HOLOMORPHIC ACTIONS OF HIGHER RANK LATTICES IN DIMENSION THREE

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ABSTRACT. We classify all holomorphic actions of higher rank lattices on compact Kähler manifolds of dimension 3. This provides a complete answer to Zimmer's program for holomorphic actions on compact Kähler manifolds of dimension at most 3.

RÉSUMÉ. Nous classons les actions holomorphes des réseaux des groupes de Lie semi-simple de rang au moins 2 sur les variétés complexes compactes kählériennes de dimension 3. Ceci répond positivement au programme de Zimmer pour les actions holomorphes en petite dimension.

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1. INTRODUCTION

1.1. Zimmer's program and automorphisms.

This article is inspired by two questions concerning the structure of groups of diffeomorphisms of compact manifolds.

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The first one is part of the so called Zimmer's program. Let G be a semi-simple real Lie group. The **real rank** $\mathsf{rk}_{\mathbf{R}}(G)$ of G is the dimension of a maximal abelian subgroup A of G such that $\mathrm{ad}(A)$ acts by simultaneously $\mathbf{R}$-diagonalizable endomorphisms on the Lie algebra $\mathfrak{g}$ of G. When $\mathsf{rk}_{\mathbf{R}}(G)$ is at least 2, we shall say that G is a **higher rank** semi-simple Lie group. Let Γ be a lattice in G; by definition, Γ is a discrete subgroup of G such that G/Γ has finite Haar volume. Margulis superrigidity theorem implies that all finite dimensional linear representations of Γ are built from representations in unitary groups and representations of the Lie group G itself. Zimmer's program predicts that a similar picture should hold for actions of Γ by diffeomorphims on compact manifolds, at least when the dimension of the manifold is close to the minimal dimension of non trivial linear representations of G (see [24] for an introduction to Zimmer's program).

The second problem that we have in mind concerns the structure of groups of holomorphic diffeomorphisms, also called **automorphisms**. Let M be a compact complex manifold of dimension n. According to Bochner and Montgomery [7, 10], the group of automorphisms $\mathsf{Aut}(M)$ is a complex Lie group, the Lie algebra of which is the algebra of holomorphic vector fields on M. The connected component of the identity $\mathsf{Aut}(M)^0$ can be studied by classical means, namely Lie theory concerning the action of Lie groups on manifolds. The group of connected components

$$\mathsf{Aut}(M)^\sharp = \mathsf{Aut}(M)/\mathsf{Aut}(M)^0$$

is much harder to describe, even for projective manifolds.

In this article, we provide a complete picture of all holomorphic actions of lattices Γ in higher rank simple Lie groups on compact Kähler manifolds M with $\dim(M) \leq 3$. The most difficult part is the study of morphisms $\Gamma \to \mathsf{Aut}(M)$ for which the natural projection onto $\mathsf{Aut}(M)^\sharp$ is injective. As a consequence, we hope that our method will shed light on both Zimmer's program and the structure of $\mathsf{Aut}(M)^\sharp$.

1.2. Examples: Tori and Kummer orbifolds.

Classical examples come from carefully chosen complex tori.

Example 1.1. Let $E = \mathbf{C}/\Lambda$ be an elliptic curve and n be a positive integer. Let A be the torus $E^n = \mathbf{C}^n/\Lambda^n$. The group $\mathsf{Aut}(A)$ contains all affine transformations $z \mapsto B(z) + c$ where B is in $\mathsf{SL}_n(\mathbf{Z})$ and c is in A. The connected component $\mathsf{Aut}(A)^0$ coincides with the group of translations. Similarly, if Λ is the lattice of integers O_d in an imaginary quadratic number field $\mathbf{Q}(\sqrt{d})$, where d is a squarefree negative integer, then $\mathsf{Aut}(A)$ contains a copy of $\mathsf{SL}_n(O_d)$.

Example 1.2. Starting with the previous example, one can change Γ in a finite index subgroup Γ_0, and change A into a quotient A/G where G is a finite subgroup of $\mathsf{Aut}(A)$ which is normalized by Γ_0. In general, A/G is an orbifold (a compact manifold with quotient singularities), and one needs to resolve the singularities in order to get an action on a smooth manifold M. The second operation that can be done is blowing up finite orbits of Γ. This provides infinitely many compact Kähler manifolds with actions of lattices $\Gamma \subset \mathsf{SL}_n(\mathbf{R})$ (resp. $\Gamma \subset \mathsf{SL}_n(\mathbf{C})$).

In these examples, the group Γ is a lattice in a real Lie group of rank $(n-1)$, namely $\mathsf{SL}_n(\mathbf{R})$ or $\mathsf{SL}_n(\mathbf{C})$, and Γ acts on a manifold M of dimension n. Moreover, the action of Γ on the cohomology of M has finite kernel and a finite index subgroup of Γ embeds in $\mathsf{Aut}(M)^\sharp$. Since this kind of construction is at the heart of the article, we introduce the following definition.

Definition 1.3. *Let Γ be a group, and $\rho : \Gamma \to \mathsf{Aut}(M)$ a morphism into the group of automorphisms of a compact complex manifold M. This morphism is a* **Kummer example** *(or, equivalently, is of* **Kummer type***) if there exists*

- *a birational morphism $\pi : M \to M_0$ onto an orbifold M_0,*
- *a finite cover $\varepsilon : A \to M_0$ of M_0 by a torus A, and*
- *a morphism $\eta : \Gamma \to \mathsf{Aut}(A)$*

such that $\varepsilon \circ \eta(\gamma) = (\pi \circ \rho(\gamma) \circ \pi^{-1}) \circ \varepsilon$ for all γ in Γ.

The name Kummer comes from the fact that the orbifolds A/G, G a finite group, are known as **Kummer orbifolds**. Examples 1.1 and 1.2 are both Kummer examples. If $A = \mathbf{C}^n/\Lambda$ is a torus of dimension n, every element of $\mathsf{Aut}(A)$ is induced by an affine transformation of $\mathbf{C}^n$. Hence, actions of Kummer type are covered by affine actions on $\mathbf{C}^{\dim(M)}$.

1.3. **Groups and lattices.**

Before stating our results, a few classical definitions need to be given. Let H be a group. A property is said to hold **virtually** for H if a finite index subgroup of H satisfies this property. For example, an action of a group Γ on a compact complex manifold M is virtually a Kummer example if this action is of Kummer type after restriction to a finite index subgroup Γ_0 of Γ.

A connected real Lie group G is said to be **almost simple** if its center is finite and its Lie algebra $\mathfrak{g}$ is a simple Lie algebra. Let G be a connected semi-simple real Lie group with finite center; then G is isogenous to a product of almost simple Lie groups G_i. The groups G_i are the **factors** of G. A lattice Γ in G is said to be **irreducible** when its projection on every simple factor of G is dense. A **higher rank lattice** is a lattice in a connected semi-simple real Lie group G with finite center and rank $\mathsf{rk}_{\mathbf{R}}(G) \geq 2$.

1.4. Main results.

Let Γ be an irreducible lattice in a higher rank almost simple Lie group. As we shall see, all holomorphic faithful actions of Γ on connected compact Kähler manifolds of dimension 1 or 2 are actions on the projective plane $\mathbb{P}^2(\mathbf{C})$ by projective transformations. As a consequence, we mostly consider actions on compact Kähler manifolds of dimension 3.

Theorem A. *Let G be a connected semi-simple real Lie group with finite center and without nontrivial compact factor. Let Γ be an irreducible lattice in G. Let M be a connected compact Kähler manifold of dimension 3, and $\rho : \Gamma \to \mathsf{Aut}(M)$ be a morphism. If the real rank of G is at least 2, then one of the following holds*

- *the image of ρ is virtually contained in the connected component of the identity $\mathsf{Aut}(M)^0$, or*
- *the morphism ρ is virtually a Kummer example.*

In the second case, G is locally isomorphic to $\mathsf{SL}_3(\mathbf{R})$ or $\mathsf{SL}_3(\mathbf{C})$ and Γ is commensurable to $\mathsf{SL}_3(\mathbf{Z})$ or $\mathsf{SL}_3(O_d)$, where O_d is the ring of integers in an imaginary quadratic number field $\mathbf{Q}(\sqrt{d})$ for some negative integer d.

Remark 1.4. In the Kummer case, the action of Γ on M comes virtually from a linear action of Γ on a torus A. We shall prove that A is isogenous to the product $B \times B \times B$, where B is an elliptic curve; for example, one can take $B = \mathbf{C}/O_d$ if Γ is commensurable to $\mathsf{SL}_3(O_d)$.

When the image of ρ is virtually contained in $\mathsf{Aut}(M)^0$, one gets a morphism from a sublattice $\Gamma_0 \subset \Gamma$ to the connected complex Lie group $\mathsf{Aut}(M)^0$. If the image is infinite, one can then find a non trivial morphism of Lie groups from G to $\mathsf{Aut}(M)^0$, and use Lie theory to describe all types of manifolds M that can possibly arise. This leads to the following result.

Theorem B. *Let G be a connected, almost simple, real Lie group with rank $\mathsf{rk}_{\mathbf{R}}(G) \geq 2$. Let Γ be a lattice in G. Let M be a connected compact Kähler manifold of dimension 3. If there is a morphism $\rho : \Gamma \to \mathsf{Aut}(M)$ with infinite image, then M has a birational morphism onto a Kummer orbifold, or M is isomorphic to one of the following*

(1) *a projective bundle $\mathbb{P}(E)$ for some rank 2 vector bundle $E \to \mathbb{P}^2(\mathbf{C})$,*
(2) *a principal torus bundle over $\mathbb{P}^2(\mathbf{C})$,*
(3) *a product $\mathbb{P}^2(\mathbf{C}) \times B$ of the plane by a curve of genus $g(B) \geq 2$,*
(4) *the projective space $\mathbb{P}^3(\mathbf{C})$,*
(5) *the smooth quadric hypersurface $Q \subset \mathbb{P}^4(\mathbf{C})$.*

In all cases except the last one, G is isogenous to $\mathsf{SL}_n(\mathbf{K})$ with $n = 3$ or 4 and $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$; in the last case, G is isogenous to $\mathsf{SO}_5(\mathbf{C})$.

One feature of our article is that we do not assume that the group $\Gamma \subset \mathsf{Aut}(M)$ preserves a geometric structure or a volume form. The existence of invariant structures comes as a byproduct of the classification. For example, section 9.2 shows that any holomorphic action of a lattice in a higher rank simple Lie group G on a compact Kähler threefold either extends virtually to an action of G or preserves a volume form (this volume form may have poles, but is locally integrable).

Both theorem A and theorem B can be extended to arbitrary semi-simple Lie groups G with $\mathsf{rk}_{\mathbf{R}}(G) \geq 2$. Section 9.4 explains how to remove the extra hypothesis concerning the center and the compact factors of G. The literature on lattices in semi-simple Lie groups almost always assume that G has finite center and no compact factor; this is the reason why we first focus on more restrictive statements.

1.5. **Organization of the paper.**

Section 4 explains how to deduce theorem B from theorem A. Let us now sketch the proof of theorem A and describe the organization of the paper. Let Γ and M be as in theorem A.

(1) Section 2 describes Lieberman-Fujiki results, Hodge index theorem, and classical facts concerning lattices, including Margulis superrigidity theorem. Assuming that the image of Γ in $\mathsf{Aut}(M)$ is not virtually contained in $\mathsf{Aut}(M)^0$, we deduce that the action of Γ on the cohomology of M extends virtually to a

linear representation

$$(*) \quad G \to \mathsf{GL}\,(H^*(M,\mathbf{R})),$$

preserving the Hodge structure, the Poincaré duality, and the cup product.

(2) Section 3 describes the geometry of possible Γ-invariant analytic subsets of M. In particular, there is no Γ-invariant curve and all Γ-periodic surfaces can be blown down simultaneously by a birational morphism

$$\pi : M \to M_0$$

onto an orbifold. This section makes use of basic fundamental ideas from holomorphic dynamics and complex algebraic geometry.

(3) Section 5 classifies linear representations of G on the cohomology of a compact Kähler threefold, as given by the representation $(*)$. It turns out that G must be locally isomorphic to $\mathsf{SL}_3(\mathbf{R})$ or $\mathsf{SL}_3(\mathbf{C})$ when this representation is not trivial. The proof uses highest weight theory for linear representations together with Hodge index theorem.

(4) In section 6 and 7, we assume that G is locally isomorphic to $\mathsf{SL}_3(\mathbf{R})$. The representation of G on the vector space

$$W := H^{1,1}(M,\mathbf{R})$$

and the cup product $\wedge$ from $W \times W$ to its dual $W^* = H^{2,2}(M,\mathbf{R})$ are described in section 6. The representation W splits as a direct sum of a trivial factor T^m of dimension m, k copies of the standard representation of G on $E = \mathbf{R}^3$, and one copy of the representation on the space of quadratic forms $\mathrm{Sym}_2(E^*)$. We then study the position of $H^2(M,\mathbf{Z})$ with respect to this direct sum, and show that it intersects $\mathrm{Sym}_2(E^*)$ on a cocompact lattice. Section 6 ends with a study of the Kähler cone of M. This cone is Γ-invariant, but is not G-invariant a priori.

(5) Section 7 shows that the trivial factor T^m is spanned by classes of Γ-periodic surfaces. In particular, the map $\pi : M \to M_0$ contracts T^m. To prove this, one pursues the study of the Kähler, nef, and big cones. This requires a characterization of Kähler classes due to Demailly and Paun.

(6) The conclusion follows from the following observation: On M_0, both Chern classes $c_1(M)$ and $c_2(M)$ vanish, and this ensures that M_0 is a Kummer orbifold. One then prove that Γ is commensurable to $\mathsf{SL}_3(\mathbf{Z})$.

(7) Section 8 concerns the case when G is locally isomorphic to $\mathsf{SL}_3(\mathbf{C})$. We just give the main lemmas that are needed to adapt the proof given for $\mathsf{SL}_3(\mathbf{R})$.
All together, this concludes the proof of theorem A.

1.6. Notes and references.

This article is independant from [12], but may be considered as a companion to this article in which the first author proved that a lattice in a simple Lie group G can not act faithfully by automorphisms on a compact Kähler manifold M if $\mathsf{rk}_{\mathbf{R}}(G) > \dim_{\mathbf{C}}(M)$.

Several arguments are inspired by previous works concerning actions of lattices by real analytic transformations (see [26], [21, 23]), or morphisms from lattices to mapping class groups (see [22, 20]).

Concerning actions of lattices on tori, the interesting reader may consult [32, 33, 6]. These references contain interesting examples that can not appear in the holomorphic setting. A nice introduction to Zimmer's program is given in [24].

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2. Automorphims, rigidity, and Hodge theory

In this section, we collect important results from Hodge theory and the theory of discrete subgroups of Lie groups, which will be used systematically in the next sections.

In what follows, M is a compact Kähler manifold. Unless otherwise specified, M is assumed to be connected.

2.1. Automorphims.

As explained in the introduction, $\mathsf{Aut}(M)$ is a complex Lie group, but the group $\mathsf{Aut}(M)^\sharp$ of its connected components can be infinite, as in example 1.1. The following theorem shows that $\mathsf{Aut}(M)^\sharp$ embeds virtually into the linear group $\mathsf{GL}\left(H^2(M,\mathbf{R})\right)$, and thus provides a way to study it.

Theorem 2.1 (Lieberman, Fujiki, see [37])**.** *Let M be a compact Kähler manifold. Let κ be the cohomology class of a Kähler form on M. The connected component of the identity $\mathsf{Aut}(M)^0$ has finite index in the stabilizer of κ in $\mathsf{Aut}(M)$.*

2.2. **Hodge structure.**

2.2.1. *Hodge decomposition.* Hodge theory implies that the cohomology groups $H^k(M,\mathbf{C})$ decompose into direct sums

$$H^k(M,\mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(M,\mathbf{C}),$$

where cohomology classes in $H^{p,q}(M,\mathbf{C})$ are represented by closed forms of type (p,q). This bigraded structure is compatible with the cup product. Complex conjugation permutes $H^{p,q}(M,\mathbf{C})$ with $H^{q,p}(M,\mathbf{C})$. In particular, the cohomology groups $H^{p,p}(M,\mathbf{C})$ admit a real structure, the real part of which is $H^{p,p}(M,\mathbf{R}) = H^{p,p}(M,\mathbf{C}) \cap H^{2p}(M,\mathbf{R})$. If κ is a Kähler class (i.e. the cohomology class of a Kähler form), then $\kappa^p \in H^{p,p}(M,\mathbf{R})$ for all p.

2.2.2. *Dimension three.* Assume, now, that M has dimension 3. In what follows, we shall denote by W the cohomology group $H^{1,1}(M,\mathbf{R})$. Poincaré duality provides an isomorphism between $H^{2,2}(M,\mathbf{R})$ and W^*, which is defined by

$$\langle \alpha | \beta \rangle = \int_M \alpha \wedge \beta$$

for all $\alpha \in H^{2,2}(M,\mathbf{R})$ and $\beta \in W$. Modulo this isomorphism, the cup product defines a symmetric bilinear application

$$\wedge : \begin{cases} W \times W & \to \quad W^* \\ (\beta_1,\beta_2) & \mapsto \quad \beta_1 \wedge \beta_2 \end{cases}$$

Lemma 2.2 (Hodge Index Theorem)**.** *If $\wedge$ vanishes identically along a subspace V of W, the dimension of V is at most* 1.

Proof. Let V be a linear subspace of W along which $\wedge$ vanishes identically: $\alpha \wedge \beta = 0$, for all α and β in V. Let $\kappa \in W$ be a Kähler class. Let Q_κ be the quadratic form which is defined on W by

$$Q_\kappa(\alpha_1,\alpha_2) = -\int_M \alpha_1 \wedge \alpha_2 \wedge \kappa,$$

and $P^{1,1}(\kappa)$ be the orthogonal complement of κ with respect to this quadratic form:

$$P^{1,1}(\kappa) = \{\beta \in W\,; \quad Q_\kappa(\beta,\kappa) = 0\}\,.$$

Since κ is a Kähler class, $\kappa \wedge \kappa$ is different from 0 and $P^{1,1}(\kappa)$ has codimension 1 in W. Hodge index theorem implies that Q_κ is positive definite on $P^{1,1}(\kappa)$ (see [45], theorem 6.32). In particular, $P^{1,1}(\kappa)$ does not intersect V, and the dimension of V is at most 1. □

2.3. **Classical results concerning lattices.**

In this paragraph, we list a few important facts concerning lattices in Lie groups. The reader may consult [44] and [4] for two nice introductions to lattices. One feature of the theory of semi-simple Lie groups is that we can switch viewpoint from Lie groups to linear algebraic groups. We shall use this almost permanently.

2.3.1. *Borel and Harish-Chandra (see* [4, 44]*).* Let G be a linear algebraic group defined over the field of rational numbers $\mathbf{Q}$. Let $\mathbf{G}_m$ denote the multiplicative group. If G does not have any character $G \to \mathbf{G}_m$ defined over $\mathbf{Q}$, the group of integer points $G(\mathbf{Z})$ is a lattice in G. If there is no morphism $\mathbf{G}_m \to G$ defined over $\mathbf{Q}$, this lattice is cocompact.

2.3.2. *Borel density theorem (see* [44]*, page 37, or* [4]*).* Let G be a linear algebraic semi-simple Lie group with no compact normal subgroup of positive dimension. If Γ is a lattice in G, then Γ is Zariski-dense in G.

2.3.3. *Proximality (see* [5]*, appendix).* Let G be a real reductive linear algebraic group and P be a minimal parabolic subgroup of G. An element g in G is **proximal** if it has an attractive fixed point in G/P. For example, when $G = \mathsf{SL}_n(\mathbf{R})$, an element is proximal if and only if its eigenvalues are n real numbers with pairwise distinct absolute values.

Let Γ be a Zariski dense subgroup of G. Then, the set of proximal elements γ in Γ is Zariski dense in G. More precisely, when G is not commutative, there exists a Zariski dense non abelian free subgroup $F < \Gamma$ such that all elements γ in $F \setminus \{\mathrm{Id}\}$ are proximal.

