DYNAMICAL DEGREES, ENTROPY, AND FATOU SETS

SERGE CANTAT

ABSTRACT. In 2002, the Journal für die reine und angewandte Mathematik published an article of C. T. McMullen on the dynamics of holomorphic diffeomorphisms of K3 surfaces, nine years later, a second one followed; both of them construct automorphisms of K3 surfaces with interesting dynamical features. The following pages present some of the main developments around the themes of research initiated in these two publications.

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INTRODUCTION

Our primary concern will be holomorphic diffeomorphisms $f\colon X \to X$ of compact, complex manifolds, in particular of complex projective surfaces. When the manifold X is projective, Chow's theorem implies that f is an algebraic transformation of X. Thus, for simplicity, holomorphic diffeomorphisms will be called *automorphisms* and holomorphic transformations will be called *endomorphisms*.

Let $X \subset \mathbb{P}^3_{\mathbf{C}}$ be a smooth projective surface defined by a homogeneous equation of degree d. When $d = 1$, X is a plane $\mathbb{P}^2_{\mathbf{C}}$ and its automorphism group $\mathsf{Aut}(X)$ is the group of linear projective transformations $\mathsf{PGL}_3(\mathbf{C})$. When $d = 2$, X is isomorphic to $\mathbb{P}^1_{\mathbf{C}} \times \mathbb{P}^1_{\mathbf{C}}$ and $\mathsf{Aut}(X)$ is the semi-direct product of $\mathsf{PGL}_2(\mathbf{C}) \times \mathsf{PGL}_2(\mathbf{C})$ and the involution $(x,y) \mapsto (y,x)$ permuting the two factors of $\mathbb{P}^1_{\mathbf{C}} \times \mathbb{P}^1_{\mathbf{C}}$. When $d = 3$ or $d \geq 5$, $\mathsf{Aut}(X)$ is finite. Thus, if $d \neq 4$, (a) $\mathsf{Aut}(X)$ is a complex algebraic group (with only finitely many components) and (b) the dynamics of any automorphism $f\colon X \to X$ is well understood. The situation changes drastically for some specific quartic surfaces, for instance for the Fermat quartic defined in homogeneous coordinates by the equation $x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0$. Another example is mentionned by A. Weil in [58]: "*Early on*, he wrote, *I was intrigued by Severi's example, by means of a quartic in* $\mathbb{P}^3$, *of a surface with an infinite group of automorphisms related to the group of units of a real quadratic field; for a while, I even hoped to find a way there of generating abelian extensions of this field.*" To this, as we shall see, one can now add that the automorphisms of Segre's quartic surface have a rich and chaotic dynamics, a fact already suggested by Y. Manin in [44] when he studied algebraic transformations of cubic surfaces.

This quote from Weil comes from his commentary on his 1958 presentation of K3 surfaces. A *K3 surface* is a compact, complex surface X with first Betti number $b_1(X) = 0$ and with a non-vanishing holomorphic 2-form Ω_X (see [3]). Such a 2-form can be chosen to satisfy $\int_X \Omega_X \wedge \overline{\Omega}_X = 1$ and is then unique up to multiplication by a complex number of modulus 1. K3 surfaces are simply connected. They form a continuous family of surfaces and, in particular, are all C^∞-diffeomorphic to the same manifold. The first examples are smooth quartic surfaces in $\mathbb{P}^3$, double covers of $\mathbb{P}^2$ ramified along a smooth sextic curve, and smooth surfaces of degree $(2,2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Another example is given by Kummer's construction. One starts with a torus $\mathbf{C}^2/\Lambda$ and takes the quotient by the involution $\eta(x,y) = (-x,-y)$; the quotient is a singular complex analytic manifold, but its minimal desingularization is a K3 surface (it is projective if, and only if the torus is projective, which depends on Λ). As this example shows, K3 surfaces are not all projective. But all are Kähler.

There is a classification of compact complex surfaces by Enriques and Kodaira, and of projective surfaces over algebraically closed field of arbitrary characteristic by Bombieri and Mumford: K3 surfaces occupy a central position in both classifications.

Understanding the dynamical features of automorphisms of K3 surfaces is at the heart of the article of McMullen published in 2002 in the Journal für die reine und angewandte Mathematik, and of its sibling published in the same journal in 2011 (see [46, 48]). Instead of focusing on their precise content, I will survey two of the main questions addressed by McMullen: which values can be taken by the topological entropy of automorphisms of complex surfaces, or by dynamical degrees of birational maps? Can one find examples of intermingled dynamical features that is, surfaces in which regions with chaotic dynamics and regions with tame dynamics coexist?

1. From topological entropy to dynamical degrees

1.1. **Topological entropy.** Consider a continuous transformation $f\colon X \to X$ of a compact metric space X. The orbit of a point $x \in X$ under the dynamics generated by f is the sequence defined recursively by $x_0 = x$ and $x_{n+1} = f(x_n)$; thus, $x_n = f^n(x)$ where f^n denotes the n-th iterate of f. In other words, X is the phase space, time takes discrete values $n \in \mathbf{N}$, and f determines the evolution of the system over time; one speaks of the *discrete dynamical system* generated by f.

As an example, take f to be the transformation of the unit circle

$$\mathbb{S}^1 = \{z \in \mathbf{C}\,;\ |z| = 1\} \tag{1}$$

defined by $f(z) = z^d$, where d is some integer ≥ 2. Writing $z = \exp(2i\pi\theta)$ with $\theta \in \mathbf{R}/\mathbf{Z}$, this transformation reads $\theta \mapsto d\theta \bmod (1)$. The orbit of z is finite if and only if its angle θ is rational; it is dense if and only if the d-adic expansion $\theta = \sum_{i\geq 1} c_i d^{-i}$ corresponds to a sequence $(c_i) \in \{0, 1, \ldots, d-1\}^{\mathbf{N}}$ that contains every finite sequence. Thus, finite orbits, dense orbits, and orbits with a closure homeomorphic to a Cantor set coexist; in any non-empty open subset of $\mathbb{S}^1$, one finds starting points whose orbits realize any of these three possibilities. As a consequence, the behavior of an orbit is highly sensitive to the precise value of its initial condition, i.e. to the starting point x.

Topological entropy is a way to measure the complexity of such a system. To define it, we fix a distance dist on X which is compatible with its topology. Then, for any $\varepsilon > 0$ and $t \geq 1$, we count the number of orbits that can be distinguished at scale ε during a period of observation t. To make this precise, we say that two points x and y are (t, ε)-distinguishable if

$$\mathrm{dist}(f^n(x), f^n(y)) > \varepsilon \tag{2}$$

for some $0 \leq n < t$ and we define the *number of orbits* $N(t,\varepsilon)$ as the maximal number of elements in a subset of X made of points which are pairwise distinguishable in time $\leq t$ (this number is finite because X is compact). The quantity

$$h(f;\varepsilon) = \limsup_{t\to+\infty} \frac{1}{t}\log(N(t,\varepsilon)) \tag{3}$$

gives the exponential growth rate of $N(t,\varepsilon)$ as the period of observation goes to $+\infty$. And the *topological entropy* is obtained by making finer and finer observations:

$$h_{\rm top}(f) = \lim_{\varepsilon\to 0} h(f;\varepsilon). \tag{4}$$

This quantity is non-negative and does not depend on the initial choice of distance dist. When X is a manifold and f is Lipschitz, $h_{\rm top}(f) \leq \dim(X)\log(\mathrm{Lip}(f))$. In the case of $f(z) = z^d$ on the unit circle, one gets easily $h_{\rm top}(f) = \log(d)$.

1.2. **Theorems of Yomdin and Gromov.** Assume now that X is a compact and connected manifold, and denote its cohomology groups by

$$H^*(X;\mathbf{Z}) = \bigoplus_{k=0}^{\dim(X)} H^k(X;\mathbf{Z}). \tag{5}$$

Then, f induces an endomorphism f^* of $H^*(X;\mathbf{Z})$ which preserves each factor $H^k(X;\mathbf{Z})$. Taking real coefficients, $H^*(X;\mathbf{R})$ becomes a finite dimensional real vector space. Given any norm $\parallel\cdot\parallel$ on $\mathsf{End}(H^*(X;\mathbf{R}))$, the *spectral radius* of f^* is defined by

$$\rho(f^*) = \lim_{n\to+\infty} \parallel (f^*)^n \parallel^{1/n} \tag{6}$$

$$= \max\{|\lambda| \ ; \ \lambda\in\mathbf{C} \text{ is an eigenvalue of } f^* \text{ on } H^*(X;\mathbf{C})\}. \tag{7}$$

Y. Yomdin proved in 1986 (see [59, 37]) that a C^∞-transformation f of a compact manifold X satisfies $h_{\rm top}(f) \geq \log(\rho(f^*))$, a result that had been conjectured by M. Shub around 1974. One can get a similar lower bound by looking at the action of f on the fundamental group of X, even when f is just continuous (see [16, 43]). On the other hand, in 1976, M. Gromov had obtained an upper bound for $h_{\rm top}(f)$ when

- X is a compact Kähler manifold. (That is, X is a compact complex manifold which admits at least one Kähler form κ, see [52].)
- f is a holomorphic transformation of X.

