RANDOM DYNAMICS ON REAL AND COMPLEX PROJECTIVE SURFACES

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ABSTRACT. We initiate the study of random iteration of automorphisms of real and complex projective surfaces, as well as compact Kähler surfaces, focusing on the fundamental problem of classification of stationary measures. We show that, in a number of cases, such stationary measures are invariant, and provide criteria for uniqueness, smoothness and rigidity of invariant probability measures. This involves a variety of tools from complex and algebraic geometry, random products of matrices, non-uniform hyperbolicity, as well as recent results of Brown and Rodriguez Hertz on random iteration of surface diffeomorphisms.

CONTENTS

1. INTRODUCTION

1.1. **Random dynamical systems.** Consider a compact manifold M and a probability measure ν on $\mathsf{Diff}(M)$; to simplify the exposition we assume throughout this introduction that the support $\mathrm{Supp}(\nu)$ is finite. The data (M, ν) defines a **random dynamical system**, obtained by randomly composing independent diffeomorphisms with distribution ν. In this paper, these random dynamical systems are studied from the point of view of *ergodic theory*, that is, we are mostly interested in understanding the asymptotic *distribution* of orbits.

Let us first recall some basic vocabulary. A probability measure μ on M is ν**-invariant** if $f_*\mu = \mu$ for ν-almost every $f \in \mathsf{Diff}(M)$, and it is ν**-stationary** if it is invariant on average: $\int f_*\mu \, d\nu(f) = \mu$. A simple fixed point argument shows that stationary measures always exist. On the other hand, the existence of an invariant measure should hold only under special circumstances, for instance when the group Γ_ν generated by $\mathrm{Supp}(\nu)$ is amenable, or has a finite orbit, or preserves an invariant volume form.

According to Breiman's law of large numbers, the asymptotic distribution of orbits is described by stationary mesures. More precisely, for every $x \in M$ and $\nu^{\mathbf{N}}$-almost every $(f_j) \in \mathsf{Diff}(M)^{\mathbf{N}}$, every cluster value of the sequence of empirical measures

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f_j \circ \cdots \circ f_0(x)} \tag{1.1}$$

is a stationary measure. Thus a classification of stationary measures gives an essentially complete understanding of the asymptotic distribution of such random orbits, as n goes to $+\infty$.

When Γ_ν is a cyclic group, the set of invariant measures is typically too large to be amenable to a complete description. On the other hand a number of recent works have shown that stationary measures, even if they always exist, tend to satisfy some *rigidity properties* when Γ_ν is large. Our goal in this article is to combine tools from algebraic and holomorphic dynamics together with these recent results from random dynamics to study the case when M is a real or complex projective surface and the action is by algebraic diffeomorphisms. Before describing the state of the art and stating a few precise results, let us highlight a nice geometric example to which our techniques can be applied.

1.2. **Randomly folding pentagons.** Let $\ell_0, \ldots, \ell_4$ be five positive real numbers such that there exists a pentagon with side lengths ℓ_i. Here a pentagon is just an ordered set of points $(a_i)_{i=0,\ldots,4}$ in the Euclidean plane, such that $\mathrm{dist}(a_i, a_{i+1}) = \ell_i$ for $i = 0, \ldots, 4$ (with $a_5 = a_0$ by definition); pentagons are not assumed to be convex, and two distincts sides $[a_i, a_{i+1}]$ and $[a_j, a_{j+1}]$ may intersect at a point which is not one of the a_i's.

Let $\mathrm{Pent}(\ell_0, \ldots, \ell_4)$ be the set of pentagons with side lengths ℓ_i. Note that $\mathrm{Pent}(\ell_0, \ldots, \ell_4)$ may be defined by polynomial equations of the form $\mathrm{dist}(a_i, a_{i+1})^2 = \ell_i^2$, so it is naturally a real algebraic variety. For every i, a_i is one of the two intersection points $\{a_i, a_i'\}$ of the circles of respective centers a_{i-1} and a_{i+1} and radii ℓ_{i-1} and ℓ_i. The transformation exchanging these two points a_i and a_i', while keeping the other vertices fixed, defines an involution s_i of $\mathrm{Pent}(\ell_0, \ldots, \ell_4)$. It commutes with the action of the group $\mathsf{SO}_2(\mathbf{R}) \ltimes \mathbf{R}^2$ of positive isometries of the plane, hence, it induces an involution σ_i on the quotient space

$$\mathrm{Pent}^0(\ell_0, \ldots, \ell_4) = \mathrm{Pent}(\ell_0, \ldots, \ell_4)/(\mathsf{SO}_2(\mathbf{R}) \ltimes \mathbf{R}^2). \tag{1.2}$$

Each element of $\mathrm{Pent}^0(\ell_0, \ldots, \ell_4)$ admits a unique representative with $a_0 = (0,0)$ and $a_1 = (\ell_0, 0)$, so as before $\mathrm{Pent}^0(\ell_0, \ldots, \ell_4)$ is a real algebraic variety, which is easily seen to be of dimension 2 (see [42, 108]). When it is smooth, this is an example of K3 surface, and the five involutions σ_i act by algebraic diffeomorphisms on this surface, preserving a canonically defined area form (see §3.2); and for a general choice of lengths, the group generated by these involutions generates a rich dynamics. Now, start with some pentagon P and at every unit of time, apply randomly one of the σ_i. This creates a random sequence of pentagons, and our results explain how this sequence is asymptotically distributed on $\mathrm{Pent}^0(\ell_0, \ldots, \ell_4)$. (The dynamics of the folding maps acting on plane quadrilaterals was studied for instance in [57, 10].)

1.3. **Stiffness.** Let us present a few landmark results that shape our understanding of these problems. First, suppose that ν is a finitely supported probability measure on $\mathsf{SL}_2(\mathbf{C})$, which we view as acting by projective linear transformations on $M = \mathbb{P}^1(\mathbf{C})$. Suppose that the group Γ_ν generated by the support of ν is **non-elementary**, that is, Γ_ν is non-compact and acts strongly irreducibly on $\mathbf{C}^2$ (in the non-compact case, this simply means that Γ_ν does not have any orbit of cardinality 1 or 2 in $\mathbb{P}^1(\mathbf{C})$). Then, *there is a unique ν-stationary probability measure μ on $\mathbb{P}^1(\mathbf{C})$, and this measure is not invariant*. This is one instance of a more general result due to Furstenberg [63].

Temporarily leaving the setting of diffeomorphisms, let us consider the semigroup of transformations of the circle $\mathbf{R}/\mathbf{Z}$ generated by m_2 and m_3, where $m_d(x) = dx \mod 1$. Since the multiplications by 2 and 3 commute, the so-called Choquet-Deny theorem asserts that any stationary measure is invariant. Furstenberg's famous "$\times 2 \times 3$ conjecture" asserts that any atomless probability measure μ invariant under m_2 and m_3 is the Lebesgue measure (see [64]). This question is still open so far, and has attracted a lot of attention. Rudolph [106] proved that the answer is positive when μ is of positive entropy with respect to m_2 or m_3.

Back to diffeomorphisms, let ν be a finitely supported measure on $\mathsf{SL}_2(\mathbf{Z})$, and consider the action of $\mathsf{SL}_2(\mathbf{Z})$ on the torus $M = \mathbf{R}^2/\mathbf{Z}^2$.

In that case, the Haar measure $dx \wedge dy$ of $\mathbf{R}^2/\mathbf{Z}^2$, as well as the atomic measures equidistributed on finite orbits $\Gamma_\nu(x,y)$, for $(x,y) \in \mathbf{Q}^2/\mathbf{Z}^2$, are examples of Γ_ν-invariant measures. By using Fourier analysis and additive combinatorics techniques, Bourgain, Furman, Lindenstrauss and Mozes [21] proved that if Γ_ν is non-elementary, then *every stationary measure μ on $\mathbf{R}^2/\mathbf{Z}^2$ is Γ_ν-invariant, and furthermore it is a convex combination of the above mentioned invariant measures.* This can be viewed as an affirmative answer to a non-Abelian version of the $\times 2 \times 3$ conjecture. This property of automatic invariance of stationary measures was called **stiffness** (or more precisely ν-stiffness) by Furstenberg [65], who conjectured it to hold in this setting. Soon after, Benoist and Quint [11] gave an ergodic theoretic proof of this result, which allowed them to extend the stiffness property to certain actions of discrete groups on homogeneous spaces. They also derived the following equidistribution result for the action of $\mathsf{SL}_2(\mathbf{Z})$ on the torus: *for every $(x,y) \notin \mathbf{Q}^2/\mathbf{Z}^2$, the random trajectory of (x,y) determined by ν almost surely equidistributes towards the Haar measure.*

Finally, Brown and Rodriguez-Hertz [22], building on the work of Eskin and Mirzakhani [58], managed to recast these measure rigidity results in terms of smooth ergodic theory to obtain a version of the stiffness theorem of [21] for general C^2 diffeomorphisms of compact surfaces. We shall describe their results in due time, so for the moment we will content ourselves with one illustrative consequence of [22]. As above, let $\nu = \sum \alpha_j \delta_{f_j}$ be a finitely supported probability measure on $\mathsf{SL}_2(\mathbf{Z})$ and consider perturbations $\{f_{i,\varepsilon}\}$ of the f_i in the group $\mathsf{Diff}^2_{\mathsf{vol}}(\mathbf{R}^2/\mathbf{Z}^2)$ of C^2 diffeomorphisms of $\mathbf{R}^2/\mathbf{Z}^2$ preserving the Haar measure. Set $\nu_\varepsilon = \sum \alpha_j \delta_{f_{j,\varepsilon}}$. Then, for sufficiently small perturbations, stiffness still holds, that is: *any ν_ε-stationary measure on $\mathbf{R}^2/\mathbf{Z}^2$ is invariant, and is a combination of the Haar measure and measures supported on finite Γ_{ν_ε}-orbits.*

In this paper, we obtain a new generalization of the stiffness theorem of [21], for algebraic diffeomorphisms of real algebraic surfaces. Before entering into specifics, let us emphasize that the article [22], by Brown and Rodriguez-Hertz, is our main source of inspiration and a key ingredient for some of our main results.

1.4. **Sample results: stiffness, classification, and rigidity.** Let X be a smooth complex projective surface, or more generally a compact Kähler surface. Denote by $\mathsf{Aut}(X)$ its group of holomorphic diffeomorphisms, referred to in this paper as **automorphisms**. When $X \subset \mathbb{P}^N(\mathbf{C})$ is defined by polynomial equations with real coefficients, the complex conjugation induces an anti-holomorphic involution $s\colon X \to X$, whose fixed point set is the real part of X: $X(\mathbf{R}) = \mathrm{Fix}(s) \subset X$. We denote by $X_\mathbf{R}$ the surface X viewed as an algebraic variety defined over $\mathbf{R}$, and by $\mathsf{Aut}(X_\mathbf{R})$ the group of automorphisms defined over $\mathbf{R}$; $\mathsf{Aut}(X_\mathbf{R})$ coincides with the subgroup of $\mathsf{Aut}(X)$ that centralizes s. When $X(\mathbf{R}) \neq \varnothing$, the elements of $\mathsf{Aut}(X_\mathbf{R})$ are the real-analytic diffeomorphisms of $X(\mathbf{R})$ admitting a holomorphic extension to X. Note that in stark contrast with groups of smooth diffeomorphisms, the groups $\mathsf{Aut}(X_\mathbf{R})$ and $\mathsf{Aut}(X)$ are typically *discrete and at most countable*.

The group $\mathsf{Aut}(X)$ acts on the cohomology $H^*(X;\mathbf{Z})$. By definition, a subgroup $\Gamma \subset \mathsf{Aut}(X)$ is **non-elementary** if its image $\Gamma^* \subset \mathsf{GL}(H^*(X;\mathbf{C}))$ contains a non-Abelian free group; equivalently, Γ^* is not virtually Abelian. When Γ is non-elementary, there exists a pair $(f,g) \in \Gamma^2$ generating a free group of rank 2 such that the topological entropy of every element in that group is positive (see Lemma A.1). Pentagon foldings provide examples for which $\mathsf{Aut}(X_\mathbf{R})$ is non-elementary.

Let ν be a finitely supported probability measure on $\mathsf{Aut}(X)$. As before we denote by Γ_ν the subgroup generated by $\mathrm{Supp}(\nu)$.

Theorem A. *Let $X_\mathbf{R}$ be a real projective surface and ν be a finitely supported symmetric probability measure on $\mathsf{Aut}(X_\mathbf{R})$. If Γ_ν preserves an area form on $X(\mathbf{R})$, then every ergodic ν-stationary measure μ on $X(\mathbf{R})$ is either invariant or supported on a proper Γ_ν-invariant subvariety. In particular if there is no Γ_ν-invariant algebraic curve, the random dynamical system (X,ν) is stiff.*

This theorem is mostly interesting when Γ_ν is non-elementary and we will focus on this case in the remainder of this introduction.

Stationary measures supported on invariant curves are rather easy to analyse (see §10.4). Moreover, it is always possible to contract all Γ_ν-invariant curves, creating a complex analytic surface X_0 with finitely many singularities. Then on $X_0(\mathbf{R})$, stiffness holds unconditionally.

This result applies to many interesting examples, because Abelian, K3, and Enriques surfaces, which concentrate most of the dynamically interesting automorphisms on compact complex surfaces, admit a canonical $\mathsf{Aut}(X)$-invariant 2-form. In particular, it applies to the dynamics of pentagon foldings. Note also that linear Anosov maps on $\mathbf{R}^2/\mathbf{Z}^2$ fall into this category, so Theorem A contains the stiffness statement of [21]. While not directly covered by this article, the character variety of the once punctured torus (or the four times punctured sphere) should be amenable to the same strategy (see [27, 66, 67]).

Once stiffness is established, the next step is to *classify invariant measures*. When X is a K3 surface and Γ_ν contains a **parabolic** automorphism, Γ_ν-invariant measures were classified by the first named author in [26]. A parabolic automorphism acts by translations along the fiber of some genus 1 fibration with a shearing property between nearby fibers (see below §11.1 for details). An example is given by the composition of the foldings σ_i and σ_{i+1} of two adjacent vertices in the space of pentagons. In a companion paper [32] we generalize and make more precise the results of [26]. A nice consequence is that for a non-elementary group of $\mathsf{Aut}(X_{\mathbf{R}})$ containing parabolic elements and preserving an area form, any invariant measure is either atomic, or concentrated on a Γ_ν-invariant algebraic curve, or is the restriction of the area form on some open subset of $X(\mathbf{R})$ bounded by a piecewise smooth curve.

For random pentagon foldings, these results give a complete answer to the equidistribution problem raised in §1.1. Indeed, assume for simplicity that the group generated by the five involutions σ_i of $\mathrm{Pent}^0(\ell_0, \ldots, \ell_4)$ does not preserve any proper Zariski closed set, and that $\mathrm{Pent}^0(\ell_0, \ldots, \ell_4)$ is connected. Then the stiffness and classification theorems imply that the only stationary measure is the canonical area form. Therefore by Breiman's law of large numbers, for every initial pentagon $P \in \mathrm{Pent}^0(\ell_0, \ldots, \ell_4)$ and almost every sequence $(m_j) \in \{0, \ldots, 4\}^{\mathbf{N}}$, the random sequence $P_n = (\sigma_{m_{n-1}} \circ \cdots \circ \sigma_{m_0})(P)$ equidistributes with respect to the area form. Thus, quantities like the asymptotic average of the diameter are given by explicit integrals of semi-algebraic functions, independently of the starting pentagon P.

Another example widely studied in the literature is the family of **Wehler surfaces**. These are the smooth surfaces $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by an equation of degree $(2, 2, 2)$. Then for each index $i \in \{1, 2, 3\}$, the projection $\pi_i \colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ which "forgets the variable x_i" has degree 2, so that there is an involution σ_i of X that permutes the two points in the generic fiber of π_i.

Corollary. *Let $X_{\mathbf{R}} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a real Wehler surface such that $X(\mathbf{R})$ is non empty. If $X_{\mathbf{R}}$ is generic, then:*

(1) *the surface X is a K3 surface and there is a unique (up to choosing an orientation of $X(\mathbf{R})$) algebraic 2-form $\mathsf{vol}_{X_{\mathbf{R}}}$ on $X(\mathbf{R})$ such that $\int_{X(\mathbf{R})} \mathsf{vol}_{X_{\mathbf{R}}} = 1$;*
(2) *the group $\mathsf{Aut}(X_{\mathbf{R}})$ is generated by the three involutions σ_i and coincides with $\mathsf{Aut}(X)$; furthermore it preserves the probability measure defined by $\mathsf{vol}_{X_{\mathbf{R}}}$;*
(3) *if ν is finitely supported and Γ_ν has finite index in $\mathsf{Aut}(X_{\mathbf{R}})$ then $(X(\mathbf{R}), \nu)$ is stiff: the only ν-stationary measures on $X(\mathbf{R})$ are convex combinations of the probability measures defined by $\mathsf{vol}_{X_{\mathbf{R}}}$ on the connected components of $X(\mathbf{R})$.*

Here by generic we mean that the equation of X belongs to the complement of at most countably many hypersurfaces in the set of polynomial equations of degree $(2, 2, 2)$ (see §3.1 for details). This result follows from Theorem A together with Proposition 3.3 and Corollary 11.5. Actually in Assertion (3), it is only shown in this paper that the ν-stationary measures are convex combinations of volume forms on components of $X(\mathbf{R})$, together with measures supported on finite orbits. The generic non-existence of finite orbits will be established in a forthcoming paper [31] dedicated to this topic.

Without assuming the existence of parabolic elements in Γ_ν we establish a measure rigidity result in the spirit of Rudolph's theorem on the $\times 2 \times 3$ conjecture.

Theorem B. *Let $X_{\mathbf{R}}$ be a real projective surface and Γ a non-elementary subgroup of $\mathsf{Aut}(X_{\mathbf{R}})$. If all elements of Γ preserve a probability measure μ supported on $X(\mathbf{R})$ and if μ is ergodic and of positive entropy for some $f \in \Gamma$, then μ is absolutely continuous with respect to any area measure on $X(\mathbf{R})$.*

In particular if Γ is a group of area preserving automorphisms, then up to normalization μ will be the restriction of the area form on some Γ-invariant set. **Kummer examples** are a generalization of linear Anosov diffeomorphisms of tori to other projective surfaces (see [33, 37] for more on such mappings). When Γ contains a real Kummer example, we can derive an exact analogue of the classification of invariant measures of [21], that is the assumption "μ has positive entropy" can be replaced by "μ has no atoms" (Theorem 12.5). We also obtain a version of Theorem B for polynomial automorphisms of the affine plane $\mathbb{A}^2_{\mathbf{R}}$(see Theorem 12.6).

1.5. **Some ingredients of the proofs.** The proofs of Theorems A and B rely on the deep results of Brown and Rodriguez-Hertz [22]. To be more precise, recall that an ergodic stationary measure μ on X admits a pair of Lyapunov exponents $\lambda^+(\mu) \geqslant \lambda^-(\mu)$, and that μ is said **hyperbolic** if $\lambda^+(\mu) > 0 > \lambda^-(\mu)$. In this case the (random) Oseledets theorem shows that for μ-almost every x and $\nu^{\mathbf{N}}$-almost every $\omega = (f_j)_{j\in\mathbf{N}}$ in $\mathsf{Aut}(X)^{\mathbf{N}}$, there exists a stable direction $E^s_\omega(x) \subset T_xX_{\mathbf{R}}$. In [22], stiffness is established for area preserving C^2 random dynamical systems on surfaces, under the condition that the stable direction $E^s_\omega(x) \subset T_xX_{\mathbf{R}}$ depends non-trivially on the random itinerary $\omega = (f_j)_{j\in\mathbf{N}}$, or equivalently that stable directions do not induce a measurable Γ_ν-invariant line field. One of our main contributions is to take care of this possibility in our setting: for this we study the dynamics on the *complex* surface X.

Theorem C. *Let X be a complex projective surface and ν be a finitely supported probability measure on $\mathsf{Aut}(X)$. If Γ_ν is non-elementary, then any hyperbolic ergodic ν-stationary measure μ on X satisfies the following alternative:*

(a) *either μ is invariant, and its fiber entropy $h_\mu(X;\nu)$ vanishes;*
(b) *or μ is supported on a Γ_ν-invariant algebraic curve;*
(c) *or the field of Oseledets stable directions of μ is not Γ_ν-invariant; in other words, it genuinely depends on the itinerary $(f_j)_{j\geqslant 0} \in \mathsf{Aut}(X)^{\mathbf{N}}$.*

As opposed to Theorems A and B, this result holds in full generality, without assuming the existence of an invariant volume form nor an invariant real structure. Understanding this somewhat technical result requires a substantial amount of material from the smooth ergodic theory of random dynamical systems, which will be introduced in due time. When μ is not invariant, nor supported by a proper Zariski closed subset, Assertion (c) precisely says that the above mentioned condition on stable directions used in [22] is satisfied. This is our key input towards Theorems A and B.

The arguments leading to Theorem C involve an interesting blend of Hodge theory, pluripotential analysis, and Pesin theory. They rely on the following well-known principle in higher dimensional holomorphic dynamics: if μ is an ergodic hyperbolic stationary measure, μ-almost every point admits a Pesin stable manifold biholomorphic to $\mathbf{C}$; then, according to a classical construction going back to Ahlfors and Nevanlinna, to any immersion $\phi : \mathbf{C} \to X$ is associated a (family of) closed positive $(1,1)$-current(s) describing the asymptotic distribution of $\phi(\mathbf{C})$ in X, hence also a cohomology class in $H^2(X,\mathbf{R})$. These currents provide a link between the infinitesimal dynamics along μ, more precisely its stable manifolds, and the action of Γ_ν on $H^2(X;\mathbf{R})$, which itself can be analyzed by combining tools from complex algebraic geometry with Furstenberg's theory of random products of matrices.

Theorem D. *Let X be a compact projective surface and ν be a finitely supported probability measure on $\mathsf{Aut}(X)$, such that Γ_ν is non-elementary. Let κ_0 be a fixed Kähler form on X.*

(1) *If κ is any Kähler form on X, then for $\nu^{\mathbf{N}}$-almost every $\omega := (f_j)_{j\geqslant 0} \in \mathsf{Aut}(X)^{\mathbf{N}}$ the limit*

$$T^s_\omega := \lim_{n\to+\infty} \frac{1}{\int_X \kappa_0 \wedge (f_n \circ \cdots \circ f_0)^*\kappa} (f_n \circ \cdots \circ f_0)^*\kappa$$

exists as a closed positive $(1,1)$-current. Moreover this current T^s_ω does not depend on κ and has Hölder continuous potentials.

(2) *If the ν-stationary measure μ is ergodic, hyperbolic and not supported on a Γ_ν-invariant proper Zariski closed set, then for μ-almost every x and $\nu^{\mathbf{N}}$-almost every ω, the only Ahlfors-Nevanlinna current of mass 1 (with respect to κ_0) associated to the stable manifold $W^s_\omega(x)$ coincides with T^s_ω.*

The right setting for such a statement is certainly that of a compact Kähler surface. We actually show in §3.6 that any compact Kähler surface supporting a non-elementary group of automorphisms is projective (see also Appendix A for the non-Kähler case). The algebraicity of X is, in fact, a crucial technical ingredient in the proof of assertion (2), because we use techniques of laminar currents which are available only on projective surfaces. Theorem D enters the proof of Theorem C as follows: since Γ_ν is non-elementary, Furstenberg's description of the random action on $H^2(X,\mathbf{R})$ implies that the cohomology class $[T^s_\omega]$ depends non-trivially on ω; therefore for μ-almost every x, $W^s_\omega(x)$ also depends non-trivially on ω.

Remark 1.1. Beyond finitely supported measures, Theorem A, B, C, and D hold under optimal moment conditions on ν (this adds several technicalities, notably in Sections 5 and 6).

1.6. **Organization of the article.** Let X be a compact Kähler surface and ν be a probability measure on $\mathsf{Aut}(X)$.

- In Section 2 we describe the action of $\mathsf{Aut}(X)$ on $H^*(X;\mathbf{Z})$, in particular on the Dolbeault cohomology group $H^{1,1}(X;\mathbf{R})$. The Hodge index theorem endows it with a Minkowski structure, which is essential in our understanding of the dynamics of Γ_ν acting on the cohomology. This section prepares the ground for the analysis of random products of matrices done in Section 5. A delicate point to keep in mind is that the action of a non-elementary subgroup of $\mathsf{Aut}(X)$ on $H^{1,1}(X;\mathbf{R})$ may be reducible.
- Section 3 describes several classes of examples, including pentagon foldings and Wehler's surfaces. It is also shown there that a compact Kähler surface with a non-elementary group of automorphims is necessarily projective (see Theorem E in §3.6).
- After a short Section 4 introducting the vocabulary of random products of diffeomorphisms, Furstenberg's theory of random products of matrices is applied in Section 5 to the study of the action on $H^{1,1}(X;\mathbf{R})$. This, combined with the theory of closed positive currents, leads to the proof of the first assertion of Theorem D in Section 6. The continuity of the potentials of the currents T^s_ω, which plays a key role in the subsequent analysis of Section 8, relies on a recent result of Gouëzel and Karlsson [68].

– Pesin theory enters into play in Section 7, in which the basics of the smooth ergodic theory of random dynamical systems (specialized to complex surfaces) are described in some detail. This is used in Section 8 to relate the Pesin stable manifolds to the currents T^s_ω, using techniques of laminar currents.
– Theorem C is proven in Section 9 by combining ideas of [22] with Theorem D and an elementary fact from local complex geometry inspired by a lemma from [7].
– Theorem A is finally established in Section 10. When Γ_ν is non-elementary (Theorem 10.10) it follows rather directly from [22], Theorem C, and a result of Avila and Viana [2]. Elementary groups are handled separately by using the classification of automorphism groups of compact Kähler surfaces (see Theorems 10.3 and Proposition 10.5). Note that the symmetry of ν is used only in the elementary case.
– Sections 11 and 12 are devoted to the classification of invariant measures. In Section 11, after recalling the results of [26, 32], we show that when Γ_ν contains a parabolic element, any invariant measure giving no mass to subvarieties is hyperbolic. Our approach is inspired by the work of Barrientos and Malicet [5]. This provides an interesting connection with some classical problems in conservative dynamics (see §11.3 for a discussion). In Section 12 we prove Theorem B, as well as several related results. This relies on a measure rigidity theorem of [22], together with ideas similar to the ones involved in the proof of Theorem C.

This article is part of a series of papers dedicated to the dynamics of groups of automorphisms of compact Kähler surfaces, notably K3 and Enriques surfaces. The article [32] is focused on the classification of invariant measures in presence of parabolic elements. In [31] we study the existence of finite orbits for non-elementary group actions; tools from arithmetic dynamics are used to study the case where X and its automorphisms are defined over a number field. In a forthcoming work, we plan to extend the techniques of Brown and Rodriguez-Hertz to the complex setting; with Theorem C at hand, this would extend Theorem A from the real to the complex case.

1.7. **Conventions.** Throughout the paper C stands for a "constant" which may change from line to line, independently of some asymptotic quantity that should be clear from the context (typically an integer n corresponding to the number of iterations of a dynamical system). Using this convention, we write $a \lesssim b$ if $a \leqslant Cb$ and $a \asymp b$ if $a \lesssim b \lesssim a$. All complex manifolds are considered to be connected, so from now on "complex manifold" stands for "connected complex manifold". For a random dynamical system on a disconnected complex manifold, there is a finite index sugbroup Γ' of Γ_ν which stabilizes each connected component, and an induced measure ν' on Γ' with properties qualitatively similar to those of ν (see §10.2), so the problem is reduced to the connected case.

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2. Hodge index theorem and Minkowski spaces

In this section we define the notion of a non-elementary group of automorphisms of a compact Kähler surface X. We study the action of such a group on the cohomology of X, and in particular the question of (ir)reducibilty. We refer to Appendix A for a discussion of the non-Kähler case.

2.1. **Cohomology.**

2.1.1. *Hodge decomposition.* Denote by $H^*(X;R)$ the cohomology of X with coefficients in the ring R; we shall use $R = \mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ or $\mathbf{C}$. The group $\mathsf{Aut}(X)$ acts on $H^*(X;\mathbf{Z})$, and $\mathsf{Aut}(X)^*$ will denote the image of $\mathsf{Aut}(X)$ in $\mathsf{GL}(H^2(X;\mathbf{Z}))$.The Hodge decomposition

$$H^k(X;\mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(X;\mathbf{C}) \tag{2.1}$$

is $\mathsf{Aut}(X)$-invariant. On $H^{0,0}(X;\mathbf{C})$ and $H^{2,2}(X;\mathbf{C})$, $\mathsf{Aut}(X)$ acts trivially. Throughout the paper we denote by $[\alpha]$ the cohomology class of a closed differential form (or current) α.

The intersection form on $H^2(X;\mathbf{Z})$ will be denoted by $\langle \cdot \mid \cdot \rangle$; the self-intersection $\langle a|a\rangle$ of a class a will also be denoted by a^2 for simplicity. This intersection form is $\mathsf{Aut}(X)$-invariant. By the Hodge index theorem, it is positive definite on the real part of $H^{2,0}(X;\mathbf{C}) \oplus H^{0,2}(X;\mathbf{C})$ and it is non-degenerate and of signature $(1, h^{1,1}(X) - 1)$ on $H^{1,1}(X;\mathbf{R})$.

Lemma 2.1. *The restriction of* $\mathsf{Aut}(X)^*$ *to the subspace* $H^{2,0}(X;\mathbf{C})$ *(resp.* $H^{0,2}(X;\mathbf{C})$*) is contained in a compact subgroup of* $\mathsf{GL}(H^{2,0}(X;\mathbf{C}))$ *(resp.* $\mathsf{GL}(H^{0,2}(X;\mathbf{C}))$*).*

Proof. This follows from the fact that $\langle\cdot|\cdot\rangle$ is positive definite on the real part of $H^{2,0}(X;\mathbf{C}) \oplus H^{0,2}(X;\mathbf{C})$. An equivalent way to describe this argument it to identify $H^{2,0}(X;\mathbf{C})$ with the space of holomorphic 2-forms on X. Then, there is a natural, $\mathsf{Aut}(X)$-invariant, hermitian form on this space: given two holomorphic 2-forms Ω_1 and Ω_2, the hermitian product is the integral

$$\int_X \Omega_1 \wedge \overline{\Omega_2}. \tag{2.2}$$

Thus, the image of $\mathsf{Aut}(X)$ in $\mathsf{GL}(H^{2,0}(X;\mathbf{C}))$ is relatively compact. □

The **Néron-Severi group** $\mathrm{NS}(X;\mathbf{Z})$ is, by definition, the discrete subgroup of $H^{1,1}(X;\mathbf{R})$ defined by $\mathrm{NS}(X;\mathbf{Z}) = H^{1,1}(X;\mathbf{R}) \cap H^2(X;\mathbf{Z})$; more precisely, it is the intersection of $H^{1,1}(X;\mathbf{R})$ with the image of $H^2(X;\mathbf{Z})$ in $H^2(X;\mathbf{R})$, i.e. with the torsion free part of the Abelian group $H^2(X;\mathbf{Z})$. The Lefschetz theorem on $(1,1)$-classes identifies $\mathrm{NS}(X;\mathbf{Z})$ with the subgroup of $H^{1,1}(X;\mathbf{R})$ given by Chern classes of line bundles on X. The Néron-Severi group is $\mathsf{Aut}(X)$-invariant, as well as $\mathrm{NS}(X;R) := \mathrm{NS}(X;\mathbf{Z}) \otimes_{\mathbf{Z}} R$ for $R = \mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$. The dimension of $\mathrm{NS}(X;\mathbf{R})$ is the **Picard number** $\rho(X)$.

2.1.2. *Norm of* f^*. Let $|\cdot|$ be any norm on the vector space $H^*(X;\mathbf{C})$. If L is a linear transformation of $H^*(X;\mathbf{C})$ we denote by $\|L\|$ the associated operator nom and if $W \subset H^*(X;\mathbf{C})$ is an L-invariant subspace of $H^*(X;\mathbf{C})$, we denote by $\|L\|_W$ the operator norm of $L|_W$.

If u is an element of $H^{1,0}(X;\mathbf{C})$, then $u \wedge \overline{u}$ is an element of $H^{1,1}(X;\mathbf{R})$ such that $|u|^2 \leqslant C\,|u \wedge \overline{u}|$ for some constant C that depends only on the choice of norm on the cohomology; in particular, the norm of f^* on $H^{1,0}(X;\mathbf{C})$ is controlled by the norm of f^* on $H^{1,1}(X;\mathbf{C})$.

Using complex conjugation, the same results hold on $H^{0,1}(X;\mathbf{C})$; by Poincaré duality we also control $\|f^*\|_{H^{p,q}(X;\mathbf{C})}$ for $p+q>2$. Together with Lemma 2.1, we obtain:

Lemma 2.2. *Let X be a compact Kähler surface. There exists a constant $C_0>1$ such that*

$$C_0^{-1}\|f^*\|_{H^*(X;\mathbf{C})} \leqslant \|f^*\|_{H^{1,1}(X;\mathbf{R})} \leqslant \|f^*\|_{H^*(X;\mathbf{C})}$$

for every automorphism $f \in \mathsf{Aut}(X)$.

2.2. **The Kähler, nef, and pseudo-effective cones.** (See [18, 83] for details on the notions introduced in this section.)

Let $\mathrm{Kah}(X) \subset H^{1,1}(X;\mathbf{R})$ be the **Kähler cone**, i.e. the cone of classes of Kähler forms. Its closure $\overline{\mathrm{Kah}}(X)$ is a salient, closed, convex cone, and

$$\mathrm{Kah}(X) \subset \overline{\mathrm{Kah}}(X) \subset \{v \in H^{1,1}(X;\mathbf{R})\ ;\ \langle v\,|\,v\rangle \geqslant 0\}. \tag{2.3}$$

The intersection $\mathrm{NS}(X;\mathbf{R})\cap\mathrm{Kah}(X)$ is the **ample cone** $\mathrm{Amp}(X)$, while $\mathrm{NS}(X;\mathbf{R})\cap\overline{\mathrm{Kah}}(X)$ is the **nef cone** $\mathrm{Nef}(X)$. They are all invariant under the action of $\mathsf{Aut}(X)$ on $H^{1,1}(X;\mathbf{R})$. We shall also say that the elements of $\overline{\mathrm{Kah}}(X)$ are nef classes, but the notation $\mathrm{Nef}(X)$ will be reserved for $\mathrm{NS}(X;\mathbf{R})\cap\overline{\mathrm{Kah}}(X)$. The set of classes of closed positive currents is the **pseudo-effective cone** $\mathrm{Psef}(X)$. This cone is an $\mathsf{Aut}(X)$-invariant, salient, closed, convex cone. It is dual to $\overline{\mathrm{Kah}}(X)$ for the intersection form (see [18, Lem. 4.1]):

$$\overline{\mathrm{Kah}}(X) = \{u \in H^{1,1}(X;R)\ ;\ \langle u\,|\,v\rangle \geqslant 0 \quad \forall v \in \mathrm{Psef}(X)\} \tag{2.4}$$

and vice-versa.

We fix once and for all a reference Kähler form κ_0 with $[\kappa_0]^2 = \int \kappa_0\wedge\kappa_0 = 1$. Then we define the **mass** of a pseudo-effective class a by $\mathbf{M}(a) = \langle a\,|\,[\kappa_0]\rangle$, or equivalently the mass of a closed positive current T by $\mathbf{M}(T) = \int T\wedge\kappa_0$; we may also extend this definition to any class, pseudo-effective or not (but then $\mathbf{M}(a) = \langle a\,|\,[\kappa_0]\rangle$ may be negative). The compactness of the set of closed positive currents of mass 1 implies that, for any norm $|\cdot|$ on $H^{1,1}(X,\mathbf{R})$, there exists a constant C such that

$$\forall a \in \mathrm{Psef}(X),\ \ C^{-1}\,|a| \leqslant \mathbf{M}(a) \leqslant C\,|a|\,. \tag{2.5}$$

If v is an element of $\mathrm{Psef}(X)$ and $v^2 \geqslant 0$, then by the Hodge index theorem we know that $\langle u\,|\,v\rangle \geqslant 0$ for every class $u \in H^{1,1}(X;\mathbf{R})$ such that $u^2 \geqslant 0$ and $\langle u\,|\,[\kappa_0]\rangle \geqslant 0$ (see Equation (2.7)). So, in Equation (2.4), the most important constraints come from the classes $v \in \mathrm{Psef}(X)$ with $v^2<0$. If v is such a class, its Zariski decomposition expresses v as a sum $v = p(v)+n(v)$ with the following properties (see [18]):

(1) this decomposition is orthogonal: $\langle p(v)\,|\,n(v)\rangle = 0$;
(2) $p(v)$ is a nef class, i.e. $p(v) \in \overline{\mathrm{Kah}}(X)$;
(3) $n(v)$ is negative: it is a sum $n(v) = \sum_i a_i[D_i]$ with positive coefficients $a_i \in \mathbf{R}_+^*$ of classes of irreducible curves $D_i \subset X$ such that the Gram matrix $(\langle D_i\,|\,D_j\rangle)$ is negative definite.