2.3.4. *Limit sets (see* [40]*, lemma 8.5).* Let G be a semi-simple analytic group having no compact normal subgroup of positive dimension. Let P be a parabolic subgroup of G. If Γ is a lattice in G, the closure of ΓP coincides with G: $\overline{\Gamma P} = G$. In particular, if Γ is a lattice in $\mathsf{SL}_3(\mathbf{K})$, $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, all orbits of Γ in $\mathbb{P}^2(\mathbf{K})$ are dense.

2.3.5. *Kazhdan property (T) (see* [15, 3]*).* We shall say that a topological group F has Kazhdan property (T) if F is locally compact and every continuous action of F by affine unitary motions on a Hilbert space has a fixed point (see [3] for equivalent definitions). If F has Kazhdan property (T) and Λ is a lattice in F, then Λ inherits property (T). If G is a simple real Lie group with rank $\mathsf{rk}_{\mathbf{R}}(G) \geq 2$, then G and all its lattices satisfy Kazhdan property (T); moreover, the same property holds for the universal covering of G.

If F is a discrete group with property (T), then

- F is finitely generated;
- every morphism from F to $\mathsf{GL}_2(k)$, k any field, has finite image (see [30] and [49]);
- every morphism from Γ to a solvable group has finite image.

Lemma 2.3. *Every morphism from a discrete Kazhdan group F to the group of automorphisms of a compact Riemann surface has finite image.*

Proof. The automorphisms group of a connected Riemann surface X is either finite (when the genus $g(X)$ is > 1), virtually abelian (when $g(X) = 1$), or isomorphic to $\mathsf{PGL}_2(\mathbf{C})$. □

2.4. **Margulis superrigidity and action on cohomology.**

2.4.1. *Superrigidity.* The following theorem is one version of the superrigidity phenomenum for linear representations of lattices (see [38] or [44]).

Theorem 2.4 (Margulis)**.** *Let G be a semi-simple connected Lie group with finite center, with rank at least* 2, *and without non trivial compact factor. Let $\Gamma \subset G$ be an irreducible lattice. Let $h : \Gamma \to \mathsf{GL}_k(\mathbf{R})$ be a linear representation of Γ.*

The Zariski closure of $h(\Gamma)$ is a semi-simple Lie group; if this Lie group does not have any infinite compact factor, there exists a continuous representation $\hat{h} : G \to \mathsf{GL}_k(\mathbf{R})$ which coincides with h on a finite index subgroup of Γ.

In other words, if the Zariski closure of $h(\Gamma)$ does not have any non trivial compact factor, then h virtually extends to a linear representation of G.

Remark 2.5. Similar results hold also for representations of lattices in $\mathsf{Sp}(1,n)$ and F_4 (see [14]).

Corollary 2.6. *If the representation h takes values into the group $\mathsf{GL}_k(\mathbf{Z})$ and has an infinite image, then h extends virtually to a continuous representation of G with finite kernel.*

If G is a linear algebraic group, any continuous linear representation of G on a finite dimensional vector space is algebraic. As a consequence, up to finite indices, representations of Γ into $\mathsf{GL}_k(\mathbf{Z})$ with infinite image are restrictions of algebraic linear representations of G.

2.4.2. *Normal subgroups (see* [38] *or* [44]*).* According to another result of Margulis, if Γ is an irreducible higher rank lattice, then Γ is almost simple: All normal subgroups of Γ are finite or cofinite (i.e. have finite index in Γ).

In particular, if $\alpha : \Gamma \to H$ is a morphism of groups, either α has finite image, or α is virtually injective, which means that we can change the lattice Γ in a sublattice Γ_0 and assume that α is injective.

2.4.3. *Action on cohomology groups.* From Margulis superrigidity and Lieberman-Fujiki theorem, one gets the following proposition.

Proposition 2.7. *Let G and Γ be as in theorem 2.4. Let $\rho : \Gamma \to \mathsf{Aut}(M)$ be a representation into the group of automorphisms of a compact Kähler manifold M. Let $\rho^* : \Gamma \to \mathsf{GL}\,(H^*(M,\mathbf{Z}))$ be the induced action on the cohomology ring of M.*

- a.- *If the image of ρ^* is infinite, then ρ^* virtually extends to a representation $\hat{\rho^*} : G \to \mathsf{GL}\,(H^*(M,\mathbf{R}))$ which preserves the cup product and the Hodge decomposition.*
- b.- *If the image of ρ^* is finite, the image of ρ is virtually contained in $\mathsf{Aut}(M)^0$.*

Hence, to prove theorem A, we can assume that the action of the lattice Γ on the cohomology of M extends to a linear representation of G.

Proof. In the first case, Margulis superrigidity implies that the morphism ρ^* extends to a linear representation $\hat{\rho^*}$ of G. Since Γ acts by holomorphic diffeomorphisms on M, Γ preserves the Hodge decomposition and the cup product. Since lattices of semi-simple Lie groups are Zariski dense (see §2.3.2), the same is true for $\hat{\rho^*}(G)$.

The second assertion is a direct consequence of theorem 2.1 □

For compact Kähler threefolds, section 2.2.2 shows that the Poincaré duality and the cup product define a bilinear map $\wedge$ on $W = H^{1,1}(M,\mathbf{R})$ with values into the dual space W^*.

Lemma 2.8. *In case (a) of proposition 2.7, the bilinear map $\wedge : W \times W \to W^*$ is G-equivariant:*

$$(\hat{\rho^*}(g)\alpha) \wedge (\hat{\rho^*}(g)\beta) = (\hat{\rho^*}(g))^*(\alpha \wedge \beta)$$

for all α, β in W.

2.4.4. *Notations.* In the proof of theorem A, we shall simplify the notation, and stop refering explicitly to ρ, ρ^*, $\hat{\rho}^*$. If the action of Γ on W extends virtually to an action of G, this action is denoted by $(g,v) \mapsto g(v)$. This representation preserves the Hodge structure, the Poincaré duality and the cup product.

2.5. **Dynamical degrees and Hodge decomposition.**

2.5.1. *Dynamical degrees (see* [28, 29] *and* [19]*).* Let f be an automorphism of a compact Kähler manifold M. Let $1 \leq p \leq \dim(M)$ be a positive integer. The dynamical degree of f in dimension p is the spectral radius of

$$f^* : H^{p,p}(M,\mathbf{R}) \to H^{p,p}(M,\mathbf{R}).$$

We shall denote it by $d_p(f)$. One easily shows that $d_p(f)$ coincides with the largest eigenvalue of f^* on $H^{p,p}(M,\mathbf{R})$. Hodge theory also implies that the following properties are equivalent:

(i) one of the dynamical degrees $d_p(f)$, $1 \leq p \leq \dim(M)-1$, is equal to 1;
(ii) all dynamical degrees $d_p(f)$ are equal to 1;
(iii) the spectral radius of $f^* : H^*(M,\mathbf{C}) \to H^*(M,\mathbf{C})$ is equal to 1.

From theorems due to Gromov and Yomdin follows that the topological entropy of $f : M \to M$ is equal to the maximum of the logarithms $\log(d_p(f))$ (see [27]):

$$\mathsf{h_{top}(f)} = \max_{1\leq \mathsf{p} \leq \dim(\mathsf{M})} \{\log(\mathsf{d_p(f)})\}.$$

2.5.2. *Invariant fibrations.* Assume that $f : M \to M$ is an automorphism of a compact Kähler manifold M, that $\pi : M \dashrightarrow B$ is a meromorphic fibration, and that f permutes the fibers of π, by which we mean that there is a bimeromorphic map $g : B \dashrightarrow B$ such that $\pi \circ f = g \circ \pi$. In this setting, one can define dynamical degrees $d_p(f\,|\,\pi)$ along the fibers of π and dynamical degrees for g on B, and relate all these degrees to those of f (see [19]).

Theorem 2.9 (Dinh and Nguyen)**.** *Let M be a complex projective manifold. Under the previous assumptions, the dynamical degrees satisfy $d_p(f) = \max_{i+j=p}(d_i(g)d_j(f\,|\,\pi))$ for all $0 \leq p \leq \dim(M)$.*

The proof given by Dinh and Nguyen works only for projective manifolds; nevertheless, the following "corollary" works in dimension 3.

Proposition 2.10. *Let M be a compact Kähler manifold of dimension* 3, *and f be an automorphism of M. Assume that f preserves a holomorphic fibration $\pi : M \to B$, inducing an automorphism $f_B : B \to B$ such that $\pi \circ f = f_B \circ \pi$, and that the topological entropy of f_B vanishes.*

(1) *If* $\dim(B) \geq 2$, *then* $\mathsf{h}_{\mathsf{top}}(\mathsf{f}) = 0$.
(2) *If* $\dim(B) = 1$ *and the generic fibers of* π *are complex projective planes, then* $\mathsf{h}_{\mathsf{top}}(\mathsf{f}) = 0$.

In both cases, Yomdin theorem implies that the dynamical degrees of f are all equal to 1; thus f^* does not have any eigenvalue > 1 on $H^*(M, \mathbf{C})$.

Proof. If B has dimension 3, then the generic fibers of π are finite; since $\mathsf{h}_{top}(f_B) = 0$, the same is true for f. If B has dimension 2, then the fibers of π are curves; since automorphisms of curves all have an iterate which is either the identity, a translation of an elliptic curve, or a homography of $\mathbb{P}^1(\mathbf{C})$, the topological entropy of f must also vanish.

If B has dimension 1, f can have positive entropy. But, if the generic fibers are projective planes, then f acts as linear projective transformations on them; moreover, an iterate of f_B is the identity, a translation, or a homography; hence $\mathsf{h}_{\mathsf{top}}(\mathsf{f}) = 0$. □

2.5.3. *Dynamical degrees for actions of lattices.* Consider an action of an irreducible higher rank lattice Γ on a compact Kähler manifold M, and assume that the action of Γ on the cohomology of M does not factor through a finite group. Then, according to proposition 2.7, the action of Γ on $H^{1,1}(M, \mathbf{R})$ extends virtually to a non trivial linear representation of the Lie group G. In particular, there are elements γ in Γ such that the spectral radius of

$$\gamma^* : H^{1,1}(M, \mathbf{R}) \to H^{1,1}(M, \mathbf{R})$$

is larger than 1 (see [12], §3.1 for more precise results). This remark and the previous paragraph 2.5.2 provide obstructions for the existence of meromorphic fibrations $\pi : M \dashrightarrow B$ that can be Γ-invariant. We shall use this fact in the next section.

3. Invariant analytic subsets

This section classifies possible Γ-invariant analytic subspaces $Z \subset M$, where Γ is a discrete Kazhdan group, or a higher rank lattice in a simple Lie group, acting faithfully on a (connected) compact Kähler threefold M.

3.1. **Fixed points and invariant curves.**

Theorem 3.1. *Let Γ be a discrete group with Kazhdan property (T), and $\Gamma \to \mathsf{Aut}(M)$ be a morphism into the group of automorphisms of a connected compact Kähler threefold M. If the fixed points set of Γ in M is infinite, the image of Γ in $\mathsf{Aut}(M)$ is finite.*

Proof. The set of fixed points of Γ is the set

$$\{m \in M \,|\, \gamma(m) = m, \; \forall \gamma \in \Gamma\}.$$

Since Γ acts holomorphically, this set is an analytic subset of M; we have to show that its dimension is 0. Let Z be an irreducible component of this set with positive dimension, and p be a smooth point of Z. Since p is a fixed point, we get a linear representation $D : \Gamma \to \mathsf{GL}\,(T_pM)$ defined by the differential at p

$$D : \gamma \mapsto D_p\gamma.$$

If Z has dimension 1, then D takes its values in the pointwise stabilizer of a line. This group is isomorphic to the semi-direct product $\mathsf{SL}_2(\mathbf{C}) \ltimes \mathbf{C}^2$. If Z has dimension 2, D takes its values in the pointwise stabilizer of a plane. This group is solvable. Since Γ has property (T), the image of D must be finite (see §2.3.5), so that all elements in a finite index subgroup Γ_0 of Γ are tangent to the identity ar p. For every $k > 1$, the group of k-jets of diffeomorphisms which are tangent to the identity at order $k-1$ is an abelian group, isomorphic to the sum of three copies of the group of homogenous polynomials of degree k (in 3 variables). Since Γ_0 has property (T), there is no non trivial morphism from Γ_0 to such an abelian group, and the Taylor expansion of every element of Γ_0 at p is trivial. Since the action of Γ_0 is holomorphic and M is connected, Γ_0 acts trivially on M, and the morphism $\Gamma \to \mathsf{Aut}(M)$ has finite image. □

Corollary 3.2. *If Γ is an infinite discrete Kazhdan group acting faithfully on a connected compact Kähler threefold M, there is no Γ-invariant curve in M.*

Proof. Let C be an invariant curve. Replacing Γ by a finite index subgroup, we can assume that Γ preserves all connected components C_i of C. The conclusion follows from the previous theorem and lemma 2.3. □

3.2. **Invariant surfaces.**

Theorem 3.3 (see [12, 13])**.** *Any holomorphic action with infinite image of a discrete Kazhdan group Γ on a compact Kähler surface S is holomorphically conjugate to an action on $\mathbb{P}^2(\mathbf{C})$ by projective transformations.*

We just sketch the proof of this result. Complete details are given in [13] in a more general context, namely actions by birational transformations (see also [12] for hyperkähler manifolds).

Sketch of the proof. Let Γ be a Kazhdan group, acting on a compact Kähler surface S. The action of Γ on $H^{1,1}(M,\mathbf{R})$ preserves the intersection form. Hodge index theorem asserts that this quadratic form has signature $(1,\rho-1)$, where ρ is the dimension of $H^{1,1}(M,\mathbf{R})$. Let $\mathcal{K}(S)\subset H^{1,1}(S,\mathbf{R})$ be the Kähler cone, i.e. the open convex cone of cohomology classes of Kähler forms of S. This cone is Γ-invariant and is contained in the cone of cohomology classes with positive self intersection. The fixed point property for actions of Kazhdan groups on hyperbolic spaces implies that there exists a Kähler class $[\kappa]$ which is Γ-invariant (see lemma 2.2.7 page 78 in [3]). Lieberman-Fujiki theorem now implies that the image of a finite index subgroup Γ_0 of Γ is contained in the connected Lie group $\mathsf{Aut}(S)^0$. In particular, all curves with negative self intersection on S are Γ_0-invariant, so that we can blow down a finite number of curves in S and get a birational morphism $\pi: S\to S_0$ onto a minimal model of S wich is Γ_0-equivariant: $\pi\circ\gamma=\Psi(\gamma)\circ\pi$ where $\Psi:\Gamma_0\to\mathsf{Aut}(S_0)^0$ is a morphism.

From lemma 2.3, we know that $\Psi(\Gamma_0)$ is finite as soon as it preserves a fibration $S\to B$, with $\dim(B)=1$. Let us use this fact together with a little bit of the classification of compact complex surfaces (see [2]).

If the Kodaira dimension $\mathrm{kod}(S_0)$ is equal to 2, then $\mathsf{Aut}(S_0)$ is finite. We can therefore assume that $\mathrm{kod}(S_0)\in\{1,0,-\infty\}$. If $\mathrm{kod}(S_0)=1$, then S_0 fibers over a curve in an $\mathsf{Aut}(S_0)$-equivariant way (thanks to the Kodaira-Iitaka fibration), and $\Psi(\Gamma_0)$ must be finite. If $\mathrm{kod}(S_0)=0$, then $\mathsf{Aut}(S_0)^0$ is abelian, so that the image of Γ_0 in $\mathsf{Aut}(S_0)$ is also finite (see §2.3.5). Assume that $\mathrm{kod}(S_0)=-\infty$. If S_0 is a ruled surface over a curve of genus $g(B)\geq 1$, the ruling is an invariant fibration over the base B. If S_0 is an Hirzebruch surface, $\mathsf{Aut}(S_0)$ also preserves a non trivial fibration in this case. In both cases, $\Psi(\Gamma_0)$ is finite. The unique remaining case is the projective plane. In particular, if the image of Γ in $\mathsf{Aut}(S)$ is infinite, then S_0 is isomorphic to $\mathbb{P}^2(\mathbf{C})$.

Assume that $S_0=\mathbb{P}^2(\mathbf{C})$, so that $\Psi(\Gamma_0)$ is a Kazhdan subgroup of $\mathsf{PGL}_3(\mathbf{C})$. The set of critical values of π is a Γ_0-invariant subset of S_0. If this set is not empty, $\Psi(\Gamma_0)$ is contained in a strict, Zariski closed subgroup of $\mathsf{PGL}_3(\mathbf{C})$. All morphisms from a discrete Kazhdan group to such a subgroup of $\mathsf{PGL}_3(\mathbf{C})$

have finite image (see §2.3.5). As a consequence, the critical locus of π is empty, and π is an isomorphism from S to $\mathbb{P}^2(\mathbf{C})$. $\square$

Corollary 3.4. *Let M be a connected compact Kähler threefold, and $\Gamma \subset \mathsf{Aut}(M)$ be a discrete Kazhdan group. Let $S \subset M$ be a Γ-invariant surface. Then S is smooth and all its connected components are isomorphic to $\mathbb{P}^2(\mathbf{C})$.*

Proof. Assuming that the singular locus of S is not empty, this set must have dimension 0 (corollary 3.2). Let p be such a singular point and Γ_0 be the finite index subgroup of Γ fixing p. The differential

$$\gamma \mapsto D_p(\gamma)$$

provides a morphism $D : \Gamma_0 \to \mathsf{GL}(T_pM)$ which preserves the tangent cone to S at p. This cone defines a curve in $\mathbb{P}(T_pM)$ which is invariant under the linear action of Γ_0. Lemma 2.3 shows that Γ_0 acts trivially on this curve, and therefore on the cone. As a consequence, the morphism D is trivial. The proof of theorem 3.1 now provides a contradiction. It follows that S is smooth. Theorems 3.3 and 3.1 imply that the connected components of S are isomorphic to $\mathbb{P}^2(\mathbf{C})$. $\square$

Theorem 3.5. *Let M be a compact Kähler manifold of dimension 3. Assume that $\mathsf{Aut}(M)$ contains a subgroup Γ such that*

(i) *Γ is an infinite discrete Kazhdan group;*
(ii) *there is an element γ in Γ with a dynamical degree $d_p(\gamma) > 1$.*

If Z is a Γ-invariant analytic subset of M, it is made of a finite union of isolated points and a finite union of disjoint smooth surfaces $S_i \subset M$ that are all isomorphic to the projective plane. Moreover, all of them are contractible to quotient singularities.

Remark 3.6. More precisely, we shall prove that each component S_i can be blown down to a singularity which is locally isomorphic to the quotient of $\mathbf{C}^3$ by scalar multiplication by a root of unity.

Remark 3.7. We shall apply this result and its corollaries for lattices in higher rank simple Lie groups in section 7.

Proof. From corollary 3.2 we know that Z is made of surfaces and isolated points, and corollary 3.4 shows that all connected, 2-dimensional components of Z are smooth projective planes. Let S be one of them. Let N_S be its normal bundle, and let r be the integer such that $N_S \simeq O(r)$. Let L be the line bundle on M which is defined by S. The adjunction formula shows that N_S is the restriction $L_{|S}$ of L to S.

