on such a parameter. We thus get a one parameter family of tori $Y = \mathbf{H}_{a,b}(\mathbf{R})/\mathbf{H}_{a,b}(\mathbf{Z})$, with complex structure J. For a generic J, the complex torus Y is indecomposable (see [10], §9), but we shall see in Section 7.3.3 that specific choices of J lead to surfaces Y that can be isogenous to a product $E \times E$ of the same elliptic curve.

7.3.2. *Complex tori of dimension* 2. Let us now assume that $n = 2m$ is even, and $T = Y^m$ for some indecomposable complex torus Y of dimension 2. From Lemma 7.3, we know that the Lie group G is (locally isomorphic to) $\mathsf{SL}_n(\mathbf{R})$. Since G has dimension $n^2 - 1$ we deduce that

$$n^2 \leq (n/2)^2 \dim(\mathsf{End}_{\mathbf{Q}}(Y) \otimes_{\mathbf{Q}} \mathbf{R}))$$

and thus $\mathsf{End}_{\mathbf{Q}}(Y)$ is a division algebra of dimension at least 4; since Y is a surface, the dimension of $\mathsf{End}_{\mathbf{Q}}(Y)$ is equal to 4. This implies (see [10, 50]) that $\mathsf{End}_{\mathbf{Q}}(Y)$ is a quaternion algebra or a complex multiplication field $\mathbf{k}$ of degree 4 over $\mathbf{Q}$.

In the latter case, Γ embeds virtually in $\mathsf{SL}_m(O_{\mathbf{k}})$ where $O_{\mathbf{k}}$ is the ring of integers in the quartic field $\mathbf{k}$. This contradicts Margulis Theorem, i.e. Theorem 3.2, because Γ is a lattice in $\mathsf{SL}_n(\mathbf{R})$ and $n = 2m > m$; as a consequence, $\mathsf{End}_{\mathbf{Q}}(Y)$ is a quaternion algebra. If this quaternion algebra is definite, then $\mathsf{End}_{\mathbf{Q}}(Y) \otimes_{\mathbf{Q}} \mathbf{R}$ is isomorphic to the quaternion algebra $\mathbf{H}$ and Γ embeds into $\mathsf{GL}_m(\mathbf{H})$, which is a real Lie group of rank m. This contradicts Theorem 3.2 and shows that $\mathsf{End}_{\mathbf{Q}}(Y)$ is an indefinite quaternion algebra. In particular, Y is an abelian surface and T is an abelian variety (see [52, 50]). Thus, we have proved the following proposition.

Proposition 7.5. *Let Y be an indecomposable complex torus of dimension* 2. *Let m be a positive integer and T be the complex torus Y^m. Let Γ be a lattice in $\mathsf{SL}_n(\mathbf{R})$. If Γ embeds into $\mathsf{Aut}(T)$, then Y is isogenous to an abelian surface such that $\mathsf{End}_{\mathbf{Q}}(Y)$ contains $\mathbf{H}_{a,b}(\mathbf{Q})$ for some indefinite quaternion algebra, and Γ is commensurable to the group $\mathsf{SL}_m(H_{a,b}(\mathbf{Z}))$.*

7.3.3. *An example.* Let L be the lattice $H_{a,b}(\mathbf{Z}) \subset \mathsf{Mat}_2(\mathbf{R}) \simeq \mathbf{H}_{a,b}(\mathbf{R})$, where $H_{a,b}(\mathbf{Q})$ is an indefinite quaternion algebra, with a and b positive integers. The group $\mathsf{GL}_1(H_{a,b}(\mathbf{Z}))$ acts by left multiplication on $\mathbf{H}_{a,b}(\mathbf{R})$, preserves the lattice L, and the one parameter family of complex structures defined at the end of Section 7.3.1. As a complex structure on $\mathbf{H}_{a,b}(\mathbf{R})$, take

$$J(w) = w \cdot (c\mathsf{k})$$

where c is the real number $1/\sqrt{ab}$. Let Y be the torus $\mathbf{H}_{a,b}(\mathbf{R})/L$, with complex structure J. The planes $\mathbf{R}+\mathbf{R}\mathsf{k}$ and $\mathbf{R}\mathsf{i}+\mathbf{R}\mathsf{j}$ are J-invariant and intersect the lattice L on cocompact lattices. Moreover, the elliptic curves $(\mathbf{R}+\mathbf{R}k)/L$ and $(\mathbf{R}i+\mathbf{R}j)/L$ are isogenous to $\mathbf{C}/\mathbf{Z}[i\sqrt{ab}]$. This shows that Y is isogenous to $(\mathbf{C}/O_d)^2$ where $d=-ab$, and that $\mathsf{GL}_1(H_{a,b}(\mathbf{Z}))$ embeds virtually into $\mathsf{GL}_2(O_d)$.