Proposition 2.3. *If a ray $\mathbf{R}_+v$ of the cone $\mathrm{Psef}(X)$ is extremal, then either $v^2 \geqslant 0$ or $\mathbf{R}_+v = \mathbf{R}_+[D]$ for some irreducible curve D such that $D^2<0$. The cone $\mathrm{Psef}(X)$ contains at most countably many extremal rays $\mathbf{R}_+v$ with $v^2<0$.*

Let u be an isotropic element of $\overline{\mathrm{Kah}}(X)$. If $\mathbf{R}_+u$ is not an extremal ray of $\mathrm{Psef}(X)$, then u is proportional to an integral class $u' \in \mathrm{NS}(X;\mathbf{Z})$.

Proof. If $\mathbf{R}_+ v$ is extremal, the Zariski decomposition $v = p(v) + n(v)$ involves only one term. If $v = p(v)$ then $v^2 \geqslant 0$. Otherwise $v = n(v)$ and by extremality $n(v) = a[D]$ for some irreducible curve D with $D^2 < 0$. The countability assertion follows, because $\mathrm{NS}(X; \mathbf{Z})$ is countable. For the last assertion, multiply u by $\langle u | [\kappa_0] \rangle^{-1}$ to assume $\langle u | [\kappa_0] \rangle = 1$ and write u as a convex combination $u = \int v \, d\alpha(v)$, where α is a probability measure on $\mathrm{Psef}(X)$ such that α-almost every v satisfies

- $\langle v | [\kappa_0] \rangle = 1$,
- $\mathbf{R}_+ v$ is extremal in $\mathrm{Psef}(X)$ and does not contain u.

Since u is nef, $\langle u \,|\, v \rangle \geqslant 0$ for each v; and u being isotropic, we get $v \in u^\perp \backslash \mathbf{R} u$ for α-almost every v. By the Hodge index theorem, $v^2 < 0$ almost surely. Now, the first assertion of this proposition implies that $v \in \mathbf{R}_+[D_v]$ for some irreducible curve $D_v \subset X$ with negative self-intersection; there are only countably many classes of that type, thus α is purely atomic, and u belongs to $\mathrm{Vect}([D_v]; \alpha(v) > 0)$, a subspace of $\mathrm{NS}(X; \mathbf{R})$ defined over $\mathbf{Q}$. On this subspace, q_X is semi-negative, and by the Hodge index theorem its kernel is $\mathbf{R}u$. Since $\mathrm{Vect}([D_v]; \alpha(v) > 0)$ and q_X are defined over $\mathbf{Q}$, we deduce that u is proportional to an integral class. □

2.3. **Non-elementary subgroups of** $\mathsf{Aut}(X)$**.** When X is a compact Kähler surface, the action of $\mathsf{Aut}(X)$ on $H^{1,1}(X, \mathbf{R})$ is subject to several constraints: the Hodge index theorem implies that it must preserve a Minkowski structure and in addition it preserves the lattice given by the Neron-Severi group. In this section we review the first consequences of these constraints.

2.3.1. *Isometries of Minkowski spaces.* Consider the Minkowski space $\mathbf{R}^{m+1}$, endowed with its quadratic form q of signature $(1, m)$ defined by

$$q(x) = x_0^2 - \sum_{i=1}^{m} x_i^2. \tag{2.6}$$

The corresponding bilinear form will be denoted $\langle \cdot | \cdot \rangle$. For future reference, note the following reverse Schwarz inequality:

$$\text{if } \; q(x) \geqslant 0 \text{ and } q(x') \geqslant 0 \; \text{ then } \; \langle x \,|\, x' \rangle \geqslant q(x)^{1/2} q(x')^{1/2} \tag{2.7}$$

with equality if and only if x and x' are collinear. We say that a subspace $W \subset \mathbf{R}^{m+1}$ is of **Minkowski type** if the restriction $q_{|W}$ is non-degenerate and of signature $(1, \dim(W) - 1)$.

In this section, we review some well-known facts concerning isometries of $\mathbf{R}^{1,m} = (\mathbf{R}^{m+1}, q)$ (see e.g. [101, 75, 61] for details). We denote by $|\cdot|$ the Euclidean norm on $\mathbf{R}^{m+1}$, and by $\mathbb{P} \colon \mathbf{R}^{m+1} \backslash \{0\} \to \mathbb{P}(\mathbf{R}^{m+1})$ the projection on the projective space $\mathbb{P}(\mathbf{R}^{m+1}) = \mathbb{P}^m(\mathbf{R})$.

The hyperboloid $\{x \; ; \; q(x) = 1\}$ has two components, and we denote by $\mathsf{O}^+_{1,m}(\mathbf{R})$ the subgroup of the orthogonal group $\mathsf{O}_{1,m}(\mathbf{R})$ that preserves the component $\mathcal{Q} = \{q(x) = 1 \; ; \; x_0 > 0\}$. Endowed with the distance $d_{\mathbb{H}}(x, y) = \cosh^{-1} \langle x \,|\, y \rangle$, $\mathcal{Q}$ is a model of the real hyperbolic space $\mathbb{H}^m$ of dimension m. The boundary at infinity of $\mathbb{H}^m$ will be identified with $\partial \mathbb{P}(\mathcal{Q}) \subset \mathbb{P}(\mathbf{R}^{m+1})$ and will be denoted by $\partial \mathbb{H}^m$. It is the set of isotropic lines of q.

Any isometry γ of $\mathbb{H}^m$ is induced by an element of $\mathsf{O}^+_{1,m}(\mathbf{R})$, and extends continuously to $\partial \mathbb{H}^m$: its action on $\partial \mathbb{H}^m$ is given by its linear projective action on $\mathbb{P}(\mathbf{R}^{m+1})$. Isometries are classified in three types, according to their fixed point set in $\mathbb{H}^m \cup \partial \mathbb{H}^m$:

- γ is **elliptic** if γ has a fixed point in $\mathbb{H}^m$;

– γ is **parabolic** if γ has no fixed point in $\mathbb{H}^m$ and a unique fixed point in $\partial\mathbb{H}^m$;
– γ is **loxodromic** if γ has no fixed point in $\mathbb{H}^m$ and exactly two fixed points in $\partial\mathbb{H}^m$.

A subgroup Γ of $\mathsf{O}^+_{1,m}(\mathbf{R})$ is **non-elementary** if it does not preserve any finite subset of $\mathbb{H}^m \cup \partial\mathbb{H}^m$. Equivalently Γ is non-elementary if and only if it contains two loxodromic elements with disjoint fixed point sets.

The group $\mathsf{O}^+_{1,m}(\mathbf{R})$ admits a **Cartan** or **KAK decomposition** (see [61, §I.5]). To state it, denote by $e_0 = (1, 0, \ldots, 0)$ the first vector of the canonical basis of $\mathbf{R}^{m+1}$; this vector is an element of $\mathbb{H}^m$, and its stabilizer $\mathrm{Stab}(e_0)$ in $\mathsf{O}^+_{1,m}(\mathbf{R})$ is a maximal compact subgroup, isomorphic to $\mathsf{O}_{m-1}(\mathbf{R})$.

Lemma 2.4. *Every $\gamma \in \mathsf{O}^+_{1,m}(\mathbf{R})$ can be written (non-uniquely) as $\gamma = k_1 a k_2$, where $k_i \in \mathrm{Stab}(\mathbf{e}_0)$ and a is a matrix of the form*

$$\begin{pmatrix} \cosh r & \sinh r & 0 \\ \sinh r & \cosh r & 0 \\ 0 & 0 & \mathrm{id}_{m-1} \end{pmatrix}$$

with $r = d_{\mathbb{H}}(e_0, \gamma e_0)$.

Proof. Note that $K := \mathrm{Stab}(e_0)$ acts transitively on the set of hyperbolic geodesics through e_0. Denote by L the hyperbolic geodesic $\mathbb{H}^m \cap \mathrm{Vect}(e_0, e_1)$, where $e_1 = (0, 1, 0, \ldots, 0)$ is the second element of the canonical basis of $\mathbf{R}^{m+1}$. If $\gamma(e_0) = e_0$ then γ belongs to K and we are done. Otherwise choose $k_1, k_2 \in K$ such that $k_1^{-1}(\gamma(e_0)) \in L$, $k_2(\gamma^{-1}(e_0)) \in L$, and e_0 lies in between $k_2(\gamma^{-1}(e_0))$ and $k_1^{-1}(\gamma(e_0))$; then e_0 is in fact the middle point of $[k_2(\gamma^{-1}(e_0)), k_1^{-1}(\gamma(e_0))]$ because $d_{\mathbb{H}}(e_0, \gamma(e_0)) = d_{\mathbb{H}}(e_0, \gamma^{-1}(e_0)) > 0$. The isometry $a := k_1^{-1}\gamma k_2^{-1}$ maps $k_2(\gamma^{-1}(e_0)) \in L$ to e_0 and e_0 to $k_1^{-1}(\gamma(e_0)) \in L$. It follows that a is a hyperbolic translation along L of translation length $d_{\mathbb{H}}(e_0, k_1^{-1}(\gamma(e_0)) = d_{\mathbb{H}}(e_0, \gamma(e_0))$. To conclude, change a into $a \circ k^{-1}$ and k_2 into $k \circ k_2$ where k is the element of K that preserves e_1 and acts like a on the orthogonal complement of $\mathrm{Vect}(e_0, e_1)$. □

Corollary 2.5. *If $\|\cdot\|$ denotes the operator norm associated to the euclidean norm in $\mathbf{R}^{m+1}$, then $\|\gamma\| = \|a\|$, where $\gamma = k_1 a k_2$ is any Cartan decomposition of γ. In particular $\|\gamma\| = \|\gamma^{-1}\|$ and*

$$\|\gamma\| \asymp \cosh d_{\mathbb{H}}(e_0, \gamma(e_0)) \asymp |\gamma e_0|\,.$$

Furthermore for every $e \in \mathbb{H}^m$ and any $\gamma \in \mathsf{O}^+_{1,m}(\mathbf{R})$

$$\|\gamma\| \asymp \cosh d_{\mathbb{H}}(e, \gamma(e)),$$

where the implied constant depends only on the base point e.

This is an immediate corollary of the previous lemma.

2.3.2. *Irreducibility.* A non-elementary subgroup of $\mathsf{O}^+_{1,m}(\mathbf{R})$ does not need to act irreducibly on $\mathbf{R}^{m+1}$. Proposition 2.8, below, clarifies the possible situations.

Lemma 2.6. *Let Γ be a non-elementary subgroup of $\mathsf{O}^+_{1,m}(\mathbf{R})$ (resp. γ be an element of $\mathsf{O}^+_{1,m}(\mathbf{R})$). Let W be a subspace of $\mathbf{R}^{1,m}$.*

(1) *If W is Γ-invariant, then either $(W, q|_W)$ is a Minkowski space and $\Gamma|_W$ is non-elementary, or $q|_W$ is negative definite and $\Gamma|_W$ is contained in a compact subgroup of $\mathsf{GL}(W)$.*

(2) *If W is γ-invariant and contains a vector w with $q(w) > 0$, then $\gamma|_W$ has the same type (elliptic, parabolic, or loxodromic) as γ; in particular, W contains the γ-invariant isotropic lines if γ is parabolic or loxodromic.*

Proof. The restriction $q|_W$ is either a Minkowski form or is negative definite. Indeed, it cannot be positive definite, because W would then be a Γ-invariant line intersecting the hyperbolic space $\mathbb{H}^m$ in a fixed point; and it cannot be degenerate, since otherwise its kernel would give a Γ-invariant point on $\partial\mathbb{H}^m$. If $q|_W$ is a Minkowski form and $\Gamma|_W$ is elementary, then Γ preserves a finite subset of $(\mathbb{H}^m \cup \partial\mathbb{H}^m) \cap V$ and Γ itself is elementary. This proves the first assertion. The proof of the second one is similar. □

Let Γ be a non-elementary subgroup of $\mathsf{O}^+_{1,m}(\mathbf{R})$. Let $\mathsf{Zar}(\Gamma) \subset \mathsf{O}_{1,m}(\mathbf{R})$ be the Zariski closure of Γ, and

$$G = \mathsf{Zar}(\Gamma)^{\mathrm{irr}} \tag{2.8}$$

the neutral component of $\mathsf{Zar}(\Gamma)$, for the Zariski topology. Note that the Lie group $G(\mathbf{R})$ is not necessarily connected for the euclidean topology.

Lemma 2.7. *The group $\Gamma \cap G(\mathbf{R})$ has finite index in Γ. If Γ_0 is a finite index subgroup of Γ, then $\mathsf{Zar}(\Gamma_0)^{\mathrm{irr}} = G$.*

Proof. The index of G in $\mathsf{Zar}(\Gamma)$ is equal to the number ℓ of irreducible components of the algebraic variety $\mathsf{Zar}(\Gamma)$, and the index of $\Gamma \cap G(\mathbf{R})$ in Γ is at most ℓ. Now, let Γ_0 be a finite index subgroup of Γ. Then, $\Gamma_0 \cap G(\mathbf{R})$ has finite index in $\Gamma \cap G(\mathbf{R})$, and we can fix a finite subset $\{\alpha_1, \ldots, \alpha_k\} \subset \Gamma \cap G(\mathbf{R})$ such that $\Gamma \cap G(\mathbf{R}) = \bigcup_j \alpha_j(\Gamma_0 \cap G(\mathbf{R}))$. So

$$\mathsf{Zar}(\Gamma \cap G(\mathbf{R})) \subset \bigcup_j \alpha_j \mathsf{Zar}(\Gamma_0 \cap G(\mathbf{R})) \subset G(\mathbf{R}). \tag{2.9}$$

Because $\Gamma \cap G(\mathbf{R})$ is Zariski dense in the irreducible group G we find $G = \mathsf{Zar}(\Gamma_0 \cap G(\mathbf{R}))$. So $G \subset \mathsf{Zar}(\Gamma_0)$ and the Lemma follows as $G = \mathsf{Zar}(\Gamma)^{\mathrm{irr}}$. □

Proposition 2.8. *Let $\Gamma \subset \mathsf{O}^+_{1,m}(\mathbf{R})$ be non-elementary.*

(1) *The representation of $\Gamma \cap G(\mathbf{R})$ (resp. of $G(\mathbf{R})$) on $\mathbf{R}^{1,m}$ splits as a direct sum of irreducible representations, with exactly one irreducible factor of Minkowski type:*
$$\mathbf{R}^{1,m} = V_+ \oplus V_0;$$
here V_+ is of Minkowski type, and V_0 is an orthogonal sum of irreducible representations $V_{0,j}$ on which the quadratic form q is negative definite.
(2) *The restriction $G|_{V_+}$ coincides with $\mathsf{SO}(V_+; q|_{V_+})$.*
(3) *The subspaces V_+ and V_0 are Γ-invariant, and the representation of Γ on V_+ is strongly irreducible.*

Proof. A group Γ is non-elementary if and only if any of its finite index subgroups is non-elementary. So, we can apply Lemma 2.6 to $\Gamma \cap G(\mathbf{R})$: if $W \subset \mathbf{R}^{1,m}$ is a non-trivial $(\Gamma \cap G(\mathbf{R}))$-invariant subspace, $q|_W$ is non-degenerate. As a consequence, $\mathbf{R}^{1,m}$ is the direct sum $W \oplus W^\perp$, where $W^\perp$ is the orthogonal complement of W with respect to q. This implies that the representation of $\Gamma \cap G(\mathbf{R})$ on $\mathbf{R}^{1,m}$ splits as a direct sum of irreducible representations, with exactly one irreducible factor of Minkowski type, as asserted in (1).

The group G preserves this decomposition, and by Proposition 1 of [9], the restriction $G|_{V_+}$ coincides with $\mathsf{SO}(V_+; q|_{V_+})$; this group is isomorphic to the almost simple group $\mathsf{SO}_{1,k}(\mathbf{R})$, with $1 + k = \dim(V_+)$. This proves the second assertion.

Since G is normalized by Γ, we see that for any $\gamma \in \Gamma$, γV^+ is a G-invariant subspace of the same dimension as V^+ and on which q is of Minkowski type. Hence V_+, as well as its orthogonal complement V_0 are Γ-invariant. By Lemma 2.7, the action of Γ on V_+ is strongly irreducible; indeed, if a finite index subgroup Γ_0 in Γ preserves a non-trivial subspace of V_+ then, by Zariski density of $\Gamma_0 \cap G(\mathbf{R})$ in $G(\mathbf{R})$, this subspace must be V_+ itself. On V_0, Γ permutes the irreducible factors $V_{0,j}$. □

Now, set $V = \mathbf{R}^{1,m}$ and assume that there is a lattice $V_{\mathbf{Z}} \subset V$ such that

(i) $V_{\mathbf{Z}}$ is Γ-invariant;
(ii) the quadratic form q is an integral quadratic form on $V_{\mathbf{Z}}$.

In other words, there is a basis of V with respect to which q and the elements of Γ are given by matrices with integer coefficients. In particular, V has a natural $\mathbf{Q}$-structure, with $V(\mathbf{Q}) = V_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Q}$. This situation naturally arises for the action of automorphisms of compact Kähler surfaces on $\mathrm{NS}(X; \mathbf{R})$. The next lemma will be useful in [31].

Lemma 2.9. *If Γ contains a parabolic element, the decomposition $V_+ \oplus V_0$ is defined over $\mathbf{Q}$, $\Gamma|_{V_0}$ is a finite group, and G is the subgroup $\mathsf{SO}(V_+; q) \times \{\mathrm{id}_{V_0}\}$ of $\mathsf{O}(V; q)$.*

Proof. If $\gamma \in \Gamma$ is parabolic, it fixes pointwise a unique isotropic line, therefore this line is defined over $\mathbf{Q}$. In addition it must be contained in V_+ because $(\gamma^n(u))_{n \geqslant 0}$ converges to the boundary point determined by this line for every $u \in \mathbb{H}^m$. So, V_+ contains at least one non-zero element of $V_{\mathbf{Z}}$. Since the action of Γ on V_+ is irreducible, the orbit of this vector generates V_+ and is contained in $V_{\mathbf{Z}}$, so V_+ is defined over $\mathbf{Q}$. Its orthogonal complement V_0 is also defined over $\mathbf{Q}$, because q itself is defined over $\mathbf{Q}$. As a consequence, $\Gamma|_{V_0}$ preserves the lattice $V_0 \cap V_{\mathbf{Z}}$ and the negative definite form $q|_{V_0}$; hence, it is finite. Thus $G|_{V_0}$ is trivial and the last assertion follows from the above mentioned equality $G|_{V_+} = \mathsf{SO}(V_+; q|_{V_+})$. □

Example 2.10. The purpose of this example is to show that the existence of a parabolic element in Γ is indeed necessary in Lemma 2.9, even for a group of automorphisms of a K3 surface.

Let a be a positive square free integer, for instance $a = 7$ or 15. Let α be the positive square root $\sqrt{a}$, K be the quadratic field $\mathbf{Q}(\alpha)$, and η be the unique non-trivial automorphism of K, sending α to its conjugate $\overline{\alpha} := \eta(\alpha) = -\sqrt{a}$. We view η as a second embedding of K in $\mathbf{C}$. Let $\mathcal{O}_K$ be the ring of integers of K.

Let ℓ be an integer $\geqslant 2$. Consider the quadratic form in $\ell + 1$ variables defined by

$$q_\ell(x_0, x_1, \ldots, x_\ell) = \alpha x_0^2 - x_1^2 - \cdots - x_\ell^2. \tag{2.10}$$

It is non-degenerate and its signature is $(1, \ell)$. The orthogonal group $\mathsf{O}(q_\ell; \mathcal{O}_K)$ is a lattice in the real algebraic group $\mathsf{O}(q_\ell, \mathbf{R})$. The conjugate quadratic form $\overline{q_\ell} = \overline{\alpha} x_0^2 - x_1^2 - \cdots - x_\ell^2$ is negative definite.

Embed $\mathcal{O}_K^{\ell+1}$ into $\mathbf{R}^{2\ell+2}$ by the map $(x_i) \mapsto (x_i, \eta(x_i))$, to get a lattice $\Lambda \subset \mathbf{R}^{2\ell+2}$ and consider the quadratic form $Q_\ell := q_\ell \oplus \overline{q_\ell}$. Then embed $\mathsf{O}(q_\ell; \mathcal{O}_K)$ into $\mathsf{O}(Q_\ell; \mathbf{R})$ by the homomorphism $A \in \mathsf{O}(q_\ell, \mathcal{O}_K) \mapsto A \oplus \eta(A)$; we denote its image by $\Gamma_\ell^* \subset \mathsf{O}(Q_\ell; \mathbf{R})$. It is shown in [97], Chapter 6.4, that

– Q_ℓ is defined over $\mathbf{Z}$ with respect to Λ,
– $\Gamma_\ell^* \subset \mathsf{O}(Q_\ell;\mathbf{Z})$ (with respect to this integral structure),
– the group $G = \mathsf{Zar}(\Gamma_\ell^*)^{\rm irr}$ coincides with $\mathsf{SO}(q_\ell;\mathbf{R}) \times \mathsf{SO}^0(\overline{q_\ell};\mathbf{R})$ (and the group $\eta(\mathsf{O}(q_\ell;\mathcal{O}_K))$ is dense in the compact group $\mathsf{O}(\overline{q_\ell};\mathbf{R})$).

Now, assume $2 \leqslant \ell \leqslant 4$, so that $2\ell+2 \leqslant 10$, and change Q_ℓ into $4Q_\ell$: it is an even quadratic form on the lattice $\Lambda \simeq \mathbf{Z}^{2\ell+2}$. According to [98, Corollary 2.9], there is a complex projective K3 surface X for which $(\mathrm{NS}(X;\mathbf{Z}), q_X)$ is isometric to $(\Lambda, 4Q_\ell)$. On such a surface, the self-intersection of every curve is divisible by 4 and consequently there is no (-2)-curve. So, by the Torelli theorem for K3 surfaces (see [6]), $\mathsf{Aut}(X)^*_{|\mathrm{NS}(X;\mathbf{Z})}$ has finite index in $\mathsf{O}(4Q_\ell;\mathbf{Z})$.

Since $\mathsf{O}(4Q_\ell;\mathbf{Z}) = \mathsf{O}(Q_\ell;\mathbf{Z})$ we can view Γ_ℓ^* as a subgroup of $\mathsf{O}(4Q_\ell;\mathbf{Z})$. Set $\Gamma^* = \mathsf{Aut}(X)^* \cap \Gamma_\ell^*$ and let Γ denote its pre-image in $\mathsf{Aut}(X)$. Then, Γ is a subgroup of $\mathsf{Aut}(X)$ for which the decomposition $\mathrm{NS}(X;\mathbf{R})_+ \oplus \mathrm{NS}(X;\mathbf{R})_0$ is non-trivial (here, both have dimension $\ell+1$) while the representation is irreducible over $\mathbf{Q}$.

2.3.3. *The hyperbolic space $\mathbb{H}_X$.* Let X be a compact Kähler surface. By the Hodge index theorem, the intersection form on $H^{1,1}(X,\mathbf{R})$ has signature $(1, h^{1,1}(X)-1)$. The hyperboloid

$$\{u \in H^{1,1}(X,\mathbf{R}),\ \langle u \,|\, u\rangle\} = 1$$

has two connected components, one of which intersecting the Kähler cone. The hyperbolic space $\mathbb{H}_X$ is by definition this connected component, which is thus a model of the hyperbolic space of dimension $h^{1,1}(X)-1$. We denote by $d_{\mathbb{H}}$ the hyperbolic distance, which is defined as before by $\cosh(d_{\mathbb{H}}(u,v)) = \langle u \,|\, v\rangle$. From Lemma 2.2 and Corollary 2.5 we see that if $\|\cdot\|$ is any norm on $H^*(X,\mathbf{C})$, then $\|f^*\| \asymp \|(f^*)^{-1}\| \asymp \langle [\kappa_0] \,|\, f^*[\kappa_0]\rangle$ (here κ_0 is the fixed Kähler form introduced in Section 2.2).

According to the classification of isometries of hyperbolic spaces, there are three types of automorphisms: **elliptic**, **parabolic** and **loxodromic**. An important fact for us is that the type of isometry is related to the dynamics on X; for instance, every parabolic automorphism preserves a genus 1 fibration, every loxodromic automorphism has positive topological entropy (see [28] for more details). A subgroup Γ of $\mathsf{Aut}(X)$ is said **non-elementary** if its action on $\mathbb{H}_X$ is non-elementary. As we shall see below, the existence of such a subgroup forces X to be projective:

Theorem 2.11. *If X is a compact Kähler surface such that $\mathsf{Aut}(X)$ is non-elementary, then X is projective.*

For expository reasons, the proof of this result is postponed to §3.6.2, Theorem E.

2.3.4. *Automorphisms and Néron-Severi groups.* Let X be a compact Kähler surface and Γ be a non-elementary subgroup of $\mathsf{Aut}(X)$. Let $\Gamma^*_{p,q}$ be the image of Γ in $\mathsf{GL}(H^{p,q}(X;\mathbf{C}))$, and Γ^* be its image in $\mathsf{GL}(H^2(X;\mathbf{C}))$. If we combine Proposition 2.8 together with Lemma 2.1 for $\Gamma^*_{1,1}$, we get an invariant decomposition

$$H^{1,1}(X;\mathbf{R}) = H^{1,1}(X;\mathbf{R})_+ \oplus H^{1,1}(X;\mathbf{R})_0. \tag{2.11}$$

Denote by $H^2(X;\mathbf{R})_0$ the direct sum of $H^{1,1}(X;\mathbf{R})_0$ and of the real part of $H^{2,0}(X;\mathbf{C}) \oplus H^{0,2}(X;\mathbf{C})$; then

$$H^2(X;\mathbf{R}) = H^{1,1}(X;\mathbf{R})_+ \oplus H^2(X;\mathbf{R})_0 \tag{2.12}$$

and $\Gamma^*|_{H^2(X;\mathbf{R})_0}$ is contained in a compact group (see Lemma 2.1). The Néron-Severi group is Γ-invariant, and since X is projective it contains a vector with positive self-intersection. Then Proposition 2.8 and Lemma 2.6 imply:

Proposition 2.12. *Let X be a compact Kähler surface and Γ be a non-elementary subgroup of $\mathsf{Aut}(X)$. Then $H^{1,1}(X;\mathbf{R})_+ = \mathrm{NS}(X;\mathbf{R})_+$ is a Minkowski space, and the action of Γ on this space is non-elementary and strongly irreducible.*

Since non-elementary groups of isometries of $\mathbb{H}^m$ occur only for $m \geqslant 2$, we get:

Corollary 2.13. *Under the assumptions of Proposition 2.12, the Picard number $\rho(X)$ is greater than or equal to 3. If equality holds then $\mathrm{NS}(X;\mathbf{R})_+ = \mathrm{NS}(X;\mathbf{R})$ and the action of Γ on $\mathrm{NS}(X;\mathbf{R})$ is strongly irreducible.*

From now on we set:

$$\Pi_\Gamma := H^{1,1}(X;\mathbf{R})_+ = \mathrm{NS}(X;\mathbf{R})_+. \tag{2.13}$$

This is a Minkowski space on which Γ acts strongly irreducibly; the intersection form is negative definite on the orthogonal complement

$$\Pi_\Gamma^\perp \subset H^{1,1}(X;\mathbf{R}). \tag{2.14}$$

Moreover by Proposition 2.8.(2) the group $G = \mathsf{Zar}(\Gamma)^{\mathrm{irr}}$ satisfies $G(\mathbf{R})|_{\Pi_\Gamma} = \mathsf{SO}(\Pi_\Gamma)$. If Γ contains a parabolic element, then Π_Γ is rational with respect to the integral structures of $\mathrm{NS}(X;\mathbf{Z})$ and $H^2(X;\mathbf{Z})$, and $G(\mathbf{R}) = \mathsf{SO}(\Pi_\Gamma) \times \{\mathrm{id}_{\Pi_\Gamma^\perp}\}$ (see Lemma 2.9).

2.3.5. *Invariant algebraic curves I.* Assume that Γ is non-elementary and let $C \subset X$ be an irreducible algebraic curve with a finite Γ-orbit. Then the action of Γ on $\mathrm{Vect}_{\mathbf{Z}}\,\{f^*[C]; f \in \Gamma\} \subset \mathrm{NS}(X;\mathbf{Z})$ factors through a finite group. From Propositions 2.8 and 2.12 we deduce that the intersection form is negative definite on $\mathrm{Vect}_{\mathbf{Z}}(\Gamma \cdot [C])$, thus $\mathrm{Vect}_{\mathbf{R}}(\Gamma \cdot [C])$ is one of the irreducible factors of $\mathrm{NS}(X,\mathbf{R})_0$. This argument, together with Grauert's contraction theorem, leads to the following result (we refer to [28, 77] for a proof; the result holds more generally for subgroups containing a loxodromic element):

Lemma 2.14. *Let X be a compact Kähler surface and Γ be a non-elementary group of automorphisms on X. Then, there are at most finitely many Γ-periodic irreducible curves. The intersection form is negative definite on the subspace of $\mathrm{NS}(X;\mathbf{Z})$ generated by the classes of these curves. There is a compact complex analytic surface X_0 and a Γ-equivariant bimeromorphic morphism $X \to X_0$ that contracts these curves and is an isomorphism in their complement.*

The next result follows from [46].

Proposition 2.15. *Let X be a compact Kähler surface and Γ a non-elementary subgroup of $\mathsf{Aut}(X)$. Then any Γ-periodic curve has arithmetic genus 0 or 1.*

Note if C is Γ-periodic, this result applies to $\widetilde{C} = \Gamma \cdot C$, which is invariant. Then, the normalization of any irreducible component of $\widetilde{C}$ has genus 0 or 1, and the incidence graph of the components of $\widetilde{C}$ obeys certain restrictions (see [28, §4.1] for details). If furthermore X is a K3 or Enriques surface, each component is a smooth rational curve of self-intersection -2.

2.3.6. *The limit set.* Let $\Gamma \subset \mathsf{Aut}(X)$ be non-elementary. The **limit set** of Γ is the closed subset $\mathrm{Lim}(\Gamma) \subset \partial\mathbb{H}_X \subset \mathbb{P}\left(H^{1,1}(X;\mathbf{R})\right)$ defined by one of the following equivalent assertions:

(a) $\mathrm{Lim}(\Gamma)$ is the smallest, non-empty, closed, and Γ-invariant subset of $\mathbb{P}(\overline{\mathbb{H}_X})$;
(b) $\mathrm{Lim}(\Gamma) \subset \partial\mathbb{H}_X$ is the closure of the set of fixed points of loxodromic elements of Γ in $\partial\mathbb{H}_X$ (these fixed points correspond to isotropic lines on which the loxodromic isometry act as a dilation or contraction);
(c) $\mathrm{Lim}(\Gamma)$ is the accumulation set of any Γ-orbit $\Gamma(\mathbb{P}(v)) \subset \mathbb{P}(H^{1,1}(X;\mathbf{R}))$, for any $v \notin \Pi_\Gamma^\perp$.

We refer to [75, 101] for a study of such limit sets. From the second characterization we get:

Lemma 2.16. *The limit set* $\mathrm{Lim}(\Gamma)$ *of a non-elementary group is contained in* $\mathbb{P}(\Pi_\Gamma) \cap \partial\mathbb{H}_X$.

From the third characterization, $\mathrm{Lim}(\Gamma)$ is contained in the closure of $\Gamma(\mathbb{P}([\kappa]))$ for every Kähler form κ on X. Since X must be projective, we can chose $[\kappa]$ in $\mathrm{NS}(X;\mathbf{Z})$. As a consequence, $\mathrm{Lim}(\Gamma)$ is contained in $\mathrm{Nef}(X)$:

Lemma 2.17. *Let X be a compact Kähler surface. If Γ is a non-elementary subgroup of* $\mathsf{Aut}(X)$ *its limit set satisfies* $\mathrm{Lim}(\Gamma) \subset \mathbb{P}(\mathrm{Nef}(X)) \subset \mathbb{P}(\mathrm{NS}(X;\mathbf{R}))$.

2.4. **Parabolic automorphisms.** We collect a few basic facts on parabolic automorphisms: they will be used in the next section to describe explicit examples, and then in Sections 10 and 11.

Let f be a parabolic automorphism of a compact Kähler surface. Then f^* preserves a unique point on $\partial\mathbb{H}_X$, and f preserves a unique genus 1 fibration $\pi_f\colon X \to B$ onto some Riemann surface B. The fixed point of f^* on $\partial\mathbb{H}_X$ is given by the class $[F]$ of any fiber of π_f (see [28]). The fibers of π_f are the elements of the linear system $|F|$, π_f is uniquely determined by $[F]$, and if g is another automorphism of X that preserves a smooth fiber of π_f (resp. the point $\mathbb{P}[F] \in \mathbb{P}\mathrm{NS}(X;\mathbf{R})$), then g preserves the fibration and is either elliptic or parabolic.

Lemma 2.18. *Let X be a K3 or Enriques surface, and $\pi\colon X \to B$ be a genus 1 fibration. If $g \in \mathsf{Aut}(X)$ maps some fiber F of π to a fiber of π, then g preserves the fibration and either g is parabolic or it is periodic of order $\leqslant 66$.*

Proof. Since g maps F to some fiber F', it maps the complete linear system $|F|$ to $|F'|$, but both linear systems are made of the fibers of π. So g preserves the fibration and is not loxodromic. If g is not parabolic it is elliptic, and its action on cohomology has finite order since it preserves $H^2(X,\mathbf{Z})$. On a K3 or Enriques surface every holomorphic vector field vanishes identically, so $\mathsf{Aut}(X)^0$ is trivial and the kernel of the homomorphism $\mathsf{Aut}(X) \ni f \mapsto f^*$ is finite (see [28, Theorem 2.6]); as a consequence, any elliptic automorphism has finite order. The upper bound on the order of g was obtained in [78]. □

Proposition 2.19. *Let X be a compact Kähler surface and let f be a parabolic automorphism of X, preserving the genus 1 fibration $\tau\colon X \to B$. Consider the group* $\mathsf{Aut}(X;\tau) := \{g \in \mathsf{Aut}(X)\ ;\ \exists g_B \in \mathsf{Aut}(B),\ \tau \circ g = g_B \circ \tau\}$, *and assume that the image of the homomorphism* $g \in \mathsf{Aut}(X;\tau) \to g_B \in \mathsf{Aut}(B)$ *is infinite. Then, X is a torus.*

This result directly follows from the proof of Proposition 3.6 in [34]. In particular the automorphism $f_B \in \mathsf{Aut}(B)$ such that $\pi_f \circ f = f_B \circ \pi_f$ has finite order when X is a K3, an Enriques, or a rational surface. The dynamics of these automorphisms is described in Section 11.1.

Lemma 2.20. *If Γ is a subgroup of $\mathsf{Aut}(X)$ containing a parabolic automorphism g, then Γ is non-elementary if and only if it contains another parabolic automorphism h such that the invariant fibrations π_g and π_h are distinct. Then, the tangency locus of the two fibrations is either empty or a curve, and there are positive integers m, n such that g^m and h^n generate a free group of rank 2.*

Proof. Let F be a fiber of π_g. If Γ is non-elementary, there is an element f in Γ that does not fix $[F]$; in particular f does not preserve π_g. Then, $h := f^{-1} \circ g \circ f$ is another parabolic automorphism with a distinct invariant fibration, namely $\pi_h = \pi_g \circ f$. Being distinct, π_g and π_h have a tangency locus of codimension $\geqslant 1$.

Conversely, if Γ contains two parabolic automorphisms with distinct fixed point in $\partial \mathbb{H}_X$, then the ping-pong lemma proves that there are powers m, $n \geqslant 1$ such that $\langle g^m, h^n \rangle$ is a free group of rank 2; in particular, Γ is non-elementary. (See [28] for more precise results.) $\square$

3. EXAMPLES AND CLASSIFICATION

This section may be skipped in a first reading. It describes a few examples, and proves that a compact Kähler surface X is projective when its automorphism group is non-elementary.

