AUTOMORPHISMS OF SURFACES: KUMMER RIGIDITY AND MEASURE OF MAXIMAL ENTROPY

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ABSTRACT. We classify complex projective surfaces X with an automorphism f of positive entropy for which the unique measure of maximal entropy is absolutely continuous with respect to the Lebesgue measure. As a byproduct, if X is a K3 surface and is not a Kummer surface, the periodic points of f are equidistributed with respect to a probability measure which is singular with respect to the canonical volume of X. The proof is based on complex algebraic geometry and Hodge theory, Pesin's theory and renormalization techniques. A crucial argument relies on a new compactness property of entire curves parametrizing the invariant manifolds of the automorphism.

1. INTRODUCTION

1.1. **Automorphisms and absolutely continuous measures.** Let X be a complex projective surface and f be an element of $\mathsf{Aut}(X)$, the group of holomorphic diffeomorphisms (also called automorphisms) of X. By definition, the **dynamical degree** of f is equal to the spectral radius λ_f of the linear endomorphism

$$f^*\colon H^2(X;\mathbf{Z}) \to H^2(X;\mathbf{Z}),$$

where $H^2(X;\mathbf{Z})$ denotes the second cohomology group of X. As a root of the characteristic polynomial of $f^*\colon H^2(X;\mathbf{Z}) \to H^2(X;\mathbf{Z})$, λ_f is an algebraic integer; more precisely, λ_f is equal to 1, to a reciprocal quadratic integer, or to a Salem number (see [27]). By the Gromov-Yomdin theorem, the topological entropy $\mathsf{h}_{top}(f)$ of an automorphism f is equal to the logarithm of its dynamical degree:

$$\mathsf{h}_{top}(f) = \log(\lambda_f).$$

Thus, f has positive entropy if, and only if there is an eigenvalue λ of f^* with $|\lambda| > 1$. If it exists, such an eigenvalue is in fact unique and is equal to λ_f.

When the entropy is positive, there is a natural f-invariant probability measure μ_f on X which satisfies the following properties (see [8, 23]):

- μ_f is the unique f-invariant probability measure with maximal entropy; it is ergodic;

- if μ_n denotes the average on the set of isolated fixed points of f^n, then μ_n converges towards μ_f as n goes to $+\infty$.

Thus μ_f encodes some of the most interesting features of the dynamics of f. In the sequel we shall call μ_f the **measure of maximal entropy** of f.

The main goal of this paper is to study the regularity properties of this measure. Since it is ergodic, it is either **singular** or **absolutely continuous** with respect to the Lebesgue measure: by definition, it is singular if there exists a Borel subset A of X satisfying $\mu_f(A) = 1$ and $\mathrm{vol}(A) = 0$ (the volume is taken with respect to any smooth volume form on X); it is absolutely continuous if $\mu_f(B) = 0$ for every Borel subset $B \subset X$ such that $\mathrm{vol}(B) = 0$.

Classical examples of pairs (X, f) for which μ_f is absolutely continuous are described in Section 1.2 below. The first examples are linear Anosov automorphisms of complex tori; one derives new examples from them by performing equivariant quotients under finite group actions and by blowing up periodic orbits. Our main theorem, stated in Section 1.3, establishes that these are the only possibilities.

This theorem allows to exhibit automorphisms of complex projective surfaces for which μ_f is singular (see Section 1.4). All previously known examples were constructed on rational surfaces whereas here, we focus on K3 surfaces. A complex projective surface X is a **K3 surface** if it is simply connected and if it supports a holomorphic 2-form Ω_X that does not vanish. Such a form is unique up to multiplication by a non-zero complex number; thus, if one imposes the constraint

$$\int_X \Omega_X \wedge \overline{\Omega_X} = 1,$$

the volume form $\mathrm{vol}_X := \Omega_X \wedge \overline{\Omega_X}$ is uniquely determined by the complex structure of X; in particular, this volume form is $\mathsf{Aut}(X)$-invariant. A byproduct of our main theorem is a characterization of the pairs (X, f) for which $\mu_f = \mathrm{vol}_X$; this occurs if and only if μ_f is absolutely continuous. For instance, if X is not a Kummer surface (see below), the periodic points of any automorphism f with $\lambda_f > 1$ are equidistributed with respect to a measure μ_f which is singular with respect to the canonical volume vol_X; this means that the measure provided by algebraic geometry is "orthogonal" to the measure which is dynamically meaningful.

Example 1.1. A good example to keep in mind is the family of (smooth) surfaces of degree $(2,2,2)$ in $\mathbb{P}^1_{\mathbf{C}} \times \mathbb{P}^1_{\mathbf{C}} \times \mathbb{P}^1_{\mathbf{C}}$. Such a surface X comes with three double covers $X \to \mathbb{P}^1_{\mathbf{C}} \times \mathbb{P}^1_{\mathbf{C}}$, hence with three holomorphic involutions σ_1, σ_2, and σ_3. If X is generic in the family of such surfaces, the composition $f = \sigma_1 \circ \sigma_2 \circ \sigma_3$ is an automorphism of X of positive entropy and μ_f is singular with respect to vol_X.

1.2. Examples with an absolutely continuous measure of maximal entropy.

1.2.1. *Abelian surfaces.* Let A be a complex abelian surface and let vol_A denote the Lebesgue (i.e. Haar) measure on A, normalized by $\mathrm{vol}_A(A) = 1$. Every $f \in \mathsf{Aut}(A)$ preserves vol_A, and the measure of maximal entropy μ_f is equal to vol_A when $\lambda_f > 1$ (see [23, 70]). Complex abelian surfaces with automorphisms of positive entropy have been classified in [53]. To describe the simplest example, start with an elliptic curve $E = \mathbf{C}/\Lambda_0$ and consider the product $A = E \times E$. The group $\mathsf{GL}_2(\mathbf{Z})$ acts on $\mathbf{C}^2$ linearly, preserving the lattice $\Lambda = \Lambda_0 \times \Lambda_0$; thus, it acts also on the quotient $A = \mathbf{C}^2/\Lambda$. This gives rise to a homomorphism $M \in \mathsf{GL}_2(\mathbf{Z}) \mapsto f_M \in \mathsf{Aut}(A)$. The spectral radius of $(f_M)^*$ on $H^2(A;\mathbf{Z})$ is equal to the square of the spectral radius of M. In particular, $\lambda_f > 1$ as soon as the trace of M satisfies $|\mathrm{tr}(M)| > 2$.

1.2.2. *Classical Kummer surfaces.* Consider the complex abelian surface $A = E \times E$ as in Section 1.2.1. The center of $\mathsf{GL}_2(\mathbf{Z})$ is generated by the involution $\eta = -\mathrm{Id}$; it acts on A by

$$\eta(x,y) = (-x,-y).$$

The quotient $X' = A/\eta$ is a singular surface. Its singularities are sixteen ordinary double points; they can be resolved by simple blow-ups, each one giving rise to a smooth rational curve with self-intersection -2. Denote by X this minimal regular model of X'. Since $\mathsf{GL}_2(\mathbf{Z})$ commutes to η, one gets an injective homomorphism $M \mapsto g_M$ from $\mathsf{PGL}_2(\mathbf{Z})$ to $\mathsf{Aut}(X)$. The topological entropy of g_M (on X) is equal to the topological entropy of f_M (on A). The holomorphic 2-form $\Omega_A = dx \wedge dy$ is η-invariant and determines a non-vanishing holomorphic 2-form Ω_X on X. The volume form $\Omega_X \wedge \overline{\Omega_X}$ is invariant under g_M: when $\lambda_{g_M} > 1$, the probability measure corresponding to this form coincides with the measure of maximal entropy μ_{g_M}. Hence, again, the measure of maximal entropy is absolutely continuous. The surface X is a Kummer surface and provides a famous example of K3 surface (see [4]).

Remark 1.2. There are explicit families $(X_t, f_t)_{t\in\mathbb{D}}$ of automorphisms of K3 surfaces such that (X_t, f_t) is a Kummer example if and only if $t = 0$ (see [26], §8.2). However, Kummer surfaces A/η with A a complex torus and $\eta(x,y) = (-x,-y)$ form a dense subset of the moduli space of K3 surfaces, see [4].

1.2.3. *Rational quotients.* Consider the complex abelian surface $A = E \times E$, as in Section 1.2.1, with the lattice $\Lambda_0 = \mathbf{Z}[\tau]$ with $\tau^2 = -1$ or $\tau^3 = 1$ (and $\tau \neq 1$). The group $\mathsf{GL}_2(\mathbf{Z}[\tau])$ acts on A and its center contains

$$\eta_\tau(x,y) = (\tau x, \tau y).$$

The quotient space $X' = A/\eta_\tau$ is singular and rational. Resolving the singularities, one gets examples of smooth rational surfaces X with automorphisms h of positive entropy. The image of the Lebesgue measure on A provides a probability measure on X which is smooth on a Zariski open subset of X and has integrable poles along the exceptional divisor of the projection $\pi\colon X \to X'$; hence, it is absolutely continuous with respect to the Lebesgue measure (see [26]). Note that most of these examples $h\colon X \to X$ are rigid (see [55], Corollaries 4.11 and 4.20)

1.3. **Main theorem.** The examples of Section 1.2 lead to the following definition (see [25, 33, 83]).

Definition 1.3. *Let X be a complex projective surface and let f be an automorphism of X. The pair (X, f) is a* **Kummer example** *if there exist*

- *a birational morphism $\pi : X \to X'$ onto an orbifold X',*
- *a finite orbifold cover $\varepsilon : Y \to X'$ by a complex torus Y,*
- *an automorphism $f_{X'}$ of X' and an automorphism f_Y of Y such that*

$$f_{X'} \circ \pi = \pi \circ f \quad \text{and} \quad f_{X'} \circ \varepsilon = \varepsilon \circ f_Y.$$

If (X, f) is a Kummer example with $\lambda_f > 1$ one easily proves that μ_f is absolutely continuous with respect to the Lebesgue measure.

Main Theorem. *Let X be a complex projective surface and f be an automorphism of X with positive entropy. Let μ_f be the measure of maximal entropy of f. This measure is absolutely continuous with respect to the Lebesgue measure if and only if (X, f) is a Kummer example.*

This answers a question raised by the first author in his thesis [22] and solves Conjecture 3.31 of McMullen in [71]. Moreover, the surfaces X that can occur are specified by the following theorem (see [29, 30]).

Classification Theorem. *Let X be a complex projective surface and f be an automorphism of X of positive entropy. If (X, f) is a Kummer example, then:*

(1) *either X is a rational surface, or there is a birational morphism $\pi : X \to Y$ onto an abelian or a K3 surface Y, and there is an automorphism f_Y of Y such that $\pi \circ f = f_Y \circ \pi$.*
(2) *If X is a K3 surface, it is a classical Kummer surface, i.e. the minimal resolution of the quotient of an abelian surface A by the involution $\eta(x, y) = (-x, -y)$; in particular, the Picard number of X is not less than* 17.
(3) *If X is rational, then λ_f is contained in $\mathbf{Q}(\zeta_l)$ where ζ_l is a primitive root of unity of order $l =$* 3, 4, *or* 5.

Remark 1.4 (see [12, 14, 69, 82])**.** Consider a holomorphic endomorphism g of the projective space $\mathbb{P}^k_{\mathbf{C}}$ of topological degree > 1. There is also an invariant probability measure μ_g that describes the distribution of periodic points and is the unique measure of maximal entropy; if μ_g is absolutely continuous with respect to the Lebesgue measure, g is a Lattès example: it lifts to an endomorphism of an abelian variety via an equivariant ramified cover. This statement is analogous to our main theorem. It is due to Zdunik for $k = 1$ and to Berteloot, Loeb and the second author for $k \geq 2$.

Remark 1.5 (see [25], Théorème C)**.** There are examples of rational transformations $h\colon X \dashrightarrow X$ of K3 surfaces with topological degree > 1 such that the topological entropy of h is positive, h preserves a unique measure of maximal entropy μ_h, μ_h coincides with the canonical volume form $\mathrm{vol}_X = \Omega_X \wedge \overline{\Omega_X}$ on X, but h is not topologically conjugate to a Kummer-Lattès example. In particular the Kummer and Lattès rigidities do not extend to non injective rational mappings. (The rigidity is recovered if one imposes that the topological degree of h is equal to the square of the spectral radius of h^* on $H^{1,1}(X;\mathbf{R})$, see [25].)

Remark 1.6. In [48], Filip and Tosatti extend our Main Theorem to all compact Kähler surfaces. The only case not covered by our theorem is the one of non-projective K3 surfaces. The main new input of [48] is a beautiful control of the Lyapunov exponents which is obtained via the existence of a Ricci flat metric on any K3 surface (see § 3.1 in [48]). Our strategy does not apply in the non-projective case: it makes use of results on the geometric intersection of currents which are not available in the non-projective case (see § 5.1 in [46], and the end of § 6.3 in [27]).

1.4. **Applications.**

1.4.1. *Lyapunov exponents and Hausdorff dimension.* The first consequence of our main theorem relies on theorems due to Ledrappier, Ruelle and Young.

Corollary 1.7. *Let X be a complex projective surface and f be an automorphism of X with positive entropy $\log\lambda_f$. Let $\lambda_s < \lambda_u$ denote the negative and positive Lyapunov exponents of the measure μ_f. The following properties are equivalent*

(1) *μ_f is absolutely continuous with respect to the Lebesgue measure;*
(2) *$\lambda_s = -\frac{1}{2}\log\lambda_f$ and $\lambda_u = \frac{1}{2}\log\lambda_f$;*
(3) *for μ_f-almost every $x \in X$,*
$$\lim_{r\to 0} \frac{\log \mu_f(B_x(r))}{\log r} = 4;$$
(4) *(X, f) is a Kummer example.*

If X is a K3 surface or, more generally, if there is an f-invariant volume form on X, the Lyapunov exponents of μ_f are opposite ($\lambda_u = -\lambda_s$), and one can replace the second item by either one of the two equalities.

1.4.2. *K3 and Enriques surfaces.* Recall that the **Néron-Severi group** $\mathrm{NS}(X)$ of a complex projective surface X is the subgroup of the second homology group $H_2(X;\mathbf{Z})$ generated by the homology classes of algebraic curves on X. The rank of this abelian group is the **Picard number** $\rho(X)$. When $\rho(X)$ is equal to 1, the entropy of every automorphism f of X vanishes (see [26, 27]); thus, the first interesting case is $\rho(X) = 2$. Examples of such K3 surfaces with an infinite group of automorphisms are described in [32, 79].

Corollary 1.8. *Let X be a complex projective K3 surface with Picard number* 2*. Assume that the intersection form does not represent* 0 *and* -2 *on* $\mathrm{NS}(X)$*. Then*

(1) $\mathsf{Aut}(X)$ *contains an infinite cyclic subgroup of index at most* 2*;*
(2) *if $f \in \mathsf{Aut}(X)$ has infinite order its entropy is positive and its measure of maximal entropy μ_f is singular with respect to the Lebesgue measure.*

An **Enriques surface** Z is the quotient of a K3 surface by a fixed point free involution. If $\pi\colon X \to Z$ is such a quotient, the canonical volume form vol_X of X determines a smooth $\mathsf{Aut}(Z)$-invariant volume form vol_Z on Z. Thus, Enriques surfaces have a natural invariant volume form that is uniquely determined by the complex structure.

If X is an Enriques surface, every global section of the canonical bundle K_X vanishes identically; thus, there is no dominant morphism from X to a K3 or abelian surface, and the Classification Theorem provides the following corollary.

Corollary 1.9. *If f is an automorphism of an Enriques surface with positive entropy, then μ_f is singular with respect to the Lebesgue measure. In particular, the periodic points of f do not equidistribute with respect to the canonical volume form of the surface.*

Let now Z be a general Enriques surface. Up to finite indices, $\mathsf{Aut}(Z)$ is isomorphic to the group of isometries of the lattice $H^2(Z;\mathbf{Z}) \simeq \mathbb{U} \oplus (-\mathbb{E}_8)$ (see [5, 35]); thus, it contains automorphisms f with positive topological entropy and Corollary 1.9 implies that μ_f is singular. On the other hand, the volume form vol_Z is the only probability measure that is invariant under the action of the full group $\mathsf{Aut}(Z)$, as shown in [24].

1.4.3. *Dynamical degrees and rational surfaces.* For the next statement, recall that the dynamical degree λ_f is an algebraic integer (see Section 1.1).

Corollary 1.10. *Let X be a complex projective surface and f be an automorphism of X with positive entropy. If the degree of λ_f (as an algebraic integer) is at least* 5 *then μ_f is singular with respect to the Lebesgue measure.*

This can be applied to the examples constructed by Bedford and Kim (see [6, 7]) and McMullen (see [72]). They exhibit families of automorphisms of rational surfaces $f_m \colon X_m \to X_m$ for which the degree λ_{f_m} increases with m. Thus μ_{f_m} is singular with respect to the Lebesgue measure when m is large enough. This may also be applied to examples constructed by Blanc. (See Sections 9.3.1 and 9.3.2)

1.5. Related problems.

1.5.1. *Geodesic flows.* A similar question of entropy rigidity concerns the geodesic flow $(\theta_t)_{t\in\mathbf{R}}$ on a negatively curved riemannian manifold (M,g). Negative curvature implies that this flow is Anosov with a unique invariant probability measure ν of maximal entropy (i.e. with metric entropy $\mathsf{h}(\theta_1,\nu)$ equal to $\mathsf{h}_{top}(\theta_1)$). The flow preserves also the Liouville measure λ_g, and the *Entropy conjecture* predicts that ν_g is absolutely continuous with respect to λ_g if and only if the riemannian manifold (M,g) is locally symmetric. This was proved by Katok for surfaces (see [61]). See [66] for a nice survey on this type of problem and [50, 9] for its relationship to rigidity properties of Anosov flows with smooth stable and unstable foliations.

1.5.2. *Random walks.* Another related question concerns the regularity of harmonic measures. Consider the fundamental group Γ_g of an orientable closed surface of genus $g \geq 2$, and identify the boundary $\partial\Gamma_g$ to the unit circle $\mathbb{S}^1$. Let ν be a probability measure on Γ_g whose support is finite and generates Γ_g. The measure ν determines a random walk on Γ_g. Given a starting point x in Γ_g and a subset A of the boundary $\partial\Gamma_g$, the harmonic measure $\omega_x(A)$ is the probability that a random path which starts at x converges to a point of A when the time goes to $+\infty$. It is conjectured that ω_x is singular with respect to the Lebesgue measure on $\partial\Gamma_g = \mathbb{S}^1$. We refer to [60, 15] for an introduction to this topic and to [18] for a recent example.

1.5.3. *Non-archimedean fields.* In [47], §1.1.2, Filip suggests that the rigidity result given by our Main Theorem may hold for the dynamics of automorphisms of K3 surfaces over certain non-archimedean valued fields.

1.6. **Organization of the paper.** Fix a complex projective surface X and an automorphism f of X with positive entropy. Section 2 recalls classical facts concerning the dynamics of f. In particular, we explain that μ_f is the product of two closed positive currents T_f^+ and T_f^-. Assume, now, that μ_f is absolutely continuous with respect to the Lebesgue measure.

1.6.1. *Renormalization along the invariant manifolds.* We first show that

(1) the absolute continuity of μ_f can be transferred to the currents T_f^+ and T_f^-;
(2) if $x \in X$ is μ_f-generic, its stable manifold is parametrized by an injective holomorphic map $\xi\colon \mathbf{C} \to\subset X$ and such a parametrization satisfies $\xi^* T_f^- = a\frac{\mathrm{i}}{2} dz \wedge d\bar{z}$ for some constant $a > 0$.

These two steps occupy Sections 3 to 5. The proof of Property (1) builds on the local product structure of μ_f and on the weak laminarity properties of $T_f^{\pm}$. The proof of (2) relies on a renormalization argument (along μ_f-generic orbits).

Remark 1.11. The renormalization techniques already appear in the proof of Lattès rigidity for endomorphisms of $\mathbb{P}^k_{\mathbf{C}}$ (see [69] and [12]). Our context is actually closer to the conformal case $k = 1$ since the renormalization is done along the stable and unstable manifolds. For endomorphisms of $\mathbb{P}^k_{\mathbf{C}}$, the unstable manifolds cover open subsets of $\mathbb{P}^k_{\mathbf{C}}$; hence, a property analogous to (2) is a strong constraint which rigidifies the dynamics on open subsets (see Lemma 3 in [12]). Here, the stable and unstable manifolds have co-dimension 1, and it is much harder to relate Property (2) to a rigidity property of the pair (X, f).

1.6.2. *Normal families of entire curves.* In Section 6, we combine Zalcman's reparametrization lemma, the Hodge index theorem, and a result of Dinh and Sibony to derive a compactness property, in the sense of Montel, for entire parametrizations of stable and unstable manifolds. This crucial step provides a new strategy to control entire curves (given by stable or unstable manifolds), which may be useful for other questions regarding the dynamics of automorphisms.

1.6.3. *Laminations, foliations, and conclusion.* Thanks to the previous step, we prove that the stable (resp. unstable) manifolds of f are organized in a (singular) lamination by holomorphic curves (Sections 7 and 8). Then, an argument of Ghys can be coupled to Hartogs phenomenon to show that this lamination extends to an f-invariant, singular, holomorphic foliation of X. At this stage of the proof, the starting hypothesis on μ_f has been upgraded to a regularity property for T_f^+ and T_f^-: these currents are smooth, and correspond to transverse invariant measures for two holomorphic foliations. To conclude, we refer to a previous theorem of the first author and Favre concerning symmetries of foliated surfaces (Theorem 3.1 in [29], p.

209-210). One could also construct by hands the complex torus and the commutative diagrams defining a Kummer example by following the arguments of the proof of Théorème 7.4 in [23].

1.6.4. *Consequences and Appendices.* Section 9 contains the proofs of the main corollaries and consequences.

The strategy described in § 1.6.2 relies on Proposition 6.9; it requires to blow down all periodic curves of f and to work on a singular surface X_0. The proof of this proposition is contained in Section 10. It is based on a precise study of the projection of T_f^+ and T_f^- on X_0; we show that $T_f^+ + T_f^-$ is cohomologous to a semi-Kähler current which is bounded from below by the pull-back of a positive $(1,1)$-form κ_0 on X_0. It also makes use of an unpublished result of Dinh and Sibony concerning entire holomorphic curves; we include a proof of their result. One difficulty comes from the fact that X_0 is not always projective: we provide such an example in Section 11.

1.6.5. *A special case.* If one assumes that the automorphism f, with positive entropy, has no periodic curve, then one can skip most of the appendices. In that case, our Main Theorem reads: *μ_f is absolutely continuous with respect to the Lebesgue measure if and only if X is an abelian surface* (and f is an Anosov linear map of X).

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2. INVARIANT CURRENTS $T_f^{\pm}$ AND THE MEASURE μ_f

We collect general results concerning the dynamics of automorphisms of compact Kähler surfaces X (see [27], and the references therein, for a complete exposition).

2.1. **Cohomology groups.** Let X be a compact Kähler surface. Let $H^k(X;\mathbf{R})$ and $H^k(X;\mathbf{C})$ denote the real and complex de Rham cohomology groups of X. If η is a closed differential form (or a closed current, see below), its cohomology class is denoted by $[\eta]$. For $0 \leq p,\, q \leq 2$, let $H^{p,q}(X;\mathbf{C})$ denote the subspace of $H^{p+q}(X;\mathbf{C})$ of all classes represented by closed (p,q)-forms, and let $h^{p,q}(X)$ be its dimension. Hodge theory implies that

$$H^k(X;\mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(X;\mathbf{C}).$$

Complex conjugation exchanges $H^{p,q}(X;\mathbf{C})$ and $H^{q,p}(X;\mathbf{C})$; thus, $H^{1,1}(X;\mathbf{C})$ inherits a real structure, with real part $H^{1,1}(X;\mathbf{R}) := H^{1,1}(X;\mathbf{C}) \cap H^2(X;\mathbf{R})$.

The intersection form is an integral quadratic form on $H^2(X;\mathbf{Z})$. It satisfies

$$\forall u, v \in H^2(X;\mathbf{R}) \ , \ \langle u|v\rangle := \int_X \tilde{u} \wedge \tilde{v},$$

where $\tilde{u}$ and $\tilde{v}$ are closed 2-forms on X representing u and v. By the Hodge index theorem $\langle\cdot|\cdot\rangle$ is non-degenerate and of signature $(1, h^{1,1}(X)-1)$ on $H^{1,1}(X;\mathbf{R})$. This endows $H^{1,1}(X;\mathbf{R})$ with the structure of a Minkowski space.

The **Kähler cone** is the subset of $H^{1,1}(X;\mathbf{R})$ consisting of classes of Kähler forms. By definition, the closure of the Kähler cone is the **nef cone**. These cones intersect only one of the two connected components of $\{u \in H^{1,1}(X;\mathbf{R}) \ , \ \langle u|u\rangle = 1\}$ and we denote by $\mathbf{H}(X)$ this component; $\mathbf{H}(X)$ is a model of the hyperbolic space of dimension $h^{1,1}(X)-1$ (see [27]).

2.2. **Action on cohomology groups (see** [27]**, §2).** Every $f \in \mathsf{Aut}(X)$ induces a linear invertible mapping f^* on $H^{1,1}(X;\mathbf{R})$ which is an isometry for the intersection product. Since the Kähler cone is f^*-invariant, so is $\mathbf{H}(X)$. Let λ_f denote the spectral radius of $f^* \colon H^{1,1}(X;\mathbf{R}) \to H^{1,1}(X;\mathbf{R})$.

- λ_f coincides with the spectral radius of f^* acting on the full cohomology group $\oplus_{k=0}^4 H^k(X;\mathbf{C})$.
- The topological entropy of f is equal to $\log \lambda_f$.
- When $\lambda_f > 1$ the eigenvalues of f^* on $H^{1,1}(X;\mathbf{R})$ (resp. on $H^2(X;\mathbf{R})$) are precisely λ_f, λ_f^{-1} (which are both simple), and complex numbers of modulus 1. The eigenlines corresponding to λ_f and λ_f^{-1} are isotropic, and they intersect the nef cone.

From the third item, we can fix a nef eigenvector θ_f^+ (resp. θ_f^-) for the eigenvalue λ_f (resp. λ_f^{-1}); we shall impose extra conditions on these classes in Section 2.4. Let Π_f be the subspace generated by θ_f^+ and θ_f^-, and let $\Pi_f^\perp$ be its orthogonal complement in $H^{1,1}(X;\mathbf{R})$. The intersection form has signature $(1,1)$ on Π_f and is negative definite on $\Pi_f^\perp$. Classes of irreducible periodic curves are in $\Pi_f^\perp$; hence, there are only finitely many of them, and we can contract them by the Grauert-Mumford criterion: this produces a (normal) singular complex analytic surface X_0 (see Section 6.1). Starting with Section 6, it will be necessary to work on such a singular surface.

2.3. **Invariant currents and continuous potentials.** We refer to [54, Chapter 3] and [38] for an account concerning currents on complex manifolds. Let T be a closed positive current (of bidegree $(1,1)$) on X. It is locally equal to $dd^c u$ where u

is a pluri-subharmonic function. By definition u is a **local potential** of T, it is unique up to addition of a pluri-harmonic function. Every closed positive current T has a cohomology class $[T]$ in $H^{1,1}(X,\mathbf{R})$. For instance, if $C \subset X$ is a complex curve and T_C is the current of integration on C, then $[T_C] = [C]$. If $f \in \mathsf{Aut}(X)$ one can define f^*T by duality: the value of f^*T on a $(1,1)$-form η is equal to the value of T on $(f^{-1})^*\eta$. If u is a local potential for T, then $u \circ f$ is a local potential for f^*T.