Combining Yomdin's and Gromov's theorems, we obtain the following statement.

Theorem 1.1 (Gromov-Yomdin)**.** *The topological entropy of a holomorphic transformation f of a compact Kähler manifold X is equal to the logarithm of the spectral radius of the linear map $f^*\colon H^*(X;\mathbf{R}) \to H^*(X;\mathbf{R})$. That is, $h_{\rm top}(f) = \log(\rho(f^*))$.*

This is an amazing theorem: it provides a close formula for a quantity which is usually extremely hard to estimate and it shows that the entropy is constant along holomorphic deformations (X_t, f_t).

1.3. **Dynamical degrees.** The proof of Gromov's upper bound gives more than just $h_{\rm top}(f) \leq \log\rho(f^*)$ (see [38]). To describe it, we endow X with the Riemannian metric induced by some fixed Kähler form κ. In a first step, Gromov introduces the iterated graph $\Gamma(f;n) = \left\{(x, f(x), \ldots, f^{n-1}(x))\ ;\ x \in X\right\} \subset X^n$ and proves that

$$h_{\rm top}(f) \leq \limsup_{n\to+\infty} \frac{1}{n} \log\left(\mathsf{vol}(\Gamma(f;n))\right) \tag{8}$$

where the volume is computed with respect to the product metric on X^n. This relies on the fact that complex analytic subsets of Kähler manifolds are minimal in the sense of Riemannian geometry[1]. In particular, we have the extraordinary fact that the iterated graphs $\Gamma(f;n)$ are always the "geometrically simplest representatives" of their homology classes in X^n.

The second step is a computation of $\mathsf{vol}(\Gamma(f;n))$. To describe it, recall that the cohomology group of a compact Kähler manifold comes with a Hodge decomposition: for each integer k between 0 and $2\dim_{\mathbf{C}}(X)$,

$$H^k(X;\mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(X;\mathbf{C}) \tag{9}$$

where $H^{p,q}(X;\mathbf{C})$ is the Dolbeault cohomology group of (closed forms of) type (p,q). The Kähler form κ is of type $(1,1)$ and its exterior powers κ^p are of type (p,p). The main input is Bishop's theorem that says that the volume of a complex analytic subset $Y \subset X$ of dimension ℓ is equal to the integral of κ^ℓ over Y; thus, it is also equal to the product $[Y]\cdot[\kappa]^\ell$ between the classes of Y in $H_{2\ell}(X;\mathbf{Z})$ and of κ^ℓ in $H^{\ell,\ell}(X;\mathbf{C})$. Gromov applies this theorem in X^n together with Künneth's formula, and computes $\mathsf{vol}(\Gamma(f;n))$ as an intersection product that involves only the action of f^* on $H^*(X;\mathbf{C})$ and exterior powers of κ. Since $[\kappa^p] \in H^{p,p}(X;\mathbf{C})$, one gets

$$\rho(f^*) = \max_p \lambda_p(f) \tag{10}$$

where $\lambda_p(f)$ is the spectral radius of f^* on $H^{p,p}(X;\mathbf{C})$.

[1]This means that a small C^1-smooth perturbation V_ε of a complex submanifold V in a Kähler manifold will always have a volume larger than or equal to the volume of V.

Example 1.2. Consider an endomorphism g of $\mathbb{P}^m(\mathbf{C})$. It can be defined in homogeneous coordinates by $m+1$ homogeneous polynomial functions $g_i \in \mathbf{C}[\mathbf{z}_0, \ldots, \mathbf{z}_m]$ of the same degree d and with no non-trivial common zeros:

$$g[\mathbf{z}_0 : \ldots : \mathbf{z}_m] = [g_0(\mathbf{z}_0, \ldots, \mathbf{z}_m) : \cdots : g_m(\mathbf{z}_0, \ldots, \mathbf{z}_m)]. \tag{11}$$

Each cohomology group $H^k(\mathbb{P}^m(\mathbf{C}); \mathbf{Z})$ is either trivial (when k is odd) or cyclic (when $k = 2p$ is even) and the action of g^* on $H^{2p}(\mathbb{P}^m(\mathbf{C}); \mathbf{Z})$ is given by multiplication by d^p. In particular, the topological degree of g is d^m, it is the largest eigenvalue of g^*, and

$$h_{\mathrm{top}}(g) = \log(\deg_{\mathrm{top}}(g)) = m \log(d). \tag{12}$$

For instance, if $g(\mathbf{z}) \in \mathbf{C}(\mathbf{z})$ is a rational fraction, viewed as an endomorphism of the Riemann sphere $\overline{\mathbf{C}} = \mathbb{P}^1(\mathbf{C})$, its topological entropy is the logarithm of its degree.

1.4. **Meromorphic transformations and dynamical degrees.** Gromov's inequality
$h_{\mathrm{top}}(f) \leq \max_p \log \lambda_p(f)$ can be extented to meromorphic transformations. The first issue is to define topological entropy when f has indeterminacy points, the second is to define the spectral radii $\lambda_p(f)$. For topological entropy, the idea is to focus on orbits that never visit the indeterminacy locus (for this we refer to [40]). Let us describe the definition of $\lambda_p(f)$ when $f \colon X \dashrightarrow X$ is a dominant meromorphic transformation of a compact Kähler manifold.

The graph of f provides a correspondence $\Gamma_f \subset X \times X$ such that the projections $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$ are surjective; $\pi_1 : \Gamma_f \to X$ is a bimeromorphic map, while the topological degree of π_2 coincides with the one of f. A differential form α on X can be pulled back to Γ_f by π_2 and then pushed forward as a current by π_1. This defines a linear operator $f^*\alpha = (\pi_1)_* \pi_2^* \alpha$, which induces a linear transformation of $H^*(X; \mathbf{Z})$ preserving the Hodge decomposition. Note, however, that if $g \colon X \dashrightarrow X$ is a meromorphic map, $(f \circ g)^*$ usually differs from $g^* \circ f^*$. For instance, the involution $s \colon \mathbb{P}^2(\mathbf{C}) \dashrightarrow \mathbb{P}^2(\mathbf{C})$ defined by $s[x_0 : x_1 : x_2] = [x_1 x_2 : x_2 x_0 : x_0 x_1]$ acts on $H^2(\mathbb{P}^2(\mathbf{C}); \mathbf{Z})$ by multiplication by 2; but $s \circ s = \mathrm{id}$ acts by multiplication by 1 (not by 4).

With such a definition at hand, the p-th *dynamical degree* of f is defined by

$$\lambda_p(f) = \lim_{n \to +\infty} \parallel (f^n)^*_{H^{p,p}(X;\mathbf{C})} \parallel^{1/n} \tag{13}$$

where $\| \cdot \|$ is any norm on $\mathsf{End}(H^{p,p}(X; \mathbf{C}))$. The existence of this limit and the following properties are obtained in [29].

(1) $\lambda_p(f) = \lambda_p(\varphi^{-1} \circ f \circ \varphi)$ for any bimeromorphic map $\varphi \colon Y \dashrightarrow X$;

(2) For any Kähler form κ on X,

$$\lambda_p(f) = \lim_{n\to+\infty} \left(\int_X (f^n)^*\kappa^p \wedge \kappa^{\dim(X)-p} \right)^{1/n} \tag{14}$$

where $(f^n)^*\kappa^p$ and the integral are computed on the complement of the indeterminacy locus of f^n. In particular, this limit does not depend on κ.

(3) The sequence $p \mapsto \lambda_p(f)$ is log-concave, that is

$$\log(\lambda_p(f)) \geq \frac{1}{2}\log(\lambda_{p-1}(f)) + \frac{1}{2}\log(\lambda_{p+1}(f)) \tag{15}$$

for any $1 \leq p \leq m-1$.

T. C. Dinh and N. Sibony proved in [29] that the upper bound

$$h_{\rm top}(f) \leq \max_p \log(\lambda_p(f)) \tag{16}$$

still holds for dominant meromorphic transformations of compact Kähler manifolds. On the other hand, one cannot expect equality as in Theorem 1.1 (see [40] for a simple example).