A similar construction works for the lattice L^m and the group $\mathsf{SL}_m(H_{a,b}(\mathbf{Z}))$ when $m>1$. This example should be kept in mind when reading the proof of Proposition 7.6 below.

7.4. **Elliptic curves.**

Proposition 7.6. *Let E be an elliptic curve. Let n be a positive integer and T the abelian variety E^n.*

(1) *The automorphisms group $\mathsf{Aut}(T)$ contains a copy of $\mathsf{SL}_n(\mathbf{Z})$.*
(2) *If $\mathsf{End}(T)$ contains a lattice Γ in $\mathsf{SL}_n(\mathbf{C})$ then Γ is commensurable to $\mathsf{SL}_n(O_d)$ where O_d is the ring of integers in $\mathbf{Q}(\sqrt{d})$ for some negative integer d, and E is isogenous to the elliptic curve $\mathbf{C}/O_d$.*
(3) *If $\mathsf{Aut}(T)$ contains a copy of a lattice Γ of $\mathsf{SL}_n(\mathbf{R})$, either this lattice is commensurable to $\mathsf{SL}_n(\mathbf{Z})$ or $n=2m$ is even, the lattice is commensurable to $\mathsf{SL}_m(H_{a,b}(\mathbf{Z}))$ for some indefinite quaternion algebra, and E is isogenous to the elliptic curve $\mathbf{C}/O_d$ for $d=-ab$.*

Proof. Let E be an elliptic curve. If E has complex multiplication, it is isogenous to $\mathbf{C}/O_d$ for some imaginary quadratic extension $\mathbf{Q}(\sqrt{d})$. If E does not have complex multiplication, then $\mathsf{End}_{\mathbf{Q}}(E)$ is isomorphic to $\mathbf{Q}$. In all cases, $\mathsf{Mat}_n(\mathsf{End}(E))$ contains $\mathsf{SL}_n(\mathbf{Z})$; this proves the first assertion.

Let Γ be a lattice in $\mathsf{SL}_n(\mathbf{R})$ or $\mathsf{SL}_n(\mathbf{C})$, and assume that Γ embeds into $\mathsf{Aut}(T)\subset\mathsf{Mat}_n(\mathsf{End}(E))$. If E does not have complex multiplication, $\mathsf{End}(E)$ is equal to $\mathbf{Z}$ and the image of Γ in $\mathsf{Mat}_n(\mathbf{Z})$ is commensurable to $\mathsf{SL}_n(\mathbf{Z})$; in particular, Γ is a lattice in $\mathsf{SL}_n(\mathbf{R})$. We can therefore assume that E is isomorphic to $\mathbf{C}/O_d$ for some square free negative integer d. Then, Γ embeds virtually into $\mathsf{SL}_n(O_d)$. If Γ is a lattice in $\mathsf{SL}_n(\mathbf{C})$, this implies that Γ is commensurable to $\mathsf{SL}_n(O_d)$.

Assume now that $E=\mathbf{C}/O_d$ and $G=\mathsf{SL}_n(\mathbf{R})$; we want to show that Γ is commensurable to $\mathsf{SL}_n(\mathbf{Z})$, or $n=2m$ and Γ is commensurable to $\mathsf{SL}_m(H_{a,b}(\mathbf{Z}))$ for an indefinite quaternion algebra.

The action of G on $\mathbf{C}^n$ preserves the complex structure; therefore, after conjugacy by an element B of $\mathsf{GL}_n(\mathbf{C})$, we can assume that ρ_G is the standard embedding of $G = \mathsf{SL}_n(\mathbf{R})$ in $\mathsf{SL}_n(\mathbf{C})$ as the subgroup of matrices with real coefficients. In particular, G and Γ preserve the decomposition $\mathbf{C}^n = \mathbf{R}^n + i\mathbf{R}^n$. Let Λ be the Γ-invariant lattice $B(O_d^n)$.

Let W be the vector space $\mathbf{C}^n$, viewed as a real vector space, and V be the subspace $\mathbf{R}^n$, so that $W = V + J(V)$ where J denotes the complex structure (multiplication by $i = \sqrt{-1}$). Let H be the subgroup of $\mathsf{SL}(W)$ preserving J (i.e. $H = \mathsf{SL}_n(\mathbf{C})$).