Theorem 2.1 (see [41, 73, 27])**.** *Let f be an automorphism of a compact Kähler surface X with positive entropy* $\log\lambda_f$. *There exists a unique closed positive current T_f^+ of bidegree $(1,1)$ on X such that $[T_f^+]$ coincides with the nef class θ_f^+. Its local potentials are Hölder continuous. Moreover, $f^*T_f^+ = \lambda_f T_f^+$.*

Similarly, there exists a unique closed positive current T_f^- of bidegree $(1,1)$ such that $[T_f^-]$ coincides with the nef class θ_f^-; this current satisfies $f^*T_f^- = \lambda_f^{-1} T_f^-$.

Remark 2.2. Dinh and Sibony strengthen Theorem 2.1 in [43] by showing that $T_f^{\pm}$ is the unique dd^c-closed positive current whose cohomology class is $[T_f^{\pm}]$.

Let C be a Riemann surface and $\theta\colon C \to X$ be a non-constant holomorphic mapping. The pull-back $\theta^*(T_f^+)$ is locally defined as $dd^c(u^+ \circ \theta)$ where u^+ is a local potential; by definition, this measure (resp. its image on $\theta(C)$) is called the **slice** of T_f^+ by θ. The same definition applies for $\theta^*(T_f^-)$.

Example 2.3. The following classical argument illustrates this notion. Let $\xi\colon \mathbb{D} \to X$ be a holomorphic map. Assume that there is a sequence of integers $n_i \to +\infty$ such that $f^{n_i} \circ \xi$ converges locally uniformly towards a holomorphic map $\eta\colon \mathbb{D} \to X$. Let $\varphi\colon \mathbb{D} \to \mathbf{R}_+$ be a smooth function with compact support in $\mathbb{D}$. Then, on one hand $\langle (f^{n_i} \circ \xi)^* T_f^+ | \varphi \rangle = \lambda_f^{n_i} \langle \xi^* T_f^+ | \varphi \rangle$, and on the other hand $\langle (f^{n_i} \circ \xi)^* T_f^+ | \varphi \rangle$ converges towards $\langle \eta^* T_f^+ | \varphi \rangle$. Since $\lambda_f^{n_i}$ goes to $+\infty$, we get $\xi^* T_f^+ = 0$. For instance, if $\xi(\mathbb{D})$ is contained in a stable manifold of f, then $\xi^* T_f^+ = 0$ (see below Theorem 5.1).

2.4. **Definition and properties of μ_f (see § 1.1 and** [27]**).** In what follows, we assume that X is projective and we fix the following data: a Kähler form κ on X and eigenvectors θ_f^+ and θ_f^- with respect to the eigenvalues λ_f and λ_f^{-1} for the endomorphism f^* of $H^{1,1}(X;\mathbf{R})$ such that

$$\langle \theta_f^+ | \theta_f^- \rangle = 1, \quad \langle \theta_f^+ | [\kappa] \rangle = \langle \theta_f^- | [\kappa] \rangle = 1. \tag{2.1}$$

Then, consider the currents $T_f^{\pm}$ provided by Theorem 2.1. The wedge product $T_f^+ \wedge T_f^-$ is locally defined as the dd^c-derivative of $u^+ dd^c u^-$, where u^+ and u^- are local potentials for T^+ and T^-. This product defines an f-invariant probability measure

$$\mu_f := T_f^+ \wedge T_f^- .$$

The dynamics of f with respect to μ_f is ergodic and mixing; moreover, μ_f is the unique invariant measure with maximal entropy.

3. Stable manifolds: parametrization and currents

In this section we recall Pesin theory and state that f can be linearized along the stable manifolds, as in [13, 59]. The stable manifolds are isomorphic to $\mathbf{C}$ and we deal with their Ahlfors currents (see also [8, Section 2.6]).

3.1. The Oseledets theorem and Lyapunov exponents (see [81, 62] and [8, 27]).

Let X be a complex projective surface and f be an automorphism of X with positive entropy $\log \lambda_f$. Let $T_f^{\pm}$ be the invariant currents introduced in Section 2.3. The normalization chosen in Equation (2.1) implies that T_f^+ and T_f^- have mass 1 with respect to the Kähler form κ, and that $\mu_f = T_f^+ \wedge T_f^-$ is an f-invariant probability measure. Since μ_f has positive entropy and is ergodic, it has one negative and one positive Lyapunov exponent; we denote them by λ_s and λ_u, with $\lambda_s < 0 < \lambda_u$ (each of them has multiplicity 2 if f is viewed as a diffeomorphism of the 4-dimensional real manifold X).

In what follows, ε denotes a positive real number that satisfies $\varepsilon \ll \min(|\lambda_s|, \lambda_u)$. The set Λ will be a Borel subset of X of total μ_f-measure; its precise definition depends on ε and may change from one paragraph to another. By construction, we can (and do) assume that Λ is invariant: indeed, Λ can always be replaced by $\cap_{n\in\mathbf{Z}} f^n(\Lambda)$. A measurable function $\alpha : \Lambda \to]0,1]$ is **ε-tempered** if it satisfies $e^{-\varepsilon}\alpha(x) \leq \alpha(f(x)) \leq e^{\varepsilon}\alpha(x)$ for every $x \in \Lambda$.

We use the same notation $\| \cdot \|$ for the standard hermitian norm on $\mathbf{C}^2$ and for the hermitian norm on the tangent bundle TX induced by the Kähler form κ. The distance on X is denoted dist_X; $\mathrm{B}_x(r)$ is the ball of radius r centered at x.

Theorem 3.1 (Oseledets-Pesin)**.** *Let X be a complex projective surface and let f be an automorphism of X with positive entropy* $\log \lambda(f) > 0$.

There exist an f-invariant Borel subset $\Lambda \subset X$ with $\mu_f(\Lambda) = 1$, two ε-tempered functions $q : \Lambda \to]0,1]$, $\beta : \Lambda \to]0,1]$, and a family of holomorphic mappings $(\Psi_x)_{x\in\Lambda}$ satisfying the following properties.

(1) *Ψ_x is defined on the bidisk $\mathbb{D}(q(x)) \times \mathbb{D}(q(x))$, takes values in X, maps the origin to the point x, and is a diffeomorphism onto its image.*

(2) $\beta(x) \parallel z_1 - z_2 \parallel \leq \mathsf{dist}_X(\Psi_x(z_1), \Psi_x(z_2)) \leq \parallel z_1 - z_2 \parallel$ *for all pairs of points* z_1 *and* z_2 *in the bidisk.*

(3) *The local diffeomorphism* $f_x := \Psi_{f(x)}^{-1} \circ f \circ \Psi_x$ *is well defined near the origin in* $\mathbb{D}(q(x)) \times \mathbb{D}(q(x))$*, and the matrix of* $D_0 f_x$ *is diagonal with coefficients* $a(x)$ *and* $b(x)$ *that satisfy*

$$|a(x)| \in e^{\lambda_u} \cdot [e^{-\varepsilon}, e^{\varepsilon}], \quad |b(x)| \in e^{\lambda_s} \cdot [e^{-\varepsilon}, e^{\varepsilon}].$$

Moreover f_x *is* ε*-close to the linear mapping* $D_0 f_x$ *in the* $\mathcal{C}^1$ *topology.*

The global **stable manifold** of a point x is the set $W^s(x)$ of points x' such that $\mathsf{dist}_X(f^n(x), f^n(x'))$ goes to 0 as n goes to $+\infty$. The **local stable manifold** $W^{s,loc}(x)$ is the connected component of $W^s(x) \cap \Psi_x(\mathbb{D}(q(x)) \times \mathbb{D}(q(x)))$ that contains x.

In $\mathbb{D}(q(x)) \times \mathbb{D}(q(x))$, the inverse image by Ψ_x of the local stable manifold is a vertical graph; in other words, it can be parametrized by a holomorphic map

$$\gamma_x : \mathbb{D}(q(x)) \to \mathbb{D}(q(x)) \times \mathbb{D}(q(x)), \quad z \mapsto (g_x(z), z),$$

where g_x is holomorphic and satisfies $g_x(0) = 0$, $g_x'(0) = 0$, and $\mathrm{Lip}\, g_x \leq 1$. We have $W^s(x) = \cup_{n \geq 0} f^{-n}(W^{s,loc}(f^n(x)))$.

Notation 3.2. *For every* x *in* Λ*, we set* $\sigma_x := \Psi_x \circ \gamma_x$*. This is a holomorphic parametrization of the local stable manifold* $W^{s,loc}(x)$*. By construction,*

- $\beta(x) \leq \parallel \sigma_x'(0) \parallel \leq 1$,
- $f(W^{s,loc}(x)) \subset W^{s,loc}(f(x))$ *and* $W^{s,loc}(x) \subset \mathrm{B}_x(1)$,
- $\lim_{n \to +\infty} \mathsf{dist}_X(f^n(x), f^n(y)) = 0$ *for every* $y \in W^{s,loc}(x)$.

We denote by $F_x : \mathbb{D}(q(x)) \to \mathbb{D}(q(f(x)))$ the mapping that satisfies

$$f \circ \sigma_x = \sigma_{f(x)} \circ F_x,$$

and by $M_x : \mathbb{D}(q(x)) \to \mathbb{D}(q(f(x)))$ the linear mapping given by

$$M_x(z) := m_x \cdot z, \text{ with } \quad m_x := F_x'(0).$$

By construction, $|m_x| \in e^{\lambda_s} \cdot [e^{-\varepsilon}, e^{\varepsilon}]$ (note that m_x is equal to the complex number $b(x)$ of Theorem 3.1).

3.2. **Linearization along the stable manifolds and entire parametrizations.** The following proposition provides a linearization of f along the stable manifolds, we refer to [13, 59] for a statement in arbitrary dimension. We keep the same notations as in the previous paragraph: ε is fixed and Λ is given by Theorem 3.1.

Proposition 3.3. *Let f be an automorphism of a complex projective surface X with positive entropy. If ε is small enough, there exist a real number $c = c(\varepsilon, \lambda_s) \in]0,1]$ and holomorphic injective functions $(\eta_x)_{x\in\Lambda}$, such that*

(1) *η_x is defined on $\mathbb{D}(cq(x))$, with values in $\mathbb{D}(q(x))$, and it satisfies $\eta_x(0) = 0$ and $\eta_x'(0) = 1$;*
(2) *$f \circ (\sigma_x \circ \eta_x) = (\sigma_{f(x)} \circ \eta_{f(x)}) \circ M_x$ on $\mathbb{D}(cq(x))$.*

Definition 3.2 and Proposition 3.3 allow us to introduce the following notation.

Notation 3.4. *For every $x \in \Lambda$, define $\xi_x^{s,loc} : \mathbb{D}(cq(x)) \to W^{s,loc}(x)$ by*

$$\xi_x^{s,loc} := \sigma_x \circ \eta_x.$$

By construction, $\xi_x^{s,loc}$ is holomorphic, injective and satisfies

- *$\xi_x^{s,loc}(0) = x$ and $\beta(x) \leq \| (\xi_x^{s,loc})'(0) \| \leq 1$;*
- *$2\beta(x)/3 \leq \| (\xi_x^{s,loc})' \| \leq 2$ on $\mathbb{D}(cq(x))$;*
- *$f \circ \xi_x^{s,loc} = \xi_{f(x)}^{s,loc} \circ M_x$ on $\mathbb{D}(cq(x))$.*

The second item follows from the construction of σ_x and Koebe's inequality for η_x.

Since the global stable manifolds satisfy $W^s(x) = \cup_{n\geq 0} f^{-n}\left(W^{s,loc}(f^n(x))\right)$, they are simply connected Riemann surfaces.

Proposition 3.5. *For every $x \in \Lambda$, the Riemann surface $W^s(x)$ is biholomorphic to $\mathbf{C}$. Moreover, there exists a biholomorphism $\xi_x^s : \mathbf{C} \to W^s(x)$ such that*

- *$\xi_x^s(0) = x$,*
- *$\xi_x^s = \xi_x^{s,loc}$ on $\mathbb{D}(cq(x))$,*
- *$f \circ \xi_x^s = \xi_{f(x)}^s \circ M_x$ on $\mathbf{C}$.*

Proof. Set $M_x^m := M_{f^{m-1}(x)} \circ \ldots \circ M_x : \mathbf{C} \to \mathbf{C}$ for every $m \geq 1$, and observe that $|M_x^m(z)| \in e^{m\lambda_s} \cdot [e^{-m\varepsilon}, e^{m\varepsilon}] \cdot |z|$. Then, define

$$\forall z \in \mathbf{C},\ \ \xi_x^s(z) := f^{-m(z)} \circ \xi_{f^{m(z)}(x)}^{s,loc} \circ M_x^{m(z)}(z),$$

where $m(z)$ is a large positive integer, so that $M_x^{m(z)}(z) \in \mathbb{D}(cq(f^{m(z)}(x)))$; such an integer exists because the function q is ε-tempered. One easily verifies that (i) the definition of ξ_x^s does not depend on $m(z)$ and (ii) $f \circ \xi_x^s = \xi_{f(x)}^s \circ M_x$ on $\mathbf{C}$ by analytic continuation. The map $\xi_x^s : \mathbf{C} \to W^s(x)$ is a biholomorphism by the definition of $W^s(x)$ and the fact that f has empty critical set. □

Remark 3.6. The definition of $\xi_x^s : \mathbf{C} \to W^s(x)$ depends on the local parametrizations $(\xi_x^{s,loc})_{x\in\Lambda}$, hence on Pesin's theory. In particular, the derivative $(\xi_x^s)'(0)$ depends measurably on x. Two biholomorphisms $\mathbf{C} \to W^s(x)$ differ by an affine automorphism $z \mapsto az+b$ where $(a,b) \in \mathbf{C}^* \times \mathbf{C}$. Thus, (i) the stable manifold $W^s(x)$ inherits a natural affine structure and (ii) every biholomorphism $\mathbf{C} \to W^s(x)$ sending 0 to x is equal to ξ_x^s modulo composition with a homothety $z \mapsto az$.

3.3. **Currents associated to entire curves.**

3.3.1. *Ahlfors-Nevanlinna currents.* Let X be a compact Kähler surface and $\| v \|$ the norm of a tangent vector v with respect to this metric. Let $\xi\colon \mathbf{C} \to X$ be a non-constant entire curve. The area of $\xi(\mathbb{D}_r)$ and the length of $\xi(\partial\mathbb{D}_r)$ are defined by

$$\begin{aligned} A(r;\xi) &:= \int_0^r \int_0^{2\pi} \| \xi'(te^{i\theta}) \|^2 \, t dt d\theta \\ L(r;\xi) &:= \int_0^{2\pi} \| \xi'(re^{i\theta}) \| \, r d\theta. \end{aligned}$$

Let $\{\xi(\mathbb{D}_s)\}$ be the current of integration on $\xi(\mathbb{D}_s)$ counting multiplicities and let

$$S(r;\xi) := \frac{1}{A(r;\xi)} \{\xi(\mathbb{D}_r)\}.$$

Ahlfors proved that there exist sequences (r_n) tending to infinity such that the ratio $L(r_n;\xi)/A(r_n;\xi)$ tends to zero (see [74, §7.6.4] and Section 10.4.2 below). Hence $(S(r;\xi))_r$ has closed limits; these limits are the **Ahlfors currents** associated to ξ. Let us also define

$$N(r;\xi) := \frac{1}{T(r;\xi)} \int_{s=0}^r \{\xi(\mathbb{D}_s)\} \frac{ds}{s}$$

where $T(r;\xi) := \int_{s=0}^r A(s;\xi)\frac{ds}{s}$. As before there exist sequences (r_n) such that $N(r_n;\xi)$ converges to a closed positive current on M, see [20]. These limits are called the **Ahlfors-Nevanlinna currents** associated to ξ.

Proposition 3.7 (Brunella [20]; see also § 10.4.2)**.** *Let X be a compact Kähler surface and $\xi\colon \mathbf{C} \to X$ be a non-constant entire curve. Let A be an Ahlfors-Nevanlinna current determined by ξ, and let $[A] \in H^{1,1}(X,\mathbf{R})$ be its cohomology class.*

(1) *If $\xi(\mathbf{C})$ is contained in an irreducible curve E, then the genus of E is equal to 0 or 1 and A is equal to $Area(E)^{-1}\{E\}$.*
(2) *If the area $A(r;\xi)$ is bounded by a constant which does not depend on r, then $\xi(\mathbf{C})$ is contained in a compact curve $E \subset X$.*

If $\xi(\mathbf{C})$ is not contained in a compact curve, then

(3) $\langle [A] | [C] \rangle \geq 0$ *for every curve $C \subset X$;*

(4) $[A]$ *is in the nef cone of X and therefore* $\langle [A] | [A] \rangle \geq 0$.

Properties (3) and (4) may a priori fail for Ahlfors currents (though we do not know of any example).

3.3.2. *Stable and unstable Ahlfors currents.* By the following theorem, the currents associated to the stable manifolds of f are equal to T_f^+; to state it, recall that Λ is the set of full μ_f-measure introduced in Theorem 3.1.

Theorem 3.8 (see [8, 23, 27])**.** *Let f be an automorphism of a complex projective surface X with positive entropy. Let $\xi_x^s \colon \mathbf{C} \to X$ be a parametrization of the stable manifold $W^s(x)$ of a point $x \in \Lambda$. If $\xi_x^s(\mathbf{C})$ is not contained in an algebraic periodic curve, then all Ahlfors-Nevanlinna currents associated to ξ_x^s coincide with T_f^+.*

For ξ_x^s one can take the parametrization of Proposition 3.5. A similar result holds for unstable manifolds and the current T_f^-.

Remark 3.9. In [43], Dinh and Sibony proved the following strengthening of Theorem 3.8 (we shall not need it): *if $\xi \colon \mathbf{C} \to X$ is an entire curve such that*

- *$\xi(\mathbf{C})$ is not contained in an algebraic periodic curve of f, and*
- *the family of entire curves $f^n \circ \xi$, $n \geq 1$, is locally equicontinuous (i.e. is a normal family of entire curves),*

then all Ahlfors-Nevanlinna currents of ξ coincide with T_f^+.

4. Product structure and absolute continuity

The currents $T_f^{\pm}$ have a geometric property called weak laminarity. We relate it to the dynamical notion of Pesin boxes, and explain that μ_f has a product structure in these boxes. This leads to Proposition 4.5 saying that almost all slices of T_f^- and T_f^+ by stable and unstable manifolds give rise to absolutely continuous measures; in other words, one can transfer the regularity assumption on μ_f to a (rather weak) regularity property of T_f^+ and T_f^-. We refer to [8, 23, 27, 46] for the proofs of some of the results used in this section.

4.1. **Laminations and quasi-conformal homeomorphisms.**

4.1.1. *Quasi-conformal homeomorphisms (see* [1, Chapter II]*).* Let $h \colon \mathcal{U} \to \mathcal{V}$ be an orientation preserving homeomorphism between two Riemann surfaces. One says that h is K-quasi-conformal, for some real number $K \geq 1$, if h is absolutely continuous on lines and

$$|\partial_{\bar{z}} h| \leq \frac{K-1}{K+1} |\partial_z h|$$

almost everywhere. A 1-quasi-conformal mapping is holomorphic. We shall say that a homeomorphism h between two (open subsets of) Riemann surfaces is **absolutely continuous** with respect to the Lebesgue measure if h and h^{-1} map sets of measure 0 to sets of measure 0 or, equivalently, if h preserves the Lebesgue class (see [68], § III.3, pages 190-191).

Lemma 4.1 (see [1], Chapter II, Theorem 3)**.** *If h is a quasi-conformal homeomorphism, then h is absolutely continuous with respect to the Lebesgue measure.*

To be more precise, fix a local co-ordinate z on $\mathcal{U}$, with Lebesgue measure Leb given by $\frac{\mathsf{i}}{2}dz\wedge d\bar{z}$. Then, the partial derivatives of h are well defined almost everywhere because h is absolutely continuous on lines. Its jacobian determinant $\mathsf{Jac(h)}$ is locally integrable, it is positive Lebesgue almost everywhere, and for every Borel subset $A\subset\mathcal{U}$ it satisfies

$$\mathsf{Leb}(h(A))=\int_A\mathsf{Jac}(h)(z)\,\frac{\mathsf{i}}{2}dz\wedge d\bar{z}.$$

4.1.2. *Laminations in bidisks (see* [45, 52]*).* By definition, a **horizontal graph** in the bidisk $\mathbb{D}\times\mathbb{D}$ is the graph $\{(z,\varphi(z));z\in\mathbb{D}\}$ of a holomorphic function $\varphi\colon\mathbb{D}\to\mathbb{D}$; thus, horizontal graphs are smooth analytic subsets of $\mathbb{D}\times\mathbb{D}$ that intersect every vertical disk $\{z\}\times\mathbb{D}$ in exactly one point. Vertical graphs are images of horizontal graphs by permutation of the co-ordinates.

Consider a family of disjoint, horizontal graphs in $\mathbb{D}\times\mathbb{D}$. If m is a point on $\{0\}\times\mathbb{D}$ which is contained in one of these graphs, one denotes by $\varphi_m\colon\mathbb{D}\to\mathbb{D}$ the holomorphic function such that $z\mapsto(z,\varphi_m(z))$ parametrizes the graph through m. By the Montel and Hurwitz theorems, one can extend this family of graphs in a unique way into a lamination $\mathcal{L}$ of a compact subset $\mathcal{K}$ of $\mathbb{D}\times\mathbb{D}$ by disjoint horizontal graphs. The leaf of $\mathcal{L}$ through a point m is denoted $\mathcal{L}(m)$.

If z_1 and z_2 are two points on $\mathbb{D}$, the vertical disks

$$\Delta_j=\{(z,w)\,|\,z=z_j\}\quad(j=1,2)$$

are transverse to the lamination $\mathcal{L}$. Denote by h_{z_1,z_2} the holonomy of the lamination $\mathcal{L}$ from Δ_1 to Δ_2; more generally, if Δ and Δ' are two complex analytic transversals (intersecting each leaf into exactly one point), one gets a holonomy map from Δ to Δ'. By the Λ-Lemma (see [45]), *the holonomy is automatically quasi-conformal* (one also says *quasi-symmetric*); by this, we mean that it extends to a quasi-conformal homeomorphism of a neighborhood of $\Delta_1\cap\mathcal{K}$ to a neighborhood of $\Delta_2\cap\mathcal{K}$; in particular, it is absolutely continuous with respect to the Lebesgue measure. Moreover, the quasi-conformal constant $K(z_1,z_2)$ of h_{z_1,z_2} satisfies

$$0 \le K(z_1,z_2) - 1 \le |z_2 - z_1|.$$

It converges to 1 when Δ' converges to Δ in the $\mathcal{C}^1$-topology.

Let $\mathcal{K}_0$ be the intersection of the support $\mathcal{K}$ of $\mathcal{L}$ with the vertical disk $\{0\} \times \mathbb{D}$. Define $\Phi\colon \mathbb{D} \times \mathcal{K}_0 \to \mathbb{D} \times \mathbb{D}$ by

$$\Phi(z,m) = (z, \varphi_m(z));$$

this map realizes a homeomorphism from $\mathbb{D} \times \mathcal{K}_0$ to $\mathcal{K}$ that maps the trivial horizontal lamination to the lamination $\mathcal{L}$.

Lemma 4.2. *The homeomorphism Φ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{D} \times \mathbb{D}$, i.e. Lebesgue subsets with zero Lebesgue measure are mapped to subsets with zero Lebesgue measure.*

Proof. Let $B \subset \mathbb{D} \times \mathbb{D}$ be a Borel subset with Lebesgue measure 0. Let B_z denote the vertical slices of B, $B_z := B \cap \{z\} \times \mathbb{D}$. By the Fubini theorem, almost every B_z is negligeable for the Lebesgue measure on the vertical disk. The image of B_z by the homeomorphism Φ is the set of points $\{(z, \varphi_m(z)) \mid (z,m) \in B_z\}$, and it coincides with the image of $\{(0,m) \mid (z,m) \in B_z\}$ under the holonomy map $h_{0,z}$. Since the holonomy maps are quasi-conformal, they are absolutely continuous, and $\Phi(B_z)$ is Lebesgue negligeable for almost every z. By the Fubini theorem, the Lebesgue measure of $\Phi(B)$ in $\mathbb{D} \times \mathbb{D}$ vanishes. □

4.2. **Laminarity and Pesin boxes.**

4.2.1. *Pesin boxes (see* [8, Section 4] *and* [23, 27, 46]*).* A **Pesin box** $\mathcal{P}$ for the automorphism $f\colon X \to X$ consists in an open subset $\mathcal{U}$ of X which is biholomorphic to a bidisk $\mathbb{D} \times \mathbb{D}$ together with two transverse laminations $\mathcal{L}^s$ and $\mathcal{L}^u$ satisfying properties that we now describe.

By convention, the lamination $\mathcal{L}^u$ is horizontal: its leaves $\mathcal{L}^u(m)$ are horizontal graphs. These graphs $\mathcal{L}^u(m)$ intersect the vertical transversal $\{0\} \times \mathbb{D}$ onto a compact set $\mathcal{K}^-$ and the union of these graphs is homeomorphic to the product $\mathbb{D} \times \mathcal{K}^-$. Similarly, $\mathcal{L}^s$ is a lamination by vertical graphs with support homeomorphic to $\mathcal{K}^+ \times \mathbb{D}$.

Given a point $w \in \mathcal{K}^-$ and a point $w' \in \mathcal{K}^+$, the horizontal leaf $\mathcal{L}^u((0,w))$ intersects the vertical leaf $\mathcal{L}^s((w',0))$ in a unique point $[w,w'] \in \mathcal{U}$. This provides a homeomorphism h between the product $\mathcal{K}^- \times \mathcal{K}^+$ and the intersection $\mathcal{K} \subset \mathcal{U}$ of the supports of $\mathcal{L}^s$ and $\mathcal{L}^u$. Moreover, by definition, a Pesin box $\mathcal{P} = (\mathcal{U}, \mathcal{L}^u, \mathcal{L}^s)$ must satisfy the following properties.

(0) – For μ_f-almost every point $x \in \mathcal{K}$, the leaf $\mathcal{L}^u(x)$ (resp. $\mathcal{L}^s(x)$) is contained in the global stable manifold $W^u(x)$ (resp. $W^s(x)$).

(1) – There is a measure ν^+ whose support is $\mathcal{K}^+$ such that the laminar current

$$T^+_{\mathcal{P}} := \int_{w \in \mathcal{K}^+} \{\mathcal{L}^s(w)\} d\nu^+(w)$$

is dominated by the restriction of T^+_f to $\mathcal{U}$ and coincides with T^+_f on the set of continuous $(1,1)$-forms whose support is a compact subset of the support of $\mathcal{L}^s$.

(2) – There is, similarly, a uniformly laminar current $T^-_{\mathcal{P}}$ associated to the lamination $\mathcal{L}^u$ and a transverse measure ν^- whose support is $\mathcal{K}^-$; this current is the restriction of T^-_f to the support of the unstable lamination $\mathcal{L}^u$.

(3) – Via the homeomorphism $h\colon \mathcal{K}^- \times \mathcal{K}^+ \to \mathcal{K}$, the measure μ_f corresponds to the product measure $\nu^+ \otimes \nu^-$, i.e. $\mu_{f|\mathcal{K}} = h_*(\nu^+ \otimes \nu^-)$.

In a Pesin box $\mathcal{P}$, the measure ν^+ can be identified to the conditional measure of μ_f with respect to the lamination $\mathcal{L}^s$ (see Property (3)). One way to specify this fact is the following. By Property (1), one can slice T^+_f with an unstable leaf $\mathcal{L}^u(m)$ to get a measure $(T^+_f)_{|\mathcal{L}^u(m)}$ (see Section 2.3), then restrict this measure to the intersection of $\mathcal{L}^u(m)$ with the support of the stable lamination, and then push it on $\mathcal{K}^+$ (using the holonomy of $\mathcal{L}^s$); again, one gets ν^+.

