SOME RIGIDITY RESULTS FOR POLYNOMIAL AUTOMORPHISMS OF $\mathbf{C}^2$

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ABSTRACT. We prove several new rigidity results for automorphisms of $\mathbf{C}^2$ with positive entropy. A first result is that a complex slice of the (forward or backward) Julia set is never a smooth, or even rectifiable, curve. We also show that such an automorphism cannot preserve a global holomorphic foliation, nor a real-analytic foliation with complex leaves.

For mappings defined over a number field, we also study the fields of definition of multipliers of saddle periodic orbits.

These results are used to show that under mild assumptions, two real-analytically conjugate automorphisms are polynomially conjugate.

CONTENTS

1. INTRODUCTION

1.1. **Rigidity problems.** In our previous paper [16], answering a question of Friedland and Milnor [31], we established the following result: if two polynomial automorphisms f and g of $\mathbf{C}^2$ of positive entropy are conjugated by a biholomorphism of $\mathbf{C}^2$, then they are conjugate in the group $\mathsf{Aut}(\mathbf{C}^2)$ of polynomial automorphisms. In [31], the authors study more generally what happens when f and g are conjugated by a real diffeomorphism, and prove that if f and g are complex Hénon maps of degree 2 which are conjugated by a real C^1 diffeomorphism then f is conjugate to g or $\overline{g}$ in $\mathsf{Aut}(\mathbf{C}^2)$, where $\overline{g}$ is obtained from g by applying complex conjugation to the coefficients. An obvious approach to this problem would be to show that the conjugacy class of the differential df^n at periodic orbits of period n characterizes an automorphism modulo

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conjugacy in $\mathsf{Aut}(\mathbf{C}^2)$ (possibly up to complex conjugation if the conjugacy class is considered in the real sense). This is an instance of the classical *multiplier rigidity problem*, itself part of the celebrated *spectral rigidity problem*. For instance, a Hénon map of small degree (say 2 or 3) is indeed characterized by its eigenvalues (or multipliers) at fixed points, which allows Friedland and Milnor to proceed with this case in [31]. But, despite recent advances in one-dimensional dynamics [38, 39], this problem is essentially untouched in our two-dimensional context, so we take a different path.

As a matter of fact, we solve the real-analytic conjugacy problem under a generic hypothesis.

Theorem A. *Let f and g be polynomial automorphisms of $\mathbf{C}^2$ of positive entropy, which are conjugated by a real-analytic diffeomorphism $\varphi : \mathbf{C}^2 \to \mathbf{C}^2$. Assume that f admits a saddle periodic point p, of some period $n \geqslant 1$, at which the two eigenvalues of df^n_p are both non-real. Then f is conjugate to g or $\overline{g}$ in $\mathsf{Aut}(\mathbf{C}^2)$.*

Theorem A relies on a number of rigidity properties *concerning a single map*, which are interesting for their own sake. In particular, we deal with the classical problem of *smoothness of the stable lamination* which has been extensively studied in hyperbolic dynamics, notably in connection with the classification of Anosov diffeomorphisms and flows: see for instance [29, Chapter 9] for an introduction to this topic, and [51] for a recent contribution in a holomorphic context. Another key step of the proof is *the non-existence of automorphisms with smooth Julia sets*, a problem that was previously addressed by Bedford and Kim [3, 4]. The underlying philosophy behind these results is that there is no "integrable" automorphism, which would play a role analogous to monomial, Chebychev or Lattès mappings in one-dimensional dynamics.

1.2. **Invariant foliations.** Let us be more specific. From now on, a polynomial automorphism of $\mathbf{C}^2$ of positive topological entropy will be called *loxodromic*; such an automorphism f is conjugate, in $\mathsf{Aut}(\mathbf{C}^2)$, to a composition of Hénon maps. We denote the forward and backward Julia sets of f by J^+ and J^-, respectively. The closure of the union of all saddle periodic orbits of f is denoted by $J^\star$, it is a subset of the Julia set $J := J^+ \cap J^-$. The stable (resp. unstable) manifold of any saddle periodic point is an immersed Riemann surface, biholomorphic to $\mathbf{C}$, which is dense in J^+ (resp. J^-) and endows J^+ (resp. J^-) with some kind of laminar structure. When f is hyperbolic (that is, when J is hyperbolic as an f-invariant set) these are actual laminations by Riemann surfaces. Most of these facts are due to Bedford and Smillie, and we refer to the original papers [7, 8, 6] for details.

We shall say that J^+ is subordinate to a foliation if there is a neighborhood U of J^+ and a foliation $\mathcal{F}$ of U by Riemann surfaces such that J^+ is saturated by $\mathcal{F}$: if $x \in J^+$, the leaf $\mathcal{F}(x)$ is contained in J^+. We allow foliations to have isolated singularities. Another possible definition is that every disk contained in a stable manifold is contained in a leaf of $\mathcal{F}$ (see §4.1 for more details).

Theorem B. *If f is a loxodromic automorphism of $\mathbf{C}^2$, then J^+ (resp. J^-) cannot be subordinate to a global real-analytic (in particular to a holomorphic) foliation.*

We could also formulate this result as the non-existence of an f-invariant real-analytic foliation with complex leaves (see Remark 4.5). Brunella [14] proved Theorem B when the global foliation is defined by an algebraic 1-form, and his theorem is actually a key step in the proof. The chain of arguments runs as follows:

no invariant algebraic foliation (Brunella [14]) $\rightsquigarrow$ no invariant holomorphic foliation (Theorem 4.3) $\rightsquigarrow$ no invariant real-analytic foliation (Theorem B).

Theorem B provides a partial answer to Question 31 in [50], which asks whether J^+ can be *locally* subordinate to a holomorphic foliation. For instance, for a dissipative and hyperbolic map, the stable lamination always admits a local extension to a $C^{1+\varepsilon}$ foliation with complex leaves (see [42, Lem. 5.3]). In § 4.3, we show in particular that for an automorphism with disconnected Julia set (e.g. a horseshoe), such a foliation is never holomorphic, nor even real analytic, in a neighborhood of $J^\star$ (see Theorem 5.1).

1.3. **Smooth Julia slices.** In one variable, it was shown by Fatou [27] that if the Julia set is contained in a curve then it is contained in a circle, and that in this case it is either a Cantor set, a segment, or the circle itself; and these last two cases correspond to integrable maps. Our study leads to a similar problem. Indeed, along the proof of Theorem B we have to consider a local slice of J^+ by some holomorphic transversal (e.g. by some disk in the unstable manifold of a saddle point) and to study the possibility that such a slice is contained in a smooth curve. Such a slice is a kind of relative to a one-dimensional Julia set.

To state an analogue of Fatou's result in our situation, recall that if p is a saddle periodic point of period n, then its unstable manifold $W^u(p)$ is biholomorphic to $\mathbf{C}$. Thus, we can fix a parametrization $\psi_p^u : \mathbf{C} \to W^u(p)$. If λ^u denotes the unstable eigenvalue of df_p^n, then $f^n \circ \psi_p^u(\zeta) = \psi_p^u(\lambda^u\zeta)$. The unstable Lyapunov exponent of p is by definition $\chi^u(p) = \frac{1}{n}\log|\lambda^u(p)|$.

It is not difficult to show that if a local holomorphic slice of J^+ is contained in a C^1 curve, then for any saddle p, $(\psi_p^u)^{-1}(J^+)$ is contained in a line ([1]); likewise if a local slice is a C^1 curve, then $(\psi_p^u)^{-1}(J^+)$ is a line. We say that f is *unstably real* in the former case, and *unstably linear* in the latter. Thus, in the previous analogy, unstably linear automorphisms correspond to integrable one variable polynomials.

Theorem C. *Unstably linear loxodromic automorphisms do not exist among loxodromic automorphisms of* $\mathbf{C}^2$.

This is an essential tool towards Theorem B. A weaker result was obtained by Bedford and Kim in [3] (see also [4]), where it was shown that J^+ itself cannot be a smooth 3-manifold. The contradiction in [3] comes from a global topological argument together with the following multiplier rigidity statement: for a generalized Hénon map of degree d it is not possible that $d-1$ of its d fixed points are saddles with the same unstable eigenvalue (for technical reasons, in [3] it needs to be assumed that f is a composition of at least 3 Hénon maps). This result also plays a key role in our proof. Another important tool to prove Theorem C is the theory of quasi-expansion developed by Bedford and Smillie in [11]. More precisely, the results of [11] show that an unstably linear map is quasi-expanding; adapting an argument from [12], and using

[1] There is a subtle point about the exact definition of a local slice of J^+, so we are abusing slightly here, see § 2.1 for the exact definition

the hyperbolicity criteria of [1], we obtain that it is actually uniformly hyperbolic. Then, a generalization of [3] leads to the desired contradiction.

Coming back to Theorem A, the bulk of the proof is to show that φ must be holomorphic or anti-holomorphic; then we invoke our previous result [16] to conclude. Assuming, by way of contradiction, that φ is neither holomorphic nor anti-holomorphic, then, pulling back the complex structure by φ gives an "exotic" f-invariant real-analytic complex structure on $\mathbf{C}^2$. Analyzing this complex structure at saddle points ultimately produces an f-invariant real-analytic foliation with complex leaves, which contradicts Theorem B, and we are done.

1.4. **The multiplier field.** The proof of Theorem A requires also a lemma on the multipliers associated to periodic orbits shadowing a homoclinic orbit: see Theorem 6.1. This statement is inspired by a one dimensional result due to Eremenko-Van Strien [25] and Ji-Xie [38], which was used by Huguin [37] to characterize rational maps on $\mathbb{P}^1(\mathbf{C})$ whose multipliers belong to a fixed number field. Adapting Huguin's argument, we obtain the following (see Theorems 8.1 and 8.2):

Theorem D. *Let $f \in \mathsf{Aut}(\mathbf{C}^2)$ be a loxodromic automorphism defined over a number field. Assume that*

- *either f is hyperbolic and admits two saddle periodic points p and p' with distinct Lyapunov exponents $\chi^u(p) \neq \chi^u(p')$;*
- *or f admits a saddle periodic point whose unstable Lyapunov exponent is larger than that of the maximal entropy measure.*

Then the unstable (resp. stable) multipliers of f cannot lie in a fixed number field.

The assumptions of Theorem D are easily checkable in a perturbative setting, and it follows that its conclusion holds for (most) Hénon maps with sufficiently small Jacobian (see Theorem 8.4 for a precise statement).

1.5. **Plan of the paper.** In Section 2, we study unstably real and linear maps and prove Theorem C. Several natural questions are discussed in § 2.5. In Section 3, which apart from some technical lemmas is independent from the rest of the paper, we push this study one step further and show that a holomorphic slice of J^+ cannot be a rectifiable curve. Sections 4 and 5 are devoted to Theorem B, as well as some local versions of it. In Section 6 we prove Theorem 6.1 on the multipliers of periodic orbits shadowing a homoclinic orbit, and Theorem A is finally obtained in Section 7. Lastly, in Section 8 we investigate some arithmetic properties of saddle point multipliers, and establish Theorem D, as well as its application to perturbations of one-dimensional maps (Theorem 8.4).

1.6. **Notation.** We use the standard notation and vocabulary of the field, as listed for instance in [9, §1] or [24, §2.1]. For instance, given a loxodromic automorphism f of $\mathbf{C}^2$, we denote by G^+ its dynamical Green function (or rate of escape function), and by $T^+ = dd^cG^+$ the associated current (in [9], the invariant currents are denoted by $\mu^\pm$ instead of $T^\pm$). The Jacobian determinant of an automorphism f of $\mathbf{C}^2$ is constant: f is dissipative if $|\mathrm{Jac}(f)| < 1$, volume expanding if $|\mathrm{Jac}(f)| > 1$, and conservative if $|\mathrm{Jac}(f)| = 1$.

*By convention the Zariski (resp. $\mathbf{R}$-Zariski) topology is the complex analytic (resp. real-analytic) Zariski topology. We sometimes refer to these simply as the (real) analytic topology.

SC: Reprendre? Cf. Huguin qui est gêné...

2. UNSTABLY REAL AND UNSTABLY LINEAR MAPS

2.1. **Transversals.** Let $f \in \mathsf{Aut}(\mathbf{C}^2)$ be a loxodromic automorphism. We say that a holomorphic disk Γ is a *transversal to* J^+ if it satisfies one of the following equivalent conditions:

(a) $\Gamma \cap J^+ \neq \varnothing$ and $\Gamma \not\subset K^+$;
(b) $G^+|_\Gamma$ is not harmonic;
(c) $G^+|_\Gamma$ vanishes and is not identically 0.

It follows from the work of Bedford, Lyubich and Smillie [6, §§8-9] that for any saddle periodic point p, $W^s(p)$ admits transverse intersections with Γ (see [24, Lem. 5.1] and [23, Lem. 3.3] for more details). We set

$$J^+_\Gamma = \mathrm{Supp}(T^+ \wedge [\Gamma]) = \mathrm{Supp}(dd^c(G^+|_\Gamma)) = \partial_\Gamma(K^+ \cap \Gamma), \tag{2.1}$$

where in the last equality ∂_Γ refers to the boundary as a subset of Γ. Note that $J^+_\Gamma \subset \Gamma \cap J^+$ but this inclusion could be strict. Indeed, Γ could a priori contain a disk that is entirely contained in J^+: such a disk would be part of $\Gamma \cap J^+$ but not of J^+_Γ. If in addition $\Gamma = \Delta^u_p$ is contained in the unstable manifold of a saddle point p, we have (with the same caveat for reverse inclusions)

$$J^+_{\Delta^u_p} \subset \Delta^u_p \cap J^\star \subset \Delta^u_p \cap J^+. \tag{2.2}$$

The first inclusion follows from the fact that $J^+_{\Delta^u_p}$ is the closure of the homoclinic intersections contained* in Δ^u_p (cf. [24, Lem. 5.1]). Recall that a homoclinic intersection is a point of $W^s(p) \cap W^u(p) \setminus \{p\}$ and that every homoclinic intersection is contained in $J^\star$ (see [6, Thm 9.9]).

SC: Ajouté "contained in Δ^u_p, cf Huguin.

As for Julia sets in one complex variable, the sets J^+_Γ vary lower semicontinuously in the Hausdorff topology, in the following sense:

Lemma 2.1. *Let Γ be a transversal to J^+. If (Γ_n) is a sequence of disks converging in the C^1 topology to Γ, then*

(1) $J^+_\Gamma \subset \liminf_{n\to\infty} J^+_{\Gamma_n}$;
(2) $J^+ \cap \Gamma \supset \limsup_{n\to\infty} J^+ \cap \Gamma_n$.

Proof. Property (1) follows from the continuity of G^+: for any $z \in J^+_\Gamma$, pick a small disk $U \subset \Gamma$ around z, then by definition $G^+|_U$ is not harmonic; so if we lift U to some disk $U_n \subset \Gamma_n$ then for large n, $G^+|_{U_n}$ cannot be harmonic and we are done. Note that we may extend this argument to the case where (Γ_n) converges to Γ with some finite multiplicity (i.e. in the sense of analytic sets). Property (2) follows directly from J^+ being closed. □

2.2. **Unstably real automorphisms.** Let p be a saddle periodic point of f, of exact period n. As in Section 1.3, we denote by $\psi^u_p : \mathbf{C} \to W^u(p)$ a parametrization of its unstable manifold by an injective entire curve that maps 0 to p. If a unit unstable vector e^u_p is given, one may normalize $\psi^u_p : \mathbf{C} \to W^u(p)$ by fixing some $\alpha \in \mathbf{C}^\times$ and imposing $(\psi^u_p)'(0) = \alpha e^u_p$. The parametrization

ψ^u_p semi-conjugates f^n to a linear map, that is, $f^n \circ \psi^u_p(\zeta) = \psi^u_p(\lambda^u \zeta)$, where $\lambda^u \in \mathbf{C}^\times$ is the unstable multiplier.