Let C be the image of $\mathsf{SL}_2(\mathbf{R})$ into $\mathsf{SL}(W)$ given by the representation which maps a 2 by 2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to the transformation

$$x + Jy \in V + J(V) \mapsto ax + by + J(cx + dy).$$

The group C has finite index in the centralizer of Γ in $\mathsf{SL}(W)$. Let $\mathcal{J}$ be the subset of C given by elements K in $\mathsf{SL}_2(\mathbf{R})$ with $K^2 = -\mathrm{Id}$; each element $K \in \mathcal{J}$ define a complex structure on W which is invariant by the actions of G and Γ. The complex structure J is an element of $\mathcal{J}$ and is the unique element which is invariant by H (resp. by a lattice in H). Choose $K \in \mathcal{J}$ different from J. Let T_K be the complex torus given by the complex structure K on W/Λ. The group $\mathsf{Aut}(T_K)$ contains Γ and does not contain any lattice in $\mathsf{SL}_n(\mathbf{C})$. As a consequence, either T_K is isogenous to a product E^n for some elliptic curve without complex multiplication, and thus Γ is commensurable to $\mathsf{SL}_n(\mathbf{Z})$, or $n = 2m$ is even and T_K is isogenous to a product Y^m for some indecomposable torus Y of dimension 2, and then Proposition 7.5 shows that Y and Γ are derived from a quaternion algebra. This concludes the proof of the proposition. $\square$

Theorem 7.1 is now a direct consequence of Lemma 7.3, Proposition 7.6 and Proposition 7.5.

7.5. **Kummer examples and singularities.** Once we have the list of possible tori and lattices, Kummer examples are obtained by a quotient with respect to a finite group of automorphisms of the torus.

Let $T = \mathbf{C}^n/\Lambda$ be a torus and Γ be a lattice in $\mathsf{SL}_n(\mathbf{R})$ or $\mathsf{SL}_n(\mathbf{C})$ acting faithfully on T. Let F be a finite group of automorphisms of T which is normalized by the action of Γ. From Lemma 6.8, we can assume that F is a finite cyclic group of homotheties.

If T is isogenous to $(\mathbf{C}/\Lambda_0)^n$, with Λ_0 a lattice in $\mathbf{C}$ and $\Lambda = \Lambda_0^n$, the order of F is 1, 2, 3, 4 or 6 (see [19]). If $n = 2k$ is even and T is isogenous to

$(\mathbf{C}^2/\mathbf{H}_{a,b}(\mathbf{Z}))^k$, the same conclusion holds: The finite group F is contained in the centralizer of Γ and preserves Λ; this group is isomorphic to the lattice of $\mathsf{SL}_2(\mathbf{R})$ given by quaternions of norm 1. Thus, F can be identified to a finite cyclic subgroup of $\mathbf{H}_{a,b}(\mathbf{Z})$. Viewed as a subgroup of $\mathsf{SL}_2(\mathbf{R})$, the traces are even integers, and thus finite order elements have trace in $\{-2,0,2\}$. Thus the order of the cyclic group F is bounded by 2 in this case. This proves the following fact.

Proposition 7.7. *Let M_0 be Kummer orbifold T/F where $T = \mathbf{C}^n/\Lambda$ is a torus of dimension n and F is a finite group of automorphisms of T. Assume that there is a faithful action of a lattice in an almost simple Lie group G of a rank $n-1$ on M_0. Then M_0 is the quotient T'/F' of a torus T' isogenous to T by a finite cyclic group F' which is generated by a scalar multiplication*

$$(x_1,\dots,x_n)\mapsto(\eta x_1,\dots,\eta x_n)$$

where η is a root of unity of order 1*,* 2*,* 3*,* 4 *or* 6*.*

7.6. **Volume forms.** Let us start with an example. Let M_0 be the Kummer orbifold T/F where T is $(\mathbf{C}/\mathbf{Z}[i])^n$ and F is the finite cyclic group generated by $f(x_1,x_2,\dots x_n) = (ix_1,ix_2,\dots,ix_n)$, where $i=\sqrt{-1}$. Let M be the smooth manifold obtained by blowing up the singular points of M_0. Let $\Omega = dx_1\wedge dx_2\wedge\dots\wedge dx_n$ be the standard "holomorphic volume form" on T. Then $f^*\Omega = i^n\Omega$ and $\Omega^{q(n)}$ is f-invariant, where $q(n)$ is equal to 1, 2 or 4 according to the residue of n modulo 4 (Ω^q is a section of $K_T^{\otimes q}$, where K_T is the canonical bundle of T). In order to resolve the singularities of M_0, one can proceed as follows. First one blows up all fixed points of elements in $F\setminus\{\mathrm{Id}\}$. For example, one needs to blow up the origin $(0,\dots,0)$. This provides a compact Kähler manifold $\hat{T}$ together with a birational morphism $\alpha:\hat{T}\to T$. The automorphism f lifts to an automorphism $\hat{f}$ of $\hat{T}$; since the differential Df is a homothety, $\hat{f}$ acts trivially on each exceptional divisor, and acts as $z\mapsto iz$ in the normal direction. As a consequence, the quotient $\hat{T}/\hat{F}$ is smooth.