Pesin boxes exist, and their union has full μ_f-measure: this comes from Pesin theory of non-uniformly hyperbolic dynamical systems, and from the fact that T^+_f and T^-_f are Ahlfors currents of entire curves parametrizing generic stable and unstable manifolds. See [8, Section 4] (and also [23, 27, 46]).

4.2.2. *Laminar structure of $T^\pm_f$ (see* [8, 46, 27]*).* The previous section says that $T^\pm_f$ is uniformly laminar in each Pesin box. In fact, $T^\pm_f$ is a sum of such currents. More precisely, there is a countable family of Pesin boxes $\mathcal{P}_i = (\mathcal{U}_i, \mathcal{L}^u_i, \mathcal{L}^s_i)$, with transverse measures $\nu^\pm_i$, such that the support of the stable laminations $\mathcal{L}^s_i$ are disjoint, and T^+_f is the sum

$$T^+_f = \sum_i T^+_{\mathcal{P}_i}$$

where $T^+_{\mathcal{P}_i}$ is the laminar current associated to an atomless measure ν^+_i:

$$T^+_{\mathcal{P}_i} = \int_{w \in \mathcal{K}^+} \{\mathcal{L}^s_i(w)\}\, d\nu^+_i(w).$$

Similarly, T^-_f is a sum of uniformly laminar currents $T^-_{\mathcal{P}_j}$ with respect to disjoint Pesin boxes. The generic disks in these laminar structures are pieces of unstable manifolds of f (resp. of stable manifolds for T_f^+). Therefore, $\xi^*(T_f^-) = 0$ for all parametrizations of unstable manifolds of f (see Example 2.3).

4.3. **Absolute continuity of the slices of the invariant currents.**

4.3.1. *Absolute continuity of the transverse measures $\nu^\pm$ in Pesin boxes.*

Lemma 4.3. *Let $\mathcal{P} = (\mathcal{U}, \mathcal{L}^u, \mathcal{L}^s)$ be a Pesin box with transverse measures ν^+ and ν^- as in Section 4.2. If μ_f is absolutely continuous with respect to the Lebesgue measure, then ν^+ and ν^- are absolutely continuous with respect to the Lebesgue measure.*

As was the case with Lemma 4.2, the proof resides on the absolute continuity of the holonomy of $\mathcal{L}^u$ and $\mathcal{L}^s$, which we obtained from the Λ-lemma. We provide this proof because it is closely related to the arguments of Section 8.1. There is a more general approach, due to Pesin, which necessitates a direct proof of the absolute continuity of the stable and unstable laminations, see [3, Chapter 8] and [75, Chapter 7].

Proof. Let Δ be the vertical disk $\{0\} \times \mathbb{D}$; it is transverse to $\mathcal{L}^u$. Let $A \subset \Delta$ be a Borel subset with Lebesgue measure 0. Let $\mathcal{L}^u(A)$ be the union of the leaves of $\mathcal{L}^u$ that intersect A. Since the holonomy maps are absolutely continuous (see Section 4.1.2), every slice of $\mathcal{L}^u(A)$ by a vertical disk has Lebesgue measure 0. Thus, by the Fubini theorem and the absolute continuity of μ_f, $\mu_f(\mathcal{L}^u(A)) = 0$. Since $\mu_f = h_*(\nu^+ \otimes \nu^-)$ in $\mathcal{K}$ (see Property (3) of Pesin boxes), one concludes that $\nu^+(A) = 0$. This shows that ν^+ is absolutely continuous with respect to the Lebesgue measure. The argument is similar for ν^-. □

4.3.2. *Slices of the invariant currents.*

Lemma 4.4. *Let M be a complex manifold. Let T be a closed positive $(1,1)$-current with local continuous potentials on M. Let $\mathcal{U}$ be an open subset of $\mathbf{C}$. Let $\nu_n\colon \mathcal{U} \to M$ be a sequence of holomorphic mappings that converges uniformly to $\nu\colon \mathcal{U} \to M$ on compact subsets of $\mathcal{U}$. Then, the sequence of measures $\nu_n^* T$ converges weakly to $\nu^* T$ as n goes to $+\infty$.*

Proof. Let $\mathcal{V}$ be an open subset of M on which T is given by a continuous potential u. If ν maps $\mathcal{U}' \subset \mathcal{U}$ into $\mathcal{V}$, then for every test function φ with support contained in $\mathcal{U}'$, the dominated convergence theorem implies that

$$\langle \nu^* T | \varphi \rangle = \int_{\mathcal{U}'} u \circ \nu(z) dd^c \varphi(z) = \lim_{n\to\infty} \int_{\mathcal{U}'} u \circ \nu_n(z) dd^c \varphi(z) = \lim_{n\to\infty} \langle \nu_n^* T | \varphi \rangle.$$

The result follows. □

Let us now consider a uniformly laminar current S, in a subset $\mathcal{U}$ of X which is biholomorphic to the bidisk $\mathbb{D}\times\mathbb{D}$; more precisely, S is determined by a lamination $\mathcal{L}$ of a compact subset $\mathcal{K}$ of $\mathcal{U}$ by horizontal graphs and by a transverse measure ν_S: for every smooth test form α whose support is contained in $\mathcal{U}$,

$$\langle S|\alpha\rangle = \int_{\mathcal{K}_0}\int_{\mathcal{L}(m)} \alpha \, d\nu_S(m) = \int_{\mathcal{K}_0} \langle \mathcal{L}(m)|\alpha\rangle \, d\nu_S(m)$$

where $\mathcal{K}_0$ is the intersection of $\mathcal{K}$ with the vertical disk $\{0\}\times\mathbb{D}\subset\mathcal{U}$. We denote by Φ the homeomorphism from $\mathbb{D}\times\mathcal{K}_0$ to $\mathcal{K}\subset\mathcal{L}$ which is defined by $\Phi(z,m) = (z,\varphi_m(z))$; fixing m, we get a holomorphic parametrization

$$\Phi_m(z) := \Phi(z,m) = (z,\varphi_m(z))$$

of the leaf $\mathcal{L}(m)$ (see the notation from Section 4.1.2).

Proposition 4.5. *Let S be such a uniformly laminar current, and assume that $S\leq T_f^+$ in $\mathcal{U}$. If μ_f is absolutely continuous with respect to the Lebesgue measure (on X) then, for ν_S almost every $m\in\mathcal{K}_0$, the measure $\Phi_m^*(T_f^-)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{D}$.*

Proof. First, recall that T_f^- has continuous potentials: on the bidisk $\mathcal{U}$, one can find a continuous function u^- such that $T_f^- = dd^c(u^-)$ in $\mathcal{U}$. The pull-back $\Phi_m^*(T_f^-)$ of T_f^- is defined by $\Phi_m^*(T_f^-) = dd^c(u^-\circ\Phi_m)$ for every leaf $\mathcal{L}(m)$.

Since $S\leq T_f^+$, one knows that S has continuous potentials (see [8, Lemmas 8.2 and 8.3]), and that $S\wedge T_f^-\leq T_f^+\wedge T_f^-$ is absolutely continuous with respect to the Lebesgue measure, because $\mu_f = T_f^+\wedge T_f^-$ is absolutely continuous.

The product $S\wedge T_f^-$ is defined by its values on smooth functions $\alpha\colon\mathcal{U}\to\mathbf{R}$ with compact support:

$$\langle S\wedge T_f^-|\alpha\rangle = \langle S|u^- dd^c(\alpha)\rangle = \int_{\mathcal{K}_0}\int_{\mathcal{L}(m)} u^- dd^c(\alpha)\, d\nu_S(m).$$

Moreover,

$$\int_{\mathcal{L}(m)} u^- dd^c(\alpha) = \int_{\mathbb{D}} u^-\circ\Phi_m\cdot dd^c(\alpha\circ\Phi_m) = \langle\Phi_m^* T_f^-|\alpha\circ\Phi_m\rangle.$$

In other words, the measure $\Phi^*(S\wedge T_f^-)$ satisfies

$$\langle\Phi^*(S\wedge T_f^-)|\psi\rangle = \int_{\mathcal{K}_0}\int_{\mathbb{D}} \psi(z,m) d\Phi_m^*(T_f^-)(z)\, d\nu_S(m). \tag{4.1}$$

From Lemma 4.2, we know that $\Phi^*(S\wedge T_f^-)$ is absolutely continuous with respect to the Lebesgue measure, because $S\wedge T_f^-$ is. Consider the projection $\pi\colon \mathbb{D}\times\mathbb{D}\to\mathbb{D}$ defined by $\pi(z,m)=m$, and define $M(m)$ to be the total mass of $\Phi_m^*(T_f^-)$ on the disk $\mathbb{D}$; Equation (4.1) says that the marginal of the measure $\Phi^*(S\wedge T_f^-)$ with respect to the projection π is the measure

$$M(m)d\nu_S(m)$$

and that the conditional measures along the fibers $\mathbb{D}\times\{m\}$ coincide with the measures $M(m)^{-1}\Phi_m^*(T_f^-)$. Thus, for a ν_S-generic point m, the marginal $\Phi_m^*(T_f^-)$ is absolutely continuous with respect to the Lebesgue measure, as desired. □

Corollary 4.6. *Let $\mathcal{P}=(\mathcal{U},\mathcal{L}^u,\mathcal{L}^s)$ be a Pesin box with transverse measures ν^+ and ν^-, as in Section 4.2. If μ_f is absolutely continuous with respect to the Lebesgue measure (on X), then ν^+-almost every slice $\{\mathcal{L}^s(m)\}\wedge T_f^-$ is absolutely continuous with respect to the Lebesgue mesure (on $\mathcal{L}^s(m)\simeq\mathbb{D}$). The same result holds for ν^--almost every slice $\{\mathcal{L}^u(m)\}\wedge T_f^+$.*

This follows from Proposition 4.5 and the inequality $T_{\mathcal{P}}^+\leq T_f^+$.

Remark 4.7. Another strategy to prove this corollary is to use the decomposition $T_f^\pm=\sum_i T_{\mathcal{P}_i}^\pm$, Lemma 4.3, and the theory of geometric intersection, as described in [46], in order to write the intersection $T_f^+\wedge T_f^-$ as a sum of intersection of uniformly laminar currents.

4.4. **Lebesgue density points.** Assume that μ_f is absolutely continuous with respect to the Lebesgue measure. Consider the pull-back of T_f^- by a holomorphic curve $\theta\colon C\to X$ such that $\theta^*T_f^-$ is absolutely continuous: in local co-ordinates

$$\theta^*(T_f^-)=\varphi(z)\frac{\mathsf{i}}{2}dz\wedge d\bar{z}$$

where φ is a non-negative locally integrable function, *i.e.* $\varphi\in L^1_{loc}(\mathsf{Leb})$. Recall that a **Lebesgue density point** for a function $\varphi\in L^1_{\mathrm{loc}}(\mathsf{Leb})$ is a point z such that

$$\frac{1}{\pi r^2}\int_{\mathbb{D}(z,r)}|\varphi(w)-\varphi(z)|\frac{\mathsf{i}}{2}dw\wedge d\overline{w}\;\longrightarrow\;0$$

as r goes to 0. This notion does not depend on the choice of local co-ordinates. Thus, on the curve $\theta(C)\subset X$, there is a well defined set of density points

$$\mathrm{Dens}(\theta(C);T_f^-)=\{\theta(z)\in\theta(C)\,|\,z\text{ is a density point of }\theta^*T_f^-\}.$$

Moreover, the set of density points has full Lebesgue measure in $\theta(C)$ because $\theta^*T_f^-$ is absolutely continuous. The same notion applies for slices of T_f^+.

The goal of this section is the following proposition; to state it, recall that, for every $x \in \Lambda$, we defined injective parametrizations $\xi_x^{s/u} : \mathbb{D}(cq(x)) \to W^{s/u,loc}(x)$ of the local stable and unstable manifolds (see Section 3.2).

Proposition 4.8. *Let G be the set of points $x \in \Lambda$ for which there exist $0 < t(x) \leq cq(x)$ and a measurable function $\varphi_x \in L^1_{loc}(\mathbb{D}(t(x)))$ satisfying the two conditions*

(1) $(\xi_x^u)^* T_f^+ = \varphi_x(z)\frac{i}{2}dz \wedge d\bar{z}$ *on* $\mathbb{D}(t(x))$,

(2) 0 *is a Lebesgue density point for* φ_x.

Then $\mu_f(G) = 1$.

A similar statement holds for T_f^-. The proof of Proposition 4.8 relies on the following lemma, the proof of which is closely related to the proof of Lemma 4.2.

Lemma 4.9. *Let $\mathcal{P} = (\mathcal{U}, \mathcal{L}^u, \mathcal{L}^s)$ be a Pesin box. Let $A \subset \mathcal{K}^-$ be a Borel subset with positive ν^--measure. If μ_f is absolutely continuous, then*

$$D := \cup_{w \in A} \mathrm{Dens}(\mathcal{L}^u(w); T_f^+)$$

has positive μ_f-measure.

Proof. Identify $\mathcal{U}$ with the bidisk $\mathbb{D} \times \mathbb{D}$, with co-ordinates (z, w). We also use co-ordinates (z, w) on $\mathbb{D} \times \mathcal{K}^-$ (thus, $\mathcal{K}^-$ is viewed alternatively as a subset of $\{0\} \times \mathbb{D}$ or $\mathbb{D}$). As explained in Section 4.1.2, there is a homeomorphism $\Phi : \mathbb{D} \times \mathcal{K}^- \to \mathcal{L}^u$ which maps the horizontal lamination of $\mathbb{D} \times \mathcal{K}^-$ to the lamination $\mathcal{L}^u$, and is the identity map on $\{0\} \times \mathcal{K}^-$. There are two ways of looking at this homeomorphism:

- Φ maps each horizontal disk to a graph in $\mathcal{U} = \mathbb{D} \times \mathbb{D}$:

$$\Phi\colon (z, w) \in \mathbb{D} \times \mathcal{K}^- \mapsto (z, \varphi_w(z))$$

  where $\varphi_w : \mathbb{D} \to \mathbb{D}$ is the holomorphic function whose graph is the leaf of $\mathcal{L}^u$ through $(0, w)$.
- Φ is given by the holonomy maps $h_{0,z}$ from the vertical $\{0\} \times \mathbb{D}$ to the vertical $\{z\} \times \mathbb{D} \subset \mathcal{U}$; more precisely, $\Phi(z, w) = (z, h_{0,z}(w))$.

Let Leb denote the standard Lebesgue measure on $\mathbb{D}$; since μ_f is absolutely continuous, there exists an L^1-function Δ_{μ_f} such that $\mu_f = \Delta_{\mu_f}(z, w) d\mathsf{Leb}(z) \otimes d\mathsf{Leb}(w)$ on $\mathbb{D} \times \mathbb{D}$.

Let us assume that $\mu_f(D) = 0$ and find a contradiction. The vanishing of $\mu_f(D)$ implies that for Leb-almost every point $z \in \mathbb{D}$,

$$\int_{(z,w) \in D} \Delta_{\mu_f}(z, w)\, d\mathsf{Leb}(w) = 0.$$

Let us use the homeomorphism $\Phi\colon \mathbb{D}\times\mathcal{K}^- \longrightarrow \mathcal{L}^u$. Since for every $z\in\mathbb{D}$ the map $w\mapsto h_{(0,z)}(w)$ is absolutely continuous, we get

$$\int_{(z,w)\in\Phi^{-1}(D)} \Delta_{\mu_f}\circ\Phi(z,w)\cdot \mathsf{Jac}(h_{(0,z)})(w)\cdot d\mathsf{Leb}(w)=0$$

for Leb-almost every z. Let $\mathbb{1}_{\Phi^{-1}(D)}$ be the characteristic function of the set $\Phi^{-1}(D)$. For every $z\in\mathbb{D}$, the function $\mathsf{Jac}(h_{(0,z)})$ is positive Leb-almost everywhere on A (see Section 4.1.1); hence $\mathbb{1}_{\Phi^{-1}(D)}\cdot\Delta_{\mu_f}\circ\Phi(z,w)=0$ for Leb-almost every $z\in\mathbb{D}$ and for Leb-almost every $(0,w)\in A$.

Now we use that $\Phi(z,w)=(z,\varphi_w(z))$ is holomorphic in the variable z. There is a subset A' of A with $\nu^-(A')=\nu^-(A)>0$ such that D has full Lebesgue measure in every leaf $\mathcal{L}^u(0,w)$, $(0,w)\in A'$; thus, the function $\mathbb{1}_{\Phi^{-1}(D)}$ is equal to 1 almost everywhere on every horizontal disk $\mathbb{D}\times\{w\}$, for $(0,w)$ in A'. We obtain

$$\Delta_{\mu_f}\circ\Phi(z,w)=0$$

for Leb-almost every $z\in\mathbb{D}$ and every $(0,w)\in A'$. Consequently, the product of $\Delta_{\mu_f}\circ\Phi(z,w)$ by the jacobian of $\Phi(z,w)$ vanishes almost everywhere on $\mathbb{D}\times\{w\}$. Thus, the Fubini theorem implies that the integral of $\Phi^*\mu_f$ on $\mathbb{D}\times A'$ is equal to 0, while $\nu^-(A')>0$. But μ_f has a product structure $\nu^+\otimes\nu^-$ in $\mathcal{K}=\mathcal{K}^+\times\mathcal{K}^-$, and $\nu^+(\mathcal{K}^+)>0$ because the measure $\mu_f(\mathcal{K})$ of the Pesin box is positive; hence $\Phi^*\mu_f(\mathbb{D}\times A')$ should be positive. This contradiction concludes the proof. □

Let us complete the proof of Proposition 4.8. The set G has positive μ_f-measure by Lemma 4.9. Since μ_f is ergodic, it suffices to verify that G is f-invariant. This property is a consequence of the relation

$$(\xi^u_{f(x)})^*T_f^+=\lambda_f(M_x^{-1})^*(\xi^u_x)^*T_f^+.$$

To prove this relation, it suffices to combine $f^*T_f^+=\lambda_f T_f^+$ and $f\circ\xi^u_x=\xi^u_{f(x)}\circ M_x$ on a neighborhood of the origin, where M_x is the multiplication by the non zero complex number m_x (see Notation 3.2).

5. Renormalization along stable manifolds

Our main goal in this section is the following theorem.

Theorem 5.1. *Let f be an automorphism of a complex projective surface X with positive entropy. Assume that the measure of maximal entropy $\mu_f=T_f^+\wedge T_f^-$ is absolutely continuous with respect to the Lebesgue measure. Then there exists a measurable subset $\Lambda\subset X$ such that (i) $\mu_f(\Lambda)=1$ and (ii) every stable manifold $W^s(x)$*

for $x \in \Lambda$ is parametrized by an injective entire curve $\xi_x^s \colon \mathbf{C} \to W^s(x)$ satisfying

$$\xi_x^s(0) = x, \quad (\xi_x^s)^* T_f^+ = 0, \quad \textit{and} \quad (\xi_x^s)^* T_f^- = \frac{\mathsf{i}}{2} dz \wedge d\bar{z}. \tag{5.1}$$

Remark 5.2. The parametrization of an unstable manifold $W^s(x)$ by $\mathbf{C}$ is unique up to composition by an affine transformation $z \mapsto az + b$ of $\mathbf{C}$. Thus,

(1) every biholomorphism $\mathbf{C} \to W^s(x)$ with Properties (5.1) is equal to ξ_x^s up to composition by a homothety $z \mapsto az$ with $|a| = 1$;
(2) the parametrization ξ_x^s is the same as the parametrization defined in Section 3.2 up to composition by a dilation $z \mapsto az$, $a \neq 0$. This is the reason why we do not introduce a new notation.

The equalities $\xi_x^s(0) = x$ and $(\xi_x^s)^* T_f^+ = 0$ in Equations (5.1) are automatically satisfied (see Example 2.3), so all we need to prove in Theorem 5.1 is the third property. There are three steps:

- (see Section 5.1) We exhibit local parametrizations ξ_x of a neighborhood of x in $W^s(x)$ such that $\xi_x^* T_f^- = \alpha(x) \frac{\mathsf{i}}{2} dz \wedge d\bar{z}$.
- (see Section 5.2) Let $\xi_x^s \colon \mathbf{C} \to W^s(x)$ be the global parametrization of $W^s(x)$ defined in Section 3.2. Using the first step we obtain that
$$(\xi_x^s)^* T_f^- = \alpha(x) |h_x(z)|^2 \frac{\mathsf{i}}{2} dz \wedge d\bar{z}$$
for some holomorphic function h_x on $\mathbb{D}(\beta(x)\rho(x)/4)$.
- To conclude, we show that $|h_x|$ is indeed constant by using recurrence and exhaustion arguments.

5.1. **First step: smoothness of a local density.** In the following proposition Λ, β and q are respectively the measurable set and the ε-tempered functions introduced in Theorem 3.1. Let $c > 0$ be the constant introduced in Proposition 3.3.

Proposition 5.3. *Let f be an automorphism of a complex projective surface X with positive entropy* $\log \lambda_f$. *Assume that μ_f is absolutely continuous with respect to the Lebesgue measure.*

(1) *Then for every $x \in \Lambda$ there exist $\rho(x) > 0$ and an injective holomorphic mapping $\xi_x : \mathbb{D}(\rho(x)) \to W^{s,loc}(x)$ such that*
   - (i) $\xi_x(0) = x$ *and* $\beta(x) \leq \| \xi_x'(0) \| \leq 1$,
   - (ii) $2\beta(x)/3 \leq \| \xi_x'(z) \| \leq 2$ *on* $\mathbb{D}(\rho(x))$,
   - (iii) $\xi_x^* T_f^- = \alpha(x) \cdot \frac{\mathsf{i}}{2} dz \wedge d\bar{z}$ *on* $\mathbb{D}(\rho(x))$ *for some* $\alpha(x) > 0$.

(2) *The Lyapunov exponents of* μ_f *satisfy*

$$\lambda_s = -\frac{1}{2}\log\lambda_f \ \textit{ and } \ \lambda_u = \frac{1}{2}\log\lambda_f.$$

Proof. We prove the first assertion together with the estimate $|\log\lambda_f + 2\lambda_s| \leq 2\varepsilon$. The second assertion follows from this estimate, applied to both f and f^{-1} for arbitrary small $\varepsilon > 0$.

• *Good subsets* $Q_{l,m}$.– To prove the proposition it suffices to work in a fixed Pesin box $\mathcal{P}$ because the union of all Pesin boxes has full μ_f-measure. Let us recall that $\sigma_x : \mathbb{D}(q(x)) \to W^{s,loc}(x)$ is the injective parametrization of the local stable manifold introduced in Section 3.2. Let $\eta_x : \mathbb{D}(cq(x)) \to \mathbb{D}(q(x))$ be the holomorphic function of Proposition 3.3; it satisfies $\eta_x(0) = 0$, $\eta_x'(0) = 1$, and

$$f \circ (\sigma_x \circ \eta_x) = (\sigma_{f(x)} \circ \eta_{f(x)}) \circ M_x.$$

We recall that $\xi_x^{s,loc} = \sigma_x \circ \eta_x$ and that it is equal to the restriction of ξ_x^s on $\mathbb{D}(cq(x))$. Changing Λ in another invariant subset of full measure if necessary, Proposition 4.8 implies that for every $x \in \mathcal{P} \cap \Lambda$ there exists a function $\varphi_x \in L^1_{loc}(\mathbb{D}(t(x)))$ with $0 < t(x) \leq cq(x)$ such that 0 is a Lebesgue density point of φ_x and

$$(\xi_x^s)^* T_f^- = \varphi_x(z) \cdot \frac{i}{2} dz \wedge d\bar{z} \ \text{ on } \ \mathbb{D}(t(x)). \tag{5.2}$$

Since the origin 0 is a Lebesgue density point of φ_x, the value $\varphi_x(0)$ is a well defined non-negative number. Let us define for every $l \geq 1$:

$$Q_l := \mathcal{P} \cap \Lambda \cap \{1/l \leq t(x)\} \cap \{1/l \leq \beta(x) \leq 1\} \cap \{1/l \leq \varphi_x(0) \leq l\}.$$

Then apply the Lusin theorem to find for every $m \geq 1$ a subset $Q_{l,m} \subset Q_l$ of measure $(1 - 1/m)\mu_f(Q_l)$ on which β is continuous. One may assume $Q_{l,m} \subset Q_{l,m+1}$, and we have

$$\mu_f(\cup_{l,m\geq 1} Q_{l,m}) = \mu_f(\mathcal{P}).$$

Fix a pair of integers (l,m) and denote $Q_{l,m}$ by Q in what follows. Since the union of the sets $Q_{l,m}$ has full μ_f-measure, we only need to prove the proposition for μ_f-generic points $x \in Q$.

• *Montel property.*– Let $\tilde{f} : Q \to Q$ be the first return map defined as $\tilde{f}(x) := f^{r(x)}(x)$ where $r(x)$ is the smallest integer $r \geq 1$ satisfying $f^r(x) \in Q$. The induced measure $\tilde{\mu}(\cdot) := \mu(Q \cap \cdot)/\mu(Q)$ is $\tilde{f}$-invariant and ergodic. Let x be a generic point of Q. The Birkhoff ergodic theorem, applied to $\tilde{f}$, yields a sequence $(n_j)_j$ depending on x such that $f^{-n_j}(x)$ is contained in Q and converges to x. To simplify the exposition we avoid the indices of the subsequence and write $f^{-n}(x)$ instead of $f^{-n_j}(x)$. Define $x_n := f^{-n}(x)$, $\eta_n = \eta_{x_n}$, and $\sigma_n = \sigma_{x_n}$.

Since $t \geq 1/l$ on $Q = Q_{l,m}$, the restriction $\eta_n : \mathbb{D}(1/l) \to \mathbb{D}(q(x_n)) \subset \mathbb{D}$ makes sense. The Montel and Hurwitz theorems provide a subsequence such that

(i) η_n converges towards an injective, holomorphic function $\eta : \mathbb{D}(1/l) \to \mathbb{D}$ such that $\eta(0) = 0$ and $\eta'(0) = 1$. Moreover, $|\eta(z)| \leq l|z|$ on $\mathbb{D}(1/l)$ by the Schwarz lemma.

We can also consider the restriction $\sigma_n : \mathbb{D}(1/l) \to W^{s,loc}(x_n)$. For large enough n, its image is contained in the ball $B_x(2)$, and the image of $\Psi_x \circ \Psi_{x_n}^{-1} \circ \sigma_n$ is a graph above the vertical axis. Moreover $|\sigma_n'(0)| \geq \beta(x_n) \geq 1/l$. The Montel and Hurwirtz theorems and the continuity of β on $Q = Q_{l,m}$ then yield

(ii) σ_n converges towards an injective holomorphic mapping $\sigma : \mathbb{D}(1/l) \to W^{s,loc}(x)$ such that $\sigma(0) = x$ and $|\sigma'(0)| \geq \beta(x) \geq 1/l$.