1.5. **Rational transformations of projective varieties.** When X is projective, one can replace κ in Property (2) above by a hyperplane section H of X and the integral of $(f^n)^*\kappa^p \wedge \kappa^{\dim(X)-p}$ in Equation (14) by the intersection product $((f^n)^*H^p \cdot H^{\dim(X)-p})$. Doing so, the definition of $\lambda_p(f)$ makes sense any rational transformation f, defined over any field $\mathbf{k}$:

$$\lambda_p(f) = \lim_{n\to+\infty} \left((f^n)^*H^p \cdot H^{\dim(X)-p} \right)^{1/n}. \tag{17}$$

This is consistent with our initial definition when $\mathbf{k} = \mathbf{C}$, and Properties (1) to (3) are still satisfied in this context (see [27, 56]). Moreover, $\max_p \lambda_p(f)$ is equal to the limit of $\parallel [\Gamma_{f^n}] \parallel^{1/n}$ where Γ_{f^n} is the graph of f^n in $X \times X$, $[\Gamma_{f^n}]$ is its numerical class and $\parallel \cdot \parallel$ is any norm on the space of numerical classes. Thus, dynamical degrees are basic invariants describing the asymptotic complexity of the n-th iterate f^n. When $\mathbf{k}$ is a local field and f is regular, f defines a continuous transformation of the compact space $X(\mathbf{k})$. It turns out that the topological entropy of f on $X(\mathbf{k})$ is always bounded from above by $\log \max \lambda_p(f)$ (see [34] for a better statement), but this upper bound is usually sharp (see Figure 1 below, and [51]).

The first question we shall look at, in the next sections, is: what are the possible values of $\lambda_1(f)$ when f is an automorphism or a birational transformation of a surface?

2. FROM ENTROPY TO SALEM AND PISOT NUMBERS

Let f be a dominant rational transformation of a closed Riemann surface X. Then f is an endomorphism and its dynamical degrees are $\lambda_0(f) = 1$ and $\lambda_1(f) = \deg_{\rm top}(f)$; in particular, they are positive integers and $h_{\rm top}(f)$ is equal to $\log(\deg_{\rm top}(f))$. When the genus of X is ≥ 2, f is an automorphism of finite order. Endomorphisms of elliptic curves $\mathbf{C}/\Lambda$ are induced by affine maps $z \mapsto az+b$, and their dynamical features are well understood. Endomorphisms of $\mathbb{P}^1$ provide some of the most interesting examples of dynamical systems (see [50, 1]); subgroups of $\mathsf{Aut}(\mathbb{P}^1) = \mathsf{PGL}_2(\mathbf{C})$ are central to topology, geometry, and dynamics, but individually, each automorphism is just a homography $z \mapsto (az+b)/(cz+d)$. Thus, the study of invertible algebraic transformations and of dynamical degrees really starts in dimension 2.

We shall see that in this dimension Salem and Pisot numbers play an important role.

2.1. Salem and Pisot numbers [12]**.** A *Pisot number* is a real algebraic integer $\lambda > 1$, the Galois conjugates of which are all in the open unit disk. A *Salem number* is a real algebraic integer $\lambda > 1$, the Galois conjugates of which are all in the closed unit disk, with at least one of them on the unit circle. The minimal polynomial of a Salem number is reciprocal and its degree is even; in particular, $1/\lambda$ is one of the conjugates of λ. We include reciprocal quadratic integers into Pisot numbers ([46] makes a different choice).

R. Salem proved that Pisot numbers form a closed subset of $\mathbf{R}$, and C. L. Siegel proved that the smallest Pisot number is the root $\lambda \simeq 1.324717$ of the cubic equation $t^3 = t+1$. On the other hand, the set of Salem numbers is not closed (every Pisot number is a limit of Salem numbers on both sides) and a famous conjecture of D. Lehmer asks whether there is a Salem number below the Lehmer number $\lambda_{10} \simeq 1.17628$, which is given by the unique root > 1 of

$$t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1 = 0. \tag{18}$$

More generally, we shall denote by λ_d *the smallest Salem number of degree d* (d even). This number is well defined; indeed, one sees easily that Pisot and Salem numbers of degree $\leq d$, for any fixed d, form a discrete subset of $\mathbf{R}$.

2.2. Intersection form. Let X be a compact Kähler surface. Let q_X denote the intersection form on $H^2(X;\mathbf{Z})$. On $H^{1,1}(X;\mathbf{R})$, by the Hodge index theorem, q_X is non-degenerate and of signature $(1,m)$ where $1+m = h^{1,1}(X)$; thus, its orthogonal group is isomorphic to $\mathsf{O}_{1,m}(\mathbf{R})$. The Kähler classes satisfy $q_X([\kappa],[\kappa']) > 0$ and the set $\mathrm{Kah}(X)$ of all Kähler classes is contained in exactly one component, denoted by $\mathrm{Pos}^+(X)$, of the set $\{u \in H^{1,1}(X;\mathbf{R})\,;\, q_X(u,u) > 0\}$.

We denote by $\mathrm{NS}(X)$ the Néron-Severi group of X, defined as the $\mathbf{Z}$-module of all Chern classes of holomorphic line bundles on X. Its rank is the *Picard number* $\rho(X)$. For projective surfaces defined over some field $\mathbf{k}$, one can replace $\mathrm{NS}(X)$ by numerical classes of divisors, and $\mathrm{Pos}^+(X)$ by the component of the cone $q_X(u,u) > 0$ containing ample classes.

2.3. **Dynamical degrees are algebraic numbers.** The group $\mathsf{Aut}(X)$ acts linearly on $H^2(X;\mathbf{Z})$ and preserves q_X. In particular, it acts by isometries on $H^{1,1}(X;\mathbf{R})$, and it preserves the Kähler cone. If f is an automorphism with positive entropy, f^* has an eigenvalue $\lambda_1(f) > 1$ on $H^{1,1}(X;\mathbf{R})$, and the corresponding isometry is *loxodromic* [54]:

(a) f^* has an eigenvalue $\lambda_1(f) > 1$, another one $\lambda_1(f)^{-1} < 1$, and its remaining eigenvalues $\alpha \in \mathbf{C}$ have modulus 1;
(b) the multiplicity of $\lambda_1(f)$ is 1, and the corresponding eigenspace E_f^+ is a line contained in the isotropic cone of $(q_X)_{|H^{1,1}(X;\mathbf{R})}$; one can write $E_f^+ = \mathbf{R}\Theta_f^+$ for some Θ_f^+ in the boundary of the Kähler cone of X;
(c) if $u \in \mathrm{Pos}^+(X)$, the sequence $\lambda_1(f)^{-n}(f^n)^*(u)$ converges towards a non-zero multiple of Θ_f^+ as n goes to $+\infty$.

Analogous objects E_f^-, Θ_f^- are associated to $\lambda_1(f)^{-1}$. If κ is some fixed Kähler form on X, Θ_f^+ and Θ_f^- can be chosen to satisfy

$$q_X(\Theta_f^+, \Theta_f^-) = 1 \quad \text{and} \quad q_X(\Theta_f^\pm, [\kappa]) = 1. \tag{19}$$

With such a choice, $\lambda_1(f)^{-n}(f^n)^*u$ converges towards $q_X(u,\Theta_f^-)\Theta_f^+$ in Property (c).

If Ω is a non-zero holomorphic 2-form on X, $\int_X \Omega\wedge\overline{\Omega} > 0$; thus $q_X(v,\overline{v})$ is a positive definite Hermitian form on $H^{2,0}(X;\mathbf{C})$, and the image of $\mathsf{Aut}(X)$ in $\mathsf{GL}(H^{2,0}(X;\mathbf{C}))$ is contained in a compact group. The same holds for $H^{0,2}(X;\mathbf{C})$ and, consequently, Property (a) extends to the action of f^* on $H^2(X;\mathbf{C})$.

Since f^* preserves $H^2(X;\mathbf{Z})$, we see that $\lambda_1(f)$ is an algebraic integer of degree $\leq b_2(X)$ with Galois conjugates equal to $1/\lambda_1(f)$ or of modulus 1. In other words, $\lambda_1(f)$ *is a Salem number of degree* $\leq b_2(X)$ *or a reciprocal quadratic integer*. By Kronecker's Lemma, the eigenvalues of f^* that are not Galois conjugate to $\lambda_1(f)$ are roots of unity.

This analysis has been extended to birational transformations by J. Diller and C. Favre. If f is a birational transformation of a projective surface X, defined over an algebraically closed field $\mathbf{k}$, they find a birational model $\varphi\colon Y \dashrightarrow X$ of X such that $f_Y := \varphi^{-1}\circ f\circ\varphi$ satisfies $(f_Y^*)^n = (f_Y^n)^*$ for all $n \geq 1$, where f_Y^* is defined as in Section 1.4. Then, $\lambda_1(f)$ coincides with the eigenvalue of the linear operator f_Y^* on the Néron-Severi group $\mathrm{NS}(X)$. The intersection form is not invariant, but $q_X(f_Y^* u, f_Y^* u) \geq q_X(u,u)$ if the class $u \in \mathrm{NS}(X)$ is big and nef. This inequality turns out to be sufficient to show the following theorem (see [28] and [14]).