3.1. **Wehler surfaces (see** [36, 103, 113, 114]**).** Consider the variety $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and let π_1, π_2, and π_3 be the projections on the first, second, and third factor: $\pi_i(z_1, z_2, z_3) = z_i$. Denote by L_i the line bundle $\pi_i^*(\mathcal{O}(1))$ and set

$$L = L_1^2 \otimes L_2^2 \otimes L_3^2 = \pi_1^*(\mathcal{O}(2)) \otimes \pi_2^*(\mathcal{O}(2)) \otimes \pi_3^*(\mathcal{O}(2)). \tag{3.1}$$

Since $K_{\mathbb{P}^1} = \mathcal{O}(-2)$, this line bundle L is the dual of the canonical bundle K_M. By definition, $|L| \simeq \mathbb{P}(H^0(M, L))$ is the linear system of surfaces $X \subset M$ given by the zeroes of global sections $P \in H^0(M, L)$. Using affine coordinates (x_1, x_2, x_3) on $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, such a surface is defined by a polynomial equation $P(x_1, x_2, x_3) = 0$ whose degree with respect to each variable is $\leqslant 2$ (see [25, 95] for explicit examples). These surfaces will be referred to as **Wehler surfaces** or **(2,2,2)-surfaces**; modulo $\mathsf{Aut}(M)$, they form a family of dimension 17.

Fix $k \in \{1, 2, 3\}$ and denote by $i < j$ the other indices. If we project X to $\mathbb{P}^1 \times \mathbb{P}^1$ by $\pi_{ij} = (\pi_i, \pi_j)$, we get a 2 to 1 cover (the generic fiber is made of two points, but some fibers may be rational curves). As soon as X is smooth the involution σ_k that permutes the two points in each (general) fiber of π_{ij} is an involutive automorphism of X; indeed X is a K3 surface and any birational self-map of such a surface is an automorphism.

Proposition 3.1. *There is a countable union of proper Zariski closed subsets $(W_i)_{i \geqslant 0}$ in $|L|$ such that*

1. *if X is an element of $|L| \backslash W_0$, then X is a smooth K3 surface and X does not contain any fiber of the projections π_{ij};*
2. *if X is an element of $|L| \backslash (\bigcup_i W_i)$, the restriction morphism $\mathrm{Pic}(M) \to \mathrm{Pic}(X)$ is surjective. In particular its Picard number is $\rho(X) = 3$.*

From the second assertion, we deduce that for a very general X, $\mathrm{Pic}(X)$ is isomorphic to $\mathrm{Pic}(M)$: it is the free Abelian group of rank 3, generated by the classes

$$c_i := [(L_i)_{|X}]. \tag{3.2}$$

The elements of $|(L_i)_{|X}|$ are the curves of X given by the equations $z_i = \alpha$ for some $\alpha \in \mathbb{P}^1$. The arithmetic genus of these curves is equal to 1: in other words the projection $(\pi_i)_{|X} \colon X \to \mathbb{P}^1$ is a genus 1 fibration. Moreover, for a general choice of X in $|L|$, $(\pi_i)_{|X}$ has 24 singular fibers of type I_1, i.e. isomorphic to a rational curve with exactly one simple double point. The intersection form is given by $c_i^2 = 0$ and $\langle c_i | c_j \rangle = 2$ if $i \neq j$, so that its matrix is given by

$$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}. \tag{3.3}$$

Proof of Proposition 3.1. By Bertini's theorem, X is smooth as soon as it is in the complement of some proper Zariski closed subset $W_0 \subset |L|$. Now, let us assume that X is smooth. The adjunction formula implies that the canonical bundle K_X is trivial. From the hyperplane section theorem of Lefschetz [96], we know that X is simply connected. So, X is a K3 surface (see [6]). Write the equation of X as $A(x_1, x_2)x_3^2 + B(x_1, x_2)x_3 + C(x_1, x_2) = 0$. Then, X contains a fiber $\pi_{12}^{-1}(a_1, a_2)$ if and only if the three curves given by $A = 0$, $B = 0$, and $C = 0$ contain the point (a_1, a_2). This imposes a non-trivial algebraic condition on X; hence, enlarging W_0, the first assertion is satisfied.

For the second assertion, we apply a general form of the Noether-Lesfchetz theorem [112, Théorème 15.33]. We know that L is very ample, that $H^{2,0}(X)$ is isomorphic to $\mathbf{C}$. Indeed X is a K3 surface, and $H^{2,0}(X)$ is contained in the vanishing cohomology since X may degenerate on six copies of $\mathbb{P}^1 \times \mathbb{P}^1$ (taking the equation $(x_1^2 - 1)(x_2^2 - 1)(x_3^2 - 1) = 0$). So, the Noether-Lefschetz theorem says precisely that the restriction morphism is surjective for a very general choice of $X \in |L|$. $\square$

Lemma 3.2. *Assume that X does not contain any fiber of the projection π_{ij}. Then, the involution σ_k^* preserves the subspace $\mathbf{Z}c_1 \oplus \mathbf{Z}c_2 \oplus \mathbf{Z}c_3$ of $\mathrm{NS}(X; \mathbf{Z})$ and*

$$\sigma_k^* c_i = c_i, \ \sigma_k^* c_j = c_j, \ \sigma_k^* c_k = -c_k + 2c_i + 2c_j.$$

Equivalently, the action of σ_k^* on $\mathrm{Vect}_{\mathbf{R}}(c_1, c_2, c_3)$ preserves the classes c_i and c_j and acts as a reflexion with respect to the hyperplane $\mathrm{Vect}(c_i, c_j) \subset \mathrm{NS}(X; \mathbf{R})$. In other words, $\sigma_k(v) = v + \frac{1}{2}\langle v | u_k \rangle u_k$ for all v in $\mathbf{Z}c_1 \oplus \mathbf{Z}c_2 \oplus \mathbf{Z}c_3$.

Proof. Since σ_k preserves π_{ij} it preserves the fibers of π_i and π_j, hence σ_k^* fixes c_i and c_j. Now, consider a fiber $C = \{z_k = w\} \subset X$ of π_k. Then, $\sigma_k(C) \cup C = \pi_{ij}^{-1}(\pi_{ij}(C))$ because there is no curve in the fibers of π_{ij}. On the other hand, $\pi_{ij}(C) \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a (2,2)-curve so it is rationally equivalent to the union of two vertical and two horizontal projective lines. This gives $\sigma_k^* c_k = -c_k + 2c_i + 2c_j$. $\square$

Combining this lemma with the previous proposition, we see that a very general Wehler surface has Picard number 3, $\mathbb{H}_X$ has dimension 2, $\mathrm{NS}(X; \mathbf{Z}) = \mathrm{Vect}_{\mathbf{Z}}(c_1, c_2, c_3)$ and the matrices of the σ_i^* in the basis (c_i) are

$$\sigma_1^* = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \ \sigma_2^* = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \ \sigma_3^* = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.4}$$

Proposition 3.3. *If X is a very general Wehler surface then:*

(1) *X is a smooth K3 surface with Picard number 3;*
(2) $\mathsf{Aut}(X)$ *is equal to $\langle \sigma_1, \sigma_2, \sigma_3\rangle$, it is a free product of three copies of $\mathbf{Z}/2\mathbf{Z}$, and* $\mathsf{Aut}(X)^*$ *is a finite index subgroup in the group of integral isometries of* $\mathrm{NS}(X;\mathbf{Z})$*;*
(3) $\mathsf{Aut}(X)^*$ *acts strongly irreducibly on* $\mathrm{NS}(X;\mathbf{R})$*;*
(4) $\mathsf{Aut}(X)$ *does not preserve any algebraic curve $D \subset X$;*
(5) *the limit set of* $\mathsf{Aut}(X)^*$ *is equal to $\partial\mathbb{H}_X$;*
(6) *the compositions $\sigma_i \circ \sigma_j$ and $\sigma_i \circ \sigma_j \circ \sigma_k$ are respectively parabolic and loxodromic for every triple (i, j, k) with $\{i, j, k\} = \{1, 2, 3\}$.*

Proof. The first three assertions follow from Proposition 3.1, [25, §1.5] and [36, Thm 3.6]. For the fourth one, note that any invariant curve D would yield a non-trivial fixed point $[D]$ in $\mathrm{NS}(X;\mathbf{Z})$, contradicting assertion (3). The fifth one follows from the second because the limit set of a lattice in $\mathsf{Isom}(\mathrm{NS}(X;\mathbf{R}))$ is always equal to $\partial\mathbb{H}_X$. To prove the last assertion, it suffices to compute the corresponding product of matrices given in Equation (3.4) (see [25]). □

Remark 3.4. In [4], Baragar gives examples of smooth surfaces $X \in |L|$ for which $\rho(X) \geqslant 4$ and the limit set of $\mathsf{Aut}(X)^*$ in $\partial\mathbb{H}_X$ is a genuine fractal set.

3.2. **Pentagons.** The dynamics on the space of pentagons with given side lengths, introduced in §1.2, shares important similarities with the dynamics on Wehler surfaces. A pentagon with side lengths $\ell_0, \ldots, \ell_4$ modulo translations of the plane is the same as the data of a 5-tuple of vectors $(v_i)_{i=0,\ldots,4}$ in $\mathbf{R}^2$ (identified with $\mathbf{C}$) of respective length ℓ_i such that $\sum_i v_i = 0$. Write $v_i = \ell_i t_i$ with $|t_i| = 1$. Then the action of $\mathsf{SO}_2(\mathbf{R})$ can be identified to the diagonal multiplicative action of $\mathsf{U}_1 = \{\alpha \in \mathbf{C}\ ;\ |\alpha| = 1\}$ on the t_i:

$$\alpha \cdot (t_0, \ldots, t_4) = (\alpha t_0, \ldots \alpha t_4). \tag{3.5}$$

Now, following Darboux [43], we consider the surface X in $\mathbb{P}^4_{\mathbf{C}}$ defined by the equations

$$\begin{cases} \ell_0 z_0 + \ell_1 z_1 + \ell_2 z_2 + \ell_3 z_3 + \ell_4 z_4 = 0 \\ \ell_0/z_0 + \ell_1/z_1 + \ell_2/z_2 + \ell_3/z_3 + \ell_4/z_4 = 0 \end{cases} \tag{3.6}$$

where $[z_0 : \ldots : z_4]$ is some fixed choice of homogeneous coordinates, and the second equation must be multiplied by $z_0z_1z_2z_3z_4$ to obtain a homogeneous equation of degree 4.

Remark 3.5. This surface is isomorphic to the Hessian of a cubic surface (see [50, §9]). More precisely, consider a cubic surface $S \subset \mathbb{P}^3_{\mathbf{C}}$ whose equation F can be written in Sylvester's pentahedral form that is, as a sum $F = \sum_{i=0}^4 \lambda_i F_i^3$ for some complex numbers λ_i and linear forms F_i with $\sum_{i=0}^4 F_i = 0$. By definition, its Hessian surface H_F is defined by $\det(\partial_i\partial_j F) = 0$. Then, using the linear forms F_i to embed H_F in $\mathbb{P}^4_{\mathbf{C}}$, we obtain the surface defined by the pair of equations $\sum_{i=0}^4 z_i = 0$ and $\sum_{i=0}^4 \frac{1}{\lambda_i z_i} = 0$. Thus, H_F is our surface X, for $\ell_i^2 = \lambda_i$. We refer to [49, 44, 48, 105] for an introduction to these surfaces and their birational transformations.

For completeness, we prove some of its basic properties.

Lemma 3.6. *Let $\ell = (\ell_0, \ldots, \ell_4)$ be an element of $(\mathbf{C}^*)^5$. The surface $X \subset \mathbb{P}^4_{\mathbf{C}}$ defined by the system* (3.6) *has 10 singularities at the points q_{ij} determined by the system of equations $\ell_i z_i + \ell_j z_j = 0$, $z_k = z_l = z_m = 0$ with $i < j$ and $\{i, j, k, l, m\} = \{0, 1, 2, 3, 4\}$. In the* *complement of these ten isolated singularities, X is smooth if and only if*

$$\sum_{i=0}^{4} \varepsilon_i \ell_i \neq 0 \quad \forall \varepsilon_i \in \{\pm 1\}\,. \tag{3.7}$$

Proof. We first look for singularities in the complement of the hyperplanes $z_i = 0$, and work in the chart $z_0 = 1$. Then $z_4 = -(\ell_0 + \ell_1 z_1 + \ell_2 z_2 + \ell_3 z_4)/\ell_4$ and we replace in the second equation of (3.6) to obtain an affine equation of X in this chart, namely:

$$\frac{\ell_1}{z_1} + \frac{\ell_2}{z_2} + \frac{\ell_3}{z_3} - \frac{\ell_4^2}{\ell_0 + \ell_1 z_1 + \ell_2 z_2 + \ell_3 z_3} + \ell_0 = 0. \tag{3.8}$$

Singularities are determined by the system of equations $z_1^2 = z_2^2 = z_3^2 = \ell_4^{-2}(\ell_0 + \ell_1 z_1 + \ell_2 z_2 + \ell_3 z_3)^2$. So, by symmetry, at a singularity where none of the coordinates vanishes we must have $z_i = \varepsilon_i z$ for some $\varepsilon_i = \pm 1$ and a common factor $z \neq 0$; this is precisely Condition (3.7).

Looking for singularities with one coordinate equal to 0, say $z_1 = 0$ in the chart $z_0 = 1$, we obtain the system of equations

$$\begin{cases} 0 = (\ell_0 z_2 z_3 + \ell_3 z_2 + \ell_2 z_3)(\ell_0 + \ell_2 z_2 + \ell_3 z_3) + (\ell_1^2 - \ell_4^2) z_2 z_3 \\ 0 = \ell_1 z_3 (\ell_0 + 2\ell_2 z_2 + \ell_3 z_3) \\ 0 = \ell_1 z_2 (\ell_0 + \ell_2 z_2 + 2\ell_3 z_3) \end{cases} \tag{3.9}$$

together with $\ell_0 + \ell_2 z_2 + \ell_3 z_3 + \ell_4 z_4 = 0$ and $\ell_1 z_2 z_3 z_4 = 0$ (in particular, z_2, z_3 or z_4 must vanish). The solutions of this system are given by $z_1 = z_2 = z_3 = 0$, which gives the point $q_{04} = [\ell_4 : 0 : 0 : 0 : -\ell_0]$, or $z_1 = z_2 = 0$ and $\ell_0 + \ell_3 z_3 = 0$, which corresponds to $q_{03} = [\ell_3 : 0 : 0 : -\ell_0 : 0]$, or $z_1 = z_3 = 0$ which gives q_{02}, or $z_1 = z_4 = 0$ but then either $z_2 = 0$ or $z_3 = 0$ and we end up again with q_{02} and q_{03}. The result follows by symmetry. $\square$

Lemma 3.7. *If $\ell \in (\mathbf{C}^*)^5$ satisfies Condition* (3.7)*, then the ten singularities are simple nodes (Morse singularities) and the surface X is a (singular) K3 surface: a minimal resolution $\hat{X}$ of X is a K3 surface, which is obtained by blowing-up its ten nodes, thereby creating ten rational (-2)-curves.*

Proof. Working in the chart $z_0 = 1$ and replacing z_4 by $-(\ell_0 + \ell_1 z_1 + \ell_2 z_2 + \ell_3 z_3)/\ell_4$, the quadratic term of the equation of X at the singularity $(z_1, z_2, z_3) = (0,0,0)$ is $(-\ell_0/\ell_4)Q$, where

$$Q(z_1, z_2, z_3) = \ell_1 z_2 z_3 + \ell_2 z_1 z_3 + \ell_3 z_1 z_2 \tag{3.10}$$

is a non-degenerate quadratic form (its determinant is $2\ell_1\ell_2\ell_3 \neq 0$). So locally X is holomorphically equivalent to the quadratic cone $\{Q = 0\}$, hence to a quotient singularity $(\mathbf{C}^2, 0)/\eta$ with $\eta(x, y) = (-x, -y)$. The minimal resolution of such a singularity is obtained by a simple blow-up of the ambient space, the exceptional divisor being a (-2)-curve in the smooth surface $\hat{X}$. The adjunction formula shows that there is a holomorphic 2-form Ω_X on the regular part of X; locally, Ω_X lifts to an η-invariant form Ω'_X on $\mathbf{C}^2 \setminus \{0\}$, which by Hartogs extends at the origin to a non-vanishing 2-form. To recover $\hat{X}$, one can first blow-up $\mathbf{C}^2$ at the origin and then take the quotient by (the lift of) η: a simple calculation shows that Ω'_X determines a non-vanishing 2-form on $\hat{X}$. After such a surgery is done at the ten nodes, $\hat{X}$ is a smooth surface with a non-vanishing section of $K_{\hat{X}}$; since it contains at least ten rational curves, it can not be an Abelian surface, so it must be a K3 surface. $\square$

Remark 3.8. Let L_{ij} be the line defined by the equations $z_i = 0$, $z_j = 0$, $\ell_0 z_0 + \cdots + \ell_4 z_4 = 0$; each of these ten lines is contained in X, each of them contains 3 singularities of X (namely q_{kl}, q_{lm}, q_{km} with obvious notations), and each singularity is contained in three of these lines. If one projects them on a plane, the ten lines L_{ij} form a Desargues configuration (see [48, 49]).

All this works for any choice of complex numbers $\ell_i \neq 0$. Now, since the ℓ_i are real, X is endowed with two real structures. First, one can consider the complex conjugation $c\colon [z_i] \mapsto [\overline{z_i}]$ on $\mathbb{P}^4(\mathbf{C})$ and restrict it to X: this gives a first antiholomorphic involution c_X. Another one is given by $s_X\colon [z_i] \mapsto [1/\overline{z_i}]$. To be more precise, consider first, the quartic birational involution $J \in \mathsf{Bir}(\mathbb{P}^4_{\mathbf{C}})$ defined by $J([z_i]) = [1/z_i]$; J preserves X, it determines a birational transformation $J_X \in \mathsf{Bir}(X)$, and on $\hat{X}$ it becomes an automorphism because every birational transformation of a K3 surface is regular. Thus, $s_X = J_X \circ c_X$ determines a second antiholomorphic involution $s_{\hat{X}}$ of $\hat{X}$. In what follows, we denote by (X, s_X) this real structure (even if it would be better to study it on $\hat{X}$); its real part is the fixed point set of s_X, i.e. the set of points in $X(\mathbf{C})$ with coordinates of modulus 1: the real part does not contain any of the singularities of X, this is why we prefer to stay in X rather than lift everything to $\hat{X}$. Thus, *with the real structure defined by s_X, the real part of X coincides with* $\mathrm{Pent}^0(\ell_0, \ldots, \ell_4)$ if $(\ell_i) \in (\mathbf{R}_+^*)^5$.

Remark 3.9. When $\ell_i > 0$ for all indices $i \in \{0, \ldots, 4\}$, a complete description of the possible homeomorphism types for the real locus (in the smooth and singular cases) is given in [42]: *in the smooth case, it is an orientable surface of genus $g = 0, \ldots, 4$ or the union of two tori.*

Remark 3.10. The involution J preserves X and the two real structures (X, c_X) and (X, s_X). It lifts to a fixed point free involution $\hat{J}_X$ on $\hat{X}$, and $\hat{X}/\hat{J}_X$ is an Enriques surface. On pentagons, J corresponds to the symmetry $(x, y) \in \mathbf{R}^2 \mapsto (x, -y)$ that reverses orientation. Thus we see that the space of pentagons modulo affine isometries is an Enriques surface. When X acquires an eleventh singularity which is fixed by J_X, then $\hat{X}/\hat{J}_X$ becomes a Coble surface: see [48, §5] for nice explicit examples. This happens for instance when all lengths are 1, except one which is equal to 2 (this corresponds to $t = 1/4$ in [48, §5.2]).

Finally, let us express the folding transformations in coordinates. Given $i \neq j$ in $\{0, \ldots, 4\}$ (consecutive or not) we define an involution $(t_i, t_j) \mapsto (t_i', t_j')$ preserving the vector $\ell_i t_i + \ell_j t_j$ by taking the symmetric of t_i and t_j with respect to the line directed by $\ell_i t_i + \ell_j t_j$. In coordinates, $t_k' = u/t_k$ for some u of modulus 1, and equating $\ell_i t_i + \ell_j t_j = \ell_i t_i' + \ell_j t_j'$ one obtains

$$(t_i', t_j') = \left(\frac{u}{t_i}, \frac{u}{t_j}\right), \text{ with } u = \frac{\ell_i t_i + \ell_j t_j}{\ell_i t_i^{-1} + \ell_j t_j^{-1}}. \tag{3.11}$$

Observe that these computations also make sense when the ℓ_i are complex numbers, or when we replace the t_i by the complex numbers z_i. This defines a birational involution $\sigma_{ij} : X \dashrightarrow X$,

$$\sigma_{ij}[z_0 : \ldots : z_4] = [z_0' : \ldots : z_4'] \tag{3.12}$$

with $z_k' = z_k$ if $k \neq i, j$, $z_i' = v z_j$, and $z_j' = v z_i$ with $v = (\ell_i z_i + \ell_j z_j)/(\ell_i z_j + \ell_j z_i)$. Again, since every birational self-map of a K3 surface is an automorphism, these involutions σ_{ij} are elements of $\mathsf{Aut}(\hat{X})$ that commute with the antiholomorphic involution $s_{\hat{X}}$; hence, they generate a subgroup of $\mathsf{Aut}(\hat{X}; s_{\hat{X}})$. Thus we have constructed a family of projective surfaces $\hat{X}$, depending on a parameter $\ell \in \mathbb{P}^4(\mathbf{C})$, endowed with a group of automorphisms generated by involutions. Note that this group can be elementary: for instance when the five lengths are all equal the group is finite because in that case $(z_i', z_j') = (z_j, z_i)$. When $j = i+1$ modulo 5, σ_{ij} corresponds to the folding transformation described in the introduction.

Remark 3.11. Pick a singular point q_{ij}, and project X from that point onto a plane, say the plane $\{z_i = 0\}$ in the hyperplane $P = \{\ell_0 z_0 + \cdots + \ell_4 z_4 = 0\}$. One gets a 2 to 1 cover $X \to \mathbb{P}^2_{\mathbf{C}}$, ramified along a sextic curve (this curve is the union of two cubics, see [105]). The involution σ_{ij} permutes the points in the fibers of this 2 to 1 cover: if x is a point of X, the line joining q_{ij} and x intersects X in the third point $\sigma_{ij}(x)$. The singularity q_{ij} is an indeterminacy point, mapped by σ_{ij} to the opposite line L_{ij}.

Proposition 3.12. *For a general parameter $\ell \in \mathbb{P}^4(\mathbf{C})$:*

(1) *X is a K3 surface with ten nodes, with two real structures c_X and s_X when $\ell \in \mathbb{P}^4(\mathbf{R})$;*
(2) *if i, $j = i+1$, $k = i+2$ are distinct consecutive indices (modulo 5), then $\sigma_{ij} \circ \sigma_{jk}$ is a parabolic transformation on $\hat{X}$;*
(3) *if i, j, k, and l are four distinct indices (modulo 5), then σ_{ij} commutes to σ_{kl}.*
(4) *the group Γ generated by the involutions σ_{ij} is a non-elementary subgroup of $\mathsf{Aut}(\hat{X}; s_{\hat{X}})$ that does not preserve any algebraic curve.*

In [48], Dolgachev computes the action of σ_{ij} on $\mathrm{NS}(\hat{X})$. This contains a proof of this proposition. He also describes, up to finite index, the Coxeter group generated by the σ_{ij}. The automorphism groups of $\hat{X}$ and of the Enriques surface $\hat{X}/\hat{J}_X$ are described in [49] and [107].

Proof. We already established Assertion (1) in the previous lemmas. For Assertion (2), denote by l, m the indices for which $\{i,j,k,l,m\} = \{0,\ldots,4\}$, and consider the linear projection $\pi_{lm} \colon \mathbb{P}^5(\mathbf{C}) \dashrightarrow \mathbb{P}^1(\mathbf{C})$ defined by $[z_0 : \ldots : z_4] \mapsto [z_l : z_m]$. The fibers of π_{lm} are the hyperplanes containing the plane $\{z_l = z_m = 0\}$, which intersects X on the line L_{lm}. This line is a common component of the pencil of curves cut out by the fibers of π_{lm} on X, and the mobile part of this pencil determines a fibration $\pi_{lm|X} \colon X \to \mathbb{P}^1$ whose fibers are the plane cubics

$$(\ell_l z_l + \ell_m z_m)(\ell_m z_l + \ell_l z_m) z_i z_j z_k = z_l z_m (\ell_i z_j z_k + \ell_j z_i z_k + \ell_k z_i z_j)(\ell_i z_i + \ell_j z_j + \ell_k z_k), \tag{3.13}$$

with $[z_l : z_m]$ fixed. The general member of this fibration is a smooth cubic, hence a curve of genus 1.

Then σ_{ij} and σ_{jk} preserve $\pi_{lm|X}$, and along the general fiber of $\pi_{lm|X}$ each of them is described by Remark 3.11; for instance, $\sigma_{ij}(x)$ is the third point of intersection of the cubic with the line (q_{ij}, x). Thus, writing such a cubic as $\mathbf{C}/\Lambda_{[z_l:z_m]}$, σ_{ij} acts as $z \mapsto -z + b_{ij}$, for some $b_{ij} \in \mathbf{C}/\Lambda_{[z_l:z_m]}$ that depends on $[z_l : z_m]$ and the parameter ℓ; it has four fixed points on the cubic curve, which are the points of intersection of the cubic (3.13) with the hyperplanes $z_i = z_j$ and $z_i = -z_j$; equivalently, the line (q_{ij}, x) is tangent to the cubic at these four points.

By Lemma 2.18, either $\sigma_{ij} \circ \sigma_{jk}$ is of order $\leqslant 66$ (in fact of order $\leqslant 12$ because it preserves $\pi_{lm|X}$ fiber-wise), or it is parabolic. Due to this bound on the order, and the fact that there do exist pentagons for which $\sigma_{ij} \circ \sigma_{jk}$ is of infinite order (indeed, this reduces to the corresponding fact for quadrilaterals, see the example below), $\sigma_{ij} \circ \sigma_{jk}$ is parabolic for general ℓ.

Example 3.13. Take $\ell = 1$ and $m = 2$, and normalize our pentagons to assume that $t_0 = 1$, which means that the first vertices are $a_0 = (0,0)$ and $a_1 = (\ell_0, 0)$; in homogeneous coordinates this corresponds to the normalization $[1 : z_1 : z_2 : z_3 : z_4]$ with $z_i = t_i$. Now, the pentagon in a fiber of $\pi_{12|X}$ have three fixed vertices, namely a_0, a_1 and a_2. The remaining vertices a_3 and a_4 move on the circles centered at a_2 and a_0 and of respective radii ℓ_2 and ℓ_4, with the constraint $a_3a_4 = \ell_3$. The circles are two conics, the fiber is a 2 to 1 cover of each of these two conics, and the automorphisms σ_{23} and σ_{34} preserve these fibers. Forgetting the vertex a_1, and looking at the quadrilateral (a_0, a_2, a_3, a_4), one recovers the involutions described in [10]. The fixed points of σ_{23} correspond to configurations with tangent circles, i.e. a_3 on the segment $[a_2, a_4]$.

Assertion (3) follows directly from the fact that σ_{ij} changes the coordinates z_i and z_j but keeps the other three fixed.

Finally, for a general parameter ℓ, Γ contains two such parabolics associated to distinct fibrations π_{lm} and $\pi_{l'm'}$ so it is non-elementary (see Lemma 2.20). In addition Γ does not preserve any curve in $\hat{X}$. Indeed, let $E \subset \hat{X}$ be a Γ-periodic irreducible curve, and denote by F its image in $\mathbb{P}^4_{\mathbf{C}}$ under the projection $\hat{X} \to X$. If F is a point, it is one of the singularities q_{ij}, and changing E into its image under (the lift of) σ_{ij} the curve F becomes the line L_{ij}. So, we may assume that F is an irreducible curve. Now, the orbit of F is periodic under the action of the parabolic automorphisms $g_i = \sigma_{ij} \circ \sigma_{jk}$ with $k = j+1$ and $j = i+1$. Since the invariant curves of a parabolic automorphisms are contained in the fibers of its invariant fibration, we deduce that F is contained in the fibers of each of the projections π_{lm}; this is obviously impossible. □

3.3. **Enriques surfaces (see** [40, 51]**).** Enriques surfaces are quotients of K3 surfaces by fixed point free involutions. According to Horikawa and Kondō ([73, 74, 81]), the moduli space $\mathcal{M}_E$ of complex Enriques surfaces is a rational quasi-projective variety of dimension 10. An Enriques surface X is nodal if it contains a smooth rational curve; such rational curves have self-intersection -2, and are called nodal-curves or (-2)-curves. Nodal Enriques surfaces form a hypersurface in $\mathcal{M}_E$.

For any Enriques surface X, the lattice $(\mathrm{NS}(X;\mathbf{Z}), q_X)$ is isomorphic to the orthogonal direct sum $E_{10} = U \oplus E_8(-1)$, ([1]). Let $W_X \subset \mathsf{O}(\mathrm{NS}(X;\mathbf{Z}))$ be the subgroup generated by reflexions about classes u such that $u^2 = -2$, and $W_X(2)$ be the subgroup of W_X acting trivially on $\mathrm{NS}(X;\mathbf{Z})$ modulo 2. Both W_X and $W_X(2)$ have finite index in $\mathsf{O}(\mathrm{NS}(X;\mathbf{Z}))$. The following result is due independently to Nikulin and Barth and Peters (see [51] for details and references).

Theorem 3.14. *If X is an Enriques surface which is not nodal, the homomorphism $\mathsf{Aut}(X) \ni f \mapsto f^* \in \mathsf{GL}(H^2(X,\mathbf{Z}))$ is injective, and its image satisfies $W_X(2) \subset \mathsf{Aut}(X)^* \subset W_X$.*

In particular, for any unnodal Enriques surface, $\mathsf{Aut}(X)$ is non-elementary, contains parabolic elements, and acts irreducibly on $\mathrm{NS}(X;\mathbf{R})$; thus, it does not preserve any curve.

3.4. **Examples on rational surfaces: Coble and Blanc.** Closely related to Enriques surfaces are the examples of Coble, obtained by blowing up the ten nodes of a general rational sextic curve $C_0 \subset \mathbb{P}^2$. The result is a rational surface X with a large group of automorphisms. To be precise, consider the canonical class $K_X \subset \mathrm{NS}(X;\mathbf{Z})$; its orthogonal complement $K_X^\perp$ is a lattice of dimension 10, isomorphic to E_{10}, and we define $W_X(2)$ exactly in the same way as for Enriques surfaces. Then, $\mathsf{Aut}(X)^*$ preserves the decomposition $K_X \oplus K_X^\perp$, and $\mathsf{Aut}(X)^*$

[1]Here, U is the standard 2-dimensional Minkowski lattice, $(\mathbf{Z}^2, x_1x_2)$, and E_8 is the root lattice given by the corresponding Dynkin diagram; so $E_8(-1)$ is negative definite, and E_{10} has signature $(1,9)$ (see [40, Chap. II]). Also, recall that in this paper $\mathrm{NS}(X;\mathbf{Z})$ denotes the torsion free part of the Néron-Severi group, which is sometimes denoted by $\mathsf{Num}(X;\mathbf{Z})$ in the literature on Enriques surfaces.

contains $W_X(2)$ when X does not contain any smooth rational curve of self-intersection -2 (see [30], Theorem 3.5). Also, Coble surfaces may be thought of as degeneracies of Enriques surfaces: an interesting difference is that $[K_X]$ is non trivial; in particular, $\mathrm{NS}(X; \mathbf{Z})_0$ is always non-trivial, for any $\Gamma \subset \mathsf{Aut}(X)$. There is a holomorphic section of $-2K_X$ vanishing exactly along the strict transform $C \subset X$ of the rational sextic curve C_0; this means that there is a meromorphic section $\Omega_X = \xi(x,y)(dx \wedge dy)^2$ of $K_X^{\otimes 2}$ that does not vanish and has a simple pole along C. Thus, the formula

$$(3.14) \quad \mathsf{vol}_X(U) = \int_U |\xi(x,y)|\, dx \wedge dy \wedge d\overline{x} \wedge d\overline{y} = \int_U |\xi(x,y)|\, (\mathsf{i}dx \wedge d\overline{x}) \wedge (\mathsf{i}dy \wedge d\overline{y})$$

determines a finite measure([2]) $\mathsf{vol}_X =$ "$\Omega_X^{1/2} \wedge \overline{\Omega_X^{1/2}}$", which we may assume to be a probability after multiplying Ω_X by some adequate constant; this measure is $\mathsf{Aut}(X)$-invariant (because vol_X is uniquely determined by the complex structure; see also Remark 3.15 below).

Another family of examples has been described by Blanc in [14]. One starts with a smooth cubic curve $C_0 \subset \mathbb{P}^2$. If q_1 is a point of C_0, there is a unique birational involution s_1 of $\mathbb{P}^2$ that fixes C_0 pointwise and preserves the pencil of lines through q_1. The indeterminacy points of s_1 are q_1 and the four tangency points of C_0 with this pencil (one of them may be "infinitely near q_1" and in that case it corresponds to the tangent direction of C_0 at q_1); thus the indeterminacies of s_1 are resolved by blowing-up points of C_0 (or points of its strict transform). After such a sequence of blow-ups s_1 becomes an automorphism of a rational surface X_1 that fixes pointwise the strict transform of C_0. So, if we blow-up other points of this curve, s_1 lifts to an automorphism of the new surface. In particular, we can start with a finite number of points $q_i \in C_0$, $i = 1, \ldots, k$, and resolve simultaneously the indeterminacies of the involutions s_i determined by the q_i. The result is a surface X, with a subgroup $\Gamma := \langle s_1, \ldots, s_k \rangle$ of $\mathsf{Aut}(X)$. Blanc proves that (1) there are no relations between these involutions, that is, Γ is a free product $\langle s_1, \ldots, s_k \rangle \simeq *_{i=1}^k \mathbf{Z}/2\mathbf{Z}$, (2) the composition of two distinct involutions $s_i \circ s_j$ is parabolic, and (3) the composition of three distinct involutions is loxodromic. There is a meromorphic section Ω_X of K_X with a simple pole along the strict transform of C_0, but the form $\mathsf{vol}_X := \Omega_X \wedge \overline{\Omega_X}$ is not integrable.

Remark 3.15. If $\Gamma \subset \mathsf{Aut}(X)$ is generated by involutions and there is a meromorphic form Ω such that $f^*\Omega = \xi(f)\Omega$ for every $f \in \Gamma$, then $\xi(f) = \pm 1$: this is the case for Blanc's examples or general Coble surfaces, since $W_X(2)$ is also generated by involutions (see [51]).

3.5. **Real forms.** For each of the examples described in Sections 3.1 to 3.4, we may ask for the existence of an additional real structure on X, and look at the group of automorphisms $\mathsf{Aut}(X_{\mathbf{R}})$ that preserve the real structure (automorphisms commuting with the anti-holomorphic involution describing the real structure). Note that if X is a smooth projective variety with a real structure, then $X(\mathbf{R})$ is either empty or a compact, smooth, and totally real surface in X.

If X is a Wehler surface defined by a polynomial equation $P(x_1, x_2, x_3)$ with real coefficients the σ_i are automatically defined over $\mathbf{R}$. If X is a Blanc surface for which C_0 is defined over $\mathbf{R}$ and the points q_i are chosen in $C_0(\mathbf{R})$, then again $\langle s_1, \ldots, s_k \rangle \subset \mathsf{Aut}(X_{\mathbf{R}})$. Real Enriques and Coble surfaces provide also many examples for which $\mathsf{Aut}(X_{\mathbf{R}})$ is non-elementary (see [45]).

[2] if locally $C = \{x = 0\}$ then $\xi(x,y) = \eta(x,y)/x$ where η is regular; thus, $|\xi| = |\eta|\, |x|^{-1}$ is locally integrable because $\frac{1}{r^\alpha}$ is integrable with respect to $rdrd\theta$ when $\alpha < 2$

3.6. **Surfaces admitting non-elementary groups of automorphisms.** The surfaces in the previous examples are all projective. This is a general fact, which we prove in this paragraph: we rely on the Kodaira-Enriques classification to describe compact Kähler surfaces which support a non-elementary group of automorphisms and prove Theorem 2.11.

3.6.1. *Minimal models.* We refer to Theorem 10.1 of [28] for the following result:

Theorem 3.16. *If X is a compact Kähler surface with a loxodromic automorphism, then*

- *either X is a rational surface, and there is a birational morphism $\pi\colon X \to \mathbb{P}^2_{\mathbf{C}}$;*
- *or the Kodaira dimension of X is equal to 0, and there is an $\mathsf{Aut}(X)$-equivariant bimeromorphic morphism $\pi\colon X \to X_0$ such that X_0 is a compact torus, a K3 surface, or an Enriques surface.*

In particular, $h^{2,0}(X)$ equals 0 or 1.