Thus, the function $\xi := \sigma \circ \eta : \mathbb{D}(1/l^2) \to W^{s,loc}(x)$ is well defined, injective and satisfies $\sigma \circ \eta(0) = x$; by construction, the sequence of parametrizations $\xi_n^s := \sigma_n \circ \eta_n$ converges towards the holomorphic map ξ on $\mathbb{D}(1/l^2)$. This map will be our desired ξ_x, it is a non-constant holomorphic mapping with values in the stable manifold $W^s(x)$ (we shall compare it with ξ_x^s in Section 5.2). Since all ξ_n^s satisfy the Lipschitz property listed in Section 3.2, we get

$$2\beta(x)/3 \leq |\xi'(z)| \leq 2 \text{ on } \mathbb{D}(1/l^2).$$

• *Renormalization.–* The identity $(f^n)^* T_f^- = \lambda_f^{-n} T_f^-$ for $n \geq 1$ implies

$$(f^n \circ \xi_n^s)^* T_f^- = \lambda_f^{-n} (\xi_n^s)^* T_f^- \quad \text{on} \quad \mathbb{D}(1/l^2).$$

On the other hand, Proposition 3.3 yields

$$(f^n \circ \xi_n^s)^* T_f^- = (\xi_x^s \circ M_n)^* T_f^- \quad \text{on} \quad \mathbb{D}(1/l^2),$$

where M_n is defined by $M_n := M_{x_{-1}} \circ \cdots \circ M_{x_{-n}}$. Note that $M_n(z) = m_n \cdot z$ on $\mathbb{D}(1/l^2)$ with $|m_n| \in e^{n\lambda_s} \cdot [e^{-n\varepsilon}, e^{n\varepsilon}]$. Combining these two equations, one gets

$$(\xi_n^s)^* T_f^- = \lambda_f^n M_n^* \, (\xi_x^s)^* T_f^- \quad \text{on} \quad \mathbb{D}(1/l^2). \tag{5.3}$$

Now, denote by φ_n the density of $(\xi_n^s)^* T_f^-$ and by φ_x the density for $(\xi_x^s)^* T_f^-$, as in Equation (5.2). Equation (5.3) gives

$$\varphi_n(z) = \lambda_f^n |m_n|^2 \varphi_x(z).$$

Since x_n and x are in the set Q, the origin is a Lebesgue density point for the densities φ_n and φ_x, and $l^{-1} \leq \varphi_n(0),\ \varphi_x(0) \leq l$. Thus, $l^{-2} \leq \lambda_f^n |m_n|^2 \leq l^2$. Taking logarithms, dividing by n, and letting n go to $+\infty$ lead to

$$|\log \lambda_f + 2\lambda_s| < 2\varepsilon,$$

as desired. Moreover, taking a subsequence, one may assume that $\lambda_f^n |m_n|^2$ converges to a positive real number $\theta \in [l^{-2}, l^2]$.

Now, we come back to the Equation (5.3) which can be written

$$(\xi_n^s)^* T_f^- = \lambda_f^n |m_n|^2 \cdot \varphi_x(m_n z) \cdot \frac{\mathsf{i}}{2} dz \wedge d\bar{z} \ \text{ on } \ \mathbb{D}(1/l^2). \tag{5.4}$$

The left hand side converges to $\xi^* T_f^-$ in the sense of distributions on $\mathbb{D}(1/l^2)$, because T_f^- has continuous potentials (see Lemma 4.4). The right hand side converges in the sense of distributions to

$$\theta \cdot \varphi_x(0) \cdot \frac{\mathsf{i}}{2} dz \wedge d\bar{z},$$

because M_n converges locally uniformly to the constant mapping 0 on compact subsets of $\mathbb{D}(1/l^2)$ and 0 is a Lebesgue density point for φ_x. As a consequence,

$$\xi^* T_f^- = \theta \cdot \varphi_x(0) \cdot \frac{\mathsf{i}}{2} dz \wedge d\bar{z} \ \text{ on } \ \mathbb{D}(1/l^2).$$

Setting $\xi_x = \xi$, $\rho(x) = 1/l^2$ and $\alpha(x) = \theta \varphi_x(0)$ we get $\xi_x^* T_f^- = \alpha(x) \cdot \frac{\mathsf{i}}{2} dz \wedge d\bar{z}$ on $\mathbb{D}(\rho(x))$ ($\rho(x)$ can be defined as the best constant l for which $x \in Q_{l,m}$). $\square$

5.2. **Second step: from ξ_x to ξ_x^s.** We need to translate Proposition 5.3 in terms of the global parametrization $\xi_x^s \colon \mathbf{C} \to W^s(x)$. Proposition 5.3 asserts that there is a parametrization $\xi_x \colon \mathbb{D}(\rho(x)) \to W^s(x)$ of a small neighborhood of x in $W^s(x)$ such that $\xi_x^* T_f^- = \alpha(x) \frac{\mathsf{i}}{2} dz \wedge d\bar{z}$ on $\mathbb{D}(\rho(x))$. Both ξ_x and ξ_x^s satisfy the Lipschitz property

$$2\beta(x)/3 \leq \| \xi'(z) \| \leq 2$$

on their domain of definition (see Notation 3.2), thus $(\xi_x^s)^{-1} \circ \xi_x$ is defined on $\mathbb{D}(\rho(x))$ and the modulus of its derivative is bounded from below by $\beta(x)/3$ and from above by $3\beta(x)^{-1}$; moreover, its derivative at the origin is in $[\beta(x), \beta(x)^{-1}]$. This function is injective and its domain of definition is $\mathbb{D}(\rho(x))$. By Koebe's (1/4)-theorem, its image contains the disk of radius $\beta(x)\rho(x)/4$, so that its reciprocal function is defined on $\mathbb{D}(\beta(x)\rho(x)/4)$ and has derivative in $[\beta(x)/3, 3\beta(x)^{-1}]$ on this disk.

In order to pull back T_f^- by ξ_x^s, one can first compute its pull-back by ξ_x, the result being $\alpha(x) \frac{\mathsf{i}}{2} dz \wedge d\bar{z}$, and then take its pull-back under $(\xi_x)^{-1} \circ \xi_x^s$. This gives

$$(\xi_x^s)^* T_f^- = \alpha(x) |h_x(z)| \frac{\mathsf{i}}{2} dz \wedge d\bar{z} \tag{5.5}$$

on $\mathbb{D}(\beta(x)\rho(x)/4)$, where h_x is holomorphic and

$$\beta(x)/3 \leq |h_x(z)| \leq 3\beta(x)^{-1}. \tag{5.6}$$

5.3. **Final step for the proof of Theorem 5.1.** Now, fix a Pesin box $\mathcal{P}$, and a subset $Q = Q_{l,m}$ as in the proof of Proposition 5.3. Then, there exists $N \geq 1$ such that

$$1/N \leq (\beta\rho/4)^2 \leq 1, \quad 1/N \leq \beta/3 \leq 1, \quad \text{and} \quad 1/N \leq \alpha \leq N$$

on the set Q. Apply the Birkhoff ergodic theorem: for a generic point x of Q there is a sequence of points $f^{-n_i}(x) \in Q$ that converges towards x. As above, we drop the index i from n_i, and write x_n for $f^{-n}(x)$, ξ^s_n for $\xi^s_{x_n}$. As in Equation (5.3), we obtain

$$(\xi^s_x)^* T^-_f = \lambda^n_f M^*_n \, (\xi^s_{x_n})^* T^-_f \text{ on } \mathbf{C}, \tag{5.7}$$

where M_n is a linear map $z \mapsto m_n \cdot z$ with $|m_n| \in e^{n\lambda_s} \cdot [e^{-n\varepsilon}, e^{n\varepsilon}]$. Equation (5.5) shows that the right hand side of Equation (5.7) has density

$$\lambda^n_f |M'_n|^2 \cdot \alpha(x_n) \cdot |h_n(M_n(z))| \tag{5.8}$$

on $M_n^{-1}(\mathbb{D}(1/N))$, where h_n is holomorphic with modulus in $[1/N, N]$. Again, evaluation at $z = 0$ gives $\lambda^n_f |m_n|^2 \cdot \alpha(x_n) \in [1/N^3, N^3]$, so that a subsequence of $\lambda^n_f |m_n|^2 \cdot \alpha(x_n)$ converges to some θ in $[1/N^3, N^3]$. Moreover the Montel theorem and Equation (5.6) imply that the sequence $(h_n)_n$ is equicontinuous on $\mathbb{D}(1/N)$; thus a subsequence converges locally uniformly to a function $h : \mathbb{D}(1/N) \to \mathbf{C}$ with modulus in $[1/N, N]$.

Let $\gamma(x) := \theta |h(0)|^2$. Let K be a compact subset of $\mathbf{C}$ and restrict the study to integers $n \geq 1$ such that $K \subset M_n^{-1}(\mathbb{D}(1/N))$. Since $(M_n)_n$ converges uniformly to zero on K, we get

$$(\xi^s_x)^* T^-_f = \gamma(x) \frac{\mathrm{i}}{2} dz \wedge d\bar{z} \quad \text{on} \quad K.$$

Taking an exhaustion by compact subsets, the same formula holds on $\mathbf{C}$. Changing ξ^s_x into $\xi^s_x(az)$ with $|a|^{-2} := \gamma(x)$, we obtain the parametrization promised by Theorem 5.1.

6. THE SINGULAR SURFACE X_0 AND COMPACTNESS OF ENTIRE CURVES

A crucial tool for the proof of the Main Theorem is a compactness property of a family of entire curves related to unstable manifolds. To obtain it, we need to contract all f-periodic algebraic curves: the result is a singular surface X_0. We start with a description of X_0 before proving the compactness theorem (Theorem 6.13).

6.1. **Construction of X_0.** We use the notations of Section 2. If $C \subset X$ is a complex curve, one denotes by $[C] \in H^2(X;\mathbf{Z})$ the dual of the homology class of C for the natural pairing between $H^2(X;\mathbf{Z})$ and $H_2(X;\mathbf{Z})$. We call it the cohomology class of C, this is an element of $H^{1,1}(X;\mathbf{R})$.

Proposition 6.1 (see [27], §4.1)**.** *Let f be an automorphism of a compact Kähler surface X with positive entropy. There exist a (compact, singular, complex analytic) normal surface X_0, a bimeromorphic morphism $\pi\colon X \to X_0$ and an automorphism f_0 of X_0 such that*

(1) $\pi \circ f = f_0 \circ \pi$.
(2) *A curve $C \subset X$ is contracted by π if and only if C is periodic, if and only if $[C] \in \Pi_f^{\perp}$ (resp. $[C] \in (\theta_f^+)^{\perp}$, resp. $[C] \in (\theta_f^-)^{\perp}$).*
(3) *If C is a connected periodic curve, then the arithmetic genus of C is* 0 *or* 1.

The surface X_0 is not always projective, even if X is (see Section 11). If (X, f) is a Kummer example on a K3 or rational surface, then X_0 is automatically singular.

Sketch of the proof. The intersection form being negative definite on $\Pi_f^{\perp}$, the Grauert-Mumford criterion allows us to contract all irreducible curves C with $[C] \in \Pi_f^{\perp}$, and only those curves. This yields the desired morphism $\pi\colon X \to X_0$, where X_0 is a normal, compact, complex analytic surface (see [4], § III.2 and [57], Remark 5.7.2 and Example 5.7.3).

Let C be an irreducible curve. If C is periodic, then $(f^*)^n[C] = [C]$ for some positive iterate of f^*, so that $[C]$ is in $\Pi_f^{\perp}$. Now, if $[C]\cdot\theta_f^- = 0$, then the positive orbit $((f^*)^n[C])_{n\geq 0}$ is a bounded subset of $(\theta_f^-)^{\perp} \cap H^2(X;\mathbf{Z})$ because $(\theta_f^-)^{\perp}$ is the direct sum of $\Pi_f^{\perp}$ and $\mathbf{R}\theta_f^-$, and f^* acts as a rotation on $\Pi_f^{\perp}$ and as a contraction on $\mathbf{R}\theta_f^-$. This implies that the orbit of $[C]$ is a finite subset of $\Pi_f^{\perp}$. As a consequence, C is constracted by π, C is the unique curve representing the cohomology class $[C]$, and C is periodic. This proves the second assertion when $[C] \in (\theta_f^-)^{\perp}$; the other cases are obtained in a similar way.

Property (3) is more difficult to establish. It is due to Castelnuovo for irreducible curves and to Diller, Jackson, and Sommese for the general case (see [39]). □

Remark 6.2. The class $\Sigma := \theta_f^+ + \theta_f^- \in H^{1,1}(X;\mathbf{R})$ has positive self-intersection and is in the nef cone. After contraction of all periodic curves by the morphism $\pi\colon X \to X_0$, there is no curve E with $\langle \pi_*(\theta_f^+ + \theta_f^-)|[E]\rangle = 0$. Hence, $\pi_*(\theta_f^+ + \theta_f^-)$ "behaves as a Kähler class" on X_0, a statement which is made precise in Proposition 10.7.

6.2. **Analysis on X_0.** The material of this section will be used in §10.3, §10.4 and §10.5. We present it for general (reduced and pure dimensional) compact complex spaces M (X_0 is such a space); our references are [2], [36], [56], and [77].

Let M^{reg} and M^{sing} denote the regular and singular parts of M. There is a finite collection of open sets $M_\alpha \subset M$ covering M such that each M_α is isomorphic, via an embedding j_α, to an analytic subset of the unit ball in $\mathbf{C}^N$ (for some N). By definition, a (p,q)-form of class $\mathcal{C}^k$ on M_α is a (p,q)-form on M_α^{reg} which is the pull back by j_α of a (p,q)-form of class $\mathcal{C}^k$ on the unit ball. A form is positive if it comes from of a positive form. A smooth function is the restriction to $j_\alpha(M_\alpha)$ of a smooth function on the unit ball of $\mathbf{C}^N$. These notions do not depend on the local embeddings (see [56, §5.A, Theorems 14 and 16], and [36, §1]). By [2], Proposition 2.4.4, one can find a partition of unity subordinated to the M_α: smooth functions $\varphi_\alpha \geq 0$ with compact support $\subset M_\alpha$ such that $\sum_\alpha \varphi_\alpha = 1$ on M.

Let ω be the standard hermitian $(1,1)$-form of $\mathbf{C}^N$ and set $\kappa_\alpha := j_\alpha^* \omega$. Then

$$\kappa_0 := \sum_\alpha \varphi_\alpha \kappa_\alpha$$

is a smooth positive $(1,1)$-form on M. If ν is a holomorphic disk drawn on M_α, the norm $\parallel \nu'(z) \parallel$ with respect to κ_0 is defined as $\parallel \nu'(z) \parallel^2 := \sum_\alpha \varphi_\alpha(z) \parallel (j_\alpha \circ \nu)'(z) \parallel^2_{\mathbf{C}^N}$; this makes sense even if $\nu(z) \in M^{sing}$ (this definition is used in §6.4 and §7.2). From the above definitions, and from the compactness of M, we deduce that if κ' is another positive $(1,1)$-form then $a^{-1}\kappa_0 \leq \kappa' \leq a\kappa_0$ for some $a > 0$, i.e. $a^{-1} \parallel \nu'(z) \parallel \leq \parallel \nu'(z) \parallel_{\kappa'} \leq a \parallel \nu'(z) \parallel$ for every holomorphic disk.

A current of bidegree $(n-p, n-q)$ on M_α (where $n = \dim M$) is an element of the dual of the space of smooth (p,q)-forms with compact support in M_α; a current on M is an element of the dual of the space of smooth (p,q)-forms (see [36]). Every current T on M_α induces a current $(j_\alpha)_* T$ on $\mathbf{C}^N$ by the formula $\langle (j_\alpha)_* T | \phi \rangle := \langle T | (j_\alpha)^* \phi \rangle$. By definition, a current T is positive if $(j_\alpha)_* T$ is positive for every α. The trace measure of a positive $(n-1, n-1)$-current T on M is defined by

$$T \wedge \kappa_0 := \sum_\alpha \langle T | \varphi_\alpha \kappa_\alpha \rangle,$$

it is equal to zero if and only if T is equal to zero.

A real valued function u on M_α is pluri-subharmonic if it is the restriction to M_α of a pluri-subharmonic function on the unit ball of $\mathbf{C}^N$. One has the following criterion : let $u \colon M_\alpha \to [-\infty, +\infty[$ be an upper semi-continuous function which is not identically $-\infty$ on any open subset of M_α; then u is pluri-subharmonic if and only if $u \circ \nu$ is sub-harmonic or identically $-\infty$ for every holomorphic disk $\nu : \mathbb{D} \to M_\alpha$ (see

Theorem 5.3.1 of [49], or [36]). Moreover, if u is continuous and pluri-subharmonic on M_α, then u is the restriction of a continuous and pluri-subharmonic function $\tilde{u}$ on the unit ball of $\mathbf{C}^N$ (see [77], Satz 2.4). For such a function, the current $dd^c u$ is defined on M_α by

$$\langle dd^c u, \phi \rangle := \int_{j_\alpha(M)} \tilde{u}\, dd^c \tilde{\phi}$$

for every $(n-1,n-1)$-test form ϕ on M_α, coming from an $(n-1,n-1)$-test form $\tilde{\phi}$ on $\mathbf{C}^N$; the integral is computed on the smooth part (see [36, 38], and § 10.5 below) and is therefore equal to $\int_{M_\alpha^{reg}} u dd^c \phi$. A function defined on an open set $\mathcal{U}$ of M is pluri-subharmonic if its restriction to the intersections $\mathcal{U} \cap M_\alpha$ are pluri-subharmonic.

We shall need the following definition to state Theorem 10.8: a closed positive $(1,1)$-current T on M has a *continuous potential around* x if $T = dd^c u$ on some open neighborhood $\mathcal{U}$ of x, for some continuous pluri-subharmonic function $u \colon \mathcal{U} \to \mathbf{R}$. If T has a continuous potential around x, its product with a closed positive $(1,1)$-current S on $\mathcal{U}$ is denoted by $S \wedge T$ (or $dd^c(uS)$) and is defined by

$$S \wedge T(\varphi) = \langle (j_\alpha)_* S | \tilde{u}\, dd^c \tilde{\varphi} \rangle$$

for every smooth real valued function φ on $\mathcal{U}$ (and any smooth extension $\tilde{\varphi}$ of φ to the unit ball). This is a positive measure on $\mathcal{U}$, see [38], Chapter III.3.

Notation 6.3. *We set $T_0^\pm := \pi_* T_f^\pm$, where $T_f^\pm$ are the invariant currents of § 2.4.*

Remark 6.4. We do not know whether $T_0 := T_0^+ + T_0^-$ always has continuous potentials near the singularities of X_0 (this holds if X_0 has quotient singularities, and in particular if X is a K3 surface); but we will not need that property. In Section 10.5 (in which Proposition 6.9 will be proved), Theorem 10.8 is applied on compact subsets K of X_0^{reg}, and on such sets, T_0 has local continuous potentials.

6.3. **Slices of $T_0^\pm$.**

Lemma 6.5. *Let C be a Riemann surface and $\nu \colon C \to X_0$ be a non-constant holomorphic map. There is a unique holomorphic map $\hat{\nu} \colon C \to X$ such that $\pi \circ \hat{\nu} = \nu$.*

Proof. Since this is a local statement, we can replace C by the unit disk $\mathbb{D}$ and assume that (i) X_0 is an analytic subset of the open unit ball $B \subset \mathbf{C}^N$ with a unique singularity $q_0 \in B$, (ii) $D = \nu(\mathbb{D})$ is an analytic subset of B of dimension 1, (iii) $\nu^{-1}(q_0) = \{0\}$, and (iv) the germ (D, q_0) is irreducible. Then, $\pi^{-1}(D)$ is an analytic subset of $\pi^{-1}(B)$, and this analytic subset decomposes into finitely many irreducible components. One, and exactly one of these components F projects onto D and the other components are contracted by π. The component F intersects the exceptional divisor $\pi^{-1}(q_0)$ in a unique point m, because otherwise (D,q_0) would not be an irreducible germ. One can then define $\hat{\nu}$ by $\hat{\nu}=\pi^{-1}\circ\nu$ on $\mathbb{D}\setminus\{0\}$ and $\hat{\nu}(0)=m$. $\square$

Notation 6.6. *We define $\nu^*T_0^{\pm}$ locally by $dd^c(u^{\pm}\circ\hat{\nu})$, where $u^{\pm}$ are continuous potentials of $T_f^{\pm}$.*

We say that a sequence of holomorphic disks $\nu_n\colon\mathbb{D}\to X_0$ converges towards $\nu\colon\mathbb{D}\to X_0$ if ν_n converges locally uniformly towards ν. This convergence can be locally described in local charts $\subset\mathbf{C}^N$.

Lemma 6.7. *Let $\nu_n\colon\mathbb{D}\to X_0$ be a sequence of holomorphic maps, converging towards a non-constant holomorphic map $\nu\colon\mathbb{D}\to X_0$. Then*

(1) *$\nu^{-1}(X_0^{sing})$ is a discrete subset of $\mathbb{D}$;*
(2) *the measures $\nu^*(T_0^+)$ and $\nu_n^*(T_0^+)$ have no atom;*
(3) *$\nu_n^*T_0^+$ converges towards $\nu^*T_0^+$ on $\mathbb{D}\setminus\nu^{-1}(X_0^{sing})$;*
(4) *if $\nu_n^*T_0^+$ converges towards an atomless measure μ, then $\nu^*T_0^+=\mu$.*

Assertion (4) will be used in the following case: *let (a_n) be a sequence of non-negative real numbers converging towards $a\in\mathbf{R}_+$; if $\nu_n^*T_0^+=a_n\frac{\mathrm{i}}{2}dz\wedge d\bar{z}$ for all $n\geq 1$, then $\nu^*T_0^+=a\frac{\mathrm{i}}{2}dz\wedge d\bar{z}$.*

Proof. Since zeroes of analytic functions are isolated, we get (1). To prove (2), it suffices to notice that if μ is a positive measure with a continous potential u on $\mathbb{D}$, then μ has no atom at the origin. This is well known, but we sketch the proof because it will be used again in Lemma 10.13. Changing u into $u-u(0)$ we assume that $u(0)=0$. Then, we denote by $\mathbb{D}_r$ the disk of radius r and we let $\chi:\mathbb{D}_3\to[0,1]$ be a smooth non-negative function which is equal to 1 on $\mathbb{D}_1$ and to 0 on $\mathbb{D}_3\setminus\mathbb{D}_2$. For $\varepsilon\leq 1/3$, define $\chi_\varepsilon(x):=\chi(x/\varepsilon)$. Then

$$0\leq\langle\mu|\chi_\varepsilon\rangle=\int_{\mathbb{D}}u\,dd^c\chi_\varepsilon\leq\max_{\mathbb{D}_{3\varepsilon}}|u|\cdot\max_{\mathbb{D}_{3\varepsilon}}\parallel dd^c\chi_\varepsilon\parallel\cdot\mathrm{Area}(\mathbb{D}_{3\varepsilon}).$$

But the maximum of $|u|$ goes to 0 with ε because u is continuous, the maximum of $\parallel dd^c\chi_\varepsilon\parallel$ is bounded from above by $c^{ste}\varepsilon^{-2}$, and $\mathrm{Area}(\mathbb{D}_{3\varepsilon})=9\pi\varepsilon^2$. Thus $\langle\mu|\chi_\varepsilon\rangle$ goes to 0 with ε, and μ has no atom at the origin.

On $\mathbb{D}\setminus\nu^{-1}(X_0^{sing})$ we know that $\hat{\nu}_n$ converges locally towards $\hat{\nu}$, because they are obtained from ν_n and ν by composition with π^{-1} (for n large). Thus, Assertion (3) follows from Lemma 4.4. Assertion (4) follows from (2) and (3). $\square$

6.4. **Entire curves on X_0.**

6.4.1. *Definitions.* A sequence of entire curves $\xi_n\colon \mathbf{C} \to X_0$ converges towards $\xi\colon \mathbf{C} \to X_0$ if ξ_n converges locally uniformly to ξ. A family $\mathcal{E}$ of entire curves on X_0 is **closed** if the limit of every converging sequence $(\xi_n) \in \mathcal{E}^{\mathbf{N}}$ is an element of $\mathcal{E}$. It is **normal** if every sequence of elements of $\mathcal{E}$ contains a converging subsequence. It is **compact** if it is non empty, normal and closed. A **Brody curve** $\xi\colon \mathbf{C} \to X_0$ is a non-constant entire curve such that $\| \xi'(z) \|$ is uniformly bounded.

6.4.2. *Zalcman reparametrization.* We have the following classical Lemma for general compact complex spaces.

Lemma 6.8 (Zalcman)**.** *Let M be a compact complex space. If a sequence of entire curves $\xi_n\colon \mathbf{C} \to M$ is not normal then there exists a sequence of affine automorphisms $z \mapsto a_n z + b_n$ such that*

(1) $\lim_n a_n = 0$,
(2) *the sequence $\nu_n\colon \mathbf{C} \to M$ defined by $z \mapsto \xi_n(a_n z + b_n)$ converges towards a Brody curve $\nu\colon \mathbf{C} \to M$.*

The proof is the same as for compact complex manifolds, in particular it relies on a control of the derivatives of holomorphic disks in M (see for instance [11], and [63, Chapter III]).

6.4.3. *Non existence of degenerate entire curves.* The following proposition is crucial to show the compactness of the family $\mathcal{A}^u_f$, see Theorem 6.13.

Proposition 6.9. *Let f be an automorphism of a complex projective surface X with positive entropy. Let $\pi\colon X \to X_0$ be the contraction of the f-periodic curves and let $T_0^{\pm} := \pi_* T_f^{\pm}$. There is no non-constant entire curve $\xi\colon \mathbf{C} \to X_0$ satisfying*

$$\xi^*(T_0^+ + T_0^-) = 0.$$

The proof is given in Section 10, it relies on a result of Dinh-Sibony [40], which we extend to the context of (singular) complex spaces. Let us explain the strategy when f has no periodic curve, so that $X_0 = X$ is smooth. In that case, $T_0^+ + T_0^- = T_f^+ + T_f^-$ is cohomologous to a Kähler form κ (see § 10.2 and [76]). Starting with ξ, Brody's lemma provides an entire curve $\tilde{\xi}$ which is not constant, has a uniformly bounded derivative, and satisfies $\tilde{\xi}^*(T_f^+ + T_f^-) = 0$. Then, the above mentioned result of [40] provides an Ahlfors current S for $\tilde{\xi}$ such that $S \wedge (T_f^+ + T_f^-)$ vanishes identically. This is a contradiction since $T_f^+ + T_f^-$ is cohomologous to a Kähler form and S is non trivial closed and positive current.

6.4.4. *The family $\mathcal{A}^u_f$.*

Definition 6.10. *Let $\mathcal{A}^u_f$ be the family of entire curves $\xi\colon \mathbf{C} \to X_0$ such that*

$$\xi^*(T_0^+) = \frac{\mathrm{i}}{2} dz \wedge d\bar{z} \quad \text{and} \quad \xi^*(T_0^-) = 0.$$

If μ_f is absolutely continuous, Theorem 5.1 and Lemma 6.11 below show that almost every unstable manifold $W^u_f(x)$ can be parametrized by an injective entire curve $\xi^u_x\colon \mathbf{C} \to X_0$ that belongs to $\mathcal{A}^u_f$. In particular $\mathcal{A}^u_f$ is not empty.

Lemma 6.11. *For μ_f-almost every $x \in \Lambda$, the global stable manifold $W^s(x)$ does not intersect the f-periodic curves. In particular, the injective parametrization $\xi^s_x\colon \mathbf{C} \to X$ remains injective when one projects it into X_0.*

Proof. Let $\xi\colon \mathbf{C} \to X$ be an injective parametrization of $W^s(x)$. Assume that ξ is not contained in any f-periodic curve. Let D be an irreducible periodic curve, and suppose that $f(D) = D$ for simplicity. If $W^s(x)$ intersects D, the forward orbit of x converges towards D. On the other hand, $\mu_f(D) = 0$ because μ_f does not charge any proper analytic subset of X. Thus, the forward orbit of x does not equidistribute, and by Birkhoff's theorem x is not a generic point with respect to μ_f. □

Notation 6.12. *In what follows, we keep the same notation $\xi^{u/s}_x$ for the unstable and stable manifolds, but considered as entire curves in X_0.*

6.4.5. *Compactness of $\mathcal{A}^u_f$.*

Theorem 6.13. *Let f be an automorphism of a complex projective surface X with positive entropy. If non-empty, the family $\mathcal{A}^u_f$ is compact.*

Proof. Lemma 6.7(4) implies that $\mathcal{A}^u_f$ is closed. Let us prove that $\mathcal{A}^u_f$ is a normal family. If not, Lemma 6.8 provides a sequence of curves $\xi_n \in \mathcal{A}^u_f$ and affine automorphisms $g_n\colon z \mapsto a_n z + b_n$ such that $\lim_n a_n = 0$ and $\nu_n := \xi_n \circ g_n$ converges towards a Brody curve $\nu\colon \mathbf{C} \to X_0$. By Lemma 6.7, this curve satisfies

$$\nu^*(T_0^+) = \lim_{n\to\infty} \nu_n^*(T_0^+) = \lim_{n\to\infty} g_n^* \left(\frac{\mathrm{i}}{2} dz \wedge d\bar{z} \right) = \lim_{n\to\infty} |a_n|^2 \frac{\mathrm{i}}{2} dz \wedge d\bar{z} = 0.$$

Similarly, $\nu^*(T_0^-) = 0$, and this contradicts Proposition 6.9. □

Remark 6.14. Theorem 6.13 does not need any assumption on μ_f.