Proposition 2.2. *Let $f \in \mathsf{Aut}(\mathbf{C}^2)$ be a loxodromic automorphism. Assume that for some transversal Γ to J^+, J^+_Γ is contained in a C^1 smooth curve. Then for every saddle periodic point p, $(\psi^u_p)^{-1}(J^+_{W^u(p)})$ is contained in a line through the origin. In particular $\lambda^u(p)$ is real and $J^+_{W^u(p)}$ is contained in a real-analytic curve. In addition*

- *either J^+_Γ is not a Cantor set, and $(\psi^u_p)^{-1}(J^+_{W^u(p)})$ is a line through the origin for every saddle periodic point p;*
- *or $J^+_{W^u(p)}$ is a Cantor set and $J^+_{W^u(p)} = J^+ \cap W^u(p)$ for every saddle periodic point p.*

If the assumption of the proposition holds, we say that f is *unstably real*. Thus, for such a map, all unstable multipliers are real. By symmetry we have a similar result for transversals to J^-, yielding the notion of *stably real* automorphism.

Proof. The argument goes back to Fatou [27, §46]. Pick a saddle periodic point p, and replace f by a positive iterate to assume that p is fixed. As already explained, $W^s(p)$ admits transversal intersections with Γ, and more precisely with J^+_Γ. As in [23, Lem. 1.12], we can find holomorphic coordinates $(x, y) \in \mathbb{D}^2$ near p in which $p = (0, 0)$, $W^s_{\rm loc}(p) = \{x = 0\}$, $W^u_{\rm loc}(p) = \{y = 0\}$ and

$$f(x, y) = (\lambda^u x(1 + xyg_1(x, y)), \lambda^s y(1 + xyg_2(x, y))) \ , \tag{2.3}$$

with $|\lambda^s| < 1 < |\lambda^u|$, and $\|g_1\|$, $\|g_2\|$ as small as we wish. In these coordinates, $f|_{\{y=0\}}$ is linear, so $\zeta \mapsto (\zeta, 0)$ is an unstable parametrization near the origin. Changing Γ in $f^m(\Gamma)$ for a large positive integer m, and taking the connected component of $f^m(\Gamma) \cap \mathbb{D}^2$ containing p, we may assume that Γ is a graph over the first coordinate, of the form $y = \gamma(x)$. Let $\sigma \mapsto t(\sigma)$ be a germ of C^1 curve at $0 \in \mathbf{C}$, with $t'(0) \neq 0$, such that J^+_Γ is contained in the image of $\sigma \mapsto (t(\sigma), \gamma(t(\sigma)))$. With our choice of coordinates, Lemma 4.2 in [23] asserts that for some $\delta > 0$, if $|x| \leqslant \delta$, then $f^n(x/(\lambda^u)^n, y) \to (x, 0)$ as $n \to \infty$. Actually the proof says a little more: if $|x_n| \leqslant \delta$ and $|y_n| < 1$, then

$$f^n\left(\frac{x_n}{(\lambda^u)^n}, y_n\right) = (x_n, 0) + o(1). \tag{2.4}$$

Fix a subsequence n_j such that $\frac{(\lambda^u)^{n_j}}{|\lambda^u|^{n_j}} \to 1$. Since $t(|\lambda^u|^{-n}\sigma) = |\lambda^u|^{-n}(t'(0)\sigma + o(1))$, Equation (2.4) for sufficiently small $\sigma \in \mathbf{R}$ implies

$$f^{n_j}\left(t\left(|\lambda^u|^{-n_j}\sigma\right), \gamma\left(t\left(|\lambda^u|^{-n_j}\sigma\right)\right)\right) \underset{j\to\infty}{\longrightarrow} \left(t'(0)\sigma, 0\right). \tag{2.5}$$

Now we use the lower semi-continuity of J^+_Γ: if z belongs to $J^+_{\{y=0\}}$, then it must be accumulated by $J^+_{f^{n_j}(\Gamma)}$, hence $z \in t'(0)\mathbf{R}$, and we conclude that $J^+_{\{y=0\}}$ is contained in a line, as asserted. We also get that if (n'_j) is any other subsequence such that $(\lambda^u)^{n'_j}/|\lambda^u|^{n'_j}$ converges to $e^{i\theta}$, then $z \in e^{i\theta}t'(0)\mathbf{R}$. Since $J^+_{\{y=0\}}$ is not reduced to $\{0\}$, this argument shows that $e^{i\theta} = \pm 1$ so $(\lambda^u)^n/|\lambda^u|^n$ converges to $\{\pm 1\}$, and therefore λ^u is real.

To get the last assertion of the proposition, we observe that since $G^+|_\Gamma$ is continuous, $J^+_\Gamma = \mathrm{Supp}(dd^c(G^+|_\Gamma))$ has no isolated points, so if it is contained in a line, it is either a Cantor set or it contains a non-trivial arc. In the latter case, by reducing Γ, we may assume that J^+_Γ is a C^1 curve, and in this case the argument shows that $J^+_{\{y=0\}}$ is a line through the origin. And if $J^+_{W^u(p)}$ is a Cantor set, then $J^+_{W^u(p)} = J^+ \cap W^u(p)$, since in this case the unique connected component of $W^u(p)\backslash J^+_{W^u(p)}$ must be contained in $\mathbf{C}^2\backslash K^+$. □

Remark 2.3. The proof shows more generally that if for some saddle periodic point q, J^+_Γ admits a tangent at some transverse intersection $W^s(q) \cap \Gamma$ (so in particular if $J^+_{W^u(q)}$ admits a tangent at q), then the conclusion of the proposition holds at every saddle periodic point p. The same is true for general hyperbolic measures and Pesin stable manifolds, with a slightly different argument (see Lemma 3.4).

2.3. **Unstably linear automorphisms: multipliers.** In this subsection we revisit and extend some results of Bedford and Kim [3, 4]. These will be used in the next subsection to prove the non-existence of unstably linear automorphisms.

Let p be a saddle periodic point of period n such that $(\psi^u_p)^{-1}\big(J^+_{W^u(p)}\big)$ is a line through the origin. Then, by Proposition 2.2, the same property holds at any other saddle point; in this situation, we say that f is *unstably linear*. Since $W^u(p)$ is not contained in K^+, at least one side of $J^+_{W^u(p)}$ in $W^u(p) \simeq \mathbf{C}$ must be contained in $\mathbf{C}^2\backslash K^+$. In particular f is unstably connected in the sense of [9]. Corollary 7.4 in [9, Cor. 7.4]) shows that a volume expanding map cannot be unstably connected, hence necessarily $|\mathrm{Jac}(f)| \leqslant 1$. Lemma 4.2 in [3] shows that the unstable multiplier equals $\pm d^n$. For convenience we recall the argument: $G^+ \circ \psi^u_p$ is harmonic and positive in some half plane (of slope, say, $\tan(\theta)$) and zero on its boundary, hence in this half plane $G^+ \circ \psi^u_p(\zeta)$ is proportional to $\mathrm{Im}(e^{-i\theta}\zeta)$; in particular it is $\mathbf{R}$-linear. Then the invariance relation $G^+ \circ \psi^u_p(\lambda^u\zeta) = d^n G^+ \circ \psi^u_p(\zeta)$ forces $\lambda^u = \pm d^n$. Note that $\lambda^u = d^n$ (resp. $\lambda^u = -d^n$) iff f^n preserves (resp. exchanges) the two half planes. Let us summarize this discussion in a lemma.

Lemma 2.4. *Assume that f is unstably linear. Then for any saddle periodic point p, of period n, $(\psi^u_p)^{-1}(J^+_{W^u(p)})$ is a line through the origin, which cuts $\mathbf{C}$ in two half-planes. Moreover:*

(1) *each component of $(\psi^u_p)^{-1}(\mathbf{C}^2\backslash K^+)$ is a half plane in which $G^+ \circ \psi^u_p(\zeta)$ is proportional to $\mathrm{Im}(e^{-i\theta}\zeta)$, where $\tan(\theta)$ is the slope of the line $(\psi^u_p)^{-1}(J^+_{W^u(p)})$;*
(2) *the unstable multiplier at p is equal to $\pm d^n$, where d is the degree of f; it is equal to d^n iff f^n preserves the two half planes;*
(3) *f is unstably connected and $|\mathrm{Jac}(f)| \leqslant 1$;*

Our primary focus in the next proposition is on unstably linear maps; nevertheless, it may be useful to remark that it holds in the unstably real case as well so we state it in this generality.

Proposition 2.5. *Let f be a loxodromic automorphism which is unstably real and dissipative. Then every periodic point p of f is a saddle. In addition, in the unstably linear case, the two sides of $W^u(p)\backslash J_{W^u(p)}$ are contained in $\mathbf{C}^2\backslash K^+$.*

Proof. The assumptions and conclusions of the proposition are not affected if we replace f by some iterate. For technical reasons, we replace f by f^3 (while still denoting its degree by d).

In a first stage, assume that f is unstably linear.

Step 1.– Assume that for some saddle point p, the two sides of $W^u(p)\backslash J_{W^u(p)}$ are contained in $\mathbf{C}^2\backslash K^+$. We claim that the same property holds for every other saddle periodic point q. Indeed, $W^s(p)$ intersects transversally $W^u(q)$ at some point $\tau \in J_{W^u(q)}$. By the inclination lemma, there is a sequence of neighborhoods U_n of τ in $W^u(q)$ such that $f^n(U_n)$ is a sequence of disks converging in the C^1 sense to $W^u_{\mathrm{loc}}(p)$. We may assume that U_n is a topological disk such that $U_n\backslash J_{W^u(q)}$ has two components. Since $\mathbf{C}^2\backslash K^+$ is open, for large n, the two sides of $f^n\big(U_n\backslash J_{W^u(q)}\big)$ intersect $\mathbf{C}^2\backslash K^+$, therefore they are contained in $\mathbf{C}^2\backslash K^+$; pulling back by f^n, we conclude that the same holds for $W^u(q)\backslash J_{W^u(q)}$.

This discussion also shows that if for some saddle point p, one side of $W^u(p)\backslash J_{W^u(p)}$ is contained in K^+, then the same holds for every other saddle point.

Step 2.– We claim that f admits at most one non-saddle fixed point.

The argument is similar to that of [21, §2.2]. Consider a non-saddle periodic point q. Without loss of generality, assume that q is fixed. Since f is dissipative, q is a sink or q is semi-neutral. In the latter case, the neutral eigenvalue is either a root of unity and q is said to be semi-parabolic, or it is not and q is either semi-Siegel or semi-Cremer according to the existence of an invariant holomorphic disk containing q. In all these cases, there exists an invariant manifold through q which is biholomorphic to $\mathbf{C}$ and contracted by the dynamics; for instance, when q has eigenvalues of distinct moduli we can take the so-called strong stable manifold $W^{ss}(q)$, associated to the most contracting eigenvalue. The theory of Ahlfors currents associated to entire curves shows that this invariant manifold must intersect $W^u(p)$ (see [21, Lem. 2.2] for this very statement and [24, Lem. 5.4] for the details of the proof). If q is a sink or if it is semi-Siegel, this stable manifold is contained in $\mathrm{Int}(K^+)$, so it must intersect $W^u(p)$ in a component of $\mathrm{Int}(K^+) \cap W^u(p)$. It then follows that $G^+ \equiv 0$ on the corresponding component of $W^u(p)\backslash J^+_{W^u(p)}$, which is then a Fatou disk entirely contained in the Fatou component of q. Since this is true for every such q, it follows that there can be at most one sink or semi-neutral point.

If q is semi-parabolic, the argument is the same except that instead of $W^{ss}(q)$, we consider any entire curve contained in the semi-parabolic basin of q (such curves exist since the basin is known to be biholomorphic to $\mathbf{C}^2$).

The last possibility is that q is semi-Cremer. Then by [28, 43], to any local center manifold $W^c_{\mathrm{loc}}(q)$ of q we can associate a hedgehog $\mathcal{H}$:

- $\mathcal{H}$ is not locally connected since it is homeomorphic to the hedgehog of a non-linearizable one-dimensional germ.
- By [43, Thm E], $\mathcal{H} \subset J^\star$, and every point in $\mathcal{H}$ admits a strong stable manifold which is biholomorphic to $\mathbf{C}$, and has uniform geometry near $\mathcal{H}$ (because $f|_{\mathcal{H}}$ admits a dominated splitting, see [28, Thm A]).

Therefore, as above, this strong stable manifold admits transverse intersections with $W^u(p)$ and we can transport by holonomy a non-trivial relatively open subset of $\mathcal{H}$ to $W^u(p)$. The resulting piece is contained in $J^+_{W^u(p)}$, as follows from the proof of [43, Thm E]([2]). This is a contradiction because $J^+_{W^u(p)}$ is locally a smooth curve while $\mathcal{H}$ is not locally connected, which shows that Semi-Cremer points cannot exist in this context.

Step 3.–We can now complete the proof of the proposition in the unstably linear case. Since f is conjugated to a composition of Hénon maps of total degree d, it admits d fixed points. By the second step, at most one of them is not a saddle, and if such a non saddle point exists, for every saddle point p, one of the components of $W^u(p)\backslash J_{W^u(p)}$ is contained in K^+. Therefore by Lemma 2.4.(3), $\lambda^u(p) = d$. In particular f has d distinct fixed points and at least $d-1$ of them are saddles with unstable multiplier equal to d. Since f is conjugated to a product of at least 3 Hénon maps and f is dissipative, Proposition 5.1 in [3] shows that this is impossible. Thus, all periodic points are saddles.

Then, by Step 1, if there is such a saddle point p for which a component of $W^u(p)\backslash J_{W^u(p)}$ is contained in K^+, then the same is true for each of the fixed points of f, so that the unstable multiplier at each of the fixed points is equal to d. Again, [3, Prop. 6.1] provides a contradiction. Thus, the two components of $W^u(p)\backslash J_{W^u(p)}$ are contained in $\mathbf{C}^2\backslash K^+$ for every periodic point, and the proof for the unstably linear case is complete.

Step 4.– If f is unstably real but not unstably linear, then by Proposition 2.2, for every saddle point p, $J^+_{W^u(p)}$ is a Cantor set contained in a line. Thus $W^u(p)\backslash J^+_{W^u(p)}$ is connected, hence contained in $\mathbf{C}^2\backslash K^+$. Applying the arguments of Step 2 above shows that f cannot possess any sink, semi-Siegel or semi-parabolic point for it would give rise to a Fatou disk contained in $W^u(p) \cap K^+$; in the case of a semi-Cremer point we would obtain a non-trivial (connected) continuum contained in $J^+_{W^u(p)}$, which again is a contradiction. □

Corollary 2.6. *A hyperbolic loxodromic automorphism is never unstably linear.*

Proof. Assume that f is unstably linear and hyperbolic. By Lemma 2.4.(4), f is unstably connected, so J is connected, and also $|\mathrm{Jac}(f)| \leqslant 1$. If f is conservative and hyperbolic, then J cannot be connected (see [10, Cor. A.3] or [19, Cor. 3.2]), so f is dissipative. By Proposition 2.5 all periodic points are saddles. On the other hand, Theorem 3.1 in [19], asserts that f has an attracting point. This contradicts Proposition 2.5. □

2.4. **Unstably linear automorphisms: non-existence.** We are now ready to prove Theorem C.