Remark 3.17. If X is a torus or K3 surface, there is a holomorphic 2-form Ω_X on X that does not vanish and satisfies $\int_X \Omega_X \wedge \overline{\Omega_X} = 1$. It is unique up to multiplication by a complex number of modulus 1. A consequence of utmost importance to us is that the volume form

$$\Omega_X \wedge \overline{\Omega_X} \tag{3.15}$$

is $\mathsf{Aut}(X)$-invariant. Furthermore for every f we can write $f^*\Omega_X = J(f)\Omega_X$, where the Jacobian $f \in \mathsf{Aut}(X) \mapsto J(f) \in \mathbb{U}_1$ is a unitary character on the group $\mathsf{Aut}(X)$. Since $H^{2,0}(X;\mathbf{C})$ is generated by $[\Omega_X]$, we obtain

$$f^*w = J(f)w \quad \forall w \in H^{2,0}(X;\mathbf{C}). \tag{3.16}$$

If Y is an Enriques surface, and $X \to Y$ is its universal cover, then X is a K3 surface: the volume form $\Omega_X \wedge \overline{\Omega_X}$ is invariant under the group of deck transformations, and determines an $\mathsf{Aut}(Y)$-invariant volume form on Y. So, if X is not rational, the dynamics of $\mathsf{Aut}(X)$ is conservative: it preserves a **canonical volume form** which is uniquely determined by the complex structure of X.

It follows from Theorem 3.16 that, in most cases, $\mathsf{Aut}(X)$ is countable (see [28, Rmk 3.3]).

Proposition 3.18. *Let X be a compact Kähler surface. If $\mathsf{Aut}(X)$ contains a loxodromic element, then the kernel of the homomorphism $\mathsf{Aut}(X) \to \mathsf{Aut}(X)^* \subset \mathsf{GL}(\mathrm{NS}(X;\mathbf{Z}))$ is finite unless X is a torus. So, if $\mathsf{Aut}(X)$ is non-elementary, then $\mathsf{Aut}(X)$ is discrete or X is a torus.*

3.6.2. *Projectivity.*

Theorem E. *Let X be a compact Kähler surface and Γ be a non-elementary subgroup of $\mathsf{Aut}(X)$. Then X is projective, and is birationally equivalent to a rational surface, an Abelian surface, a K3 surface, or an Enriques surface.*

From the discussion in §§3.1–3.4 we see that there exist examples with a non-elementary group of automorphisms for each of these four classes of surfaces. Theorem E is a direct consequence of Theorem 3.16 and the following lemmas.

Lemma 3.19. *Let f be a loxodromic automorphism of a compact Kähler surface X. The following properties are equivalent:*

(1) on $H^{2,0}(X;\mathbf{C})$, f^* acts by multiplication by a root of unity;

(2) X is projective.

If X supports a loxodromic automorphism, then $\dim(H^{2,0}(X;\mathbf{C})) \leqslant 1$; and with notation as in Remark 3.17, the first assertion is equivalent to

(1') either $H^{2,0}(X;\mathbf{C}) = 0$ or $J(f)$ is a root of unity.

Proof of Lemma 3.19. The characteristic polynomial χ_f of $f^*\colon H^2(X;\mathbf{Z}) \to H^2(X;\mathbf{Z})$ is a monic polynomial with integer coefficients. Since f is loxodromic, f^* has a real eigenvalue $\lambda(f) > 1$. Besides $\lambda(f)$ and $\lambda(f)^{-1}$, all other roots of χ_f have modulus 1, so $\lambda(f)$ is a reciprocal quadratic integer or a Salem number (see § 2.4.3 of [28] for more details). Thus, the decomposition of χ_f into irreducible factors can be written as

$$\chi_f(t) = S_f(t) \times R_f(t) = S_f(t) \times \prod_{i=1}^{m} C_{f,i}(t) \tag{3.17}$$

where S_f is a Salem polynomial or a reciprocal quadratic polynomial, and the $C_{f,i}$ are cyclotomic polynomials. In particular if ξ is an eigenvalue of f^* and a root of unity, we see that ξ is a root of $R_f(t)$ but not of $S_f(t)$.

The subspace $H^{2,0}(\mathbf{C}) \subset H^2(X;\mathbf{C})$ is f^*-invariant and, by Lemma 2.1, all eigenvalues of f^* on that subspace have modulus 1; if an eigenvalue of $f^*|_{H^{2,0}(X;\mathbf{C})}$ is not a root of unity, then it is a root of S_f.

Assume that all eigenvalues of f^* on $H^{2,0}(X;\mathbf{C})$ are roots of unity. Then $\mathrm{Ker}(S_f(f^*)) \subset H^2(X;\mathbf{R})$ is a f^*-invariant subspace of $H^{1,1}(X;\mathbf{R})$. This subspace is defined over $\mathbf{Q}$ and is of Minkowski type; in particular, it contains integral classes of positive self-intersection, and by the Kodaira embedding theorem, X is projective. Conversely, assume that X is projective. The Néron-Severi group $\mathrm{NS}(X;\mathbf{Q}) \subset H^{1,1}(X;\mathbf{R})$ is f^*-invariant and contains vectors of positive self-intersection, so by Proposition 2.8 it contains all isotropic lines associated to loxodromic automorphisms. Now any f^* invariant subspace defined over $\mathbf{Q}$ and containing the eigenspace associated to $\lambda(f^*)$ contains $\mathrm{Ker}(S_f(f^*))$, so we deduce that $\mathrm{Ker}(S_f(f^*)) \subset \mathrm{NS}(X;\mathbf{Q})$. In particular, $\mathrm{Ker}(S_f(f^*))$ does not intersect $H^{2,0}(X;\mathbf{C})$, which is invariant, and we conclude that all eigenvalues of f^* on $H^{2,0}(X;\mathbf{C})$ are roots of unity. □

Lemma 3.20. *Let X be a compact Kähler surface. If X is not projective, then $\mathsf{Aut}(X)^*$ is virtually Abelian and if it contains a loxodromic element it is virtually cyclic.*

Proof. Assume that $\mathsf{Aut}(X)^*$ is not virtually Abelian, or that it contains a loxodromic element without being virtually cyclic. According to Theorem 3.2 of [28], $\mathsf{Aut}(X)^*$ contains a non-Abelian free group Γ such that all elements of $\Gamma\setminus\{\mathrm{id}\}$ are loxodromic; from Theorem 3.16, either $h^{2,0}(X) = 0$ or X is the blow-up of a torus or a K3 surface. In the first case, $H^2(X;\mathbf{R}) = H^{1,1}(X;\mathbf{R})$ so, by the Hodge index theorem, $H^{1,1}(X;\mathbf{R})$ contains an integral class with positive self-intersection; then, the Kodaira embedding theorem shows that X is projective. In the second case, by uniqueness of the minimal model, the morphism $X \to X_0$ onto the minimal model of X is $\mathsf{Aut}(X)$-equivariant, so we can assume that $X = X_0$ is minimal and $h^{2,0}(X) = 1$. Consider the homomorphism $J\colon \mathsf{Aut}(X) \to \mathbb{U}_1$, as in Remark 3.17. Since $\mathbb{U}_1$ is Abelian $\ker(J|_\Gamma)$ contains loxodromic elements: indeed if $f, g \in \Gamma$ and $f \neq g$ then $[f,g] = fgf^{-1}g^{-1}$ is loxodromic and $J([f,g]) = 1$. From Lemma 3.19 we deduce that X is projective. □

4. Glossary of random dynamics, I

We now initiate the random iteration by introducing a probability measure on $\mathsf{Aut}(X)$. In this section we introduce a first set of ideas from the theory of random dynamical systems, as well as some notation that will be used throughout the paper.

4.1. **Random holomorphic dynamical systems.** Let X be a compact Kähler surface, such that $\mathsf{Aut}(X)$ is non-elementary. Note that $\mathsf{Aut}(X)$ is locally compact for the topology of uniform convergence –in many interesting cases it is actually discrete (see Proposition 3.18)– so it admits a natural Borel structure. We fix some Riemannian structure on X, for instance the one induced by the Kähler form κ_0. For $f \in \mathsf{Aut}(X)$, we denote by $\|f\|_{C^1}$ the maximum of $\|Df_x\|$ where the norm of $Df_x \colon T_xM \to T_{f(x)}M$ is computed with respect to this Riemannian metric.

We consider a probability measure ν on $\mathsf{Aut}(X)$ satisfying the **moment condition** (or integrability condition)

$$\int \left(\log \|f\|_{C^1(X)} + \log \left\|f^{-1}\right\|_{C^1(X)}\right) d\nu(f) < +\infty. \tag{4.1}$$

The norm $\|\cdot\|_{C^1(X)}$ is relative to our choice of Riemannian metric, but the finiteness of the integral in (4.1) does not depend on this choice. In many interesting situations the support of ν will be finite, in which case the integrability (4.1), as well as stronger moment conditions which will appear later (see Conditions (5.26) and (5.27)), are obviously satisfied.

Lemma 4.1. *The measure ν satisfies the moment condition* (4.1) *if and only if it satisfies the higher moment conditions*

$$\int \left(\log \|f\|_{C^k(X)} + \log \left\|f^{-1}\right\|_{C^k(X)}\right) d\nu(f) < \infty, \tag{4.2}$$

for all $k \geqslant 1$.

Here the C^k norm is relative to the expression of f in a system of charts (we don't need to be precise here because only the finiteness in (4.2) matters). This lemma follows from the Cauchy estimates. In particular, if ν satisfies (4.1), then it satisfies a similar moment condition for the C^2 norm, a property required to apply Pesin's theory.

Given ν, we shall consider independent, identically distributed sequences $(f_n)_{n\geqslant 0}$ of random automorphisms of X with distribution ν, and study the dynamics of random compositions of the form $f_{n-1} \circ \cdots \circ f_0$. The data (X, ν) will be referred to as a **random holomorphic dynamical system** on X. Many properties of (X, ν) depend on the properties of the subgroup

$$\Gamma = \Gamma_\nu := \langle \mathrm{Supp}(\nu) \rangle \tag{4.3}$$

generated by (the support of) ν in $\mathsf{Aut}(X)$. If in addition Γ_ν is non-elementary, we say that (X, ν) is **non-elementary**.

4.2. **Invariant and stationary measures.** Let G be a topological group and ν be a probability measure on G. Consider a measurable action of G on some measurable space $(M, \mathcal{A})$. Every $f \in G$ determines a push-forward operator $\mu \mapsto f_\star\mu$, acting on positive (resp. probability) measures μ on $(M, \mathcal{A})$. By definition, a probability measure μ on $(M, \mathcal{A})$ is **ν-stationary** if

$$\int f_\star \mu \, d\nu(f) = \mu, \tag{4.4}$$

and it is **ν-almost surely invariant** if $f_\star\mu = \mu$ for ν-almost every f. Let us stress that we only deal with probability measures in this definition; slightly abusing terminology, most often we drop the mention to ν and the mention that μ is a probability. A stationary measure is **ergodic** if it is an extremal point of the closed convex set of stationary measures (see [12, §2.1.3]).

If μ is almost surely invariant then it is stationary but the converse is generally false. If M is compact, the action $G \times M \to M$ is continuous, and $\mathcal{A}$ is the Borel σ-algebra, the Kakutani fixed point theorem implies the existence of at least one stationary measure. On the other hand the existence of an invariant measure is a very restrictive property. For instance, proximal, strongly irreducible linear actions on projective spaces have no (almost surely) invariant probability measure (see Sections 1.3 and 5.3). Following Furstenberg [65] we say that an action is **stiff** (or ν-stiff) if any ν-stationary measure is ν-almost surely invariant.

We shall consider several measurable actions of $\mathsf{Aut}(X)$: its tautological action on X, but also its action on the projectivized tangent bundle $\mathbb{P}(TX)$, on cohomology groups of X and their projectivizations, on spaces of currents, etc. In all cases, M will be a locally compact space and $\mathcal{A}$ its Borel σ-algebra, which will be denoted by $\mathcal{B}(M)$.

Remark 4.2. Since X is compact and the action $\mathsf{Aut}(X) \times X \to X$ is continuous, a probability measure μ on $(X, \mathcal{B}(X))$ is ν-almost surely invariant if and only if it is invariant under the action of the closure of Γ_ν in $\mathsf{Aut}(X)$; this follows from the dominated convergence theorem.

4.3. **Random compositions.** Set $\Omega = \mathsf{Aut}(X)^{\mathbf{N}}$, endowed with its product topology. The associated Borel σ-algebra coincides with the product σ-algebra, and it is generated by cylinders (see § 7.1). We endow Ω with the product measure $\nu^{\mathbf{N}}$. Choosing a random element in Ω with respect to $\nu^{\mathbf{N}}$ is equivalent to choosing an independent and identically distributed random sequence of automorphisms in $\mathsf{Aut}(X)$ with distribution ν. For $\omega \in \Omega$, we let $f_\omega = f_0$ and denote by f_ω^n the left composition of the n first terms of ω, that is

$$f_\omega^n = f_{n-1} \circ \cdots \circ f_0 \tag{4.5}$$

for $n > 0$. By definition $f_\omega^0 = \mathrm{id}$. Let us record for future reference the following consequence of the Borel-Cantelli lemma. We denote by $\sigma\colon \Omega \to \Omega$ the unilateral shift, i.e. the continuous transformation defined by $\sigma(f_0, f_1, \ldots) = \sigma(f_1, f_2, \ldots)$.

Lemma 4.3. *If (X, ν) is a random dynamical system satisfying the moment condition (4.1), then for $\nu^{\mathbf{N}}$-almost every sequence $\omega = (f_n) \in \Omega$,*

$$\frac{1}{n}\left(\log \|f_n\|_{C^1} + \log \|f_n^{-1}\|_{C^1}\right) \xrightarrow[n\to\infty]{} 0.$$

Remark 4.4. We are **not** considering the most general version of random holomorphic dynamical systems: one might consider compositions $f_{\vartheta^{n-1}(\xi)} \circ \cdots \circ f_{\vartheta(\xi)} \circ f_\xi$ where $\vartheta : \Sigma \to \Sigma$ is some measure preserving transformation of a probability space and $\Sigma \ni \xi \mapsto f_\xi \in \mathsf{Aut}(X)$ is measurable. The methods developed below do not apply to this more general setting.

5. FURSTENBERG THEORY IN $H^{1,1}(X;\mathbf{R})$

Consider a non-elementary random holomorphic dynamical system (X, ν) on a compact Kähler surface, satisfying the moment condition (4.1). The main purpose of this section is to analyze the linear action of (X, ν) on $H^{1,1}(X, \mathbf{R})$ by way of the theory of random products of matrices. Basic references for this subject are the books by Bougerol and Lacroix [19] and by Benoist and Quint [12].

5.1. **Moments and cohomology.** We start with a general discussion on the dilatation of cohomology classes under smooth transformations. Let M be a compact connected manifold of dimension m, endowed with some Riemannian metric g. If $f\colon M \to M$ is a smooth map, $\|f\|_{C^1}$ denotes the maximum norm of its tangent action, computed with respect to g (see Section 4.1). Thus, f is a Lipschitz map with $\mathrm{Lip}(f) = \|f\|_{C^1}$ for the distance determined by g; in particular $\|f\|_{C^1} \geqslant 1$ whenever f is onto. Fix a norm $|\cdot|_{H^k}$ on each cohomology group $H^k(M;\mathbf{R})$, for $0 \leqslant k \leqslant m$.

Lemma 5.1. *There is a constant $C > 0$, that depends only on M, g, and the norms $|\cdot|_{H^k}$, such that $|f^*[\alpha]|_{H^k} \leqslant C^k \,\mathrm{Lip}(f)^k \,|[\alpha]|_{H^k}$ for every class $[\alpha] \in H^k(M;\mathbf{R})$ and every map $f\colon M \to M$ of class C^1. In other words, the operator norm $\|f^*\|_{H^k}$ is controlled by the Lipschitz constant:*

$$\|f^*\|_{H^k} \leqslant C^k \,\mathrm{Lip}(f)^k \leqslant C^k \|f\|_{C^1}^k.$$

Proof. Pick a basis of the homology group $H_k(M;\mathbf{R}) \simeq H^k(M;\mathbf{R})^*$ given by smoothly immersed, compact, k-dimensional manifolds $\iota_i\colon N_i \to M$, and a basis of $H^k(M;\mathbf{R})$ given by smooth k-forms α_j. Then, the integral $\int_{N_i} \iota_i^*(f^*\alpha_j)$ is bounded from above by $C^k\|f\|_{C^1}^k$ for some constant C, because

$$|(f^*\alpha_j)_x(v_1,\ldots,v_k)| = |\alpha_j(f_*v_1,\ldots,f_*v_k)| \leqslant c_j\|f\|_{C^1}^k \prod_{\ell=1}^k |v_\ell|_{\mathrm{g}} \tag{5.1}$$

for every point $x \in M$ and every k-tuple of tangent vectors $v_\ell \in T_xM$; here, c_j is the supremum of the norm of the multilinear map $(\alpha_j)_x$ over $x \in M$. □

If ν is a probability measure on $\mathsf{Diff}(M)$ satisfying the moment condition (4.1), then

$$\forall 1 \leqslant k \leqslant m, \quad \int_{\mathsf{Diff}(M)} \log\left(\|f^*\|_{H^k}\right) + \log\left(\left\|(f^{-1})^*\right\|_{H^k}\right)\, d\nu(f) < +\infty. \tag{5.2}$$

If we specialize this to automorphisms of compact Kähler surfaces we get

$$\int_{\mathsf{Aut}(X)} \log\left(\|f^*\|_{H^{1,1}}\right) + \log\left(\left\|(f^{-1})^*\right\|_{H^{1,1}}\right)\, d\nu(f) < +\infty, \tag{5.3}$$

which is actually equivalent to (5.2) by Lemma 2.2. We saw in §2.3.3 that $\|f^*\|_{H^{1,1}} \asymp \left\|(f^{-1})^*\right\|_{H^{1,1}}$, so this last condition is in turn equivalent to

$$\int_{\mathsf{Aut}(X)} \log\left(\|f^*\|_{H^{1,1}}\right)\, d\nu(f) < +\infty. \tag{5.4}$$

5.2. **Cohomological Lyapunov exponent.** From now on we denote by $|\cdot|$ a norm on $H^{1,1}(X,\mathbf{R})$ and by $\|\cdot\|$ the associated operator norm. The linear action induced by the random dynamical system (X,ν) on $H^{1,1}(X,\mathbf{R})$ defines a random product of matrices. Since the moment condition (5.4) is satisfied, we can define the **upper Lyapunov exponent** $\lambda_{H^{1,1}}$ (or $\lambda_{H^{1,1}}(\nu)$) by

$$\lambda_{H^{1,1}} = \lim_{n\to\infty} \frac{1}{n} \int \log(\|(f_\omega^n)^*\|) d\nu^{\mathbf{N}}(\omega) \tag{5.5}$$

$$= \lim_{n\to+\infty} \frac{1}{n} \log \|(f_\omega^n)^*\| \tag{5.6}$$

where the second equality holds almost surely, i.e. for $\nu^{\mathbf{N}}$-almost every $\omega \in \Omega$. This convergence follows from Kingman's subadditive ergodic theorem, since $\|\cdot\|$ being an operator norm, $(\omega, n) \mapsto \log(\|(f_\omega^n)^*\|)$ defines a subadditive cocycle (see [12, Thm 4.28] or [19, Thm I.4.1]). Note that $(f_\omega^n)^* = f_0^* \circ \cdots \circ f_{n-1}^*$, so we are dealing with right compositions instead of the usual left composition. However since $f_0^* \circ \cdots \circ f_{n-1}^*$ has the same distribution as $f_{n-1}^* \circ \cdots \circ f_0^*$, the Lyapunov exponent in (5.5) corresponds to the usual definition of the upper Lyapunov exponent of the random product of matrices. We refer to [19, 85] for the definition and main properties of the subsequent Lyapunov exponents (see also [12, §10.5]).

Proposition 5.2. *Let (X,ν) be a non-elementary holomorphic dynamical system on a compact Kähler surface, satisfying the moment condition* (4.1)*, or more generally* (5.4)*. Then the cohomological Lyapunov exponent $\lambda_{H^{1,1}}$ is positive and the other Lyapunov exponents of the linear action on $H^{1,1}(X,\mathbf{R})$ are $-\lambda_{H^{1,1}}$, with multiplicity* 1*, and* 0*, with multiplicity $h^{1,1}(X)-2$.*

Proof. Consider the Γ_ν-invariant decomposition $\Pi_{\Gamma_\nu} \oplus \Pi_{\Gamma_\nu}^\perp$ given by Proposition 2.12 and Equation (2.13). Since the intersection form is negative definite on $\Pi_{\Gamma_\nu}^\perp$, the group $\Gamma_\nu^*|_{\Pi_{\Gamma_\nu}^\perp}$ is bounded and all Lyapunov exponents of $\Gamma_\nu^*|_{\Pi_{\Gamma_\nu}^\perp}$ vanish. The linear action of Γ_ν on Π_{Γ_ν} is strongly irreducible and non-elementary, hence not relatively compact. Therefore Furstenberg's theorem asserts that $\lambda_{H^{1,1}} > 0$ (see e.g. [19, Thm III.6.3] or [12, Cor 4.32]), and the remaining properties of the Lyapunov spectrum on Π_{Γ_ν} follow from the KAK decomposition in $\mathsf{O}_{1,m}^+(\mathbf{R})$, with $1+m = \dim(\Pi_{\Gamma_\nu})$ (see Lemma 2.4). □

Lemma 5.3. *If $a \in H^{1,1}(X;\mathbf{R})$ satisfies $a^2 > 0$, for instance if a is a Kähler class, then*

$$\lim_{n\to+\infty} \frac{1}{n} \log |(f_\omega^n)^* a| = \lambda_{H^{1,1}}$$

for $\nu^{\mathbf{N}}$-almost every ω.

Proof. Corollary 2.5 implies that if $a \in \mathbb{H}_X$ then for every $f \in \mathsf{Aut}(X)$, $|f^*a| \asymp \|f^*\|$, where the implied constants depend only on a. Thus the result follows from Equation (5.6). □

Remark 5.4. It is natural to expect that Lemma 5.3 holds for any $a \in \Pi_\Gamma \setminus \{0\}$; this is true under the more stringent moment assumption (5.26) (see the proof of Proposition 5.15 below).

If the order of compositions is reversed (which is less natural from the point of view of iterated pull-backs), then Lemma 5.3 indeed holds for any a in Π_{Γ_ν} (see [19, Cor. III.3.4.i]):

Lemma 5.5. *For any $a \in \Pi_{\Gamma_\nu}$ and for $\nu^{\mathbf{N}}$-almost every $\omega = (f_n)_{n\geqslant 0} \in \Omega$ we have*

$$\lim_{n\to+\infty} \frac{1}{n} \log |f_n^* \cdots f_1^* a| = \lambda_{H^{1,1}}.$$

5.3. **The measure** μ_∂**.** By Furstenberg's theory the linear projective action of the random dynamical system (X, ν) on $\mathbb{P}\Pi_{\Gamma_\nu} \subset \mathbb{P}H^{1,1}(X; \mathbf{R})$ admits a unique stationary measure $\mu_{\mathbb{P}\Pi_{\Gamma_\nu}}$; this measure does not charge any proper projective subspace of $\mathbb{P}\Pi_{\Gamma_\nu}$. Recall that the mass of a class a is defined by $\mathbf{M}(a) = \langle a | [\kappa_0] \rangle$ (see § 2.2).

Lemma 5.6. *For $\nu^{\mathbf{N}}$-almost every ω, there exists a unique nef class $e(\omega)$ such that $\mathbf{M}(e(\omega)) = 1$ and*

$$\frac{1}{\mathbf{M}((f_\omega^n)^* a)} (f_\omega^n)^* a \underset{n \to \infty}{\longrightarrow} e(\omega) \tag{5.7}$$

for any pseudo-effective class a with $a^2 > 0$ (in particular for any Kähler class). In addition, the class $e(\omega)$ is almost surely isotropic and $\mathbb{P}(e(\omega))$ is a point of the limit set $\mathrm{Lim}(\Gamma_\nu) \subset \partial\mathbb{H}_X$.

Before starting the proof, note that $\Gamma_\nu^*|_{\Pi_{\Gamma_\nu}}$ is proximal, in the sense of [12, §4.1]; equivalently, $\Gamma_\nu^*|_{\Pi_{\Gamma_\nu}}$ is contracting, in the sense of [19, Def III.1.3]. In other words, there are sequences of elements $g_n \in \Gamma_\nu$ such that $\|g_n^*\|^{-1} g_n^*|_{\Pi_{\Gamma_\nu}}$ converges to a matrix of rank 1: for instance one can take $g_n = f^n$, where $f \in \Gamma_\nu$ is any loxodromic automorphism.

Proof. For $f \in \mathsf{Aut}(X)$, we use the notation $\underline{f}^*$ for its action on $\mathbb{P}H^{1,1}(X; \mathbf{R})$. Since the action of Γ_ν on Π_{Γ_ν} is strongly irreducible and proximal, its projective action satisfies the following contraction property (see [19, Thm III.3.1]): there is a measurable map $\omega \in \Omega \mapsto \underline{e}(\omega) \in \mathbb{P}\Pi_{\Gamma_\nu}$ such that for almost every ω, any cluster value $L(\omega)$ of

$$\frac{1}{\|f_0^* \cdots f_n^*\|} f_0^* \cdots f_n^* \tag{5.8}$$

in $\mathsf{End}(\Pi_{\Gamma_\nu})$ is an endomorphism of rank 1 whose range is equal to $\mathbf{R}\underline{e}(\omega)$.

Let $e(\omega)$ be the unique vector of mass 1 in the line $\mathbf{R}\underline{e}(\omega)$. If $a \in \Pi_{\Gamma_\nu}$ satisfies $a^2 > 0$ and $\mathbf{M}(a) > 0$, then any cluster value of $\mathbf{M}((f_\omega^n)^* a)^{-1} (f_\omega^n)^* a$ must coincide with $e(\omega)$ because by Corollary 2.5 the mass $\mathbf{M}((f_\omega^n)^* a)$ is comparable to the norm $\|f_0^* \cdots f_n^*\|$. Thus, the convergence (5.7) is satisfied. Furthermore $e(\omega)$ is nef, because we can apply this convergence to a nef class a and $\mathsf{Aut}(X)$ preserves the nef cone. Also, $e(\omega)$ belongs to $\mathrm{Lim}(\Gamma_\nu)$, hence it is isotropic. Now, let a and a' be two classes of $\mathbb{H}_X$ with $a \in \Pi_{\Gamma_\nu}$. Since the hyperbolic distance between $(f_\omega^n)^*(a)$ and $(f_\omega^n)^*(a')$ remains constant and the convergence (5.7) holds for a, it also holds for a'. This concludes the proof, for every class with positive self-intersection is proportional to a unique class in $\mathbb{H}_X$. □

Remark 5.7. As in Remark 5.4, under the exponential moment condition (5.26), the convergence in Equation (5.7) holds for any $a \in \Pi_\Gamma \setminus \{0\}$ and almost every $\omega \in \Omega$; to be precise, $\frac{1}{\mathbf{M}((f_\omega^n)^* a)} (f_\omega^n)^* a$ converges towards $e(\omega)$ or its opposite. Then, we actually get the convergence for any $a \in H^{1,1}(X; \mathbf{R}) \setminus \Pi_\Gamma^\perp$ (write $a = a_+ + a_0$ and use that Γ_ν acts by isometries on $\Pi_\Gamma^\perp$)

Here is a summary of the properties of the stationary measure $\mu_{\mathbb{P}\Pi_{\Gamma_\nu}}$; from now on, we view it as a measure on $\mathbb{P}H^{1,1}(X; \mathbf{R})$ and rename it as μ_∂ because it is supported on $\partial\mathbb{H}_X$.

Theorem 5.8. *The probability measure defined on $\mathbb{P}H^{1,1}(X; \mathbf{R})$ by*

$$\mu_\partial = \int \delta_{\mathbb{P}(e(\omega))}\, d\nu^{\mathbf{N}}(\omega) \tag{5.9}$$

is ν-stationary and ergodic. It is the unique stationary measure on $\mathbb{P}H^{1,1}(X;\mathbf{R})$ such that $\mu_\partial(\mathbb{P}(\Pi_{\Gamma_\nu}^\perp)) = 0$. The measure μ_∂ has no atoms and is supported on $\mathrm{Lim}(\Gamma_\nu)$; in particular, if $\Lambda' \subset \mathrm{Lim}(\Gamma_\nu)$ is such that $\mu_\partial(\Lambda') > 0$ then Λ' is uncountable.

The top Lyapunov exponent satisfies the so-called Furstenberg formula:

$$\lambda_{H^{1,1}} = \int \log\left(\frac{|f^*\tilde{u}|}{|\tilde{u}|}\right) d\nu(f)\, d\mu_\partial(u), \tag{5.10}$$

where $\tilde{u} \in H^{1,1}(X,\mathbf{R})\setminus\{0\}$ denotes any lift of $u \in \mathrm{Lim}(\Gamma_\nu) \subset \mathbb{P}H^{1,1}(X,\mathbf{R})$.

Proof. The ergodicity of $\mu_\partial = \mu_{\mathbb{P}\Pi_{\Gamma_\nu}}$ as well as its representation (5.9) follow from the properties of the action of Γ_ν on $\mathbb{P}(\Pi_\Gamma)$ (see [19, Chap. III]). Also, we know that $\lambda_{H^{1,1}}$ is equal to the top Lyapunov exponent of the restriction of the action to $\mathbb{P}(\Pi_{\Gamma_\nu})$, so the formula (5.10) follows from the strongly irreducible case (see [19, Cor III.3.4]).

Let now μ be a stationary measure on $\mathbb{P}H^{1,1}(X;\mathbf{R})$ such that $\mu(\mathbb{P}\Pi_{\Gamma_\nu}^\perp) = 0$. A martingale convergence argument shows that $(f_\omega^n)^*\mu$ converges to some measure μ_ω for almost every ω (see [19, Lem. II.2.1]). Since Γ_ν preserves the decomposition $\Pi_{\Gamma_\nu} \oplus \Pi_{\Gamma_\nu}^\perp$ and $\|(f_\omega^n)^*\|$ tends to infinity while $\| (f_\omega^n)^*|_{\Pi_{\Gamma_\nu}^\perp} \|$ stays uniformly bounded, we get that $(f_\omega^n)^*u$ converges to $\mathbb{P}\Pi_{\Gamma_\nu}$ for μ-almost every u and $\nu^{\mathbf{N}}$-almost every ω; thus μ_ω is almost surely supported on $\mathbb{P}\Pi_{\Gamma_\nu}$. Since by stationarity $\mu = \int \mu_\omega d\nu^{\mathbf{N}}(\omega)$ we conclude that μ gives full mass to $\mathbb{P}(\Pi_{\Gamma_\nu})$, hence $\mu = \mu_\partial$. □

Remark 5.9. If $\mathrm{Supp}(\nu)$ generates Γ_ν as a semi-group, then $\mathrm{Supp}(\mu_\partial) = \mathrm{Lim}(\Gamma_\nu)$, otherwise the inclusion can be strict: take a Schottky group $\Gamma = \langle f, g\rangle \subset \mathsf{PSL}(2,\mathbf{R})$ and $\nu = (\delta_f + \delta_g)/2$.

Remark 5.10. Since $\mathrm{Lim}(\Gamma_\nu) \subset \mathrm{Psef}(X)$, for every $u \in \mathrm{Lim}(\Gamma_\nu)$ there exists a unique $\tilde{u}$ such that $\mathbb{P}\tilde{u} = u$ and $\langle \tilde{u} \,|\, [\kappa_0]\rangle = \mathbf{M}(\tilde{u}) = 1$. Then the following formula holds:

$$\lambda_{H^{1,1}} = \int \log\left(\mathbf{M}(f^*\tilde{u})\right) d\nu(f)\, d\mu_\partial(u) = \int \log\left(\frac{\mathbf{M}(f^*\tilde{u})}{\mathbf{M}(\tilde{u})}\right) d\nu(f)\, d\mu_\partial(u). \tag{5.11}$$

Indeed set $r(w) = \mathbf{M}(w)/|w|$. On the limit set this function satisfies $1/C \leqslant r(\tilde{u}) \leqslant C$, where C is the positive constant from Equation (2.5). Then, for all $m \geqslant 1$,the stationarity of μ_∂ implies

$$\int \log\left(\frac{r(f^*\tilde{u})}{r(\tilde{u})}\right) d\nu(f)\, d\mu_\partial(u) = \int \log\left(\frac{r(f_m^* \cdots f_0^*\tilde{u})}{r(f_{m-1}^* \cdots f_0^*\tilde{u})}\right) d\nu(f_m)\cdots d\nu(f_0)\, d\mu_\partial(u).$$

Summing from $m = 0$ to $n-1$, telescoping the sum, and dividing by n gives

$$\int \log\left(\frac{r(f^*\tilde{u})}{r(\tilde{u})}\right) d\nu(f)\, d\mu_\partial(u) = \frac{1}{n}\int \log\left(\frac{r(f_{n-1}^* \cdots f_0^*\tilde{u})}{r(\tilde{u})}\right) d\nu(f_{n-1})\cdots d\nu(f_0)\, d\mu_\partial(u).$$

Finally since $1/C \leqslant r \leqslant C$, the right hand side tends to zero as $n \to \infty$. Hence the integral of $\log(r \circ f^*/r)$ vanishes, and (5.11) follows from Furstenberg's formula. □

Proposition 5.11. *The point $\mathbb{P}(e(\omega))$ is $\nu^{\mathbf{N}}$-almost surely extremal in $\mathbb{P}(\overline{\mathrm{Kah}}(X))$ and in $\mathbb{P}(\mathrm{Psef}(X))$.*

Proof. The class $e(\omega)$ almost surely belongs to $\overline{\mathrm{Kah}}(X)$ and to the isotropic cone. By the Hodge index theorem –more precisely, by the case of equality in the reverse Schwarz Inequality (2.7)– $e(\omega)$ cannot be a non-trivial convex combination of classes with non-negative intersection and mass 1; so $\mathbb{P}(e(\omega))$ is an extremal point of the convex set $\mathbb{P}(\overline{\mathrm{Kah}}(X)) \subset \mathbb{P}H^{1,1}(X;\mathbf{R})$.

From Proposition 2.3, there are at most countably many points $\mathbb{P}(u)$ in $\mathbb{P}(\overline{\mathrm{Kah}}(X))$ such that $u^2 = 0$ and $\mathbb{P}(u)$ is not extremal in $\mathbb{P}(\mathrm{Psef}(X))$. Therefore the second assertion follows from the fact that μ_∂ is atomless. ☐

5.4. **Some estimates for random products of matrices.**

5.4.1. *Sequences of good times.* We now describe a theorem of Gouëzel and Karlsson, specialized to our specific context. Fix a base point e_0 in the hyperbolic space $\mathbb{H}_X$, for instance $e_0 = [\kappa_0]$ with κ_0 a fixed Kähler form (as in Section 2.2). Consider the two functions of $(n, \omega) \in \mathbf{N} \times \Omega$ defined by

$$T(n,\omega) = d_{\mathbb{H}}(e_0, (f^n_\omega)^* e_0), \quad N(n,\omega) = \log \|(f^n_\omega)^*\|. \tag{5.12}$$

They satisfy the subadditive cocycle property

$$a(n+m,\omega) \leqslant a(n,\omega) + a(m, \sigma^n(\omega)), \tag{5.13}$$

where σ is the unilateral shift on Ω (see § 4.3). Let $a(n, \omega)$ be such a subadditive cocycle; if $a(1, \omega)$ is integrable the asymptotic average is defined to be the limit

$$A = \lim_{n\to+\infty} \frac{1}{n} \int a(n,\omega)\, d\nu^{\mathbf{N}}(\omega); \tag{5.14}$$

it exists in $[-\infty, +\infty)$, and we say it is finite if $A \neq -\infty$. The functions T and N are examples of ergodic subadditive cocycles and from Theorem 5.8, Remark 5.10, and Corollary 2.5, we deduce that the asymptotic average of each of these cocycles is equal to $\lambda_{H^{1,1}}$.