Denote by $E\subset\hat{T}$ the exceptional divisor corresponding to the blowing up of the origin, and fix local coordinates $(\hat{x_1},\hat{x_2},\dots\hat{x_n})$ in $\hat{T}$ such that the local equation of E is $\hat{x_n}=0$. In these coordinates, the form $\alpha^*\Omega$ is locally given by

$$\alpha^*\Omega = \hat{x_n}^{n-1}d\hat{x_1}\wedge d\hat{x_2}\dots\wedge d\hat{x_n}.$$

The projection $\varepsilon:\hat{T}\to M=\hat{T}/\hat{F}$ is given by $(\hat{x_1},\hat{x_2},\dots\hat{x_n})\mapsto(u_1,u_2,\dots,u_n) = (\hat{x_1},\hat{x_2},\dots,\hat{x_n}^4)$, and the projection of $\alpha^*\Omega$ on M is

$$\varepsilon_*\alpha^*\Omega = \frac{1}{4}u_n^{(n-4)/4}du_1\wedge du_2\dots\wedge du_n.$$

The fourth power of this form is a well defined meromorphic section of $K_M^{\otimes 4}$; $\varepsilon_*\alpha^*\Omega$ has a pole of order $1/2$ if $n=2$ and of order $1/4$ if $n=3$; it is smooth and does not vanish if $n=4$, and vanishes along the exceptional divisor $\varepsilon(E)$ when $n>4$.

A similar study can be made for all Kummer examples. More precisely, a local computation shows that

$$\varepsilon_*\alpha^*\Omega = \frac{1}{r}u_n^{(n-r)/r}du_1\wedge du_2\ldots\wedge du_n.$$

where $r\in\{2,3,4,6\}$ is the order of the root η and the u_i are local coordinates on the quotient $\hat{\mathbf{C}}^n/\eta$. For all pairs (n,r) the real volume form $\varepsilon_*\alpha^*(\Omega\wedge\overline{\Omega})$ on M is integrable and Γ-invariant. This form does not have any zeros or poles if and only if $(n,r)\in\{(2,2),(3,3),(4,4),(6,6)\}$; in this case, $\varepsilon_*\alpha^*\Omega$ trivializes the canonical bundle of M. Recall that Calabi-Yau manifolds are simply connected, Kähler manifolds with a trivial canonical bundle. Our study admits the following corollary.

Corollary 7.8. *If M is a Calabi-Yau manifold of dimension n and $\mathsf{Aut}(M)$ contains a lattice of an almost simple Lie group of rank $n-1$, then* $\dim(M) =$ 2, 3, 4, *or* 6. *In dimension* 3, 4 *and* 6, *all examples are of type $\widehat{T/\eta}$ where*

- *η is one of the roots $e^{2i\pi/3}, e^{-2i\pi/3}$ (for $n=3$), $i,-i$ (for $n=4$), or $e^{i\pi/3}, e^{-i\pi/3}$ (for $n=6$),;*
- *$T=E^n$, with $E=\mathbf{C}/\mathbf{Z}[\eta]$.*

Another consequence is the existence of invariant volume forms as a byproduct of the classification (while standard conjectures in Zimmer's program a priori assume the existence of such an invariant volume).

Corollary 7.9. *Let M be a compact Kähler manifold of dimension $n\geq 3$. Let Γ be a lattice in a simple Lie group G with $\mathsf{rk}_{\mathbf{R}}(G)\geq n-1$. If Γ acts faithfully on M, then the action of Γ on M*

- *virtually extends to an action of G, or*
- *preserves an integrable volume form μ which is locally smooth or the product of a smooth volume form by $|w|^{-a}$, where w is a local coordinate and $a\in\{-1/2,-1\}$ for $n=3$, or $a=-2/3$ for $n=4$ or $a=-1/3$ for $n=5$.*

Proof. If the action of Γ on the cohomology of M factors through a finite group, then Γ is virtually contained in $\mathsf{Aut}(M)^0$ and two cases may occur. In the first case, the morphism $\Gamma\to\mathsf{Aut}(M)$ virtually extends to a morphism $G\to\mathsf{Aut}(M)$. In the second case, Γ is virtually contained in a compact subgroup of

$\mathsf{Aut}(M)^0$, and then Γ preserves a Kähler metric. In particular, it preserves a smooth volume form. If the action of Γ on the cohomology is almost faithful, then it is a Kummer example, and the result follows from what has just been said. □

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