Theorem 2.1. *Let $f\colon X \dashrightarrow X$ be a birational transformation of a compact Kähler surface (resp. a projective surface defined over an algebraically closed field $\mathbf{k}$). Then,*

1. *$\lambda_1(f)$ is equal to 1, to a Salem number or to a Pisot number;*
2. *if $\lambda_1(f)$ is a Salem number, there is a birational model $\varphi\colon Y \dashrightarrow X$ such that the map $f_Y := \varphi^{-1} \circ f \circ \varphi$ is an automorphism of Y.*

This result is specific to invertible transformations of surfaces [8, 9]: transcendental dynamical degrees appear for non-invertible maps (resp. varieties of dimension ≥ 3).

Remark 2.2. For automorphisms of projective surfaces in arbitrary characteristic, $\lambda_1(f)$ was defined in Section 1.5 using hyperplane sections; equivalently, $\lambda_1(f)$ is the spectral radius of f^* on $\mathrm{NS}(X)$. Algebraic geometry provides other cohomology groups - for instance étale ℓ-adic, de Rham, or crystalline cohomologies - hence also new concepts of dynamical degrees. H. Esnault and V. Srinivas proved in [33] that they all coincide (the characteristic polynomial of f^* is always the product of the minimal polynomial of $\lambda_1(f)$ and cyclotomic polynomials).

3. Possible entropies, possible dynamical degrees

We now arrive to one of the main topics initiated in [46] and [48]: among Pisot and Salem numbers, which ones are realized as dynamical degrees of automorphisms of surfaces? For complex surfaces, the problem is to describe the possible values taken by the topological entropy of automorphisms. We shall enlarge the point of view by considering also birational transformations, defined over any algebraically closed field; in other words, we want to describe the *dynamical spectrum*

$$\Lambda = \{\lambda \in \mathbf{R} \,;\, \lambda \text{ is the dynamical degree } \lambda_1(f) \text{ of a birational transformation of a surface}\}.$$

Elements of Λ larger than 1 are Pisot or Salem numbers. As we shall see below, many Pisot and Salem numbers are not in Λ, and the structure of this set is somewhat mysterious.

3.1. The classification of surfaces.

The existence of an automorphism with positive entropy on a compact complex surface X imposes strong constraints on the geometry of X. This can be stated in two ways (see [22, 24, 25]).

(1) If X is a compact complex surface with an automorphism of positive entropy, then X is a blow up of $\mathbb{P}^2$, of a torus $\mathbf{C}^2/\Lambda$, of a K3 surface, or of an Enriques surface[2]; moreover, in the last three cases, the automorphism comes from an automorphism of the torus, the K3 surface, or the Enriques surface. Thus, one only needs to study automorphisms of rational surfaces, tori, and K3 surfaces to understand automorphisms with positive entropy.

(2) If X is a projective surface over an algebraically closed field $\mathbf{k}$ with a birational transformation f such that $\lambda_1(f) > 1$, then X is birationally equivalent to a rational surface, an abelian surface, a K3 surface or an Enriques surface. Moreover, any birational transformation of an abelian surface (resp. a torus), a K3 surface, or an Enriques surface is in fact an automorphism. Thus, to describe the possible values of $\lambda_1(f)$ one only needs to look at birational transformations of $\mathbb{P}^2$, automorphisms of tori, and automorphisms of Enriques and K3 surfaces.

3.2. **The lowest dynamical degrees.**

Theorem 3.1 (McMullen)**.** *The Lehmer number λ_{10} is the smallest dynamical degree larger than* 1 *realized by birational transformations of compact Kähler surfaces. It is realized by an automorphism of a non-projective K3 surface, by an automorphism of a projective K3 surface, and by an automorphism of a rational (hence projective) surface.*

This statement concatenates Theorem 2.1(2) and the main results of [46, 45, 47, 49]. As a byproduct, $\log(\lambda_{10})$ is the minimal positive entropy for automorphisms of compact complex surfaces. For projective K3 surfaces, examples of automorphisms with $\lambda_1(f) = \lambda_{10}$ have been found over any algebraically closed field $\mathbf{k}$ of characteristic $\neq 2,3,7$ (see [19]).

The table below shows the smallest dynamical degree for the different types in the classification of Enriques and Kodaira (see [48, 53]). For surfaces not birationally equivalent to one of this table, $\lambda_1(f) = 1$ for every birational transformation of $f\colon X \dashrightarrow X$.

Type of surface	Minimal dynamical degree > 1
rational	Lehmer number $\lambda_{10} \simeq 1.17628$
Abelian	$\lambda_4 \simeq 1.72208$, root of $t^4 - t^3 - t^2 - t + 1$
compact torus	$\lambda_6 \simeq 1.40126$, root of $t^6 - t^4 - t^3 - t^2 + 1$
(non-)projective K3	Lehmer number $\lambda_{10} \simeq 1.17628$
Enriques	$\lambda_E \simeq 1.59234$, root of $t^6 - t^4 - 2t^3 - t^2 + 1$

[2]An Enriques surface X is, by definition, the quotient of some K3 surface X' by a fixed point free involution. The surface X' is the universal cover of X, and automorphisms of X lifts to automorphisms of X'.

This provides a complete description of the bottom of the dynamical spectrum Λ; its structure above λ_{10}, and in particular above its first accumulation points, is more mysterious.

3.3. **The case of K3 surfaces.** If X is a K3 surface, $\mathsf{Bir}(X)$ coincides with $\mathsf{Aut}(X)$ and, as shown above, the dynamical degree of any loxodromic automorphism $f\colon X \to X$ is a root of a quadratic or Salem polynomial of degree at most $22 = b_2(X)$, and at most the Picard number $\rho(X)$ when X is projective. More precisely,

- if X is projective over a field of characteristic 0, then $\rho(X) \leq 20$ hence $\deg \lambda_1(f) \leq 20$;
- McMullen constructs automorphisms of K3 surfaces for which the degree of $\lambda_1(f)$ is 22.
- in any characteristic $p > 0$, $\rho(X)$ reaches 22 for the so-called supersingular K3 surfaces. There are automorphisms of K3 surfaces in any characteristic $p > 0$ for which $\deg \lambda_1(f) = 22$; those do not lift to characteristic 0 (see [14, 17, 32, 55] for instance).

One does not know yet which ones of the Salem numbers of degree ≤ 22 are realized by automorphisms of K3 surfaces, but a complete answer is provided by S. Brandhorst if we *stabilize* the question by allowing taking powers (see [18]):

Theorem 3.2 (Brandhorst)**.** *Let λ be a reciprocal quadratic number or a Salem number of degree at most 20. Then, there is an integer $k \geq 1$, a complex projective K3 surface X, and an automorphism f of X such that $\lambda_1(f) = \lambda^k$.*

Tori, non-projective K3 surfaces, and Enriques surfaces are also dealt with in [18].

3.4. **The case of rational surfaces.** Let $\pi\colon X \to \mathbb{P}^2$ be any surface obtained by blowing up the projective plane m times. Pulling back a line from $\mathbb{P}^2$ to X and taking its class, one obtains an element $\mathbf{e}_0 \in \mathrm{NS}(X)$. Pulling back the exceptional divisors of the blow-ups, one obtains m additional classes $\mathbf{e}_i \in \mathrm{NS}(X)$.

Denote simply by $(\,\cdot\,)$, instead of $q_X(\ ,\ )$, the intersection form on $\mathrm{NS}(X)$. The classes $\mathbf{e}_j$ form an orthogonal basis of $\mathrm{NS}(X)$ for the intersection form, with $(\mathbf{e}_0 \cdot \mathbf{e}_0) = +1$ and $(\mathbf{e}_i \cdot \mathbf{e}_i) = -1$ if $1 \leq i \leq m$. The vectors $r_0 = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$ and $r_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ for $1 \leq i \leq m-1$, satisfy $(r_j \cdot r_j) = -2$. The reflection with respect to the orthogonal complement $r_j^{\perp}$ is $s_j\colon u \mapsto u + (u \cdot r_j) r_j$, and these reflections generate a subgroup $W_m \subset O_{1,m}(\mathbf{Z})$ isomorphic to the Coxeter group attached to the diagram of type E_m. Moreover, W_m preserves the canonical class $k_m = -3\mathbf{e}_0 + \sum_{i \geq 1} \mathbf{e}_i$ and its action on $k_m^{\perp}$ is the Tits representation of W_m. By a theorem of M. Nagata, *the image of* $\mathsf{Aut}(X)$ *in* $\mathsf{GL}(\mathrm{NS}(X))$ *is contained in*

W_m. In particular, dynamical degrees of automorphisms of rational surfaces are eigenvalues of elements of W_m (for some m).