If $r > 0$, then S moves in a linear system $|S|$ of positive dimension (see [31]); equivalently, the space of sections $H^0(M,L)$ has dimension ≥ 2. Moreover the line bundle $L_{|S} \simeq O(r)$ is very ample, so that the base locus of the linear system $|S|$ is empty. As a consequence, this linear system determines a well defined morphism

$$\Phi_L : M \to \mathbb{P}(H^0(M,L)^*)$$

where $\Phi_L(x)$ is the linear form which maps a section s of L to its value at x (this is well defined up to a choice of a coordinate along the fiber L_x, i.e. up to a scalar multiple). The self intersection L^3 being positive, the dimension of $\Phi_L(M)$ is equal to 3 and Φ_L is generically finite. Since S is Γ-invariant, Γ permutes linearly the sections of L. This gives a morphism $\eta : \Gamma \to \mathsf{PGL}\,(H^0(M,L)^*)$ such that $\Phi_L \circ \gamma = \eta(\gamma) \circ \Phi_L$ for all γ in Γ. Let γ be an element of Γ. The action of $\eta(\gamma)$ on $\Phi_L(M)$ is induced by a linear mapping. This implies that $(\eta(\gamma))^*$ has finite order on the cohomology groups of $\Phi_L(M)$. Since Φ_L is generically finite to one, all eigenvalues of γ on $H^{1,1}(M,\mathbf{R})$ have modulus 1 (section 2.5.2), contradicting assumption (ii).

Assume now that $r = 0$. From [36], we know that S moves in a pencil of surfaces. In other words, $H^0(M,L)$ has dimension 2 and defines a holomorphic fibration $\Phi_L : M \to B$ where B is a curve. This fibration is Γ-invariant, and replacing Γ by a finite index subgroup, we can assume that the action on the base B is trivial (lemma 2.3). All fibers of Φ_L are Γ-invariant surfaces and, as such, are smooth projective planes. This implies that Φ_L is a locally trivial fibration: There is a covering U_i of B such that $\Phi_L^{-1}(U_i)$ is isomorphic to $U_i \times \mathbb{P}^2(\mathbf{C})$. In such coordinates, Γ acts by

$$\gamma(u,v) = (u, A_u(\gamma)(v))$$

where $(u,v) \in U_i \times \mathbb{P}^2(\mathbf{C})$ and $u \mapsto A_u$ is a one parameter representation of Γ into $\mathsf{Aut}(\mathbb{P}^2(\mathbf{C}))$. Once again, section 2.5.2 implies that all eigenvalues of γ on $H^{1,1}(M,\mathbf{R})$ have modulus 1, a contradiction.

As a consequence, the normal bundle N_S is isomorphic to $O(r)$ with $r < 0$. Grauert's theorem shows that S can be blown down to a quotient singularity of type $(\mathbf{C}^3,0)/\xi$ where ξ is a root of unity of order r. $\square$

Corollary 3.8. *If M and Γ are as in theorem 3.5, there is no Γ-equivariant fibration $p : M \to B$ with* $\dim(B) \in \{1,2\}$.

Proof. If there is such a fibration with $\dim(B) = 1$, then the action of Γ on the base B is virtually trivial, and the fibers are virtually invariant. We then get a contradiction with the previous theorem. If the dimension of B is 2, then two cases may occur. The action of Γ on the base is virtually trivial, and we get a similar contradiction, or B is isomorphic to the plane and Γ acts by projective transformations on it. Once again, the contradiction comes from section 2.5.2. $\square$

3.3. **Albanese variety and invariant classes.**

The Picard variety $\mathrm{Pic}^0(M)$ is the torus $H^1(M,O)/H^1(M,\mathbf{Z})$ parametrizing line bundles with trivial first Chern class (see [45], section 12.1.3). If this torus is reduced to a point, the trivial line bundle is the unique line bundle on M with trivial first Chern class. If its dimension is positive, the first Betti number of M is positive; in that case, the Albanese map provides a morphism $\alpha : M \to A_M$, where A_M is the torus $H^0(M,\Omega^1_M)^*/H_1(M,\mathbf{Z})$, of dimension $b_1(M)/2$ (see [45]). This map satisfies a universal property with respect to morphisms from M to tori.

Proposition 3.9. *Let M and Γ be as in theorem 3.5. If $Pic^0(M)$ has positive dimension, the Albanese map $\alpha : M \to A_M$ is a birational morphism.*

Remark 3.10. When M is a torus $\mathbf{C}^3/\Lambda$, Γ acts by affine transformations on it, because every automorphism of a torus lifts to an affine transformation of its universal cover $\mathbf{C}^3$.

Proof. If $\mathrm{Pic}^0(M)$ has positive dimension, the image of the Albanese map α has dimension 1, 2, or 3. From the universal property of the Albanese variety, Γ acts on A_M and α is Γ-equivariant. Corollary 3.8 implies that the image of α has dimension 3, so that α has generically finite fibers.

Let V be the image of α. Either V has general type, or V is degenerate, i.e. V is invariant under translation by a subtorus $K \subset A$ with $\dim(K) \geq 1$ (see [16], chapter VIII). In the first case, the action of Γ on V, and then on M, would factor through a finite group, a contradiction. In the second case, we may assume that K is connected and $\dim(K)$ is maximal among all subtori preserving V. Then, Γ preserves the fibration of V by orbits of K. From corollary 3.8, we deduce that V coincides with one orbit, so that V is isomorphic to the torus K. Since V is a torus, it coincides with the Albanese variety A_M and α is a ramified covering. Since A_M does not contain any projective plane, the ramification locus of α is empty (theorem 3.5), and α is a birational morphism. $\square$

Corollary 3.11. *If Z is an effective divisor, the homology class of which is Γ-invariant, then Z is uniquely determined by its homology class and Z is Γ-invariant.*

Proof. Let Z be a divisor, and let L be the line bundle associated to Z. A divisor Z' is the zero set of a holomorphic section of L if and only if Z' is linearly equivalent to Z. Since $\mathrm{Pic}^0(M)$ is trivial (proposition 3.9), the line bundle L is determined by the homology class $[Z]$.

If Γ preserves Z, then Γ linearly permutes the sections of L. Let now $\Phi_L : M \dashrightarrow \mathbb{P}(H^0(M,L)^*)$ be the rational map defined by L. This map is Γ equivariant: $\Phi_L \circ \gamma = {}^t\eta(\gamma) \circ \Phi_L$ where $\eta(\gamma)$ denotes the linear action of $\gamma \in \Gamma$ on the space of sections of L. From corollary 3.8, we know that the image of Φ_L has dimension 0 or 3, and since there are elements in Γ with degrees $d_p(\gamma) > 1$, the dimension must be 0 (see the proof of theorem 3.5). This implies that L has a unique section up to a scalar factor, which means that the divisor Z is uniquely determined by its homology class. This implies that Z is Γ-invariant. □

3.4. **Contracting invariant surfaces.**

From section 3.2, we know that every Γ-invariant or periodic surface $S \subset M$ is a disjoint union of copies of the projective plane $\mathbb{P}^2(\mathbf{C})$. Let S_i, $i = 1, \ldots, k$, be Γ-periodic planes, and O_j, $j = 1, \ldots, l$, be the orbits of these planes. If the number l is bigger than the dimension of $H^2(M, \mathbf{Z})$, there is a linear relation between their cohomology classes $[O_i]$, that can be written in the form

$$\sum_{i \in I} a_i [O_i] = \sum_{j \in J} b_j [O_j]$$

where the sets of indices I and J are disjoint and the coefficients a_i and b_j are positive integers. We obtain two distinct divisors $\sum a_i O_i$ and $\sum b_j O_j$ in the same invariant cohomology class, contradicting corollary 3.11. This contradiction proves the following assertion.

Theorem 3.12. *Let M and Γ be as in theorem 3.5. The number of Γ-periodic irreducible surfaces $S \subset M$ is finite. All of them are smooth projective planes with negative normal bundle, and these surfaces are pairwise disjoint. There is a birational morphism $\pi : M \to M_0$ to an orbifold M_0 with quotient singularities which contracts simultaneously all Γ-periodic irreducible surfaces, and is a local biholomorphism in the complement of these surfaces. In particular, the group Γ acts on M_0 and π is Γ-equivariant with respect to the action of Γ on M and M_0.*

Remark 3.13. Of course, this result applies when Γ is a lattice in an almost simple Lie group of rank at least 2.

4. Actions of Lie groups

In this section, we show how theorem A implies theorem B. By assumption, Γ is a lattice in an almost simple higher rank Lie group G, and Γ acts on a connected compact Kähler manifold M. Since this action has infinite image, it has finite kernel (§2.4.2). Now, theorem A implies that either the action is of Kummer type, and then M is birational to a Kummer orbifold, or the image of Γ is virtually contained in $\mathsf{Aut}(M)^0$. Changing Γ in a finite index subgroup, we can now assume that the action of Γ on M maps Γ into $\mathsf{Aut}(M)^0$.

According to Lieberman and Fujiki, $\mathsf{Aut}(M)^0$ acts on the Albanese variety of M by translations (see section 3.3), and the kernel of this action is a linear algebraic group L; thus, $\mathsf{Aut}(M)^0$ is an extension of a compact torus by a linear algebraic group L (see [37] and [10] for these results). Since Γ has property (T), the projection of Γ onto the torus has finite image; once again, changing Γ into a finite index subgroup, we may assume that the image of Γ is contained in L. As explained in [12], section 3.3 (see also [49], §3, or [38], chapter VII and VIII), this implies that there is a non trivial morphism $G \to L$. As a consequence, L contains a simple complex Lie group H, the Lie algebra of which has the same Dynkin diagram as $\mathfrak{g}$. The problem is now to classify compact Kähler threefolds M with a holomorphic faithful action of an almost simple complex Lie group H, the rank of which is at least 2. Hence, theorem B is a consequence of the following proposition

Proposition 4.1. *Let H be an almost simple complex Lie group with rank $\mathsf{rk}(H) \geq 2$. Let M be a compact Kähler threefold. If there exists an injective morphism $H \to \mathsf{Aut}(M)^0$, then M is one of the following:*

(1) *a projective bundle $\mathbb{P}(E)$ for some rank 2 vector bundle $E \to \mathbb{P}^2(\mathbf{C})$, and then H is locally isomorphic to $\mathsf{PGL}_3(\mathbf{C})$;*
(2) *a principal torus bundle over $\mathbb{P}^2(\mathbf{C})$, and H is locally isomorphic to $\mathsf{PGL}_3(\mathbf{C})$;*
(3) *a product $\mathbb{P}^2(\mathbf{C}) \times B$ of the plane by a curve of genus $g(B) \geq 2$, and then H is locally isomorphic to $\mathsf{PGL}_3(\mathbf{C})$;*
(4) *the projective space $\mathbb{P}^3(\mathbf{C})$, and H is locally isomorphic to a subgroup of $\mathsf{PGL}_4(\mathbf{C})$, so that its Lie algebra is either $\mathfrak{sl}_3(\mathbf{C})$ or $\mathfrak{sl}_4(\mathbf{C})$*

(5) *the smooth quadric hypersurface* $Q \subset \mathbb{P}^4(\mathbf{C})$*, and* H *is locally isomorphic to* $\mathsf{SO}_5(\mathbf{C})$ *(and to* $\mathsf{Sp}_4(\mathbf{C})$*, see* [25]*, page 278).*

Example 4.2. The group $\mathsf{GL}_3(\mathbf{C})$ acts on $\mathbb{P}^2(\mathbf{C})$, and its action lifts to an action on the total space of the line bundles $O(k)$ for every $k \geq 0$; sections of $O(k)$ are in one-to-one correspondence with homogenous polynomials of degree k, and the action of $\mathsf{GL}_3(\mathbf{C})$ on $H^0(\mathbb{P}^2(\mathbf{C}), O(k))$ is the usual action on homogenous polynomials in three variables. Let p be a positive integer and E the vector bundle of rank 2 over $\mathbb{P}^2(\mathbf{C})$ defined by $E = O \oplus O(p)$. Then $\mathsf{GL}_3(\mathbf{C})$ acts on E, by isomorphisms of vector bundles. From this we get an action on the projectivized bundle $\mathbb{P}(E)$, i.e. on a compact Kähler manifold M which fibers over $\mathbb{P}^2(\mathbf{C})$ with rational curves as fibers.

A similar example is obtained from the $\mathbf{C}^*$-bundle associated to $O(k)$. Let λ be a complex number with modulus different from 0 and 1. The quotient of this $\mathbf{C}^*$-bundle by multiplication by λ along the fibers is a compact Kähler threefold, with the structure of a torus principal bundle over $\mathbb{P}^2(\mathbf{C})$. Since multiplication by λ commutes with the $\mathsf{GL}_3(\mathbf{C})$-action on $O(k)$, we obtain a (transitive) action of $\mathsf{GL}_3(\mathbf{C})$ on this manifold.

Remark 4.3. Embed $\mathsf{GL}_3(\mathbf{C})$ into $\mathsf{GL}_4(\mathbf{C})$ as the subgroup preserving the vector $(0,0,0,1)$. Then $\mathsf{GL}_3(\mathbf{C})$ acts on $\mathbb{P}^3(\mathbf{C})$, preserving the point $[0:0:0:1]$. Blow up this point: This provides a projective threefold M together with a birational morphism $\pi : M \to \mathbb{P}^3(\mathbf{C})$. The action of $\mathsf{GL}_3(\mathbf{C})$ lifts to M. This provides a link between case (4) and case (1).

Proposition 4.1 is easily deduced from the classification of homogenous complex manifolds of dimension at most 3, as described in the work of Winkelmann (see [46]). Let us sketch its proof.

Sketch of the proof. First, H contains a Zariski dense lattice with property (T), so that we can apply the results on invariant analytic subsets from section 3 to the group H. If H has a fixed point p, one can locally linearize the action of H in a neighborhood of p (see for example theorems 2.6 and 10.4 in [9]). This provides a regular morphism $H \to \mathsf{SL}_3(\mathbf{C})$. Since all complex Lie subalgebras of $\mathfrak{sl}_3(\mathbf{C})$ with rank ≥ 2 are equal to $\mathfrak{sl}_3(\mathbf{C})$, this morphism is onto. In particular, if H has a fixed point, H has an open orbit.

Assume, first, that H does not have any open orbit; thus, no orbit of H is a point. Let O be an orbit of H. Its closure is an invariant analytic subset; since it can not be a point, it must be a projective plane (corollary 3.4). The action of H on $\overline{O}$ gives a map $H \to \mathsf{Aut}(\mathbb{P}^2(\mathbf{C})) = \mathsf{PGL}_3(\mathbf{C})$, and this map is surjective because the rank of H is at least 2. Hence, H does not preserve any strict subset of $\overline{O}$, and O coincides with its closure. From this follows that all orbits of H are closed, isomorphic to $\mathbb{P}^2(\mathbf{C})$, and the action of H on each orbit coincides with the action of $\mathsf{PGL}_3(\mathbf{C})$ on $\mathbb{P}^2(\mathbf{C})$. In that case, there is an invariant fibration $\pi : M \to B$, where B is a curve, with orbits of H as fibers. Let A be a generic diagonal matrix in $\mathsf{PGL}_3(\mathbf{C})$. The action of A on every fiber of π has exactly three fixed points: One saddle, one sink and one source. This gives three sections of the fibration $\pi : M \to B$. The action of H is transitive along the fibers, and permutes the space of sections of B. From this follows easily that the fibration is trivial. According to the value of the genus of B, this case falls in one of the three possibilities (1), (2), (3).

Now assume that H has an open orbit $M_0 \subset M$, which does not coincide with M. According to section 3, its complement is a disjoint union of points and of projective planes, H is locally isomorphic to $\mathsf{PGL}_3(\mathbf{C})$, and acts as $\mathsf{PGL}_3(\mathbf{C})$ on each of these invariant planes. The open orbit M_0 is a homogenous complex manifold of dimension 3. Chapter 4, §5, p. 46–49 of [46], shows that M_0 is a $\mathbf{C}^*$-bundle over $\mathbb{P}^2(\mathbf{C})$; more precisely, there exists a bimeromorphic map $\varphi : M \dashrightarrow \mathbb{P}(E)$ where E is a rank 2 vector bundle over $\mathbb{P}^2(\mathbf{C})$ such that

- $E = O(r) \oplus O(s)$, with $r, s \in \mathbf{Z}$; hence $\mathbb{P}(E)$ has two natural sections corresponding to the line bundles $O(r) \oplus \{0\}$ and $\{0\} \oplus O(s)$; denote by V_0 and V_∞ the images of these sections;
- φ is a regular isomorphism from M_0 to the complement of $V_0 \cup V_\infty$ in $\mathbb{P}(E)$;
- φ is equivariant with respect to the action of H on M and the action of $\mathsf{PGL}_3(\mathbf{C})$ on $\mathbb{P}(E)$.

This implies that $M \setminus M_0$ has two connected components, which are mapped to V_0 and V_∞ by φ. If these two components are isomorphic to $\mathbb{P}^2(\mathbf{C})$, then φ is in fact an isomorphism because the action of H (resp. $\mathsf{PGL}_3(\mathbf{C})$) on M (resp. $\mathbb{P}(E)$) is regular; once again, M falls in case (1). If one component is a point, φ blows up this point to V_0 (or V_∞). Since M is smooth, φ^{-1} contracts V_0 to a smooth point, and the normal bundle of V_0 is $O(-1)$. This implies that M is isomorphic to $\mathbb{P}^3(\mathbf{C})$, as in case (4), and φ is the link described in remark 4.3.

The last case corresponds to transitive actions, when M is isomorphic to a quotient H/N where N is a closed subgroup of H. A classical result due to Tits (see [43]) classifies all homogenous compact complex manifolds of dimension 3. For compact Kähler threefolds with a transitive action of an almost simple Lie group, the list reduces to the projective space $\mathbb{P}^3(\mathbf{C})$, the smooth quadric $Q_3 \subset \mathbb{P}^4(\mathbf{C})$, and principal torus bundles as in (2). As a consequence, homogenous manifolds fall in case (2), (4), and (5). □

5. HODGE STRUCTURES AND HIGHER RANK LIE GROUPS

In this section, we use Hodge theory, Lie theory and Margulis rigidity to rule out several kind of lattices and Lie groups. In what follows, G is a connected semi-simple Lie group with finite center, without non trivial compact factor, and with rank at least 2. The Lie algebra of G is denoted by $\mathfrak{g}$. Let Γ be an irreducible lattice in G and $\Gamma \to \mathsf{Aut}(M)$ be an almost faithfull representation of Γ into the group of automorphisms of a compact Kähler threefold M. The first statement that we want to prove is the following.

Theorem 5.1. *If the action of Γ on the cohomology of M does not factor through a finite group, then G is locally isomorphic to $\mathsf{SL}_3(\mathbf{R})$ or $\mathsf{SL}_3(\mathbf{C})$.*

The proof is given in sections 5.1 to 5.6.

5.1. Preliminaries.

In order to prove theorem 5.1, we first apply section 2.4.2: Since the action of Γ on $H^*(M,\mathbf{Z})$ does not factor through a finite group, this action is almost faithful. Let W denote $H^{1,1}(M,\mathbf{R})$. From section 2.5.1, we also know that the action of Γ on W is faithful, with discrete image. Apply corollary 2.6, proposition 2.7, and lemma 2.8. The action of Γ on $H^*(M,\mathbf{R})$ extends virtually to a linear representation of G on $H^*(M,\mathbf{R})$ that preserves the Hodge decomposition, the cup product, and Poincaré duality. Hence, theorem 5.1 is a corollary of the following proposition and of Hodge index theorem.

Proposition 5.2. *Let $\mathfrak{g}$ be a semi-simple Lie algebra, and $\mathfrak{g} \to \mathsf{End}(W)$ a faithful finite dimensional representation of $\mathfrak{g}$. If*

(i) *there exists a symmetric bilinear $\mathfrak{g}$-equivariant mapping $\wedge : W \times W \to W^*$, where W^* is the dual representation,*
(ii) *$\wedge$ does not vanish identically on any subspace of dimension 2 in W, and*
(ii) *the real rank $\mathsf{rk}_{\mathbf{R}}(\mathfrak{g})$ is at least 2,*

then $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_3(\mathbf{R})$ or $\mathfrak{sl}_3(\mathbf{C})$.

Our main goal now, up to section 5.6, is to prove this proposition; $\mathfrak{g}$ will be a semi-simple Lie algebra acting on W, and $\wedge$ a $\mathfrak{g}$ equivariant bilinear map with values in the dual representation W^*, as in the statement of proposition 5.2.