Theorem 2.7. *A loxodromic automorphism of $\mathbf{C}^2$ is never unstably linear.*

Corollary 2.8. *If Γ is a transversal to J^+, then J^+_Γ is never a C^1 curve. More generally, a non trivial component of J^+_Γ does not admit a tangent line at any transverse intersection of Γ with $W^s(p)$, for any saddle point p.*

[2]For the reader's convenience we explain the argument of [43, Thm E]: if for some $z \in \mathcal{H}$, some transverse intersection point of $W^{ss}(z) \cap W^u(p)$ were not contained in $J^+_{W^u(p)}$, then one side of $W^u\backslash J^+_{W^u(p)}$ would be a Fatou disk Ω. It can be shown that if (n_j) is a subsequence such that $f^{n_j}(\Omega)$ converges to a holomorphic disk Γ (possibly reduced to a point), then Γ should be contained in every local center manifold of q, hence in $\mathcal{H}$ (by definition of the hedgehog). So by [43, Thm C] the whole of Ω would be contained in $W^{ss}(\mathcal{H})$, which contradicts the fact that $\mathcal{H}$ has relative zero interior in any center manifold of q.

The remainder of this subsection is devoted to the proof of the theorem. Henceforth we assume that f is an unstably linear loxodromic automorphism. Throughout the proof, the unstable parametrizations $\psi^u_p : \mathbf{C} \to W^u(p)$ are normalized by (see [11]):

$$\psi^u_p(0) = p, \quad \max_{|\zeta|\leqslant 1} G^+ \circ \psi^u_p(\zeta) = 1, \quad \text{and} \quad (\psi^u_p)^{-1}(J^+_{W^u(p)}) = \mathbf{R}. \tag{2.6}$$

Lemma 2.9. *Let p be a periodic point of f, and ψ^u_p be normalized as above. Then up to replacing $\psi^u_p(\zeta)$ by $\psi^u_p(-\zeta)$, we have*

$$G^+ \circ \psi^u_p(\zeta) = \begin{cases} \mathrm{Im}(\zeta) \text{ for } \mathrm{Im}(\zeta) \geqslant 0 \\ c\,|\mathrm{Im}(\zeta)| \text{ for } \mathrm{Im}(\zeta) < 0 \end{cases}$$

for some $0 \leqslant c \leqslant 1$. In particular $\left\{G^+ \circ \psi^u_p \leqslant 1\right\}$ is the strip $\{-1/c \leqslant \mathrm{Im}(\zeta) \leqslant 1\}$ (it is a half plane when $c = 0$). When f is dissipative, $c > 0$.

Proof. We already know from Lemma 2.4 that $G^+ \circ \psi^u_p(\zeta)$ is proportional to $|\mathrm{Im}(\zeta)|$ on both sides of the real axis, say $G^+ \circ \psi^u_p(\zeta) = c_+ |\mathrm{Im}(\zeta)|$ on the upper half plane and $G^+ \circ \psi^u_p(\zeta) = c_- |\mathrm{Im}(\zeta)|$ on the lower half plane. After replacing ζ by $-\zeta$, we may assume $c_+ \geqslant c_- \geqslant 0$, and then $c_+ > 0$ because G^+ can not vanish identically on $W^u(p)$. Since $\max_{\overline{\mathbb{D}}} G^+ \circ \psi^u_p(\zeta) = 1$, we conclude that $c_+ = 1$. The last assertion follows from Proposition 2.5. ☐

Remark 2.10. If p is of period n and $\lambda^u = -d^n$, then the relation $G^+ \circ \psi^u(\lambda^u\zeta) = d^n G^+ \circ \psi^u(\zeta)$ forces $c = 1$.

The following result is essentially (but not exactly) contained in [11, Thm 4.8]. Before stating it, let us introduce the concept of quasi-expansion. Let $\mathcal{S}$ be the set of saddle periodic points of f. This is a dense, f-invariant subset of $J^\star$. Let $\Psi_{\mathcal{S}}$ be the set of parametrizations ψ^u_p, $p \in \mathcal{S}$, normalized as in Equation (2.6). For p in $\mathcal{S}$, there is a linear map of the form $L\colon \zeta \mapsto \lambda\zeta$ such that $f \circ \psi^u_p = \psi^u_{f(p)} \circ L$. One of the equivalent definitions of *quasi-expansion* is to require the existence of a constant $\kappa > 1$ such that $|\lambda| \geqslant \kappa$ uniformly for all $p \in \mathcal{S}$, and then κ is a *quasi-expansion factor*. We refer to [11] for this notion, in particular to Theorem 1.2 there.

Lemma 2.11. *An unstably linear loxodromic automorphism is quasi-expanding, with quasi-expansion factor $\kappa = d$.*

Proof. Fix $p \in \mathcal{S}$ and consider the map $L\colon \zeta \mapsto \lambda\zeta$ introduced above. The relation $G^+ \circ f = dG^+$ implies that L maps the closed unit disk to a disk $\overline{D}(0, r)$ such that $\max_{\overline{D}(0,r)} G^+ \circ \psi^u_{f(p)} = d$. By Lemma 2.9, $r = d$, so $|\lambda| = d$ is uniformly bounded from below. ☐

In the next two paragraphs we assume that f is quasi-expanding and state some general facts. Quasi-expansion implies that $\Psi_{\mathcal{S}}$ is a normal family (and vice versa). We denote by $\widehat{\Psi}$ the set of all its normal limits (i.e. $\widehat{\Psi}$ is the closure of $\Psi_{\mathcal{S}}$ with respect to local uniform convergence). For any $x \in J^\star$, and any sequence $(p_n) \in \mathcal{S}^{\mathbf{N}}$ converging to x, one can extract a subsequence such that $(\psi^u_{p_n})$ converges towards an element $\widehat{\psi} \in \widehat{\Psi}$ with $\widehat{\psi}(0) = x$. It is a non-constant entire curve because $\max_{\overline{\mathbb{D}}} G^+ \circ \widehat{\psi} = 1$, but it is not necessarily injective. For $x \in J^\star$, we let

$$\tau(x) = \max_{\widehat{\psi}\in\widehat{\Psi},\ \widehat{\psi}(0)=x} \mathrm{ord}_0(\widehat{\psi}), \tag{2.7}$$

where $\mathrm{ord}_0(\widehat{\psi}) = \min\left\{k \geqslant 0\,;\, \psi^{(k)}(0) \neq 0\right\} < \infty$ is the vanishing order of $\widehat{\psi}$ at 0. By [34, Lem 3.1], τ is uniformly bounded (in our setting, we give a direct proof of this fact in Lemma 2.13 below).

For every $x \in J^\star$ we set $\mathcal{W}^u(x) = \hat{\psi}(\mathbf{C})$. It does not depend on the choice of normal limit $\hat{\psi} \in \hat{\Psi}$ with $\hat{\psi}(0) = x$ in the above construction; it is contained in K^-, and, in fact, it is the image of an injective entire curve (see [2, §1] [3]). If x is a saddle point, then $\mathcal{W}^u(x) = W^u(x)$. As usual we denote by $\psi^u_x : \mathbf{C} \to \mathcal{W}^u(x)$ an injective parametrization of $\mathcal{W}^u(x)$ such that $\psi^u_x(0) = x$. If we consider a sequence $p_n \to x$ as above, then the the limit $\widehat{\psi}$ satisfies $\widehat{\psi} = \psi^u_x \circ h$ for some polynomial function h (see [11, Lem. 6.5], and [34, §3]) such that $h(\zeta) = c\zeta^k + \text{h.o.t.}$, $k = \mathrm{ord}_0\widehat{\psi}$. In particular, G^+ does not vanish identically on $\mathcal{W}^u(x)$.

We now resume the proof of Theorem 2.7.

Lemma 2.12. *If f is quasi-expanding, then for every $x \in J^\star$, $G^+ \circ \psi^u_x$ does not vanish identically in a neighborhood of the origin.*

Proof. If p is a saddle periodic point, the maximum of $G^+ \circ \psi^u_p$ on $\mathbb{D}_1$ is equal to 1. Fix a radius $r \in]0,1[$. Then, by Property (3) in Theorem 1.2 of [11], there is a constant $\alpha(r) > 0$ that does not depend on p such that the maximum of $G^+ \circ \psi^u_p$ on $\mathbb{D}_r$ is at least $\alpha(r)$. Since the family $(\psi^u_p)_{p\in\mathcal{S}}$ is normal and G^+ is continuous, this property is satisfied by all $\hat{\psi}_x$ in $\hat{\Psi}$, and the lemma follows. □

Second proof, specific to unstably linear maps. Pick a sequence (p_n) in $\mathcal{S}^{\mathbf{N}}$ converging to x. On the upper half plane $\mathbb{H}$ we have that $G^+ \circ \psi^u_{p_n}(\zeta) = \mathrm{Im}(\zeta)$. After possible extraction, with notation as above, $\psi^u_{p_n}$ converges uniformly to $\widehat{\psi}$, so by continuity of G^+ we get that

$$G^+ \circ \widehat{\psi}(\zeta) = G^+ \circ \psi^u_x \circ h = \mathrm{Im}(\zeta) \text{ on } \mathbb{H}, \tag{2.8}$$

so $G^+ \circ \psi^u_x$ takes positive values arbitrary close to 0. □

The following result is reminiscent from [12, Prop. 2.2].

Lemma 2.13. *If f is unstably linear, then $\tau(x) \leqslant 2$ for every $x \in J^\star$.*

Proof. Assume that $x \in J^\star$ is such that $\tau(x) = k$. Then we have a sequence $\psi^u_{p_n}$ converging to $\widehat{\psi} = \psi^u_x \circ h$ with $h(\zeta) = c\zeta^k + \text{h.o.t.}$ at the origin. From Equation (2.8), we have $G^+ \circ \widehat{\psi}(\zeta) = \mathrm{Im}(\zeta)$. On the other hand, locally near the origin, $\left\{G^+ \circ \widehat{\psi} = 0\right\}$ has k-fold rotational symmetry. This is possible only if $k \leqslant 2$. □

For $i = 1, 2$, set $J^\star_i = \{x \in J^\star\,;\, \tau(x) = i\}$, which is an invariant set (denoted by $\mathcal{J}_i$ in [11]). Theorem 6.7 in [11] shows that $J^\star_1$ is open and dense in $J^\star$, so $J^\star_2$ is closed. From the comments preceding Lemma 6.4 in [11], we see that (in their notation) $\mathcal{J}'_2 = \mathcal{J}_2$, so it follows from Lemma 6.5 there that for every $x \in J^\star_2$, if $\widehat{\psi}$ is a non-injective parametrization of $\mathcal{W}^u(x)$, then $\widehat{\psi}$ is of the form $\zeta \mapsto \psi^u_x(c\zeta^2)$ with ψ^u_x injective (see also [12, Prop. 2.6] for a related argument).

Lemma 2.14. *If f is unstably linear, then $J^\star = J^\star_1$; in other words $\tau(x) \leqslant 1$ for every $x \in J^\star$.*

[3] This is essentially contained in [11] but a missing ingredient there was that $\mathcal{W}^u(x)$ is smooth at x.

First proof of the lemma. Assume that $J_2^\star$ is non-empty. We claim that for every $x \in J_2^\star$, $(\psi_x^u)^{-1}(J)$ is contained in a half line. Indeed, if $\widehat{\psi} \in \widehat{\Psi}$ is a non-injective parametrization of $\mathcal{W}^u(x)$ with $\widehat{\psi}(0) = x$, then as in Equation (2.8) we have $G^+ \circ \widehat{\psi}(\zeta) = \mathrm{Im}(\zeta)$ in $\overline{\mathbb{H}}$, and by the previous comments $G^+ \circ \widehat{\psi}(\zeta) = G^+ \circ \psi_x^u(c\zeta^2)$. If we rotate ψ_x^u so that $c = 1$, we see that $G^+ \circ \psi_x^u = 0$ is the positive real axis, as asserted.

Since $J_2^\star$ is a closed invariant set, it supports an ergodic invariant measure ν. By [11, Thm 6.2], ν has a positive Lyapunov exponent, hence also a negative one since $|\mathrm{Jac}(f)| \leqslant 1$. If y is a ν-generic point, its stable manifold intersects transversally the unstable manifold of some saddle point p, and at such a point $J^+_{W^u(y)}$ has a tangent line. Since y is recurrent, from Lemma 3.4 below, $(\psi_y^u)^{-1}\big(J^+_{W^u(y)}\big)$ is a line, which is a contradiction. □

Second proof of the lemma. As in the first proof, if $J_2^\star$ is non-empty, then it supports a hyperbolic, ergodic, invariant probability measure ν. Theorem 6.2 in [11] asserts that the positive exponent of ν is not smaller that $2\log\kappa = 2\log d$ (see Lemma 2.11). It is a general fact from Pesin theory that the Lyapunov exponents of hyperbolic measures are approximated by those of periodic orbits (see [49]). But all Lyapunov exponents of periodic points are equal to $\log d$: this contradiction finishes the proof. □

We are now ready to complete the proof of Theorem 2.7. The previous lemma shows that if f is unstably linear, then $J_1^\star = J^\star$. Therefore the local manifolds $\mathcal{W}^u_{\mathrm{loc}}(x)$ form a lamination in a neighborhood of $J^\star$ (see [11, Prop. 5.3]). In the vocabulary of [1], every point in $J^\star$ is u-regular. In addition by Lemma 2.12, G^+ does not vanish identically along any of the $\mathcal{W}^u_{\mathrm{loc}}(x)$, $x \in J^\star$, and Lemma 2.4 shows that $|\mathrm{Jac}(f)| \leqslant 1$. Therefore, [1, Prop 2.16] (or [1, Thm 2.18]) shows that f is hyperbolic. This contradicts Corollary 2.6, and the proof is complete. □

2.5. **Questions on unstably real automorphisms.** Up to conjugacy, the only examples of unstably real mappings that we know of are real automorphisms with maximal entropy $h_{\mathrm{top}}(f|_{\mathbf{R}^2}) = h_{\mathrm{top}}(f)$. In particular they are stably real as well.

Question 2.15.

(1) Is every unstably real automorphism conjugate to a real automorphism with maximal entropy?
(2) If f is unstably real, then is $\mathrm{Jac}(f)$ real?
(3) Assume that all unstable multipliers of f are real, then is f unstably real?

The second question is a weak form of the first one. We refer to [25] for a positive answer to the third question in dimension 1.

2.6. **Comments on other affine surfaces.** Consider the affine surface $S = \mathbf{C}^\times \times \mathbf{C}^\times$, i.e. the complex multiplicative group of dimension 2. The group $\mathsf{GL}_2(\mathbf{Z})$ acts by automorphisms on S: if $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{GL}_2(\mathbf{Z})$, the corresponding automorphism is $f_M\colon (x, y) \mapsto (x^a y^b, x^c y^d)$. Then, f_M is loxodromic if and only if M has an eigenvalue λ with $|\lambda| > 1$ (the second eigenvalue is $\det(M)/\lambda$, and $\det(M) = \pm 1$). Such a loxodromic monomial automorphism has positive entropy, preserves a pair of holomorphic foliations, and has a Julia set which is a real analytic surface, namely $\{(x, y) \; ; \; |x| = 1 = |y|\}$. A similar example is obtained by looking at the Cayley cubic surface S/η, where $\eta(x, y) = (1/x, 1/y)$ (see [15, 17]). Thus, Theorems B and C do not extend directly to arbitrary affine surface. Following the arguments presented in this article in view of such a generalization, the main missing input to characterize unstably linear automorphims of affine surfaces would be a version of the Bedford-Kim theorem on multipliers of fixed points (Proposition 6.1 in [3]).

Question 2.16. *Let f be an automorphism of a complex affine surface X, with first dynamical degree $\lambda_1(f) > 1$. Assume that for every sadlle periodic point, except finitely many of them, the unstable multiplier λ_p^u is equal to $\pm\lambda_1(f)^n$, where n is the period of p. Is X isomorphic to $\mathbf{C}^\times \times \mathbf{C}^\times$ or its quotient by $(x, y) \mapsto (1/x, 1/y)$?*

3. Rectifiable Julia sets

In this section we push the techniques of the previous section one step further.