Theorem 3.3 (T. Uehara [57])**.** *A real number $\lambda \geq 1$ is the dynamical degree of some automorphism of some complex rational surface if and only if there is an integer $m \geq 1$ and an element w in the Coxeter group W_m such that λ is the largest eigenvalue of w in its Tits representation.*

The integer m is arbitrary large and the numbers λ obtained in this way do not form a discrete subset of $\mathbf{R}$. But some structure naturally shows up: *the dynamical spectrum Λ is a well ordered subset of* $\mathbf{R}$ (every decreasing sequence of dynamical degrees is finite), *its order type is* ω^ω, and *every number in this set is a limit of dynamical degrees of automorphisms* (see [14, 15]). For instance, the golden mean $\gamma = (1+\sqrt{5})/2$ is the dynamical degree of the birational map $(x,y) \mapsto (y,xy)$; it is a limit from below of Pisot and Salem numbers contained in Λ; and there is an $\varepsilon_0 > 0$ such that $]\gamma, \gamma+\varepsilon_0[$ contains infinitely many Pisot and Salem numbers, none of which is a dynamical degree.

4. Periodic points and the canonical volume form

Let X be a compact Kähler surface and f an automorphism of X with positive topological entropy. Let Θ_f^+ and Θ_f^- be the eigenvectors of f^* in $H^{1,1}(X;\mathbf{R})$ constructed in Section 2.3, with the normalization from Equation (19) for some Kähler form κ. Then, one easily shows that there is a unique closed positive current T_f^+ (resp. T_f^-) of type $(1,1)$ such that $[T_f^+] = \Theta_f^+$ (resp. $[T_f^-] = \Theta_f^-$). These currents satisfy $f^*T_f^\pm = \lambda_1(f)^{\pm 1} T_f^\pm$ and the value of $T_f^\pm$ on a smooth 2-form ω can be obtained dynamically as the limit

$$\langle T_f^\pm | \omega \rangle = \lim_{n\to+\infty} \frac{1}{\lambda_1(f)^n} \int_X \omega \wedge (f^{\pm n})^* \kappa. \tag{20}$$

Moreover, locally $T_f^\pm = dd^c g^\pm$ for some continuous function, which implies that the product $\mu_f = T_f^+ \wedge T_f^-$ is a well defined probability measure; this measure is invariant by f. Thus, Hodge's theory and complex analysis provide a natural f-invariant probability measure. As we shall see now, μ_f describes the equidistribution of periodic points of f.

Denote by $\mathrm{Per}(f;n)$ the set of periodic points of f of period n: by definition $x \in \mathrm{Per}(f;n)$ if $f^n(x) = x$ and $f^k(x) \neq x$ for $1 \leq k \leq n-1$. This algebraic set might contain a curve, and one denotes by $\mathrm{Per}^0(f;n)$ the finite subset of its isolated points. A point $q \in \mathrm{Per}(f;n)$ is a *saddle* if the eigenvalues α and β of the tangent map Df_q^n satisfy $|\alpha| > 1 > |\beta|$. Saddle periodic points are isolated (for, if 1 is not an eigenvalue of Df_q^n the graph of f^n in $X \times X$ is transverse

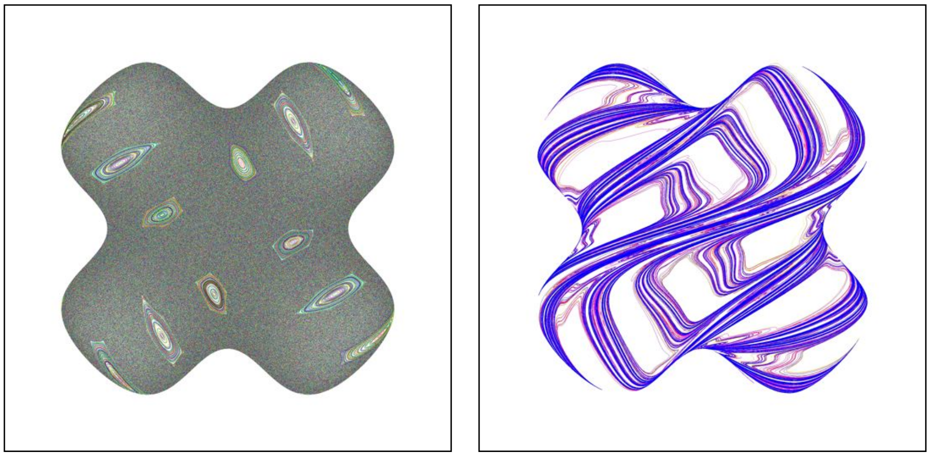

FIGURE 1. This, is the real part of a K3 surface with an automorphism acting on it. On the left, distinct colors correspond to distinct orbits. Some are trapped in invariant (KAM) islands; some might a priori fill densely a subset of positive Lebesgue measure. On the right, the curve is (the real part of) a stable manifold $W^{s}(q)$, for some saddle fixed point q. [COURTESY C. T. MCMULLEN AND V. PIT]

to the diagonal). Averaging on the set of isolated periodic points one gets a probability measure

$$\mu_n = \frac{1}{|\mathrm{Per}^0(f;n)|} \sum_{q \in \mathrm{Per}^0(f;n)} \delta_q. \tag{21}$$

Averaging over the set $\mathrm{Per}^{\mathrm{s}}(f;n)$ of saddle periodic points, one gets a second probability measure μ_n^{s}. The following equidistribution result is proven in [23, 24, 30].

Theorem 4.1. *Let X be a complex projective surface and f be an automorphism of X with positive entropy $h_{\mathrm{top}}(f) = \log(\lambda_1(f))$. Then, $|\mathrm{Per}^0(f;n)| \sim \lambda_1(f)^n$ and the sequence of probability measures (μ_n) converges towards μ_f as n goes to $+\infty$. Similarly $|\mathrm{Per}^{\mathrm{s}}(f;n)| \sim \lambda_1(f)^n$ and (μ_n^{s}) converges towards μ_f.*

In particular, there are infinitely many isolated periodic points, something that can be derived from the Lefschetz fixed point formula (see [24]), and most of them are saddle periodic points. Moreover, saddle periodic points are dense in the support of μ_f, a compact set with positive Hausdorff dimension. On the support of μ_f, the dynamics is chaotic: the behaviour of an orbit $(f^n(x))$ is sensitive to the initial condition x, as in the prototypical example $z \mapsto z^d$ from Section 1.1. To each saddle periodic point q are associated a stable manifold $W^{\mathrm{s}}(q)$ and an unstable manifold $W^{\mathrm{u}}(q)$; they are f^n-invariant, where n is the period of q; and they are parametrized by injective holomorphic entire curves $\xi^{s/u}\colon \mathbf{C} \to X$ such that $f^n \circ \xi^s(z) = \xi^s(\beta z)$ (resp. $f^n \circ \xi^u(z) = \xi^u(\alpha z)$) where α and β are the eigenvalues of Df^n_q. Stable manifolds intersect unstable ones, and create horseshoe pictures (see [21, 43]).

As explained in the introduction, a K3 surface admits a natural holomorphic 2-form Ω_X which is uniquely determined up to a scalar factor of modulus 1. Thus, the volume form $\mathsf{vol}_X = \Omega_X \wedge \overline{\Omega_X}$ is preserved by $\mathsf{Aut}(X)$. Kummer surfaces are the only one for which $\mu_f = \mathsf{vol}_X$ (see [26, 35]), and natural questions arise. What are the possible stochastic properties of vol_X with respect to loxodromic automorphisms? What can be said on μ_f and its support? The study of Fatou sets is related to these questions.

5. Fatou sets and Siegel balls

If f is an automorphism of a complex projective surface X with positive entropy, its dynamics is chaotic. At least, this is true on the support of the invariant measure μ_f. But it might occur that this support be a “thin” subset of X, for instance with empty interior. Note, however, that Figure 1 is misleading: in this example $\mu_f(X(\mathbf{R})) = 0$, most periodic points are not real, and it might very well be the case that the support of μ_f be equal to $X(\mathbf{C})$. The notion of Fatou sets, described below, is the right one to understand the complement of the support of μ_f or, more accurately, the complement of the supports of T^+_f and T^-_f.

5.1. **The Fatou set.** Let $f\colon X \to X$ be a dominant endomorphism of a compact complex manifold X. A point $x \in X$ is in the *Fatou set* $\mathrm{Fat}(f)$ if there is a compact neighborhood U of x on which the iterates $f^n\colon U \to X$ form a normal family in the sense of Montel. In other words, the family $((f^n)_{|U})$ is pre-compact: for any subsequence $((f^{n_i})_{|U})$, one can extract a further subsequence that converges uniformly to a holomorphic map $U \to X$; equivalently, on U the differential Df^n is bounded by a constant that does not depend on n. The Fatou set is open and $f^{-1}(\mathrm{Fat}(f)) = \mathrm{Fat}(f)$. The connected components of $\mathrm{Fat}(f)$ are called *Fatou components*.

By definition, the dynamics of f on $\mathrm{Fat}(f)$ is tame; for instance, $\mathrm{Fat}(f) = X$ if and only if f is invertible and contained in a compact subgroup of $\mathsf{Aut}(X)$; in that case, there is an f-invariant Riemannian metric on X, in particular the entropy of f vanishes. Thus, in the most interesting cases, $\mathrm{Fat}(f)$ is a proper subset of X, but it is usually hard, on a given example, to determine whether $\mathrm{Fat}(f)$ is empty or not.