Following [68], we say that $a(n, \omega)$ is **tight along the sequence of positive integers** (n_i) if there is a sequence of real numbers $(\delta_\ell) = (\delta_\ell(\omega))_{\ell \geqslant 0}$ such that

(i) δ_ℓ converges to 0 as ℓ goes to $+\infty$;
(ii) for every i, and for every $0 \leqslant \ell \leqslant n_i$,
$$\left| a(n_i,\omega) - a(n_i - \ell, \sigma^\ell(\omega)) - A\ell \right| \leqslant \ell \delta_\ell;$$
(iii) for every i and for every $0 \leqslant \ell \leqslant n_i$
$$a(n_i,\omega) - a(n_i - \ell, \omega) \geqslant (A - \delta_\ell)\ell.$$

Theorem 5.12 (Gouëzel and Karlsson [68])**.** *Let $a(n, \omega)$ be an ergodic subadditive cocycle, with a finite asymptotic average A. Then, for almost every ω, the cocycle is tight along a subsequence $(n_i(\omega))$ of positive upper density.*

Recall that the (asymptotic) upper density of a subset S of $\mathbf{N}$ is the non-negative number defined by $\overline{\mathrm{dens}}(S) = \limsup_{k\to+\infty} \left(\frac{1}{k} |S \cap [0, k-1]|\right)$. A sequence $(n_i)_{i\geqslant 0}$ is said to have positive upper density if the set of its values $S = \{n_i \,;\, i \geqslant 0\}$ satisfies $\overline{\mathrm{dens}}(S) > 0$.

Proof. Let us explain how this result follows from [68]. First, fix a small positive real number $\rho > 0$, and apply Theorem 1.1 and Remark 1.2 of [68] to get a set Ω_ρ of measure $1 - \rho$ such that the first two properties (i) and (ii) are satisfied for every $\omega \in \Omega_\rho$ with respect to a sequence (δ_ℓ) that does not depend on ω, and for a sequence of times $(n_i(\omega))$ of upper density $\geqslant 1 - \rho$. To get (iii), we apply Lemma 2.3 of [68] to the sub-additive cocycle $a(n, \omega)$ (not to the cocycle $b(n, \omega) = a(n, \sigma^{-n}(\omega))$ as done in [68]). For every $\varepsilon > 0$, there is a subset $\Omega'_\varepsilon \subset \Omega$ and a sequence $(\delta'_\ell)_{\ell\geqslant 0}$ such that

(a) $\nu^{\mathbf{N}}(\Omega'_\varepsilon) > 1-\varepsilon$, and δ'_ℓ converges towards 0 as ℓ goes to $+\infty$;
(b) for every $\omega \in \Omega'_\varepsilon$, there is a set of bad times $B(\omega) \subset \mathbf{N}$ such that for every $k \geqslant 0$ $|B(\omega) \cap [0, k-1]| \leqslant \varepsilon k$, and for every $n \notin B(\omega)$ and every $0 \leqslant \ell \leqslant n$,

$$a(n,\omega) - a(n-\ell,\omega) \geqslant (A - \delta'_\ell)\ell.$$

If ω belongs to $\Omega_\rho \cap \Omega'_\varepsilon$, the set of indices i for which $n_i(\omega) \notin B(\omega)$ is infinite. More precisely, the set $S(\omega) = \{n_j(\omega)\ ;\ n_j(\omega) \notin B(\omega)\}$ has asymptotic upper density $\geqslant 1-\rho-\varepsilon$. Along this subsequence, the three properties (i), (ii), and (iii) are satisfied. Since this holds for all $\omega \in \Omega'_\varepsilon \cap \Omega_\rho$ and the measure of this set is $\geqslant 1-\rho-\varepsilon$, this holds for $\nu^{\mathbf{N}}$-almost every ω. $\square$

Corollary 5.13. *For $\nu^{\mathbf{N}}$-almost every $\omega \in \mathsf{Aut}(X)^{\mathbf{N}}$, there is an increasing sequence of integers $(n_i(\omega))$ going to $+\infty$ and a real number $A(\omega)$ such that*

$$\sum_{j=0}^{n_i(\omega)} \frac{\left\|\left(f_\omega^j\right)^*\right\|}{\left\|\left(f_\omega^{n_i(\omega)}\right)^*\right\|} \leqslant A(\omega) \ \text{ and } \ \sum_{j=0}^{n_i(\omega)} \frac{\left\|\left(f_{\sigma^j(\omega)}^{n_i(\omega)-j}\right)^*\right\|}{\left\|\left(f_\omega^{n_i(\omega)}\right)^*\right\|} \leqslant A(\omega)$$

for all indices $i \geqslant 0$.

Proof. Apply Theorem 5.12 to the subadditive cocyle $N(n,\omega)$ and note that

$$\sum_{j=0}^{n_i(\omega)} \frac{\left\|\left(f_\omega^j\right)^*\right\|}{\left\|\left(f_\omega^{n_i(\omega)}\right)^*\right\|} = \sum_{\ell=0}^{n_i(\omega)} \frac{\left\|\left(f_\omega^{n_i-\ell}\right)^*\right\|}{\left\|\left(f_\omega^{n_i}\right)^*\right\|} = \sum_{\ell=0}^{n_i(\omega)} \frac{e^{N(n_i-\ell,\omega)}}{e^{N(n_i,\omega)}} \leqslant \sum_{\ell=0}^{n_i(\omega)} e^{-\ell(\lambda_{H^{1,1}}-\delta_\ell)} \tag{5.15}$$

which is bounded as $n_i(\omega) \to \infty$. The second estimate is similar. $\square$

5.4.2. *A mass estimate for pull-backs.* Assume that (X,ν) is non-elementary and satisfies the condition (4.1). Recall from Lemma 5.5 that $\mathbf{M}((f_\omega^n)^*a)^{-1}(f_\omega^n)^*a$ converges to the pseudo-effective class $e(\omega)$ for almost every ω and every Kähler class a. Thus, on a set of total $\nu^{\mathbf{N}}$-measure, this convergence holds for all $\sigma^k(\omega)$, $k \geqslant 0$. Since $\mathbf{M}(e(\omega)) = 1$, we obtain

$$f_0^* e(\sigma\omega) = \mathbf{M}(f_0^* e(\sigma\omega)) e(\omega); \tag{5.16}$$

more generally, for every $k \geqslant 1$,

$$(f_\omega^k)^* e(\sigma^k\omega) = \mathbf{M}((f_\omega^k)^* e(\sigma^k\omega)) e(\omega). \tag{5.17}$$

Lemma 5.14. *For $\nu^{\mathbf{N}}$-almost every ω, we have*

$$\frac{1}{n}\log \mathbf{M}((f_\omega^n)^* e(\sigma^n\omega)) \underset{n\to\infty}{\longrightarrow} \lambda_{H^{1,1}}.$$

This does *not* follow from Lemma 5.3 because $e(\sigma^n\omega)$ depends on n. Our argument relies on Theorem 5.12 for convenience but other strategies could certainly be applied.

Proof. For almost every ω, for every $k \geqslant 1$, and for every Kähler class a, we have

$$e(\sigma^k\omega) = \lim_{n\to\infty} \frac{f_k^* \cdots f_{n-1}^* a}{\mathbf{M}(f_k^* \cdots f_{n-1}^* a)}. \tag{5.18}$$

So

$$f_0^* \cdots f_{k-1}^* e(\sigma^k(\omega)) = \left(\lim_{n\to\infty} \frac{\mathbf{M}(f_0^* \cdots f_{n-1}^* a)}{\mathbf{M}(f_k^* \cdots f_{n-1}^* a)}\right) e(\omega) =: \zeta(k,\omega) e(\omega) \tag{5.19}$$

where $\zeta(k,\omega)$ is both equal to $\mathbf{M}((f_\omega^k)^* e(\sigma^k(\omega)))$ and to the limit

$$\zeta(k,\omega) = \lim_{n\to\infty} \frac{\mathbf{M}(f_0^* \cdots f_{n-1}^* a)}{\mathbf{M}(f_k^* \cdots f_{n-1}^* a)} = \lim_{n\to\infty} \frac{\mathbf{M}((f_\omega^n)^* a)}{\mathbf{M}((f_{\sigma^k(\omega)}^{n-k})^* a)}. \tag{5.20}$$

We want to show that, $\nu^{\mathbf{N}}$-almost surely, $(1/k)\log\zeta(k,\omega)$ converges to $\lambda_{H^{1,1}}$.

Before starting the proof, note that ζ is a multiplicative cocycle: $\zeta(k,\omega) = \prod_{\ell=1}^k \zeta(1,\sigma^\ell\omega)$; in particular, $\log\zeta(k,\omega)$ is equal to the Birkhoff sum $\sum_{\ell=1}^k \log\zeta(1,\sigma^\ell\omega)$. Since

$$C^{-1}\left\|(f_0^{-1})^*\right\|_{H^{1,1}} \leqslant \mathbf{M}(f_0^* e(\sigma(\omega))) \leqslant C\|f_0^*\|_{H^{1,1}}, \tag{5.21}$$

our moment condition shows that $\log(\zeta(1,\omega))$ is integrable. So, by the ergodic theorem of Birkhoff, $\lim_k \frac{1}{k}\log\zeta(k,\omega)$ exists $\nu^{\mathbf{N}}$-almost surely.

Pick a sequence (n_i) of good times for ω, as in Theorem 5.12. If we compute the limit in Equation (5.20) along the subsequence (n_i) we see that $\zeta(k,\omega) \geqslant C\exp((\lambda_{H^{1,1}} - \delta(k))k)$ for some constant $C > 0$, and some sequence $\delta(k)$ converging to 0 as k goes to $+\infty$. This gives

$$\limsup_{k\to+\infty} \frac{1}{k}\log\zeta(k,\omega) \geqslant \lambda_{H^{1,1}}. \tag{5.22}$$

Now, consider the linear cocycle $\Upsilon : \Omega \times H^{1,1}(X,\mathbf{R}) \to \Omega \times H^{1,1}(X,\mathbf{R})$ defined by

$$\Upsilon(\omega,u) = (\sigma(\omega), (f_\omega^1)_* u) \tag{5.23}$$

and let $\mathbb{P}\Upsilon$ be the associated projective cocycle on $\Omega \times \mathbb{P}H^{1,1}(X,\mathbf{R})$. The Lyapunov exponents of Υ are $\pm\lambda_{H^{1,1}}$, each with multiplicity 1, and 0, with multiplicity $h^{1,1}(X) - 2$. Since $\mathbb{P}((f_\omega^1)^* e(\sigma(\omega))) = \mathbb{P}(e(\omega))$, the measurable section $\{(\omega, \mathbb{P}(e(\omega)))\ ;\ \omega \in \Omega\}$ is $\mathbb{P}\Upsilon$-invariant. Therefore, by ergodicity of σ with respect to $\nu^{\mathbf{N}}$, $m = \int \delta_{\mathbb{P}(e(\omega))}\, d\nu^{\mathbf{N}}(\omega)$ defines an invariant and ergodic measure for $\mathbb{P}\Upsilon$. It follows from the invariance of the decomposition into characteristic subspaces in Oseledets' theorem that $e(\omega)$ is contained in a given characteristic subspace of the cocycle Υ; thus, if λ denotes the Lyapunov exponent of Υ in that characteristic subspace, we get (as in Remark 5.10) that

$$\lambda = \int \log\frac{\left|(f_\omega^1)_* u\right|}{|u|}\, dm(\omega,u) = \int \log\frac{\mathbf{M}((f_\omega^1)_*(e(\omega)))}{\mathbf{M}(e(\omega))}\, d\nu^{\mathbf{N}}(\omega) \tag{5.24}$$

$$= \int \log\zeta(1,\omega)^{-1}\, d\nu^{\mathbf{N}}(\omega) \tag{5.25}$$

(see Ledrappier [85, §1.5]). Birkhoff's ergodic theorem implies that $\lim_k \frac{1}{k}\log\zeta(k,\omega) = -\lambda$, with $\lambda \in \{\pm\lambda_{H^{1,1}}, 0\}$, therefore the Inequality (5.22) concludes the proof. □

5.4.3. *Exponential moments.* The result of this section will only be used in Theorem 6.17 so this paragraph may be skipped on a first reading. Consider the exponential moment condition

$$\exists\tau > 0,\ \int \left(\|f\|_{C^1} + \left\|f^{-1}\right\|_{C^1}\right)^\tau\, d\nu(f) < +\infty. \tag{5.26}$$

As in Section 5.1, this upper bound implies the cohomological moment condition

$$\exists\tau > 0,\ \int \left(\|f^*\|_{H^{1,1}} + \left\|(f^{-1})^*\right\|_{H^{1,1}}\right)^\tau\, d\nu(f) < +\infty. \tag{5.27}$$

Proposition 5.15. *Assume that ν satisfies the Condition* (5.26). *Let $D\colon \mathsf{Aut}(X) \to \mathbf{R}_+$ be a measurable function such that $\int D(f)^{\tau'} d\nu(f) < \infty$ for some $\tau' > 0$. Then, there is a measurable function $B\colon \Omega \to \mathbf{R}_+$ satisfying*

$$\int \log^+(B(\omega))\, d\nu^{\mathbf{N}}(\omega) < \infty$$

such that for $\nu^{\mathbf{N}}$-almost every $\omega = (f_n)$ and every $n \geqslant 0$

$$\sum_{j=1}^{n-1} D(f_{j-1}) \frac{\left\| f_j^* \cdots f_{n-1}^* \right\|}{\left\| f_0^* \cdots f_{n-1}^* \right\|} \leqslant B(\omega), \quad \textit{and} \quad \sum_{j=1}^{n-1} D(f_j) \frac{\left\| f_0^* \cdots f_{j-1}^* \right\|}{\left\| f_0^* \cdots f_{n-1}^* \right\|} \leqslant B(\omega).$$

This is a refined version of Corollary 5.13. The result is stated in our setting, but it holds for more general random products of matrices.

Proof. We are grateful to Sébastien Gouëzel for explaining this argument to us. We temporarily use the notation $\mathbb{P}(\cdot)$ for probability with respect to ν^n or $\nu^{\mathbf{N}}$ (so, here, $\mathbb{P}$ does not denote projectivisation).

First Estimate.– We start with the first estimate: $\sum_{j=1}^{n-1} D(f_{j-1}) \frac{\left\| f_j^* \cdots f_{n-1}^* \right\|}{\left\| f_0^* \cdots f_{n-1}^* \right\|} \leqslant B(\omega)$.

Step 1.– For every $0 < \varepsilon < \lambda_{H^{1,1}}$ there exists constants $c, C > 0$ such that

$$\mathbb{P}\left(|(f_\omega^n)^* b| \leqslant e^{\varepsilon n} \right) \leqslant C e^{-cn}. \tag{5.28}$$

for every $b \in \Pi_\Gamma$ with $|b| = 1$. This large deviation result, which is uniform in n and b, follows from condition (5.27) (see for instance [19, §V.6], and [12, §12]).

Step 2.– Let us prove that

$$\mathbb{P}\left(\frac{\left\| f_j^* \cdots f_{n-1}^* \right\|}{\left\| f_0^* \cdots f_{n-1}^* \right\|} > e^{-\varepsilon j} \right) \leqslant C e^{-cj}. \tag{5.29}$$

For this, fix $f_j, \ldots, f_{n-1}$. Then, there is a point $a \in \Pi_\Gamma$ with $|a| = 1$ such that $\left\| f_j^* \cdots f_{n-1}^* \right\| = \left| f_j^* \cdots f_{n-1}^* a \right|$. Hence, if $\left\| f_0^* \cdots f_{n-1}^* \right\| < \left\| f_j^* \cdots f_{n-1}^* \right\| e^{\varepsilon j}$, we infer that

$$\left| f_0^* \cdots f_{n-1}^* a \right| < \left\| f_j^* \cdots f_{n-1}^* \right\| e^{\varepsilon j} = \left| f_j^* \cdots f_{n-1}^* a \right| e^{\varepsilon j}. \tag{5.30}$$

Thus, if we set

$$b = \frac{1}{\left| f_j^* \cdots f_{n-1}^* a \right|} f_j^* \cdots f_{n-1}^* a, \tag{5.31}$$

we obtain that $\left| f_0^* \cdots f_{j-1}^* b \right| < e^{\varepsilon j}$; this happens with (conditional) probability $\leqslant C e^{-cj}$ (relative to ν^{*j}), for the uniform constants given in Step 1. Averaging over $f_j, \ldots, f_{n-1}$, we get the result.

Step 3.– The moment condition satisfied by D and Markov's inequality imply $\mathbb{P}(D > K) \leqslant C_1 K^{-\tau'}$ for some constant $C_1 > 0$. Fix $\varepsilon \in \mathbf{R}_+^*$ small with respect to $\lambda_{H^{1,1}}$ and τ'. Then, on a set $\Omega(\varepsilon, J)$ of measure

$$\nu^{\mathbf{N}}(\Omega(\varepsilon, J)) \geqslant 1 - C_2(e^{-(\varepsilon\tau'/2)J} + e^{-\varepsilon c J}), \tag{5.32}$$

for some $C_2 = C_2(\varepsilon) > 0$, we have both $D(f_{j-1}) \leqslant e^{\varepsilon j/2}$ and $\frac{\|f_j^* \cdots f_{n-1}^*\|}{\|f_0^* \cdots f_{n-1}^*\|} \leqslant e^{-\varepsilon j}$ for all $j \geqslant J$. For $\omega = (f_n)$ in $\Omega(\varepsilon, J)$, we get

$$\begin{aligned}\sum_{j=1}^{n-1} D(f_{j-1}) \frac{\|f_j^* \cdots f_{n-1}^*\|}{\|f_0^* \cdots f_{n-1}^*\|} &\leqslant \sum_{j=1}^{J} D(f_{j-1}) \frac{\|f_j^* \cdots f_{n-1}^*\|}{\|f_0^* \cdots f_{n-1}^*\|} + \sum_{j=J+1}^{n-1} e^{-\varepsilon j/2} \\ &\leqslant \sum_{j=1}^{J} D(f_{j-1}) \left\|(f_{j-1}^{-1})^* \cdots (f_0^{-1})^*\right\| + C_3 \\ &= C_3 + \sum_{j=0}^{J-1} \|f_0^*\| \cdots \|f_j^*\| D(f_j).\end{aligned} \tag{5.33}$$

The moment condition (5.26) gives $\mathbb{P}(\|f^*\| > K) \leqslant C_4 K^{-\tau}$ and as already noticed, we also have $\mathbb{P}(D(f) > K) \leqslant C_1 K^{-\tau'}$. So, with $\eta = \min(\tau, \tau')$, there is a set of probability at least $1 - C_5 J K^{-\eta}$ on which

$$\sum_{j=0}^{J-1} D(f_j) \|f_0^*\| \cdots \|f_j^*\| \leqslant C_6 J K^{J+2}. \tag{5.34}$$

Taking $K = J^{3/\eta}$, we have $JK^{-\eta} = J^{-2}$, and we obtain

$$\mathbb{P}\left(\sum_{j=0}^{J-1} D(f_j) \|f_0^*\| \cdots \|f_j^*\| > J^{1+3(J+2)/\eta}\right) \leqslant C_7 J^{-2}. \tag{5.35}$$

Also, note that $J^{1+(3J+6)/\eta} \leqslant \exp\left(CJ^{3/2}\right)$.

By the Borel-Cantelli lemma, the sum in (5.33) is almost surely bounded by some constant $B(\omega)$ which satisfies $\mathbb{P}\left(\log B > J^{3/2}\right) \leqslant CJ^{-2}$; in particular $\mathbb{E}\left(\log^+ B\right) < \infty$.

Second Estimate.– To obtain the second estimate of Proposition 5.15, we apply the above proof to the reversed random dynamical system, induced by $\check{\nu} : f \mapsto \nu(f^{-1})$. Indeed, the core of the argument is the inequality (5.33) which is not sensitive to the order of compositions. □

6. LIMIT CURRENTS

Our goal in this section is to prove the counterpart of the convergence (5.7) at the level of closed positive currents on X. Throughout this section we fix a non-elementary random holomorphic dynamical system (X, ν) satisfying the moment condition (4.1), so that all results of §5 apply. We refer the reader to [71] (in particular Chapter 8) for basics on pluripotential theory on compact Kähler manifolds (see also [**?**]).

6.1. **Potentials and cohomology classes of positive closed currents.** Let us fix once and for all a family of Kähler forms $(\kappa_i)_{1 \leqslant i \leqslant h^{1,1}(X)}$ such that $[\kappa_i]^2 = 1$ and the $[\kappa_i]$ form a basis of $H^{1,1}(X; \mathbf{R})$; in addition we require that the κ_i satisfy

$$\kappa_0 = \beta \sum_i \kappa_i \tag{6.1}$$

for some $\beta > 0$, where κ_0 is the Kähler form chosen in Section 2.2 (note that necessarily $\beta < 1$). We also fix a smooth volume form vol_X on X, normalized by $\int_X \mathrm{vol} = 1$. On tori, K3 and Enriques surfaces, we choose vol_X to be the canonical $\mathsf{Aut}(X)$-invariant volume form (see Remark 3.17). It is convenient to assume in all cases that vol_X is also the volume form associated with the Kähler metric κ_0 (up to scaling). On tori, K3 and Enriques surfaces this implies that κ_0 is the unique Ricci-flat Kähler metric in its Kähler class; its existence is guaranteed by Yau's theorem (see [60] for the interest of such a choice in holomorphic dynamics).

Unless otherwise specified, the currents we shall consider will be of type $(1,1)$. The action of a current T on a test form φ will be denoted by $\langle T, \varphi\rangle$ or $\int T \wedge \varphi$. If T is closed, we denote its cohomology class by $[T]$; so, if φ is a closed form, $\langle T, \varphi\rangle = \langle [T] \,|\, [\varphi]\rangle$. By definition the **mass** of a current is the quantity $\mathbf{M}(T) = \int T \wedge \kappa_0$; so $\mathbf{M}(T) = \langle [T] | [\kappa_0]\rangle$ when T is closed.

6.1.1. *Normalized potentials.* If a is an element of $H^{1,1}(X;\mathbf{R})$, we denote by $(c_i(a))_{1\leqslant i\leqslant h^{1,1}(X)}$ its coordinates in the basis $([\kappa_i])$, so that $a = \sum_i c_i(a)[\kappa_i]$. Then, we set

$$\Theta(a) = \sum_i c_i(a)\kappa_i. \tag{6.2}$$

Likewise, given a closed $(1,1)$-form α or a closed current of bidegree $(1,1)$, we set $c_i(\alpha) = c_i([\alpha])$ and $\Theta(\alpha) = \Theta([\alpha])$; hence, $[\Theta(\alpha)] = [\alpha]$. It is worth keeping in mind that some coefficients $c_i(\alpha)$ can be negative and $\Theta(\alpha)$ need not be semi-positive, even if α is a Kähler form. If T is a closed positive current of bidegree $(1,1)$ on X we define its **normalized potential** to be the unique function $u_T \in L^1(X)$ such that

$$T = \Theta(T) + dd^c(u_T) \ \text{ and } \ \int_X u_T \,\mathsf{vol} = 0 \tag{6.3}$$

(see [71, §8.1]). The function u_T is locally given as the difference $v - w$ of a psh potential v of T and a smooth potential w of $\Theta(T)$.

Lemma 6.1. *There is a constant $A > 0$ such that the following properties are satisfied for every closed positive current T of mass 1*

(1) $-A \leqslant c_i(T) \leqslant A$ *for all* $1 \leqslant i \leqslant h^{1,1}(X)$, *and* $-A\kappa_0 \leqslant \Theta(T) \leqslant A\kappa_0$.
(2) *the function u_T is $(A\kappa_0)$-psh: $dd^c(u_T) + A\kappa_0$ is a positive current.*

Proof. Since the coefficients $T \mapsto c_i(T)$ are continuous functions on the space of currents and closed positive currents of mass 1 form a compact set K, the functions $|c_i|$ are bounded by some uniform constant A' on K. Setting $A = A'\beta^{-1}$, with β as in Equation (6.1), we get $-A\kappa_0 \leqslant \Theta(T) \leqslant A\kappa_0$ for all $T \in K$. Then $dd^c u_T = T - \Theta(T) \geqslant -A\kappa_0$ and (2) follows. □

Corollary 6.2. *The set of potentials $\{u_T \mid T$ is a closed positive current of mass 1 on $X\}$ is a compact subset of $L^1(X;\mathsf{vol})$.*

Proof. Since this is a set of $(A\kappa_0)$-psh functions which are normalized with respect to a smooth volume form, the result follows from Proposition 8.5 and Remark 8.6 in [71]. □

Remark 6.3. Another usual normalization imposes the condition $\sup_{x\in X} u_T(x) = 0$; by compactness this would only change u_T by some uniformly bounded constant. However since many of our dynamical examples preserve a natural volume form it is more convenient for us to normalize as in (6.3).

6.1.2. *The diameter of a pseudo-effective class.* For a class $a \in \mathrm{Psef}(X)$ we define

$$\mathrm{Cur}(a) = \{T \;;\; T \text{ is a closed positive current with } [T] = a\}, \tag{6.4}$$

This is a compact convex subset of the space of currents. If S and T are two elements of $\mathrm{Cur}(a)$, then $\Theta(S) = \Theta(T) = \Theta(a)$ and $T - S = dd^c(u_T - u_S)$. We set

$$\mathrm{dist}(S, T) = \int_X |u_S - u_T| \,\mathsf{vol}\;. \tag{6.5}$$

This is a distance that metrizes the weak topology on $\mathrm{Cur}(a)$: this follows for instance from the fact that by Corollary 6.2 $(\mathrm{Cur}(a), \mathrm{dist})$ is compact. By definition, the **diameter** of a is

$$\mathrm{Diam}(a) = \mathrm{Diam}(\mathrm{Cur}(a)) = \sup\{\mathrm{dist}(S, T) \;;\; S,\, T \text{ in } \mathrm{Cur}(a)\}, \tag{6.6}$$

If $a \in \mathrm{Psef}(X)$, then $\mathrm{Diam}(a)$ is a non-negative real number which is finite by Corollary 6.2. If $\mathrm{Cur}(a) = \emptyset$, we set $\mathrm{Diam}(a) = -\infty$. Note that Diam is homogeneous of degree 1: $\mathrm{Diam}(ta) = t\,\mathrm{Diam}(a)$ for every $a \in \mathrm{Psef}(X)$ and $t > 0$.

Example 6.4. Let $\pi\colon X \to B$ be a fibration of genus 1. Let a be the cohomology class of any fiber $X_w = \pi^{-1}(w)$, $w \in B$. Then, to every probability measure μ_B on B corresponds a closed positive current $T_{\mu_B} \in \mathrm{Cur}(a)$, defined by $\langle T_{\mu_B}, \varphi\rangle = \int_B \int_{X_w} \varphi d\mu_B(w)$, and any closed positive current in $\mathrm{Cur}(a)$ is of this form. In this case $\mathrm{Diam}(a) > 0$. Now, assume that f is a loxodromic automorphism of X, and denote by θ_f the unique $(1,1)$-class of mass 1 that satisfies $f^*\theta_f = \lambda_f \theta_f$, where λ_f is the spectral radius of $f^* \in \mathsf{GL}(H^{1,1}(X; \mathbf{R}))$; then $\mathrm{Cur}(\theta_f)$ is represented by a unique closed positive current T_f^+ and $\mathrm{Diam}(\theta_f) = 0$. For generic Wehler surfaces, these two types of classes, given by eigenvectors of loxodromic automorphisms and classes of genus 1 fibrations, are dense in the boundary of $\mathbb{H}_X \cap \mathrm{NS}(X; \mathbf{R})$ (see [28]).

Lemma 6.5. *On* $\mathrm{Psef}(X)$*,* $a \mapsto \mathrm{Diam}(a)$ *is upper semi-continuous, hence measurable.*

Proof. Let (a_n) be a sequence of pseudo-effective classes converging to a. For every n we choose a pair of currents (S_n, T_n) in $\mathrm{Cur}(a_n)^2$ such that $\mathrm{dist}(S_n, T_n) \geqslant \mathrm{Diam}(a_n) - 1/n$. The masses of S_n and T_n are uniformly bounded because they depend only on a_n. By Corollary 6.2, we can extract a subsequence such that S_n and T_n converge towards closed positive currents $S, T \in \mathrm{Cur}(a)$, and u_{S_n} and u_{T_n} converge towards their respective potentials u_S and u_T in $L^1(X, \mathsf{vol})$. Then, $\mathrm{dist}(S, T) = \int_X |u_S - u_T| \mathsf{vol} = \lim_n \mathrm{dist}(S_n, T_n)$, which shows that $\mathrm{Diam}(a) \geqslant \limsup_n (\mathrm{Diam}(a_n))$. □

6.2. **Action of** $\mathsf{Aut}(X)$**.**

6.2.1. *A volume estimate.* Let X be a compact, complex manifold, and let vol be a C^0-volume form on X with $\mathsf{vol}(X) = 1$. If f is an automorphism of X, let $\mathrm{Jac}(f)\colon X \to \mathbf{R}$ denote its Jacobian determinant with respect to the volume form vol: $f^*\mathsf{vol} = \mathrm{Jac}(f)\mathsf{vol}$. The following lemma is a variation on well-known ideas in holomorphic dynamics (see for instance [70]).

Lemma 6.6. *Let* κ *be a hermitian form on* X*. Let* h *be a* κ*-psh function on* X *such that* $\int_X h\,\mathsf{vol} = 0$*, and let* f *be an automorphism of* X*. Then,*

$$\int_X |h \circ f| \;\mathsf{vol} \leqslant C \log(C \| \mathrm{Jac}(f^{-1}) \|_\infty)$$

for some positive constant C *that depends on* (X, κ) *but neither on* f *nor on* h*.*

Proof. We first observe that there is a constant $c > 0$ such that $\mathsf{vol}\{|h| \geqslant t\} \leqslant c\exp(-t/c)$; this follows from Lemma 8.10 and Theorem 8.11 in [71], together with Chebychev's inequality (see Remark 6.3 for the normalization). Then, we get

$$\begin{aligned} \int_X |h\circ f|\,\mathsf{vol} &= \int_0^\infty \mathsf{vol}\{|h\circ f| \geqslant t\}dt && (6.7)\\ &= \int_0^\infty \mathsf{vol}(f^{-1}\{|h|\geqslant t\})dt \\ &\leqslant \int_0^s \mathsf{vol}(X)dt + \left\|\mathrm{Jac}(f^{-1})\right\|_\infty \int_s^\infty c\exp(-t/c)dt \\ &\leqslant s\,\mathsf{vol}(X) + \left\|\mathrm{Jac}(f^{-1})\right\|_\infty c^2\exp(-s/c) && (6.8)\end{aligned}$$

where the inequality in the third line follows from the change of variable formula. Now, we minimize (6.8) by choosing $s = c\log(c\left\|\mathrm{Jac}(f^{-1})\right\|_\infty/\mathsf{vol}(X))$ and we infer that

$$\int_X |h\circ f|\,\mathsf{vol} \leqslant c\,\mathsf{vol}(X)\left(1+\log\left(\frac{c\left\|\mathrm{Jac}(f^{-1})\right\|_\infty}{\mathsf{vol}(X)}\right)\right). \tag{6.9}$$

Since the total volume is invariant, $\left\|\mathrm{Jac}(f)\right\|_\infty \geqslant 1$, and the asserted estimate follows. □

6.2.2. *Equivariance.* Let us come back to the study of (X,ν). If f is an automorphism of X, then $f^*\,\mathrm{Cur}(a) = \mathrm{Cur}(f^*(a))$ for every class $a \in H^{1,1}(X,\mathbf{R})$. If $a \in \mathrm{Psef}(X)$ and $T \in \mathrm{Cur}(a)$, then $T = \Theta(a) + dd^c(u_T)$ and

$$f^*T = f^*\Theta(a) + dd^c(u_T\circ f) = \Theta(f^*a) + dd^c(u_{f^*\Theta(a)} + u_T\circ f). \tag{6.10}$$

This shows that the normalized potential of f^*T is given by

$$u_{f^*T} = u_{f^*\Theta(a)} + u_T\circ f + E(f,T) \tag{6.11}$$

where $E(f,T) \in \mathbf{R}$ is the constant for which the integral of u_{f^*T} vanishes; since $u_{f^*\Theta(a)}$ has mean 0, we get

$$E(f,T) = -\int_X \left(u_{f^*\Theta(a)} + u_T\circ f\right)\,\mathsf{vol} = -\int_X u_T\circ f\,\mathsf{vol}. \tag{6.12}$$

Remark 6.7. If vol is f-invariant, for instance if it is the canonical volume on a K3 or Enriques surface, then $E(f,T) = 0$, which simplifies a little bit the analysis of the potentials below.

Lemma 6.8. *On the set of closed positive currents of mass* 1*, the function* $(f,T) \mapsto E(f,T)$ *satisfies*

$$|E(f,T)| \leqslant C\log\left(C\left\|\mathrm{Jac}(f^{-1})\right\|_\infty\right)$$

where the implied positive constant C *depends neither on* f *nor on* T*.*

Proof. From Lemma 6.1, the potentials u_T are uniformly $(A\kappa_0)$-psh, so the conclusion follows from Equation (6.12) and Lemma 6.6. □

Lemma 6.9. *There exists a constant* C *such that if* a *is any pseudo-effective of mass* 1*, and* f *is any automorphism of* X*, then*

$$\mathrm{Diam}(f^*a) \leqslant C\log\left(C\left\|\mathrm{Jac}(f^{-1})\right\|_\infty\right).$$

Proof. Indeed, if S and T belong to $\mathrm{Cur}(a)$, by Equation (6.11) we have $u_{f^*T} - u_{f^*S} = (u_T - u_S) \circ f + E(f,T) - E(f,S)$, so

$$\mathrm{dist}(f^*T, f^*S) \leqslant \int |u_T \circ f| \,\mathsf{vol} + \int |u_S \circ f| \,\mathsf{vol} + |E(f,T)| + |E(f,S)| \, ; \tag{6.13}$$

and the result follows from Lemmas 6.6 and 6.8, since u_S and u_T are uniformly $(A\kappa_0)$-psh. □

6.2.3. *An estimate for canonical potentials.*

Lemma 6.10. *For any Kähler form κ on X there exists a positive constant $C(\kappa)$ such that for every $f \in \mathrm{Aut}(X)$,*

$$\left\| u_{f^*\kappa} \right\|_{C^1} \leqslant C(\kappa) \|f\|^2_{C^1}.$$

In addition $C(\kappa) \leqslant C' \|\kappa\|_\infty$, where $\|\kappa\|_\infty$ is the sup norm of the coefficients of κ in a system of coordinate charts, and C' depends only on X (and the choice of these coordinate charts).

Recall the choice of Kähler forms (κ_i) from § 6.1 and the definition of $\Theta(\cdot)$ from § 6.1.1.

Corollary 6.11. *If $\kappa = \sum_i c_i \kappa_i$ in Lemma 6.10, then the constant $C(\kappa)$ satisfies $C(\kappa) \leqslant C'' \mathbf{M}(\kappa)$. Likewise, $\left\| u_{f^*\Theta(a)} \right\|_{C^1} \leqslant C''' \mathbf{M}(a) \|f\|^2_{C^1}$ for all $a \in \mathrm{Psef}(X)$.*

Indeed $C(\kappa) \leqslant C' \|\kappa\|_\infty \leqslant C'' \sum_i |c_i|$ and $u_{f^*\Theta(a)} = \sum c_i(a) u_{f^*\kappa_i}$.