Theorem 3.1. *Let $f \in \mathsf{Aut}(\mathbf{C}^2)$ be a loxodromic automorphism. If Γ is a transversal to J^+, then J^+_Γ is not a rectifiable curve.*

Examples of automorphisms such that J^+_Γ is locally a curve include perturbations of 1-dimensional hyperbolic polynomials with quasi-circle Julia sets.

Corollary 3.2. *If Γ is a transversal to J^+ such that J^+_Γ is a Jordan arc, then its 1-dimensional Hausdorff measure is infinite.*

Proof of the corollary. A Jordan arc has finite 1-dimensional measure if and only if it is rectifiable (see e.g. [26, Lem. 3.2]). □

This generalizes some results of Hamilton [35] in 1-dimensional dynamics to plane polynomial automorphisms. In dimension 1, a deeper result states that if a polynomial Julia set is not totally disconnected, then either it is smooth (a circle or an interval) or its Hausdorff dimension is greater than 1. In the connected case, this follows from Zdunik's results in [53] together with Makarov's celebrated theorem [44] on the dimension of plane harmonic measure; the general case is established by Przytycki and Zdunik in [47]. In our setting, one may expect that such an alternative holds for unstable slices (except that, as we saw, smooth examples do not exist).

3.1. An extension of Proposition 2.2.

Recall that a subset $F \subset \mathbf{R}^2$ has a tangent at x if for every $\theta > 0$ there exists $r > 0$ such that $F \cap B(x, r)$ is contained in a (two-sided) angular sector of width θ. This notion is invariant under diffeomorphisms so it makes sense on any real surface. The first step is a generalization of Proposition 2.2 (see also Remark 2.3).

Proposition 3.3. *Let $f \in \mathsf{Aut}(\mathbf{C}^2)$ be a loxodromic automorphism. Let Γ be a transversal to J^+. If J^+_Γ admits a tangent on a set of positive measure for $T^+ \wedge [\Gamma]$, then f is unstably real.*

Before starting the proof, let us fix some vocabulary. Since the canonical measure $\mu = \mu_f = T^+ \wedge T^-$ is hyperbolic, by Pesin's theory, μ-almost every point admits a stable and an unstable manifold, biholomorphic to $\mathbf{C}$ (see e.g. [6] for details). We choose a measurable family of unstable parametrizations $\psi^u_x : \mathbf{C} \to W^u(x)$, normalized by

$$\psi^u_x(0) = x \quad \text{and} \quad \big\|(\psi^u_x)'(0)\big\| = 1. \tag{3.1}$$

In these parametrizations, f acts linearly. We say that $W^u(x)$ *has size* r *at* x if it is a graph of slope at most 1 over the disk of radius r in $E^u(x)$ relative to the orthogonal projection π from $\mathbf{C}^2$ to $E^u(x)$. Then the Koebe distortion theorem gives universal bounds on the distortion of $\pi \circ \psi^u_x$ on any disk of radius $< r/4$ (see [13, Lem. 3.7]). We denote by $W^u_r(x)$ the connected component of $W^u(x) \cap B(x, r)$ containing x, and by $e^u(x)$ a unit vector tangent to $W^u(x)$ at x.

Proof.
Step 1: an intermediate case.– In a first stage, we assume that for some μ-generic x (that is, satisfying a finite number of full measure conditions to be made clear below), $(\psi^u_x)^{-1}\big(J^+_{W^u(x)}\big)$ has a tangent at x. We claim that f is unstably real.

Indeed, set $I(x) = (\psi^u_x)^{-1}\big(J^+_{W^u(x)}\big)$, fix some $\rho > 0$, and let $A(x) \subset \mathbf{N}$ be the set of integers n such that $W^u(f^n(x))$ has size at least ρ at $f^n(x)$. For generic x and sufficiently small ρ, the lower density of $A(x)$ in $\mathbf{N}$ is arbitrary close to 1. Fix such a ρ, and assume furthermore that $\|df^n_x(e^u(x))\|$ tends to infinity, which again is a generic property.

Viewed in the unstable parametrizations, f^n acts linearly; so, by the normalization (3.1), it maps a disk of radius r in $(\psi^u_x)^{-1}(W^u(x))$ to a disk of radius $\|df^n_x(e^u(x))\| r$ in $(\psi^u_{f^n(x)})^{-1}(W^u_{f^n(x)})$. Thus, if $I(x)$ has a tangent at x, it follows that for every $\theta > 0$ and large enough n, $I(f^n(x)) \cap \mathbb{D}(0;\rho)$ is contained in an angular sector S_n of width θ.

Since $A(x)$ has positive density, we may fix a subsequence n_j in $A(x)$ such that $f^{n_j}(x)$ converges to a Pesin generic point x_0, whose unstable manifold has size $4\rho'$, with $\rho' < \rho/4$ and $W^u_{\rho'}(f^{n_j}(x))$ converges in the C^1 sense to $W^u_{\rho'}(x_0)$. Choose local coordinates $(x, y) \in \mathbb{D}(0, \rho')^2$ such that $W^u_{2\rho'}(x_0) \cap \mathbb{D}(0, \rho')^2 = \{y = 0\}$ and $W^u_{2\rho'}(f^{n_j}(x)) \cap \mathbb{D}(0, \rho')^2$ is a graph over the first coordinate, and denote by $\pi\colon (x, y) \mapsto x$ the first projection. As explained above, since $n_j \in A(x)$, by the Koebe distortion theorem, $(\pi \circ \psi^u_{f^{n_j}(x)})$ is a sequence of univalent mappings in some fixed disk, say $\mathbb{D}(0, \rho'/10)$. Upon extraction we may assume that it converges to some univalent holomorphic map $\gamma : \mathbb{D}(0, \rho'/10) \to \mathbb{D}(0, \rho')$, and extracting further if necessary, the images of sectors $\pi \circ \psi^u_{f^{n_j}(x)}(S_{n_j})$ converge to the analytic curve $\gamma(e^{i\theta}\mathbf{R})$. Exactly as in Proposition 2.2, the lower semicontinuity of $\Gamma \mapsto J^+_\Gamma$ implies that this limit curve is unique: if not, we would get at point of $J^+_{\{y=0\}}$ outside $\gamma(e^{i\theta}\mathbf{R})$, which would prevent $J^+_{W^u_{\rho'/10}(f^{n_j}(x))}$ to be contained in an arbitrary small angular sector for large j. Hence, $J^+_{W^u_{\rho'/10}(x_0)}$ is contained in a smooth curve, and since $W^u_{\rho'/10}(x_0)$ is a transversal to J^+ we conclude from Proposition 2.2 that f is unstably real.

For further reference, let us record a mild generalization of what we just proved, where we incorporate an additional inclination lemma argument.

Lemma 3.4. *Let ν be a hyperbolic, ergodic, f-invariant measure. There exists a set E of full measure made of Pesin regular points, with the following property. Let Γ be a transversal to J^+, let x be an element of E , and assume that for some sequence (n_j), $f^{n_j}(x)$ converges to* *a point $y \in E$. If J^+_Γ admits a tangent at a transverse intersection point $t \in W^s(x) \cap \Gamma$, then $(\psi^u_y)^{-1}\big(J^+_{W^u(y)}\big)$ is a line.*

Step 2: conclusion.– We first state a lemma which will be proven afterwards.

Lemma 3.5. *Let Γ be any transversal to J^+. For every subset A of full μ_f-measure there exists $E \subset \Gamma$ of full $(T^+ \wedge [\Gamma])$-measure such that if $t \in E$, there exists $x \in A$ such that*

- *$t \in W^s(x)$,*
- *$W^s(x)$ intersects Γ transversally at t.*

Thus, if J^+_Γ has a tangent at t for t in a set of positive $(T^+ \wedge [\Gamma])$-measure, applying Birkhoff's theorem together with Lemma 3.4, we obtain that $(\psi^u_y)^{-1}\big(J^+_{W^u(y)}\big)$ has a tangent at y for y in a set of total μ_f-measure. The conclusion then follows from the first step. □

Proof of Lemma 3.5. This is identical to [23, Lem. 3.3], except that the algebraic curve C in that lemma needs to be replaced by the transversal Γ. To adapt the proof we make use of [8, Thm 3]: since $\int T^+ \wedge [\Gamma] > 0$ and, reducing Γ slightly if necessary, the trace measure σ_{T^+} gives no mass to $\partial\Gamma$, the sequence of currents $d^{-n} f^n_\star[\Gamma]$ converges to some positive multiple of T^-. □

3.2. **Proof of Theorem 3.1.** Assume that for some transversal disk Γ, J^+_Γ is a rectifiable curve. We will show that f is unstably linear, which is impossible by Theorem 2.7.

Let $\Omega \subset \mathbf{C}$ be a domain. In what follows, we denote by $\omega(x, A, \Omega)$ the harmonic measure of a subset $A \subset \partial\Omega$ viewed from x in Ω. In all the cases considered below, Ω will be a Jordan domain, and $\omega(x, A, \Omega)$ is the value at x of the solution of the Dirichlet problem in Ω with boundary value $\mathbf{1}_A$. In other words, if φ is a continuous function on $\partial\Omega$ and u_φ is its harmonic extension to Ω, then $\int \varphi(y)\omega(x, dy, \Omega) = u_\varphi(x)$ (see [32] for a detailed account). If $\phi : \Omega \to \phi(\Omega)$ is a biholomorphism that extends to a homeomorphism $\overline{\Omega} \to \overline{\phi(\Omega)}$, then $\omega(x, A, \Omega) = \omega(\phi(x), \phi(A), \phi(\Omega))$ (conformal invariance).

Lemma 3.6. *For any $x_0 \in \Gamma$ such that $G^+(x_0) > 0$, there exists an open subset Ω of Γ containing x_0 whose boundary contains a relatively open subset of the Jordan curve J^+_Γ, an open subset U of $\overline{\Omega}$ intersecting J^+_Γ and a positive constant c such that $\omega(x_0, \cdot, \Omega)|_{U \cap J^+_\Gamma} \leqslant c\Delta G^+|_U$.*

We provide below a proof of this lemma, which is certainly known to some experts.

Let us complete the proof of the theorem. Since J^+_Γ is a rectifiable curve, a classical theorem of F. and M. Riesz asserts that the harmonic measure is mutually absolutely continuous with the 1-dimensional Hausdorff measure $\mathcal{H}^1$ on J^+_Γ (see [32, Thm VI.1.2]). In addition, J^+_Γ admits a tangent at $\mathcal{H}^1$-almost every point. Thus, from Lemma 3.6 and Proposition 3.3, we infer that for every saddle or Pesin generic point p, $(\psi^u_p)^{-1}(J^+_{W^u(p)})$ is contained in a line. Furthermore, $(\psi^u_p)^{-1}\big(J^+_{W^u(p)}\big)$ has no compact component, for such a compact component would give rise to small components of J^+_Γ under stable holonomy (see [19, Prop. 1.8]). Therefore we conclude that $(\psi^u_p)^{-1}\big(J^+_{W^u(p)}\big)$ is a line, as asserted.

3.3. **Proof of Lemma 3.6.** To explain the idea, suppose for a moment that Γ is a piece of unstable manifold of some saddle periodic point p. Since J^+_Γ is a rectifiable curve, $(\psi^u_p)^{-1}\big(J^+_{W^u(p)}\big)$ is a line that cuts $\mathbf{C}$ in two half-planes and it follows from Lemma 2.4 that $\Delta(G^+ \circ \psi^u_p)$ is proportional to the Lebesgue measure on the line $(\psi^u_p)^{-1}\big(J^+_{W^u(p)}\big)$. Now, if Ω is a domain with piecewise smooth boundary intersecting $(\psi^u_p)^{-1}\big(J^+_{W^u(p)}\big)$ along a segment, then the harmonic measure has a smooth density along this segment (see [32, §II.4], and the result follows.

For a general transversal Γ we need a local and rectifiable version of this picture. Let $\Omega_0 \subset \Gamma$ be a smoothly bounded simply connected domain containing x_0 such that $\Omega_0 \cap J^+_\Gamma$ is an arc separating Ω_0 into two components; such a set exists because J^+_Γ is rectifiable. Let Ω be the component containing x_0. Consider a Riemann uniformization $\phi : \mathbb{H} \to \Omega$ that maps ∞ outside J^+_Γ. Since J^+_Γ is rectifiable, ϕ extends continuously to the boundary, and $\phi^{-1}(J^+_\Gamma \cap \partial\Omega) =: I$ is a segment of the real axis. Set $\tilde G = G^+ \circ \phi$. Since $\tilde G = 0$ on I, by Schwarz reflexion, $\tilde G$ extends to a harmonic function across the interior $\mathrm{Int}(I)$ of I; the extension is positive on one side and negative on the other side. Let $\zeta \in \mathrm{Int}(I)$ be a point at which the differential of $\tilde G$ does not vanish. In some neighborhood $\tilde U$ of ζ, the level sets $\big\{\tilde G = \varepsilon\big\} \cap \tilde U$, with $\epsilon > 0$, are graphs over the real axis, converging in the C^1 topology to $\tilde U \cap I$ as $\varepsilon \to 0$. We then shrink Ω as in Figure 1 to insure that it coincides with $\phi(\tilde U)$ near $\phi(\zeta)$.

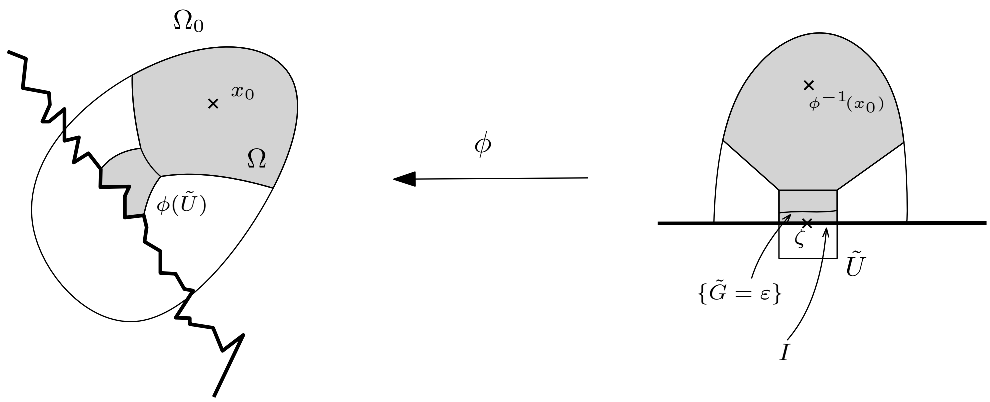

FIGURE 1. Sets appearing in the proof of Lemma 3.6 (Ω is the shaded region).

Then, for small ε, the harmonic measure of $\big\{\tilde G = \varepsilon\big\} \cap \tilde U$ viewed from $\phi^{-1}(x_0)$ is equivalent to the arclength measure on $\big\{\tilde G = \varepsilon\big\} \cap \tilde U$, and the implied constants are uniform. Likewise, $\Delta(\max(\tilde G, \varepsilon))$ is equivalent to the arclength measure on $\big\{\tilde G = \varepsilon\big\} \cap \tilde U$, with uniform constants: indeed since $\tilde G$ is locally the imaginary part of a univalent holomorphic function, there is a holomorphic coordinate $z = x + iy$ in which $\max(\tilde G, \varepsilon)$ becomes $\max(y, \varepsilon)$; thus, its Laplacian is a multiple of the Lebesgue measure on $\{y = \varepsilon\}$ and when ε goes to 0, these measures converge to a measure supported on $\tilde U \cap I$, which is equivalent to Lebesgue measure, with uniform constants (see [32, Cor. 4.7 of Chapter II] for similar computations).