7. LAMINATIONS BY UNSTABLE MANIFOLDS

7.1. **The compact family $\mathcal{B}^u_f$ of unstable manifolds.** We assume that μ_f is absolutely continuous, so that $\mathcal{A}^u_f$ is not empty.

Definition 7.1. *Let $\mathcal{B}^u_f$ be the smallest compact subset of $\mathcal{A}^u_f$ that contains all injective parametrizations of unstable manifolds which are in $\mathcal{A}^u_f$. We set*

$$\mathcal{B}^u_f(X_0) = X_0^{reg} \bigcap \bigcup_{\xi \in \mathcal{B}^u_f} \xi(\mathbf{C}).$$

Note that $\mathcal{A}^u_f$ and $\mathcal{B}^u_f$ are invariant under translation and rotation: if ξ is an element of one of these sets, then $z \mapsto \xi(e^{i\theta}z + b)$ is an element of the same set for all $b \in \mathbf{C}$ and $\theta \in \mathbf{R}$. Using this remark, one verifies that $\mathcal{B}^u_f(X_0)$ is a closed subset of X_0^{reg}, because $\mathcal{B}^u_f$ is a compact family of entire curves. The sets $\mathcal{B}^s_f$ and $\mathcal{B}^s_f(X_0)$ are defined in a similar way, with parametrizations of stable manifolds such that $(\xi^s_x)^* T^-_f = \frac{i}{2} dz \wedge d\bar{z}$. Let us now derive further properties of $\mathcal{B}^u_f$ and $\mathcal{B}^u_f(X_0)$.

Lemma 7.2. *Let η_1, η_2 be elements of $\mathcal{B}^u_f$. Then either $\eta_1(\mathbf{C})$ and $\eta_2(\mathbf{C})$ are disjoint or $\eta_1(\mathbf{C}) = \eta_2(\mathbf{C})$. In the latter case $\eta_1(z) = \eta_2(e^{i\theta}z + b)$ on $\mathbf{C}$ for some $b \in \mathbf{C}$ and $\theta \in \mathbf{R}$.*

Proof. First, note that two unstable manifolds $W^u(x)$ and $W^u(x')$ of f either coincide or are disjoint; indeed, if y is a point of intersection, then the distances $\mathsf{dist}(f^n(y), f^n(x))$ and $\mathsf{dist}(f^n(y), f^n(x'))$ go to 0 as n goes to $-\infty$; thus the distance $\mathsf{dist}(f^n(x), f^n(x'))$ also goes to 0, so that x and x' are in the same unstable manifold. The first property of the lemma follows from the Hurwitz Theorem:

Lemma 7.3 (Hurwitz, see [8]). *Let C_n and D_n be two families of irreducible curves in the unit ball of $\mathbf{C}^2$. Assume that $C_n \cap D_n$ is empty for all n, that C_n converges to an irreducible curve C uniformly and that D_n converges to an irreducible curve D uniformly. Then either $C \cap D$ is empty or C coincides with D.*

To prove the second property, assume that η_1 and η_2 have the same image W. Let m be a point of W and fix two points z_1 and z_2 such that $\eta_1(z_1) = \eta_2(z_2) = m$. Assume that $(\eta_2)'(z_2) \neq 0$; one can always find such pairs (m, z_2) because η_2 is not constant. Then η_2 determines a local diffeomorphism from a neighborhood of z_2 in $\mathbf{C}$ to a neighborhood of m in W. The map $\varphi = \eta_2^{-1} \circ \eta_1$ is defined on a small disk centered at z_1, is holomorphic, and preserves $\frac{i}{2} dz \wedge d\bar{z}$. Thus, $\varphi(z) = e^{i\theta}z + b$ for some $b \in \mathbf{C}$ and $\theta \in \mathbf{R}$. As a consequence, there is a non-empty open subset of $\mathbf{C}$ on which $\eta_2(e^{i\theta}z + b) = \eta_1(z)$; this property holds on $\mathbf{C}$ by analytic continuation. □

Lemma 7.4. *Let x be an element of $\mathcal{B}^u_f(X_0)$ and ξ be an element of $\mathcal{B}^u_f$ with $x = \xi(0)$. Let $\mathcal{W} \subset X_0^{reg}$ be a neighborhood of x and r_0 be a positive radius such that $\xi(\mathbb{D}_{r_0}) \subset \mathcal{W}$. Let ε be a positive real number. There exists a neighborhood $\mathcal{V} \subset \mathcal{W}$ of x such that: for every $\eta \in \mathcal{B}^u_f$ with $\eta(0) \in \mathcal{V}$, there exists $\theta \in \mathbf{R}$ satisfying*

$$\forall z \in \mathbb{D}_{r_0}, \quad \mathsf{dist}_{X_0}(\eta(e^{i\theta}z), \xi(z)) \leq \varepsilon.$$

Proof. If not, there exists a sequence $\eta_n \in \mathcal{B}^u_f$ such that $\eta_n(0) \in B_x(\frac{1}{n})$ and

$$\forall \theta \in \mathbf{R}/2\pi\mathbf{Z}, \exists z_{n,\theta} \in \mathbb{D}_{r_0}, \mathsf{dist}(\eta_n(e^{i\theta}z_{n,\theta}), \xi(z_{n,\theta})) > \varepsilon.$$

For every angle $\theta \in \mathbf{R}$, choose a limit point $z_\theta \in \overline{\mathbb{D}}_{r_0}$ of the sequence $z_{n,\theta}$. By compactness of $\mathcal{B}^u_f$, one can assume that η_n converges to some $\eta \in \mathcal{B}^u_f$ with $\eta(0) = x$. By construction,

$$\forall \theta \in \mathbf{R}, \mathsf{dist}_{X_0}(\eta(e^{i\theta}z_\theta), \xi(z_\theta)) \geq \varepsilon.$$

In particular η is not equal to ξ up to a rotation, contradicting Lemma 7.2. □

Proposition 7.5. *Assume that the measure μ_f is absolutely continuous with respect to the Lebesgue measure. Let ξ be an element of $\mathcal{B}^u_f$. Then*

(1) *$\overline{\xi(\mathbf{C})} \cap X_0^{reg}$ is contained in $\mathcal{B}^u_f(X_0) \cap \mathcal{B}^s_f(X_0)$;*
(2) *$\forall z \in \mathbf{C}$, $\xi(z)$ is contained in the image of an entire curve $\nu \in \mathcal{B}^s_f$;*
(3) *the set of points $z \in \mathbf{C}$ such that ξ intersects a stable manifold ξ^s_y of f transversely at $\xi(z)$ is dense in $\mathbf{C}$.*

The same result holds if one permutes the roles of stable and unstable parametrizations.

Proof. Assertion (2) is weaker than assertion (1). Thus, we only prove (1) and (3). First, by definition, $\xi(\mathbf{C}) \cap X_0^{reg}$ is contained in $\mathcal{B}^u_f(X_0)$. Since $\mathcal{B}^u_f(X_0)$ is a closed subset of X_0^{reg}, it contains $\overline{\xi(\mathbf{C})} \cap X_0^{reg}$; similarly, $\mathcal{B}^s_f(X_0)$ contains $\overline{\eta(\mathbf{C})} \cap X_0^{reg}$ for all curves $\eta \in \mathcal{B}^s_f$. Since $\xi^* T_0^+$ coincides with the Lebesgue measure, it has full support in $\mathbf{C}$. But T_0^+ is an Ahlfors-Nevanlinna current for every stable manifold $\xi^s_x \colon \mathbf{C} \to X_0$ (see Section 3.8). Taking ξ^s_y in $\mathcal{B}^s_f$, we obtain

$$\xi(\mathbf{C}) \subset \overline{\xi^s_y(\mathbf{C})} \subset \mathcal{B}^s_f(X_0),$$

and the first assertion follows from these inclusions.

To prove (3), we apply (2): given any open subset $\mathcal{V}$ of $\mathbf{C}$, the Riemann surface $\xi(\mathcal{V})$ is contained in the family of entire curves $\eta(\mathbf{C})$, $\eta \in \mathcal{B}^s_f$; by Lemma 7.2, these curves $\eta(\mathbf{C})$ are pairwise disjoint. Now, apply the following lemma to conclude that the generic intersections between $\xi(\mathcal{V})$ and the curves $\eta \in \mathcal{B}^s_f$ are transversal. □

Lemma 7.6 ([8], Lemma 6.4). *Let C and D be complex submanifolds of $\mathbf{C}^2$ such that (i) $C \cap D = \{p\}$ and (ii) $T_pC = T_pD$. Let U be a bounded neighborhood of p. If D' is sufficiently close to D but $D \cap D' = \emptyset$, then the intersection of D' and C in U is non-empty and non-tangential at all intersection points.*

7.2. **Local laminations.** Every $\xi \in \mathcal{B}_f^u$ is locally a uniform limit of (generic) unstable manifolds $\xi_n^u \in \mathcal{A}_f^u$; such ξ_n^u take values in X_0^{reg} and $(\xi_n^u)'(z) \neq 0$ for every $z \in \mathbf{C}$. It may happen that $\xi'(z) = 0$ for some $z \in \mathbf{C}$. For instance, in (singular) Kummer examples, the velocity vanishes for the stable and unstable manifolds when they pass through the singularities of X_0. To keep another example in mind, consider the curve $\xi(z) = (z^2, 0)$ for $z \in \mathbb{D}$; this curve is not injective but its image is the (smooth) horizontal disk $\mathbb{D} \times \{0\} \subset \mathbb{D}^2$. Moreover, ξ is the limit of the pairwise disjoint and injective curves $\xi_n(z) = (z^2, 3^{-n}(1 + z/5))$. Such a non-injective limit may a priori arise in $\mathcal{B}_f^u$.

Remark 7.7. Let $\xi \in \mathcal{B}_f^u$ and $z \in \mathbf{C}$ satisfy $\xi(z) \in X_0^{reg}$ and $\xi'(z) = 0$. Since ξ is the limit of pairwise disjoint entire curves, the germ $\xi((\mathbf{C}, z))$ is not singular at $\xi(z)$ by [67, Proposition 12]. We thank Misha Lyubich for this reference; see also [10].

Recall that $\| \cdot \|$ is defined in Section 6.2.

Definition 7.8. *The* **velocity** $v(x)$ *at a point* $x \in \mathcal{B}_f^u(X_0)$ *is defined by*

$$v(x) := \| \xi'(0) \|,$$

where ξ is any element of $\mathcal{B}_f^u$ such that $\xi(0) = x$. The set $\mathcal{B}_f^u(X_0)$ is partitioned into

$$\mathcal{B}_f^{u,+}(X_0) := \left\{ x \in \mathcal{B}_f^u(X_0) \, , \, v(x) > 0 \right\}$$

and

$$\mathcal{B}_f^{u,0}(X_0) := \left\{ x \in \mathcal{B}_f^u(X_0) \, , \, v(x) = 0 \right\}.$$

The fact that the velocity is well defined, i.e. does not depend on the choice of the parametrization $\xi \in \mathcal{B}_f^u$, follows from Lemma 7.2. We first study the local geometry of the unstable manifolds near points of $\mathcal{B}_f^{u,+}(X_0)$; points of $\mathcal{B}_f^{u,0}(X_0)$ are dealt with in Section 8.2.

Proposition 7.9. *Assume that μ_f is absolutely continuous. Let x be a point in $\mathcal{B}_f^{u,+}(X_0)$ and ξ be an element of $\mathcal{B}_f^u$ such that $\xi(0) = x$. There are neighborhoods $\mathcal{U} \subset \mathcal{U}'$ of x in X_0^{reg} such that*

(1) *$\mathcal{U}'$ is isomorphic to a bidisk $\mathbb{D} \times \mathbb{D}$.*

(2) *The connected component of* $\xi(\mathbf{C}) \cap \mathcal{U}'$ *that contains* x *is a horizontal graph in* $\mathcal{U}$.
(3) *There is a lamination* $\mathcal{L}^u$ *of the whole open set* $\mathcal{U}$ *by horizontal graphs, each of which is contained in the image of a curve* $\eta \in \mathcal{B}^u_f$. *In particular* x *is an interior point of* $\mathcal{B}^u_f(X_0)$ *in the complex surface* X_0^{reg}.
(4) *If* η *is an element of* $\mathcal{B}^u_f$, Δ *is an open subset of* $\mathbf{C}$ *and* $\eta(\Delta)$ *is contained in* $\mathcal{U}$ *then* $\eta(\Delta)$ *is contained in a leaf of this lamination.*
(5) *There is a transversal to the lamination* $\mathcal{L}^u$ *which is a piece of a stable manifold of* f.
(6) *The support of* μ_f *in* $\mathcal{U}$ *coincides with* $\mathcal{U}$.

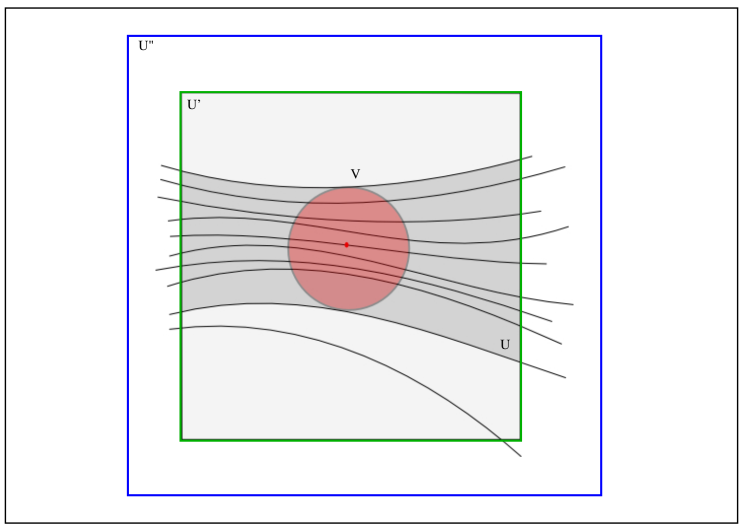

FIGURE 1. The point x is the red point right in the middle of the picture. The open set $\mathcal{V}$ is the red shaded ball, the open set U is grey; U' is light grey with a green contour, and U" is white with a blue contour. The horizontal curves represent pieces of unstable manifolds (one of them is not a graph above $\mathbb{D}$).

Proof. By definition, $\xi'(0) \neq 0$. Hence, there exist $r_0 > 0$ and neighborhoods $\mathcal{U}' \subset \mathcal{U}''$ of x in X_0^{reg} such that

- $\xi(\mathbb{D}_{r_0})$ is a smooth curve,
- the pair of open sets $(\mathcal{U}', \mathcal{U}'')$ is isomorphic to a pair of bidisks $(\mathbb{D} \times \mathbb{D}, \mathbb{D}_R \times \mathbb{D}_R)$, with $R > 1$,
- $\xi(\partial \mathbb{D}_{r_0}) \subset X_0^{reg} \setminus \mathcal{U}'$ and $\xi(\mathbb{D}_{r_0})$ is contained in $\mathcal{U}''$,
- $\xi(\mathbb{D}_{r_0}) \cap \mathcal{U}'$ is a horizontal graph in the bidisk $\mathcal{U}'$.

Changing $(\mathcal{U}', \mathcal{U}'')$ if necessary there exists $r < r_0$ such that $\xi(\mathbb{D}_{r_0}) \cap \mathcal{U}' = \xi(\mathbb{D}_r)$.

Lemma 7.10. *Let $r_0 > 0$ and $\mathcal{U}'$ as above. There exists a neighborhood $\mathcal{V}$ of x in X_0^{reg} such that for every $\eta \in \mathcal{B}_f^u$ with $\eta(0) \in \mathcal{V}$, the curve $\eta(\mathbb{D}_{r_0}) \cap \mathcal{U}'$ is a horizontal graph in $\mathcal{U}'$.*

Proof. Apply Lemma 7.4: given $\varepsilon > 0$, there is a neighborhood $\mathcal{V}$ of $\xi(0)$ such that every curve $\eta \in \mathcal{B}_f^u$ with $\eta(0) \in \mathcal{V}$ is ε-close to ξ on $\mathbb{D}_{r_0}$ after composition with a rotation $z \mapsto e^{i\theta} z$. All we need to prove, is that η is also a horizontal graph if ε is small enough.

Let $\pi\colon \mathcal{U}'' \simeq \mathbb{D}_R \times \mathbb{D}_R \to \mathbb{D}_R$ denote the first projection. The holomorphic mappings $\pi \circ \xi$ and $\pi \circ \eta$ are ε-close on $\mathbb{D}_{r_0}$. On the smaller disk $\mathbb{D}_r$, the map $\pi \circ \xi$ is one-to-one, with image equal to the unit disk $\mathbb{D}$. If ε is small enough, $|\pi \circ \eta - \pi \circ \xi| < |\pi \circ \xi|$ on the boundary of $\mathbb{D}_{r_0}$. Thus, the Rouché theorem implies that $\pi \circ \eta : \mathbb{D}_r \to \mathbb{D}$ is also one-to-one, and the result follows. □

Fix $\mathcal{V}$ as in the previous lemma. The curves $\eta(\mathbb{D}_{r_0}) \cap \mathcal{U}'$ with $\eta \in \mathcal{B}_f^u$ and $\eta(0) \in \mathcal{V}$ form a family of horizontal graphs in $\mathcal{U}'$. Let $\mathcal{U}$ be the union of these graphs. By Lemma 7.2 it is laminated by disjoint horizontal graphs; we denote this lamination by $\mathcal{L}^u$. Apply Proposition 7.5, Assertion (3), to the curve ξ. One can find an element ξ_y^s of $\mathcal{B}_f^s$ such that $\xi_y^s(0) \in \xi(\mathbb{D}_r) \cap \mathcal{V}$, with transverse intersection. Let Δ be a disk centered at the origin for which $\xi_y^s(\Delta)$ is contained in $\mathcal{V}$. Apply Proposition 7.5, Assertion (1), but to the stable manifold ξ_y^s: every point of $\xi_y^s(\Delta)$ is contained in the image of a curve $\eta \in \mathcal{B}_f^u$. Thus, the set $\mathcal{U}$ contains $\xi_y^s(\Delta)$. But $\mathcal{U}$ is laminated, and the Λ-lemma implies that the holonomy maps of a lamination are (quasi-conformal) homeomorphisms. Thus, $\mathcal{U}$ contains a neighborhood of $\xi(\mathbb{D}_r)$, and we replace $\mathcal{U}$ by such a laminated open neighborhood.

For Property (5), shrink Δ and $\mathcal{U}$ to assure that $\xi_y^s(\Delta)$ is transverse to $\mathcal{L}^u$.

The previous argument shows that the support of the restriction $T^-_{f|\mathcal{U}}$ coincides with $\mathcal{U}$ because its slice $(\xi_y^s)^*(T_f^-)_{|\Delta}$ is the Lebesgue measure (see the proof of Proposition 7.5). To prove Property (6), fix a small ball $B \subset \mathcal{U}$. Since T_f^- charges B, there is an unstable manifold that enters B. Take a point $x' \in B$ on this unstable manifold which intersects a stable manifold transversally, and apply Properties (1) to (5) for the stable manifolds: we get a stable lamination $\mathcal{L}^s$ of a neighborhood $\mathcal{W}$ of x' which is transverse to $\mathcal{L}^u$. In $\mathcal{W}$, the product of T_f^- and T_f^+ is strictly positive; hence $\mu_f(B) > 0$ (see § 4.2.1). □

8. HOLOMORPHIC FOLIATION AND HARTOGS EXTENSION

We conclude the proof of the main theorem. The first step is to promote the local laminations obtained in Proposition 7.9 into local holomorphic foliations. Then we use the Hartogs phenomenon to extend the foliation into a global, singular foliation of X. The theorem eventually follows from a classification of automorphisms preserving holomorphic foliations.

8.1. **From laminations to holomorphic foliations.** Proposition 7.9 asserts that $\mathcal{B}_f^u$ determines near every $x \in \mathcal{B}_f^{u,+}(X_0)$ a local lamination denoted by $\mathcal{L}^u$. In this section we promote these local laminations into local holomorphic foliations.

Proposition 8.1. *The local laminations $\mathcal{L}^u$ are holomorphic.*

Proof. The following argument is due to Ghys (see [51, 23]).

Let $\mathcal{U}$ and $\mathcal{U}'$ be connected open subsets of X_0^{reg} such that $\mathcal{U}'$ is biholomorphic to a bidisk $\mathbb{D} \times \mathbb{D}$, $\mathcal{U}$ is contained in $\mathcal{U}'$, and the lamination $\mathcal{L}^u$ of $\mathcal{U}$ is made of disjoint horizontal graphs (as in Fig. 1). Denote by (x_1, x_2) the coordinates in $\mathcal{U}' \simeq \mathbb{D} \times \mathbb{D}$ and by π_1 and π_2 the two projections ($\pi_i(x_1, x_2) = x_i$, $i = 1, 2$). Let $p = (p_1, p_2)$ be a point of $\mathcal{U}$ and $\Delta_p \subset \mathcal{U}'$ be the vertical curve $\{x_1 = p_1\}$. Denote by γ a continuous path in the leaf through x that starts at x and ends at another point $q = (q_1, q_2)$. The holonomy map from the transversal Δ_p to the transversal $\Delta_q := \{x_1 = q_1\}$ is a homeomorphism h_γ from $\Delta_{p,\mathcal{U}} := \Delta_p \cap \mathcal{U}$ to its image $\Delta_{q,\mathcal{U}} := \Delta_q \cap \mathcal{U}$. According to the Λ-lemma, this homeomorphism $h_{p,q}$ is $K(p,q)$-quasi-conformal, with a constant $K(p,q)$ satisfying

$$0 \leq K(p,q) - 1 \leq \mathsf{dist}(p,q).$$

Recall that a 1-quasi-conformal map is conformal, hence holomorphic (see Section 4.1.1). Instead of looking at vertical disks, we may choose pairs of disks Δ and Δ' which are contained in stable manifolds of f and are transverse to the lamination. Let γ be a path in a leaf $\mathcal{L}^u(m)$ that joins the intersection points $x := \Delta \cap \mathcal{L}^u(m)$ and $x' := \mathcal{L}^u(m) \cap \Delta'$, let h_γ be the (germ of) holonomy from Δ to Δ'. Applying f^{-n} and using Remark 3.6 there exists $\alpha_n \in \mathbf{C}^*$ such that

$$f^{-n} \circ \xi_x^u(z) = \xi_{f^{-n}(x)}^u(\alpha_n z).$$

Proposition 3.5 precisely asserts that $\lim_n \alpha_n = 0$ for almost every point x.

Using the Poincaré recurrence theorem we can assume that $f^{-n}(x) \in \mathcal{U}$. This and the compactness of $\mathcal{B}_f^u$ imply that the path γ is shrunk uniformly under the action of f^{-n}; simultaneously, the disks Δ and Δ' are mapped to large disks $f^{-n}(\Delta)$ and $f^{-n}(\Delta')$: the connected components of $f^{-n}(\Delta) \cap \mathcal{U}$ and $f^{-n}(\Delta') \cap \mathcal{U}$ containing

$f^{-n}(x)$ and $f^{-n}(x')$ are vertical disks in $\mathcal{U}$, and their relative distance goes to 0 with n. Thus, the holonomy h_{-n} between these disks is K_n-quasi-conformal, with $\lim_n K_n = 1$. But f^n conjugates h_{-n} with h_γ on a small neighborhood of x in Δ. Hence, for almost every point $x \in \Delta$ (with respect to the conditional measure of μ_f and therefore also with respect to the Lebesgue measure), and for every $\varepsilon > 0$, the holonomy h_γ is $(1+\varepsilon)$-quasi-conformal on a small neighborhood $\Delta_\varepsilon \subset \Delta$ of x. This implies that h_γ is $(1+\varepsilon)$-quasi-conformal for all $\varepsilon > 0$; hence, h_γ is indeed conformal.

We have proved that the holonomy between two transversal disks which are contained in stable manifolds is holomorphic. By Proposition 7.5 these transversal disks form a dense subset of transversals. Consequently, the holonomy between all pairs of vertical disks Δ_x and Δ_y is holomorphic. This implies that the lamination $\mathcal{L}^u$ is holomorphic. □

Lemma 8.2. *The local holomorphic foliations $\mathcal{L}^u$ defined near every point of $\mathcal{B}_f^{u,+}(X_0)$ can be glued together to provide a holomorphic foliation $\mathcal{F}^u$ of the open set $\mathcal{B}_f^{u,+}(X_0)$.*

Proof. Propositions 7.9 and 8.1 show that if $x \in \mathcal{B}_f^{u,+}(X_0)$, then there exist a neighborhood $\mathcal{U}$ of x and a holomorphic foliation $\mathcal{L}^u$ of $\mathcal{U}$ such that all entire curves $\eta \in \mathcal{B}_f^u$ are tangent to $\mathcal{L}^u$. More precisely, if η is an element of $\mathcal{B}_f^u$ and $\eta(\Delta)$ is contained in $\mathcal{U}$, then $\eta(\Delta)$ is contained in a leaf of $\mathcal{L}^u$; in other words, if ω^u is a holomorphic 1-form on $\mathcal{U}$ which defines the foliation $\mathcal{L}^u$, then $\eta^*\omega^u = 0$ on Δ. The holomorphic foliation $\mathcal{F}^u$ can thus be defined as the unique holomorphic foliation of $\mathcal{B}_f^{u,+}(X_0)$ such that the generic unstable manifolds of f are leaves of $\mathcal{F}^u$. □

8.2. **Hartogs extension of the holomorphic foliation $\mathcal{F}^u$.** The foliation $\mathcal{F}^u$ is defined on the open set $\mathcal{B}_f^{u,+}(X_0)$. We extend it to X_0^{reg} (i.e. through $\mathcal{B}_f^{u,0}(X_0)$).

Proposition 8.3. *Let x be an element of $\mathcal{B}_f^{u,0}(X_0)$. There exists a neighborhood $\mathcal{V}$ of x in X_0^{reg} such that the holomorphic foliation $\mathcal{F}^u$ extends as a (singular) holomorphic foliation of $\mathcal{V}$.*

To prove this proposition, fix a curve $\xi \in \mathcal{B}_f^u$ with $\xi(0) = x$ (and $\xi'(0) = 0$). Remark 7.7 tells us that $\xi(\mathbb{D})$ is smooth, but we shall not need this result. Even if $\xi(\mathbb{D})$ is singular, there exists an open neighborhood $\mathcal{V}$ of x such that

(i) $\mathcal{V} \subset \mathbf{C}^2$ up to a local choice of co-ordinates,
(ii) the connected component of $\xi^{-1}(\mathcal{V})$ containing 0 contains a disk $\mathbb{D}_r$,
(iii) $\xi'(0) = 0$ but ξ' does not vanish on $\mathbb{D}_r \setminus \{0\}$.

The existence of such a neighborhood follows, for instance, from the Weierstrass preparation theorem (applied to the curve $\xi(\mathbb{D})$); it follows also from Remark 7.7 (since $\xi(\mathbb{D}_s)$ is smooth for s small).