5.2. $\mathfrak{sl}_2(\mathbf{R})$-representations.

For all positive integers n, the Lie algebra $\mathfrak{sl}_2(\mathbf{R})$ acts linearly on the space of degree n homogeneous polynomials in two variables. Up to isomorphism, this representation is the unique irreducible linear representation of $\mathfrak{sl}_2(\mathbf{R})$ in dimension $n+1$. The weights of this representation with respect to the Cartan subalgebra of diagonal matrices are

$$-n, -n+2, -n+4, ..., n-4, n-2, n,$$

and the highest weight n characterizes this irreducible representation. These representations are isomorphic to their own dual representations.

Lemma 5.3. *Let $\mu : \mathfrak{sl}_2(\mathbf{R}) \to \mathfrak{g}$ be an injective morphism of Lie algebras. Then the highest weights of the representation*

$$\mathfrak{sl}_2(\mathbf{R}) \to \mathfrak{g} \to \mathsf{End}\,(W)$$

are bounded from above by 4, *and the weight* 4 *appears at most once.*

Proof. Let V be an irreducible subrepresentation of $\mathfrak{sl}_2(\mathbf{R})$ in W, and let m be its highest weight ; assume that m is the highest possible weight among all irreducible subrepresentations of W. Let u_m and u_{m-2} be elements in $V \setminus \{0\}$ with respective weights m and $m-2$. From Hodge index theorem (cf. lemma 2.2), we know that one of the cup products

$$u_m \wedge u_m,\ u_m \wedge u_{m-2},\ u_{m-2} \wedge u_{m-2}$$

is different from 0. The weight of this vector is at least $2(m-2)$, and is bounded from above by the highest weight of W^*, that is by m; consequently, $2(m-2) \leq m$, and $m \leq 4$.

Assuming that the weight $m = 4$ appears twice, there are 2 linearly independant vectors u_4 and v_4 of weight 4. Since the highest weight of W^* is also 4, the cup products $u_4 \wedge u_4$, $u_4 \wedge v_4$, and $v_4 \wedge v_4$ vanish, contradicting Hodge index theorem (lemma 2.2). □

5.3. Actions of $\mathfrak{sl}_2(\mathbf{R}) \times \mathfrak{sl}_2(\mathbf{R})$.

Lemma 5.4. *If there is a faithful representation $\mathfrak{g} \to \mathsf{End}\,(W)$ as in proposition 5.2, then $\mathfrak{g}$ does not contain any copy of $\mathfrak{sl}_2(\mathbf{R}) \oplus \mathfrak{sl}_2(\mathbf{R})$.*

Proof. Suppose that $\mathfrak{g}$ contains $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ with both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ isomorphic to $\mathfrak{sl}_2(\mathbf{R})$. Let n_1 be the highest weight of $\mathfrak{g}_1$ and n_2 be the highest weight of $\mathfrak{g}_2$ in W. Since the representation of $\mathfrak{g}$ is faithful, both n_1 and n_2 are positive integers. After permutation of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, we shall assume that $n_1 = \max(n_1, n_2)$.

Let $\mathfrak{h} \leq \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be the diagonal copy of $\mathfrak{sl}_2(\mathbf{R})$; the highest weight of $\mathfrak{h}$ is n_1. Let $u_i \in W$ be a vector of weight n_i for $\mathfrak{g}_i$. Since $\mathfrak{g}_1$ and $\mathfrak{g}_2$ commute, u_1 is not colinear to u_2. If $u_i \wedge u_j$ is not zero, its weight for $\mathfrak{h}$ is $n_i + n_j$. This implies that $u_1 \wedge u_1 = 0$ and $u_1 \wedge u_2 = 0$. Since n_2 is a highest weight for $\mathfrak{g}_2$, we also know that $u_2 \wedge u_2 = 0$ because $2n_2$ does not appear as a weight for $\mathfrak{g}_2$ on W^*. Hence, $\wedge$ should vanish identically on the vector space spanned by u_1 and u_2, contradicting Hodge index theorem. This concludes the proof. □

5.4. **Actions of $\mathfrak{sl}_3(\mathbf{R})$.**

We now assume that $\mathfrak{g}$ contains a copy of the Lie algebra $\mathfrak{sl}_3(\mathbf{R})$, and we restrict the faithful representation $\mathfrak{g} \to \mathsf{End}(W)$ to $\mathfrak{sl}_3(\mathbf{R})$. Doing that, we assume in this section that $\mathfrak{g} = \mathfrak{sl}_3(\mathbf{R})$. In what follows, we choose the diagonal subalgebra

$$\mathfrak{a} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} ; \quad a_1 + a_2 + a_3 = 0 \right\}$$

as a Cartan algebra in $\mathfrak{g}$. We shall denote by $\lambda_i : \mathfrak{a} \to \mathbf{R}$ the linear forms $\lambda_i(a_1, a_2, a_3) = a_i$, and by $\mathfrak{a}^+$ the Weyl chamber $a_1 \geq a_2 \geq a_3$. With such a choice, highest weights are linear forms on $\mathfrak{a}$ of type

$$\lambda = a\lambda_1 - b\lambda_3,$$

where a and b are non negative integers. Irreducible representations are classified by their highest weight. For example, $(a,b) = (0,0)$ corresponds to the trivial 1-dimensional representation (denoted T in what follows), $(a,b) = (1,0)$ corresponds to the standard representation on the vector space $E = \mathbf{R}^3$, while $(0,1)$ is the dual representation E^*. When $(a,b) = (0,2)$, the representation is $\mathrm{Sym}_2(E^*)$, the space of quadratic forms on E. The weight $(a,b) = (1,1)$ corresponds to the adjoint representation $(E \otimes E^*)_0$, i.e. to the action of $\mathsf{SL}_3(\mathbf{R})$ by conjugation on 3×3 matrices with trace zero. We refer to [25] for highest weight theory and a detailed description of standard representations of $\mathsf{SL}_3(\mathbf{R})$. If V is a representation of a Lie group G, we shall denote by V^k the direct sum of k copies of V.

Proposition 5.5. *If $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_3(\mathbf{R})$, then:*

a.- *The possible highest weights of irreducible subrepresentations in* W *are* $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, $(2,0)$, $(0,2)$*;*

b.- W *contains at most one irreducible subrepresentation with highest weight* $a\lambda_1 - b\lambda_3$ *such that* $a+b=2$.

Proof. The action of $\mathfrak{sl}_2(\mathbf{R})$ on the space of quadratic homogenous polynomial $ax^2+bxy+cy^2$ provides an embedding $\mu_2 : \mathfrak{sl}_2(\mathbf{R}) \to \mathfrak{g}$. If we restrict it to the chosen Cartan subalgebras, the diagonal matrix $diag(s,-s)$ is mapped to $diag(2s,0,-2s)$. Let $\lambda = a\lambda_1 - b\lambda_3$ be a highest weight for the representation W. After composition with μ_2, we see that $2(a+b)$ is one of the weight of $\mathfrak{sl}_2(\mathbf{R})$. From lemma 5.3, we get $a+b \leq 2$. The list of possible weights follows. Point (b.-) follows from the last assertion in lemma 5.3. □

Lemma 5.6. *If* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{sl}_3(\mathbf{R})$, *then:*

a.- *The representation* W *contains at least one weight* $a\lambda_1 - b\lambda_3$ *with* $a+b=2$*;*

b.- *The adjoint representation (corresponding to* $(a,b)=(1,1)$*) does not appear in* W*;*

c.- *If* W *contains a factor of type* $Sym_2(E^*)$ *(resp.* $Sym_2(E)$*), then* W *does not contain any factor of type* E^* *(resp.* E*).*

Proof. The cup product defines a symmetric, $\mathfrak{g}$-equivariant, bilinear map $\wedge : W \times W \to W^*$. We interprete it as a linear map from $\mathrm{Sym}_2(W)$ to W^*, and deduce the lemma from its study.

a.- If W does not contain any weight with $a+b=2$, then W is isomorphic to a direct sum

$$E^k \oplus (E^*)^l \oplus T^m$$

where T is the trivial one-dimensional representation of $\mathfrak{g}$ and k, l, and m are non-negative integers. If $k>0$, the cup product determines a $\mathfrak{g}$-equivariant linear map $\wedge_E : \mathrm{Sym}_2(E) \to W^*$. Since $\mathrm{Sym}_2(E)$ is irreducible and is not isomorphic to E, E^*, or T, this map $\wedge_E$ vanishes identically, contradicting lemma 2.2. From this we deduce that $k=0$, and similarly that $l=0$; this means that W is the trivial representation, contradicting the fact that $\mathfrak{g} \to \mathsf{End}(W)$ is a faithful representation.

b.- The weights of the adjoint representation $(E \otimes E^*)_0$ are $\lambda_i - \lambda_j$ where i, j $\in \{1,2,3\}$ are distinct numbers. This representation is self-dual. If W contains it as a factor, then W^* also, and this factor is the unique one with highest weight $a\lambda_1 - b\lambda_3$ such that $a+b \geq 2$. Let u_{ij} be a non-zero eigenvector of $\mathfrak{a}$ corresponding to the weight $\lambda_i - \lambda_j$. Then $u_{13} \wedge u_{13}$ has weight $2(\lambda_1 - \lambda_3)$, which does not appear in W^*. As a consequence, $u_{13} \wedge u_{13} = 0$, and similarly $u_{23} \wedge u_{23} = u_{13} \wedge u_{23} = 0$. This implies that $\wedge$ vanishes on the two-dimensional vector space $\mathrm{Vect}(u_{13}, u_{23})$, contradicting lemma 2.2.

c.- Assume that W decomposes as

$$\mathrm{Sym}_2(E) \oplus E^k \oplus (E^*)^l \oplus T^m,$$

with $k > 0$. Choose u in $\mathrm{Sym}_2(E) \setminus \{0\}$ a vector of weight $2\lambda_1$, and v in $E \setminus \{0\}$ of weight λ_1. The weights of $u \wedge u$, $u \wedge v$, and $v \wedge v$ are equal to $n\lambda_1$, with $n = 4$, 3, and 2 respectively. None of them appears in W^*, and lemma 2.2 provides the desired contradiction. □

Theorem 5.7. *If the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_3(\mathbf{R})$, and if $\mathfrak{g} \to \mathsf{End}\,(W)$ is a linear representation as in proposition 5.2, there exist two integers $k \geq 0$ and $m \geq 0$ such that the representation W is isomorphic to $Sym_2(E^*) \oplus E^k \oplus T^m$, or to its dual.*

5.5. Actions of $\mathfrak{sl}_3(\mathbf{C})$ and $\mathfrak{sl}_3(\mathbf{H})$.

Assume, now, that $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_3(\mathbf{C})$. Let $E_{\mathbf{C}}$ denote the vector space $\mathbf{C}^3$ with its standard action of the Lie algebra $\mathfrak{sl}_3(\mathbf{C})$. We denote by $\overline{E_{\mathbf{C}}}$ the complex conjugate representation, and by $\mathsf{Her}\,(E^*_{\mathbf{C}})$ the representation of $\mathfrak{sl}_3(\mathbf{C})$ on the space of hermitian forms. As before, T will denote the trivial one dimensional representation of $\mathfrak{sl}_3(\mathbf{C})$.

Theorem 5.8. *If the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_3(\mathbf{C})$, and if $\mathfrak{g} \to \mathsf{End}\,(W)$ is a linear representation as in proposition 5.2, there exist three integers $k \geq 0$, $l \geq 0$, and $m \geq 0$ such that the representation W is isomorphic to*

$$W = \mathsf{Her}\,(E^*_{\mathbf{C}})) \oplus E^k_{\mathbf{C}} \oplus \overline{E_{\mathbf{C}}}^l \oplus T^m$$

after composition by an automorphism of $\mathfrak{g}$.

Remark 5.9. The automorphism $b \mapsto -{}^t b$ of $\mathfrak{sl}_3(\mathbf{C})$ turns a representation into its dual, and $b \mapsto \overline{b}$ into its complex conjugate. If $\mathfrak{g}$ is not isomorphic to $\mathfrak{sl}_3(\mathbf{C})$ but contains a subalgebra $\mathfrak{g}' \simeq \mathfrak{sl}_3(\mathbf{C})$, theorem 5.8 holds for $\mathfrak{g}'$.

Proof. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_3(\mathbf{R})$. With respect to the action of $\mathfrak{h}$, we may assume that W splits as

$$W = \mathrm{Sym}_2(E^*) \oplus E^k \oplus T^m.$$

Let $\mathfrak{a}$ be the real diagonal subalgebra in $\mathfrak{g}$: This is a Cartan subalgebra for both $\mathfrak{h}$ and $\mathfrak{g}$. The highest weight for the representation of $\mathfrak{a}$ on W is

$$-2\lambda_3 : (a_1, a_2, a_3) \mapsto -2a_3.$$

The eigenspace of $\mathfrak{a}$ corresponding to this weight has dimension 1; it is spanned by the element $dx_3 \otimes d\overline{x_3}$ of $\mathsf{Her}\,(E^*_{\mathbf{C}})$. Its orbit under $\mathfrak{h}$ spans $\mathsf{Sym}_2(E^*)$, and one easily checks that its orbit under $\mathfrak{g}$ determines a unique copy of the representation $\mathsf{Her}\,(E^*_{\mathbf{C}})$. Since $\mathfrak{g}$ is a simple Lie algebra, W decomposes as a $\mathfrak{g}$-invariant direct sum $W = \mathsf{Her}\,(E^*_{\mathbf{C}}) \oplus W_0$ where W_0 is isomorphic to $E^{k-1} \oplus T^m$ as a $\mathfrak{h}$-module. In particular, all weights of $\mathfrak{a}$ on W_0 are equal to 0 or L_1. This shows that W_0 is isomorphic to

$$E_{\mathbf{C}}^{k'} \oplus \overline{E_{\mathbf{C}}}^{l'} \oplus T^m$$

as a $\mathfrak{g}$-module, with $k = 2(k'+l')+1$. □

Let $\mathbf{H}$ be the field of quaternions.

Proposition 5.10. *If there is a faithful representation* $\mathfrak{g} \to \mathsf{End}\,(W)$ *as in proposition 5.2, then* $\mathfrak{g}$ *does not contain any copy of* $\mathfrak{sl}_3(\mathbf{H})$.

Proof. Let us switch to representations of Lie groups, and assume that $\mathsf{SL}_3(\mathbf{H})$ acts on W, as in proposition 5.2. Let Δ be the subgroup $\mathsf{SL}_3(\mathbf{H})$ which is made of diagonal matrices with coefficents h_1, h_2, and h_3 in $\mathbf{H}$ (with the condition that the real determinant of the matrix is 1, see [25] page 99).

Let $A = diag(a_1, a_2, a_3)$ be an element of $\Delta \cap \mathsf{SL}_3(\mathbf{C})$ with complex entries $a_j = \rho_j e^{i\phi_j}$, where $\rho_j = |a_j|$. Theorem 5.8 implies that the eigenvalues of A on $W \otimes \mathbf{C}$ are

- ρ_j^{-2}, $j = 1,2,3$, with multiplicity 1;
- $\rho_j e^{i(\phi_{j'} - \phi_{j''})}$, where $\{j, j', j''\} = \{1,2,3\}$, with multiplicity 1;
- $\rho_j e^{i\phi_j}$, $j = 1,2,3$, with multiplicity k;
- $\rho_j e^{-i\phi_j}$, $j = 1,2,3$, with multiplicity l;
- 1 with multiplicity m.

Let now B be an element of $\mathsf{SL}_3(\mathbf{H})$ of type $diag(e^{i\psi}, 1, 1)$ (resp $diag(1, e^{i\psi}, 1)$, resp. $diag(1, 1, e^{i\psi})$). Since B commutes with $\Delta \cap \mathsf{SL}_3(\mathbf{C})$, B preserves the eigenspaces of A. In particular, B preserves $\mathsf{Her}\,(E^*_{\mathbf{C}})$, acts trivially on the one dimensional eigenspaces of A corresponding to the eigenvalues ρ_j^{-2}, and as a standard rotation on the invariant planes associated to pairs of conjugate eigenvalues $\rho_j e^{\pm i(\phi_{j'} - \phi_{j''})}$. In other words, B acts on $\mathsf{Her}\,(E^*_{\mathbf{C}})$ as elements of

$\mathsf{GL}_3(\mathbf{C})$ do. As a consequence, composing elements of type B with elements of type A, we see that the action of $\Delta \cap \mathsf{GL}_3(\mathbf{C})$ on $\mathsf{Her}(E_{\mathbf{C}}^*)$ is the standard action on hermitian forms; the previous computation for the eigenspaces of matrices A holds for this larger group.

Choose $A = diag(\rho_1 e^{i\phi_1}, \rho_2, \rho_3 e^{-i\phi_3})$ with pairwise distinct ρ_j, distinct irrational ϕ_1 and ϕ_3, and $\rho_1\rho_2\rho_3 = 1$. This element of $\mathsf{SL}_3(\mathbf{H})$ commutes with the subgroup $S = \{diag(1,q,1) \,|\, q \in \mathsf{SU}_2\}$ of $\mathsf{SL}_3(\mathbf{H})$. In particular, S preserves the eigenspaces of A. But $S \simeq \mathsf{SU}_2$ does not have any non trivial representation in dimension < 3. This implies that S acts trivially on $\mathsf{Her}(E_{\mathbf{C}}^*)$. In the same way, $S' = \{diag(1,1,q) \,|\, q \in \mathsf{SU}_2\}$ acts trivially on $\mathsf{Her}(E_{\mathbf{C}}^*)$. This is a contradiction because $diag(1, e^{i\phi}, e^{-i\phi})$, $\phi \neq 0$, is an element of the product SS' that does not act trivially on $\mathsf{Her}(E_{\mathbf{C}}^3)$. □

5.6. **Proof of proposition 5.2.**

Lemma 5.11. *Let $\mathfrak{g}$ be a simple real Lie algebra. If the real rank of $\mathfrak{g}$ is at least* 3, *then $\mathfrak{g}$ contains a copy of $\mathfrak{sl}_2(\mathbf{R}) \oplus \mathfrak{sl}_2(\mathbf{R})$. If the real rank of $\mathfrak{g}$ is equal to* 2, *then*

a.- *If $\mathfrak{g}$ admits a complex structure, then $\mathfrak{g}$ is isomorphic to one of the three algebras $\mathfrak{sl}_3(\mathbf{C})$, $\mathfrak{sp}_4(\mathbf{C})$ or $\mathfrak{g}_2(\mathbf{C})$.*

b.- *If $\mathfrak{g}$ does not admit a complex structure, either $\mathfrak{g}$ contains a copy of $\mathfrak{sl}_2(\mathbf{R}) \oplus \mathfrak{sl}_2(\mathbf{R})$, or $\mathfrak{g}$ is isomorphic to one of the algebras $\mathfrak{sl}_3(\mathbf{R})$, $\mathfrak{sl}_3(\mathbf{H})$, or $\mathfrak{e}_{IV}$.*

Sketch of the proof. This is a consequence of the classification of simple Lie algebras (see [34]) and exceptional isomorphisms in small dimensions. For example, the complex Lie group E_6 has two real forms of rank 2: $\mathfrak{e}_{III}$ and $\mathfrak{e}_{IV}$. The root system of $\mathfrak{e}_{III}$ with respect to its maximal $\mathbf{R}$-diagonalizable subalgebra is isomorphic to BC_2, which contains the root system of $\mathfrak{sl}_2(\mathbf{R}) \oplus \mathfrak{sl}_2(\mathbf{R})$; the same is true for $\mathfrak{g}_2$. Another example is given by the Lie algebras $\mathfrak{so}_{2,k}$, $k \geq 2$: All of them contain $\mathfrak{so}_{2,2}$, i.e. $\mathfrak{sl}_2(\mathbf{R}) \oplus \mathfrak{sl}_2(\mathbf{R})$. □

Lemma 5.12. *The real Lie algebra $\mathfrak{e}_{IV}$ contains a copy of $\mathfrak{sl}_3(\mathbf{H})$.*

Proof. This follows from the classification of simple real Lie algebras in terms of their Vogan diagrams (see [34], chapter VI), and from the fact that the diagram of $\mathfrak{sl}_3(\mathbf{H})$ embeds into the diagram of $\mathfrak{e}_{IV}$. □

Proof of proposition 5.2. Let $\mathfrak{g}$ and $\mathfrak{g} \to \mathsf{End}(W)$ be as in proposition 5.2. If $\mathfrak{g}$ has two simple factors of rank ≥ 1, then $\mathfrak{g}$ contains a copy of $\mathfrak{sl}_2(\mathbf{R}) \oplus \mathfrak{sl}_2(\mathbf{R})$, and lemma 5.4 provides a contradiction. We can therefore assume that $\mathfrak{g}$ is simple. Lemma 5.4 implies that $\mathfrak{g}$ does not contain any copy of $\mathfrak{sl}_2(\mathbf{R})\oplus\mathfrak{sl}_2(\mathbf{R})$, and proposition 5.10 shows that it does not contain any copy of $\mathfrak{sl}_3(\mathbf{H})$. The proposition is now a consequence of lemmas 5.11 and 5.12. □

6. Lattices in $\mathsf{SL}_3(\mathbf{R})$: Part I

We pursue the proof of theorem A. From section 5, theorem 5.1, we can assume that Γ is a lattice in $\mathsf{SL}_3(\mathbf{R})$ or $\mathsf{SL}_3(\mathbf{C})$. Here we deal with the first case, namely $G=\mathsf{SL}_3(\mathbf{R})$. Our main standing assumptions are now

- Γ is a lattice in $G=\mathsf{SL}_3(\mathbf{R})$;
- Γ acts holomorphically on a compact Kähler threefold M;
- the action of Γ on $H^*(M,\mathbf{R})$ extends to a non trivial linear representation of G.