When f is an endomorphism of $\mathbb{P}^1(\mathbf{C})$ of degree $d \geq 2$, a celebrated theorem of D. Sullivan says that

(1) the orbit $f^n(V)$ of every Fatou component V is ultimately periodic: there is a Fatou component $U = f^m(V)$ such that $f^\ell(U) = U$, for some m, $\ell \geq 1$;
(2) an f-invariant Fatou component is either a Siegel disk, a Herman ring, or an immediate attracting bassin.

By a Siegel disk, we mean a domain $U \subset \mathbb{P}^1(\mathbf{C})$ for which there is a biholomorphic mapping $\varphi\colon \mathbb{D} \to U$ such that $\varphi^{-1} \circ f \circ \varphi$ is a rotation of the unit disk. Similarly, a Herman ring is a rotation domain which is biholomorphic to an annulus $\{z \in \mathbf{C}\,;\, a < |z| < b\}$. Siegel disks contain fixed points (periodic Siegel disks contain periodic points), but Herman rings do not. If q is a fixed point such that $|f'(q)| < 1$, its attracting bassin is the open set of points $z \in \mathbb{P}^1(\mathbf{C})$ such that $f^n(z)$ converges towards q as n goes to $+\infty$; its immediate attracting bassin is the component containing q.

Such a classification of Fatou components does not hold in higher dimension or for dynamics over complete non-archimedean fields. In particular, there are examples of Fatou components V such that the $f^n(V)$ are parirwise disjoint (see [2, 10, 11, 31]). (On a K3 surface, the volume form vol_X being invariant, Poincaré's recurrence theorem implies that every Fatou component is periodic.)

5.2. **Constructing Siegel balls.** Let f be a holomorphic transformation of a complex manifold X of dimension m. One way to construct a Fatou component is to conjugate locally the dynamics of f to the dynamics of a rotation around a fixed point of f. More precisely, let us denote by $\mathbb{B}^m$ the unit ball of $\mathbf{C}^m$ with respect to the standard Hermitian metric, and by U_m the unitary group. A *Siegel ball* for f is an open subset $U \subset X$ such that (i) $f(U) = U$ and (ii) there is a biholomorphism $\varphi\colon \mathbb{B}^m \to U$ that conjugates f to a rotation $R \in \mathsf{U}_m$. The center of the ball is fixed by the rotation, the point $x = \varphi(0)$ is fixed by f, and $D\varphi_0$ conjugates R to Df_x. Conversely, assume f fixes a point $x \in X$ and Df_x is conjugate to a rotation $R \in \mathsf{U}_m$ by some linear change of coordinates. One can then try to conjugate f itself to R on some neighborhood of x. Let $\alpha_1, \ldots, \alpha_m$ be the eigenvalues of Df_x, repeated according to their multiplicities (they all have modulus 1). By definition, the α_i are *multiplicatively independent* if the only solution to $\alpha_1^{k_1} \cdots \alpha_m^{k_m} = 1$ with $k = (k_1, \ldots, k_m) \in \mathbf{Z}^m$ is the trivial solution $k = 0$; and they *satisfy a Diophantine condition* if

$$|\alpha_1^{k_1} \cdots \alpha_m^{k_m} - 1| \geq \frac{C}{(\max |k_i|)^M} \tag{22}$$

for some positive constants C and M and for all $k \in \mathbf{Z}^m \setminus \{0\}$. The following theorem is due to C. L. Siegel and S. Sternberg (see [41]).

Theorem 5.1. *Let g be a germ of holomorphic diffeomorphism of $\mathbf{C}^m$ fixing the origin 0. Suppose the eigenvalues $\{\alpha_i\}_{i=1}^m$ of Dg_0 have modulus 1 and satisfy* *a Diophantine condition. Then, there is a holomorphic diffeomorphism φ from $\mathbb{B}^m$ to an open and g-invariant neighborhood U of 0 in $\mathbf{C}^m$ that conjugates g to the rotation $R(z_1,\ldots,z_m) = (\alpha_1 z_1,\ldots,\alpha_m z_m)$.*

Coming back to holomorphic transformations of complex manifolds, we see that if f fixes x, Df_x is conjugate to a rotation R, and its eigenvalues satisfy a Diophantine condition, then f is locally conjugate to R on a neighborhood of x and $\mathrm{Fat}(f)$ contains x. When a Siegel ball U is obtained in this way, the images of the coordinate hyperplanes by the conjugacy φ are f-invariant; in their complement, each orbit is dense in an m-dimensional real torus.

6. Siegel balls in compact Kähler surfaces

Theorem 6.1. *There are examples of automorphisms $f\colon X \to X$ of compact complex surfaces with positive entropy and an invariant Siegel ball. Such examples can be constructed on rational surfaces and K3 surfaces.*

Examples on rational surfaces are constructed in [6, 47], while [46] and [42] concern non-projective K3 surfaces. On the other hand, it is an open question to decide whether such an example can be constructed on a projective K3 or an Enriques surface (see below Section 7). As explained in Sections 4 and 5, this theorem provides examples of automorphisms with positive entropy for which the support of μ_f is not equal to X, so that periodic points are not dense.

6.1. **Proof strategy.** Here is how the proof goes for K3 surfaces.

The Torelli theorem. The Torelli theorem for K3 surfaces provides a way to construct a K3 surface X and an automorphism f of X from Hodge theoretic data.

The complex structure of X determines

- a Hodge decomposition $H^2(X;\mathbf{Z})\otimes_{\mathbf{Z}}\mathbf{C} = H^{2,0}(X)\oplus H^{1,1}(X)\oplus H^{0,2}(X)$, and
- a Kähler cone $\mathrm{Kah}(X) \subset H^{1,1}(X;\mathbf{R})$ (made of all classes of Kähler forms).

Both are invariant under the action of $\mathsf{Aut}(X)$. Conversely, the Torelli theorem says that if an element F of $\mathsf{GL}(H^2(X;\mathbf{Z}))$ preserves q_X, the Hodge decomposition, and the Kähler cone, then $F = f^*$ for a unique $f \in \mathsf{Aut}(X)$.

To understand the second constraint, a description of $\mathrm{Kah}(X)$ is necessary. Firstly, $\mathrm{Kah}(X)$ is contained in $\mathrm{Pos}^+(X)$ (see Section 2.2). Secondly, if $C \subset X$ is a complex curve and κ is a Kähler form, then $\int_C \kappa > 0$; thus, identifying the homology class of C with its Poincaré dual $[C] \in H^{1,1}(X;\mathbf{R})$, we obtain $q_X([\kappa],[C]) > 0$. Conversely, those constraints determine completely $\mathrm{Kah}(X)$: *a* *class in* $\mathrm{Pos}^+(X)$ *that intersects every curve positively is a Kähler class*. Moreover, by the Hodge index theorem one only needs to test this last property on irreducible curves E with negative self-intersection and, on a K3 surface, such a curve is a smooth rational curve with $q_X(E,E) = -2$; conversely if u is a class with $q_X(u,u) = -2$ then u or $-u$ is a sum of classes of smooth rational curves of self-intersection -2.

To sum up, each class $u \in \mathrm{NS}(X)$ with $q_X(u,u) = -2$ determines a hyperplane $u^\perp$. The arrangement of hyperplanes obtained in this way cuts $\mathrm{Pos}^+(X)$ into a certain (finite or infinite) number of chambers, and $\mathrm{Kah}(X)$ is one of these chambers.

Surjectivity of the period map. Since all K3 surfaces are diffeomorphic they all have the same cohomology group. With the intersection form q_X, it turns out that $H^2(X;\mathbf{Z})$ is isometric to the unique, even, unimodular lattice[3] $\mathrm{II}_{3,19}$ of rank 22 and signature $(3,19)$. For simplicity, we denote this lattice by (L,q_L) and identify $(H^2(X;\mathbf{Z}),q_X)$ with it. Doing so, the Hodge decomposition gives a decomposition $L\otimes_{\mathbf{Z}}\mathbf{C} = L^{2,0}\oplus L^{1,1}\oplus L^{0,2}$ into three complex subspaces such that $\overline{L^{2,0}} = L^{0,2}$, $\dim(L^{1,1}) = 20$, and q_L is of signature $(1,19)$ on $L^{1,1}(\mathbf{R})$. In a nutshell, the *surjectivity of the period map* tells us that all such decompositions come from an identification of L with $H^2(X;\mathbf{Z})$ for some K3 surface.

The Lefschetz formulas. In the holomorphic setting, the Lefschetz fixed point formula is the conjonction of several equalities due to M. Atiyah and R. Bott, one for each integer $\ell \leq \dim(X)$ (see [36]).