Now we transport this to Γ by ϕ. The conformal invariance of the Laplacian and the harmonic measure shows that $\Delta(\max(G^+, \varepsilon))|_{\phi(\tilde U \cap \mathbb{H})}$ converges to a measure equivalent to the harmonic measure $\omega(x_0, \cdot, \Omega)$ on a piece of J^+_Γ (namely $J^+_\Gamma \cap \overline{\phi(\tilde{U})}$). On the other hand this limit is dominated by ΔG^+ (the domination may be strict because we are considering only one side of the curve and G^+ could be positive on the other side as well), and the result follows. □

4. Invariant holomorphic foliations

Let $\mathcal{F}$ be a foliation with complex leaves in some open set $U \subset \mathbf{C}^2$. We allow $\mathcal{F}$ to have isolated singularities (see below). By definition, the leaf $\mathcal{F}(q)$ of $\mathcal{F}$ through a point $q \in U$ is just reduced to $\{q\}$ if q is a singularity, and otherwise it is the leaf of the regular folitation defined by $\mathcal{F}$ in the complement of its singularities. We say that a set $S \subset \mathbf{C}^2$ is *subordinate to* $\mathcal{F}$ in U if $S \cap U$ is not empty and is *$\mathcal{F}$-saturated*, that is, for every $q \in S$ the leaf $\mathcal{F}(q)$ of $\mathcal{F}$ through q is also contained in S.

In this section we address the following question: is it possible that J^+ be locally (resp. globally) subordinate to a holomorphic foliation? The case of real analytic foliations will be considered in the next section.

4.1. **Preliminary observations.** Let us first gather a few general facts.

Proposition 4.1. *Let $\mathcal{F}$ be a continuous, regular foliation with complex leaves in some connected open set U. The following properties are equivalent:*

(a) *the forward Julia set J^+ is subordinate to $\mathcal{F}$ in U;*
(b) *for some saddle periodic point p, every holomorphic disk contained in $W^s(p) \cap U$ is contained in a leaf of $\mathcal{F}$.*

If these properties hold, then:

(1) *in U, each of the three sets in the partition $\mathrm{Int}(K^+) \cup J^+ \cup \mathbf{C}^2 \backslash K^+$ is $\mathcal{F}$-saturated;*
(2) *$\mathcal{F}$ is f-invariant in the following sense: if $x \in J^+ \cap U$ and some iterate f^m of f maps x to another point $x' = f^m(x)$ of U, then f^m maps the leaf $\mathcal{F}(x)$ to the leaf $\mathcal{F}(x')$ (as germs of Riemann surfaces).*

Proof. Assume that J^+ is subordinate to $\mathcal{F}$ in U. Since $W^s(p)$ is dense in J^+, it intersects U. Let $L \subset W^s(p) \cap U$ be a holomorphic disk. Then L is contained in a leaf of $\mathcal{F}$, since otherwise its $\mathcal{F}$-saturation

$$\mathcal{F}(L) = \bigcup_{q \in L} \mathcal{F}(q) \tag{4.1}$$

would have non-empty interior and be contained in J^+, a contradiction. Conversely, assume that $W^s(p) \cap U$ is subordinate to $\mathcal{F}$. Let q be a point of $J^+ \cap U$ and let U' be a flow box for $\mathcal{F}$ around q. By density of $W^s(p)$, the plaque L through q is approximated in the C^1 topology by plaques contained in $W^s(p)$. Since J^+ is closed, L is contained in J^+, as asserted.

For Assertion (1), we observe that if a leaf F in U intersects $\mathrm{Int}(K^+)$ then it cannot meet $\mathbf{C}^2 \backslash K^+$, otherwise it would intersect J^+, and be entirely contained in it. For Assertion (2), note that, if $f^m(\mathcal{F}(x))$ were not contained in a leaf, then its saturation, as in Equation (4.1), would produce an open subset of J^+. □

Proposition 4.2. *Assume that, in the connected open set U, J^+ is subordinate to a continuous foliation $\mathcal{F}$ by complex leaves. Then,*

(1) *the current T^+ is uniformly laminar in U, the laminar structure of T^+ is subordinate to $\mathcal{F}$, and $T^+|_U$ is induced by a transverse measure on $\mathcal{F}$;*
(2) *if $\Gamma \subset U$ is a disk that is transverse to $\mathcal{F}$, the transverse measure is given by*
$$T^+ \wedge [\Gamma] = \Delta(G^+|_\Gamma);$$
(3) *if Γ' is another disk transverse to $\mathcal{F}$,* the $\mathcal{F}$-holonomy $h\colon \Gamma \to \Gamma'$ maps (on its domain of definition) the measure $T^+ \wedge [\Gamma]$ to the measure $T^+ \wedge [\Gamma']$.

Proof. This is a version of [6, §8-9] in a simpler context. Shrinking U, we may assume that U is biholomorphic to $\mathbb{D}^2$, with coordinates (z, w), and for $U' \subset U$ intersecting J^+, every leaf of $\mathcal{F}$ intersecting U' is a vertical disk in U. We already know that the current T^+ is laminar (in a weak sense) and its support equals J^+. Moreover, by construction (see [6]), the disks constituting its laminar structure are limits of (pieces of) stable manifolds. Since these stable pieces are leaves of $\mathcal{F}$, the asserted properties follow. □

4.2. **No invariant global holomorphic foliation.** In this subsection, we prove the holomorphic version of Theorem B, which is an extension of Brunella's theorem [14, Corollary] to arbitrary (transcendental) foliations.

Theorem 4.3. *A loxodromic automorphism $f \in \mathsf{Aut}(\mathbf{C}^2)$ does not preserve any global singular holomorphic foliation.*

We start with a lemma which is a variation on Proposition 4.1, Assertion (2), when $\mathcal{F}$ is a global holomorphic foliation.

Lemma 4.4. *Let $\mathcal{F}$ be a global, possibly singular, holomorphic foliation of $\mathbf{C}^2$. If J^+ is subordinate to $\mathcal{F}$ in some open subset, then $\mathcal{F}$ is f-invariant. Conversely, if $\mathcal{F}$ is f-invariant, then J^+ or J^- is subordinate to $\mathcal{F}$.*

Proof. Since there are infinitely many saddle periodic points in the compact set $J^\star$ and the singularities of $\mathcal{F}$ are isolated, there is a saddle periodic point p that is not a singularity of $\mathcal{F}$. Let n be its period. As observed in Proposition 4.1, the stable manifold $W^s(p)$ is a leaf of $\mathcal{F}$, and so is the stable manifold $W^s(f(p))$. Thus, $W^s(p)$ is a common leaf of the foliations $f^*\mathcal{F}$ and $\mathcal{F}$. Now, observe that $W^s(p)$ is dense for the complex analytic Zariski topology: this follows from the fact that $J^+ = \overline{W^s(p)}$ carries a unique closed positive current which gives no mass to curves (see [30]). So we deduce that $f^*\mathcal{F} = \mathcal{F}$.

For the converse, since $\mathcal{F}$ is f-invariant, $\mathcal{F}(p)$ is f^n-invariant. But then, $\mathcal{F}(p)$ must coincide with $W^s(p)$ or $W^u(p)$ near p, and by density of $W^s(p)$ in J^+ (resp. of $W^u(p)$ in J^-), we conclude that J^+ or J^- is subordinate to $\mathcal{F}$. □

Remark 4.5. In Theorem 5.6 below, we prove that J^+ is dense for the real-analytic topology. It follows that the previous result holds for real-analytic foliations with complex leaves (and possibly a locally finite singular set).

The key step in the proof is to compare the alleged invariant foliation with the canonical f-invariant holomorphic foliation $\mathcal{F}^+$ of $\mathbf{C}^2\backslash K^+$, given near infinity by the level sets of the Böttcher function φ^+ (see [36, §5]). Recall that φ^+ is defined near infinity in $\mathbf{C}^2\backslash K^+$ by

$$\varphi^+ = \lim_{n\to\infty} (f^n)^{1/d^n} \tag{4.2}$$

for some appropriate branch of the $(d^n)^{\mathrm{th}}$ root; then, by construction, $G^+ = \log|\varphi^+|$ there.

Lemma 4.6. *Let $f \in \mathsf{Aut}(\mathbf{C}^2)$ be a loxodromic automorphism. If J^+ is subordinate to a holomorphic foliation $\mathcal{F}$ in some connected open set U then $\mathcal{F}$ coincides with $\mathcal{F}^+$ in $U\backslash K^+$.*

Proof. In $\mathbf{C}^2\backslash K^+$, G^+ is positive and pluriharmonic, and each level set $\{G^+ = c\}$, $c > 0$, admits a unique foliation by Riemann surfaces, given by $\mathcal{F}^+$; more precisely, if x belongs to $\{G^+ = c\}$ then the local leaf of $\mathcal{F}^+$ through x is the unique local Riemann surface in $\{G^+ = c\}$ containing x.

By analytic continuation it is enough to prove the result locally near J^+. Thus, shrinking U as in Proposition 4.2 we may assume that $U \simeq \mathbb{D}^2$, with coordinates (z, w), the leaves of $\mathcal{F}$ are the disks $\{z = \mathrm{C}^{\mathrm{st}}\}$, and Γ is a horizontal disk. The current T^+ is uniformly laminar in U, and its transverse measure is given by $\Delta(G^+|_{\Gamma'})$ for any horizontal disk Γ' of $\mathbb{D}^2$. Fix such a disk Γ' and let $h : \Gamma \to \Gamma'$ be the holonomy along $\mathcal{F}$. Then $h_*(T^+ \wedge [\Gamma]) = T^+ \wedge [\Gamma']$ and h is holomorphic, thus $H_{\Gamma'} := G^+|_\Gamma - G^+|_{\Gamma'} \circ h$ is a harmonic function on Γ. Since J^+ is subordinate to $\mathcal{F}$, $H_{\Gamma'}$ vanishes on the infinite set $J^+ \cap \Gamma$, so either $H_{\Gamma'}$ vanishes identically on the disk Γ or $\{H_{\Gamma'} = 0\}$ is a real analytic curve containing $J^+ \cap \Gamma$.

If $H_{\Gamma'}$ vanishes identically for all Γ' (or equivalently, for a dense set of horizontals Γ'), then G^+ is invariant under holonomy, $\mathcal{F}$ coincides with $\mathcal{F}^+$, and we are done.

Thus, we can assume that $\{H_{\Gamma'} = 0\}$ is a curve for a non-empty and open set of horizontal disks Γ'. The intersection of these curves as Γ' varies determines a real analytic curve C in Γ (it can not be reduced to a finite set because $J^+ \cap \Gamma$ is infinite). If this curve contains a point x where $G^+(x) > 0$, then by definition of C, the level set $\{G^+ = G^+(x)\}$ contains the leaf of $\mathcal{F}$ through x, so this leaf must coincide with the leaf of $\mathcal{F}^+$. Changing x in nearby points $x' \in C$ we see that $\mathcal{F}$ and $\mathcal{F}^+$ have infinitely many leaves in common. Thus, $\mathcal{F}$ coincides locally with $\mathcal{F}^+$ in this case. Otherwise, $G^+ \equiv 0$ on the curve C and we must have $J^+_\Gamma = J^+ \cap \Gamma = K^+ \cap \Gamma = C$. Indeed since $\mathcal{F}$ preserves $\mathrm{Int}(K^+)$ (Proposition 4.1), $\Gamma \cap \mathrm{Int}(K^+) = \varnothing$, for otherwise C would contain an open set. So $J^+ \cap \Gamma = K^+ \cap \Gamma$. Then, since J^+ is locally foliated, $J^+ \cap \Gamma$ must have empty interior, hence $J^+_\Gamma = J^+ \cap \Gamma$ (see § 2.1). Thus, f is unstably linear, which is impossible by Theorem 2.7. □

Proof of Theorem 4.3. Denote by $\mathcal{F}$ such a hypothetical invariant foliation. By Lemma 4.4, replacing f by f^{-1} if needed, J^+ is subordinate to $\mathcal{F}$. Hence by Lemma 4.6, $\mathcal{F}$ coincides with $\mathcal{F}^+$ outside K^+. On the other hand, by [19, Lem. 3.5], $\mathcal{F}^+$ extends to a singular foliation of $\mathbb{P}^2$ by adding the line at infinity as a leaf. Thus, $\mathcal{F}$ extends to a global singular holomorphic foliation of $\mathbb{P}^2$. Such a foliation is automatically defined by a global algebraic 1-form, hence this contradicts Brunella's theorem and we are done. □

4.3. **Local subordination.** For the question of *local subordination* of $J^\pm$ to a holomorphic foliation, we only have partial answers.

Proposition 4.7. *Let f be a loxodromic automorphism such that J^+ is subordinate to a holomorphic foliation in some open set U. Then f is unstably connected. In particular J is connected and $|\mathrm{Jac}(f)| \leqslant 1$.*

Corollary 4.8. *If $|\mathrm{Jac}(f)| < 1$, J^- is not locally subordinate to a holomorphic foliation.*

Proof of the corollary. Indeed, by [9], a dissipative automorphism is not stably connected. □

Proof of Proposition 4.7. By Lemma 4.6, $\mathcal{F}$ coincides with $\mathcal{F}^+$ outside K^+. Consider a flow box of $\mathcal{F}$ intersecting J^+, together with a holomorphic transversal disk Γ to this flow box.

We claim that $G^+|_\Gamma$ has no critical points in $\Gamma \cap (\mathbf{C}^2 \backslash K^+)$. Indeed, for every $x \in \Gamma \backslash K^+$, some iterate $f^n(x)$ belongs to the region V^+ where φ^+ is well-defined and defines a non-singular fibration. Since $\mathcal{F}^+$ is transverse to Γ, there is a neighborhood N of x in Γ such that $\varphi^+|_{f^n(N)}$ is univalent, hence $G^+|_{f^n(N)}$ has no critical points, and our claim follows by invariance.

We now rely on the results of [9]. Lemma 4.9 below implies that f is unstably connected. The remaining conclusions respectively follow from [9, Thm 5.1] and the fact that a volume expanding automorphism is never unstably connected, as follows from Theorem 7.3 and Corollary 7.4 in [9]. □

Lemma 4.9. *If Γ is a transversal to J^+ such that $G^+|_\Gamma$ has no critical points, then f is unstably connected.*

Proof. This is a mild generalization of some results of [9]. Assume by way of contradiction that f is unstably disconnected. Fix a transverse intersection point of $W^s(p)$ and Γ. Iterating Γ forward and applying the inclination lemma and the continuity of G^+, as in the proof of [19, Prop. 1.8], we construct arbitrary many compact components of $K^+ \cap \Gamma$. None of these components is isolated. Indeed, otherwise f would be transversely connected in the sense of [19, Def. 1.5] hence unstably connected, by [19, Prop. 1.8]. But then, we obtain a contradiction with Lemma 7.2 in [9] ([4]). □

Proposition 4.10. *Let f be a loxodromic automorphism such that $|\mathrm{Jac}(f)| < 1$ and $\mathrm{Int}(K^+)$ is either empty or a union of sink and parabolic basins. Then J^+ cannot be subordinate to a holomorphic foliation in a neighborhood of $J^\star$.*

Proof. The closing lemma of [21] (see Corollary 1.2 there) shows that the orbit of every point of J^+ accumulates $J^\star$. So, by pulling back, $\mathcal{F}$ extends to a holomorphic foliation on a neighborhood of J^+. (Note that if $x \in J^+$ and n_1, n_2 are such that $f^{n_i}(x)$ are close to $J^\star$, then the local foliations obtained by pulling back under f^{n_1} and f^{n_2} coincide, because they coincide on $J^\star$.) Thus by Lemma 4.6, one can extend $\mathcal{F}$ holomorphically by $\mathcal{F}^+$ to $\mathbf{C}^2 \backslash K^+$.