We shall denote by W both the cohomology group $H^{1,1}(M,\mathbf{R})$ and the linear representation $G\to\mathsf{GL}\,(H^{1,1}(M,\mathbf{R}))$. This gives a representation of the Lie algebra $\mathfrak{g}=\mathfrak{sl}_3(\mathbf{R})$, and section 5.4 shows that this representation $\mathfrak{g}\to\mathsf{End}\,(W)$ decomposes into the direct sum

$$(*)\quad W=\mathrm{Sym}_2(E^*)\oplus E^k\oplus T^m,$$

or its dual; composing $\Gamma\to\mathsf{Aut}(M)$ with the automorphism $B\mapsto {}^t(B^{-1})$ of G, we shall assume that W is isomorphic to the direct sum $(*)$. The representation $H^{2,2}(M,\mathbf{R})$ is isomorphic to the dual of W. We shall denote it by W^*, with its direct sum decomposition

$$(**)\quad W^*=\mathrm{Sym}_2(E)\oplus(E^*)^k\oplus(T^*)^m$$

where T^* is just another notation for the one dimensional trivial representation of G.

Our goal is to show that the action of Γ on M is of Kummer type. In this section, we study the structure of the cup product $\wedge$ and the position of the lattice $H^2(M,\mathbf{Z})$ with respect to the Hodge decomposition of $H^2(M,\mathbf{C})$, the direct sum decomposition $(*)$ of $H^{1,1}(M,\mathbf{R})$, and the Kähler cone of M.

Most of the lemmas that we shall prove in this section are not specific to $\mathsf{SL}_3(\mathbf{R})$, and will be used for $G=\mathsf{SL}_3(\mathbf{C})$ in section 8.

6.1. The cup product.

6.1.1. *Intersections between irreducible factors.* The following lemmas are straightforward applications of representation theory, in the same spirit as what we did in section 5.

Lemma 6.1. *Up to a multiplicative scalar factor,*

a.- *there is a unique non-zero symmetric bilinear G-equivariant mapping from* $Sym_2(E^*)$ *to* $Sym_2(E)$.

b.- *All G-equivariant bilinear symmetric mappings from* $Sym_2(E^*)$ *to* $(E^*)^k \oplus (T^*)^m$ *vanish identically.*

Proof. From [25], page 189, we know that

$$\mathrm{Sym}_2(Sym_2(E^*)) \simeq \mathrm{Sym}_4(E^*) \oplus \mathrm{Sym}_2(E).$$

Both assertions follow from this isomorphism. □

In the same way, one shows that, up to multiplication by a scalar factor, there is a unique G-equivariant, symmetric, and bilinear map from E to $\mathrm{Sym}_2(E)$, but there is no non-zero map of this type from E to $(E^*)^k \oplus (T^*)^m$.

Lemma 6.2 (See [25], pages 180-181)**.** *The symmetric tensor product*

$$Sym_2(Sym_2(E^*) \oplus E^k)$$

decomposes as the direct sum of the following factors : $Sym_2(Sym_2(E^*))$, $Sym_2(E^k)$, *and k copies of* $\Gamma_{1,2} \oplus E \oplus E^*$, *where* $\Gamma_{1,2}$ *is the irreducible representation with highest weight* $\lambda_1 - 2\lambda_3$.

As a consequence, there exist non trivial symmetric and G-equivariant bilinear mappings from $\mathrm{Sym}_2(E^*) \oplus E^k$ to $(E^*)^k$, but all of them vanish identically on $\mathrm{Sym}_2(E^*)$ and E^k. In particular,

(1) if u is in $\mathrm{Sym}_2(E^*)$ then $u \wedge u \in \mathrm{Sym}_2(E)$;
(2) if v is in E^k then $v \wedge v \in \mathrm{Sym}_2(E)$;
(3) if u is in $\mathrm{Sym}_2(E^*)$ and v is in E^k then $u \wedge v \in (E^*)^k$;
(4) if t is in T then $t \wedge t \in (T^*)^m$;
(5) if u is in $\mathrm{Sym}_2(E^*) \oplus E^k$ and t is in T then $u \wedge t = 0$.

Moreover, $\wedge$ is uniquely determined up to a scalar multiple once restricted to $\mathrm{Sym}_2(E^*)$ (resp. to each factor E of E^k).

6.1.2. *Cubic form.* Let D be the cubic form which is defined on W by

$$D(u) = \int_M u \wedge u \wedge u.$$

Lemma 6.3. *When restricted to $Sym_2(E^*)$, the cubic form D coincides with a non trivial scalar multiple of the determinant* $\mathsf{det} : Sym_2(E^*) \to \mathbf{R}$. *It vanishes identically on E^k.*

The automorphism $u \mapsto -u$ of $\mathrm{Sym}_2(E^*)$ commutes with the action of G and changes det in $-$det. As a consequence, we shall assume that *there exists a positive number ε such that*

$$\int_M u \wedge u \wedge u = \varepsilon \, \mathsf{det}(u)$$

for all u in $Sym_2(E^*)$.

Proof. The trilinear mapping D is symmetric, and

$$\mathrm{Sym}_3(\mathrm{Sym}_2(E^*)) = \mathrm{Sym}_6(E^*) \oplus \Gamma_{2,2} \oplus T,$$

where the trivial factor T is generated by det (see [25], page 191). This implies that D is proportional to det. Let $\kappa_0 \in W$ be a Kähler class, and let $\kappa_0 = u_0 + v_0 + t_0$ be its decomposition with respect to $W = \mathrm{Sym}_2(E^*) \oplus E^k \oplus T^m$. By Hodge index theorem, $Q_\kappa(u,u) = -\int_M \kappa_0 \wedge u \wedge u$ does not vanish identically along a subspace of W of dimension > 1 (see the proof of lemma 2.2). This remark and property (1) above imply the existence of an element u in $\mathrm{Sym}_2(E^*)$ such that

$$\int_M u_0 \wedge u \wedge u \neq 0.$$

Since D is symmetic, D does not vanish identically along $\mathrm{Sym}_2(E^*)$, and D is a non zero scalar multiple of det.

The second assertion follows from the fact that $v \wedge v \in \mathrm{Sym}_2(E)$ for all v in E^k, and that $\int_M w \wedge e = 0$ for all w in $\mathrm{Sym}_2(E) \subset W^*$ and all e in E. $\square$

The following lemma shows how the hard Lefschetz theorem can be used in the same spirit as what we did with Hodge index theorem.

Lemma 6.4. *There are elements $t \in T^m \subset W$ such that $D(t) \neq 0$.*

Proof. Let $\kappa_0 \in W$ be a Kähler class, and let $\kappa_0 = u_0 + v_0 + t_0$ be its decomposition with respect to $W = \mathrm{Sym}_2(E^*) \oplus E^k \oplus T^m$. The hard Lefschetz theorem (see [45], page 142 theorem 6.25) shows that the linear map

$$\beta \mapsto \kappa_0 \wedge \beta$$

is an isomorphism between W and W^*. As a consequence, $\kappa_0 \wedge t \neq 0$ for all $t \in T \setminus \{0\}$. Let t_1 be a non-zero element of T^m. Then, from property (5) above, we get

$$\kappa_0 \wedge t_1 = (u_0 + v_0 + t_0) \wedge t_1 = t_0 \wedge t_1 \in (T^m)^* \setminus \{0\}.$$

Since $t_0 \wedge t_1$ is different from 0, there exists t_2 in T^m such that $t_0 \wedge t_1 \wedge t_2 \neq 0$. This implies that the symmetric trilinear form D does not vanish identically on T^m. □

6.2. **Cohomology with integer coefficients.**

Our next goal is to describe the position of $W = H^{1,1}(M, \mathbf{R})$ with respect to $H^2(M, \mathbf{Z})$. Note that $H^2(M, \mathbf{Z})$ is a lattice in $H^2(M, \mathbf{R})$, where we use the following definition: A **lattice** $\mathcal{L}$ in a real vector space V is a cocompact discrete subgroup of $(V, +)$.

6.2.1. *Invariant lattices.*

Lemma 6.5. *Let G be an almost simple Lie group and Γ be a lattice in G. Let V be a G-linear representation with no trivial factor, let T be the trivial one dimensional representation of G, and let $V \oplus T^m$ be the direct sum of V with m copies of T. If $\mathcal{L} \subset V \oplus T^m$ is a Γ-invariant lattice, then $\mathcal{L} \cap V$ is also a lattice in V.*

Proof. Let $u = v_0 + t_0$ be an element of $\mathcal{L}$, with v_0 in $V \setminus \{0\}$ and t_0 in T^m. Let us consider the subset $\mathcal{L}_0$ of $\mathcal{L} \cap V$ defined by

$$\mathcal{L}_0 = \{v - w \,|\, w, v \in \mathcal{L} \cap (V + t_0)\}.$$

This set is Γ-invariant, and contains all elements of type $\gamma(v_0) - v_0$, when γ describes Γ. Since the representation V does not contain any trivial factor, and Γ is Zariski dense in G, we know that Γ does not fix the vector v_0; in particular, $\mathcal{L}_0$ spans a non trivial subspace V_0 of V. If V_0 coincides with V, we are done, because $\mathcal{L}_0$ is a lattice in V_0. Otherwise, the codimension of V_0 in V is positive. Since $\mathcal{L}$ is a lattice in $V + T$, the projection of $\mathcal{L}$ on V spans V, so that there exists an element $v_1 + t_1$ in $\mathcal{L}$ such that v_1 is not in V_0. The same argument, once applied to $v_1 + t_1$ in place of $v_0 + t_0$, produces a new G-invariant subspace V_1 in V for which $\mathcal{L} \cap V_1$ is a lattice in V_1. Either $V_0 \oplus V_1 = V$, and the proof is complete, or the construction can be pushed further. In less than $\dim(V)$ steps, we are done. □

Lemma 6.6. *If $\mathcal{L}_0$ is a Γ-invariant lattice in $Sym_2(E^*) \oplus E^k$, the intersection $\mathcal{L}_0 \cap Sym_2(E^*)$ is a lattice in $Sym_2(E^*)$.*

Proof. Let B be an element of Γ. Since Γ is an arithmetic lattice, the eigenvalues of B are algebraic integers. Let α be such an eigenvalue, and let Q_α be its minimal polynomial: By construction, Q_α is a polynomial in one variable with integer coefficients, and the roots of Q_α are all Galois-conjugate to α.

Now assume that B is diagonalizable (over $\mathbf{C}$) with three distinct eigenvalues satisfying $|\alpha| > |\beta| > |\gamma|$. Such elements exist because Γ is Zariski dense in G (see section 2.3.3). Let P_B be the product

$$P_B(t) = Q_\alpha(t) Q_\beta(t) Q_\gamma(t);$$

its coefficients a_j are integers and $P_B(B) = \sum_j a_j B^j = 0$.

Let $\rho_1 : G \to \mathsf{GL}(E^k)$ be the diagonal representation. Let

$$\rho_2 : G \to \mathsf{GL}(\mathrm{Sym}_2(E^*))$$

be the representation on quadratic forms, and $\rho = \rho_1 \oplus \rho_2$ the direct sum of these two representations. Then $P_B(\rho_1(B)) = 0$. Assume that $P_B(\rho_2(B)) = 0$. The eigenvalues of $\rho_2(B)$ are roots of P_B, and so are their Galois-conjugates. Let μ be a root of P_B. Then, by construction, μ is conjugate to an eigenvalue of B, so that μ^{-2} is conjugate to an eigenvalue of $\rho_2(B)$. As a consequence, $P_B(\mu^{-2}) = 0$. From this we deduce the following: If μ is a root of P_B, so is μ^4. This implies that all roots of P_B are roots of unity, contradicting the choice of B. This contradiction shows that $P_B(\rho_2(B))$ is different from 0.

Choose an element $u_0 + v_0$ in $\mathcal{L}_0$ with $u_0 \in \mathrm{Sym}_2(E^*)$ and v_0 in E^k. Then $P_B(\rho(B))(u_0 + v_0)$ is an element of $\mathcal{L}_0$ because $\mathcal{L}_0$ is Γ-invariant and P_B has integer coefficients. Moreover,

$$P_B(\rho(B))(u_0 + v_0) = P_B(\rho_2(B))(u_0)$$

is an element of $\mathrm{Sym}_2(E^*)$. Since $P_B(\rho_2(B)) \in \mathsf{End}(\mathrm{Sym}_2(E^*))$ is different from 0 and $\mathcal{L}_0$ is a lattice, we can choose $u_0 + v_0$ in such a way that $P_B(\rho_2(B))(u_0) \neq 0$. This implies that the lattice $\mathcal{L}_0$ intersects $\mathrm{Sym}_2(E^*)$ non trivially. From the Zariski density of Γ in G and the irreducibility of the linear representation $\mathrm{Sym}_2(E^*)$, we conclude that $\mathcal{L}_0 \cap \mathrm{Sym}_2(E^*)$ is a lattice in $\mathrm{Sym}_2(E^*)$. □

6.2.2. *The lattice* $H^2(M, \mathbf{Z})$. Assume that $H^{2,0}(M, \mathbf{C})$ is not trivial. The sesquilinear mapping

$$(\Omega_1, \Omega_2) \mapsto \Omega_1 \wedge \overline{\Omega_2},$$

from $H^{2,0}(M, \mathbf{C})$ to $H^{2,2}(M, \mathbf{C}) = W^* \otimes_{\mathbf{R}} \mathbf{C}$ is G-equivariant. Moreover, $\Omega \wedge \overline{\Omega} = 0$ if and only if $\Omega = 0$. From this we deduce that, if $\lambda = a\lambda_1 - b\lambda_3$ is a weight for the linear representation of G on $H^{2,0}(M, \mathbf{C})$, then $2a\lambda_1 -$

$2b\lambda_3$ is a weight for W^*. This implies that the linear representation of G on $H^{2,0}(M,\mathbf{C})\oplus H^{0,2}(M,\mathbf{C})$ decomposes as a sum of standard and trivial factors: There exist two integers k' and m' such that

$$H^{2,0}(M,\mathbf{C})\oplus H^{0,2}(M,\mathbf{C}) = E^{k'}\oplus T^{m'}.$$

We can then write the linear representation of G on $H^2(M,\mathbf{R})$ as a direct sum

$$H^2(M,\mathbf{R}) = \mathrm{Sym}_2(E^*)\oplus E^{k''}\oplus T^{m''}$$

where $k'' = k+k'$ and $m'' = m+m'$.

Proposition 6.7. *The lattice $H^2(M,\mathbf{Z})$ intersects each of the three factors $Sym_2(E^*)$, $E^{k''}$, and $T^{m''}$ on lattices.*

Proof. Let $\mathcal{L}$ be the lattice $H^2(M,\mathbf{Z})$ in $H^2(M,\mathbf{R})$. We shall say that a subspace V of $H^2(M,\mathbf{R})$ is defined over the integers, if it is defined by linear forms from the dual lattice $\mathcal{L}^*$. From the two previous lemmas, we know that $\mathcal{L}$ intersects $\mathrm{Sym}_2(E^*)$ on a lattice $\mathcal{L}_1$. The cup product is defined over the cohomology with integral coefficients, so that $H^4(M,\mathbf{Z})$ intersects also the subspace $\mathrm{Sym}_2(E)$ of W^* onto a lattice. Since its orthogonal with respect to Poincaré duality coincides with $E^{k''}\oplus T^{m''}$, this subspace of $H^2(M,\mathbf{R})$ is defined over $\mathbf{Z}$, and intersects $\mathcal{L}$ on a lattice. From lemma 6.5, $\mathcal{L}\cap E^{k''}$ is a lattice in $E^{k''}$. Since $T^{m''}$ is the set of cohomology classes t such that $u\wedge t = 0$ for all u in $\mathrm{Sym}_2(E)\oplus(E^*)^{k''}$, this subspace is also defined over $\mathbf{Z}$, and intersects $\mathcal{L}$ on a lattice. □

6.3. Invariant cones.

6.3.1. *Cones from complex geometry.* Recall that a convex cone $\mathcal{C}$ in a real vector space V is **strict** when it does not contain any line. In other words, $\mathcal{C}$ is entirely contained on one side of at least one hyperplane of V. The **Kähler cone** of M is the set $\mathcal{K}(M)\subset W$ of cohomology classes of all Kähler forms of M. This set is an open convex cone; its closure $\overline{\mathcal{K}}(M)$ is a strict and closed convex cone, the interior of which coincides with $\mathcal{K}(M)$. We shall say that $\overline{\mathcal{K}}(M)$ is the cone of **nef** $(1,1)$-cohomology classes. The **pseudo-effective cone** is the set $\overline{\mathcal{P}s}(M)\subset W$ of cohomology classes $[c]$ of closed positive currents of type $(1,1)$. This is a strict and closed convex cone, that contains $\overline{\mathcal{K}}(M)$. The **cone of numerically positive classes** is the subset $\mathcal{P}(M)\subset W$ of cohomology classes $[\alpha]$ such that

$$\int_Y \alpha^{\dim(Y)} > 0$$

for all analytic submanifolds Y of M. All these cones are invariant under the action of $\mathsf{Aut}(M)$. In our context, they are Γ-invariant, but it is not clear a priori that $\mathcal{K}(M)$ or $\overline{\mathcal{P}s}(M)$ is G-invariant.

6.3.2. *Invariant cones in $E^k \oplus T^m$.*

Lemma 6.8. *The representation E^k (resp. $(E^*)^k$) does not contain any non trivial Γ-invariant strict convex cone.*

Proof. Let $\mathcal{D} \subset E$ be a Γ invariant convex cone, which is different from $\{0\}$. Its boundary provides a closed Γ-invariant subset in the projective plane $\mathbb{P}(E)$. Since the limit set of Γ in $\mathbb{P}(E)$ coincides with $\mathbb{P}(E)$ (see §2.3.4), the boundary of $\mathcal{D}$ is empty, and $\mathcal{D}$ coincides with E.

Let now $\mathcal{C} \subset E^k$ be a Γ-invariant convex cone. Let A be the smallest linear subspace of E^k which contains $\mathcal{C}$. Since Γ is Zariski dense in G and preserves A, the group G preserves A. Replacing E^k by A, we can therefore assume that $\mathcal{C}$ spans E^k; in other words, we shall assume that $\mathcal{C}$ has non empty interior in E^k. Let $\mathcal{C}'$ be the interior of $\mathcal{C}$.