Proof of Lemma 6.10. By definition $f^*\kappa - \Theta(f^*\kappa) = dd^c \left(u_{f^*\kappa} \right)$. The desired estimate will be obtained by constructing a solution ϕ to the equation

$$dd^c \phi = f^*\kappa - \Theta(f^*\kappa) \tag{6.14}$$

which satisfies $\|\phi\|_{C^1} \leqslant C \|f\|^2_{C^1}$. Then, since $u_{f^*\kappa}$ and ϕ differ by a constant and $u_{f^*\kappa}$ is known to vanish at some point, it follows that $u_{f^*\kappa}$ satisfies the same estimate. To construct the potential ϕ, we follow the method of Dinh and Sibony [47, Prop. 2.1] which is itself based on [17] (we keep the notation from [47]). Let α be a closed $(2,2)$-form on $X \times X$ which is cohomologous to the diagonal Δ. In [17], Bost, Gillet and Soulé construct an explicit $(1,1)$-form K on $X \times X$ such that $dd^c K = [\Delta] - \alpha$; they refer to it as the "Green current". It is C^∞ outside the diagonal, and along Δ, it satisfies the estimates

$$K(x,y) = O\left(\frac{\log |x-y|}{|x-y|^2} \right) \quad \text{and} \quad \nabla K(x,y) = O\left(\frac{\log |x-y|}{|x-y|^3} \right) \tag{6.15}$$

(here we mean that these estimates hold for the coefficients of K and ∇K in local coordinates). These estimates are easily deduced from the explicit expression of K as $\pi_*(\widehat{\varphi}\eta - \beta)$ given in [47, Prop. 2.1], where $\pi : \widehat{X \times X} \to X \times X$ is the blow-up of the diagonal, η and β are smooth (1,1) forms on $\widehat{X \times X}$ and $\widehat{\varphi}$ is a function with logarithmic singularities along the proper transform of Δ in $X \times X$. It is shown in [47, Prop. 2.1] that a solution to Equation (6.14) is given by

$$\phi(x) = \int_{y \in X} K(x,y) \wedge \left(f^*\kappa(y) - \Theta(f^*\kappa)(y) \right) \tag{6.16}$$

(in the notation of [47], $f^*\kappa$ and $\Theta(f^*\kappa)$ correspond to Ω^+ and Ω^- respectively). The coefficients of the smooth $(1,1)$-forms $f^*\kappa$ and $\Theta(f^*\kappa)$ have their uniform norms bounded by $C\|f\|^2_{C^1}$, where $C = C(\kappa) \leqslant C' \|\kappa\|_\infty$. The first estimate in (6.15) implies that the coefficients of K belong to L^p_{loc} for $p < 2$, so it follows from the Hölder inequality that $\|\phi\|_{C^0} \leqslant C'' \|\kappa\|_\infty \|f\|^2_{C^1}$

(for some constant C'' depending only on X). A similar estimate for $\nabla\phi$ is obtained from derivation under the integral sign and the fact that $\nabla K \in L^p_{\mathrm{loc}}$ for $p < 4/3$. This concludes the proof. □

6.3. **Convergence and extremality.**

Theorem 6.12. *Let (X, ν) be a non-elementary random holomorphic dynamical system on a compact Kähler surface X, satisfying the moment condition* (4.1)*. Then for μ_∂-almost every point $\underline{a} \in \mathrm{Lim}(\Gamma)$, the following properties hold:*

(1) *there is a unique nef and isotropic class $a \in H^{1,1}(X;\mathbf{R})$ of mass* 1 *with $\mathbb{P}(a) = \underline{a}$;*
(2) *the convex set* $\mathrm{Cur}(a)$ *is a singleton $\{T_a\}$;*
(3) *the class $\underline{a}$ is an extremal point of* $\mathbb{P}(\overline{\mathrm{Kah}}(X))$ *and of* $\mathbb{P}(\mathrm{Psef}(X))$*;*
(4) *the current T_a is extremal in the convex set of closed positive currents of mass* 1.

Combining this result with Lemma 5.6 and Equation (5.9) we obtain the first and second assertions of the following corollary; the third assertion follows from the first one and the equivariance relation (5.16).

Corollary 6.13. *The following properties are satisfied for $\nu^{\mathbf{N}}$-almost every ω:*

(1) *there exists a unique closed positive current T^s_ω in the cohomology class $e(\omega)$;*
(2) *for every Kähler form κ,*
$$\frac{1}{\mathbf{M}\left((f^n_\omega)^*\kappa\right)}(f^n_\omega)^*\kappa \underset{n\to\infty}{\longrightarrow} T^s_\omega.$$
(3) *the currents T^s_ω satisfy the equivariance property*
$$(f_\omega)^*T^s_{\sigma(\omega)} = \frac{\mathbf{M}((f_\omega)^*T^s_{\sigma(\omega)})}{\mathbf{M}(T^s_\omega)}T^s_\omega = \mathbf{M}((f_\omega)^*T^s_{\sigma(\omega)})T^s_\omega.$$

Proof of Theorem 6.12. The first and third properties were already established, respectively in Lemma 2.16 and 2.17 and Proposition 5.11. Property (4) follows from (2) and (3). It remains to prove (2). For this, we denote by $\underline{f}^*$ the projective action of f^* on $\mathbb{P}H^{1,1}(X;\mathbf{R})$. For $\underline{a} \in \mathrm{Lim}(\Gamma)$, let us set $\mathrm{diam}\,(\underline{a}) = \mathrm{Diam}(a)$, where a is the unique pseudo-effective class of mass 1 such that $\mathbb{P}(a) = \underline{a}$; this defines a measurable function on $\mathrm{Lim}(\Gamma)$, by Lemma 6.5. Our purpose is to show that $\mathrm{diam}\,(\underline{a}) = 0$ for μ_∂-almost every $\underline{a}$. The stationarity of μ_∂ reads

$$\int \mathrm{diam}\,(\underline{a})\; d\mu_\partial\,(\underline{a}) = \iint \mathrm{diam}\,\left(\underline{f}^*\,(\underline{a})\right)\; d\nu(f)d\mu_\partial\,(\underline{a}) \tag{6.17}$$

and iteratingthis relation gives

$$\int \mathrm{diam}\,(\underline{a})\; d\mu_\partial\,(\underline{a}) = \int \mathrm{diam}\,\left(\underline{f}^*_n \cdots \underline{f}^*_1\,(\underline{a})\right)\; d\nu(f_1)\cdots d\nu(f_n)d\mu_\partial\,(\underline{a}) \tag{6.18}$$

(notice the order of compositions chosen here). Since the diameter is upper-semicontinuous it is uniformly bounded on $\mathrm{Lim}(\Gamma)$. So, if we prove that

$$\lim_{n\to+\infty} \mathrm{diam}\left(\underline{f}^*_n \cdots \underline{f}^*_1\,(\underline{a})\right) = 0 \tag{6.19}$$

for $\nu^{\mathbf{N}}$-almost every (f_n) and every $\underline{a}$, then we can apply the dominated convergence theorem to infer that $\mathrm{diam}\,(\underline{a}) = 0$ μ_∂-almost surely. To derive the convergence (6.19), note that

$$\mathrm{diam}\left(\underline{f}_n^* \cdots \underline{f}_1^*\,(\underline{a})\right) = \frac{\mathrm{Diam}\,(f_n^* \cdots f_1^* a)}{\mathbf{M}\,(f_n^* \cdots f_1^* a)} \tag{6.20}$$

because Diam is homogeneous. Applying Lemma 6.9 and the multiplicativity of the Jacobian we get that

$$\mathrm{diam}\left(\underline{f}_n^* \cdots \underline{f}_1^*\,(\underline{a})\right) \leqslant \frac{C\log\left(C\left\|\mathrm{Jac}(f_1\circ\cdots\circ f_n)^{-1}\right\|_\infty\right)}{\mathbf{M}\,(f_n^* \cdots f_1^* a)} \leqslant C\frac{\sum_{i=0}^{n-1}\log\left\|f_i^{-1}\right\|_{C^1}}{\mathbf{M}(f_n^* \cdots f_1^* a)}. \tag{6.21}$$

We conclude with two remarks. Firstly, the moment condition (4.1) implies that the sequence $\frac{1}{n}\sum_{i=0}^{n-1}\log\left\|f_i^{-1}\right\|_{C^1}$ is almost surely bounded. Secondly, Lemma 5.5 shows that $\mathbf{M}(f_n^* \cdots f_1^* a)$ goes exponentially fast to infinity for $\nu^{\mathbf{N}}$-almost every $\omega = (f_n)$ (this is where the order of compositions matters). Thus $\mathrm{diam}\big(\underline{f}_n^* \cdots \underline{f}_1^*\,(\underline{a})\big) \to 0$ almost surely, and we are done. □

Remark 6.14. The uniqueness of T_a in its cohomology class implies that T_a depends measurably on a. Indeed there is a set $E \subset \mathrm{Lim}(\Gamma)$ of full measure along which the map $\underline{a} \mapsto T_a$ is continuous (recall that the space $\mathrm{Cur}_1(X)$ of positive closed currents of mass 1 on X is a compact metrizable space). This implies that $\underline{a} \mapsto T_a$ is a measurable map from $\mathrm{Lim}(\Gamma)$, endowed with the μ_∂-completion of the Borel σ-algebra, to $\mathrm{Cur}_1(X)$, endowed with its Borel σ-algebra.

6.4. **Continuous potentials.** We now study the limit currents T^s_ω introduced in Corollary 6.13.

Theorem 6.15. *Let (X,ν) be a non-elementary random holomorphic dynamical system on a compact Kähler surface X, satisfying the moment condition* (4.1)*. Then for $\nu^{\mathbf{N}}$-a.e. ω the current T^s_ω has continuous potentials.*

Lemma 6.16. *Let κ be any Kähler form on X. For $\nu^{\mathbf{N}}$-almost every ω, there exists an increasing sequence of integers $(n_i)_{i\geqslant 0} = (n_i(\omega))$ such that*

(1) *the potentials $\mathbf{M}((f_\omega^{n_i})^*\kappa)^{-1}u_{(f_\omega^{n_i})^*\kappa}$ are uniformly bounded;*
(2) *the potentials $\mathbf{M}((f_\omega^{n_i})^*\kappa)^{-1}u_{(f_\omega^{n_i})_*\kappa}$ are uniformly bounded too.*

If the exponential moment condition (5.26) *holds, these assertions hold for all n (i.e. extracting a subsequence (n_i) is not necessary); in addition the function $\omega \mapsto \log^+\left\|u_{T^s_\omega}\right\|_\infty$ is $\nu^{\mathbf{N}}$-integrable.*

Proof of the Lemma. Recall the notation $\omega = (f_n)_{n\geqslant 0}$. First,

$$\begin{aligned} f_{n-1}^*\kappa &= f_{n-1}^*\Theta(\kappa) + dd^c\left(u_\kappa\circ f_{n-1}\right) \\ &= \Theta(f_{n-1}^*\kappa) + dd^c\left(u_{f_{n-1}^*\Theta(\kappa)} + u_\kappa\circ f_{n-1}\right) \end{aligned} \tag{6.22}$$

(For the moment, we do not introduce the constants $E(f_n;\kappa)$ in the computation). We obtain

$$\begin{aligned} f_{n-2}^*f_{n-1}^*\kappa &= f_{n-2}^*\Theta(f_{n-1}^*\kappa) + dd^c\left(u_{f_{n-1}^*\Theta(\kappa)}\circ f_{n-2} + u_\kappa\circ(f_{n-1}\circ f_{n-2})\right) \\ &= \Theta(f_{n-2}^*f_{n-1}^*\kappa) + dd^c\left(u_{f_{n-2}^*\Theta(f_{n-1}^*\kappa)} + u_{f_{n-1}^*\Theta(\kappa)}\circ f_{n-2} + u_\kappa\circ(f_{n-1}\circ f_{n-2})\right). \end{aligned}$$

Setting $G_{j,k} = f_{k-1} \circ \cdots \circ f_j$, for $j \leqslant k-1$, (so in particular $G_{0,j} = f_\omega^j$ for all $j \geqslant 1$) and $G_{j,j} = \mathrm{id}_X$, we get

$$(f_\omega^n)^*\kappa = \Theta((f_\omega^n)^*\kappa) + dd^c\left(u_\kappa \circ f_\omega^n + \sum_{j=0}^{n-1} u_{f_j^*\Theta(G_{j+1,n}^*\kappa)} \circ G_{0,j}\right). \tag{6.23}$$

Let u_n denote the function in the parenthesis. We want to estimate the sup-norm $\|u_n\|_\infty$. Lemma 6.10 and Corollary 6.11 provide successively the following upper bounds

$$\left\| u_{f_j^*\Theta(G_{j+1,n}^*\kappa)} \right\|_\infty \leqslant C\|f_j\|_{C^1}^2 \mathbf{M}(G_{j+1,n}^*\kappa) \leqslant C\mathbf{M}(\kappa)\|f_j\|_{C^1}^2 \left\|G_{j+1,n}^*\right\|, \tag{6.24}$$

$$\left\| \frac{1}{\mathbf{M}((f_\omega^n)^*\kappa)} u_n \right\|_\infty \leqslant \frac{\|u_\kappa\|_\infty}{\mathbf{M}((f_\omega^n)^*\kappa)} + C\mathbf{M}(\kappa)\sum_{j=0}^{n-1} \|f_j\|_{C^1}^2 \frac{\left\|G_{j+1,n}^*\right\|}{\mathbf{M}((f_\omega^n)^*\kappa)}. \tag{6.25}$$

To estimate this sum we apply Theorem 5.12 to the subadditive cocycle $N(n,\omega) = \log\|(f_\omega^n)^*\|$, as we did for Corollary 5.13: there exists a sequence (δ_j) of positive numbers converging to 0, an increasing sequence $n_i = n_i(\omega)$ of integers, and a constant $C'(\omega)$ such that

$$\frac{\left\|G_{j+1,n_i}^*\right\|}{\mathbf{M}((f_\omega^{n_i})^*\kappa)} \asymp \frac{\left\|f_{j+1}^* \cdots f_{n_i-1}^*\right\|}{\left\|f_0^* \cdots f_{n_i-1}^*\right\|} \leqslant C' \exp(-(\lambda_1 - \delta_j)j) \tag{6.26}$$

for all $i \geqslant 1$ and all $0 \leqslant j \leqslant n_i$. Fix any real number ε with $0 < \varepsilon < \lambda_1$. Then from Lemma 4.3, we know that, for almost every ω, there is a constant $C''(\omega)$ such that $\|f_j\|_{C^1}^2 \leqslant C'' \exp(\varepsilon j)$. So from (6.25) we get

$$\left\| \frac{1}{\mathbf{M}((f_\omega^{n_i})^*\kappa)} u_{n_i} \right\|_\infty \leqslant \frac{\|u_\kappa\|_\infty}{\mathbf{M}((f_\omega^{n_i})^*\kappa)} + C'''(\omega)\mathbf{M}(\kappa)\sum_{j=0}^{n_i-1} \exp(-(\lambda_1 - \varepsilon - \delta(j))j) \tag{6.27}$$

This inequality shows that $\left\|\mathbf{M}((f_\omega^{n_i})^*\kappa)^{-1} u_{n_i}\right\|_\infty$ is uniformly bounded.

Now, note that $u_{(f_\omega^n)^*\kappa} = u_n + E_n$ with $E_n = -\int u_n \mathsf{vol}$. Since $\left\|\mathbf{M}((f_\omega^{n_i})^*\kappa)^{-1} u_{n_i}\right\|_\infty$ is uniformly bounded, so is $\mathbf{M}((f_\omega^{n_i})^*\kappa)^{-1} E_{n_i}$, and the first assertion of the lemma is established.

The second assertion is proved exactly in the same way, except that the expressions of the form $f_j^*\Theta(G_{j+1,n}^*\kappa)$ must be replaced by $(f_{n-j}^{-1})^*\Theta((f_0^{-1} \circ \cdots \circ f_{n-j-1}^{-1})^*\kappa)$; then we use the second estimate in Corollary 5.13, and the fact that for every $f \in \mathsf{Aut}(X)$, $\|f^*\| \asymp \left\|(f^{-1})^*\right\|$.

If the exponential moment condition (5.26) holds, we follow the same argument and apply Proposition 5.15 – instead of Theorem 5.12 – to (6.25), with $D(f) = \|f\|_{C^1}^2$. $\square$

Proof of Theorem 6.15. First, we prove that *the normalized potential $u_{T_\omega^s}$ is bounded, for $\nu^{\mathbf{N}}$-almost every ω.* To see this, recall that $\mathbf{M}((f_\omega^n)^*\kappa)^{-1}(f_\omega^n)^*\kappa$ converges to T_ω^s as $n \to \infty$. From Lemma 6.16, we know that the normalized potentials $\mathbf{M}((f_\omega^n)^*\kappa)^{-1}u_{(f_\omega^n)^*\kappa}$ of the currents $\mathbf{M}((f_\omega^n)^*\kappa)^{-1}\,(f_\omega^n)^*\,\kappa$ are uniformly bounded along some subsequence $n_i = n_i(\omega)$. These potentials are $A\kappa_0$-psh functions on X so, by compactness, they converge to $u_{T_\omega^s}$ in $L^1(X;\mathsf{vol})$. Thus, $u_{T_\omega^s}$ is essentially bounded. We conclude that $u_{T_\omega^s}$ is bounded because quasi-plurisubharmonic functions are usc and have a value (in $\mathbf{R} \cup \{-\infty\}$) at every point.

Now, we show that $u_{T^s_\omega}$ *is continuous*. Here, the argument is similar to the one used to prove Theorem 6.12. If T is a positive closed current with bounded potential on X, we define

$$\mathrm{Jump}(T) = \max_{x\in X}\left(\limsup_{y\to x} u_T(y) - \liminf_{y\to x} u_T(y)\right). \tag{6.28}$$

Then $0 \leqslant \mathrm{Jump}(T) \leqslant 2\|u_T\|_\infty$, and $\mathrm{Jump}(T) = 0$ if and only if u_T is continuous. In addition $\mathrm{Jump}(f^*T) = \mathrm{Jump}(T)$ for every $f \in \mathsf{Aut}(X)$ because $f^*T = \Theta(f^*a) + dd^c(u_{f^*\Theta(a)} + u_T \circ f)$ and $u_{f^*\Theta([T])}$ is continuous (see Equation (6.10)). From the equivariance relation

$$T^s_\omega = \frac{1}{\mathbf{M}\left((f^n_\omega)^* T^s_{\sigma^n\omega}\right)} T^s_{\sigma^n\omega}, \tag{6.29}$$

which follows from the third assertion of Corollary 6.13, we get

$$\mathrm{Jump}\left(T^s_\omega\right) = \frac{1}{\mathbf{M}\left((f^n_\omega)^* T^s_{\sigma^n\omega}\right)} \mathrm{Jump}\left(T^s_{\sigma^n\omega}\right). \tag{6.30}$$

Remark 6.14 says that $\omega \mapsto T^s_\omega$ is measurable; hence, $\omega \mapsto u_{T^s_\omega}$ is measurable. If C is large enough, the first step of the proof gives a subset $\Omega_C \subset \Omega$ such that $\nu(\Omega_C) > 0$ and $\left\|u_{T^s_\omega}\right\|_\infty \leqslant C$ for all $\omega \in \Omega_C$. By ergodicity of the shift, $\sigma^n\omega \in \Omega_C$ for almost every ω and infinitely many n; for such an n, $\left\|u_{T^s_{\sigma^n\omega}}\right\|_\infty \leqslant C$ and $\mathrm{Jump}\left(T^s_{\sigma^n\omega}\right) \leqslant 2C$. By Lemma 5.14, $\mathbf{M}\left((f^n_\omega)^* T^s_{\sigma^n\omega}\right)$ goes to infinity almost surely. So, $\mathrm{Jump}\left(T^s_\omega\right) = 0$, and the proof is complete. □

Theorem 6.17. *Let (X,ν) be a non-elementary random holomorphic dynamical system on a compact Kähler surface X, satisfying the exponential moment condition* (5.26)*. Then there exists $\theta > 0$ such that for $\nu^{\mathbf{N}}$-almost every ω the potential $u_{T^s_\omega}$ is Hölder continuous of exponent θ.*

The proof is a variation on the following well-known fact, applied to $u = u_{T^s_\omega}$: let u_n be a sequence of continuous functions converging uniformly to $u : M \to \mathbf{R}$ on some metric space M. If $\|u_n - u\|_\infty \leqslant A^n$ and $\mathrm{Lip}(u_n) \leqslant B^n$ with $A < 1 < B$, then u is a Hölder continuous function for the exponent $\alpha = -\log(A)/(\log(B) - \log(A))$.

Proof. The initial computations are similar (but not identical) to those used to reach Lemma 6.16. Keeping the notation $G_{j,n} = f_{n-1} \circ \cdots \circ f_j$, a descending induction starting from

$$f^*_{n-1}T^s_{\sigma^n\omega} = \Theta(f^*_{n-1}T^s_{\sigma^n\omega}) + dd^c\left(u_{f^*_{n-1}\Theta(T^s_{\sigma\omega})} + u_{T^s_{\sigma^n\omega}} \circ f_{n-1}\right) \tag{6.31}$$

yields

$$(f^n_\omega)^*T^s_{\sigma^n\omega} = \Theta\left((f^n_\omega)^*T^s_{\sigma^n\omega}\right) + dd^c\left(\sum_{j=0}^{n-1} u_{f^*_j\Theta(G^*_{j+1,n}T^s_{\sigma^n\omega})} \circ f^j_\omega + u_{T^s_{\sigma^n\omega}} \circ f^n_\omega\right). \tag{6.32}$$

Thus, there is a constant of normalization $E = E(\omega;n)$ such that

$$u_{T^s_\omega} = \frac{1}{\mathbf{M}((f^n_\omega)^*(T^s_{\sigma^n\omega}))}\left(\sum_{j=0}^{n-1} u_{f^*_j\Theta(G^*_{j+1,n}T^s_{\sigma^n\omega})} \circ f^j_\omega + u_{T^s_{\sigma^n\omega}} \circ f^n_\omega\right) + E. \tag{6.33}$$

Note that the additional term E does not affect the modulus of continuity of $u_{T^s_\omega}$. Since $\mathrm{Lip}(f_j) \leqslant \|f_j\|_{C^1}$ for all j, Lemma 6.10 and Corollary 6.11 imply $\mathrm{Lip}(u_{f^*_j\Theta(a)}) \leqslant C\|f_j\|^2_{C^1}\mathbf{M}(a)$ for every class $a \in \mathrm{Psef}(X)$; hence

$$\mathrm{Lip}\left(u_{f_j^*\Theta(G_{j+1,n}^* T^s_{\sigma^n\omega})}\right) \leqslant C\|f_j\|^2_{C^1}\mathbf{M}(G^*_{j+1,n}T^s_{\sigma^n\omega}) \leqslant C\|f_j\|^2_{C^1}\|G^*_{j+1,n}\| \tag{6.34}$$

$$\leqslant C\|f_j\|^2_{C^1}\prod_{\ell=j+1}^{n-1}\|f_\ell^*\|_{H^{1,1}} \leqslant C\prod_{\ell=j}^{n-1}\|f_\ell\|^2_{C^1}. \tag{6.35}$$

Finally, since $1 \leqslant \mathrm{Lip}(f_j)$ for every $0 \leqslant j \leqslant n-1$, we obtain

$$\mathrm{Lip}\left(u_{f_j^*\Theta(G_{j+1,n}^* T^s_{\sigma^n\omega})}\circ f_\omega^j\right) \leqslant \mathrm{Lip}\left(u_{f_j^*\Theta(G_{j+1,n}^* T^s_{\sigma^n\omega})}\right)\prod_{\ell=0}^{j-1}\mathrm{Lip}(f_\ell) \leqslant C\prod_{\ell=0}^{n-1}\|f_\ell\|^2_{C^1}. \tag{6.36}$$

Denoting the modulus of continuity by $\mathrm{modc}(u,r) = \sup_{d(x,x')\leqslant r}|u(x)-u(x')|$, we infer from Equation (6.33) that

$$\mathrm{modc}(u_{T^s_\omega}, r) \leqslant \frac{1}{\mathbf{M}\left((f^n_\omega)^*(T^s_{\sigma^n\omega})\right)}\left(Cn\prod_{\ell=0}^{n-1}\|f_\ell\|^2_{C^1}\cdot r + \left\|u_{T^s_{\sigma^n\omega}}\right\|_\infty\right). \tag{6.37}$$

To ease notation set $\lambda = \lambda_{H^{1,1}}$. Fix a small $\varepsilon > 0$. By Lemma 5.14, for almost every ω there exists $C = C_\varepsilon(\omega)$ such that $\mathbf{M}\left((f^n_\omega)^*(T^s_{\sigma^n\omega})\right)^{-1} \leqslant Ce^{-n(\lambda-\varepsilon)}$ for every n. Fix M larger than but close to $\exp\left(\mathbb{E}\left(\log\|f\|_{C^1}\right)\right)$. Applied to the $\nu^{\mathbf{N}}$-integrable function $\omega = (f_n) \mapsto \log\|f_0\|_{C^1}$, the Birkhoff ergodic theorem gives

$$\prod_{\ell=0}^{n-1}\|f_\ell\|^2_{C^1} \leqslant CM^n \quad \text{as well as} \quad n\prod_{\ell=0}^{n-1}\|f_\ell\|^2_{C^1} \leqslant CM^n \tag{6.38}$$

for some $C = C_M(\omega)$ (increase M to deduce the second inequality from the first). Thus,

$$\mathrm{modc}(u_{T^s_\omega}, r) \leqslant C_1 e^{-n(\lambda-\varepsilon)}\left(M^n r + \left\|u_{T^s_{\sigma^n\omega}}\right\|_\infty\right) \tag{6.39}$$

for some $C_1 > 0$. By Lemma 6.16, $\omega \mapsto \log^+\left\|u_{T^s_\omega}\right\|_\infty$ is integrable, so for almost every ω there exists $C_2 = C_\varepsilon(\omega)$ such that $\left\|u_{T^s_{\sigma^n\omega}}\right\|_\infty \leqslant C_2 e^{\varepsilon n}$ holds for all n, and we infer that

$$\mathrm{modc}(u_{T^s_\omega}, r) \leqslant C_3 e^{-n(\lambda-\varepsilon)}(M^n r + e^{\varepsilon n}) = C_3 e^{-n(\lambda-2\varepsilon)}\left((Me^{-\varepsilon})^n r + 1\right). \tag{6.40}$$

Choosing n so that $r \asymp (Me^{-\varepsilon})^{-n}$ we get $\mathrm{modc}(u_{T^s_\omega}, r) \leqslant C_4 r^\theta$ with $\theta = \frac{\lambda-2\varepsilon}{\log M+\varepsilon}$ and the proof of the theorem is complete. □

7. Glossary of random dynamics, II

In this section we consider a random holomorphic dynamical system (X,ν) on a compact Kähler surface, satisfying the moment condition (4.1). Our goal is to collect a number of facts from the ergodic theory of random dynamical systems, including the construction of associated skew products, fibered entropy and Lyapunov exponents of stationary measures, stable and unstable manifolds, and various measurable partitions. Here the group Γ_ν may a priori be elementary; also, the compactness assumption on X can be dropped in most of these results (in this case (4.1) should be strengthened to a C^2-moment condition). Since some subsequent arguments rely on the work [22] of Brown and Rodriguez-Hertz, we have tried to make notation consistent with that paper as much as possible.

7.1. **Skew products and stationary measures associated to** (X, ν)**.** Define:

- $\Omega = \mathsf{Aut}(X)^{\mathbf{N}}$, whose elements are denoted by $\omega = (f_n)_{n\geqslant 0}$. On Ω, the one-sided shift is denoted by $\sigma\colon \Omega \to \Omega$.
- $\Sigma = \mathsf{Aut}(X)^{\mathbf{Z}}$, whose elements are denoted by $\xi = (f_n)_{n\in\mathbf{Z}}$. On Σ, the two-sided shift is denoted by $\vartheta\colon \Sigma \to \Sigma$.
- $\mathcal{X} = \Sigma \times X$ and $\mathcal{X}_+ = \Omega \times X$, whose elements are denoted by $x = (\xi, x)$ and $x = (\omega, x)$ respectively. The natural projections are denoted by $\pi_\Sigma : \mathcal{X} \to \Sigma$ (resp. $\pi_\Omega : \mathcal{X}_+ \to \Omega$) and $\pi_X : \mathcal{X} \to X$ (resp. $\pi_X : \mathcal{X}_+ \to X$, using the same notation).

Recall that the product σ-algebra on Ω (resp. Σ) is generated by **cylinders** ([3]), and that it coincides with the Borel σ-algebra $\mathcal{B}(\Omega)$ (resp. $\mathcal{B}(\Sigma)$) (see [**?**, Lem. 6.4.2]).

7.1.1. *Skew products.* For $\omega \in \Omega$ and $n \geqslant 1$, f_ω^n is the left composition $f_\omega^n = f_{n-1} \circ \cdots \circ f_0$; in particular, $f_\omega^1 = f_0$ (see § 4.3). For $n = 0$, we set $f_\omega^0 = \mathrm{id}$. This is consistent with the notation used in the previous sections. The same notation f_ξ^n is used for $\xi \in \Sigma$ and $n \geqslant 0$. When $n < 0$, we set $f_\xi^n = (f_n)^{-1} \circ \cdots \circ (f_{-1})^{-1}$. With this definition the cocycle formula $f_\xi^{n+m} = f_{\vartheta^m\xi}^n \circ f_\xi^m$ holds for all $(m, n) \in \mathbf{Z}^2$ and $\xi \in \Sigma$. By definition, the skew products induced by the random dynamical system (X, ν) are the transformations $F_+ \colon \mathcal{X}_+ \to \mathcal{X}_+$ and $F\colon \mathcal{X} \to \mathcal{X}$ defined by

$$F_+\colon (\omega, x) \longmapsto (\sigma\omega, f_\omega^1(x)) \tag{7.1}$$

$$F\colon \ (\xi, x) \longmapsto (\vartheta\xi, f_\xi^1(x)). \tag{7.2}$$

If $\varpi\colon \mathcal{X} \to \mathcal{X}_+$ denotes the natural projection, then $\varpi \circ F = F_+ \circ \varpi$. Note that F is invertible, with $F^{-1}(x) = (\vartheta^{-1}\xi, f_{\theta^{-1}\xi}^{-1}(x))$, but F_+ is not; indeed $(\mathcal{X}, F)$ *is the natural extension of* $(\mathcal{X}_+, F_+)$.

Lemma 7.1. *The measure μ on X is stationary if and only if the product measure*

$$m_+ := \nu^{\mathbf{N}} \times \mu$$

on $\mathcal{X}_+$ is invariant under F_+.

Proof of Lemma 7.1. The invariance of m_+ is equivalent to the equality

$$m_+(F_+^{-1}(C \times A)) = m_+(C \times A) = \left(\prod_{j=0}^{N} \nu(C_j)\right) \cdot \mu(A), \tag{7.3}$$

for all cylinders $C = C_0 \times \cdots \times C_N$ in Ω and Borel sets $A \subset X$. By definition

$$F_+^{-1}(C \times A) = \{(\omega, x) \in \Omega \times X \ ; \ f_N \in C_{N-1}, \ldots, f_1 \in C_0, f_0(x) \in A\}, \tag{7.4}$$

so clearly it is enough to check (7.3) for $N = 1$. Now by Fubini's theorem

$$\begin{aligned}(\nu \times \mu)\left(\{(f_0, x) \ ; \ f_0(x) \in A\}\right) &= \int\int \mathbf{1}_{f_0^{-1}(A)}(x)\, d\nu(f_0)\, d\mu(x) \\ &= \int\int \mu(f_0^{-1}(A))\, d\nu(f_0) \qquad (7.5)\end{aligned}$$

and the result follows. □

[3]Cylinders are products $C = \prod C_j$ of Borel sets, all of which are equal to $\mathsf{Aut}(X)$ except finitely many of them. For simplicity, we denote a cylinder by $C = \prod_{j=0}^{N} C_j$ if $C_k = \mathsf{Aut}(X)$ for $|k| > N$.

A stationary measure is said **ergodic** if it is an extremal point in the convex set of stationary measures; hence, μ is ergodic if and only if m_+ is F_+-ergodic. Actually μ is ergodic if and only if every ν-almost surely invariant measurable subset $A \subset X$ (that is a measurable subset such that for ν-almost every f, $\mu(A\Delta f^{-1}(A)) = 0$) has measure $\mu(A) = 0$ or 1. This is by no means obvious since F_+-invariant sets have no reason to be of product type. This statement is part of the so-called **random ergodic theorem** (see Propositions 1.8 and 1.9 in [12]).

Proposition 7.2. *There exists a unique F-invariant probability measure m on $\mathcal{X}$ projecting on m_+ under the natural projection $\mathcal{X} \to \mathcal{X}_+$. Moreover,*

(1) *the measure m is equal to the weak-$\star$ limit*
$$m = \lim_{n\to\infty} (F^n)_\star \left(\nu^{\mathbf{Z}} \times \mu\right).$$
(2) *the projections $(\pi_\Sigma)_* m$ and $(\pi_X)_* m$ are respectively equal to $\nu^{\mathbf{Z}}$ and μ;*
(3) *the equality $m = \nu^{\mathbf{Z}} \times \mu$ holds if and only if μ is f-invariant for ν-almost every f;*
(4) *$(\mathcal{X}, F, m)$ is ergodic if and only if $(\mathcal{X}_+, F_+, m_+)$ is.*

The existence and uniqueness of m, as well as the characterization of its ergodicity, follow from the fact that $(\mathcal{X}, F)$ is the natural extension of $(\mathcal{X}_+, F_+)$ (see [80, §1.2] for a detailed explanation).

Proof of (1), (2), (3). Let us prove directely that the limit in (1) does exist, and show that this limit m satisfies (2) and (3). Since $\varpi_\star\left(\nu^{\mathbf{Z}} \times \mu\right) = \nu^{\mathbf{N}} \times \mu = m_+$ and $\varpi \circ F = F_+ \circ \varpi$, the F_+-invariance of m_+ gives $\varpi_\star(F^n)_\star\left(\nu^{\mathbf{Z}} \times \mu\right) = m_+$ for every $n \in \mathbf{Z}$. So if we prove that the limit $\lim_{n\to\infty}(F^n)_\star\left(\nu^{\mathbf{Z}} \times \mu\right)$ exists, then this limit m will be an F-invariant probability measure projecting on m_+ under ϖ; hence it will coincide with the invariant measure m.

To prove this convergence, we consider a cylinder $C = \prod_{j=-N}^{N} C_j$ in Σ and a Borel set $A \subset X$, and we show that $(\nu^{\mathbf{Z}} \times \mu)(F^{-n}(C \times A))$ stabilizes for $n > N$. Arguing as in Lemma 7.1, we see that the set $F^{-n}(C \times A)$ is equal to the set of points $\chi = (\xi, x)$ satisfying the constraints $(\theta^n\xi)_j \in C_j$ for $-N \leqslant j \leqslant N$ and $x \in (f_\xi^n)^{-1}(A)$; for $n > N$, these constraints are independent, and $\left(\nu^{\mathbf{Z}} \times \mu\right)\left(F^{-n}(C \times A)\right)$ is equal to

$$\nu^{\mathbf{Z}}(\theta^{-n}(C)) \times (\nu^n \times \mu)\left(\{(f_0, \ldots, f_{n-1}, x)\ ;\ f_{n-1} \circ \cdots \circ f_0(x) \in A\}\right). \tag{7.6}$$

Then the invariance of $\nu^{\mathbf{Z}}$ under the shift and the the stationarity of μ give (see Equation (7.5))

$$\begin{aligned}\left(\nu^{\mathbf{Z}} \times \mu\right)\left(F^{-n}(C \times A)\right) &= \nu^{\mathbf{Z}}(C) \times \int \mu\left(f_0^{-1} \circ \cdots \circ f_{n-1}^{-1} A\right) \nu(f_0) \cdots \nu(f_{n-1}) \\ &= \nu^{\mathbf{Z}}(C) \times \mu(A).\end{aligned} \tag{7.7}$$

This proves Assertions (1) and (2). For Assertion (3) it will be enough for us to consider the case where Γ is discrete. By Assertion (1) we see that $\nu^{\mathbf{Z}} \times \mu$ is F-invariant if and only if $m = \nu^{\mathbf{Z}} \times \mu$. Now assume $m = \nu^{\mathbf{Z}} \times \mu$ and let us show that μ is Γ_ν-invariant. The reverse implication is similar. Fix $f_0 \in \mathrm{Supp}(\nu)$ and consider the cylinder $C = C_0 = \{f_0\}$ (in 0^{th} position). If $A \subset X$ is a Borel subset we have

$$\left(\nu^{\mathbf{Z}} \times \mu\right)\left(F(C \times A)\right) = \left(\nu^{\mathbf{Z}} \times \mu\right)(C \times A) = \nu\left(C_0\right) \times \mu(A). \tag{7.8}$$

On the other hand $F(C \times A) = \vartheta(C) \times f_0(A)$ so the left hand side of (7.8) is equal to $\nu\left(C_0\right) \times \mu(f_0(A))$. Thus, $\mu(f_0(A)) = \mu(A)$, which proves that μ is Γ_ν-invariant. □

7.1.2. *Past, future, and partitions.* Let $\mathcal{F}$ denote the σ-algebra on $\mathcal{X}$ obtained by taking the m-completion of $\mathcal{B}(\Sigma)\otimes\mathcal{B}(X)$. It will often be important to detect objects depending only on the "future" or on the "past". To formalize this, we define two σ-algebras on Σ:

– $\hat{\mathcal{F}}^+$ is the $\nu^{\mathbf{Z}}$-completion of the σ-algebra generated by the cylinders $C=\prod_{j=0}^{N} C_j$.
– $\hat{\mathcal{F}}^-$ is the $\nu^{\mathbf{Z}}$-completion of the σ-algebra generated by the cylinders $C=\prod_{j=-N}^{-1} C_j$.