Fix a radius s with $0 < s < r/3$. Since ξ' does not vanish on $\partial\mathbb{D}_s$, there is an open neighborhood $\mathcal{U} \subset \mathcal{V}$ of $\xi(\partial\mathbb{D}_s)$ and a holomorphic foliation $\mathcal{L}^u$ on $\mathcal{U}$ such that all entire curves $\eta \in \mathcal{B}^u_f$ are tangent to $\mathcal{L}^u$ in $\mathcal{U}$.

Fix $\varepsilon > 0$. Let $\theta \in \mathbf{R}$ and define $x_\theta := \xi(se^{i\theta})$. By Lemma 7.4 there exists a neighborhood $\mathcal{U}(\theta,\varepsilon) \subset \mathcal{U}$ of x_θ such that, if $\eta \in \mathcal{B}^u_f$ and $\eta(se^{i\theta}) \in \mathcal{U}(\theta,\varepsilon)$, then there exists a complex number a with $|a| = 1$ such that $\eta(z)$ and $\xi(az)$ are ε-close on $\mathbb{D}_r$. In particular, the curve $\eta(\mathbb{D}_{2s})$ is contained in an ε-neighborhood of $\xi(\mathbb{D}_r)$.

Thus, there is a neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ of $\xi(\partial\mathbb{D}_s)$ satisfying

- $\forall m \in \mathcal{U}_0$, the leaf of $\mathcal{L}^u$ through m is contained in a disk $\eta(\mathbb{D}_{2s})$ where η is an element of $\mathcal{B}^u_f$ which is ε-close to ξ on $\mathbb{D}_r$;
- if $(x_n) \in \mathcal{U}_0^{\mathbf{N}}$ converges towards a point of $\xi(\mathbb{D}_s)$, one can choose such disks $\eta_n(\mathbb{D}_{2s})$ with $\eta_n \in \mathcal{B}^u_f$ converging towards ξ uniformly on $\mathbb{D}_r$.

Let $(x_n)_{n\geq 1}$ be such a sequence, with the additional property that x_n is contained in an unstable manifold of f for all n. Then, η_n is a parametrization of an unstable manifold, and therefore η_n' does not vanish. Hence, there is an open neighborhood $\mathcal{U}_n$ of $\eta_n(\mathbb{D}_s)$ such that $\mathcal{L}^u$ extends as a holomorphic foliation of $\mathcal{U}_0 \cup \mathcal{U}_n$. As explained in Section 8.1, the extensions of $\mathcal{L}^u$ to the open sets $\mathcal{U}_n$ are compatible, and determine a foliation of $\mathcal{U}_\infty := \cup_{n\geq 0}\mathcal{U}_n$.

Note that $\mathcal{U}_\infty$ is a subset of $\mathcal{V} \subset \mathbf{C}^2$. The slopes of the leaves of $\mathcal{L}^u$ determine a holomorphic function $\mathsf{s}^u \colon \mathcal{U}_\infty \to \mathbb{P}^1$ (with $\mathbb{P}^1$ the projective line of all possible "slopes"). Such a function extends as a meromorphic function $\widehat{\mathsf{s}^u}$ on the envelop of holomorphy $\widehat{\mathcal{U}_\infty}$ of $\mathcal{U}_\infty$ (see [58]). The function $\widehat{\mathsf{s}^u}$ determines a (singular) holomorphic foliation on $\widehat{\mathcal{U}_\infty}$ which extends $\mathcal{L}^u$.

It remains to show that $\widehat{\mathcal{U}_\infty}$ contains a neighborhood of $\xi(\mathbb{D}_s)$, hence a neighborhood of x. For this purpose, we apply the following theorem to $D = \mathcal{U}_\infty$, $S_n = \eta_n(\mathbb{D}_s)$ and $S = \xi(\mathbb{D}_s)$, and we remark that the boundaries $\eta_n(\partial\mathbb{D}_s)$ are contained in a compact neighborhood K of $\xi(\partial\mathbb{D}_s)$ with $K \subset \mathcal{U}_0$.

Theorem 8.4 (Behnke-Sommer, [34], chapter 13)**.** *Let D be a bounded domain of $\mathbf{C}^m$, $m \geq 2$. Let S_n be a sequence of complex analytic curves which are properly contained in D. Assume that S_n converges towards a curve $S \subset \mathbf{C}^m$ and that the boundaries ∂S_n converge to a curve $\Gamma \Subset D$. Then every holomorphic function $h \in O(D)$ extends to a neighborhood of S.*

Since, on a surface, the singularities of a holomorphic foliation are isolated (they correspond to the indeterminacy points of the " slope function $\hat{s^u}$ "), there is an open neighborhood of x in which $\mathcal{L}^u$ has at most one singular leaf, namely the leaf $\xi(\mathbb{D}_r)$.

Corollary 8.5. *The set $\mathcal{B}^u_f(X_0)$ coincides with X_0^{reg}, the foliation $\mathcal{F}^u$ extends to a (singular) holomorphic foliation of X_0^{reg}, and this foliation is f-invariant. Its lift to X by the bimeromorphic morphism $\pi\colon X \to X_0$ determines a (singular) f-invariant foliation of X.*

Proof. The set $\mathcal{B}^u_f(X_0)$ is closed. Propositions 7.9 and 8.3 show that $\mathcal{B}^u_f(X_0)$ is also open. Since X_0^{reg} is isomorphic to the complement of finitely many curves in X, it is connected. Thus, $\mathcal{B}^u_f(X_0)$ is equal to X_0^{reg}, and X_0^{reg} supports a (singular) holomorphic foliation $\mathcal{F}^u$, such that every unstable manifold of f is a leaf of $\mathcal{F}^u$. This implies that f preserves $\mathcal{F}^u$. Then, we lift $\mathcal{F}^u$ to a (singular) holomorphic foliation on $\pi^{-1}(X_0^{reg})$; by [21, §1], this foliation automatically extends to an f-invariant holomorphic foliation of X. □

8.3. **Proof of the main theorem.** To complete the proof of the main theorem, we refer to [23] and [29, Théorème 3.1], in which the following theorem is proved:

Theorem 8.6. *Let X be a compact Kähler surface, with a (singular) holomorphic foliation $\mathcal{F}$ and an automorphism $f\colon X \to X$ of positive entropy. If f preserves $\mathcal{F}$, then (X, f) is a Kummer example, and the stable (or unstable) manifolds of f are leaves of $\mathcal{F}$.*

In fact, [29] classifies all triples $(X, \mathcal{F}, f)$ where X is a smooth surface, $\mathcal{F}$ is a (singular) holomorphic foliation of X, and f is a birational transformation of X of infinite order preserving $\mathcal{F}$.

A posteriori, one verifies that the surface X' in Definition 1.3 coincides with the surface X_0 of Proposition 6.1 (the morphism π contracts only periodic curves of f, and all of them because f_Y has no periodic curve).

9. Consequences and applications

9.1. **Equivalent dynamical characterizations.** As before, consider an automorphism f of a complex projective surface X, with positive entropy $\log \lambda_f$.

9.1.1. *Ruelle's inequalities and absolute continuity.* The first part of the following result is due to Ruelle. The second part is proved by Ledrappier in [65, Corollaire 5.6], in a more general setting. Here the local product structure of μ_f leads to a somewhat simplified proof.

Proposition 9.1 (Ruelle, Ledrappier)**.** *Let X be a complex projective surface and f be an automorphism of X with positive entropy $\log\lambda_f$. Then the Lyapunov exponents of f with respect to μ_f satisfy*

$$\lambda_s \leq -\frac{1}{2}\log\lambda_f \quad \textit{and} \quad \lambda_u \geq \frac{1}{2}\log\lambda_f.$$

If equality holds simultaneously in these two inequalities, then μ_f is absolutely continuous with respect to the Lebesgue measure.

Proof. The Ruelle inequality states that $\log(\lambda_f) \leq 2\lambda_u$ and $\log(\lambda_f) \leq -2\lambda_s$ (see [78] and [62], p. 669). Assume that equality holds simultaneously in these two inequalities. Fix a Pesin box $\mathcal{P}$, as in Section 4.2. According to [65, Théorème 4.8], the conditional measures of μ_f with respect to the stable and unstable manifolds are absolutely continuous with respect to the Lebesgue measure. In other words, both ν^+ and ν^- are absolutely continuous with respect to the Lebesgue measure, because they coincide with the conditional measures of μ_f with respect to the stable and unstable laminations of $\mathcal{P}$. In the Pesin box, μ_f corresponds to the product measure $\nu^+ \otimes \nu^-$ via the homeomorphism h of Section 4.2.1. Since the holonomy of the stable and unstable laminations are quasi-conformal, they are absolutely continuous with respect to the Lebesgue measure (see Section 4.3.1). Hence, μ_f is absolutely continuous with respect to the Lebesgue measure in $\mathcal{P}$, and therefore in X. □

9.1.2. *Proof of Corollary 1.7.* Our main theorem proves $(1) \Rightarrow (4)$, while the reverse implication is obvious. We proved $(2) \Rightarrow (1)$ in Proposition 9.1 and $(1) \Rightarrow (2)$ in Proposition 5.3 (this equivalence follows also from [65, Corollaire 5.6]). To prove $(2) \Leftrightarrow (3)$ we use Ruelle's inequality [78], which provides $\lambda_s \leq -\frac{1}{2}\log\lambda_f$ and $\lambda_u \geq \frac{1}{2}\log\lambda_f$, and Young's theorem [80] which ensures that the generic limits in Property (3) are equal to $(1/\lambda_u - 1/\lambda_s)\log\lambda_f$. (Young's theorem is proved for $\mathcal{C}^\infty$-diffeomorphisms of compact surfaces, but her proof applies also to our context)

9.2. **K3 and Enriques surfaces.**

9.2.1. *The Classification Theorem.* The Classification Theorem stated in the introduction is proved in [29, 30]. Let us add two remarks. Its first assertion rules out the case of Enriques surfaces: *if f is an automorphism of an Enriques surface with positive entropy, then μ_f is singular with respect to the Lebesgue measure.* Assertion (3) is sharp, meaning that there are rational Kummer examples with λ_f in $\mathbf{Q}(\zeta_l)$ for all possible orders $l = 3$, 4, and 5. For instance, an example is given in [30], of an automorphism g of an abelian surface A such that $\lambda_g = |1+\zeta_5|^2$, where ζ_5 is the primitive fifth root of unity $\exp(2i\pi/5)$. The linear transformation $(x,y) \mapsto (\zeta_5 x, \zeta_5 y)$ induces an automorphism of A, and the quotient is a rational surface: this gives examples of Kummer automorphisms on rational surfaces for which λ_f has degree 4.

9.2.2. *Proof of Corollary 1.8.* This corollary is a direct consequence of the second assertion of the Classification Theorem and the following classical result.

Lemma 9.2. *Let X be a complex projective K3 surface, with Picard number equal to* 2*. The group of automorphisms of X is infinite if and only if the intersection form does not represent* 0 *and* -2 *in* $\mathrm{NS}(X)$*. If it is infinite, then it is virtually cyclic, and all elements of* $\mathsf{Aut}(X)$ *of infinite order have positive topological entropy.*

Sketch of proof. Let $\mathsf{Iso}(\mathrm{NS}(X))$ be the group of isometries of the lattice $\mathrm{NS}(X)$ with respect to the intersection form $\langle\cdot|\cdot\rangle$.

Step 1.– By the Hodge index theorem the intersection form has signature $(1,1)$ on $\mathrm{NS}_{\mathbf{R}}(X)$. Assume that this form represents 0; this means that the two isotropic lines of $\langle\cdot|\cdot\rangle$ are defined over $\mathbf{Z}$: they contain primitive elements v_1 and v_2 in $\mathrm{NS}(X)$. Since the isotropic cone is $\mathsf{Iso}(\mathrm{NS}(X))$-invariant and the automorphisms of $\mathbf{Z}$ coincides with $\pm\mathrm{Id}$, a subgroup of index at most 4 in $\mathsf{Iso}(\mathrm{NS}(X))$ preserves the two isotropic lines pointwise. Thus, $\mathsf{Iso}(\mathrm{NS}(X))$ has at most four elements. On the other hand, every element f of $\mathsf{Aut}(X)$ determines an element f^* in $\mathsf{Iso}(X)$ and the homomorphism $f \mapsto f^*$ has finite kernel (because the group of automorphisms of a K3 surface is discrete). Thus, if the intersection form represents 0, $\mathsf{Aut}(X)$ is finite.

Step 2.– Now assume that $\langle\cdot|\cdot\rangle$ does not represent 0. Let $\mathsf{Iso}(\mathrm{NS}(X))^+$ denote the subgroup $\mathsf{Iso}(\mathrm{NS}(X))$ that fixes the connected component H of $\{u \in \mathrm{NS}_{\mathbf{R}}(X);\ \langle u|u\rangle > 0\}$ containing ample classes. This group is infinite, and is either cyclic, or dihedral (this is equivalent to the resolution of the Pell-Fermat equations); more precisely, a subgroup of $\mathsf{Iso}(\mathrm{NS}(X))^+$ of index at most 2 is generated by a hyperbolic isometry ψ, which dilates one of the isotropic lines by a factor $\lambda_\psi > 1$ and contracts the other one by $1/\lambda_\psi$. If the ample cone of X coincides with H, the Torelli theorem shows that the image of $\mathsf{Aut}(X)$ in $\mathsf{Iso}(\mathrm{NS}(X))$ is a finite index subgroup of $\mathsf{Iso}(\mathrm{NS}(X))^+$ (see [4]). Since the kernel of $f \mapsto f^*$ is finite, $\mathsf{Aut}(X)$ is virtually cyclic and, if f is an automorphism of X of infinite order, f^* coincides with an iterate of ψ^l, $l \neq 0$. Thus, the topological entropy of f is equal to $|l|\log(\lambda_\psi)$ and is positive.

Step 3.– The ample cone of X is the subset of classes $a \in H$ such that $\langle a|[E]\rangle > 0$ for all irreducible curves $E \subset X$ with negative self-intersection. But, on a K3 surface, such a curve is a smooth rational curve with self-intersection -2. Thus, H is the ample cone if and only if X does not contain any -2-curve. On the other hand, the Riemann-Roch formula implies that X contains such a -2-curve if and only if the intersection form represents -2 on $\mathrm{NS}(X)$. To sum up, if $\langle\cdot|\cdot\rangle$ represents -2, the ample cone is a strict sub-cone of H. In that case, the group of isometries of $\mathrm{NS}(X)$ preserving both H and the ample cone is finite, so that $\mathsf{Aut}(X)$ is finite too. □

9.3. **Rational surfaces and Galois conjugates.**

9.3.1. *Proof of Corollary 1.10.* Let A be a complex abelian surface. Its Picard number is bounded from above by $h^{1,1}(A;\mathbf{R})$ hence by 4. The dynamical degree λ_g of every $g\in Aut(A)$ is the largest eigenvalue of g^* on $\mathrm{NS}(A)\otimes_{\mathbf{Z}}\mathbf{R}$. As such, λ_g is a root of the characteristic polynomial of $g^*\colon \mathrm{NS}(A)\to\mathrm{NS}(A)$, and it is an algebraic integer of degree at most 4. Passing to a finite g-equivariant quotient $\pi\colon A\to X_0$, one does not change the topological entropy. Thus, if (X,f) is a Kummer example, the dynamical degree λ_f is also an algebraic integer of degree at most 4.

Example 9.3. Bedford and Kim in [6, 7], and McMullen in [72], construct automorphisms $f_n\colon X_n\to X_n$ of rational surfaces with positive entropy. Most of them have a singular measure of maximal entropy because the degree of λ_{f_n} goes to $+\infty$ with n.

9.3.2. *Blanc's automorphisms.* An automorphism of an abelian surface with positive entropy does not preserve any curve of genus 1. And the resolution of an orbifold singularity only creates rational curves. Thus, our Main Theorem gives:

Lemma 9.4. *Let $f\colon X\to X$ be an automorphism of a complex projective surface with positive entropy* $\log\lambda_f$. *Assume that X contains a curve of genus* 1 *which is f-periodic. Then μ_f is singular with respect to the Lebesgue measure.*

In [16], Blanc constructs rational surfaces X_n which are obtained by blowing up a finite set of points on a smooth cubic curve $C_0\subset\mathbb{P}^2(\mathbf{C})$; on these surfaces, there are n involutions σ_i, $1\leq i\leq n$, that preserve the strict transform C of C_0 and do not satisfy any non trivial relation (they generate a group of automorphisms isomorphic to a free product $\mathbf{Z}/2\mathbf{Z}\star\cdots\star\mathbf{Z}/2\mathbf{Z}$). The composition $f_{ijk}=\sigma_i\circ\sigma_j\circ\sigma_k$ of three distinct involutions is an automorphism with positive entropy that preserves the genus 1 curve C. We conclude that the measure of maximal entropy of f_{ijk} is singular with respect to the Lebesgue measure.

10. Appendix I: Currents and potentials near periodic curves

Recall that, blowing down the periodic curves of the automorphism $f\colon X\to X$, we get a bimeromorphic morphism $\pi\colon X\to X_0$ (X is a proper modification of the normal surface X_0, see Section 6.1).

This section starts with a study of $T_f^+ + T_f^-$: Proposition 10.7 shows that this current is cohomologous to a semi-Kähler current $\hat{\tau}$ which is bounded from below by the pull-back $\pi^*(\kappa_0)$ of a positive $(1,1)$-form κ_0 on X_0 (the interested reader may also consult [39] and [73] for related results). Then, a result of Dinh and Sibony is stated, and proved, in the context of complex spaces (see Theorem 10.8). Put together, these preliminary statements lead to a proof of Proposition 6.9.

Remark 10.1. Two difficulties appear: firstly, the surface X_0 could be non-projective (see Section 11); secondly, even if the currents $T_f^\pm$ have local potentials on X, their projections $\pi_*(T_f^\pm)$ could fail to satisfy such a property at the singularities of X_0.

10.1. **Big and nef classes in a neighborhood of the class $\Sigma := \theta_f^+ + \theta_f^-$.** We keep the notations of Section 2. The tensor product $\mathrm{NS}(X) \otimes_{\mathbf{Z}} A$, for A in $\{\mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}\}$, is denoted $\mathrm{NS}(X; A)$. Consider the subspace PC_f of $\mathrm{NS}(X;\mathbf{R})$ generated by all classes of f-periodic curves. By construction, PC_f is a subspace of $\Pi_f^\perp$, because $\langle \theta_f^\pm | [C] \rangle = 0$ for every periodic curve (see Sections 2.2 and 6.1).

Denote by $\Psi_f \in \mathbf{Z}[t]$ the minimal polynomial of the algebraic integer λ_f. The characteristic polynomial of $f^* \in \mathsf{GL}(\mathrm{NS}(X))$ is a product of Ψ_f and cyclotomic factors. The vector space $\mathrm{NS}(X;\mathbf{Q})$ splits as a direct sum $\mathrm{N}_f \oplus \mathrm{N}_f^\perp$ such that

(1) N_f and $\mathrm{N}_f^\perp$ are f^*-invariant vector subspaces of $\mathrm{NS}(X;\mathbf{Q})$;
(2) the characteristic polynomial of $f^* \colon \mathrm{N}_f \to \mathrm{N}_f$ is equal to Ψ_f;
(3) the characteristic polynomial of $f^* \colon \mathrm{N}_f^\perp \to \mathrm{N}_f^\perp$ is a product of cyclotomic factors;
(4) $\mathrm{N}_f \otimes_{\mathbf{Q}} \mathbf{R}$ contains θ_f^+ and θ_f^-, and the intersection form is of Minkowski type on N_f;
(5) $\mathrm{N}_f^\perp \otimes_{\mathbf{Q}} \mathbf{R}$ contains PC_f and the intersection form is negative definite on $\mathrm{N}_f^\perp$.

Observe that if $\mathrm{PC}_f = \Pi_f^\perp$ then the plane Π_f is defined over $\mathbf{Q}$ and λ_f is a quadratic unit. Thus PC_f is usually much smaller than $\Pi_f^\perp$ (see [31, 76]).

Lemma 10.2. *There is an open neighborhood $\mathcal{W}$ of the class $\Sigma := \theta_f^+ + \theta_f^-$ in $\mathrm{N}_f \otimes_{\mathbf{Q}} \mathbf{R}$ which is contained in the big and nef cone.*

This lemma corresponds to Theorem 3 and Proposition 10 of [76]; we include a proof for completeness.

Proof. The class Σ is nef because θ_f^+ and θ_f^- are nef. Since $\Sigma^2 = 2\langle \theta_f^+ | \theta_f^- \rangle > 0$, it is also big. If C is a(n effective) curve then $\langle \Sigma | [C] \rangle \geq 0$, with equality if and only if $[C]$ is in $\Pi_f^\perp$, if and only if C is a periodic curve, if and only if $[C] \in \mathrm{PC}_f$ (see Section 6.1). We now prove the lemma by contradiction. Since the condition $u^2 > 0$ is open in $H^{1,1}(X,\mathbf{R})$, we may assume that there is a sequence (w_n) of classes $w_n \in \mathrm{N}_f$ converging towards Σ such that w_n is not nef. Since w_n is not nef, there exists an irreducible curve C_n such that $\langle w_n | [C_n] \rangle < 0$. In particular, the curve C_n is not in PC_f, because w_n is an element of N_f; thus, $\langle \Sigma | [C_n] \rangle > 0$. We may also assume that the curves C_n are pairwise distinct; otherwise, we could extract a constant subsequence $C_{n_j} = C$ and we would have $\langle \Sigma | [C] \rangle \leq 0$ because Σ is the limit of (w_n): this would contradict that C is effective but not in PC_f.

Take a subsequence of $([C_n]/ \parallel [C_n] \parallel)$ that converges to a non-zero pseudo-effective class c_∞. We have $\langle \Sigma | c_\infty \rangle = 0$, because w_n converges towards Σ. Being pseudo-effective, c_∞ is in $\Pi_f^\perp$ and consequently $c_\infty^2 < 0$. On the other hand, c_∞^2 is the limit of $\langle C_n | C_{n+1} \rangle / (\parallel [C_n] \parallel \parallel [C_{n+1}] \parallel)$ and, as such, is non-negative. This contradiction concludes the proof. □

Remark 10.3. Since the cone of nef and big classes is f^*-invariant, this lemma implies that every element of the form $a\theta_f^+ + b\theta_f^-$ with a and b positive is in the big and nef cone, and is in the relative interior of this cone in $\mathrm{N}_f \otimes_{\mathbf{Q}} \mathbf{R}$.

10.2. A current τ in the class $\Sigma = \theta_f^+ + \theta_f^-$.

10.2.1. *Semi-Kähler forms.* A smooth $(1,1)$-form κ on X is **semi-Kähler** if it is closed and the set of points $x \in X$ around which κ is (strictly) positive is the complement of a Zariski closed set $Z(\kappa) \subset X$ (hence, κ induces a Kähler form on $X \setminus Z(\kappa)$).

The main source of examples is the following (see [64, §2.1.B]). Let M be a big and semi-ample line bundle. Set

$$\mathrm{Free}(X;M) = \{m \in \mathbf{N} \mid mM \text{ is base point free}\}.$$

(We use additive notations, hence mM is also the line bundle $M^{\otimes m}$.) It is a semi-group, and we denote by $\mathrm{fr}(M)$ the largest natural number such that every element of $\mathrm{Free}(X;M)$ is a multiple of $\mathrm{fr}(M)$. Given k in $\mathrm{Free}(X;M)$, the line bundle kM determines a birational morphism

$$\Phi_{kM} \colon X \to X_{kM} \subset \mathbb{P}(H^0(X,kM)^*),$$

onto a projective (singular) surface X_{kM}. According to [64], Theorem 2.1.27, there is an algebraic fibre space $\Phi \colon X \to Y$ such that

(1) Y is a normal projective surface (see [64], Example 2.1.15);
(2) $X_{kM} = Y$ and $\Phi_{kM} = \Phi$ for sufficiently large elements k of $\mathrm{Free}(X;M)$;
(3) there is an ample line bundle A on Y such that $\Phi^*A = \mathrm{fr}(M)M$.

For large elements k of $\mathrm{Free}(X;M)$, the pull-back of a Fubini-Study form by Φ_{kM} is a semi-Kähler form κ_{kM} on X, and the Chern class $c_1(M)$ is represented by $(1/k)\kappa_{kM}$.

This semi-Kähler form κ_{kM} vanishes *along* the set of curves which are blown-down by Φ_{kM} (i.e. by Φ).

Remark 10.4. If y is a singularity of X_{kM}, one can find a ball $\mathcal{V}$ in $\mathbb{P}(H^0(X, kM)^*)$ containing y and a smooth potential $u\colon \mathcal{V} \to \mathbf{R}$ for the Fubini-Study form on $\mathcal{V}$. Then $\Phi_{kM}^{-1}(\mathcal{V})$ is a neighborhood of $\Phi_{kM}^{-1}\{y\}$ on which $u \circ \Phi_{kM}$ is a global potential for κ_{kM}.

10.2.2. *A representative of the class* $\Sigma = \theta_f^+ + \theta_f^-$.

Proposition 10.5. *There exists a closed positive current* τ *on* X *such that*

1. τ *represents the class* Σ*;*
2. $\tau = \kappa + \{F\}$ *where* κ *is a semi-Kähler form and* $\{F\}$ *is the current of integration on a real effective divisor* F*;*
3. $Z(\kappa)$ *and the support* $\mathrm{Supp}(F)$ *of* F *are both unions of* f*-periodic curves, and every irreducible periodic curve is contained in* $Z(\kappa) \cup \mathrm{Supp}(F)$*;*
4. $\mathrm{Supp}(F)$ *is a union of connected components of the union of periodic curves;*
5. *If* E *is a connected component of the union of periodic curves, then* E *is a connected component of* $\mathrm{Supp}(F)$ *or of* $Z(\kappa)$*, but not both;*
6. *the form* κ *and the currents* T_f^+ *and* T_f^- *have continuous potentials in neighborhoods of* $Z(\kappa)$*.*

Assertion (6) means that there is an open neighborhood $\mathcal{U}$ of $Z(\kappa)$ and continuous functions u_κ, u_+, and u_- on $\mathcal{U}$ such that the restrictions $\kappa_{|\mathcal{U}}$, $T^+_{|\mathcal{U}}$, $T^-_{|\mathcal{U}}$ coincide respectively with $\frac{\mathsf{i}}{\pi}\partial\overline{\partial}u_\kappa$, $\frac{\mathsf{i}}{\pi}\partial\overline{\partial}u_+$, $\frac{\mathsf{i}}{\pi}\partial\overline{\partial}u_-$. Proposition 10.5 does not assert that $Z(\kappa)$ and $\mathrm{Supp}(F)$ are disjoint. A priori, κ may vanish on an irreducible periodic curve $C \subset \mathrm{Supp}(F)$; but, according to Property (5), κ can not be identically zero on a connected component of $\mathrm{Supp}(F)$. From this property, we can classify the connected components E of the union of periodic curves in two types: the ones in $\mathrm{Supp}(F)$, and the others in $Z(\kappa)$.

Since $T = T_f^+ + T_f^-$ also represents the class Σ, we can and shall write

$$\tau = T + dd^c h.$$

From Property (2) and the existence of continuous potentials for T, the function $h\colon X \to [-\infty, \infty[$ is a continuous function with logarithmic singularities along F. Locally, h is given by a potential of the semi-Kähler form κ, plus the logarithm of an equation for F, minus a potential of T. From Properties (4), (5), and (6), we see that, if E is a connected component of $Z(\kappa) \setminus \mathrm{Supp}(F)$, there is a neighborhood of E on which h is the difference of a smooth pluri-subharmonic function and a continuous pluri-subharmonic one.