Let $\pi : E^k \to E$ be the projection onto the first factor. The convex cone $\pi(\mathcal{C}')$ is open and Γ-invariant; as such it must coïncide with E. The fiber $\pi^{-1}\{0\} \cap \mathcal{C}'$ is an open, convex cone in E^{k-1}, and is Γ-invariant. After $k-1$ steps, we end up with a Γ-invariant open convex subcone of $\mathcal{C}'$ in E. This cone must coincide with E; in particular, $\mathcal{C}$ is not strict. $\square$

Lemma 6.9. *If $\mathcal{C}$ is a strict and Γ-invariant convex cone in $E^k \oplus T^m$ (resp. $(E^*)^k \oplus (T^*)^m$) then $\mathcal{C}$ intersects $T^m \setminus \{0\}$ (resp. $(T^*)^m$).*

Proof. Let $\mathcal{D}$ be the projection of $\mathcal{C}$ onto E^k. The previous lemma shows the existence of a vector v in E^k such that v and $-v$ are both contained in $\mathcal{D}$. From this follows the existence of a pair (t, t') of elements of T^m such that both $v + t$ and $-v + t'$ are contained in $\mathcal{C}$. Since $\mathcal{C}$ is convex and strict, $t + t'$ is contained in $\mathcal{C} \cap T^m \setminus \{0\}$. $\square$

Remark 6.10. If $\mathcal{C}$ is a strict, open, and Γ-invariant convex cone and $\mathcal{L}$ is a Γ-invariant lattice in $E^k \oplus T^m$ (resp. $(E^*)^k \oplus (T^*)^m$), the same proof implies that $\mathcal{C} \cap \mathcal{L}$ intersects T^m.

6.3.3. *Invariant cones in $Sym_2(E^*)$.* We now study invariant cones in the representation $\mathrm{Sym}_2(E^*)$. Let $Q_+ \subset \mathrm{Sym}_2(E^*)$ be the cone of positive definite quadratic forms. This convex cone is open, strict, and G-invariant.

Proposition 6.11. *If $\mathcal{C}$ is a Γ-invariant, strict, and open convex cone in $Sym_2(E^*)$, $\mathcal{C}$ coïncides with Q_+ or its opposite $-Q_+$. If $\mathcal{C}$ is a Γ-invariant, strict, and closed convex cone, then $\mathcal{C}$ is either $\{0\}$, $\overline{Q}_+$, or $-\overline{Q}_+$.*

Proof. Let $\mathcal{C} \subset \mathrm{Sym}_2(E^*)$ be a Γ-invariant, strict convex cone which is different from $\{0\}$. First of all, $\mathcal{C}$ spans $\mathrm{Sym}_2(E^*)$ because Γ is Zariski dense in G. In particular, the interior $\mathcal{C}'$ of $\mathcal{C}$ is not empty.

The projective space $\mathbb{P}(\mathrm{Sym}_2(E^*))$ has dimension 5. The group Γ acts on it by projective transformations, and its limit set coïncides with the surface $S \subset \mathbb{P}(\mathrm{Sym}_2(E^*))$ of projective classes of quadratic forms of rank 1. The projection $\mathbb{P}(\partial C)$ of the boundary of $\mathcal{C}$ onto the projective space $\mathbb{P}(\mathrm{Sym}_2(E^*))$ is a closed and Γ-invariant subset of $\mathbb{P}(\mathrm{Sym}_2(E^*))$. As such it must contain the limit set S.

The proposition is now a consequence of the following observation: If a strict and closed convex cone $\mathcal{D} \subset \mathrm{Sym}_2(E^*)$ contains S in its projective boundary, then $\mathcal{D}$ contain Q_+ or its opposite $-Q_+$. □

6.3.4. *The nef cone.*

Proposition 6.12. *The intersection of the nef cone $\overline{\mathcal{K}}(M)$ with the subspace $Sym_2(E^*)$ of W is equal to $\overline{Q_+}$.*

Proof. Let γ be a proximal element of Γ (see 2.3.3). Then, γ preserves a one dimensional eigenspace $R \subset \mathrm{Sym}_2(E^*) \subset W$, with the property that the eigenvalue λ of γ along R dominates all other eigenvalues of γ on $W \otimes_{\mathbf{R}} \mathbf{C}$ strictly. If $u+v+t$ is an element of $W = \mathrm{Sym}_2(E^*) \oplus E^k \oplus T^m$, then

$$u_+ := \lim_{n\to+\infty} \frac{1}{\lambda^n}\gamma^n(u+v+t)$$

is contained in R.

Apply this remark to a generic element $u+v+t$ of $\mathcal{K}(M)$. Since the Kähler cone is open, we may assume that the vector $u_+ \in R$ in the previous limit is different from 0. The Γ-invariance of $\mathcal{K}(M)$ implies that $R \cap \overline{\mathcal{K}}(M) \neq \{0\}$. From this follows that the intersection between $\overline{\mathcal{K}}(M)$ and $\mathrm{Sym}_2(E^*)$ is a non trivial, Γ-invariant, strict, and closed convex cone of $\mathrm{Sym}_2(E^*)$. It must therefore coïncide with Q_+ or its opposite. The conclusion follows from the normalization chosen in section 6.1.2 and the inequality

$$\int_M u\wedge u\wedge u = \varepsilon \det(u) < 0$$

if u is in $-Q_+$, which is not compatible with the existence of nef classes in Q_+.
□

7. Lattices in $\mathsf{SL}_3(\mathbf{R})$: Part II

We pursue the proof of theorem A under the same assumptions as in the previous section. While section 6 focussed on the cohomology of M, we now use complex algebraic geometry more deeply to conclude.

7.1. **Contraction of the trivial factor.**

7.1.1. *Ample, big and nef classes.* The following theorem describes the Kähler cone of any compact complex manifold M (see section 6.3.1 for definitions).

Theorem 7.1 (Demailly, Paun [18])**.** *The Kähler cone $\mathcal{K}(M)$ of a (connected) compact Kähler manifold M is a connected component of the cone $\mathcal{P}(M)$ of numerically positive classes.*

As a consequence, if the cohomology class α is on the boundary of $\mathcal{K}(M)$, there exists an analytic subvariety Y of M such that $\int_Y \alpha^{\dim(Y)} = 0$; in dimension 3, this leads to three cases:

(1) $\dim(Y) = 3$, i.e. $Y = M$, and $\alpha^3 = 0$;
(2) $\dim(Y) = 2$, Y is a surface and $\int_Y \alpha^2 = 0$;
(3) $\dim(Y) = 1$, Y is a curve and $\int_Y \alpha = 0$.

Another result that we shall use is the following characterisation of **big** line bundles. *Let L be a line bundle on M. If the first Chern class $c_1(L)$ is contained in the nef cone $\overline{\mathcal{K}}(M)$, and $\int_M c_1(L)^3 > 0$, then L is big, in the sense that*

$$\dim(H^0(M, L^{\otimes k}) > c^{ste}\, k^{\dim(M)}$$

for $k > 0$ (see [17], corollary 6.8, and [18]).

7.1.2. *Base locus of Q_+.* The cone Q_+ is open, and $H^2(M, \mathbf{Z})$ intersects $\mathrm{Sym}_2(E^*)$ on a lattice. As a consequence, the set $Q_+ \cap H^2(M, \mathbf{Z})$ spans $\mathrm{Sym}_2(E^*)$ as a real vector space. Let u be an element of $Q_+ \cap H^2(M, \mathbf{Z})$. Lefschetz theorem on $(1,1)$-classes implies that u is the first Chern class of a line bundle L on M (see [45], theorem 7.2, page 150). This line bundle is big and nef because Q_+ is contained in $\overline{\mathcal{K}}(M)$ and

$$\int_M u^3 = \varepsilon \det(u) > 0.$$

Let $\mathsf{Bs}(L)$ be the base locus of L. If we choose two line bundles L_1 and L_2 with first Chern class in Q_+, the base locus $\mathsf{Bs}(L_1 \otimes L_2)$ is contained in the intersection of $\mathsf{Bs}(L_1)$ and $\mathsf{Bs}(L_2)$. From this we deduce that

$$\mathsf{Bs}(Q_+) := \bigcap_{L, c_1(L) \in Q_+} \mathsf{Bs}(L)$$

is an analytic set, that coincides with $\mathsf{Bs}(\otimes_i L_i)$ if we take sufficiently many line bundles L_i with $c_1(L_i) \in Q_+$. The set $\mathsf{Bs}(Q_+)$ will be refered to as the base locus of Q_+.

Since Q_+ is Γ-invariant, this base locus is a Γ-invariant analytic subset of M. Theorem 3.5 implies the following proposition, because Γ has property (T).

Proposition 7.2. *The base locus of Q_+ is made of a fixed component, which is a disjoint union of projective planes, and of a finite number of points. Contracting all Γ-periodic surfaces, one gets a birational morphism $\pi : M \to M_0$ onto an orbifold M_0 such that the base locus of $\pi_* Q_+$ is reduced to a finite set.*

7.1.3. *Remarks concerning orbifolds.* Let M_0 be an orbifold of dimension 3; by definition, M_0 is locally isomorphic to $\mathbf{C}^3/G_i$ where G_i is a finite linear group. A Kähler form on M_0 is a $(1,1)$-form ω which, locally in a chart $U_i \subset M_0$ of type $\mathbf{C}^3/G_i$, lifts to a G_i-invariant Kähler form $\tilde{\omega}$ in $\mathbf{C}^3$. All other classical objects are defined in a similar way (see [11] for more details): $(1,1)$-forms, closed and exact classes, cohomology groups, etc. For example, Chern classes can be defined in terms of curvature of hermitian metrics. In a neighborhood of a singularity, locally of type $\mathbf{C}^3/G_i$ where G_i is a finite linear group, the metric is supposed to lift to a G_i-invariant smooth metric on $\mathbf{C}^3$; the curvature is then invariant, and the Chern classes are well-defined on M_0.

7.1.4. *Ampleness in $\pi_* Q_+$.* The morphism π defines two linear operators $\pi^* : H^2(M_0, \mathbf{R}) \to H^2(M, \mathbf{R})$ and $\pi_* : H_2(M, \mathbf{R}) \to H_2(M_0, \mathbf{R})$. The first one is defined by tacking pull back of forms; this is compatible with the operation of pull back for Cartier divisors and line bundles. The second one is defined by taking push forward of currents; this is compatible with the usual operator π_* for divisors. Since π has degree 1, we have $\pi_* \circ \pi^* = \mathrm{Id}$ on $H^2(M_0, \mathbf{R})$.

We now study $\pi_* Q_+$ and show that this subspace of $H^{1,1}(M_0, \mathbf{R})$ is indeed contained in the Kähler cone of M_0.

The morphism π being Γ-equivariant, the linear map π^* embeds the representation of G on $H^{1,1}(M_0, \mathbf{R})$ into a subrepresentation of W. On the other hand, π_* contracts a subfactor of the trivial part T^m (corresponding to Γ-invariant surfaces), so that $H^{1,1}(M_0,\mathbf{R})$ is isomorphic, via π^*, to a direct sum $\mathrm{Sym}_2(E^*)\oplus E^k\oplus T^n$, with $n\leq m$. We shall prove that $n=0$.

The proof of theorem 7.1 can be adapted to the orbifold case almost word by word (see [11] for similar ideas), so that *if V is a (connected) compact Kähler orbifold, the Kähler cone $\mathcal{K}(V)$ is a connected component of $\mathcal{P}(V)$.*

Corollary 7.3. *The set π_*Q_+ is contained in the Kähler cone of M_0. There are ample line bundles L on M_0 with $c_1(L)\in\pi_*Q_+$.*

Proof. To simplify notations, we shall denote the projection π_*Q_+ by $Q_{+,0}$, the space $H^{1,1}(M_0,\mathbf{R})$ by W_0 and the subspace $\pi_*\mathrm{Sym}_2(E^*)$ by $\mathrm{Sym}_2(E^*)_0$, and use similar notations for the dual vector space $W_0^*\subset H^{2,2}(M_0,\mathbf{R})$.

From proposition 6.12 we know that $Q_{+,0}$ is contained in the nef cone. Assume that $Q_{+,0}$ does not intersect the Kähler cone, so that $Q_{+,0}$ is in the boundary of $\mathcal{P}(M_0)$ (theorem 7.1). Let v_1, ..., v_6 be elements of $Q_{+,0}$ that span $\mathrm{Sym}_2(E^*)_0$ as a real vector space. The vectors $v_i\wedge v_j$, $1\leq i\leq j\leq 6$, span $\mathrm{Sym}_2(E)_0\subset W_0^*$. Let v be the sum of the v_i; v is in $\partial\mathcal{P}(M_0)$ and $v^3>0$. Theorem 7.1 provides an analytic subset $Y\subset M_0$ of pure dimension $\dim(Y)=1$ or 2, such that

$$\int_Y v^{\dim(Y)}=0.$$

Since all v_i are nef, this implies that $\int_Y v_i\wedge v_j=0$ for all (i,j) if $\dim(Y)=2$ (resp. $\int_Y v_i=0$, $\forall i$, if $\dim(Y)=1$). In particular, $\int_Y w=0$ for every w in W_0^* (resp. $\int_Y w=0$ for all w in W_0).

Fix Y as above, with dimension 2 (resp. 1), and let us decompose its cohomology class $[Y]\in W_0$ (resp. W_0^*) with respect to the direct sum $W_0=\mathrm{Sym}_2(E^*)\oplus E^k\oplus T^m$ (resp. to its dual):

$$[Y]=a+b+c.$$

Since

$$\int_Y w=0$$

for all w in $\mathrm{Sym}_2(E)_0$ (resp. $\mathrm{Sym}_2(E^*)_0$) we have $a=0$. In particular, there is an effective class $[Y]$ in $(E^*)^k\oplus(T^*)^n$ that does not intersect Q_+. Let $\mathcal{C}\subset(E^*)^k\oplus(T^*)^n$ (resp. $\subset E^k\oplus T^n$) be the cone

$$\mathcal{C}=\{w\in(E^*)^k\oplus(T^*)^n|\, w \text{ is effective and } \langle w|[Y]\rangle=0\}.$$

Since this cone is strict, lemma 6.9 and remark 6.10 imply the existence of an effective divisor (resp. curve) Z, with integer coefficients a_i,

$$Z = \sum a_i Y_i,$$

such that its homology class $[Z]$ is an element of the trivial factor $(T^*)^n$ (resp. T^n).

When the dimension of Z is 2, we know that $[Z]$ is contained in T^n, so that the cohomology class of Z is Γ-invariant, and corollary 3.11 shows that Z itself must be Γ-invariant. But there is no Γ-invariant surface since all of them have been contracted by π.

Assume now that $\dim(Z) = 1$. Let $Cov([Z])$ be the set covered by all curves Z' such that $[Z'] = [Z]$. Since $[Z]$ is an element of T^n, its class is Γ-invariant. The set $Cov([Z])$ is therefore a Γ-invariant analytic set and, as such, must be all of M_0. Let D_i, $1 \leq i \leq k$, be three divisors with cohomology classes in $Q_{+,0}$ such that the intersection

$$\bigcap_{1 \leq i \leq k} D_i$$

is made of a finite number of points; such divisors exist because the base locus of $Q_{+,0}$ is reduced to a finite number of points. Let Z' be a curve with $[Z'] = [Z]$ which goes through at least one point of $\cap_i D_i$. Let Z'_0 be an irreducible component of Z' containing that point. Since $[Z'_0].[D_i] = 0$ for all i, the curve Z'_0 must be contained in all of the D_i, a contradiction.

In both cases ($\dim(Z) = 1$ or 2) we get a contradiction. This implies that $\pi_* Q_+$ intersects the Kähler cone of the orbifold M_0, and proposition 6.11 shows that $\pi_*(Q_+)$ is contained in $\mathcal{K}(M_0)$. Since Q_+ intersects $H^2(M, \mathbf{Z})$, there are ample line bundles L on M_0 with $c_1(L)$ in $\pi_*(Q_+)$. $\square$

7.1.5. *Contraction of T^m.* We shall now prove that $n = 0$, i.e. that π_* contracts all classes t in T^m. This is a consequence of Hodge index theorem.

Corollary 7.4. *The cohomology classes of all Γ-periodic surfaces $S_i \subset M$ span T^m. Since all of them are contracted by $\pi : M \to M_0$, the trivial factor T^m is mapped to 0 by π_*, and there is no non-zero Γ-invariant cohomology class on M_0.*

Proof. Let u be a Kähler class in $\pi_* Q_+$. Then $u \wedge u$ is different from 0, and $E^k \oplus T^n$ is contained in $(u \wedge u)^\perp$. Since u is in the Kähler cone, Hodge index theorem implies that the quadratic form

$$v \mapsto -\int_M u \wedge v \wedge v$$

is positive definite on $(u \wedge u)^{\perp}$. But from section 6.1.1 we know that all elements $t \in T^m$ satisfy $\int_M u \wedge t \wedge t = 0$. All together, n is equal to 0. □

Remark 7.5. We could also apply the hard Lefschetz's theorem, as in the proof of lemma 6.4, to prove this corollary (see [48] for a proof of the hard Lefschetz theorem in the orbifold case).

7.2. **Triviality of Chern classes and tori.**

Let $c_1(M_0)$ and $c_2(M_0)$ be the Chern classes of the orbifold M_0. These classes are respectively of type $(1,1)$ and $(2,2)$, and are invariant under the action of Γ on the cohomology of M_0. From corollary 7.4, any Γ-invariant class is equal to 0, proving that *the Chern classes of the orbifold M_0 are equal to* 0. We now use the following result, the proof of which is described in [35], and is easily adapted to the orbifold case (see [11] for similar extensions of classical results to the orbifold setting).

Theorem 7.6. *Let X_0 be a connected Kähler orbifold with trivial first and second Chern classes. Then X_0 is covered by a torus: There is an unramified covering $\varepsilon : A \to X_0$ where A is a torus of dimension* $\dim(X_0)$.

Let us be more precise. In our case, M_0 is a three dimensional (connected) orbifold with trivial Chern classes $c_1(M_0)$ and $c_2(M_0)$. This implies that there is a flat Kähler metric on M_0 (see [35]). The universal cover of M_0 (in the orbifold sense) is then isomorphic to $\mathbf{C}^3$ and the (orbifold) fundamental group $\pi_1^{orb}(M_0)$ acts by affine isometries on $\mathbf{C}^3$ for the standard euclidean metric. In other words, $\pi_1^{orb}(M_0)$ is identified to a cristallographic group Δ of affine motions of $\mathbf{C}^3$. Let Δ^* be the group of translations contained in Δ. Bieberbach's theorem shows that (see [47], chapter 3, theorem 3.2.9).

a.- Δ^* is a lattice in $\mathbf{C}^3$;

b.- Δ^* is the unique maximal and normal free abelian subgroup of Δ of rank 6.

The torus A in the previous theorem is the quotient of $\mathbf{C}^3$ by this group of translations. By construction, A covers M_0, Let F be the quotient group Δ/Δ^*; we identify it to the group of deck transformations of the covering $\varepsilon : A \to M_0$.

Lemma 7.7.

(1) *A finite index subgroup of* Γ *lifts to* $\mathsf{Aut}(A)$.
(2) *Either* M_0 *is singular, or* M_0 *is a torus.*
(3) *If* M_0 *is singular, then* M_0 *is a quotient of the torus* A *by a homothety* $(x,y,z) \mapsto (\eta x, \eta y, \eta z)$, *where* η *is a root of* 1.