To formulate it differently, we define **local stable and unstable sets** for the shift ϑ:

$$\Sigma^s_{\rm loc}(\xi)=\{\eta\in\Sigma\,;\ \forall i\geqslant 0,\ \eta_i=\xi_i\}\ \text{ and }\ \Sigma^u_{\rm loc}(\xi)=\{\eta\in\Sigma\,;\ \forall i<0,\ \eta_i=\xi_i\}\,. \tag{7.9}$$

Then a subset of Σ is $\hat{\mathcal{F}}^+$-measurable (resp. $\hat{\mathcal{F}}^-$ measurable) if, up to a set of zero $\nu^{\mathbf{Z}}$-measure, it is Borel and saturated by local stable sets $\Sigma^s_{\rm loc}(\xi)$ (resp. unstable sets $\Sigma^u_{\rm loc}(\xi)$). The σ-algebra $\mathcal{F}^+$ on $\mathcal{X}$ will be the m-completion of $\hat{\mathcal{F}}^+\otimes\mathcal{B}(X)$. An $\mathcal{F}^+$-measurable object should be understood as "depending only on the future", thus it makes sense on $\mathcal{X}$ and on $\mathcal{X}_+$. Actually $\mathcal{F}^+$ coincides with the completion of the pull-back of $\mathcal{B}(\mathcal{X}_+)$ under $\varpi:\mathcal{X}\to\mathcal{X}_+$. The σ-algebra $\mathcal{F}^-$ of "objects depending only on the past" is defined analogously. Consider the partition into the subsets $\mathcal{F}^-(x):=\Sigma^u_{\rm loc}(\xi)\times\{x\}$ (each of them can be naturally identified to Ω). Then, modulo m-negligible sets, the elements of $\mathcal{F}^-$ are saturated by this partition.

For $\xi\in\Sigma$ we set $X_\xi=\{\xi\}\times X=\pi_\Sigma^{-1}(\xi)$, which can be naturally identified with X via π_X. The disintegration of the probability measure m with respect to the partition into fibers of π_Σ gives rise to a family of conditional probabilities m_ξ such that $m=\int m_\xi\, d\nu^{\mathbf{Z}}(\xi)$, because $(\pi_\Sigma)_*m=\nu^{\mathbf{Z}}$.

Lemma 7.3. *The conditional measure m_ξ on X_ξ satisfies $\nu^{\mathbf{Z}}$-almost surely*

$$m_\xi=\lim_{n\to+\infty}(f_{-1}\circ\cdots\circ f_{-n})_*\mu=\lim_{n\to+\infty}(f^n_{\vartheta^{-n}\xi})_*\mu.$$

In particular, the family of measures $\xi\mapsto m_\xi$ is $\mathcal{F}^-$-measurable.

Proof. It follows from the martingale convergence theorem that the limit

$$\tilde\mu_\xi:=\lim_{n\to+\infty}(f_{-1}\circ\cdots\circ f_{-n})_*\mu \tag{7.10}$$

exists almost surely (see e.g. [12, §2.5] or [19, §II.2]). Now F^n maps $X_{\vartheta^{-n}\xi}$ to X_ξ and $F^n|_{X_{\vartheta^{-n}\xi}}=f_{-1}\circ\cdots\circ f_{-n}$, so

$$\big((F^n)_*(\nu^{\mathbf{Z}}\times\mu)\big)(\,\cdot\,|X_\xi)=(f_{-1}\circ\cdots\circ f_{-n})_*\mu. \tag{7.11}$$

Identify $\tilde\mu_\xi$ with a measure on X_ξ. For every continuous function ϕ on $\mathcal{X}$ the dominated convergence theorem gives

$$\big((F^n)_*(\nu^{\mathbf{Z}}\times\mu)\big)(\varphi)=\int\left(\int_{X_\xi}\varphi(x)\ d(f_{-1}\circ\cdots\circ f_{-n})_*\mu(x)\right)d\nu^{\mathbf{Z}}(\xi) \tag{7.12}$$

$$\underset{n\to\infty}{\longrightarrow}\int\left(\int_{X_\xi}\varphi(x)\ d\tilde\mu_\xi(x)\right)d\nu^{\mathbf{Z}}(\xi). \tag{7.13}$$

But $\big((F^n)_*(\nu^{\mathbf{Z}}\times\mu)\big)(\varphi)$ converges to $m(\varphi)$, and the marginal of m with respect to the projection $\pi_\Sigma\colon\mathcal{X}\to\Sigma$ is $\nu^{\mathbf{Z}}$, so we get the result. □

Since $\xi \mapsto m_\xi$ is $\mathcal{F}^-$-measurable, the conditional measures of m on the atoms $\mathcal{F}^-(x) = \Sigma^u_{\rm loc}(\xi) \times \{x\}$ of the partition generating $\mathcal{F}^-$ are induced by the lifts of the conditionals of $\nu^{\mathbf{Z}}$ on the $\Sigma^u_{\rm loc}(\xi)$, via the natural projection $\pi_\Sigma : \mathcal{X} \to \Sigma$. In addition we can simultaneously identify $\Sigma^u_{\rm loc}(\xi)$ to Ω and $\nu^{\mathbf{Z}}(\,\cdot \mid \Sigma^u_{\rm loc})$ to $\nu^{\mathbf{N}}$. In this way we get

$$m(\,\cdot \mid \mathcal{F}^-(x)) \;=\; \nu^{\mathbf{Z}}(\,\cdot \mid \Sigma^u_{\rm loc}(\xi)) \times \delta_x \;\simeq\; \nu^{\mathbf{N}} \tag{7.14}$$

for m-almost every $x = (\xi, x) \in \mathcal{X}$. This corresponds to Equation (9) in [22]. By [22, Prop. 4.6], this implies that $\mathcal{F}^+ \cap \mathcal{F}^-$ is equivalent, modulo m-negligible sets, to $\{\varnothing, \Sigma\} \otimes \mathcal{B}(X)$.

7.2. **Lyapunov exponents.** Let μ be a stationary measure for (X, ν); assume that μ (or equivalently m or m_+) is ergodic. The upper and lower Lyapunov exponents $\lambda^+ \geqslant \lambda^-$ are respectively defined by the almost sure limits

$$\lambda^+ = \lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n_\omega\| \;\text{ and }\; \lambda^- = \lim_{n\to\infty} \frac{1}{n} \log \left\|(D_x f^n_\omega)^{-1}\right\|^{-1}; \tag{7.15}$$

the existence of these limits is guaranteed by Kingman's subadditive ergodic theorem, thanks to the moment condition (4.1), and the convergence also holds on average. Let us now apply the Oseledets theorem successively to the tangent cocycle defined by the fiber dynamics $(\mathcal{X}_+, F_+, m_+)$, and then to the cocycle associated to $(\mathcal{X}, F, m)$.

7.2.1. *The non-invertible setting.* Define the tangent bundles $T\mathcal{X}_+ := \Omega \times TX$ and $T\mathcal{X} := \Sigma \times TX$, and denote by DF and DF_+ the natural tangent maps, that is $D_{(\xi,x)}F : \{\xi\} \times T_xX \to \{\vartheta\xi\} \times T_{f_\xi(x)}X$ is induced by $D_x f^1_\xi$:

$$D_{(\xi,x)}F(v) = D_x f^1_\xi(v) \quad (\forall v \in T_xX_\xi = T_xX) \tag{7.16}$$

For the non-invertible dynamics on $\mathcal{X}_+$, the Oseledets theorem gives: for m_+-almost every (ω, x), there exists a non-trivial complex subspace $V^-(\omega, x)$ of $\{\omega\} \times T_xX$ such that

$$\forall v \in V^-(\omega, x)\backslash\{0\}, \quad \lim_{n\to+\infty} \frac{1}{n} \log \|D_x f^n_\omega(v)\| = \lambda^- \tag{7.17}$$

$$\forall v \notin V^-(\omega, x), \quad \lim_{n\to+\infty} \frac{1}{n} \log \|D_x f^n_\omega(v)\| = \lambda^+. \tag{7.18}$$

The field of subspaces V^- is measurable and almost surely invariant. Two cases can occur: either $\lambda^- < \lambda^+$ and $V^-(\omega, x)$ is almost surely a complex line, or $\lambda^- = \lambda^+$ and $V^-(\omega, x) = \{\omega\} \times T_xX$.

7.2.2. *The invertible setting.* For the dynamical system $F\colon \mathcal{X} \to \mathcal{X}$, the statement is:

– if $\lambda^- = \lambda^+$ then for m-almost every $x = (\xi, x)$, for every non-zero $v \in T_xX_\xi \simeq T_xX$,

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \left\|D_x f^n_\xi(v)\right\| = \lambda^-; \tag{7.19}$$

– if $\lambda^- < \lambda^+$ then for m-almost every x there exists a decomposition $T_xX_\xi = E^-(\xi, x) \oplus E^+(\xi, x)$ such that for $\star \in \{-, +\}$ and every $v \in E^\star(\xi, x) \setminus \{0\}$,

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \left\|D_x f^n_\xi(v)\right\| = \lambda^\star. \tag{7.20}$$

Furthermore the line fields $E^{\pm}$ are measurable and invariant, and $\log|\angle(E^-, E^+)|$ is integrable (here, the "angle" $\angle(E^-(x), E^+(x))$ is the distance between the two lines $E^-(x)$ and $E^+(x)$ in $\mathbb{P}(T_x\mathcal{X})$).

7.2.3. *Hyperbolicity.* It can happen that λ^- and λ^+ have the same sign. If λ^- and λ^+ are both negative, the conditional measures m_ξ are atomic: this can be shown by adapting a classical Pesin-theoretic argument (see e.g. [76, Cor. S.5.2]) to the fibered dynamics of F on $\mathcal{X}$ (see [84, Prop. 2] for a direct proof and an example where the m_ξ have several atoms). Such random dynamical systems are called **proximal**. For instance, generic random products of automorphisms of $\mathbb{P}^2(\mathbf{C})$, that is of matrices in $\mathsf{PGL}(3, \mathbf{C})$, are proximal; in such examples the stationary measure is not invariant. Other examples are given by contracting iterated function systems.

When λ^+ and λ^- are both non-negative, we have the so-called **invariance principle**:

Theorem 7.4. *Let (X, ν) be a random holomorphic dynamical system satisfying the integrability condition* (4.1)*, and let μ be an ergodic stationary measure. If $\lambda^+(\mu) \geqslant \lambda^-(\mu) \geqslant 0$ then μ is almost surely invariant.*

This result was proven by Crauel, building on ideas of Ledrappier described below in §11.4 (see Theorem 5.1, Corollary 5.3 and Remark 5.6 in [41], and also Avila-Viana [2, Thm B]).

Remark 7.5. *If λ^- and λ^+ are both positive then μ is atomic.* Indeed, since μ is almost surely invariant we get $m = \nu^{\mathbf{Z}} \times \mu$. Reversing time, the Lyapunov exponents of m become negative, so as explained above the measures m_ξ are atomic. By invariance $m_\xi = \mu$, so μ is atomic too.

By definition, μ is **hyperbolic** if $\lambda^- < 0 < \lambda^+$. In this case we rather use the conventional superscripts s/u instead of $-/+$ for stable and unstable objects. We also have $E^s = V^s$ in this case (and more generally when $\lambda^- < \lambda^+$); so, it follows that the complex line field E^s on $T\mathcal{X}$ is $\mathcal{F}^+$-measurable. Conversely the unstable line field E^u is $\mathcal{F}^-$-measurable.

7.3. **Invariant volume forms.** Let us start with a well-known result.

Lemma 7.6. *Let (X, ν) be a random holomorphic dynamical system satisfying the integrability condition* (4.1)*, and μ be an ergodic stationary probability measure. Then*

$$\lambda^- + \lambda^+ = \int \log|\mathrm{Jac}\, f(x)|\, d\mu(x) d\nu(f),$$

where Jac *denotes the Jacobian determinant relative to any smooth volume form on X.*

We omit the proof, since this result is a corollary of Proposition 7.8 below. When X is an Abelian, or K3, or Enriques surface, Remark 3.17 provides an $\mathsf{Aut}(X)$-invariant volume form on X. Thus, we obtain:

Corollary 7.7. *Assume that X is an Abelian, or K3, or Enriques surface. Let ν be a probability measure on* $\mathsf{Aut}(X)$ *satisfying the integrability condition* (4.1)*, and μ be an ergodic ν-stationary measure. Then $\lambda^- + \lambda^+ = 0$.*

Let η be a non-trivial meromorphic 2-form on the surface X. There is a cocycle Jac_η, with values in the multiplicative group $\mathcal{M}(X)^\times$ of non-zero meromorphic functions, such that

$$f^*\eta = \mathrm{Jac}_\eta(f)\eta \tag{7.21}$$

for every $f \in \mathsf{Aut}(X)$. We say that η is **almost invariant** if $|\mathrm{Jac}_\eta(f)(x)| = 1$ for every $x \in X$ and ν-almost every $f \in \mathsf{Aut}(X)$ (in particular $\mathrm{Jac}_\eta(f)$ is a constant). We refer to §3.4 for examples with an invariant meromorphic 2-form.

Proposition 7.8. *Let (X, ν) be a random holomorphic dynamical system satisfying the integrability condition* (4.1)*, and μ be an ergodic stationary measure. Let η be a non-trivial meromorphic* 2*-form on X such that*

(i) $\int \log^+ |\mathrm{Jac}_\eta(f)(x)| d\mu(x) d\nu(f) < +\infty$;
(ii) *μ gives zero mass to the set of zeroes and poles of η.*

Then

$$\lambda^- + \lambda^+ = \int \log(|\mathrm{Jac}_\eta f(x)|^2) d\mu(x) d\nu(f);$$

in particular $\lambda^- + \lambda^+ = 0$ if η is almost invariant.

Proof. Fix a trivialization of the tangent bundle TX, given by a measurable family of linear isomorphisms $L(x)\colon T_xX \to \mathbf{C}^2$ such that (a) $\det(L(x)) = 1$ and (b) $1/C \leqslant \|L(x)\| + \|L(x)^{-1}\| \leqslant C$, for some constant $C > 1$; here, the determinant is relative to the volume form vol on X and the standard volume form $dz_1 \wedge dz_2$ on $\mathbf{C}^2$, and the norm is with respect to the Kähler metric $(\kappa_0)_x$ on T_xX and the standard euclidean metric on $\mathbf{C}^2$. For $(\xi, x) \in \mathcal{X}$ and $n \geqslant 0$, the differential $D_x f_\xi^n$ is expressed in this trivialization as a matrix $A^{(n)}(\xi, x) = L(f_\xi^n(x)) \circ D_x f_\xi^n \circ L(x)^{-1}$. Let $\chi_n^-(\xi, x) \leqslant \chi_n^+(\xi, x)$ be the singular values of $A^{(n)}(\xi, x)$. Then m-almost surely, $\frac{1}{n} \log \chi_n^\pm(\xi, x) \to \lambda^\pm$ as $n \to +\infty$.

The form $\eta \wedge \overline{\eta}$ can be written $\eta \wedge \overline{\eta} = \varphi(x)\mathsf{vol}$ for some function $\varphi\colon X \to [0, +\infty]$. Locally, one can write $\eta = h(x) dx_1 \wedge dx_2$ where (x_1, x_2) are local holomorphic coordinates and h is a meromorphic function; then $\varphi(x)\mathsf{vol} = |h(x)|^2 dx_1 \wedge dx_2 \wedge d\overline{x_1} \wedge d\overline{x_2}$. The jacobian Jac_η satisfies

$$|\mathrm{Jac}_\eta(f)(x)|^2 = \frac{\varphi(f(x))}{\varphi(x)} \mathrm{Jac}_{\mathsf{vol}}(f)(x) \tag{7.22}$$

for every $f \in \mathsf{Aut}(X)$ and $x \in X$. Using $\det(L(x)) = 1$, we get

$$\det(A^{(n)}(\xi, x)) = \mathrm{Jac}_{\mathsf{vol}}(f_\xi^n)(x), \tag{7.23}$$

and then

$$\frac{1}{n} \log \chi_n^-(\xi, x) + \frac{1}{n} \log \chi_n^+(\xi, x) = \frac{2}{n} \log \left|\mathrm{Jac}_\eta f_\xi^n(x)\right| - \frac{1}{n} \log(\varphi(f_\xi^n(x))/\varphi(x)). \tag{7.24}$$

By the Oseledets theorem, the left hand side of (7.24) converges almost surely to $\lambda^- + \lambda^+$. Since the Jacobian Jac_η is multiplicative along orbits, i.e. $\mathrm{Jac}_\eta f_\xi^n(x) = \prod_{k=0}^{n-1} \mathrm{Jac}_\eta f_{\vartheta^k \xi}(f_\xi^k x)$, the integrability condition and the ergodic theorem imply that, almost surely,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \log \left|\mathrm{Jac}_\eta f_\xi^n(x)\right| &= \int \log \left|\mathrm{Jac}_\eta f_\xi^1(x)\right| dm(\xi, x) \\ &= \int \log \left|\mathrm{Jac}_\eta f_\omega^1(x)\right| dm_+(\omega, x) \\ &= \int \log \left|\mathrm{Jac}_\eta f(x)\right| d\mu(x) d\nu(f). \end{aligned} \tag{7.25}$$

Let $\mathrm{div}(\eta)$ be the set of zeroes and poles of η. Since μ is ergodic and does not charge $\mathrm{div}(\eta)$, we deduce that for m-almost every (ξ, x), there is a sequence (n_j) such that $f_\xi^{n_j}(x)$ stays at positive distance from $\mathrm{div}(\eta)$; along such a sequence, $\log|\varphi(f_\xi^{n_j}(x))/\varphi(x)|$ stays bounded, and the right hand side of (7.24) tends to $2\int \log|\mathrm{Jac}_\eta f(x)|\, d\mu(x) d\nu(f)$. This concludes the proof. □

7.4. **Intermezzo: local complex geometry.** Recall that X is endowed with a Riemannian structure, hence a distance, induced by the Kähler metric κ_0. For $x \in X$, we denote by euc_x the translation-invariant Hermitian metric on T_xX (which is considered here as a manifold in its own right) associated to the Riemannian structure induced by $(\kappa_0)_x$. Given any orthonormal basis (e_1, e_2) of T_xX for this metric, we obtain a linear isometric isomorphism from T_xX to $\mathbf{C}^2$, endowed respectively with euc_x and the standard euclidean metric; we shall implicitly use such identifications in what follows.

We denote by $\mathbb{D}(z; r)$ the disk of radius r around z in $\mathbf{C}$, and set $\mathbb{D}(r) = \mathbb{D}(0; r)$.

7.4.1. *Hausdorff and C^1-convergence.* Let $U \subset \mathbf{C}$ be a domain. If $\gamma\colon U \to X$ is a holomorphic curve, we can lift it canonically to a curve $\gamma^{(1)}\colon U \to TX$ by setting $\gamma^{(1)}(z) = (\gamma(z), \gamma'(z)) \in T_{\gamma(z)}X$, where $\gamma'(z)$ denotes the velocity of γ at z. The Riemannian metric κ_0 induces a Riemannian metric and therefore a distance dist_{TX} on TX. We say that two parametrized curves γ_1 and γ_2 are δ-close in the C^1-topology if $\mathrm{dist}_{TX}(\gamma_1^{(1)}(z), \gamma_2^{(1)}(z)) \leqslant \delta$ uniformly on U. This implies that $\gamma_1(U)$ and $\gamma_2(U)$ are δ-close in the Hausdorff sense, but the converse does not hold (take $U = \mathbb{D}(1)$, $\gamma_1(z) = (z, 0)$, and $\gamma_2(z) = (z^k, \varepsilon z^\ell)$ with k and ℓ large while ε is small).

7.4.2. *Good charts.* Let R_0 be the injectivity radius of κ_0. We fix once and for all a family of charts $\Phi_x\colon U_x \subset T_xX \to X$ with the following properties (for some constant C_0):

(i) $\Phi_x(0) = x$ and $(D\,\Phi_x)_0 = \mathrm{id}_{T_xX}$;
(ii) Φ_x is a holomorphic diffeomorphism from its domain of definition U_x to an open subset V_x contained in the ball of radius R_0 around x;
(iii) on U_x, the Riemannian metrics euc_x and Φ_x^* satisfy $C_0^{-1} \leqslant \mathsf{euc}_x/\,\Phi_x^*\,\kappa_0 \leqslant C_0$;
(iv) the family of maps Φ_x depends continuously on x.

With $r_0 \leqslant R_0/(\sqrt{2}C_0)$, we can add:

(v) for every orthonormal basis (e_1, e_2) of T_xX, the bidisk $\mathbb{D}(r_0)e_1 + \mathbb{D}(r_0)e_2$ is contained in U_x; in particular, the ball of radius r_0 centered at the origin for euc_x is contained in U_x.

To make assertion (iv) more precise, fix a continuous family of orthonormal basis $(e_1(x), e_2(x))$ on some open set V of X: Assertion (iv) means that, if we compose Φ_x with the linear isomorphism $(z_1, z_2) \in \mathbf{C}^2 \mapsto z_1e_1(x) + z_2e_2(x) \in T_xX$ we obtain a continuous family of maps. If needed, we can also add the following property (see [69, pp. 107-109]):

(iii') euc_x osculates $\Phi_x^*\,\kappa_0$ up to order 2 at x.

7.4.3. *Families of disks.* A holomorphic disk $\Delta \subset X$ containing x is said to be a disk **of size (at least)** r **at** x (resp. **of size exactly** r **at** x), for some $r < r_0$, if there is an orthonormal basis (e_1, e_2) of T_xX such that $\Phi_x^{-1}(\Delta)$ contains (resp. is) the graph $\{ze_1 + \varphi(z)e_2\ ;\ z \in \mathbb{D}(r)\}$ for some holomorphic map $\varphi\colon \mathbb{D}(r) \to \mathbb{D}(r)$. By the Koebe distortion theorem if Δ has size r at x, then its geometric characteristics around x at scale smaller than $r/2$, say, are comparable to that of a flat disk. An alternative definition for the concept of disks of size $\geqslant r$ could be that

Δ contains the image of an injective holomorphic map $\gamma\colon \mathbb{D}(r) \to X$ such that $\gamma(\partial\mathbb{D}(r)) \subset X\backslash B_X(x;r)$ and $\|\gamma'\| \leqslant D$, for some fixed constant D. Then, if Δ contains a disk of size r for one of these definitions, it contains a disk of size $\varepsilon_0 r$ for the other definition, for some uniform $\varepsilon_0 > 0$; in particular, there is a constant C depending only on (X, κ_0) such that a disk of size r at x contains an embedded submanifold of $B_X(x; Cr)$.

Let (x_n) be a sequence converging to x in X, and let r be smaller than the radius r_0 introduced in Assertion (v), § 7.4.2. Let Δ_n be a family of disks of size at least r at x_n and Δ be a disk of size at least r at x. We say that Δ_n **converges towards** Δ *as a sequence of disks of size* r, if there is an orthonormal basis (e_1, e_2) of T_xX for euc_x such that

(i) $\Phi_x^{-1}(\Delta)$ contains the graph $\{ze_1 + \varphi(z)e_2; z \in \mathbb{D}(r)\}$ for some holomorphic function $\varphi\colon \mathbb{D}(r) \to \mathbb{D}(r)$;
(ii) for every $s < r$, if n is large enough, the disk $\Phi_x^{-1}(\Delta_n)$ contains the graph $\{ze_1 + \varphi_n(z)e_2; z \in \mathbb{D}(s)\}$ of a holomorphic function $\varphi_n\colon \mathbb{D}(s) \to \mathbb{D}(r)$;
(iii) for every $\varepsilon > 0$, we have $|\varphi(z) - \varphi_n(z)| < \varepsilon$ on $\mathbb{D}(s)$ if n is large enough.

By the Cauchy estimates, the convergence then holds in the C^1-topology (see § 7.4.1). It follows from the usual compactness criteria for holomorphic functions that the space of disks of size r on X is compact (for the topology induced by the Hausdorff topology in X). Likewise, if a sequence of disks of size r converges in the Hausdorff sense, then it also converges in the C^1 sense, at least as disks of size $s < r$, because two holomorphic functions φ and ψ from $\mathbb{D}(r)$ to $\mathbb{D}(r)$ whose graphs are ε-close are also $\varepsilon(r-s)^{-1}$-close in the C^1-topology.

It may also be the case that the Δ_n are contained in different fibers X_{ξ_n} of $\mathcal{X}$. By definition, we say that the sequence Δ_n converges to $\Delta \subset X_\xi$ if ξ_n converges to ξ and the projections of Δ_n converge to Δ in X.

7.4.4. *Entire curves.* An **entire curve** in X is, by definition, a holomorphic map $\psi\colon \mathbf{C} \to X$. The curve is immersed if its velocity ψ' does not vanish. Our main examples of immersed curves will, in fact, be injective and immersed entire curves. If ψ_1 and ψ_2 are two immersed entire curves with the same image, there exists a holomorphic diffeomorphism of $\mathbf{C}$, i.e. a non-constant affine map $A\colon z \mapsto az + b$, such that $\psi_2 = \psi_1 \circ A$. If ψ is an immersed entire curve and $|\psi'| \geqslant \eta$ on $\mathbb{D}(z_0, s)$, its image contains a disk of size Cs at $\psi(z_0)$, for some $C > 0$ that depends only on η and κ_0.

7.5. **Stable and unstable manifolds.** By Lemma 4.1, Condition (4.1) implies similar moment conditions for higher derivatives, so Pesin's theory applies. The following proposition summarizes the main properties of Pesin local stable and unstable manifolds. Recall that a function h is ε**-slowly varying**, relatively to some dynamical system g, if $e^{-\varepsilon} \leqslant h(g(x))/h(x) \leqslant e^{\varepsilon}$ for every x. We view the stable manifold of $\mathcal{x} = (\xi, x)$ as contained in X_ξ; it can also be viewed as a subset of X: whether we consider one or the other point of view should be clear from the context. If $\mathcal{x} = (\xi, x)$ and $\mathcal{y} = (\xi, y)$ are points of the same fiber X_ξ, we denote by $\mathrm{dist}_X(\mathcal{x}, \mathcal{y})$ the Riemannian distance between x and y computed in X.

Proposition 7.9. *Let (X, ν) be a random holomorphic dynamical system, and μ be an ergodic and hyperbolic stationary measure. Then, for every $\delta > 0$, there exists measurable positive δ-slowly varying functions r and C on $\mathcal{X}$ (depending on δ) and, for m-almost every*

$x = (\xi, x) \in \mathcal{X}$, local stable and unstable manifolds $W^s_{r(x)}(x)$ and $W^u_{r(x)}(x)$ in X_ξ such that m-almost surely:

(1) *$W^s_{r(x)}(x)$ and $W^u_{r(x)}(x)$ are holomorphic disks of size at least $2r(x)$ at x respectively tangent to $E^s(x)$ and $E^u(x)$;*

(2) *for every $y \in W^s_{r(x)}(x)$ and every $n \geqslant 0$,*

$$\mathrm{dist}_X(F^n(x), F^n(y)) \leqslant C(x)\exp((\lambda^s + \delta)n);$$

likewise for every $y \in W^u_{r(x)}(x)$ and every $n \geqslant 0$

$$\mathrm{dist}_X(F^{-n}(x), F^{-n}(y)) \leqslant C(x)\exp(-(\lambda^u - \delta)n);$$

(3) *$F(W^s_{r(x)}(x)) \subset W^s_{r(F(x))}(F(x))$ and $F^{-1}(W^u_{r(F(x))}(F(x))) \subset W^u_r(x)$.*

By Lusin's theorem, for every $\varepsilon > 0$ we can select a compact subset $\mathcal{R}_\varepsilon \subset \mathcal{X}$ with $m(\mathcal{R}_\varepsilon) > 0$ on which $r(x)$ and $C(x)$ can be replaced by uniform constants (respectively denoted by r and C) and the following additional property holds:

(4) *on $\mathcal{R}_\varepsilon$ the local stable and unstable manifolds $W^{s/u}_r(x)$ vary continuously for the C^1-topology (in the sense of § 7.4.1 and 7.4.3).*

The subsets $\mathcal{R}_\varepsilon$ are usually called **Pesin sets**, or regular sets. We also denote the local stable or unstable manifolds by $W^{s/u}_{\mathrm{loc}}(x)$, or by $W^{s/u}_r(x)$ when x is in a Pesin set on which $r(\cdot) \geqslant r$. On several occasions we will have to deal with measurability issues for $W^{s/u}_{\mathrm{loc}}(x)$ as a function of x: this will be done by exhausting $\mathcal{R}_\varepsilon$ by Pesin sets and using their continuity on $\mathcal{R}_\varepsilon$.

The global stable and unstable manifolds of x are respectively defined by the following increasing unions:

$$W^s(x) = \bigcup_{n \geqslant 0} F^{-n}\left(W^s_{r(x)}(F^n(x))\right) \quad \text{and} \quad W^u(x) = \bigcup_{n \geqslant 0} F^{n}\left(W^u_{r(x)}(F^{-n}(x))\right). \tag{7.26}$$

In particular, they are injectively immersed holomorphic curves in X_ξ. Pesin theory shows that:

$$W^s(x) = \left\{(\xi, y) \in X_\xi \ ; \ \limsup_{n\to\infty} \frac{1}{n}\log \mathrm{dist}_X(F^n(\xi, y), F^n(\xi, x)) < 0\right\} \tag{7.27}$$

$$W^u(x) = \left\{(\xi, y) \in X_\xi \ ; \ \limsup_{n\to-\infty} \frac{1}{|n|}\log \mathrm{dist}_X(F^n(\xi, y), F^n(\xi, x)) < 0\right\}. \tag{7.28}$$

Proposition 7.10. *Under the assumptions of Proposition 7.9, $W^s(x)$ and $W^u(x)$ are biholomorphic to $\mathbf{C}$ for m-almost every x.*

More precisely, $W^s(x)$ is parametrized by an injectively immersed entire curve $\psi^s_x : \mathbf{C} \to X$ such that $\psi^s_x(0) = x$ and this parametrization is unique, up to an homothety $z \mapsto az$ of $\mathbf{C}$. Likewise, $W^s(x)$ is parametrized by such an entire curve ψ^u_x.

Proof. By (7.26) and Proposition 7.9.(3), $W^s(x)$ is an increasing union of disks and is therefore a Riemann surface homeomorphic to $\mathbf{R}^2$; so, it is biholomorphic to $\mathbf{C}$ or $\mathbb{D}$. Let $A \subset \mathcal{X}$ be a set of positive measure on which $r \geqslant r_0$ and $C \leqslant C_0$. By Proposition 7.9.(2), there exists $n_0 \in \mathbf{N}$ and $m_0 > 0$ such that if $n \geqslant n_0$ and if x and $F^n(x)$ belong to A, then $W^s_r(F^n(\xi, x)) \setminus (F^n W^s_r(\xi, x))$ is an annulus of modulus $\geqslant m_0$. Now for m-almost every $x \in \mathcal{X}$

there is an infinite sequence (k_j) such that $F^{k_j}(x) \in A$ and $k_{j+1} - k_j > n_0$. For such an x, $W^s(x) \backslash W^s_r(x)$ contains an infinite nested sequence of annuli of modulus at least m_0, namely the $F^{-k_{j+1}}(W^s_r(F^{k_{j+1}}(x)) \backslash F^{k_{j+1}-k_j}(W^s_r(F^{k_j}(x))$. Thus, $W^s(x)$ is biholomorphic to $\mathbf{C}$. □

If we are only interested in stable manifolds, there is a simplified version of Proposition 7.9 which takes place on X:

Proposition 7.11. *Let (X, ν) be a random holomorphic dynamical system and μ an ergodic stationary measure, whose Lyapunov exponents satisfy $\lambda^- < 0 \leqslant \lambda^+$. Then for m_+-almost every (ω, x) the stable set*

$$W^s(\omega, x) = \left\{ y \in X \ ; \ \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{dist}_X(f^n_\omega(y), f^n_\omega(x)) < 0 \right\}$$

is an injectively immersed entire curve in X.

Indeed, stable manifolds can be obtained from a purely "one-sided" construction, that is, by considering only positive iterates (see [91, Chap. III]). This also shows that local stable manifolds in $\mathcal{X}$ are $\mathcal{F}^+$-measurable, and may be viewed as living in $\mathcal{X}_+$.

7.6. **Fibered entropy.** Here we recall the definition of the **metric fibered entropy** of a stationary measure μ (see [80, §2.1] or [91, Chap. 0 and I] for more details). If η is a finite measurable partition of X, its entropy relative to μ is $H_\mu(\eta) = -\sum_{C\in\eta} \mu(C) \log \mu(C)$. Then, we set

$$h_\mu(X, \nu; \eta) = \lim_{n\to\infty} \frac{1}{n} \int H_\mu \left(\bigvee_{k=0}^{n-1} \left(f^k_\xi\right)^{-1} (\eta) \right) d\nu^{\mathbf{N}}(\xi), \tag{7.29}$$

$$h_\mu(X, \nu) = \sup \left\{ h_\mu(X, \nu; \eta) \ ; \ \eta \text{ a finite measurable partition of } X \right\}. \tag{7.30}$$

Actually $h_\mu(X, \nu; \eta)$ can be interpreted as a conditional (or fibered) entropy for the skew-products F_+ on $\mathcal{X}_+$ and F on $\mathcal{X}$. Indeed, the so-called Abramov-Rokhlin formula holds [16]:

$$h_\mu(X, \nu) = h_{\nu^{\mathbf{N}}\times\mu}(F_+|\eta_\Omega) = h_{m_+}(F_+) - h_{\nu^{\mathbf{N}}}(\sigma) \tag{7.31}$$

$$= h_m(F|\eta_\Sigma) \ = \ h_m(F) - h_{\nu^{\mathbf{Z}}}(\vartheta), \tag{7.32}$$

where η_Ω (resp. η_Σ) denotes the partition into fibers of the first projection $\pi_\Omega \colon \mathcal{X}_+ \to \Omega$ (resp. $\pi_\Sigma \colon \mathcal{X} \to \Sigma$) and in the second and fourth equalities we assume $h_{\nu^{\mathbf{N}}}(\sigma) = h_{\nu^{\mathbf{Z}}}(\vartheta) < \infty$. The next result is the fibered version of the **Margulis-Ruelle inequality**.

Proposition 7.12. *Let (X, ν) be a random holomorphic dynamical system satisfying the moment condition* (4.1) *and μ be an ergodic stationary measure. If $h_\mu(X, \nu) > 0$ then μ is hyperbolic and* $\min(\lambda^u, -\lambda^s) \geqslant \frac{1}{2} h_\mu(X, \nu)$.

Proof. See [3] or [91, Chap. II] for the inequality $\lambda^u \geqslant \frac{1}{2} h_\mu(X, \nu)$. For $-\lambda^s \geqslant \frac{1}{2} h_\mu(X, \nu)$, we use the fact that $h_m(F|\eta_\Sigma) = h_m(F^{-1}|\eta_\Sigma)$ (see e.g. [91, I.4.2]) and apply the Margulis-Ruelle inequality to F^{-1}. Beware that there is a slightly delicate point here: (F^{-1}, m) is not associated to a random dynamical system in our sense; fortunately, the statement of the Margulis-Ruelle inequality in [3] (see also [91, Appendix A]) covers this situation. □

7.7. **Unstable conditionals and entropy.** Assume μ is ergodic and hyperbolic. By definition, an **unstable Pesin partition** η^u on $\mathcal{X}$ is a measurable partition of $(\mathcal{X}, \mathcal{F}, \mu)$ with the following properties:

– η is increasing: $F^{-1}\eta^u$ refines η^u;
– for m-almost every x, $\eta^u(x)$ is an open subset of $W^u(x)$ and

$$\bigcup_{n\geqslant 0} F^n\left(\eta^u(F^{-n}(x))\right) = W^u(x); \tag{7.33}$$

– η^u is a generator, i.e. $\bigvee_{n=0}^{\infty} F^{-n}(\eta^u)$ coincides m-almost surely with the partition into points.

Here, as usual, $\eta^u(x)$ denotes the atom of η^u containing x, and $F^{-1}\eta^u$ is the partition defined by $(F^{-1}\eta^u)(x) = F^{-1}(\eta^u(F(x)))$. The definition of a **stable Pesin partition** η^s is similar. A neat proof of the existence of such a partition is given by Ledrappier and Strelcyn in [88], which easily adapts to the random setting (see [91, §IV.2]).