10.2.3. *Proof of Proposition 10.5.* Let $BN(\mathrm{N}_f)$ be the set of big and nef line bundles with Chern class in N_f (see Lemma 10.2). Given $L \in BN(\mathrm{N}_f)$, and a sufficiently divisible integer ℓ, one can decompose ℓL as the sum $\ell L = F + M$ of its fixed component $F = \sum_{i=1}^m a_i F_i$ and a movable part M; hence, M has a finite base locus, and is big and nef. By a theorem of Zariski (see Remark 2.1.32 of [64]), there exists a positive integer k_0 such that kM is semi-ample for all sufficiently divisible $k \geq k_0$. Thus, $c_1(M)$ is represented by a semi-Kähler form $(1/k)\kappa_{kM}$, as in Section 10.2.1; for simplicity, we denote its zero locus by $Z(L)$ instead of $Z(\kappa_{kM})$: by construction, $Z(L)$ is either empty, or a curve. Denote by Z_f the intersection of the sets $Z(L)$ for L in $BN(\mathrm{N}_f)$. Since $BN(\mathrm{N}_f)$ is f-invariant, the set Z_f is also invariant. If L and L' are two elements of $BN(\mathrm{N}_f)$, their sum $L + L'$ is also in $BN(\mathrm{N}_f)$, and the movable part of $\ell(L+L')$ is larger than the sum $M + M'$ of the movable parts of ℓL and $\ell L'$. Thus,

$$Z(L+L') \subset Z(L) \cap Z(L'),$$

and we see that (i) there are line bundles $L \in BN(\mathrm{N}_f)$ with $Z(L) = Z_f$, and (ii) Z_f is either empty or a curve. Thus, Z_f is empty or a union of irreducible periodic curves.

Let us now study the fixed component F of ℓL and its decomposition $F = \sum_{i=1}^m a_i F_i$ in irreducible components; we assume that F is not empty and that $a_i > 0$ for every index $1 \leq i \leq m$. Suppose that the irreducible components $F_1, \ldots, F_s$ are not f-periodic and that $F_{s+1}, \ldots, F_m$ are all periodic. For $1 \leq i \leq s$, we obtain

- $\theta_f^+ \cdot F_i > 0$, because F_i is not periodic (see Proposition 6.1 in Section 6.1);
- $\lambda_f^{-n} (f^n)^* F_i$ converges towards a positive multiple $b_i \theta_f^+$ of θ_f^+ as n goes to infinity.

Let $D_{i,n} = c_i F_i + d_i (f^n)^* F_i$, where c_i and d_i are positive integers. Then

$$D_{i,n} \cdot F_i \simeq c_i F_i^2 + d_i b_i \lambda_f^n (\theta_f^+ \cdot F_i) > 0$$

if n is large enough. Similarly, $D_{i,n}$ intersects $(f^n)^* F_i$ positively for large values of n. Thus, $D_{i,n}$ is big and nef. The fixed component of (a large multiple of) $D_{i,n}$ is either equal to $c_i' F_i$ for some $0 \leq c_i' < c_i$, or to $d_i' (f^n)^* F_i$ for some $0 \leq d_i' < d_i$. Assume that the fixed component of $D_{1,n}$ is a multiple of F_1. Then, the fixed component of $(c_1 - c_1')\ell L + d_1 (f^n)^* (\ell L)$ does not involve the curves F_1 and $(f^n)^* F_1$ anymore, and it involves at most one of the curves F_i or $(f^n)^*(F_i)$ for $2 \leq i \leq l$. This process reduces the number of non-periodic irreducible fixed components by (at least) 1. Thus, in a finite number of steps, we construct a linear combination with positive coefficients of the $(f^n)^*(\ell L)$ with only f-periodic fixed components. To sum up, there are line bundles L satisfying the following properties

(a) L is an element of $BN(\mathrm{N}_f)$,
(b) $Z(L) = Z_f$ is a union of f-periodic curves,
(c) the fixed component F of L is a sum $\sum_j a_j F_j$ where each irreducible curve F_j is periodic.

Lemma 10.6. *The class $\Sigma = \theta_f^+ + \theta_f^-$ is a positive sum $\sum_j b_j c_1(L_j)$ of Chern classes of line bundles satisfying properties (a), (b), and (c).*

Proof. Note that there is a direct sum decomposition

$$\mathrm{N}_f \otimes \mathbf{R} = \mathbf{R}\theta_f^+ \oplus \mathbf{R}\theta_f^- \oplus \bigoplus_i V_i \tag{10.1}$$

such that each V_i is an f^*-invariant plane $\mathbf{R}^2$ on which f^* acts as an irrational rotation (the eigenvalues of f^* on $V_i \otimes \mathbf{C}$ are Galois conjugates of λ_f). Fix a line bundle L satisfying properties (a), (b), and (c), and decompose its Chern class as a sum $c_1(L) = c_+(L) + c_-(L) + \sum_i c_{\perp,i}(L)$ with respect to the direct sum (10.1). We get

$$f^*(c_1(L)) = \lambda_f c_+(L) + \lambda_f^{-1} c_-(L) + \sum_i f^*_{|V_i}(c_{\perp,i}(L)).$$

Then, remark that an irrational rotation r of a plane V has the following property: the convex hull of the orbit of any given vector $v \in V \setminus \{0\}$ under the action of r contains the origin $0 \in V$. Thus, there is a linear combination $\sigma = \sum_i \alpha_i (f^i)^*(c_1(L))$ of the classes $(f^i)^*(c_1(L))$ with coefficients $\alpha_i \in \mathbf{R}_+^*$ which is contained in $\mathbf{R}\theta_f^+ \oplus \mathbf{R}\theta_f^-$. Since the properties (a), (b), (c) are invariant under the action of f, the class $\sigma \in \mathbf{R}\theta_f^+ \oplus \mathbf{R}\theta_f^-$ is a positive sum of Chern classes of line bundles $L_i := (f^i)^* L$ satisfying (a), (b), and (c). To conclude, remark that Σ is a positive linear combination of $(f^{-m})^*\sigma$ and $(f^m)^*\sigma$ for large enough natural integers m. □

Fix line bundles L_j as in Lemma 10.6 and denote by M_j the movable part of (a sufficiently divisible multiple of) L_j. For k sufficiently divisible, kM_j provides a birational morphism

$$\Phi_j \colon X \to \Phi_j(X) \subset \mathbb{P}(H^0(X, kM_j)^*).$$

This morphism contracts $Z(L_j) = Z_f$, it is an isomorphism in the complement of $Z(L_j)$, and its image is a (singular) normal projective surface X_j; the pull-back κ_j of the Fubini-Study form is a semi-Kähler form on X, with zero set $Z(\kappa_j) = Z_f$ (see

Section 10.2.1). Thus, the form

$$\kappa = \frac{1}{k} \sum_j b_j \kappa_j \tag{10.2}$$

is a semi-Kähler form with zero set $Z(\kappa) = Z_f$. The class $\Sigma = \sum_j b_j c_1(L_j) = \theta_f^+ + \theta_f^-$ is represented by the closed positive current $\tau = \kappa + \{F\}$, where $\{F\}$ is a current of integration on a real effective divisor supported on the f-periodic curves. This proves Properties (1) and (2) of Proposition 10.5.

Now, consider an irreducible f-periodic curve C. If C intersects $\mathrm{Supp}(F)$, but is not contained in it, we obtain

$$[\tau] \cdot [C] = [F] \cdot [C] + \int_C \kappa > 0$$

because $[F] \cdot [C] > 0$ and the form κ is non-negative. But $[\tau] \cdot [C] = 0$ because C is periodic. This contradiction shows that C is contained in the support of F as soon as it intersects it. If C is not contained in $\mathrm{Supp}(F)$, the same argument shows that $[F] \cdot [C]$ and $\int_C \kappa$ both vanish. In particular, the semi-Kähler form κ vanishes along C. Thus, every irreducible periodic curve is contained in $Z_f = Z(\kappa)$ or in $\mathrm{Supp}(F)$, so that Properties (3) and (4) are satisfied.

Now, let E be a connected component of the union of periodic curves, and assume that E is contained in $\mathrm{Supp}(F)$. Consider the effective divisor F_E which is supported on E and whose coefficients a_i are the same as the coefficients of F for all irreducible components C_i of E. Then, $[F_E]^2 < 0$ because $[F_E]$ is contained in PC_f. If κ vanishes along E one gets $[\tau] \cdot [F_E] = [F_E]^2 < 0$, and this is a contradiction because F_E is supported on a set of periodic curves. Thus, E can not be contained in $Z(\kappa)$, and Property (5) is established: there are two types of connected components of the union of periodic curves, the first ones in Z_f and the others in $\mathrm{Supp}(F)$.

As above, let Φ_i be the birational morphism associated to the mobile part kM_i, let $X_i = \Phi_i(X)$ be its image, and let κ_i be the associated semi-Kähler form on X. Since the Fubini-Study metric has local smooth potentials, the forms κ_i and the form κ have smooth potentials in small neighborhoods of Z_f (see Remark 10.4). The same property holds for the forms $(f^n)^* \kappa_i$, because Z_f is f-invariant. Let us now prove that T_f^+ and T_f^- also have local continuous potentials in a neighborhood of Z_f.

The cohomology classes $[(f^n)^* \kappa_1]$ generate a finite dimensional subspace W of $H^{1,1}(X; \mathbf{R})$. Set $m = \dim(W)$. Then the classes of $\kappa_{1,1} = \kappa_1$, $\kappa_{1,2} = (f)^* \kappa_1$, ..., $\kappa_{1,m} = (f^m)^* \kappa_1$ form a basis of W, and there is a companion matrix $(a_{i,j}) \in \mathsf{GL}_m(\mathbf{R})$

and there are smooth functions $g_i \colon X \to \mathbf{R}$ such that

$$f^*\kappa_{1,i} = \sum_{j=1}^{m} a_{i,j}\kappa_{1,j} + dd^c(g_i)$$

for every index i. We already know that T_f^+ is the limit of the sequence $\frac{1}{\lambda_f^n}(f^n)^*\kappa_1$ as n goes to $+\infty$. One can now copy the proof of Proposition 2.4 in [41] (or [73], p. 58) to show that

$$T_f^+ \quad = \quad \lim_{n\to+\infty} \frac{1}{\lambda_f^n}(f^n)^*\kappa_1 = \sum_{j=1}^{m} \alpha_j\kappa_{1,j} + dd^c(u)$$

where the α_j are real numbers and $u \colon X \to \mathbf{R}$ is a Hölder continuous function. Since each $\kappa_{1,j}$ has a continuous potential in a neighborhood of Z_f, then so does T_f^+. The same argument applies to T_f^-, and this completes the proof of Proposition 10.5.

10.3. **Regularization of the current τ.** We denote by $\tau := T + dd^c h = \kappa + \{F\}$ the current which is provided by Proposition 10.5; we keep the notations from Section 10.2, hence $\kappa = (1/k)\sum_j b_j\kappa_j$ where each κ_j is the pull-back of the Fubini-Study form by the morphism Φ_j. Recall that κ_0 is a positive form on X_0; we refer to Section 6.2 for the definition of κ_0 and complex analysis on X_0.

Proposition 10.7. *There is a continuous function $\hat{h} \colon X \to \mathbf{R}$ such that:*

(1) *$\hat{\tau} := T + dd^c\hat{h}$ is a closed positive current and $\hat{\tau} \geq \varepsilon_0\pi^*\kappa_0$ for some $\varepsilon_0 > 0$;*
(2) *$\hat{h}$ is constant on every connected component E of the union of the f-periodic curves. If such a component E is contained in $Z(\kappa)$, then $\hat{h}$ is the difference of a smooth pluri-subharmonic function and a continuous pluri-subharmonic function in a neighborhood of E; if E is contained in the support of F, then $\hat{h}$ is a smooth pluri-subharmonic function in a neighborhood of E.*
(3) *if E is such a connected component, one can find a neighborhood $\mathcal{U}_0$ of $\pi(E)$ in X_0, and an embedding of $\mathcal{U}_0$ in the unit ball of $\mathbf{C}^N$ such that $(\pi_*\hat{h})_{|\mathcal{U}_0}$ is the restriction to $\mathcal{U}_0$ of the difference of two continuous pluri-subharmonic functions defined on the unit ball of $\mathbf{C}^N$.*

Proof. • First, we construct a form κ_0'. Let E be a connected component of the union of periodic curves. Consider the point $q_0 := \pi(E) \in X_0$ and choose a small neighborhood $\mathcal{W}_0$ of q_0 which does not contain any other critical value of π. We can assume that $\mathcal{W}_0$ is an analytic subset in the unit ball of $\mathbf{C}^N$.

If $E \subset \mathrm{Supp}(F)$, let ω_0 (resp. g_0) be the restriction to $\mathcal{W}_0$ of the standard $(1,1)$-Kähler form on $\mathbf{C}^N$ (resp. of a potential of the standard Kähler form); then, define $\kappa_0' = \omega_0$ on $\mathcal{W}_0$.

Now, assume $E \subset Z(\kappa)$ (hence E is not in Supp(F) by Proposition 10.5). The bimeromorphic morphisms π and Φ_j both contract E onto normal surfaces; thus, there is an open neighborhood $\mathcal{U}$ of E and a local diffeomorphism $\eta\colon \pi(\mathcal{U}) \to \Phi_j(\mathcal{U})$ such that $\eta \circ \pi = \Phi_j$. The restriction of the Fubini-Study metric to $\Phi_j(\mathcal{U})$ is a smooth positive $(1,1)$-form $\kappa_{j,0}$, and we define

$$\kappa_0' = \frac{1}{k}\sum_j b_j \eta^* \kappa_{j,0}$$

on $\pi(\mathcal{U})$; hence, $\pi^*\kappa_0' = \kappa$ on $\mathcal{U}$ (see Equation 10.2).

This done, κ_0' is defined in a small neighborhood of every singular point of X_0. We extend it to a global, smooth, positive $(1,1)$-form on X_0, that we still denote κ_0'. There exists $0 < \varepsilon_0 \leq 1$ such that

$$\kappa \geq \varepsilon_0 \pi^* \kappa_0' \text{ on } X. \tag{10.3}$$

In particular, $\pi^*\omega_0 \geq \varepsilon_0 \pi^* \kappa_0'$ on a small neighborhood of the support of F.

• Let $\tau = \kappa + \{F\} = T + dd^c h$ be the current given by Proposition 10.5. Consider a connected component E of the union of f-periodic curves.

Asssume $E \subset Z(\kappa)$ (hence $E \cap \mathrm{Supp}(F) = \emptyset$). In a neighborhood of E, the current T is defined by a continuous pluri-subharmonic potential u. The form κ also has a smooth potential v on a neighborhood of E. By the maximum principle, the restriction of u and v to E are constant. Then, on some neighborhood of E, h is the difference of the smooth pluri-subharmonic function v and the continuous pluri-subharmonic function u; in particular, it is constant along E. By construction,

$$\tau = \kappa + \{F\} \geq \kappa \geq \varepsilon_0 \pi^* \kappa_0' \ \text{ near } E.$$

Now, assume $E \subset \mathrm{Supp}(F)$. As above, let $\mathcal{W}_0 \subset \mathbf{C}^N$ be a small neighborhood of $q_0 := \pi(E)$ and let g_0 be a smooth potential for ω_0 on $\mathcal{W}_0$. We define a new continuous function on $\pi^{-1}(\mathcal{W}_0)$ by

$$h_\eta := \widetilde{\max}(g_0 \circ \pi - \eta, h),$$

where η is a large real number and $\widetilde{\max}$ is the regularized maximum, as defined in Section I.5.E of [38][1]. Since h is continuous on the boundary of $\pi^{-1}(\mathcal{W}_0)$, the function h_η coincides with h near this boundary for η large enough (see Lemma 5.18(b,c) of [38]). Since h has a pole along E, the function h_η coincides with $\pi^* g_0 -$

[1]To define it, let $\theta\colon \mathbf{R} \to \mathbf{R}$ be an even, non-negative function of class $\mathcal{C}^\infty$ with support in $[-1,1]$ such that $\int_{\mathbf{R}} \theta(t)dt = 1$. Then, fix a pair $\alpha = (\alpha_1, \alpha_2)$ of positive real numbers, and set $M_\alpha(a,b) = \int_{\mathbf{R}^2} \max(a+s, b+t)(\alpha_1\alpha_2)^{-1}\theta(s/\alpha_1)\theta(t/\alpha_2)dsdt$ for all pairs of real numbers (a,b). Then, set $\widetilde{\max}(u,v)(x) = M_\alpha(u(x), v(x))$, with α_1 and α_2 very small.

η in an open neighborhood $\mathcal{V}_E \subset \pi^{-1}(\mathcal{W}_0')$ of E. Thus, h_η extends to a continuous function on X that coincides with h outside $\pi^{-1}(\mathcal{W}_0')$, that is smooth and pluri-subharmonic in a neighborhood of E, and that is constant along E.

We reproduce the same surgery near every connected component E of the support of F. This replaces h by a continuous function $\hat{h}$ (equal to h_η near every such component E), and we define

$$\hat{\tau} := T + dd^c\hat{h}.$$

• We now prove that $\hat{\tau} \geq \varepsilon_0\pi^*\kappa_0'$ on X; this implies $\hat{\tau} \geq \varepsilon_0\pi^*\kappa_0$ on X by taking a smaller ε_0 (see §6.2) and completes the proof of Proposition 10.7. Consider a finite open cover $(\mathcal{U}_\alpha)$ of X, with potentials $u_\alpha : \mathcal{U}_\alpha \to \mathbf{R}$ for the current T. Let $E \subset \mathrm{Supp}(F)$, $\mathcal{W}_0' \subset X_0$ and $\mathcal{V}_E \subset X$ be as above. On $\mathcal{U}_\alpha \cap \pi^{-1}(\mathcal{W}_0')$ we have $dd^c(u_\alpha + g_0 \circ \pi - \eta) \geq dd^c(g_0 \circ \pi) = \pi^*\omega_0$, which is bounded from below by $\varepsilon_0\pi^*\kappa_0'$. This implies that $\hat{\tau} \geq \varepsilon_0\pi^*\kappa_0'$ on the neighborhood $\mathcal{V}_E$ of E. Let $\mathcal{V}$ be the union of these open sets $\mathcal{V}_E$, for $E \subset \mathrm{Supp}(F)$. This open set contains F.

It remains to verify $\hat{\tau} \geq \varepsilon_0\pi^*\kappa_0'$ on $X \setminus \mathcal{V}$, and for that we use the properties of the regularized maximum. Locally on $X \setminus \mathcal{V}$, the current $dd^c(u_\alpha + h)$ is equal to the semi-Kähler form κ, which is bounded from below by $\varepsilon_0\pi^*\kappa_0'$ by Equation (10.3). Since $dd^c(u_\alpha + h)$ and $dd^c(u_\alpha + g_0 \circ \pi - \eta)$ are both larger than $\varepsilon_0\pi^*\kappa_0'$ on $\pi^{-1}(\mathcal{W}_0') \setminus \mathcal{V}$, Lemma 5.18(d,e) of [38] ensures that $\hat{\tau} \geq \varepsilon_0\pi^*\kappa_0'$ on $X \setminus \mathcal{V}$.

• Let us prove the third item. Let E be a connected component of the union of the f-periodic curves. Assume that $E \subset \mathrm{Supp}(F)$ and let $\mathcal{V}_E$ be a neighborhood of E on which $\hat{h}$ is continuous and pluri-subharmonic. Since $\hat{h}$ is continuous on $\mathcal{V}_E$ and constant on E, $\pi_*\hat{h}$ is continuous on $\mathcal{V}_0' := \pi(\mathcal{V}_E)$. Now if $\nu : \mathbb{D} \to \mathcal{V}_0'$ is a holomorphic disk, then $\pi_*\hat{h} \circ \nu = \hat{h} \circ \hat{\nu}$, where $\hat{\nu} : \mathbb{D} \to \mathcal{V}_E$ is the lift of ν defined by Lemma 6.5. Since $\hat{h}$ is pluri-subharmonic on $\mathcal{V}_E$, $\pi_*\hat{h} \circ \nu = \hat{h} \circ \hat{\nu}$ is sub-harmonic on $\mathbb{D}$. Now, reducing $\mathcal{V}_0'$ to a smaller open neighborhood $\mathcal{U}_0$ of $\pi(E)$ if necessary, we can embed $\mathcal{U}_0$ in the unit ball B of $\mathbf{C}^N$ for some $N > 0$. We now refer to the results of Fornaess, Narasimhan and Richberg described in Section 6.2. By [49, Theorem 5.3.1] we deduce that $\pi_*\hat{h}$ is a pluri-subharmonic function on $\mathcal{U}_0$: it is the restriction of a pluri-subharmonic function $B \to \mathbf{R}$. Then [77, Satz 2.4] ensures that $\pi_*\hat{h}$ is the restriction to $\mathcal{U}_0$ of a continuous pluri-subharmonic function defined on B.

The proof is similar when E is contained in $Z(\kappa)$. □

10.4. **Dinh-Sibony's arguments.**

10.4.1. *Introduction.* We use the notions and notations introduced in Section 6.2. Let M be a compact complex space endowed with a positive $(1,1)$-form κ_0. A

Brody curve $\xi\colon \mathbf{C}\to M$ is a non-constant entire curve such that $\| \xi'(z) \|$ is uniformly bounded. As in Section 3.3.1, $A(r;\xi)$ denotes the area of $\xi(\mathbb{D}_r)$ with respect to the metric κ_0 (the area is counted with multiplicity).

Let $K\subset M$ be a compact subset. Let T and S be closed positive $(1,1)$-currents on M. We recall that if $x\in K$ has a neighborhood $\mathcal{U}$ in M on which $T = dd^c u$ for some continuous potential u on $\mathcal{U}$, then $S\wedge T := dd^c(uS)$ is a well defined measure on $\mathcal{U}$ (see Section 6.2). If this product vanishes on an open neighborhood of every point x of K, we say that "$S\wedge T = 0$ on K". Similarly, if $\xi\colon \mathbf{C}\to M$ is an entire curve, we write "$\xi^*T = 0$ on K" if the continuous functions $u\circ\xi$ are harmonic on $\xi^{-1}(\mathcal{U})$.

Theorem 10.8. *Let M be a compact complex space, and K a compact subset of M. Let T be a closed positive $(1,1)$-current on M such that T has a local continuous potential in a neighborhood of every point $x\in K$. Let $\xi\colon \mathbf{C}\to M$ be a Brody curve for which $\xi^*T = 0$ on K. Then there exists a sequence of radii $(r_n)_n$ such that the sequence of currents*

$$S_n := \frac{1}{A(r_n;\xi)}\{\xi(\mathbb{D}_{r_n})\}$$

converges towards a closed positive current S on M such that $S\wedge T = 0$ on K.

We have $\langle S|\kappa_0\rangle = 1$ since lengths and areas are computed with respect to κ_0. Theorem 10.8 is applied in Section 10.5 to $T_0 := \pi_*(T_f^+ + T_f^-)$ and compact subsets of X_0^{reg} (T_0 has local continuous potentials near every point $x\in X_0^{reg}$). Theorem 10.8 is due to Dinh-Sibony when M is smooth [40]. We first prove it for entire curves with finite area, then we adapt the arguments from [40] for the unbounded area case.

10.4.2. *Bounded area.*

Theorem 10.9 (Demailly, Moncet [37, 73])**.** *Let M be a compact complex space. Let $\xi\colon \mathbf{C}\to M$ be a non-constant entire curve such that the function $A(\cdot;\xi)$ is bounded, and let $A(\infty;\xi)$ be its limit as r goes to $+\infty$. Then ξ extends uniquely to a holomorphic mapping $\tilde{\xi}\colon \overline{\mathbf{C}}\to M$. In particular, every Ahlfors current S associated to ξ is equal to the current of integration $\{\tilde{\xi}(\overline{\mathbf{C}})\}/A(\infty;\xi)$.*

Before starting the proof, let us recall the Ahlfors inequality and one of its consequences. Denote by $L(r;\xi)$ the length, counted with multiplicity, of the image of the circle of radius r by ξ. If $A'(\cdot;\xi)$ is the derivative of $A(\cdot;\xi)$, the Cauchy-Schwarz inequality yields

$$L(r;\xi)^2 \leq 2\pi r A'(r;\xi). \tag{10.4}$$

If the ratio $L(r;\xi)/A(r;\xi)$ is bounded from below by a positive constant B, integrating between 1 and R, one gets $B^2\log(R)\leq 2\pi(1/A(R;\xi) - 1/A(1;\xi))$, and this is a contradiction for large values of R. Thus, there is a sequence of radii (R_n) going to infinity such that $L(R_n;\xi)/A(R_n;\xi)$ goes to 0 as n goes to ∞; consequently, the limit S of any converging subsequence of $\{\xi(\mathbb{D}_{R_n})\}/A(R_n;\xi)$ is a closed current.

Proof of Theorem 10.9. Let E_r denote the subset $\xi(\mathbf{C}\setminus\mathbb{D}_r)$ of M. The diameter $\mathrm{diam}(E_r)$ is a non-increasing function, and we denote by δ its limit as r goes to infinity. If $\delta = 0$ then ξ extends to a continuous mapping $\bar{\xi} : \overline{\mathbf{C}} \to M$, and $\bar{\xi}$ is holomorphic because it is continuous on $\overline{\mathbf{C}}$ and holomorphic on $\mathbf{C}$.

We now assume that δ is positive and derive a contradiction. Set $C(r,r') := \xi(\{r < |z| < r'\})$. Given r, we have $\mathrm{diam}\, C(r,r') \geq 2\delta/3$ for $r' \gg 1$. From the Ahlfors inequality there exists a sequence $(R_n)_n$ going to infinity along which $L(R_n;\xi)$ goes to 0. Let us extract two sequences $(r_n)_n$ and $(r'_n)_n$ from $(R_n)_n$ such that

(a) $r_n < r'_n < r_{n+1}$,
(b) $L(r_n;\xi) < \delta/6$ and $L(r'_n;\xi) < \delta/6$,
(c) $\mathrm{diam}\, C(r_n,r'_n) \geq 2\delta/3$.

If every $x \in C(r_n,r'_n)$ were at distance $< \delta/12$ from one of the two boundary curves $\xi(\{|z| = r_n\})$ and $\xi(\{|z| = r'_n\})$, the diameter of $C(r_n,r'_n)$ would be at most $4\delta/12$, contradicting item (c): indeed, $C(r_n,r'_n)$ is connected, and those two curves have diameter $< \delta/12$ by item (b). Thus, there is a point x_n in $C(r_n,r'_n)$ whose distance to $\partial C(r_n,r'_n)$ is at least $\delta/12$.

The Lelong theorem gives an $\eta > 0$ (depending on M, κ_0, δ) such that for every $x \in M$, the area of every analytic curve $V \subset B_x(\delta/12)$ containing x is larger than η. Applied to the curves $V = C(r_n,r'_n)$ and the centers x_n, it shows that the area of $C(r_n,r'_n)$ is larger than η. Since the annuli $C(r_n,r'_n)$ are pairwise disjoint subsets of $\mathbf{C}$, the area $A(r;\xi)$ goes to infinity as r goes to infinity, in contradiction with the assumed upper bound. □

Let us now show Theorem 10.8 in the bounded area case. Apply Theorem 10.9. Fix an open neighborhood $\mathcal{V}$ of K on which $\xi^*T = 0$. Let φ be a smooth function with support contained in an open subset $\mathcal{U}$ of $\mathcal{V}$ on which T has a local potential u. We want to show that $\langle S \wedge T|\varphi\rangle = 0$, where S is the current of integration $\{\bar{\xi}(\overline{\mathbf{C}})\}$. The function $\tilde{u} := u \circ \bar{\xi}$ is a (bounded) continuous function on $\bar{\xi}^{-1}(\mathcal{U})$ and is harmonic on the complement of ∞; as such, it extends to a harmonic function on $\bar{\xi}^{-1}(\mathcal{U})$. Thus, $dd^c(\tilde{u}) = 0$ on the support of $\varphi \circ \bar{\xi}$, and this implies $\langle S \wedge T|\varphi\rangle = 0$.