Proof. By property (b.) all automorphisms of M_0 lift to A. Let $\overline{\Gamma} \subset \mathsf{Aut}(A)$ be the group of automorphisms of A made of all possible lifts of elements of Γ. So, $\overline{\Gamma}$ is an extension of Γ by F:

$$1 \to F \to \overline{\Gamma} \to \Gamma \to 1.$$

Let $L : \mathsf{Aut}(A) \to \mathsf{GL}_3(\mathbf{C})$ be the morphism which applies any automorphism f of A to its linear part $L(f)$. Since A is obtained as the quotient of $\mathbf{C}^3$ by all translations of Δ, the restriction of L to F is injective. Let $N \subset \mathsf{GL}_3(\mathbf{C})$ be the normalizer of $L(F)$, and N^0 the connected component of the identity of N. The group $L(\overline{\Gamma})$ normalizes $L(F)$. Hence we have a well defined morphism $\overline{\Gamma} \to N$, and an induced morphism $\delta : \Gamma \to N/L(F)$. Changing Γ into a finite index subgroup, we may and do assume that δ is injective and $\delta(\Gamma)$ is contained in $N^0/L(F)$. Assume that $L(F)$ is not contained in the center of $\mathsf{GL}_3(\mathbf{C})$. Then there is an element g of F which is not an homothety; in particular, g has an eigenspace V of dimension 1 or 2. Since this space is N^0 invariant, N^0 embeds in the stabilizer of V. But V and $\mathbf{C}^3/V$ both have dimension at most 2. Hence, if H is a discrete group with Kazhdan property (T), all morphisms from H to N^0 (resp. to $N^0/L(F)$) have finite image (see §2.3.5). This contradicts the existence of $\delta : \Gamma \to N^0/L(F)$. As a consequence, $L(F)$ is contained in the center of $\mathsf{GL}_3(\mathbf{C})$.

Either F is trivial, and then M_0 coincides with the torus A, or F is a cyclic subgroup of $\mathbf{C}^*\mathrm{Id}$. In the first case, there is no need to lift Γ to $\mathsf{Aut}(A)$. In the second case, we fix a generator g of F, and denote by η the root of unity such that $L(g)$ is multiplication by η. The automorphism g has at least one (isolated) fixed point x_0 in A; this point is fixed by F. Changing Γ into a finite index subgroup Γ_1, we can also assume that $\overline{\Gamma_1}$ fixes x_0. The linear part L embeds $\overline{\Gamma_1}$ into $\mathsf{GL}_3(\mathbf{C})$. Selberg's lemma assures that a finite index subgroup of $\overline{\Gamma_1}$ has no torsion. This subgroup does not intersect F, hence projects bijectively onto a finite index subgroup of Γ_1. This proves that a finite index subgroup of Γ lifts to $\mathsf{Aut}(A)$. □

7.3. **Possible lattices and tori.**

We now show that Γ is commensurable to $\mathsf{SL}_3(\mathbf{Z})$ and that A is isogenous to a product $B\times B\times B$ where B is an elliptic curve. For this purpose, we shall use the following proposition, and refer to the next section for a proof.

Proposition 7.8. *Let Γ be a lattice in $\mathsf{SL}_3(\mathbf{R})$, which preserves a lattice $\mathcal{L}$ in $Sym_2(E^*)$. If there exists a real number ε such that $\mathsf{det} : Sym_2(E^*) \to \mathbf{R}$ maps $\mathcal{L}$ into $\varepsilon\mathbf{Q}$, then Γ is commensurable with $\mathsf{SL}_3(\mathbf{Z})$.*

This proposition, proposition 6.7 and lemma 6.3 show that Γ is commensurable with $\mathsf{SL}_3(\mathbf{Z})$. After a conjugacy in $\mathsf{SL}_3(\mathbf{R})$, we may assume that Γ intersects $\mathsf{SL}_3(\mathbf{Z})$ on a finite index subgroup. As a consequence, there exists a finite index subgroup F of $\mathsf{SL}_2(\mathbf{Z})$ such that

$$\begin{pmatrix} a & b & 0\\ c & d & 0\\ 0 & 0 & 1\end{pmatrix} \in \Gamma, \quad \forall \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in F.$$

The group F, viewed as a subgroup of Γ, acts on the subspace $\mathrm{Sym}_2(E^*)$ of $W = H^{1,1}(A,\mathbf{R})$, preserving the lattice $\mathcal{R} = \mathrm{Sym}_2(E^*)\cap H^2(A,\mathbf{Z})$. This lattice defines a $\mathbf{Q}$-structure on the real vector space $\mathrm{Sym}_2(E^*)$. Since F preserves $\mathcal{R}$, the intersection of all subspaces

$$V_1(\gamma) = \{v \in \mathrm{Sym}_2(E^*);\ \gamma(v) = v\}$$

where γ describes F is defined over $\mathbf{Q}$. This space has dimension one, and is generated by the cohomology class of a $(1,1)$-form of rank 1. This implies that there exists a line bundle L on A with a first Chern class $c_1(L)$ in $\mathrm{Sym}_2(E^*)$ that is invariant under the action of F. Moreover, this Chern class being proportional to a rank 1 hermitian form, L determines a morphism to an elliptic curve B:

$$\phi_L : A \to B.$$

Composing ϕ_L with elements in Γ which are not in F, we can construct two new projections $\Phi_L\circ\gamma_i : A \to B$ such that the holomorphic 1-forms $D\Phi_L$, $D\Phi_L\circ\gamma_1$, and $D\Phi_L\circ\gamma_2$ generate T^*A at every point. It follows that the map

$$\Phi_L \times (\Phi_L\circ\gamma_1)\times(\Phi_L\circ\gamma_2) : A \to B\times B\times B$$

is an isogeny. This proves that A is isogenous to a product $B\times B\times B$.

7.4. **Proof of proposition 7.8.**

We now prove proposition 7.8. By assumption, $\mathcal{L}$ is a lattice in $\mathrm{Sym}_2(E^*)$, and Γ is a lattice in $\mathsf{SL}_3(\mathbf{R})$ which preserves $\mathcal{L}$. The lattice $\mathcal{L}$ determines a

$\mathbf{Q}$-structure on the real vector space $\mathrm{Sym}_2(E^*)$, and Γ acts by matrices with integer coefficients with respect to this structure.

From the classification of lattices in $\mathsf{SL}_3(\mathbf{R})$, we know that Γ is commensurable to $\mathsf{SL}_3(\mathbf{Z})$, Γ has $\mathbf{Q}$-rank 1, or Γ is cocompact (see [39]). What we have to do, is to rule out the last two possibilities.

7.4.1. *Cocompact lattices.* Let us suppose that Γ is cocompact, and get a contradiction.

Let q_0 be an element of $\mathcal{L}$ which, as a quadratic form on E, has signature $(+,+,-)$. Then there exists a basis of E such that

$$q_0(x,y,z) = 2xy + z^2$$

(note that the value of ε may change after the choice of a new basis for E). The stabilizer H of q_0 in $\mathsf{SL}_3(\mathbf{R})$ is isomorphic to the group $\mathsf{SO}_{2,1}(\mathbf{R})$. The orbit of q_0 under $\mathsf{SL}_3(\mathbf{R})$ may be identified to $\mathsf{SL}_3(\mathbf{R})/H$. The orbit of q_0 under Γ is contained in $\mathcal{L}$, and is therefore discrete in $\mathsf{SL}_3(\mathbf{R})/H$. From this we deduce that the H-orbit of q_0 in $\Gamma\backslash\mathsf{SL}_3(\mathbf{R})$ is closed. Since $\Gamma\backslash\mathsf{SL}_3(\mathbf{R})$ is compact, the H-orbit must be compact too, so that $\Gamma\cap H$ is a cocompact lattice in H.

As a consequence, we can find an element γ in $\Gamma\cap H$ which is a proximal element of H; this means that γ has 3 distinct eigenvalues λ, $1/\lambda$, and 1, with $|\lambda| > 1$. After conjugation inside H, we can assume that the set of fixed points of the endomorphism

$$\gamma\colon \mathrm{Sym}_2(E^*) \to \mathrm{Sym}_2(E^*)$$

coincides with the plane

$$V_1 = \mathrm{Vect}\{q_1, q_2\}$$

where q_1 and q_2 are the quadratic forms $q_1 = z^2$ and $q_2 = 2xy$. Since γ preserves $\mathcal{L}$, the plane V_1 is defined over $\mathbf{Q}$ (with respect to the $\mathbf{Q}$-structure given by $\mathcal{L}$). In particular, $\mathcal{L}\cap V_1$ is a (cocompact) lattice in V_1.

Let $r_0 = 2\alpha xy + \beta z^2$ be an element of $\mathcal{L}\cap V_1$ which is not proportional to q_0, so that $\alpha \neq \beta$. Computing $\det(mq_0 + nr_0)$ we see that

$$(m+n\alpha)^2(m+n\beta) \in \varepsilon\mathbf{Q}$$

for all integers m and n. With $n = 0$, we see that ε is rational. With $m = 0$, we see that $\alpha^2\beta$ is a rational number. This implies that the affine function

$$(m,n) \mapsto \beta + 2\alpha m + (2\alpha\beta + \alpha^2)n$$

takes rational values for $(m,n)\in\mathbf{Z}^2$, $mn\neq 0$. As a consequence, both α and β are rational numbers.

From this we deduce the existence of positive integers a and b such that aq_1 and bq_2 are contained in $\mathcal{L}\cap V_1$. In particular, $\mathcal{L}$ contains a multiple of the rank 1 quadratic form q_1. The cone of quadratic forms of rank 1 is parametrized by the map

$$s:\left\{\begin{array}{rcl} E^*\setminus\{0\} & \to & \mathrm{Sym}_2(E^*) \\ f & \mapsto & s(f)=f^2 \end{array}\right.$$

This map is a 2-to-1 cover and, in particular, is proper. Since s is Γ-equivariant (i.e. $s(\gamma.f)=\gamma(s(f))$), the Γ-orbit of $s^{-1}(q_1)$ is a discrete subset of E^*. This contradicts the following theorem, due to Greenberg (see [1], appendix): *Let Γ be a cocompact lattice in $\mathsf{SL}_n(\mathbf{R})$ (resp. $\mathsf{SL}_n(\mathbf{C})$); if v if an element of $\mathbf{R}^n\setminus\{0\}$ (resp. $\mathbf{C}^n\setminus\{0\}$), the orbit Γv is a dense subset of $\mathbf{R}^n$ (resp $\mathbf{C}^n$).*

7.4.2. $\mathbf{Q}$-*rank*-1 *lattices.* Assume that the $\mathbf{Q}$-rank of Γ is equal to 1. In this case, Γ is obtained from the construction in chapter 7.E in the book [39]. In particular, after conjugacy, Γ contains matrices of type

$$\left(\begin{array}{ccc} 1 & a+b\sqrt{r} & -(a^2-rb^2)/2+c\sqrt{r} \\ 0 & 1 & -a+b\sqrt{r} \\ 0 & 0 & 1 \end{array}\right)$$

where r is a fixed square free integer, and a, b, and c are sufficiently divisible integers. The group Γ contains also the transpose of those matrices.

For the representation $\rho:\mathsf{SL}_3(\mathbf{R})\to\mathsf{GL}\left(\mathrm{Sym}_2(E^*)\right)$, we have

$$\mathsf{tr}(\rho(B^{-1}))=\mathsf{tr}(B^2)-\mathsf{tr}(B^{-1})$$

for all $B\in\mathsf{SL}_3(\mathbf{R})$. Apply this formula to $B={}^tAA'$ where A and $A'\in\Gamma$ are as above. A straightforward computation shows that the trace is not rational in general. This implies that the action of Γ on $\mathrm{Sym}_2(E^*)$ can not preserve a lattice $\mathcal{L}$.

8. Lattices in $\mathsf{SL}_3(\mathbf{C})$

We now study holomorphic actions of lattices in a real Lie group G which is locally isomorphic to $\mathsf{SL}_3(\mathbf{C})$. As before, Γ will be a lattice in G that acts faithfully on a compact Kähler threefold M, and we assume that the action of Γ on $W=H^{1,1}(M,\mathbf{R})$ extends to a non trivial linear representation

$$G\to\mathsf{GL}(W).$$

Our goal is to prove that

(1) the action of Γ on M is virtually of Kummer type (coming from an action of a finite index subgroup Γ_0 in Γ on a three dimensional compact torus A);
(2) Γ is commensurable to $\mathsf{SL}_3(O_d)$ where O_d is the ring of integers in a quadratic field $\mathbf{Q}(\sqrt{d})$ for some negative squarefree integer d;
(3) the torus A is isogenous to a product $B \times B \times B$ where B is the elliptic curve $\mathbf{C}/O_d$.

Since the proof follows the same lines as for lattices in $\mathsf{SL}_3(\mathbf{R})$, we just list the results that are required to adapt sections 6 and 7.

8.1. The linear representation on W.

From theorem 5.8, we can assume that there exist three integers $k \geq 0$, $l \geq 0$, and $m \geq 0$ such that the representation W of $\mathfrak{g} = \mathfrak{sl}_3(\mathbf{C})$ is isomorphic to

$$W = \mathsf{Her}\,(E_{\mathbf{C}}^*)) \oplus E_{\mathbf{C}}^k \oplus \overline{E_{\mathbf{C}}}^l \oplus T^m$$

after composition by an automorphism of $\mathfrak{g}$.

8.2. The cup product and the integer structure.

As in section 6.1.1, one then proves that the cup product satisfies properties (1) to (5) (§6.1.1) where $\mathrm{Sym}_2(E^*)$ must be replaced by $\mathsf{Her}\,(E_{\mathbf{C}}^*)$ and E^k by $E_{\mathbf{C}}^k \oplus E_{\mathbf{C}}^l$.

Once again, the lattice $H^2(M,\mathbf{Z})$ intersects the irreducible factor $\mathsf{Her}\,(E_{\mathbf{C}}^*)$ on a (cocompact) lattice.

8.3. Invariant cones.

The cone $Q_+ \subset \mathrm{Sym}_2(E^*)$ must now be replaced by the cone $H_+ \subset \mathsf{Her}\,(E_{\mathbf{C}}^*)$ of positive definite hermitian forms. This cone admits the following characterization.

Lemma 8.1. *If $\mathcal{C}$ is a Γ-invariant, strict, and open convex cone in $\mathsf{Her}\,(E_{\mathbf{C}}^*)$, $\mathcal{C}$ coïncides with H_+ or its opposite $-H_+$. If $\mathcal{C}$ is a Γ-invariant, strict, and closed convex cone, then $\mathcal{C}$ is either $\{0\}$, $\overline{H}_+$, or $-\overline{H}_+$.*

Sketch of the proof. Every orbit of the lattice Γ in $\mathbb{P}(E)$ is dense, so that every orbit of Γ on the cone of rank one positive hermitian forms $\xi \otimes \overline{\xi}$ is dense. As a consequence, the projectivized boundary of any strict invariant cone $\mathcal{C}$ must contain the set of rank one hermitian forms. Since the cone of rank one positive hermitian forms contains H_+ in its convex hull, the lemma follows. □

Similarly, one shows that there is no invariant convex cone $\mathcal{C}$ in $E_{\mathbf{C}}$ (resp. $\overline{E_{\mathbf{C}}}$) except the trivial ones $\{0\}$ and $E_{\mathbf{C}}$ (resp. $\overline{E_{\mathbf{C}}}$). This implies that any non trivial, strict, closed, and Γ-invariant convex cone $\mathcal{C}$ in $E_{\mathbf{C}}^k \oplus E_{\mathbf{C}}^l \oplus T^m$ intersects $T^m \setminus \{0\}$.

8.4. **Conclusion.**

The strategy used in section 7 can now be applied word by word. It shows that the action $\Gamma \times M \to M$ is virtually of Kummer type. Changing Γ in one of its finite index subgroups, this action comes from a linear action of Γ on a compact complex torus A.

Proposition 8.2. *Let Γ be a lattice in $\mathsf{SL}_n(\mathbf{C})$. If there exists a Γ-invariant lattice $\mathcal{L}$ in $\mathbf{C}^n$ then Γ is commensurable to $\mathsf{SL}_n(O_d)$ for some square free negative integer d.*

Proof. Let G be the group $\mathsf{SL}_n(\mathbf{C})$. This is an algebraic subgroup of $\mathsf{SL}_{2n}(\mathbf{R})$ acting linearly on $\mathbf{R}^{2n} = \mathbf{C}^n$ and preserving the complex structure. By assumption, the lattice Γ in G preserves a lattice $\mathcal{L} \subset \mathbf{R}^{2n}$.

Let $(e_1, ..., e_n)$ be a family of n linearly independant vectors $e_i \in \mathcal{L}$ such that

$$P := \mathrm{Vect}_{\mathbf{R}}(e_1, ..., e_n)$$

is a totally real subspace in $\mathbf{C}^n$, i.e. $P \oplus \sqrt{-1}P = \mathbf{C}^n$. Then $(e_1, ..., e_n)$ is a basis of $\mathbf{C}^n$ as a complex vector space. Changing $\mathcal{L}$ into one of its finite index lattices, we may assume that e_1, e_2, ..., and e_n are the first elements in a basis $(e_1, ..., e_{2n})$ of $\mathcal{L}$ as a $\mathbf{Z}$-module. In this new basis (i.e. after conjugation by an element B in $\mathsf{GL}_{2n}(\mathbf{R})$), $\mathcal{L}$ is the standard lattice $\mathbf{Z}^{2n}$, Γ is a subgroup of $\mathsf{SL}_{2n}(\mathbf{Z})$, and since G contains Γ as a lattice, G is defined over $\mathbf{Q}$. Borel-Harish-Chandra theorem (see section 2.3.1) shows that Γ has finite index in the lattice $G \cap \mathsf{SL}_{2n}(\mathbf{Z})$.

In this new basis, the n-plane P is obviously defined over $\mathbf{Q}$, and Borel-Harish Chandra theorem shows that $SL_{2n}(\mathbf{Z})$ intersects the stabilizer of P in G onto a lattice Γ'. Let us use $(e_1, ..., e_n)$ as a complex basis of $\mathbf{C}^n$. The stabilizer of P coincides with $\mathsf{SL}_n(\mathbf{R}) \subset \mathsf{SL}_n(\mathbf{C})$, and Γ' has finite index in $\mathsf{SL}_n(\mathbf{Z}) \subset \mathsf{SL}_n(\mathbf{R})$.

Let π be the projection of $\mathbf{R}^{2n}$ onto $\sqrt{-1}P$ parallel to P. The projection of $\mathcal{L}$ is then a lattice $\pi(\mathcal{L})$ in $\sqrt{-1}P$. The group Γ', viewed as a subgroup of $\mathsf{SL}_n(\mathbf{C})$ in the basis $(e_1, ..., e_n)$, is given by matrices with integer coefficients. As such, it preserves both P and $\sqrt{-1}P$. On the real vector space $\sqrt{-1}P$, we use the basis $(\sqrt{-1}e_1, ..., \sqrt{-1}e_n)$, and choose a matrix $M \in \mathsf{GL}(\sqrt{-1}P) = \mathsf{GL}_n(\mathbf{R})$ such that

$$\pi(L) = M(\mathbf{Z}\sqrt{-1}e_1 \oplus ... \oplus \mathbf{Z}\sqrt{-1}e_n).$$

Since Γ' preserves $\pi(\mathcal{L})$, both $M\Gamma' M^{-1}$ and Γ' are finite index subgroups of $\mathsf{SL}_n(\mathbf{Z})$. This implies that $\pi(\mathcal{L})$ is, up to finite indices, equal to $\sqrt{-1}\delta(\mathbf{Z}e_1 \oplus ... \oplus \mathbf{Z}e_n)$ for some positive real number δ. As a consequence, up to finite indices, we may assume that $\mathcal{L}$ coincides with the subgroup

$$\mathbf{Z}^n \oplus \sqrt{-1}\delta\mathbf{Z}^n$$

in the basis $(e_1, ..., e_n)$ of $\mathbf{C}^n$.