Here is a consequence of those formulas when f is an automorphism of a K3 surface constructed from the Torelli theorem, that is from $f^* \in \mathsf{GL}(H^2(X;\mathbf{Z}))$. Suppose the largest eigenvalue $\lambda_1(f)$ of f^* is a Salem number of degree 22. Then, 1 is not an eigenvalue of f^* on $H^2(X;\mathbf{Z})$, thus f does not fix any curve and its fixed points are isolated. If the trace of f^* is -1, the classical Lefschetz formula tells us that f has a unique fixed point $x_0 \in X$. The determinant of Df_{x_0} is the eigenvalue δ of f^* on $H^{2,0}(X)$ and its trace can then be obtained from the holomorphic Lefschetz formulas. In particular, *the eigenvalues* α *and* β *of* Df_{x_0} *are algebraic numbers determined by* δ. For good choices of δ, they have modulus 1.

As a general fact (assuming $\lambda_1(f) > 1$), δ not being a root of unity is equivalent to δ being Galois conjugate to $\lambda_1(f)$ and to X not being a projective surface.

[3]This lattice $\mathrm{II}_{3,19}$ is isometric to $3U \oplus 2E_8(-1)$ where U is $\mathbf{Z}^2$ with the intersection form $2xy$ and $E_8(-1)$ is the lattice or rank 8 associated to the graph of type E_8, but with sign reversed to be negative definite.

Since we supposed $\deg(\lambda_1(f)) = 22 = b_2(X)$, we are in this situation. Moreover, the characteristic polynomial of f^* is the minimal polynomial of $\lambda_1(f)$, so the trace of f^* is the trace of $\lambda_1(f)$.

Gel'fond-Baker independence of logarithms. If the algebraic numbers α and β are multiplicatively independent[4], a theorem of N. I. Feldman, based on the Gel'fond-Baker method, implies that they satisfy a Diophantine condition, as in Equation (22) (see [13] for a good introduction). From Theorem 5.1, we conclude that f has a Siegel ball centered at x_0.

Lattices. So, doing some reverse engineering, one starts with a Salem number λ and one wants to construct a triple made of

(a) an $F \in \mathsf{GL}(L;\mathbf{Z})$ preserving the intersection form q_L and with spectral radius λ;
(b) an F-invariant decomposition $L \otimes_Z \mathbf{C} = L^{2,0} \oplus L^{1,1} \oplus L^{0,2}$ into complex subspaces such that $\overline{L^{2,0}} = L^{0,2}$, $\dim(L^{1,1}) = 20$, and the signature of q_L on $L^{1,1}(\mathbf{R})$ is $(1,19)$;
(c) in $\{v \in L^{1,1}(\mathbf{R})\,;\, q_L(v,v) > 0\}$, an F-invariant component K of the complement of the hyperplanes $\{u^\perp\,;\, u \in L^{1,1} \cap L,\ q_L(u,u) = -2\}$.

Then, the surjectivity of the period map and the Torelli theorem will provide an automorphism f of a K3 surface X with $f^* \simeq F$ and $H^{1,1}(X) \simeq L^{1,1}$.

On top of that, one wants λ to have degree 22 and trace -1 and additional conditions on the eigenvalue δ of F on $L^{2,0}$ to insure that Df_{x_0} has multiplicatively independent eigenvalues of modulus 1. This is achieved in [46] with a beautiful blend of arithmetic and lattice theoretic arguments. In the construction of this example, Property (c) follows directly from $\deg(\lambda) = 22$. Indeed, since 22 is the rank of L, every proper F-invariant subspace of $L \otimes_{\mathbf{Z}} \mathbf{Q}$ is trivial. Thus, $L^{1,1} \cap L$ is empty and (c) is void.

So, the main issues are to construct adequate Salem numbers[5] and then an isometry F of L with characteristic polynomial equal to the minimal polynomial $S(t)$ of λ. Once λ is given, the lattice is chosen to be the ring of integers $\mathbf{Z}[t]/\langle S(t)\rangle$, F is multiplication by t, and the quadratic form q_L is $q_L(x,y) = \mathrm{Tr}(axy)$ where Tr is the trace function and $a \in \mathbf{Q}[t]/\langle S(t)\rangle$ is chosen to obtain an even, unimodular form of signature $(3,19)$. (See [4, 5, 39]

[4]McMullen shows in Theorem 6.1 and Lemma 7.5 of [48] that (1) the trace of Df_{x_0} is $(1+\delta+\delta^2)/(1+\delta)$, (2) α and β have modulus 1 if and only if $\delta+\delta^{-1} > 1-2\sqrt{2} \simeq -1.82842$, and (3) α and β are multiplicatively independent as soon as some Galois conjugate of $\delta+\delta^{-1}$ is $< 1-2\sqrt{2}$.

[5]It is known now, by a theorem of J. McKee and C. Smyth, that every integer is the trace of some Salem numbers. The existence of infinitely many Salem numbers with trace -1 was a recent result of C. Smyth in 2002.

for a complete description of characteristic polynomials of isometries of even unimodular lattices.)

6.2. **Other examples.** To get Theorem 3.1 for K3 surfaces, one starts with the Lehmer number λ_{10}. The natural lattices associated to it have rank $10 = \deg(\lambda_{10})$ (for instance, λ_{10} is an eigenvalue of the Coxeter element in W_{10}, the group introduced in Section 3.4). Thus, to construct the pair (L, F) one needs to glue such a lattice M of rank 10 to an additional lattice M' of rank 12 (on which F will act as a finite order isometry), so as to get a lattice isometric to L. In general, $M \oplus M'$ is not unimodular and a finite index extension $M \oplus M' \subset L$ is needed in the gluing construction: this is a classical but tricky part, perfectly described in §2-4 of [48]. What makes the construction significatively more delicate is the Constraint (c) related to the invariance of a candidate K for the Kähler cone. Typically, the hyperplanes $u^{\perp}$, where $u \in L^{1,1} \cap L$ satisfies $q_L(u,u) = -2$, cut the positive cone into infinitely many chambers and one must tune the construction of L, F, and the decomposition $L^{2,0} \oplus L^{1,1} \oplus L^{0,2}$ in such a way that at least one chamber be F-invariant. This problem is at the heart of [48, 49, 53].

This does not end the story. As the reader certainly noticed, even if the end product is an automorphism $f\colon X \to X$ of a projective surface, the surjectivity of the period map is an implicit statement that does not provide explicit models of X. Finding an embedding $X \subset \mathbb{P}^N_{\mathbf{C}}$, equations defining X, and formulas for f is an additional, nontrivial task. It has been achieved in [19] for one of the most interesting examples with $\lambda_1(f) = \lambda_{10}$ (see also [7, 20, 49]).

7. An open question

Does there exist an automorphism f of a complex projective K3 surface X such that $\lambda_1(f) > 1$ and the set of its periodic points is not dense in $X(\mathbf{C})$ for the euclidean topology?

A stronger formulation would be to require $\mathrm{Fat}(f) \neq \emptyset$. An example with a Siegel ball would provide a positive answer, but Siegel balls are much harder to construct because on a projective K3 surface the Jacobian determinant δ of any automorphism is a root of unity, which implies multiplicative dependence of the eigenvalues at periodic points.

One dimensional Herman rings suggest another strategy, which leads to the following questions. Does there exist a loxodromic automorphism $f\colon X_{\mathbf{R}} \to X_{\mathbf{R}}$ of a real projective K3 surface such that $X(\mathbf{R})$ is non-empty and contained in $\mathrm{Fat}(f)$?

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REFERENCES

[1] Daniel S. Alexander. *A history of complex dynamics*, volume E24 of *Aspects of Mathematics*. Friedr. Vieweg & Sohn, Braunschweig, 1994. From Schröder to Fatou and Julia.

[2] Matthieu Astorg, Xavier Buff, Romain Dujardin, Han Peters, and Jasmin Raissy. A two-dimensional polynomial mapping with a wandering Fatou component. *Ann. of Math. (2)*, 184(1):263–313, 2016.

[3] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 2004.

[4] Eva Bayer-Fluckiger. Isometries of lattices and Hasse principles. *J. Eur. Math. Soc. (JEMS)*, 26(9):3365–3428, 2024.

[5] Eva Bayer-Fluckiger and Lenny Taelman. Automorphisms of even unimodular lattices and equivariant Witt groups. *J. Eur. Math. Soc. (JEMS)*, 22(11):3467–3490, 2020.

[6] Eric Bedford and Kyounghee Kim. Dynamics of rational surface automorphisms: rotation domains. *Amer. J. Math.*, 134(2):379–405, 2012.

[7] Eric Bedford and Kyounghee Kim. Dynamics of (pseudo) automorphisms of 3-space: periodicity versus positive entropy. *Publ. Mat.*, 58(1):65–119, 2014.

[8] Jason P. Bell, Jeffrey Diller, and Mattias Jonsson. A transcendental dynamical degree. *Acta Math.*, 225(2):193–225, 2020.