10.4.3. *Unbounded area: preliminary estimate.* In this section the entire curve ξ has unbounded area; it will be a Brody curve in Section 10.4.4. We cover the compact set K by two finite families of open subsets $\mathcal{U}_\alpha \subset \mathcal{V}_\alpha$ such that $\xi^*T = 0$ on the union of the $\mathcal{V}_\alpha$ and T has local continuous potentials on the $\mathcal{V}_\alpha$. We want to construct an Ahlfors current $S = \lim S_n$ of ξ such that $\langle S | u_\alpha dd^c \varphi_\alpha \rangle = 0$ for every test function φ_α supported on $\mathcal{U}_\alpha$. We denote by ε the distance

$$\varepsilon = \frac{1}{2} \min_\alpha \mathsf{dist}(\mathcal{U}_\alpha, \partial \mathcal{V}_\alpha),$$

we drop the index α from the notation, and we define

$$\mathcal{U}' := \xi^{-1}(\mathcal{U}),\ \mathcal{V}' := \xi^{-1}(\mathcal{V}),\ \mathbb{D}'_r := \mathbb{D}_r \cap \xi^{-1}\overline{\mathcal{U}},\ \tilde{u} := u \circ \xi, \text{ and } \tilde{\varphi} := \varphi \circ \xi.$$

We set $d\mathrm{vol}(z) := \frac{\mathrm{i}}{2} dz \wedge d\bar{z}$ on $\mathbf{C}$. The following lemma holds for every entire curve ξ such that $\tilde{u}$ is harmonic on $\mathcal{U}'$ (ξ does not need to be a Brody curve).

Lemma 10.10. *There exists a constant $C > 0$ such that*

$$\forall r \geq 1\ ,\ \int_{\mathbb{D}'_r} \parallel \nabla \tilde{u} \parallel dvol \leq C\, r A(2r;\xi)^{1/2}.$$

Proof. Since $\tilde{u}$ is bounded and harmonic, the Harnack inequality provides a constant $c > 0$ such that $\parallel \nabla \tilde{u}(z) \parallel \leq c\, \mathsf{dist}(z, \partial \mathcal{V}')^{-1}$ for every $z \in \mathcal{V}'$. Thus, all we need to do is to exhibit a constant C' such that

$$\forall r \geq 1\ ,\ J(r) := \int_{\mathbb{D}'_r} \mathsf{dist}(z, \partial \mathcal{V}')^{-1} d\mathrm{vol}(z) \leq C'\, r A(2r;\xi)^{1/2}.$$

To obtain this upper bound, choose a positive number δ and define

$$\mathbb{D}'_r(\delta) = \{ z \in \mathbb{D}'_r\ | \mathsf{dist}(z, \partial \mathcal{V}') < \delta \}.$$

For every $\Delta > 0$, the Fubini theorem implies

$$\int_0^\Delta \mathrm{Area}_{\mathbf{C}}(\mathbb{D}'_r(\delta)) \frac{d\delta}{\delta^2} = \int_{\mathbb{D}'_r(\Delta)} \left(\frac{1}{\mathsf{dist}(z, \partial \mathcal{V}')} - \frac{1}{\Delta} \right) d\mathrm{vol}(z),$$

where $\mathrm{Area}_{\mathbf{C}}$ denotes the euclidean area in $\mathbf{C}$. Thus, if one splits the integral $J(r)$ into an integral over $\mathbb{D}'_r(\Delta)$ and another over $\mathbb{D}'_r \setminus \mathbb{D}'_r(\Delta)$ one gets the upper bound

$$\begin{aligned} J(r) &\leq \int_{\mathbb{D}'_r \setminus \mathbb{D}'_r(\Delta)} \frac{d\mathrm{vol}(z)}{\Delta} + \int_{\mathbb{D}'_r(\Delta)} \frac{d\mathrm{vol}(z)}{\Delta} + \int_0^\Delta \mathrm{Area}_{\mathbf{C}}(\mathbb{D}'_r(\delta)) \frac{d\delta}{\delta^2} \\ &\leq \frac{\pi r^2}{\Delta} + \int_0^\Delta \mathrm{Area}_{\mathbf{C}}(\mathbb{D}'_r(\delta)) \frac{d\delta}{\delta^2}. \end{aligned}$$

Lemma 10.10 follows if one takes $\Delta = r / A(2r;\xi)^{1/2}$ and if one proves the following fact: there exists a constant C'' such that

$$\forall r \geq 1\ ,\ \forall \delta > 0\ ,\ \frac{\mathrm{Area}_{\mathbf{C}}(\mathbb{D}'_r(\delta))}{\delta^2} \leq C'' A(2r;\xi). \tag{10.5}$$

Fix $r \geq 1$. If $\delta \geq r/4$ we are done because

$$\frac{\mathrm{Area}_{\mathbf{C}}(\mathbb{D}'_r(\delta))}{\delta^2} \leq \frac{\pi r^2}{(r/4)^2} = 16\pi \leq \frac{16\pi}{A(2;\xi)} A(2r;\xi).$$

Now, assume $\delta < r/4$ and fix a covering of $\mathbb{D}'_r$ by finitely many disks $\mathbb{D}(z_i;\delta)$ such that every point of $\mathbb{D}'_r$ is contained in at most 36 disks $\mathbb{D}(z_i;3\delta)$. Since each z_i is in $\mathbb{D}_{r+\delta}$ and $r+4\delta \leq 2r$, we obtain $\mathbb{D}(z_i;3\delta) \subset \mathbb{D}_{2r}$.

Lemma 10.11 (Briend-Duval). *Let M be a compact complex space endowed with a positive $(1,1)$-form κ_0. For every $\varepsilon > 0$ and $0 < a < 1$ there exists $\eta > 0$ such that*

$$\mathrm{Area}(\psi(\mathbb{D}_1)) \leq \eta \implies \mathrm{Diameter}(\psi(\mathbb{D}_a)) \leq \varepsilon$$

for every holomorphic mapping $\psi\colon \mathbb{D}_1 \to M$.

The proof which is given in the appendix of [19] and relies on the Lelong inequality and conformal moduli of annuli, as well as the one from [42, Lemma 1.55] using Cauchy estimates, both work for compact complex spaces.

Applying this lemma with the constant $\varepsilon \leq \frac{1}{2}\mathsf{dist}(\mathcal{U}, \partial\mathcal{V})$, we get $\eta > 0$ such that for every $z \in \mathbf{C}$,

$$\mathrm{Area}(\xi(\mathbb{D}(z,3\delta))) \leq \eta \implies \mathrm{Diameter}(\xi(\mathbb{D}(z,2\delta))) \leq \varepsilon$$

(here, areas and diameters are computed in M, with respect to its hermitian metric). Split the set of centers $\{z_i\}$ into two disjoint subsets $\{a_j\}$ and $\{b_k\}$ such that

$$\mathrm{Area}(\xi(\mathbb{D}(a_j;3\delta))) > \eta, \quad \mathrm{Area}(\xi(\mathbb{D}(b_k;3\delta))) \leq \eta \quad (\forall i,j).$$

In particular, the diameter of $\xi(\mathbb{D}(b_k;2\delta))$ is at most ε. Now, if z is a point in $\mathbb{D}(b_k;\delta) \cap \mathbb{D}'_r$, then $\xi(z) \in \overline{\mathcal{U}}$, $\xi(\mathbb{D}(b_k;2\delta))$ is contained in $\mathcal{V}$ by definition of ε, and thus $\mathsf{dist}(z, \partial\mathcal{V}') > \delta$; as a consequence, $\mathbb{D}(b_k;\delta)$ does not intersect $\mathbb{D}'_r(\delta)$ and $\mathbb{D}'_r(\delta)$ is covered by the disks $\mathbb{D}(a_j;\delta)$. Let N be the number of these disks, so that $\mathrm{Area}_{\mathbf{C}}(\mathbb{D}'_r(\delta)) \leq N\pi\delta^2$. Since $N\eta$ is bounded above by $36A(2r;\xi)$, we get

$$\frac{\mathrm{Area}_{\mathbf{C}}(\mathbb{D}'_r(\delta))}{\delta^2} \leq N\pi \leq \frac{36\pi}{\eta} A(2r;\xi),$$

and Equation (10.5) follows. This completes the proof of Lemma 10.10. □

10.4.4. *Unbounded area: conclusion.* We have not used that ξ is a Brody curve yet. This assumption implies that $A(r;\xi)$ grows at most quadratically. Thus, there is a sequence of radii (R_n) such that R_n goes to ∞ with n and $A(2R_n;\xi) \leq 3A(R_n;\xi)$. The next lemma is a consequence of Lemma 10.10.

Lemma 10.12. *Let σ_r denote the arc length measure on the circle of radius r in $\mathbf{C}$. There exist a constant $\tilde{D} > 0$ and a sequence of radii $(r_n)_n$ satisfying $R_n \leq r_n \leq 2R_n$ such that*

$$L(r_n;\xi) \leq \tilde{D}\, A(r_n;\xi)^{1/2} \quad \textit{and} \quad \int_{\partial \mathbb{D}'_{r_n}} \| \nabla \tilde{u} \| \, d\sigma_{r_n} \leq \tilde{D}\, A(r_n;\xi)^{1/2}.$$

Proof. We first establish that there exists $D > 0$ such that

$$\int_{R_n}^{2R_n} L(r;\xi)dr \leq D\, R_n\, A(R_n;\xi)^{1/2} \text{ and } \int_{R_n}^{2R_n} \int_{\partial \mathbb{D}'_r} \| \nabla \tilde{u} \| \, d\sigma_r dr \leq D\, R_n\, A(R_n;\xi)^{1/2}. \tag{10.6}$$

The second inequality follows from Lemma 10.10 and $A(2R_n;\xi) \leq 3A(R_n;\xi)$. For the first one, we apply the Ahlfors inequality as in § 10.4.2, and integrate from R_n to $2R_n$:

$$\int_{R_n}^{2R_n} L(r;\xi)^2 dr \leq 4\pi\, R_n\, A(2R_n;\xi) \leq 12\pi\, R_n\, A(R_n;\xi).$$

The first inequality of (10.6) is then again a consequence of the Cauchy-Schwarz inequality. We deduce from Equation (10.6) that

$$\frac{1}{R_n} \int_{R_n}^{R_{2n}} \left(L(r;\xi) + \int_{\partial \mathbb{D}'_r} \| \nabla \tilde{u} \| \, d\sigma_r \right) \, dr \leq 2D\, A(R_n;\xi)^{1/2}.$$

Thus, there exists r_n between R_n and $2R_n$ such that

$$L(r_n;\xi) + \int_{\partial \mathbb{D}'_{r_n}} \| \nabla \tilde{u} \| \, d\sigma_{r_n} \leq 2D\, A(R_n;\xi)^{1/2}.$$

This completes the proof of Lemma 10.12. □

We can now complete the proof of Theorem 10.8. Let $(r_n)_n$ be a sequence provided by Lemma 10.12. Extracting a subsequence, we may assume that the sequence of currents $S_n = \frac{1}{A(r_n;\xi)}\{\xi(\mathbb{D}_{r_n})\}$ converges towards a positive current S on M; since $L(r_n;\xi)/A(r_n;\xi)$ goes to zero as n goes to $+\infty$, the current S is closed.

By assumption, $\tilde{u}$ is harmonic on $\mathcal{V}'$, hence smooth. Since $dd^c(\tilde{u}) = 0$, an integration by parts yields

$$I(n) := \langle S_n | u dd^c \varphi \rangle = \frac{1}{A(r_n;\xi)} \int_{\mathbb{D}'_{r_n}} \tilde{u}\, dd^c \tilde{\varphi} = \frac{-1}{A(r_n;\xi)} \int_{\mathbb{D}'_{r_n}} (d(\tilde{\varphi} d^c \tilde{u}) + d^c(\tilde{u} d\tilde{\varphi})).$$

And the Stokes formula gives

$$I(n) \;=\; \underbrace{\frac{-1}{A(r_n;\xi)} \int_{\partial \mathbb{D}'_{r_n}} \tilde{\varphi}\, d^c \tilde{u}}_{I_1(n)} \;+\; \underbrace{\frac{-1}{A(r_n;\xi)} \int_{\mathbb{D}'_{r_n}} d^c(\tilde{u}\, d\tilde{\varphi})}_{I_2(n)}.$$

Let us show that $I_1(n)$ and $I_2(n)$ tend to zero. Since $2\pi d^c\tilde{u} = \partial_y(\tilde{u})dx - \partial_x(\tilde{u})dy$ one obtains

$$|I_1(n)| \leq \frac{\max|\varphi|}{2\pi}\frac{1}{A(r_n;\xi)}\int_{\partial\mathbb{D}'_{r_n}} \parallel \nabla\tilde{u} \parallel d\sigma_{r_n}.$$

The second inequality of Lemma 10.12 shows that $I_1(n)$ goes to zero. For $I_2(n)$, let us write $\tilde{u}d\tilde{\varphi} = adx + bdy$, where a and b are smooth functions. The Stokes formula gives

$$I_2(n) = \frac{1}{2\pi A(r_n;\xi)}\int_{\partial\mathbb{D}_{r_n}}(ady - bdx)$$

and if $\max(u)$ denotes the maximum of $|u|$ on the support of φ, one gets

$$|I_2(n)| \leq \frac{\max(u) \parallel d\varphi \parallel_\kappa L(r_n;\xi)}{2\pi A(r_n;\xi)}.$$

Thus, the first inequality of Lemma 10.12 shows that $I_2(n)$ tends to zero, completing the proof of Theorem 10.8.

10.5. **Proof of Proposition 6.9.** Let $T := T_f^+ + T_f^-$. We apply Proposition 10.7 to construct the current $\hat{\tau} = T + dd^c\hat{h}$. We set

$$T_0 := \pi_* T \ \ , \ \ h_0 := \pi_*\hat{h} \ \ , \ \ \hat{\tau}_0 := \pi_*\hat{\tau}.$$

By Proposition 10.7, there exists $\varepsilon_0 > 0$ such that $\hat{\tau}_0 = T_0 + dd^c h_0 \geq \varepsilon_0\kappa_0$ and h_0 is locally the difference of two continuous pluri-subharmonic functions. More precisely, every point of X_0 has a neighborhood $\mathcal{U}_0$ which embeds into the unit ball $B \subset \mathbf{C}^N$ and on which h_0 is the restriction of a difference of two continuous pluri-subharmonic functions defined on B.

Suppose that there is a non-constant entire curve $\xi : \mathbf{C} \to X_0$ such that $\xi^*(T_0) = 0$. It means that $dd^c((u^+ + u^-)\circ\hat{\xi}) = 0$, where $u^\pm$ are local continuous potentials of $T_f^\pm$ and $\hat{\xi} : \mathbf{C} \to X$ is the lift of ξ. If ξ is not a Brody curve, apply the Zalcman's Lemma 6.8 and Lemma 6.7 to replace it by a Brody curve with the same properties.

Let K_m be an increasing sequence of compact subsets of X_0^{reg}, the interiors of which exhaust X_0^{reg}. On X_0^{reg} the current $T_0 = \pi_*(T)$ has local continuous potentials, and Theorem 10.8 provides an Ahlfors current S_m on X_0 for the curve ξ such that $S_m \wedge T_0 = 0$ on (small neighborhoods of) K_m. We recall that this product is locally equal to $dd^c(uS_m)$, where u is a local potential of T_0 (near K_m). The currents S_m have mass 1 with respect to the positive $(1,1)$-form κ_0, so that we can extract a subsequence converging towards a closed positive current S on X_0. Since $S_m \wedge T_0 = 0$ on K_m, we get $S \wedge T_0 = 0$ on X_0^{reg}. Note that we do not need to define (or to extend) the product $S \wedge T_0$ on X_0; the computation of this product is only done on the regular part. All we need to know is that, on X_0^{reg}, the product $S\wedge T_0$ is a well defined (non-negative) measure and is, in fact, the zero measure.

As in Section 6.2, we define the signed measure $S\wedge dd^c h_0$ on X_0 as $dd^c(h_0 S)$. Let us be more precise. If x is a point of X_0, we consider a local embedding $j:(X_0,x)\to(B,0)$, with B the unit ball in $\mathbf{C}^N$, for some N. We know that h_0 is the restriction on (X_0,x) of a difference $\tilde{h}$ of two continuous pluri-subharmonic functions on B (this follows from Proposition 10.7(3) if $x=\pi(E)$ for some component of the periodic curves, and from 10.7(1) otherwise). If φ is a smooth real valued function on X_0 which is given by the restriction of a smooth test function $\tilde{\varphi}\colon B\to\mathbf{R}$, then

$$\langle S\wedge dd^c h_0|\varphi\rangle := \langle j_*S\wedge dd^c\tilde{h}|\tilde{\varphi}\rangle = \langle j_*S|\tilde{h}\,dd^c\tilde{\varphi}\rangle.$$

This is a signed measure on X_0, see [38, Chapter III.3].

Lemma 10.13. *Let S be a closed positive current on X_0. The signed measure $S\wedge dd^c h_0$ and the positive measure $S\wedge\kappa_0$ on X_0 have no atom.*

Proof. Let x be a point of X_0. With the above notation, let us show that x is not an atom of the measure $j_*S\wedge dd^c\tilde{h}$. Changing $\tilde{h}$ into $\tilde{h}-\tilde{h}(x)$, we can assume that $\tilde{h}(x)=0$. Let $B(r)\subset(\mathbf{C}^N,0)$ be the euclidian ball of radius r centered at the origin. Let $\chi:B(3)\to[0,1]$ be a smooth non-negative function equal to 1 on $B(1)$ and to 0 on $B(3)\setminus B(2)$. Define $\chi_\varepsilon(x):=\chi(x/\varepsilon)$. Then

$$|\langle j_*S\wedge dd^c\tilde{h}|\chi_\varepsilon\rangle| = |\langle j_*S|\tilde{h}\,dd^c\chi_\varepsilon\rangle| \leq \max_{B(3\varepsilon)}|\tilde{h}|\cdot\max_{B(3\varepsilon)}\parallel dd^c\chi_\varepsilon\parallel\cdot\int_{B(3\varepsilon)} j_*S\wedge\omega_0,$$

where ω_0 is the standard Kähler form on $(\mathbf{C}^N,0)$. But the maximum of $|\tilde{h}|$ goes to 0 with ε because $\tilde{h}$ is continuous, the maximum of $\parallel dd^c\chi_\varepsilon\parallel$ is bounded from above by $c^{ste}\varepsilon^{-2}$, and the mass of j_*S in $B(3\varepsilon)$ is bounded from above by $c^{ste}\varepsilon^2$ because j_*S is a closed positive current of bidimension $(1,1)$ (Lelong's inequality, see [38] Chapter III.5). Thus, the right hand term of this inequality goes to 0 with ε, and $j_*S\wedge dd^c\tilde{h}$ does not charge the origin, as desired. The same proof holds for $S\wedge\kappa_0$, since we compute it from $j_*S\wedge dd^c g_0$, where g_0 is a potential of κ_0 on $(\mathbf{C}^N,0)$. □

Applying Lemma 10.13 to the limit S of the currents S_m, we see that $S\wedge dd^c h_0$ and $S\wedge\kappa_0$ are atomless measures on the surface X_0. From $T_0+dd^c h_0\geq\varepsilon_0\kappa_0$ and $S\wedge T_0=0$ on X_0^{reg}, we get

$$\langle S\wedge dd^c h_0|\mathbb{1}\rangle_{X_0^{reg}} \geq \varepsilon_0\langle S\wedge\kappa_0|\mathbb{1}\rangle_{X_0^{reg}}$$

where $\mathbb{1}$ is the constant function equal to 1, and the index X_0^{reg} means that we integrate on X_0^{reg}. Since the measures $S\wedge dd^c h_0$ and $S\wedge\kappa_0$ have no atom on X_0, we can also integrate on X_0, and we obtain

$$\langle S \wedge dd^c h_0 | \mathbb{1} \rangle \geq \varepsilon_0 \langle S \wedge \kappa_0 | \mathbb{1} \rangle.$$

By definition of the product $S \wedge dd^c h_0$ and by using a partition of unity, one has

$$\langle S \wedge dd^c h_0 | \mathbb{1} \rangle = \langle S | h_0 dd^c (\mathbb{1}) \rangle = 0.$$

This provides a contradiction because $\langle S \wedge \kappa_0 | \mathbb{1} \rangle = 1$.

11. Appendix II : A family of examples with X_0 non-projective

Theorem 11.1. *There are automorphisms of rational surfaces $g \colon Y \to Y$ with a unique irreducible g-periodic curve $C \subset Y$ such that the contraction of C provides a normal surface Y_0 which is not projective. For every even $n \geq 10$, one can find such an automorphism whose entropy is the logarithm of a Salem number of degree n.*

We construct such examples by blowing-up points of a plane cuspidal cubic. When $n = 10$, one can find such an automorphism with entropy $\log(\lambda_L)$, where λ_L is the Lehmer number (see Remark 11.3 below, and Corollary 1.3 in [72]).

11.1. **Picard groups.** Let Y be a complex projective surface, and $C \subset Y$ be an irreducible curve for which the restriction

$$\mathrm{res}_C \colon \mathrm{Pic}(Y) \to \mathrm{Pic}(C)$$

is injective. If $\langle C | C \rangle < 0$, there is a bimeromorphic morphism $\pi \colon Y \to Y_0$ onto a normal surface Y_0 that contracts C to a point q and is an isomorphism from $Y \setminus \{C\}$ to $Y_0 \setminus \{q\}$. Since res_C is injective, there is no nontrivial line bundle on Y_0 and Y_0 is not projective (see [57], Example 5.7.3). We shall construct such an example, together with an automorphism g of Y which preserves C and has positive entropy.

11.2. **The groups W_n.** Let $n \geq 10$ be an even integer. Blow up n distinct points p_i of $\mathbb{P}^2(\mathbf{C})$ to get a rational surface Y. Denote by e_i, $1 \leq i \leq n$, the classes of the exceptional divisors E_i, and by e_0 the class of the total transform of a line. The canonical class of Y is $k_n = -3e_0 + e_1 + \ldots + e_n$. The group $\mathsf{Aut}(Y)$ acts on the Néron-Severi group of Y, preserving the intersection form and the class k_n. Thus, $\mathsf{Aut}(Y)$ acts by isometries on the orthogonal complement $k_n^\perp$. Nagata proved that the image of $\mathsf{Aut}(Y)$ in $\mathsf{O}(k_n^\perp)$ is contained in an explicit Coxeter group W_n (see [28, 44, 72] for instance). The group W_n is defined as follows. Denote by α_i the **simple roots**

$$\alpha_0 = e_0 - e_1 - e_2 - e_3, \quad \alpha_i = e_i - e_{i+1}, \text{ for } \quad 1 \leq i \leq n-1.$$

Each of the α_i is an element of $k_n^\perp$ and has self-intersection -2; it determines an isometric involution s_i of $k_n^\perp$, defined by $s_i(x) = x + \langle x|\alpha_i\rangle\alpha_i$. The group W_n is the group generated by those n involutions. By definition, $\cup_i W_n(\alpha_i)$ is the set of **roots**.

Lemma 11.2. *The action of W_n on $k_n^\perp$ determines a Zariski dense subgroup of the linear algebraic group $\mathsf{O}(k_n^\perp \otimes \mathbf{R})$. There are elements w of W_n such that the characteristic polynomial $\chi_w(t) \in \mathbf{Z}[t]$ is irreducible of degree n and has a real root > 1.*

Sketch of the proof. The representation of W_n on $k_n^\perp \otimes \mathbf{R}$ is irreducible; in fact, it coincides with the Tits representation of the Coxeter group W_n, and for $n \geq 10$ the Zariski closure is $\mathsf{O}(k_n^\perp \otimes \mathbf{R})$ (see [28, 44]). Thus, the proof given in [17], §3.2.2, can be applied to deduce the lemma. (One uses that n is even to say that the characteristic polynomial of a general element of $\mathsf{O}(k_n^\perp \otimes \mathbf{R})$ does not vanish on roots of unity) □

Remark 11.3. If $n = 10$ and w_n denotes the Coxeter element in W_n, then $\chi_{w_{10}}(t) \in \mathbf{Z}[t]$ is irreducible, and its unique root > 1 is the Lehmer number (see [72]).

11.3. **Cuspidal cubic curves.** From now on, we fix an element $w \in W_n$ whose characteristic polynomial is irreducible over $\mathbf{Q}$ and has a real root $\lambda > 1$, as in Lemma 11.2. This implies that every proper, w-invariant, rational subspace of $k_n^\perp$ is trivial. Since w is an isometry of a quadratic form of signature $(1, n-1)$, its Galois conjugates have modulus ≤ 1, and $1/\lambda$ is one of them. Thus, λ is a Salem number of degree n.

Suppose one can find n points p_i in $\mathbb{P}^2(\mathbf{C})$, an automorphism g of the surface Y, and an irreducible g-invariant curve C such that (i) g^* coincides with w on $k_n^\perp$ and (ii) $[C] = -k_n$. By (ii), $k_n^\perp$ coincides with the subspace of classes with degree 0 along the curve C. By (i), we get the following alternative:

- either the image of the restriction $\mathrm{res}_C \colon k_n^\perp \subset \mathrm{Pic}(Y) \to \mathrm{Pic}^0(C)$ is finite;
- or the image $\mathrm{res}_C(k_n^\perp)$ is infinite and $\mathrm{res}_C \colon \mathrm{Pic}(Y) \to \mathrm{Pic}(C)$ is injective.

Indeed, the kernel of res_C in $k_n^\perp$ being g^*-invariant, it is either co-finite or trivial. In the second case, one can blow-down C to get a surface Y_0, and by Section 11.1, Y_0 is not projective. Thus, for Theorem 11.1, we only have to construct such an example.

Start with the cuspidal cubic curve $C_0 = \{yz^2 = x^3\}$ in $\mathbb{P}^2(\mathbf{C})$. The smooth part of C_0 is parametrized by $\mathbf{C}$ via the morphism $s : t \mapsto [t : t^3 : 1]$, and the Picard group $\mathrm{Pic}^0(C_0)$ is isomorphic to $\mathbf{C}$. Fix an eigenvalue β of w (for instance $\beta = \lambda_w$, the largest one). There is a non-trivial linear form $\rho_{\mathbf{C}} \colon k_n^\perp \otimes_{\mathbf{Z}} \mathbf{C} \to \mathbf{C}$ such that

$$\rho_{\mathbf{C}}(w(x)) = \beta\rho(x).$$

for all $x \in k_n^\perp \otimes_{\mathbf{Z}} \mathbf{C}$. Let ρ be the restriction of $\rho_{\mathbf{C}}$ to $k_n^\perp$. This is an injective homomorphism of abelian groups, because the w-orbit of every $u \in \mathbf{Z}^n \setminus \{0\}$ generates a finite index subgroup of $\mathbf{Z}^n$ (every proper w-invariant subspace of $\mathbf{Q}^n$ is trivial). Once such an embedding ρ is fixed, it determines n points

$$p_i = s(\rho(e_i)), \quad 1 \leq i \leq n,$$

on the curve C_0. Blow up the p_i to construct a surface Y and denote by C the strict transform of C_0: since C_0 is a cubic and each p_i is a smooth point of C_0, we get $[C] = -k_n$. The homomorphism res_C coincides with ρ on $k_n^\perp$. Since ρ is injective, there is no root $\alpha \in k_n^\perp$ with $\mathrm{res}_C(\alpha) = 0$. Thus, a result of McMullen (see [72], § 6, 7) implies that w is realized by an automorphism g of Y that fixes C. It turns out that C is the unique periodic curve of g because $N_g = k_n^\perp$. By construction, Y_0 is not projective.

Remark 11.4 (see [39, 72])**.** If β has modulus 1, g preserves a "volume form" $\Omega \wedge \overline{\Omega}$, where Ω is a meromorphic section of K_Y that does not vanish and has a pole along C. This form is singular along C and its total volume is infinite, but it is smooth on $Y \setminus C$.

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