Let now Ω be a component of $\mathrm{Int}(K^+)$. A first possibility is that Ω is the basin of some attracting periodic point a. Since the domain of definition of $\mathcal{F}$ contains a neighborhood of $\partial\Omega$, by pulling back we see that $\mathcal{F}$ extends to an invariant holomorphic foliation of $\Omega \backslash \{a\}$, hence by

[4]This lemma is stated for an unstable manifold, but the proof evidently works in any transversal.

the Hartogs' extension theorem $\mathcal{F}$ extends to a singular holomorphic foliation of Ω.([5]) If Ω is the basin of a semi-parabolic periodic point q, since $p \in J^\star$, $\mathcal{F}$ is well defined in some neighborhood of q, and again pulling back extends $\mathcal{F}$ to Ω. Altogether, we have constructed an extension of $\mathcal{F}$ to $\mathbf{C}^2$, and we conclude by Theorem 4.3. □

Remark 4.11. The only known examples for which J^+ is locally laminated by stable manifolds near $J^\star$ are hyperbolic maps or maps with a dominated splitting on $J^\star$. For hyperbolic maps the assumption on $\mathrm{Int}(K^+)$ holds by [7]. For maps with a dominated splitting on $J^\star$ (with the additional hypothesis that $|\mathrm{Jac}(f)| < 1/(\deg(f))^2$), it follows from the structure theorem of Lyubich-Peters [42]. See [1, §2] for sufficient conditions for hyperbolicity based on the laminarity properties of $J^\pm$. Thus it is very plausible that the assumption that $\mathrm{Int}(K^+)$ is a union of sink and parabolic basins is superfluous in Proposition 4.10.

5. Real-analytic foliations

In this section we study the situation where J^+ is locally or globally subordinate to a real-analytic foliation with complex leaves.

5.1. **Proof of Theorem B.** Recall the statement of Theorem B:

Theorem 5.1. *Let f be a loxodromic automorphism of $\mathbf{C}^2$. Then J^+ is not subordinate to a global real-analytic foliation (with a possibly locally finite singular set).*

The proofs goes by showing that J^+ is locally subordinate to a holomorphic foliation near $J^\star$, which requires separate arguments in the unstably real and non unstably real cases. Altogether, the theorem is a consequence of Theorem 4.3 Corollary 5.3 and Proposition 5.5. The argument also implies that Proposition 4.10 holds in the real-analytic case.

Proposition 5.2. *Let f be a loxodromic automorphism of $\mathbf{C}^2$. Assume that*

(i) *f is not unstably real;*
(ii) *in some open set U intersecting $J^\star$, J^+ is subordinate to a real analytic foliation $\mathcal{F}$ with complex leaves.*

Then $\mathcal{F}$ is holomorphic and f is unstably connected.

Corollary 5.3. *If f is a loxodromic automorphism which is not unstably real, then J^+ cannot be subordinate to a global real analytic foliation with complex leaves.*

Proof of the corollary. By Proposition 5.2, $\mathcal{F}$ is holomorphic in a neighborhood of $J^\star$. This property propagates to $\mathbf{C}^2$ by real analyticity, and Theorem 4.3 finishes the proof. □

The proof of Proposition 5.2 relies on a variation on a well-known result of Ghys [33]. The essence of Ghys' argument is contained in the following lemma.

[5]More precisely, on a small neighborhood of a, the foliation is defined by a line field $z \mapsto L(z)$. Denoting by $\ell(z)$ the slope of $L(z)$, we obtain a meromorphic function ℓ on a small neighborhood of a. By Levi's extension theorem [46, p. 133], this function extends to a meromorphic function on some neighborhood of a.

Lemma 5.4. *Let f be a loxodromic automorphism. Assume that in some open set U intersecting $J^\star$, J^+ is subordinate to an f-invariant C^1 foliation $\mathcal{F}$ with complex leaves. Let p be a saddle periodic point in U and L be a leaf of $\mathcal{F}$ contained in $W^s(p)$. Let x, x' be two points of L, τ and τ' be local holomorphic transversals to L, and $h_{\tau,\tau'} : (\tau, x) \to (\tau', x')$ the germ of holonomy along $\mathcal{F}$. Then $h_{\tau,\tau'}$ is holomorphic at x.*

Recall that the notion of invariance for a local foliation was explained in Proposition 4.1, Assertion (2).

Proof of the lemma. Let k be the period of p. Fix a path γ in $L \cap U$ joining x and x'. For large enough $n \in k\mathbf{Z}$, $f^n(\gamma)$ is contained in U and close to p. By the inclination lemma, there are two disks $V_n \subset \tau$ and $V'_n \subset \tau'$, containing respectively x and x', such that $f^n(V_n)$ and $f^n(V'_n)$ are close to $W^u_{\rm loc}(p)$. Then the holonomy between $f^n(V_n)$ and $f^n(V'_n)$ is a C^1 quasiconformal map whose distortion is bounded by ε_n, with $\varepsilon_n \to 0$. Since f, τ and τ' are holomorphic, it follows that the quasiconformal distortion of $h_{\tau,\tau'}$ in V_n is bounded by ε_n, so by letting n tend to infinity we conclude that the distortion at x vanishes, that is, $h_{\tau,\tau'}$ is holomorphic at x, as was to be shown. □

Proof of Proposition 5.2. By Proposition 4.1, if x and x' belong to $J^+ \cap U$ and f^n maps x to x', then it must locally send $\mathcal{F}(x)$ to $\mathcal{F}(x')$. Let τ be some transversal to $\mathcal{F}$ at x'. Since f is not unstably real, Proposition 2.2, shows that the local $\mathcal{F}$ saturation of J^+_τ is $\mathbf{R}$-Zariski dense. Therefore we infer that $\mathcal{F}$ is invariant (this is a real analytic version of Lemma 4.4).

To prove that $\mathcal{F}$ is holomorphic, we fix two holomorphic transversals τ, τ' in some flow box of $\mathcal{F}$; we have to show that the holonomy $h_{\tau,\tau'}$ between these transversals is holomorphic. Fix a saddle point $p \in J^\star \cap U$. Then $W^s(p) \cap U$ is dense in $J^+ \cap U$, so $W^s(p) \cap \tau$ is dense in J^+_τ. By Lemma 5.4, $h_{\tau,\tau'}$ is holomorphic at every such point. Since in addition J^+_τ is $\mathbf{R}$-Zariski dense in τ, and $h_{\tau,\tau'}$ is real analytic, this entails that $h_{\tau,\tau'}$ is holomorphic everywhere. Thus, $\mathcal{F}$ is holomorphic, and the unstable connectedness follows from Proposition 4.7. □

In the unstably real case we readily get a contradiction.

Proposition 5.5. *If f be a loxodromic automorphism which is unstably real, then J^+ cannot be subordinate to a real analytic foliation $\mathcal{F}$ with complex leaves in an open set U intersecting $J^\star$.*

Proof. Let $\mathcal{F}$ be the alleged foliation and p be a saddle periodic point in U. Since f is unstably real, $J^+_{W^u(p)}$ is contained in a real-analytic curve, therefore, saturating by $\mathcal{F}$, we deduce that J^+ is contained in a real-analytic hypersurface Σ, which must be Levi-flat. A classical theorem of Cartan [18] asserts that the Levi foliation of a real analytic Levi flat hypersurface locally extends to a holomorphic foliation. Thus, in some neighborhood U of p, there is a holomorphic foliation $\mathcal{F}'$ extending $\mathcal{F}|_\Sigma$ (which may not coincide with $\mathcal{F}$), hence J^+ is subordinate to $\mathcal{F}'$ in U. Then, Proposition 4.7 implies that f is unstably connected, hence unstably linear, which is impossible by Theorem 2.7. □

5.2. **An application.** The idea underlying Proposition 5.5 leads to the following result in the spirit of [3].

Theorem 5.6. *Let $f \in \mathsf{Aut}(\mathbf{C}^2)$ be loxodromic. Then*

(1) *J^+ (resp. J^-) is not contained in a proper real-analytic subvariety;*
(2) *the dimension of the $\mathbf{R}$-Zariski closure of $J^\star$ cannot be equal to* 0*,* 1*, or* 3*.*

On the other hand, the $\mathbf{R}$-Zariski closure of $J^\star$ can be of dimension 2: this happens for mappings with $J^\star \subset \mathbf{R}^2$ (or equivalently $h_{\rm top}(f|_{R^2}) = h_{\rm top}(f)$) and their conjugates. Therefore the $\mathbf{R}$-Zariski closure of $J^\star$ has dimension 2 or 4; it is easy to show that if one saddle point has a non-real stable or unstable eigenvalue, then it cannot be of dimension 2 (see below Propositions 6.4 and 6.5 for stronger results in this spirit). Thus, the second statement of the theorem is an argument in favor of the implication "unstably real implies stably real" (see § 2.5).

Proof. Assume that J^+ is contained in a proper real-analytic subvariety V. Only finitely many components of V intersect $J^\star$, so these components are periodic and one of them has positive (not necessarily full) μ_f-measure[6]. Let V_1 be such a component. Since μ_f is hyperbolic, by Pesin theory, V_1 contains a flow box of positive measure $\mathcal{L}^s$ made of Pesin local stable manifolds. Since $\mathcal{L}^s$ contains infinitely many holomorphic disks, it is $\mathbf{R}$-Zariski dense in V_1. At every point $x \in \mathcal{L}^s$, since V_1 contains a holomorphic disk through x, the Levi form of V_1 vanishes. By Zariski density, we conclude that V_1 is Levi flat. In addition, the intersection of V_1 with a local stable manifold $W^s_{\rm loc}(x)$ is contained in an analytic curve, so f must be unstably real. Thus, Proposition 5.5 yields the desired contradiction.

Now assume that the $\mathbf{R}$-Zariski closure V of $J^\star$ has dimension 3, and fix a component V_1 of V of dimension 3. By Zariski density, there is a saddle point p together with an open subset $U \ni p$ such that $J^\star \cap U \subset \mathrm{Reg}(V_1)$. By the local f^n-invariance of V_1 at p, we see that either $W^s_{\rm loc}(p)$ or $W^u_{\rm loc}(p)$ is contained in V_1: indeed, if T_pV_1 does not contain the stable direction $E^s(p)$ (resp. $E^u_{\rm loc}(p)$), the Inclination Lemma shows that V_1 contains $W^u_{\rm loc}(p)$ (resp. $W^s_{\rm loc}(p)$). In this way we can find an $\mathbf{R}$-Zariski dense subset of V_1 made of local unstable (resp. stable) manifolds $W^u_{\rm loc}(p) \subset V_1$, and we complete the argument similarly as in the previous case.

To conclude, note that $J^\star$ is infinite, so the dimension of its $\mathbf{R}$-Zariski closure is $\geqslant 1$. If its Zariski closure were a curve, a component of it would be contained in the stable (or unstable) manifold of a saddle periodic point. But then, this stable manifold would contain more than one periodic point, a contradiction. □

6. HOMOCLINIC ORBITS AND THE ARGUMENTS OF PERIODIC POINT MULTIPLIERS

In this section we extend to two dimensions a technical result which has recently proven useful in certain multiplier rigidity results in dimension 1 (see [25, 38, 37]).

Theorem 6.1. *Let f be a holomorphic diffeomorphism on a complex surface. Assume that p is a saddle fixed point admitting a homoclinic intersection. Then there exists a sequence of saddle periodic points p_n of period n accumulating p, whose stable and unstable multipliers satisfy $\lambda^s(p_n) \sim c(\lambda^s(p))^n$ and $\lambda^u(p_n) \sim c'(\lambda^u(p))^n$ for some c, $c' \in \mathbf{C}^*$. Moreover, the sequence (p_n) can be chosen to be Zariski dense for the complex analytic topology*

[6]At this stage it could be that these components are strictly periodic, and intersect along a 2-dimensional subset which must be fixed because μ_f is mixing.

Recall that for a loxodromic automorphism of $\mathbf{C}^2$, every saddle periodic point generates (transverse) homoclinic intersections (see [6, Thm. 9.6]).

Proof.

Step 1.– Horseshoes. By [22, Prop. 3.2], there is a point τ of transverse intersection between $W^u(p)$ and $W^s(p)$. If one adds $\{p\}$ to the orbit of such a point one gets a compact hyperbolic set $\mathcal{O}(\tau) \cup \{p\}$; and for any small enough neighborhood V' of $\mathcal{O}(\tau) \cup \{p\}$, there exists a smaller neighborhood V such that the invariant set

$$\Lambda = \bigcap_{n \in \mathbf{Z}} f^{-n}(V) \tag{6.1}$$

is a hyperbolic set with positive entropy (see e.g. [48, §7.4.2]), which is locally maximal by construction. We shall refer to it as *a horseshoe* Λ *(adapted to* τ*).* Then for sufficiently large positive M, M', $\left\{f^{-M}(\tau), \ldots, f^{M'}(\tau)\right\}$ is an ε-pseudo-orbit, with $\varepsilon = O(|\lambda^u|^{-M} + |\lambda^s|^{M'})$, so by the Closing Lemma it is shadowed by a unique periodic orbit of length $M + M' + 1$ contained in Λ.

Step 2.– Shadowing. As in the proof of Proposition 2.2, we consider local holomorphic coordinates (x, y) near p in which $p = (0,0)$ and f is of the form

$$f(x,y) = (\lambda^u x(1 + xyg_1(x,y)), \lambda^s y(1 + xyg_2(x,y)))\,. \tag{2.3}$$

In these coordinates, the axes coincide with the local unstable and stable manifolds. Renormalizing if needed, we assume that these coordinates are well defined over the unit bidisk $\mathbb{D}^2$.

From Step 1, there is a point $\tau_0 = (x_0, 0) \in W^u_{\rm loc}(p)$, with $x_0 \neq 0$, a neighborhood U of x_0 in $\mathbf{C}$, and an integer $k \geqslant 1$ such that the piece of unstable manifold $f^k(U \times \{0\})$

- intersects $W^s_{\rm loc}(p) = \{x = 0\}$ transversally at $f^k(\tau_0) = (0, y_0) \in \mathbb{D}^2$, with $y_0 \neq 0$, and
- is a graph (relative to the first coordinate) over some neighborhood of the origin.

If we fix a small neighborhood V'' of $\left\{f(\tau_0), \ldots, f^{k-1}(\tau_0)\right\}$ and put $V' = \mathbb{D}^2 \cup V''$, the first step shows that, for every large enough N, there is a unique saddle periodic orbit, of length $n := 2N + k$, shadowing

$$f^{-N}(\tau_0), \ldots, \tau_0, \ldots, f^k(\tau_0), \ldots, f^{k+N-1}(\tau_0). \tag{6.2}$$

We label this periodic orbit by q_{-N}, …, q_0, …, q_{N-1}, …, q_{N+k-1} in such a way that q_{-N}, …, q_N are in $\mathbb{D}^2$, q_{-N} is close to $f^k(\tau_0)$, q_0 is close to the origin, and q_N is close to τ_0.The period of q_0 is equal to $2N + k$.

At the end of the proof, for a given large n, if n of the form $2N + k$ (that is $n \equiv k \bmod. 2$) we shall set, $p_n = q_0$ where q_0 is as above; otherwise, for n of the form $2N + k + 1$, the reader can reproduce the same proof with the pseudo-orbit $f^{-N+1}(\tau_0), \ldots, \tau_0, \ldots, f^{k+N-1}(\tau_0)$.

Step 3.– Estimating the multiplier. In the following we study the action of f on tangent vectors in $\mathbb{D}^2$ which are close to the horizontal direction. These vectors will be written as $v = \nu(1, m)$, where $\nu \in \mathbf{C}^*$ and $m \in \mathbf{C}$ is the complex slope. In particular,

$$df^k_{\tau_0}(1, 0) = \nu_0(1, m) \tag{6.3}$$

for some $\nu_0 \in \mathbf{C}^*$, where $(1, m)$ is tangent to $f^k(U \times \{0\})$ at $f^k(\tau_0)$. In other words, if we write $f^k = ((f^k)_1, (f^k)_2)$, ν_0 is given by the partial derivative[7]

$$\nu_0 = \frac{\partial (f^k)_1}{\partial x}(x_0, 0).$$

We will complete the proof of the theorem by establishing the following more precise lemma. The estimate for $\lambda^s(q_0)$ is obtained in the same way.