Let Γ_1 be the finite index subgroup of Γ that preserves $\mathbf{Z}^n \oplus \sqrt{-1}\delta\mathbf{Z}^n$. Let $A + \sqrt{-1}B$ be an element of Γ_1 where A and B are $n \times n$ real matrices. Then

$$Ax - \delta By \in \mathbf{Z}^n \text{ and } \delta Ay + Bx \in \delta\mathbf{Z}^n \text{ for all } (x, y) \in \mathbf{Z}^n \times \mathbf{Z}^n.$$

Hence, A is an element of $\mathsf{Mat}_n(\mathbf{Z})$, B is an element of $\delta\mathsf{Mat}_n(\mathbf{Z})$, and δ^2 is a positive integer. Let d be the negative integer $-\delta^2$. Then $\mathcal{L}$ coincides with $\mathbf{Z}^n \oplus \sqrt{d}\mathbf{Z}^n$ and Γ with $\mathsf{SL}_3(O_d)$, up to finite indices. $\square$

We apply this proposition to our context. Since we can change Γ into finite index subgroups and A into isogenous tori, we may assume that A is the quotient of $\mathbf{C}^3$ by the lattice $\mathbf{Z}^3 \oplus \sqrt{d}\mathbf{Z}^3$, and Γ has finite index in the lattice $\mathsf{SL}_3(O_d)$. In particular, A is (isogenous to) $B \times B \times B$ where $B = \mathbf{C}/O_d$. This proves theorem A when $G = \mathsf{SL}_3(\mathbf{C})$, which was the last remaining case.

9. Complements

9.1. **Kummer examples.**

Let A be a torus of dimension 3 with a faithful action of an irreducible higher rank lattice Γ. We proved that A is isogenous to a product $B \times B \times B$ and that Γ contains a subgroup Γ_0 which is commensurable to $\mathsf{SL}_3(\mathbf{Z})$ (note that $\mathsf{SL}_3(O_d)$ contains $\mathsf{SL}_3(\mathbf{Z})$).

Let now F be a finite subgroup of $\mathsf{Aut}(A)$. If the orbits of F are permuted by the action of Γ, then Γ normalizes F: $\gamma F \gamma^{-1} = F$ for all γ in Γ. Changing Γ_0 into a finite index subgroup, we can and do assume that Γ_0 commutes with all elements of F. Let F_0 be the group of translations contained in F:

$$F_0 = \mathsf{Aut}(A)^0 \cap F.$$

This group is normal, and commutes to Γ_0. Changing A into the torus $A' = A/F_0$, we can assume that F_0 is trivial or, equivalently, that the morphism

$$f \mapsto L(f)$$

which maps an automorphism onto its linear part, is injective. Under these assumption, we proved in lemma 7.7 that F acts as a finite cyclic group of homotheties: F is generated by $(x,y,z) \mapsto (\eta x, \eta y, \eta z)$ where η is a root of 1 (for this we put the origin of A at a fixed point of F). Multiplication by η must preserve a lattice of type $\Lambda \times \Lambda \times \Lambda$ where $B = \mathbf{C}/\Lambda$ is an elliptic curve, and thus preserves a lattice $\Lambda \subset \mathbf{C}$. As a consequence, η is equal to -1, $i = \sqrt{-1}$, $e^{i\pi/3}$, or to one of their powers (to prove it, remark that multiplication by η is a finite order element of $\mathsf{SL}(\Lambda) \simeq \mathsf{SL}_2(\mathbf{Z})$).

Proposition 9.1. *Let M_0 be a Kummer orbifold A/F where A is a torus of dimension* 3 *and F is a finite subgroup of* $\mathsf{Aut}(A)$. *Assume that*

(i) *M_0 is not a torus, and*
(ii) *F is normalized by an irreducible higher rank lattice $\Gamma \subset \mathsf{Aut}(A)$.*

Then M_0 is isomorphic to a quotient A'/F' where A' is a torus and F' is a cyclic subgroup of $\mathsf{Aut}(A')$ *generated by a homothety $f(x,y,z) = (\eta x, \eta y, \eta z)$, where η is a root of* 1 *of order* 1, 2, 3, 4 *or* 6.

9.2. **Volume forms.**

Let us start with an example. Let M_0 be the Kummer orbifold A/F where A is $(\mathbf{C}/\mathbf{Z}[i])^3$ and F is the finite cyclic group generated by

$$f(x,y,z) = (ix, iy, iz),$$

where $i = \sqrt{-1}$. Let M be the smooth manifold obtained by blowing up the singular points of M_0. Let

$$\Omega = dx \wedge dy \wedge dz$$

be the standard volume form on A. Then $f^*\Omega = i\Omega$ and Ω^4 is f-invariant (Ω^4 may be viewed as a section of $K_A^{\otimes 4}$, where $K_A = \det(T^*A)$ is the canonical bundle of A). In order to resolve the singularities of M_0, one can proceed as follows. First one blows up all fixed points of elements in $F \setminus \{\mathrm{Id}\}$. For example, one needs to blow up the origin $(0,0,0)$. This provides a compact Kähler manifold $\hat{A}$ together with a birational morphism $\alpha : \hat{A} \to A$. The automorphism f lifts to an automorphism $\hat{f}$ of $\hat{A}$; since the differential Df is a homothety,

f acts trivially on each exceptional divisor, and acts as $z \mapsto iz$ in the normal direction. As a consequence, the quotient $\hat{A}/\hat{F}$ is smooth.

Denote by $E \subset \hat{A}$ the exceptional divisor corresponding to the blowing up of the origin, and fix local coordinates $(\hat{x}, \hat{y}, \hat{z})$ in $\hat{A}$ such that the local equation of E is $\hat{z} = 0$. In these coordintates, the form $\alpha^*\Omega$ is locally given by

$$\alpha^*\Omega = \hat{z}^2 \hat{x} \wedge \hat{y} \wedge \hat{z}.$$

The projection $\varepsilon : \hat{A} \to M = \hat{A}/\hat{F}$ is given by $(\hat{x}, \hat{y}, \hat{z}) \mapsto (u, v, w) = (\hat{x}, \hat{y}, \hat{z}^4)$, and the projection of $\alpha^*\Omega$ on M is

$$\varepsilon_* \alpha^* \Omega = \frac{1}{4w^{1/4}} du \wedge dv \wedge dw.$$

This form is locally integrable, and its fourth power is a well defined meromorphic section of $K_M^{\otimes 4}$.

A similar study can be made for all Kummer examples. More precisely, a local computation shows that, after one blow up, the volume form $\varepsilon_*\alpha^*\Omega$ on the quotient $\hat{\mathbf{C}}^3/\eta$

- vanishes along the exceptional divisor $\varepsilon(E)$ with multiplicity $1/2$, if $\eta = -1$,
- has a pole of type $1/w^{1/4}$ if η has order 4,
- is smooth and does not vanish if η has order 3,
- has a pole of type $1/w^{1/2}$ if η has order 6.

As a consequence, *the real volume form* $\varepsilon_*\alpha^*(\Omega \wedge \overline{\Omega})$ *on* M *is integrable and* Γ*-invariant* (this form is not smooth if η is not in $\{1, -1, e^{2i\pi/3}, e^{2i\pi/3}\}$).

Corollary 9.2. *Let M be a compact Kähler manifold of dimension 3. Let Γ be a lattice in a simple Lie group G with $\mathsf{rk}_{\mathbf{R}}(G) \geq 2$. If Γ acts faithfully on M, then the action of Γ on M*

- *virtually extends to an action of G, or*
- *preserves an integrable volume form μ which is locally smooth or the product of a smooth volume form by $|w|^{-1/N}$, where w is a local coordinate and $N = 1$ or 2.*

Proof. If the action of Γ on the cohomology of M factors through a finite group, then Γ is virtually contained in $\mathsf{Aut}(M)^0$ and two cases may occur. In the first case, the morphism $\Gamma \to \mathsf{Aut}(M)$ virtually extends to a morphism $G \to \mathsf{Aut}(M)$. In the second case, Γ is virtually contained in a compact subgroup of $\mathsf{Aut}(M)^0$, and then Γ preserves a Kähler metric. In particular, it preserves a smooth volume form.

If the action of Γ on the cohomology is almost faithful, then it is a Kummer example, and the result follows from what has just been said. $\square$

9.3. **Calabi-Yau examples.**

Let M be a compact Kähler manifold of dimension 3. By definition, M is a (irreducible) Calabi-Yau manifold if its fundamental group is finite and its first Chern class is trivial. Another definition, which is not equivalent to the previous one, requires a trivial first Chern class and a trivial first Betti number. The difference between the two definitions comes from the existence of smooth quotients of tori A/F with trivial first Betti number (the fundamental group has a finite abelianization, see [41]). Both definitions work for the following corollary.

Corollary 9.3. *Let M be a Calabi-Yau manifold of dimension* 3. *If* $\mathsf{Aut}(M)$ *contains an irreducible higher rank lattice, then M is birational to the quotient of* $(\mathbf{C}/\mathbf{Z}[j])^3$ *by multiplication by j, where $j = e^{2i\pi/3}$.*

Proof. Let M be a Calabi-Yau manifold of dimension 3 such that $\mathsf{Aut}(M)$ contains an irreducible higher rank lattice Γ. Since $\mathsf{Aut}(M)$ is discrete, we can and do assume that Γ acts faithfully on the cohomology of M. Contracting all Γ periodic surfaces as in section 3.4, theorem A shows that we get a birational morphism $\pi : M \to M_0$ onto a Kummer orbifold. In particular, there exists a torus A and a finite subgroup F of $\mathsf{Aut}(A)$ such that M_0 is isomorphic to the quotient A/F. Sections 9.1 and 9.2 show that we can write M_0 as A'/F' where F' acts on the torus A' by multiplication by η, with $\eta \in \{-1, i, e^{2i\pi/3}, -e^{2i\pi/3}\}$. The unique case which leads to a Calabi-Yau manifold is $\eta = e^{2i\pi/3}$. $\square$

9.4. **Infinite center and compact factors.**

In this section, our goal is to remove the hypothesis concerning the center (resp. the compact factors) of G in theorem A and theorem B. We first start with the center.

9.4.1. *Infinite center.* Let G be a connected semi-simple Lie group without non trivial compact factor. Let Z be its center, $G' = G/Z$ the quotient, and $\pi : G \to G'$ the natural projection. Let Γ be a lattice in G and $\Gamma' = \pi(\Gamma)$ its projection. The following fact is well known but is hard to localize. We present a proof using Borel density theorem.

Lemma 9.4. *The group Γ' is a lattice in G' and $\Gamma \cap Z$ has finite index in Z.*

Proof. Let 1 denote the neutral element in G and $1' = \pi(1)$. Let B be a neighborhood of 1 in G such that $\Gamma \cap B = \{1\}$ and π defines a diffeomorphism from B to its image B'. Assume that Γ' is not discrete; then there exists a sequence (γ'_n) of pairwise distinct elements of $\Gamma' \cap B'$ which converges toward $1'$. After extraction of a subsequence, we can assume that $\gamma'_n/\mathsf{dist}(\gamma'_n, 1')$ converges toward an element $v \neq 0$ of the Lie algebra $\mathfrak{g}$. Let γ_n be elements of Γ such that $\pi(\gamma_n) = \gamma'_n$. Write $\gamma_n = z_n \varepsilon_n$ with z_n in Z and ε_n in B. Let β be an element of Γ. If n is large enough,

$$[\beta, \gamma_n] = [\beta, \varepsilon_n] \in \Gamma \cap B$$

and therefore $[\beta, \gamma_n] = 1$. Thus, v is invariant under the adjoint representation $ad : \Gamma \to \mathsf{GL}(\mathfrak{g})$. Borel density theorem (see theorem 5.5 of [42]) implies that v is invariant under the adjoint representation of G'. Since G' is semi-simple, we obtain a contradiction with $v \neq 0$, proving that Γ' is indeed discrete.

Let $\tilde{\Gamma}$ be the group $\pi^{-1}(\Gamma')$. Since π is a covering, $\tilde{\Gamma}$ is a discrete subgroup of G containing both Γ and Z. Since Γ is a lattice in G, $\tilde{\Gamma}$ is also a lattice and Γ has finite index in $\tilde{\Gamma}$. In particular, $\Gamma \cap Z$ has finite index in Z. Moreover, $G'/\Gamma' = G/\tilde{\Gamma}$, so that Γ' is a lattice in G'. □

Let now $\rho : \Gamma \to \mathsf{GL}(V)$ be a finite dimensional linear representation of Γ. Let L be the Zariski closure of $\rho(\Gamma)$, A be the center of L, and $\pi_A : L \to L/A$ the natural projection.

Lemma 9.5. *If the image of* ρ *is discrete, its natural projection in* L/A *is also discrete.*

The proof is along the same lines as the previous one. For the sake of simplicity, we slightly change it, using a finite generating set $\{\beta_i, 1 \leq i \leq l\}$ for Γ.

Proof. Let B be a neighborhood of 1 in L such that $\Gamma \cap B = \{1\}$. Let $U \subset B$ be a neighborhood of 1 such that $[\rho(\beta_i), U] \subset B$ for all $1 \leq i \leq l$. Let $U' = \pi_A(U)$. Let γ be an element of Γ such that $\pi_A(\rho(\gamma))$ is contained in U'. Write $\rho(\gamma) = a\varepsilon$ where $a \in A$ and $\varepsilon \in U$. Since a is in the center of L, we see that $[\rho(\beta_i), \rho(\gamma)]$ is contained in $B \cap \Gamma$, and is thus equal to 1, for all l generators β_i. This implies that $\rho(\gamma)$ is in the center of $\rho(\Gamma)$. Since L is the Zariski closure of $\rho(\Gamma)$, $\rho(\gamma)$ is in the center A of L. As a consequence, $\pi_A(\rho(\Gamma))$ intersects U' trivially, proving the lemma. □

Assume now that the rank of G is at least 2. Margulis's arithmeticity theorem implies that Γ' is an arihtmetic lattice in G'. We can then apply a result due to Millson, Deligne and Raghunathan (see [38], remark 6.18 (A), page 333), according to which any subgroup Λ of G with $\pi(\Lambda) = \Gamma'$ is a lattice in G. Since $[\Gamma', \Gamma']$ has finite index in Γ', this implies that $[\Gamma, \Gamma]$ has finite index in Γ.

Let now $\rho : \Gamma \to \mathsf{Aut}(M)$ be a morphism into the group of automorphisms of a compact Kähler manifold M. Let ρ^* be the morphism given by the action on the cohomology $H^*(M, \mathbf{Z})$. Let $V = H^*(M, \mathbf{Z}) \otimes \mathbf{R}$. The image of ρ^* is a discrete subgroup of $\mathsf{GL}(V)$. With the same notations as above, the morphism $\pi_A \circ \rho^* : \Gamma \to L/A$ is trivial on $\Gamma \cap Z$ because $\rho^*(Z \cap \Gamma)$ is contained in A. As a consequence, this morphism factors through Γ'. Since its image is discrete, it extends virtually to a morphism of Lie groups $G' \to L/A$ with Zariski dense image (cf. theorem 2.4). As a consequence, L/A is finite or locally isomorphic to G, because G is semi-simple.

If L/A is finite, the image of ρ^* is virtually abelian, thus finite because $[\Gamma, \Gamma]$ has finite index in Γ. In other words, a finite index subgroup of Γ acts trivially on the cohomology of M. Lieberman-Fujiki's theorem implies that $\rho(\Gamma)$ is virtually contained in $\mathsf{Aut}(M)^0$.

If L/A is locally isomorphic to G, then L is locally isomorphic to $G \times A$ because $A \to L \to L/A$ is a central extension. Since $[\Gamma, \Gamma]$ has finite index in Γ, A is a finite group and $\rho^*(Z \cap \Gamma)$ is finite too. In other words, ρ^* factors virtually to an almost faithful representation of Γ'. Section 5 implies that G' is locally isomorphic to $\mathsf{SL}_3(\mathbf{R})$ or $\mathsf{SL}_3(\mathbf{C})$. In particular, the center of G is finite, and Γ is commensurable to Γ'.

This proves that theorem A holds even if the center of G is infinite. Theorem B follows from theorem A, section 3, and classical algebraic geometry. Section 3 needs property (T) for Γ, and this property holds for lattices in Lie groups G as soon as $\mathsf{rk}_{\mathbf{R}}(\mathfrak{g}) \geq 2$ and $\mathfrak{g}$ is simple (the center of G can be infinite, see [3]). As a consequence, theorem B can be generalized as follows.

Theorem B'. *Let G be a connected real Lie group with a simple, higher rank Lie algebra $\mathfrak{g}$. Let Γ be a lattice in G. Let M be a connected compact Kähler manifold of dimension 3. If there is a morphism $\rho : \Gamma \to \mathsf{Aut}(M)$ with infinite image, then M has a birational morphism onto a Kummer orbifold, or M is isomorphic to one of the following*

(1) *a projective bundle $\mathbb{P}(E)$ for some rank 2 vector bundle $E \to \mathbb{P}^2(\mathbf{C})$,*
(2) *a principal torus bundle over $\mathbb{P}^2(\mathbf{C})$,*
(3) *a product $\mathbb{P}^2(\mathbf{C}) \times B$ of the plane by a curve of genus $g(B) \geq 2$,*

(4) *the projective space* $\mathbb{P}^3(\mathbf{C})$.

In all cases, $\mathfrak{g}$ *is isomorphic to* $\mathfrak{sl}_n(\mathbf{K})$ *with* $n = 3$ *or* 4 *and* $\mathbf{K} = \mathbf{R}$ *or* $\mathbf{C}$.

9.4.2. *Compact factors.* Let now G be a higher rank semi-simple Lie group. Let K be a maximal, connected, normal and compact subgroup of G. Let $\overline{G}$ be the quotient G/K and $\pi : G \to \overline{G}$ be the natural projection. Let Γ be a lattice in G. Since Γ is discrete, $\Gamma \cap K$ is finite and $\overline{\Gamma} = \pi(\Gamma)$ is a lattice in $\overline{G}$. Let $\rho : \Gamma \to \mathsf{Aut}(M)$ be a morphism from Γ to the group of automorphisms of a connected compact Kähler manifold M. Let $\rho^* : \Gamma \to \mathsf{GL}(H^*(M,\mathbf{Z}))$ be the action on the cohomology of M. From Selberg's lemma, there is a finite index subgroup Γ_1 of Γ such that $\rho^*(\Gamma_1)$ is torsion free. Changing Γ into Γ_1, we have $\rho^*(K \cap \Gamma) = \{1\}$. In other words, ρ^* factors through $\overline{\Gamma}$. Sections 5 and 9.4.1 imply that

- the image of ρ^* is finite, or
- $\overline{G}$ is locally isomorphic to $\mathsf{SL}_3(\mathbf{R})$ or $\mathsf{SL}_3(\mathbf{C})$ and the action of $\overline{\Gamma}$ on $H^*(M,\mathbf{R})$ extends virtually to a non trivial representation of $\overline{G}$.

In the first case, the image of ρ is virtually contained in $\mathsf{Aut}(M)^0$. In the second case, there is a section s of $\pi : \Gamma \to \overline{\Gamma}$ over a finite index subgroup $\overline{\Gamma}_1$ of $\overline{\Gamma}$. Changing Γ into $s(\overline{\Gamma}_1)$, we can then apply theorem A. The final form of theorem A can now be stated as follows.

Theorem A'. *Let G be a connected semi-simple real Lie group. Let K be the maximal compact, connected, and normal subgroup of G. Let Γ be an irreducible lattice in G. Let M be a connected compact Kähler manifold of dimension* 3, *and* $\rho : \Gamma \to \mathsf{Aut}(M)$ *be a morphism. If the real rank of G is at least* 2, *then one of the following holds*

- *the image of* ρ *is virtually contained in the connected component of the identity* $\mathsf{Aut}(M)^0$, *or*
- *the morphism* ρ *is virtually a Kummer example.*

In the second case, G/K is locally isomorphic to $\mathsf{SL}_3(\mathbf{R})$ *or* $\mathsf{SL}_3(\mathbf{C})$ *and* Γ *is commensurable to* $\mathsf{SL}_3(\mathbf{Z})$ *or* $\mathsf{SL}_3(O_d)$, *where O_d is the ring of integers in an imaginary quadratic number field* $\mathbf{Q}(\sqrt{d})$ *for some negative integer d.*

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