[9] Jason P. Bell, Jeffrey Diller, Mattias Jonsson, and Holly Krieger. Birational maps with transcendental dynamical degree. *Proc. Lond. Math. Soc. (3)*, 128(1):Paper No. e12573, 47, 2024.

[10] Robert L. Benedetto. *Dynamics in one non-archimedean variable*, volume 198 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2019.

[11] Pierre Berger and Sébastien Biebler. Emergence of wandering stable components. *J. Amer. Math. Soc.*, 36(2):397–482, 2023.

[12] Marie-José Bertin and Martine Pathiaux-Delefosse. *Conjecture de Lehmer et petits nombres de Salem*, volume 81 of *Queen's Papers in Pure and Applied Mathematics*. Queen's University, Kingston, ON, 1989.

[13] Yuri Bilu and Yann Bugeaud. Démonstration du théorème de Baker-Feldman via les formes linéaires en deux logarithmes. *J. Théor. Nombres Bordeaux*, 12(1):13–23, 2000.

[14] Jérémy Blanc and Serge Cantat. Dynamical degrees of birational transformations of projective surfaces. *J. Amer. Math. Soc.*, 29(2):415–471, 2016.

[15] Anna Bot. The ordinal of dynamical degrees of birational maps of the projective plane. *C. R. Math. Acad. Sci. Paris*, 362:117–134, 2024.

[16] Rufus Bowen. Entropy and the fundamental group. In *The structure of attractors in dynamical systems (Proc. Conf., North Dakota State Univ., Fargo, N.D., 1977)*, volume 668 of *Lecture Notes in Math.*, pages 21–29. Springer, Berlin-New York, 1978.

[17] Simon Brandhorst. Automorphisms of Salem degree 22 on supersingular K3 surfaces of higher Artin invariant. *Math. Res. Lett.*, 25(4):1143–1150, 2018.

[18] Simon Brandhorst. On the stable dynamical spectrum of complex surfaces. *Math. Ann.*, 377(1-2):421–434, 2020.

[19] Simon Brandhorst and Noam D. Elkies. Equations for a K3 Lehmer map. *J. Algebraic Geom.*, 32(4):641–675, 2023.

[20] Simon Brandhorst and Matthias Zach. An explicit enriques surface with an automorphism of minimum entropy, 2025.

[21] Michael Brin and Garrett Stuck. *Introduction to dynamical systems*. Cambridge University Press, Cambridge, back edition, 2015.

[22] Serge Cantat. Dynamique des automorphismes des surfaces projectives complexes. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(10):901–906, 1999.

[23] Serge Cantat. Dynamique des automorphismes des surfaces $K3$. *Acta Math.*, 187(1):1–57, 2001.

[24] Serge Cantat. Dynamics of automorphisms of compact complex surfaces. In *Frontiers in complex dynamics*, volume 51 of *Princeton Math. Ser.*, pages 463–514. Princeton Univ. Press, Princeton, NJ, 2014.

[25] Serge Cantat. The Cremona group. In *Algebraic geometry: Salt Lake City 2015*, volume 97 of *Proc. Sympos. Pure Math.*, pages 101–142. Amer. Math. Soc., Providence, RI, 2018.

[26] Serge Cantat and Christophe Dupont. Automorphisms of surfaces: Kummer rigidity and measure of maximal entropy. *J. Eur. Math. Soc. (JEMS)*, 22(4):1289–1351, 2020.

[27] Nguyen-Bac Dang. Degrees of iterates of rational maps on normal projective varieties. *Proc. Lond. Math. Soc. (3)*, 121(5):1268–1310, 2020.

[28] J. Diller and C. Favre. Dynamics of bimeromorphic maps of surfaces. *Amer. J. Math.*, 123(6):1135–1169, 2001.

[29] Tien-Cuong Dinh and Nessim Sibony. Regularization of currents and entropy. *Ann. Sci. École Norm. Sup. (4)*, 37(6):959–971, 2004.

[30] Romain Dujardin. Laminar currents and birational dynamics. *Duke Math. J.*, 131(2):219–247, 2006.

[31] Romain Dujardin. Geometric methods in holomorphic dynamics. In *ICM—International Congress of Mathematicians. Vol. 5. Sections 9–11*, pages 3460–3482. EMS Press, Berlin, [2023] ©2023.

[32] Hélène Esnault, Keiji Oguiso, and Xun Yu. Automorphisms of elliptic K3 surfaces and Salem numbers of maximal degree. *Algebr. Geom.*, 3(4):496–507, 2016.

[33] Hélène Esnault and Vasudevan Srinivas. Algebraic versus topological entropy for surfaces over finite fields. *Osaka J. Math.*, 50(3):827–846, 2013.

[34] Charles Favre, Tuyen Trung Truong, and Junyi Xie. Topological entropy of a rational map over a complete metrized field, 2022.

[35] Simion Filip and Valentino Tosatti. Kummer rigidity for k3 surface automorphisms via ricci-flat metrics. *American Journal of Mathematics*, 143(5):1431–1462, 2021.

[36] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[37] M. Gromov. Entropy, homology and semialgebraic geometry. Number 145-146, pages 5, 225–240. 1987. Séminaire Bourbaki, Vol. 1985/86.

[38] Mikhaïl Gromov. On the entropy of holomorphic maps. *Enseign. Math. (2)*, 49(3-4):217–235, 2003.

[39] Benedict H. Gross and Curtis T. McMullen. Automorphisms of even unimodular lattices and unramified Salem numbers. *J. Algebra*, 257(2):265–290, 2002.

[40] Vincent Guedj. Entropie topologique des applications méromorphes. *Ergodic Theory Dynam. Systems*, 25(6):1847–1855, 2005.

[41] Michael-R. Herman. Recent results and some open questions on Siegel's linearization theorem of germs of complex analytic diffeomorphisms of $\mathbf{C}^n$ near a fixed point. In *VIIIth international congress on mathematical physics (Marseille, 1986)*, pages 138–184. World Sci. Publishing, Singapore, 1987.
[42] Katsunori Iwasaki and Yuta Takada. K3 surfaces, Picard numbers and Siegel disks. *J. Pure Appl. Algebra*, 227(3):Paper No. 107215, 31, 2023.
[43] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.
[44] Yu.Ĭ. Manin. *Cubic forms*, volume 4 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1986. Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel.
[45] Curtis T. McMullen. Coxeter groups, Salem numbers and the Hilbert metric. *Publ. Math. Inst. Hautes Études Sci.*, (95):151–183, 2002.
[46] Curtis T. McMullen. Dynamics on $K3$ surfaces: Salem numbers and Siegel disks. *J. Reine Angew. Math.*, 545:201–233, 2002.
[47] Curtis T. McMullen. Dynamics on blowups of the projective plane. *Publ. Math. Inst. Hautes Études Sci.*, (105):49–89, 2007.
[48] Curtis T. McMullen. K3 surfaces, entropy and glue. *J. Reine Angew. Math.*, 658:1–25, 2011.
[49] Curtis T. McMullen. Automorphisms of projective K3 surfaces with minimum entropy. *Invent. Math.*, 203(1):179–215, 2016.
[50] John Milnor. *Dynamics in one complex variable*, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, third edition, 2006.
[51] Arnaud Moncet. Real versus complex volumes on real algebraic surfaces. *Int. Math. Res. Not. IMRN*, (16):3723–3762, 2012.
[52] David Mumford. *Algebraic geometry. I*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Complex projective varieties, Reprint of the 1976 edition.
[53] Keiji Oguiso and Xun Yu. Minimum positive entropy of complex Enriques surface automorphisms. *Duke Math. J.*, 169(18):3565–3606, 2020.
[54] John G. Ratcliffe. *Foundations of hyperbolic manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.
[55] Ichiro Shimada. Automorphisms of supersingular $K3$ surfaces and Salem polynomials. *Exp. Math.*, 25(4):389–398, 2016.
[56] Tuyen Trung Truong. Relative dynamical degrees of correspondences over a field of arbitrary characteristic. *J. Reine Angew. Math.*, 758:139–182, 2020.
[57] Takato Uehara. Rational surface automorphisms with positive entropy. *Ann. Inst. Fourier (Grenoble)*, 66(1):377–432, 2016.
[58] André Weil. *Oeuvres scientifiques/collected papers. II. 1951–1964*. Springer Collected Works in Mathematics. Springer, Heidelberg, 2014. Reprint of the 2009 [MR2883739] and 1979 [MR0537935] editions.
[59] Y. Yomdin. Volume growth and entropy. *Israel J. Math.*, 57(3):285–300, 1987.

SERGE CANTAT, IRMAR (UMR 6625 DU CNRS), UNIVERSITÉ DE RENNES 1, CAMPUS DE BEAULIEU, 35042 RENNES CEDEX, FRANCE

Email address: serge.cantat@univ-rennes1.fr