Lemma 6.2. *The unstable multiplier of* q_0 *satisfies*

$$\lambda^u(q_0) = \nu_0 (\lambda^u(p))^{2N} (1 + O(\theta^N))$$

for some $\theta < 1$.

Note that if instead of the pair $(\tau_0, f^k(\tau_0))$ we choose $(f^\ell(\tau_0), f^{k+\ell'}(\tau_0))$, then, as soon as these points remain in $\mathbb{D}^2$, the same result holds with ν_0 replaced by $\nu_0(\lambda^u(p))^{\ell'-\ell}$ and $(\lambda^u(p))^{2N}$ replaced by $(\lambda^u(p))^{2N+\ell-\ell'}$.

Proof. For $-N \leqslant j \leqslant N$, we single out the unit vector $e^u(q_j) = (1, m_j)$ as a complex tangent vector to the unstable direction of q_j, and write

$$df_{q_j}(e^u(q_j)) = \mu_j e^u(q_{j+1}) \quad \text{and} \quad df^k(e^u(q_N)) = \nu e^u(q_{-N}). \tag{6.4}$$

These quantities depend on N but we don't mark this dependence for notational simplicity. With these conventions, the unstable multiplier at q_0 is

$$\lambda^u(q_0) = \left(\prod_{j=-N}^{N-1} \mu_j \right) \times \nu. \tag{6.5}$$

In terms of partial derivatives, we have

$$\mu_j = \frac{\partial f_1}{\partial x}(q_j) + m_j \frac{\partial f_1}{\partial y}(q_j). \tag{6.6}$$

From Equation (2.3) we get

$$\frac{\partial f_1}{\partial x}(x, y) = \lambda^u(p) \left(1 + xyh_1(x, y)\right) \quad \text{and} \quad \frac{\partial f_1}{\partial y}(x, y) = \lambda^u(p) x^2 h_2(x, y), \tag{6.7}$$

where h_1 and h_2 are holomorphic. Thus,

$$\mu_j = \lambda^u(p) \left(1 + x_j y_j h_1(x_j, y_j) + x_j^2 m_j h_2(x_j, y_j)\right), \text{ where } q_j = (x_j, y_j). \tag{6.8}$$

Since q_0 admits $N - 1$ forward iterates and N backward iterates in $\mathbb{D}^2$, we obtain the estimates

$$|x_j| = O(\theta^{N-j}) \quad \text{and} \quad |y_j| = O(\theta^{N+j}) \tag{6.9}$$

for $-N \leqslant j \leqslant N$ and some $\theta < 1$ (any $\theta > \max(|\lambda^s(p)|, |(\lambda^u)^{-1}|)$ will do); in particular $|x_j y_j| = O(\theta^{2N})$. When $j \leqslant 0$ we also have $\left|x_j^2 m_j\right| = O(\theta^{2N})$.

It is known that on a compact hyperbolic set, the stable and unstable distributions are Hölder continuous (see [40, Thm 19.1.6]). Applying this result to the horseshoe Λ, we deduce that when

[7]This quantity is intrinsic in the sense that it takes the same value in any system of coordinates (x, y) in which f is of the form (2.3). This can be shown by a direct calculation, but it also follows a posteriori from Lemma 6.2.

$j \geqslant 0$ the slope m_j of the unstable direction at q_j satisfies $|m_j| = O(|y_j|^\alpha)$ for some $0 < \alpha \leqslant 1$ and from this we deduce that $\left|x_j^2 m_j\right| = O(\theta^{\alpha N})$. Finally, we obtain

$$\frac{\prod_{j=-N}^{N-1} \mu_j}{(\lambda^u(p))^{2N}} = \prod_{j=-N}^{N-1} (1 + O(\theta^{\alpha N})) = 1 + O(N\theta^{\alpha N}). \tag{6.10}$$

We conclude the proof by showing that ν is exponentially close to ν_0. Indeed, the precise form of the Anosov Closing Lemma given in [40, Cor. 6.4.17] shows that q_N is $O(\theta^N)$ close to τ_0. Since the slope m_N satisfies $|m_N| = O(\theta^{\alpha N})$, we get

$$\left\| df^k_{\tau_0}(1,0) - df^k_{q_N}(1, m_N)\right\| = O(\theta^{\alpha N}), \tag{6.11}$$

hence $|\nu - \nu_0| = O(\theta^{\alpha N})$, and, replacing θ by some $\theta' > \theta^\alpha$ completes the proof. □

Step 4.– Density. As said above, we choose $p_n = q_0$ as above for each $n = 2N + k$ (resp. $2N + k - 1$). It remains to show that the points p_n can be chosen to be Zariski dense for the complex analytic topology.

Indeed, replacing p_n (i.e. q_0 above) by an appropriately chosen point in its orbit (i.e. another q_m), we can arrange that the closure of (p_n) contains the whole negative orbit of τ_0. Now assume that (p_n) is contained in a complex analytic subvariety, which is decomposed into irreducible components as $V_1 \cup \cdots \cup V_k$. One of the V_i accumulates infinitely many points in $\left\{f^{-j}(\tau),\ j \geqslant 0\right\}$, hence it contains them; therefore it must locally coincide with $W^u_{\rm loc}(p)$. But since the p_n are periodic, they cannot be contained in an unstable manifold, so we arrive at a contradiction. □

Remark 6.3. Since in our situation, the stable and unstable distributions of the horseshoe are complex 1-dimensional, the bunching condition of [40, Thm 19.1.8] holds, so they are actually Lipschitz, and we could have chosen $\alpha = 1$ in the above estimates. This shows that in Lemma 6.2, any $\theta > \max(|\lambda^s(p)|, \left|(\lambda^u(p))^{-1}\right|)$ is convenient.

The following result will be used in the proof of Theorem A.

Proposition 6.4. *Let f be a loxodromic automorphism of $\mathbf{C}^2$. If f has a saddle periodic point p with non-real stable and unstable multipliers, then there is a set of such periodic points which is dense for the real analytic topology.*

Proof. Without loss of generality, assume that p is fixed. By [6], every saddle point admits homoclinic intersections; furthermore, $J^+_{W^u(p)}$ is the closure of homoclinic intersections (see [24, Lem. 5.1]). Since $\lambda^u(p) \notin \mathbf{R}$, f is not unstably real, so $J^+_{W^u(p)}$ is not contained in a C^1 curve. Likewise $J^-_{W^s(p)}$ is not contained in a C^1 curve. For any transverse homoclinic intersection point $\tau \in W^u(p) \cap W^s(p)$, by Theorem 6.1 we can construct a sequence of saddle periodic points (p_n) such that the closure of (p_n) contains τ and $\lambda^u(p_n) \sim c\lambda^u(p)^n$ and $\lambda^s(p_n) \sim c'(\lambda^s(p))^n$. Thus, we have

- if $\arg(\lambda^u(p))$ or $\arg(\lambda^s(p))$ differs from $\pm\frac{\pi}{2}$, then both $\lambda^u(p_n)$ and $\lambda^s(p_n)$ are non-real for infinitely many n;

- otherwise, $\arg(\lambda^u(p)) = \varepsilon\frac{\pi}{2}$ and $\arg(\lambda^s(p)) = \varepsilon'\frac{\pi}{2}$ with ε and ε' in $\{\pm 1\}$; in particular, $\mathrm{Jac}(f) \in \mathbf{R}$. Then it might a priori occur that $\lambda^u(p_n)$ and $\lambda^s(p_n)$ are alternatively real, depending on the parity of n, so that exactly one of them is always real; but such a case would imply $\mathrm{Jac}(f) \notin \mathbf{R}$, a contradiction.

Altogether, we infer that $\lambda^u(p_n) \notin \mathbf{R}$ and $\lambda^s(p_n) \notin \mathbf{R}$ for all but at most a proper periodic subsequence of indices n. Repeating this construction by varying τ in $W^u_{\mathrm{loc}}(p)$ and $W^s_{\mathrm{loc}}(p)$, we obtain a set of periodic points p_n with non-real multipliers that accumulates $W^u_{\mathrm{loc}}(p)$ and $W^s_{\mathrm{loc}}(p)$ on subsets which are not contained in C^1 curves.

To prove $\mathbf{R}$-Zariski density, we apply this construction to a dense subset of homoclinic intersections τ_j in $J^+_{W^u(p)}$ and we denote by (q_n) a sequence of saddle periodic points obtained from the sequences (p_n) (one for each τ_j) by a diagonal process.

Assume first that V is a proper irreducible real-analytic subset containing the q_n. Then V contains a set of homoclinic points $\tau \in W^u_{\mathrm{loc}}(p)$ (resp. $W^s_{\mathrm{loc}}(p)$) which is not contained in a C^1 curve, because $J^+ \cap W^u_{\mathrm{loc}}(p)$ (resp. $J^- \cap W^s_{\mathrm{loc}}(p)$) is not contained in such a curve. As in Remark **??**, since none of the q_n belongs to $W^u_{\mathrm{loc}}(p) \cup W^s_{\mathrm{loc}}(p)$ we see that V properly contains $W^u_{\mathrm{loc}}(p) \cup W^s_{\mathrm{loc}}(p)$. In particular, $\dim(V) = 3$ and V is singular at p.

To conclude, we can apply the above construction at each of the q_n instead of p, getting periodic points $q_{n,m}$ with non-real multipliers accumulating $W^u_{\mathrm{loc}}(q_n)$ and $W^s_{\mathrm{loc}}(q_n)$ along subsets which are not contained in C^1 curves. Let $V = V_1 \cup \cdots \cup V_k$ be a proper real-analytic subset containing these points. Then for one of the irreducible components, say V_1, there exists an infinite family of points q_n, which is not contained in a C^1 curve in both $W^u_{\mathrm{loc}}(p)$ and $W^s_{\mathrm{loc}}(p)$, and such that at each q_n, V_1 contains a set of homoclinic points in $W^u_{\mathrm{loc}}(q_n) \cup W^s_{\mathrm{loc}}(q_n)$ which is not contained in a C^1 curve in both $W^u_{\mathrm{loc}}(q_n)$ and $W^s_{\mathrm{loc}}(q_n)$. Therefore V_1 has dimension 3 and is singular at each such q_n. But the q_n are not contained in an analytic subset of dimension $\leqslant 2$, so we get a contradiction. □

Using Theorem 5.6 we get a version of Proposition 6.4 involving unstable multipliers only.

Proposition 6.5. *Let f be a loxodromic automorphism of $\mathbf{C}^2$ which is unstably disconnected. If f has a saddle periodic point p with non-real unstable multiplier, then there is a $\mathbf{R}$-Zariski dense set of periodic points with the same property.*

Proof. Arguing as in the previous proposition (and without taking stable eigenvalues into account) we can construct a sequence of saddle points (analogous to the points $q_{n,m}$ from the previous proof), whose unstable multipliers are non-real and whose $\mathbf{R}$-Zariski closure has dimension at least 3. Let V be the Zariski closure of the set of periodic points whose unstable multiplier is non-real, and assume that it has dimension 3. Note that by construction V contains $W^u(p)$. The subvariety V is periodic, of period dividing that of p. Replacing f by f^k we may assume that V is fixed. By Theorem 5.6, $J^\star$ is not contained in V so there is a saddle periodic point $q \in J^\star \setminus V$. Fix $t \in W^s(q) \cap W^u(p)$, so that t is in V. But $f^n(t) \to q$ as $n \to \infty$ and V is invariant so q belongs to V, which is a contradiction. □

7. Real analytic conjugacies

In this section, we prove Theorem A in the following slightly more precise form.

Theorem 7.1. *Suppose $\varphi\colon \mathbf{C}^2 \to \mathbf{C}^2$ is a real analytic conjugacy between two loxodromic automorphisms $f, g \in \mathsf{Aut}(\mathbf{C}^2)$. Assume that there exists a saddle periodic point for f for which both stable and unstable multipliers are not real. Then φ is a polynomial automorphism of $\mathbf{C}^2$, or the composition of an automorphism with the complex conjugation $(x, y) \mapsto (\overline{x}, \overline{y})$.*

This theorem extends the Main Theorem of [16] from the holomorphic to the real analytic case, with however one additional hypothesis (on the non-real multipliers). The case of C^1 conjugacies in degree 2 was treated in [31, Thm 7.5]. One may naturally ask whether this theorem of Friedland and Milnor extends to arbitrary degrees.

Before starting the proof, let us make a few preliminary observations, which do not depend on the assumption that there is a saddle point with non-real eigenvalues. If p is a saddle fixed point of f then $q = \varphi(p)$ is a saddle fixed point of g. Let $\psi_f\colon \mathbf{C} \to \mathbf{C}^2$ be a parametrization of the stable manifold of f at p, and let $\psi_g\colon \mathbf{C} \to \mathbf{C}^2$ be a parametrization of the stable manifold of g at q. Then $\varphi \circ \psi_f = \psi_g \circ \widetilde{\varphi}$ for some real analytic diffeomorphism $\widetilde{\varphi}\colon \mathbf{C} \to \mathbf{C}$, $\zeta \mapsto \widetilde{\varphi}(\zeta)$, with $\widetilde{\varphi}(0) = 0$. Note that $\widetilde{\varphi}$ is completely determined by its restriction to a small neighborhood of the origin. Denote by λ^s_p (resp. λ^s_q) the eigenvalue of df_p (resp. dg_q) along the stable direction.

Lemma 7.2. *With the above notation, we have that:*

(1) $\widetilde{\varphi}$ is $\mathbf{R}$-linear;
(2) either $\lambda^s_q = \lambda^s_p$ or $\lambda^s_q = \overline{\lambda^s_p}$, and in the latter case, $\widetilde{\varphi}$ reverses the orientation of $\mathbf{C}$;
(3) if λ^s_p (equivalently λ^s_q) is not real, then either $\widetilde{\varphi}(\zeta) = \alpha\zeta$ and $\lambda^s_q = \lambda^s_p$ or $\widetilde{\varphi}(\zeta) = \alpha\overline{\zeta}$ and $\lambda^s_q = \overline{\lambda^s_p}$, where α is some nonzero complex number.

Proof. To see this, it suffices to write the Taylor expansion of $\widetilde{\varphi}(\zeta)$ near the origin as $\sum_{k,\ell} a_{k,\ell}\zeta^k\overline{\zeta}^\ell$ and write that $\widetilde{\varphi}$ conjugates $\zeta \mapsto \lambda^s_p\zeta$ to $\zeta \mapsto \lambda^s_q\zeta$, which gives a relation of the form

$$\sum_{k,\ell=1}^{\infty} a_{k,\ell}(\lambda^s_p)^k \left(\overline{\lambda^s_p}\right)^\ell \zeta^k\overline{\zeta}^\ell = \lambda^s_q \sum_{k,\ell=1}^{\infty} a_{k,\ell}\zeta^k\overline{\zeta}^\ell. \tag{7.1}$$

Equating the coefficients we first infer that

$$a_{1,0}\left(\lambda^s_p - \lambda^s_q\right) = a_{0,1}\left(\overline{\lambda^s_p} - \lambda^s_q\right) = 0, \tag{7.2}$$

hence, since $\widetilde{\varphi}$ is a diffeormorphism, either $\lambda^s_p = \lambda^s_q$ or $\overline{\lambda^s_p} = \lambda^s_q$ (or both). Using that $\left|\lambda^s_p\right| = \left|\lambda^s_q\right| < 1$, we obtain $a_{k,\ell} = 0$ for $k + \ell > 1$. Finally, if λ is not real, then φ is holomorphic or antiholomorphic along the stable manifold W^s_p, as asserted. □

Proof of Theorem 7.1. Since f has a saddle periodic point p with non-real stable and unstable multipliers, Proposition 6.4 provides a $\mathbf{R}$-Zariski dense set of such points. Therefore, from the previous lemma, we deduce that: either (a) φ is holomorphic, or (b) φ is anti-holomorphic, or (c) there is a dense set of saddle periodic points p_j for the real analytic topology such that φ is holomorphic along $W^u_{p_i}$ and antiholomorphic along $W^s_{p_i}$ or vice versa.

In case (a), we conclude with the Main Theorem of [16]. In case (b), composing φ with the complex conjugation, we obtain a holomorphic conjugacy $\overline{\varphi}$ between f and $\overline{g}$. Again, the Main Theorem of [16] implies that $\overline{\varphi}$ is an automorphism, as desired.

To complete the proof it remains to show that Case (c) leads to a contradiction. Denote by J the complex structure of $\mathbf{C}^2$ and by J' the pull back of J under φ. Then, exchanging the roles of the stable and unstable manifolds if necessary, J' coincides with J along $W^u_{p_i}$ and with $-\mathrm{J}$ along $W^s_{p_i}$, for a set of saddle periodic points p_i which is dense in $\mathbf{C}^2$ for the real analytic topology.

On the real tangent spaces $T_{p_i}\mathbf{C}^2 \simeq \mathbf{R}^4$, there is a plane on which $\mathrm{J}'v = \mathrm{J}v$ and another one on which $\mathrm{J}'v = -\mathrm{J}v$. We claim that by density, this property holds everywhere. To see this, we study the properties of $\mathrm{J}'\mathrm{J} \in \mathrm{End}(T_z\mathbf{C}^2)$. Observe that for $\varepsilon = \pm 1$ and $v \in T_z\mathbf{C}^2$,

$$\mathrm{J}'\mathrm{J}v = \varepsilon v \Leftrightarrow \mathrm{J}v = -\varepsilon \mathrm{J}'v \Leftrightarrow \mathrm{J}'v = -\varepsilon \mathrm{J}v \Leftrightarrow \mathrm{J}\mathrm{J}'v = \varepsilon v,$$

and in addition this property is invariant under J and J', i.e. v satisfies it if and only if $\mathrm{J}v$ or $\mathrm{J}'v$ does. It follows that on an $\mathbf{R}$-Zariski dense set of points $z \in \mathbf{C}^2$, the characteristic polynomial $\chi_z(X)$ of $\mathrm{J}'\mathrm{J}$ is $(X-1)^2(X+1)^2$; since $\chi_z(X)$ depends analytically on z, this property holds everywhere. Likewise, the minimal polynomial of $\mathrm{J}'\mathrm{J}$ at each p_i is $(X-1)(X+1)$ so again this property holds everywhere. In this way, we obtain two distributions of planes such that

(a) $T_z\mathbf{C}^2 = P_-(z) \oplus P_+(z)$ at each point z of $\mathbf{C}^2$,
(b) $P_-(z)$ and $P_+(z)$ are invariant under J and J'.

Furthermore, $P_-(z) = T_zW^u_{p_i}$ and $P_+(z) = T_zW^s_{p_i}$ as soon as z is a point on $W^u_{p_i}$ or $W^s_{p_i}$, respectively. Indeed, $P_-(z)$ and $P_+(z)$ are the only $\mathrm{J}\mathrm{J}'$- and J-invariant planes at z, and the property follows by continuity.

These distributions of planes define two real analytic foliations, the leaves of which are holomorphic. Indeed, if a k-dimensional distribution is not integrable, then there is an open set U in which there is a pair of vector fields tangent to the distribution whose Lie bracket is everywhere transverse to the distribution. Conversely, this property cannot be satisfied if there is a dense set of points at which one can find a local submanifold of dimension k tangent to the distribution. In the real-analytic case, to guarantee integrability, it is enough to find such a set that is dense for the real-analytic topology[8]. Since the $W^u_{\mathrm{loc}}(p_i)$ and $W^s_{\mathrm{loc}}(p_i)$ provide such local submanifolds, we conclude that P_- and P_+ define two foliations $\mathcal{F}_-$ and $\mathcal{F}_+$. Since, by definition, P_- and P_+ are J-invariant, the leaves are holomorphic.

This argument shows that f preserves two global real analytic foliations by holomorphic curves, so that Theorem B completes the proof. □

[8]Let $z \mapsto P(z) \subset T_z\mathbf{C}^2$ be a real analytic distribution of real planes. Given a real analytic vector field $z \mapsto v(z)$, its orthogonal projection $v_P(z)$ on $P(z)$ for the standard euclidean metric is also real analytic. In this way, we construct many real analytic vector fields which are everywhere tangent to P. The integrability of P means that the Lie bracket of any pair of such vector fields is again tangent to P. Thus, the integrability property propagates from any $\mathbf{R}$-Zariski dense subset to $\mathbf{C}^2$.

8. MULTIPLIERS IN NUMBER FIELDS

Recall that for a saddle periodic point p of period n, the Lyapunov exponent is

$$\chi^u(p) = \frac{1}{n} \log |\lambda^u(p)| . \tag{8.1}$$

We denote by $\chi^u(\mu_f)$ the positive Lyapunov exponent of the unique measure of maximal entropy.

Huguin's theorem [37] asserts that a rational map in $\mathbb{P}^1(\mathbf{C})$ whose multipliers lie in some fixed number field must be exceptional. Here we obtain two partial generalizations of this result.

Theorem 8.1. *Let $f \in \mathsf{Aut}(\mathbf{C}^2)$ be a loxodromic automorphism. Assume that:*

(1) f is defined over a number field;
(2) f is uniformly hyperbolic;
(3) f admits two saddle periodic points with distinct Lyapunov exponents.

Then the unstable (resp. stable) multipliers of f cannot lie in a fixed number field.

All three assumptions are absent from [37] and are expected to be superfluous.

Proof. Following [37], we argue by contradiction. Let K be a number field containing the coefficients of f as well as all unstable multipliers. Note that $\mathrm{Jac}(f)$ is in K, so K contains all stable multipliers as well, and the statements on stable and unstable multipliers are equivalent.

Let p be any saddle periodic point. We show that $\chi^u(\mu_f) = \chi^u(p)$, thereby contradicting Assumption (3).

Let (q_n) be the sequence provided by Theorem 6.1, associated to some homoclinic intersection, which we assume moreover to be Zariski dense by Remark **??**. By a diagonal process (see e.g. [23, p. 3455]) we may extract a subsequence (q_{n_j}) that converges to the generic point for the $\overline{\mathbf{Q}}$-Zariski topology. Without loss of generality rename (q_{n_j}) into (q_n). Let $\mathrm{Gal}(q_n)$ be the orbit of q_n under the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/K)$. Yuan's arithmetic equidistribution theorem (see [52], and [41] for this application) implies that the sequence of probability measures

$$\mu_n := \frac{1}{\mathrm{Gal}(q_n)} \sum_{r \in \mathrm{Gal}(q_n)} \delta_r \tag{8.2}$$

converges to the equilibrium measure μ_f in the weak-$\star$ topology.

Since f is uniformly hyperbolic, the unstable line field is continuous along J. For $z \in J$, denote by $\|df^u_z\|$ the euclidean norm of $df_z(e^u(z))$ for any unit vector $e^u(z)$ tangent to the unstable direction at z; this does not depend on the choice of $e^u(z)$. Then the map $z \mapsto \log \|df^u_z\|$ is continuous. For any invariant measure ν, the average positive Lyapunov exponent is $\chi^u(\nu) = \int \log \|df^u_z\| d\nu(z)$. In particular, Yuan's equidistribution theorem implies that $\chi^u(\mu_n) \to \chi^u(\mu_f)$ as $n \to \infty$.

Now, if we decompose μ_n as a sum of ergodic invariant measures along periodic orbits, we see that all these periodic orbits are $\mathrm{Gal}(\overline{\mathbf{Q}}/K)$-conjugate to that of q_n, so their multipliers are

Galois conjugate as well. But by assumption these multipliers lie in K so they are all equal and $\chi^u(\mu_n) = \chi^u(q_n)$.

On the other hand, since $\lambda^u(q_n) \sim c\lambda^u(p)^n$, it follows that $\chi^u(q_n) \to \chi^u(p)$ as $n \to \infty$. Altogether, we conclude that $\chi^u(p) = \chi^u(\mu_f)$, as asserted, and the proof is complete. □

Theorem 8.2. *Let $f \in \mathsf{Aut}(\mathbf{C}^2)$ be a loxodromic automorphism. Assume that:*

(1) f is defined over a number field;
(2) f admits a saddle periodic point p such that $\chi^u(p) > \chi^u(\mu_f)$.

Then the unstable (resp. stable) multipliers of f cannot lie in a fixed number field.

Proof. Note that since $\chi^u(p) + \chi^s(p) = \chi^u(\mu_f) + \chi^s(\mu_f) = \log|\mathrm{Jac}(f)|$, Assumption (2) implies that $|\chi^s(p)| > |\chi^s(\mu_f)|$, so the statements on the stable and unstable multipliers are indeed equivalent.

The proof follows closely that of Theorem 8.1, the only difference being that due to the lack of hyperbolicity, the unstable directions are a priori not continuous, thus the function $\log\|df^u_z\|$ is a priori not continuous either, and we cannot assert that $\chi^u(\mu_n) \to \chi^u(\mu_f)$ as $n \to \infty$. On the other hand, the upper semi-continuity of Lyapunov exponents ([9]) shows that $\limsup_{n\to\infty} \chi^u(\mu_n) \leqslant \chi^u(\mu_f)$. But since $\chi^u(\mu_n) = \chi^u(p)$, by choosing p as in (2), we get the desired contradiction. □

Remark 8.3. If $f : \mathbb{P}^1(\mathbf{C}) \to \mathbb{P}^1(\mathbf{C})$ is a non-exceptional rational map, a result of Zdunik [54] guarantees that f admits repelling periodic points with exponents greater than $\chi^u(f)$. The contradiction in [37] is based on this property.

The assumptions of Theorems 8.1 and 8.2 can be checked on certain perturbative examples. We illustrate this on the Hénon family but more general close-to-degenerate examples could be considered. For $d \geqslant 2$, let us parameterize the space $\mathcal{H}_d$ of Hénon maps of degree d by $(a, c) \in \mathbf{C}^\times \times \mathbf{C}^{d-1}$, where $f_{a,c}(z, w) = (aw + p_c(z), z)$, with $p_c(z) = z^d + \sum_{j=0}^{d-2} c_j z^j$. Note that p_c is *exceptional* (or *integrable*) when it is conjugate to $\pm T_d$, where T_d is a Chebychev polynomial, or when $c = 0$, in which case p is monomial. We obtain the extended parameter space $\overline{\mathcal{H}_d}$ by adjoining the hypersurface $\{a = 0\}$ of *degenerate maps*: $f_{0,c}$ maps $\mathbf{C}^2$ to the curve $\{z = p_c(w)\}$ and the dynamics on that curve is conjugate to that of p_c. In this respect, a Hénon map $f_{a,c}$ with small Jacobian a may be seen as a perturbation of p_c.

Theorem 8.4. *There exists a neighborhood $\mathcal{N}$ of $\{0\} \times (\mathbf{C}^{d-1} \setminus \{0\})$ in the extended Hénon parameter space $\overline{\mathcal{H}_d}$ such that if $f \in \mathcal{N} \cap \mathcal{H}_d(\overline{\mathbf{Q}})$, then the unstable (resp. stable) multipliers of f are not contained in a fixed number field.*

It is interesting that, as opposed to [37], Chebychev polynomials do not play an exceptional role in this result.

[9]Recall the argument for upper-semicontinuity: for any invariant measure ν, the average upper Lyapunov exponent $\chi^u(\nu)$ is the limit of $\frac{1}{k}\int \log\|Df^k_x\| d\nu(x)$ as $k \to \infty$. Choosing a submultiplicative norm and taking the limit along the subsequence $k = 2^q$ realizes $\nu \mapsto \chi^u(\nu)$ as the limit of a decreasing sequence of continuous functions for the weak-* topology, hence it is upper semi-continuous.

Proof. Assume first that p_{c_0} is neither monomial nor conjugate to a Chebychev polynomial. Then by the above-mentioned theorem of Zdunik [54], p_{c_0} admits a repelling point r_0 with $\chi(r_0) > \chi(\mu_{p_{c_0}})$. For (a, c) close to $(0, c_0)$, $a \neq 0$ r_0 persists as a saddle point $r = r_{a,c}$ with unstable Lyapunov exponent close to $\chi(r_0)$. Furthermore $\chi^u(\mu_{f_{a,c}})$ is close to $\chi(\mu_{p_{c_0}})$ (see [20, §3]; actually we only need the "easy" inequality $\limsup_{(a,c)\to(0,c_0)} \chi^u(\mu_{f_{a,c}}) \leqslant \chi(\mu_{p_{c_0}})$). So if $(a, c) \in \overline{\mathbf{Q}}^{d-1}$, Theorem 8.2 applies and we are done.

If p_{c_0} is conjugate to $\pm T_d$, we observe that all Lyapunov exponents of periodic points are equal to $\log d$, except for the post-critical fixed points whose exponent is $2 \log d$ (see [45, Cor. 3.9]). Since $\pm T_d$ has a connected Julia set, $\chi(\mu_{\pm T_d}) = \log d$, so again the assumptions of Theorem 8.2 are satisfied ([10]). □

Remark 8.5. It is natural to expect that Theorem 8.4 holds also in the neighborhood of $(0, 0)$. To prove this, it would be enough to find for every parameter close to $(0, 0)$ a saddle periodic point with Lyapunov exponent $\neq \log d$. Indeed, in such a neighborhood $\mathcal{N}_0$, every f is uniformly hyperbolic, with a connected Julia set, so $\chi^u(\mu_f) = \log d$. So if r is a saddle point with $\chi^u(r) \neq \log d$, either $\chi^u(r) > \log d$ and we conclude by Theorem 8.2, or $\chi^u(r) < \log d$ and since by [5] $\chi^u(\mu_f)$ is approximated by unstable exponents of periodic orbits, there exists r' such that $\chi^u(r') > \chi^u(r)$ and we conclude by Theorem 8.1. In small degrees (say 2 or 3) we can hope to prove numerically the existence of such a periodic point, by covering U with finitely many regions where some point of small period has exponent different from $\log d$.

Remark 8.6. For Hénon maps of degree 2 and 3 the Jacobian determinant together with the traces of the differential at the fixed points determine the map: this follows from the proof of [31, Thm 7.1] ([11]). So in Theorem 8.4 it is enough to assume that $\mathrm{Jac}(f) \in \overline{\mathbf{Q}}$, instead of assuming f in $\mathcal{H}_d(\overline{\mathbf{Q}})$. Indeed in this case if the unstable multipliers of the fixed points are algebraic numbers, then so are the stable multipliers, hence so are the traces, and we conclude that f itself is defined over $\overline{\mathbf{Q}}$.

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[10] The difference between the one- and the two-dimensional situations is that in the one-dimensional case, the post-critical fixed points do not have transverse homoclinic intersections

[11] Let us give a few details. For convenience we follow the notation of [31], i.e. we write $f(x, y) = (y, p(y) - \delta x)$. Then $p(x) = (x - x_1) \ldots (x - x_d) + (1 + \delta)x$, where the fixed points are (x_j, x_j) and $\sum x_j = 0$. For $d = 2$, the trace of Df at the fixed point x_j is $2x_j$ so the result is clear. For $d = 3$, the calculations of [31, Lem. 7.4] show that

$$3x_1 = \sqrt{\frac{\mathrm{tr}_1 \, \mathrm{tr}_2}{\mathrm{tr}_3}} + \sqrt{\frac{\mathrm{tr}_1 \, \mathrm{tr}_3}{\mathrm{tr}_2}}$$

where $\mathrm{tr}_j = \mathrm{trace}(Df_{x_j})$, for some appropriate choice of square root, and we are done.

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