Lemma 7.13. *There exists a stable Pesin partition whose atoms are $\mathcal{F}^+$-measurable, that is, saturated by local stable sets $\Sigma^s_{\rm loc} \times \{x\}$.*

Proof. To justify the existence of such a partition, we briefly review the proof of Ledrappier and Strelcyn [88] and show that it can be rendered $\mathcal{F}^+$-measurable. Let E be a set of positive measure in $\mathcal{X}$ such that (a) $\pi_X(E)$ is contained in a ball of radius r_0, (b) for every $x = (\xi, x) \in E$, and every $0 < r \leqslant 2r_0$, $W^s(x)$ contains a disk of size exactly r at x, denoted by $\Delta^s(x, r)$ and (c) for every $0 < r \leqslant 2r_0$, $E \ni x \mapsto \Delta^s(x, r)$ is continuous for the C^1 topology. Then for $0 < r < r_0$ we define a measurable partition η_r whose atoms are the $\Delta^s(x, r)$ for $x \in E$ as well as $\mathcal{X} \setminus \bigcup_{x\in E} \Delta^s(x, r)$. Since stable manifolds are $\mathcal{F}^+$-measurable, we can further require that for every $\xi' \in \Sigma^s_{\rm loc}(\xi)$, with $x' = (\xi', x)$, we have $\Delta^s(x', r) = \Delta^s(x, r)$. The argument of [88] shows that for Lebesgue-almost every $r \in [0, r_0]$, the partition $\eta^s = \bigvee_{n=0}^{\infty} F^{-n}(\eta_r)$ is a Pesin stable partition. Thus with x and x' as above we infer that

$$\eta^s(x') = \bigcap_{n\geqslant 0} F^{-n}\eta_r(F^n(x')) = \bigcap_{n\geqslant 0} F^{-n}\eta_r(F^n(x)) = \eta^s(x) \tag{7.34}$$

where the middle equality comes from the fact that $\vartheta^n\xi' \in \Sigma^s_{\rm loc}(\vartheta^n\xi)$, and we are done. □

The existence of unstable partitions enables us to give a meaning to the **unstable conditionals** of m. Indeed, first observe that if η^u and ζ^u are two unstable Pesin partitions, then m-almost surely $m(\cdot|\eta^u)$ and $m(\cdot|\zeta^u)$ coincide up to a multiplicative factor on $\eta^u(x) \cap \zeta^u(x)$. Furthermore, there exists a sequence of unstable partitions η^u_n such that for almost every x, if K is a compact subset of $W^u(x)$ for the intrinsic topology (i.e. the topology induced by the biholomorphism $W^u(x) \simeq \mathbf{C}$) then $K \subset \eta^u_n(x)$ for sufficiently large n: indeed by (7.33), the sequence of partitions $F^n\eta^u$ does the job. Hence almost surely the conditional measure of m on $W^u(x)$ is well-defined up to scale; we define m^u_x by normalizing so that $m^u_x(\eta^u(x)) = 1$.

The next proposition is known as the (relative) **Rokhlin entropy formula**, stated here in our specific context.

Proposition 7.14. *Let (X, ν) be a random holomorphic dynamical system satisfying the moment condition* (4.1)*, and μ be an ergodic and hyperbolic stationary measure. Let η^u be an unstable*

Pesin partition. Then

$$h_\mu(X,\nu) = H_m(F^{-1}\eta^u|\,\eta^u) := \int \log J_{\eta^u}(x)dm(x),$$

where $J_{\eta^u}(x)$ is the "Jacobian" of F relative to η^u, that is

$$J_{\eta^u}(x) = m\left(F^{-1}\left(\eta^u(F(x))\right)|\,\eta^u(x)\right)^{-1}.$$

Sketch of proof. The argument is based on the following sequence of equalities, in which η_Σ is the partition into fibers of π_Σ, as before:

$$\begin{aligned} h_\mu(X,\nu) &= h_m(F|\eta_\Sigma) = h_m(F^{-1}|\eta_\Sigma) \\ &= h_m(F^{-1}|\eta^u \vee \eta_\Sigma) \\ &:= H_m(\eta^u|F\eta^u \vee \eta_\Sigma) = H_m(\eta^u|F\eta^u) = H_m(F^{-1}\eta^u|\eta^u) \end{aligned} \tag{7.35}$$

The equalities in the first and last line follow from the general properties of conditional entropy: see [91, Chap. 0] for a presentation adapted to our context (note that the conditional entropy would be denoted by $h_m^{\eta_\Sigma}$ there) or Rokhlin [104] for a thorough treatment. On the other hand the equality (7.35) is non-trivial. If η^u were of the form $\bigvee_{n=0}^{+\infty}\eta$, where η is a 2-sided generator with finite entropy, this equality would follow from the general theory. For a Pesin unstable partition the result was established for diffeomorphisms in [89, Cor 5.3] and adapted to random dynamical systems in [91, Cor. VI.7.1]. □

Remark 7.15. It is customary to present the Rokhlin entropy formula using unstable partitions, mostly because entropy is associated to expansion. Nonetheless, a similar formula holds in the stable direction:

$$h_\mu(X,\nu) = \int \log J_{\eta^s}(x)dm(x) \text{ where } J_{\eta^s}(x) = m\left(F\left(\eta^s(F^{-1}(x))\right)|\,\eta^s(x)\right)^{-1}.$$

The proof is identical to that of Proposition 7.14, applied to F^{-1}, with however the same caveat as in Proposition 7.12: (F^{-1}, m) is not associated to a random dynamical system in our sense. The only non-trivial point is to check that the key equality (7.35) holds in this case. Fortunately, the main purpose of [**?**] is to explain how to adapt [91, Chap. VI], hence the equality (7.35), to a more general notion of "random dynamical system" which covers the case of (F^{-1}, m) (see in particular the last lines of [**?**, §5] for a short discussion of the Rokhlin formula).

Corollary 7.16. *Under the assumptions of the previous proposition, the following assertions are equivalent:*

(a) $h_\mu(X,\nu) = 0$*;*
(b) $m(\cdot|\eta^u(x)) = \delta_x$ *for m-almost every x;*
(c) $m(\cdot|\eta^u(x))$ *is atomic for m-almost every x.*

The same result holds for the stable Pesin partition η^s.

Proof. In view of the definition of J_{η^u}, the entropy vanishes if and only if for m-almost every x, $m(\cdot|\eta^u(x))$ is carried by a single atom of the finer partition $F^{-1}\eta^u$. Now since $H_m(F^{-1}\eta^u|\,\eta^u) = \frac{1}{n}H_m(F^{-n}\eta^u|\,\eta^u)$, the same is true for $F^{-n}\eta^u$, and finally since $(F^{-n}\eta^u)$ is generating, we conclude that (a)⇔(b). That (c) implies (a) follows from the same ideas but it is slightly more delicate, see [111, §2.1-2.2] for a clear exposition in the case of the iteration a single diffeomorphism, which readily adapts to our setting.

The result for the stable Pesin partition η^s follows by changing F to F^{-1} (see Remark 7.15). □

A further result is that if the fiber entropy vanishes there is a set of full m-measure which intersects any global unstable leaf in only one point. This was originally shown for individual diffeomorphisms in [89, Thm. B].

8. Stable manifolds and limit currents

Let as before (X, ν) be a non-elementary random holomorphic dynamical system on a compact Kähler (hence projective) surface, and assume μ is an ergodic stationary measure admitting exactly one negative Lyapunov exponent, as in Proposition 7.11. Our purpose in this section is to relate the stable manifolds $W^s(\omega, x)$ to the stable currents T^s_ω constructed in §6. According to Proposition 7.11, the stable manifolds are parametrized by injective entire curves; the link between these curves and the stable currents will be given by the well-known Ahlfors-Nevanlinna construction of positive closed currents associated to entire curves.

8.1. **Ahlfors-Nevanlinna currents.** We denote by $\{V\}$ the integration current on a (possibly non-closed, or singular) curve V. Let $\phi : \mathbf{C} \to X$ be an entire curve. By definition, if α is a test 2-form, $\langle \phi_* \{\mathbb{D}(0,t)\}, \alpha \rangle = \int_{\mathbb{D}(0,t)} \phi^*\alpha$, which accounts for possible multiplicities coming from the lack of injectivity of ϕ; $\phi_* \{\mathbb{D}(0,t)\} = \{\phi(\mathbb{D}(0,t))\}$ when ϕ is injective. Set

$$A(R) = \int_{\mathbb{D}(0,R)} \phi^*\kappa_0 \text{ and } T(R) = \int_0^R A(t)\frac{dt}{t}. \tag{8.1}$$

for $R > 0$. When ϕ is an immersion, $A(R)$ is the area of $\phi(\mathbb{D}(0,R))$; in all cases, $A(R)$ is the mass of $\phi_*\{(\mathbb{D}(R))\}$.

Proposition 8.1 (see Brunella [23, §1]). *If $\phi : \mathbf{C} \to X$ is a non-constant entire curve, there exist sequences of radii (R_n) increasing to infinity such that the sequence of currents*

$$N(R_n) = \frac{1}{T(R_n)} \int_0^{R_n} \phi_* \{\mathbb{D}(0,t)\} \frac{dt}{t}$$

converges to a closed positive current T. If furthermore $\phi(\mathbf{C})$ is Zariski dense, and T is such a closed current, the class $[T] \in H^{1,1}(X,\mathbf{R})$ is nef. In particular $\langle [T] \,|\, [T] \rangle \geqslant 0$ and $\langle [T] \,|\, [C] \rangle \geqslant 0$ for every algebraic curve $C \subset X$.

Such limit currents T will be referred to as **Ahlfors-Nevanlinna currents** associated to the entire curve ϕ: $\mathbf{C} \to X$. If $\phi(\mathbf{C})$ is not Zariski dense then $\overline{\phi(\mathbf{C})}$ is a (possibly singular) curve of genus 0 or 1; if ϕ is injective, then $\overline{\phi(\mathbf{C})}$ is rational.

8.2. **Equidistribution of stable manifolds.** If μ is hyperbolic, or more generally if it admits exactly one negative Lyapunov exponent, then, for m_+-almost every $x = (\omega, x) \in \mathcal{X}_+$, the stable manifold $W^s(x)$, which is viewed here as a subset of X as in Proposition 7.11, is parametrized by an injectively immersed entire curve. Then we can relate the Ahlfors-Nevanlinna currents to the limit currents T^s_ω; here are the three main results that will be proved in this section.

Theorem 8.2. *Let (X, ν) be a non-elementary random holomorphic dynamical system on a compact Kähler surface, satisfying* (4.1). *Let μ be an ergodic stationary measure such that $\lambda^-(\mu) < 0 \leqslant \lambda^+(\mu)$. Then exactly one of the following alternative holds.*

(a) *For m_+-almost every x, the stable manifold $W^s(x)$ is not Zariski dense. Then μ is supported on a Γ_ν-invariant curve $Y \subset X$ and for m_+-almost every x, $W^s(x) \subset Y$. In addition every component of Y is a rational curve, and the intersection form is negative definite on the subspace of $H^{1,1}(X; \mathbf{R})$ generated by the classes of components of Y.*
(b) *For m_+-almost every x the stable manifold $W^s(x)$ is Zariski dense and the only normalized Ahlfors-Nevanlinna current associated to $W^s(x)$ is T^s_ω.*

Corollary 8.3. *Under the assumptions of Theorem 8.2, if in addition μ is hyperbolic and non-atomic, then the Alternative* (b) *is equivalent to*

(b') *μ is not supported on a Γ_ν-invariant curve.*

Corollary 8.4. *Under the assumptions of Theorem 8.2, assume furthermore that ν satisfies the exponential moment condition* (5.26). *Then in Alternative (b) there exists $\theta > 0$ such that for m_+-almost every $x \in \mathcal{X}_+$ the Hausdorff dimension of $\overline{W^s(x)}$ equals $2 + \theta$.*

8.3. **Proof of Theorem 8.2 and its corollaries.** We work under the assumptions of Theorem 8.2.

Lemma 8.5. *If there exists a proper Zariski closed subset of X with positive μ-measure, then:*

- *either μ is the uniform counting measure on a finite orbit of Γ_ν;*
- *or μ has no atom and it is supported on a Γ_ν-invariant algebraic curve, which is the Γ_ν-orbit of an irreducible algebraic curve.*

Proof. Consider the real number $\delta^0_{\max}(\mu) = \max_{x \in X} \mu(\{x\})$. If $\delta^0_{\max}(\mu) > 0$, there is a non-empty finite set $F \subset X$ for which $\mu(\{x\}) = \delta^0_{\max}(\mu)$. By stationarity, F is Γ_ν-invariant, and by ergodicity μ is the uniform measure on F. Now, assume that μ has no atom. Let $\delta^1_{\max}(\mu)$ be the maximum of $\mu(D)$ among all irreducible curves $D \subset X$. If $\mu(Z) > 0$ for some proper Zariski closed subset $Z \subset X$, then $\delta^1_{\max}(\mu) > 0$. Since two distinct irreducible curves intersect in at most finitely many points and μ has no atom, there are only finitely many irreducible curves E such that $\mu(E) = \delta^1_{\max}(\mu)$. To conclude, we argue as in the zero dimensional case. ☐

If $V \subset X$ is a smooth curve, possibly with boundary, if T is a closed positive $(1,1)$-current on X with a continuous normalized potential u_T (as in § 6.1.1), then by definition

$$\langle T \wedge \{V\}, \varphi \rangle = \int_V \varphi \, \Theta_T + \int_V \varphi \, dd^c(u_T|_V), \tag{8.2}$$

for every test function φ. Here is the key relation between stable manifolds and limit currents:

Lemma 8.6. *For m_+-almost every $x = (\omega, x)$, if Δ is a disk contained in $W^s(x)$, then $T^s_\omega \wedge \{\Delta\} = 0$.*

Proof. With no loss of generality we assume that the boundary of the disk Δ in $W^s(x) \simeq \mathbf{C}$ is smooth. We consider points $x = (\omega, x) \in \mathcal{X}_+$ which are generic in the following sense: they are regular from the point of view of Pesin's theory, and T^s_ω satisfies the conclusions of §6.

By Pesin's theory, for every $\varepsilon > 0$, there is a set $A_\varepsilon \subset \mathbf{N}$ of density larger than $1-\varepsilon$, such that for n in A_ε, the local stable manifold $W^s_r(F^n_+(x))$ is a disk of size $r = r(\varepsilon)$ at $f^n_\omega(x)$ and $f^n_\omega(\Delta)$ is a disk contained in an exponentially small neighborhood of $f^n_\omega(x)$. We have

$$\mathbf{M}(T^s_{\sigma^n\omega} \wedge \{f^n_\omega(\Delta)\}) = \int_{W^s_r(F^n_+(x))} \mathbf{1}_{f^n_\omega(\Delta)}\, \Theta_{T^s_{\sigma^n\omega}} + \int_{W^s_r(F^n_+(x))} \mathbf{1}_{f^n_\omega(\Delta)} dd^c u_{T^s_{\sigma^n\omega}}. \tag{8.3}$$

Since $\mathbf{M}(T^s_{\sigma^n\omega}) = 1$, Lemma 6.1 shows that $\Theta_{T^s_{\sigma^n\omega}}$ is bounded by $A\kappa_0$; so the first integral on the right hand side of (8.3) is bounded by a constant times the area of $f^n_\omega(\Delta)$, which is exponentially small. By ergodicity, there exists $A'_\varepsilon \subset A_\varepsilon$ of density at least $1-2\varepsilon$ such that if $n \in A'_\varepsilon$, $\|u_{T^s_{\sigma^n\omega}}\|_\infty$ is bounded by some contant $D_\varepsilon > 0$. For such an n, let χ be a test function in $W^s_{\mathrm{loc}}(F^n_+(x))$ such that $\chi = 1$ in $W^s_{r/2}(F^n_+(x))$. We write

$$\begin{aligned} \int_{W^s_r(F^n_+(x))} \mathbf{1}_{f^n_\omega(\Delta)} dd^c u_{T^s_{\sigma^n\omega}} &\leqslant \int_{W^s_r(F^n_+(x))} \chi dd^c u_{T^s_{\sigma^n\omega}} \\ &= \int_{W^s_r(F^n_+(x))} u_{T^s_{\sigma^n\omega}} dd^c \chi \\ &\leqslant C(r) \|\chi\|_{C^2} \|u_{T^s_{\sigma^n\omega}}\|_\infty \end{aligned} \tag{8.4}$$

where $C(r)$ bounds the area of $W^s_r(F^n_+(x))$; this last term is uniformly bounded because $n \in A'_\varepsilon$. Thus we conclude that $\mathbf{M}(T^s_{\sigma^n\omega} \wedge \{f^n_\omega(\Delta)\})$ is bounded along such a subsequence.

On the other hand, the relation $(f^n_\omega)^* T^s_{\sigma^n\omega} = \mathbf{M}((f^n_\omega)^* T^s_{\sigma^n\omega}) T^s_\omega$ gives

$$T^s_{\sigma^n(\omega)} \wedge \{f^n_\omega(\Delta)\} = \mathbf{M}\left((f^n_\omega)^* T^s_{\sigma^n(\omega)}\right) (f^n_\omega)_* (T^s_\omega \wedge \{\Delta\}). \tag{8.5}$$

The mass $\mathbf{M}((f^n_\omega)_*(T^s_\omega \wedge \{\Delta\}))$ is constant, equal to the mass of the measure $T^s_\omega \wedge \{\Delta\}$; so

$$\mathbf{M}\left(T^s_{\sigma^n(\omega)} \wedge \{f^n_\omega(\Delta)\}\right) = \mathbf{M}((f^n_\omega)^* T^s_{\sigma^n(\omega)}) \mathbf{M}(T^s_\omega \wedge \{\Delta\}). \tag{8.6}$$

By Lemma 5.14, $\mathbf{M}((f^n_\omega)^* T^s_{\sigma^n(\omega)})$ goes exponentially fast to infinity. Since the left hand side is bounded, this shows that $\mathbf{M}(T^s_\omega \wedge \{\Delta\}) = 0$, as desired. □

With Lemma 2.14, the following statement takes care of the first alternative in Theorem 8.2.

Lemma 8.7. *If there is a Borel subset $A \subset \mathcal{X}_+$ of positive measure such that for every $x \in A$, the stable manifold $W^s(x)$ is contained in an algebraic curve, then μ is supported on a Γ_ν-invariant algebraic curve. In addition, for m_+-almost every x, $\overline{W^s(x)}$ is an irreducible rational curve of negative self-intersection.*

Proof. For $x \in A$, let $D(x)$ be the Zariski closure of $W^s(x)$. Discarding a set of measure zero if needed, $W^s(x)$ is biholomorphic to $\mathbf{C}$ so $D(x)$ is a (possibly singular) irreducible rational curve, and $D(x) \setminus W^s(x)$ is reduced to a point. By Lemma 8.6, $T^s_\omega \wedge \{\Delta\} = 0$ for every disk $\Delta \subset W^s(x)$. Since T^s_ω has continuous potentials, $T^s_\omega \wedge \{D(x)\}$ gives no mass to points (see e.g. [33, Lem. 10.13] for the singular case). It follows that $T^s_\omega \wedge \{D(x)\} = 0$, hence $\langle e(\omega) \,|\, [D(x)] \rangle = 0$.

By the Hodge index theorem, either $[D(x)]^2 < 0$ or $[D(x)]$ is proportional to $e(\omega)$, however this latter case would contradict the fact that $e(\omega)$ is $\nu^{\mathbf{N}}$-almost surely irrational (see Theorem 5.8; one could also use that $\mathrm{Cur}(e(\omega))$ is reduced to T^s_ω). Thus, $[D(x)]^2 < 0$.

An irreducible curve with negative self-intersection is uniquely determined by its cohomology class; since $\mathrm{NS}(X;\mathbf{Z})$ is countable, there are only countably many irreducible curves $(D_k)_{k\in\mathbf{N}}$ with negative self intersection. Since $W^s_{\mathrm{loc}}(x) \subset D_k$ if and only if $D(x) = D_k$, and since local stable manifolds vary continuously on the Pesin regular set $\mathcal{R}_\varepsilon$ for every $\varepsilon > 0$, we infer that $\{x \in A\ ;\ \ D(x) = D_k\}$ is measurable for every k. Hence there exists an index k such that $m_+(\{x \in A\ ;\ [D(x)] = [D_k]\}) > 0$. Since x belongs to $W^s_{\mathrm{loc}}(x)$, Fubini's theorem implies that $\mu(D_k) > 0$, and Lemma 8.5 shows that μ is supported on the Γ_ν-orbit of D_k.

Finally, this argument shows that the property $W^s_{\mathrm{loc}}(x) \subset \bigcup_{k\in\mathbf{N}} D_k$, or equivalently that $W^s_{\mathrm{loc}}(x)$ is contained in a rational curve of negative self intersection, is invariant and measurable, so by ergodicity of m_+ it is of full measure. The proof is complete. □

We are now ready to conclude the proof of Theorem 8.2. Let A be the set of Pesin regular points such that $W^s(x)$ is contained in an algebraic curve. From the proof of Lemma 8.7, x belongs to A if and only if $W^s_{\mathrm{loc}}(x)$ is contained in one of the countably many irreducible curves $D_k \subset X$ of negative self-intersection. This condition determines a countable union of closed subsets in the Pesin sets $\mathcal{R}_\varepsilon$, hence A is Borel measurable. By Lemma 8.7, if A has positive m_+-measure then Alternative (a) holds. So, if (a) is not satisfied, $W^s(x)$ is almost surely Zariski dense. Pick such a generic x, which further satisfies the conclusion of Lemma 8.6, and let N be an Ahlfors-Nevanlinna current associated to $W^s(x)$. By Proposition 8.1, $[N]$ is a nef class so $[N]^2 \geqslant 0$. Thus, if we are able to show that $\langle [N] \,|\, [T^s_\omega]\rangle = 0$, we deduce from the Hodge index theorem and $\mathbf{M}(N) = 1$ that $[N] = [T^s_\omega] = e(\omega)$, hence $N = T^s_\omega$ by Theorem 6.12. So, it only remains to prove that $\langle [N] \,|\, [T^s_\omega]\rangle = 0$, or equivalently

$$N \wedge T^s_\omega = 0. \tag{8.7}$$

This is intuitively clear because N is an Ahlfors-Nevanlinna current associated to the entire curve $W^s(x)$ and $T^s_\omega \wedge \{\Delta\} = 0$ for every bounded disk $\Delta \subset W^s(x)$. However, there is a technical difficulty to derive (8.7) from $T^s_\omega \wedge \{\Delta\} = 0$, even if $W^s(x)$ is an increasing union of such disks Δ.

At least two methods were designed to deal with this situation: the first one uses the geometric intersection theory of laminar currents (see [7, 53]), and the second one was developed by Dinh and Sibony in the preprint version of [47] (details are published in [33, §10.4]). Unfortunately these papers only deal with the case of currents of the form $\lim_n \frac{1}{A(R_n)}\phi(\mathbb{D}(0,R_n))$, instead of the Ahlfors-Nevanlinna currents introduced in Section 8.1, which were designed to get the nef property stated in Proposition 8.1. So, we have to explain how to adapt the formalism of [7, 53] to the Ahlfors-Nevanlinna currents of Proposition 8.1.

Following [56] we say that T is an **Ahlfors current** if there exists a sequence (Δ_n) of *unions* of smoothly bounded holomorphic disks such that $\mathrm{length}(\partial\Delta_n) = o\,(\mathbf{M}(\Delta_n))$ and T is the limit as $n \to \infty$ of the sequence of normalized integration currents $\frac{1}{\mathbf{M}(\Delta_n)}\{\Delta_n\}$; here, $\mathrm{length}(\partial\Delta_n)$ is by definition the sum of the lengths of the boundaries of the disks constituting Δ_n, lengths which are computed with respect to the Riemannian metric induced by κ_0. We say furthermore that T is an **injective Ahlfors current** if the disks constituting Δ_n are disjoint or intersect along subsets with relative non-empty interior. By discretizing the integral defining the currents $N(R_n)$ in Proposition (8.1) we see that any Ahlfors-Nevanlinna current is an injective Ahlfors current.

Strongly approximable laminar currents are a class of positive currents introduced in [53] with geometric properties which are well suited for geometric intersection theory. In a nutshell, a current T is a strongly approximable laminar current if for every $r > 0$, there exists a uniformly laminar current T_r (non closed in general) made of disks of size r, and such that $\mathbf{M}(T - T_r) = O(r^2)$. This mass estimate is crucial for the geometric understanding of wedge products of such currents. Since these notions have been studied in a number of papers, we refer to [7, 53, 28] for definitions, the basic properties of these currents, and technical details. This presentation in terms of disks of size r is from [54, §4]. The next lemma is a mild generalization of the methods of [7, §7], [25, §4.3] and [53, §4]. For completeness we provide the details in Appendix B.

Lemma 8.8. *Any injective Ahlfors current T on a projective surface X is a strongly approximable laminar current: if $T = \lim_n \frac{1}{\mathbf{M}(\Delta_n)} \{\Delta_n\}$ where the disks Δ_n have smooth boundaries and* $\mathrm{length}(\partial\Delta_n) = o(\mathbf{M}(\Delta_n))$*, one can construct a family of uniformly laminar currents T_r, whose constitutive disks are limits of pieces of the Δ_n, and such that if S is any closed positive current with continuous potential on X, then $S \wedge T_r$ increases to $S \wedge T$ as r decreases to* 0.

With this lemma at hand, let us conclude the proof of Theorem 8.2. Since X is projective, we can apply the previous lemma to any Ahlfors-Nevanlinna current N associated to $W^s(x)$. In this way we get a family of currents N_r such that $N_r \wedge T^s_\omega$ increases to $N \wedge T^s_\omega$ as r decreases to 0. On the other hand, by Lemma 8.6, the intersection of T^s_ω with every disk contained in $W^s(x)$ vanishes, so again using the fact that T^s_ω has a continuous potential, we infer that if Δ is any disk subordinate to N_r, $T^s_\omega \wedge \{\Delta\} = 0$. Hence $N_r \wedge T^s_\omega = 0$ for every $r > 0$, and finally $N \wedge T^s_\omega = 0$, as desired. □

Proof of Corollary 8.3. Since (b') and (a) are contradictory, (b') implies (b). Conversely assume that μ is hyperbolic, non atomic and supported on a Γ_ν-invariant curve C. Since μ has no atom, it gives full mass to the regular set of C, hence $\Sigma \times T(\mathrm{Reg}(C))$ defines a DF-invariant bundle, and by the Oseledets theorem the ergodic random dynamical system (C, ν, μ) must either have a positive or a negative Lyapunov exponent. If this exponent were positive then μ would be atomic, as observed in Section 7.2.3. Hence, the Lyapunov exponent tangent to C is negative and $W^s(x)$ is contained in C for m_+-almost every x. So (b) implies (b'). □

Proof of Corollary 8.4. Since ν satisfies an exponential moment condition, Theorem 6.17 provides a $\theta > 0$ such that $u_{T^s_\omega}$ is Hölder continuous of exponent θ for $\nu^{\mathbf{N}}$-almost every ω. This implies that T^s_ω gives mass 0 to sets of Hausdorff dimension $< 2 + \theta$ (see [109, Thm 1.7.3]). Since for m_+-almost every x, $\mathrm{Supp}(T^s_\omega) \subset \overline{W^s(x)}$, we infer that $\mathrm{HDim}\big(\overline{W^s(x)}\big) \geqslant 2 + \theta$.

To conclude the proof it is enough to show that $x \mapsto \mathrm{HDim}\big(\overline{W^s(x)}\big)$ is constant on a set of full m_+-measure. Indeed, $x \mapsto \mathrm{HDim}\big(\overline{W^s(x)}\big)$ defines an F_+-invariant function, defined on the full measure set $\mathcal{R}$ of Pesin regular points. If we show that this function is measurable, then the result follows by ergodicity. This is a consequence of the following two facts:

(1) the assignment $x \mapsto \overline{W^s(x)}$ defines a Borel map from $\mathcal{R}$ to the space $\mathcal{K}(X)$ of compact subsets of X;
(2) the function $\mathcal{K}(X) \ni K \mapsto \mathrm{HDim}(K)$ is Borel (see [94, Thm 2.1]).

In both cases $\mathcal{K}(X)$ is endowed with the topology induced by the Hausdorff metric. For the first point, observe that $\mathcal{R}$ is the increasing union of the compact sets $\mathcal{R}_\varepsilon$ so it is Borel; then, on a

Pesin set $\mathcal{R}_\varepsilon$, $x \mapsto \overline{W^s_r(x)}$ is continuous, so $x \mapsto F^{-n}\big(\overline{W^s_r(F^n(x))}\big)$ is continuous as well. Since $F^{-n}\big(\overline{W^s_r(F^n(x))}\big)$ converges to $\overline{W^s(x)}$ in the Hausdorff topology, we infer that $x \mapsto \overline{W^s(x)}$ is a pointwise limit of continuous maps on $\mathcal{R}_\varepsilon$, hence Borel, and finally $x \mapsto \overline{W^s(x)}$ is Borel on $\mathcal{R}$, as claimed. □

9. NO INVARIANT LINE FIELDS

As above, let (X, ν) be a random holomorphic dynamical system satisfying the moment condition (4.1), and μ be an ergodic hyperbolic stationary measure. From §7.2 and §7.5, the local stable manifolds and stable Oseledets directions are $\mathcal{F}^+$-measurable; so, $E^s(\xi, x)$ is naturally identified to $E^s(\omega, x)$ under the projection $(\xi, x) \in \mathcal{X} \mapsto (\omega, x) \in \mathcal{X}_+$, and the same property holds for stable manifolds. Then, m_+-almost every $x \in \mathcal{X}_+$ has a Pesin stable manifold $W^s(x)$ (resp. direction $E^s(x)$). Let $V(x) = V(\omega, x)$ be such a measurable family of objects (stable manifolds, or stable directions, etc); we say that $V(x)$ is **non-random** if for μ-almost every x, $V(\omega, x)$ does not depend on ω, that is, there exists $V(x)$ such that $V_\omega(x) = V(x)$ for $\nu^{\mathbf{N}}$-almost every ω. If V is not non-random, we say that V **depends non-trivially on the itinerary**. Since stable directions depend only on the future, the random versus non-random dichotomy can be analyzed in $\mathcal{X}_+$ or in $\mathcal{X}$. Our purpose in this section is to establish the following result.

Theorem 9.1. *Let (X, ν) be a non-elementary random holomorphic dynamical system on a compact Kähler surface satisfying the Condition* (4.1)*. Let μ be an ergodic and hyperbolic stationary measure, not supported on a Γ_ν-invariant curve. Then the following alternative holds:*

(a) *either the Oseledets stable directions depend non-trivially on the itinerary;*
(b) *or μ is ν-almost surely invariant and $h_\mu(X, \nu) = 0$.*

We shall see that (a) often implies that μ is invariant (see §10). In (b), the almost-sure invariance implies that μ is in fact Γ_ν-invariant (see Remark 4.2). It turns out that (a) and (b) are mutually exclusive. Indeed the main argument of [22] ([4]) implies that the fiber entropy is positive if the Oseledets stable directions depend non-trivially on the itinerary (see [22, Rmk 12.3]). So we get the following:

Corollary 9.2. *Let (X, ν, μ) be as in Theorem 9.1. If μ is not ν-almost surely invariant, then its fiber entropy is positive.*

To motivate the following pages, let us give a heuristic explanation for the fact that $h_\mu(X, \nu) = 0$ when the stable directions are non-random. Fix a stable Pesin partition η^s; according to Corollary 7.16, we have to show that the conditional measures $m(\cdot|\eta^s(x))$ are atomic. Since the stable directions are non-random, the stable manifolds $W^s_{\mathrm{loc}}(\xi, x)$ and $W^s_{\mathrm{loc}}(\xi', x)$ are generically tangent at x. For simplicity, assume that they are tangent for μ-almost all x and *for all pairs* (ξ, ξ'), and that $W^s_{\mathrm{loc}}(\xi, x)$ *depends continuously on* (ξ, x). Take such a generic point x; if $m(\cdot|\eta^s(\xi, x))$ is not atomic, there is a sequence of generic points $x_j \in W^s_{\mathrm{loc}}(\xi, x)$ converging to x in $X \simeq X_\xi$. Fix $\xi' \neq \xi$. Then by continuity $W^s_{\mathrm{loc}}(\xi', x_j)$ converges towards $W^s_{\mathrm{loc}}(\xi', x)$, is disjoint from $W^s_{\mathrm{loc}}(\xi', x)$, and is tangent to $W^s(\xi, x)$ at x_j. This contradicts the following local geometrical result: if C and D are local smooth irreducible curves through the origin in $\mathbb{D}^2$, with

[4]This actually requires checking that the whole proof of [22] can be reproduced in our complex setting: we will come back to this issue in a forthcoming paper. Since we are just using this remark here in Corollary 9.2 we take the liberty to anticipate on that research.

an order of contact equal to k, and if $D_n \subset \mathbb{D}^2$ is a sequence of curves such that $D_n \cap D = \varnothing$ but D_n converges towards D in $\mathbb{D}^2$, then for n sufficiently large, D_n intersects C transversally in k points.

9.1. **Intersection multiplicities.** Let us start with some basics on intersection multiplicities for curves. If V_1 and V_2 are germs of curves at $0 \in \mathbf{C}^2$, with an isolated intersection at 0, the **intersection multiplicity** $\mathrm{inter}_0(V_1, V_2)$ is, by definition, the number of intersection points of V_1 and $V_2 + u$ in N for small generic $u \in \mathbf{C}^2$, where N is a neighborhood of 0 such that $V_1 \cap V_2 \cap N = \{0\}$ (see [38, §12]). It is a positive integer, and $\mathrm{inter}_0(V_1, V_2) = 1$ if and only if V_1 and V_2 are transverse at 0. We extend this definition by setting $\mathrm{inter}_0(V_1, V_2) = 0$ if V_1 or V_2 does not contain 0 and $\mathrm{inter}_0(V_1, V_2) = \infty$ if 0 is not an isolated point of $V_1 \cap V_2$, that is locally V_1 and V_2 share an irreducible component. The intersection multiplicity extends to analytic cycles (that is, formal integer combinations of analytic curves).

Lemma 9.3. *The multiplicity of intersection* $\mathrm{inter}_0(\cdot, \cdot)$ *is upper semi-continuous for the Hausdorff topology on analytic cycles.*

In our situation we will only apply this result to holomorphic disks with multiplicity 1, in which case the topology is just the usual local Hausdorff topology.

Proof. Assume $\mathrm{inter}_0(V_1, V_2) = k$ and $V_{1,n} \to V_1$ (resp. $V_{2,n} \to V_2$) as cycles; we have to show that $\limsup \mathrm{inter}_0(V_{1,n}, V_{2,n}) \leqslant k$. If $k = \infty$ there is nothing to prove. Otherwise, $\{0\}$ is isolated in $V_1 \cap V_2$, so we can fix a neighborhood U of 0 such that $V_1 \cap V_2 \cap U = \{0\}$; then, the result follows from [38, Prop 2 p.141] (stability of proper intersections). ☐

9.2. **Generic intersection multiplicity of stable manifolds.** Recall from §7.5 that for m-almost every $\mathscr{x} = (\xi, x) \in \mathcal{X}$ there exists a local stable manifold $W^s_{r(\mathscr{x})}(\mathscr{x}) \subset X_\xi \simeq X$, depending measurably on $\mathscr{x}$; we might simply denote it by $W^s_{\mathrm{loc}}(\mathscr{x})$.

Let us cover a subset of full measure in $\mathcal{X}$ by Pesin subsets $\mathcal{R}_{\varepsilon_n}$. Take a point $x \in X$, and consider the set of points $((\xi, x), (\zeta, x)) \in \mathcal{R}_{\varepsilon_n} \times \mathcal{R}_{\varepsilon_m}$, for some fixed pair of indices (n, m); Lemma 9.3 shows that the intersection multiplicity $\mathrm{inter}_x \left(W^s_{\mathrm{loc}}(\xi, x), W^s_{\mathrm{loc}}(\zeta, x)\right)$ is an upper semi-continuous function of $((\xi, x), (\zeta, x))$ on that compact set. Thus, the intersection multiplicity $\mathrm{inter}_x \left(W^s_{\mathrm{loc}}(\xi, x), W^s_{\mathrm{loc}}(\zeta, x)\right)$ is a measurable function of (ξ, ζ). Recall that

- the σ-algebra $\mathcal{F}^-$ on $\mathcal{X}$ is generated, modulo m-negligible sets, by the partition into subsets of the form $\Sigma^u_{\mathrm{loc}}(\xi) \times \{x\}$ (see § 7.1, Equation (7.9));
- $\xi \mapsto m_\xi$ is $\mathcal{F}^-$-measurable, i.e $m_\xi = m_\zeta$ almost surely when $\zeta \in \Sigma^u_{\mathrm{loc}}(\xi)$;
- the conditional measures of m with respect to this partition satisfy (see Equation (7.14))

$$m(\,\cdot \mid \mathcal{F}^-(\mathscr{x})) = \nu^{\mathbf{Z}}(\,\cdot \mid \Sigma^u_{\mathrm{loc}}(\xi)) \times \delta_x. \tag{9.1}$$

The next lemma can be seen as a complex analytic version of [22, Lemma 9.9].

Lemma 9.4. *Let $k \geqslant 1$ be an integer. Exactly one of the following assertions holds:*

(a) *for m-almost every $\mathscr{x} = (\xi, x)$ and for $m(\,\cdot \mid \mathcal{F}^-(\xi, x))$-almost every η*

$$\mathrm{inter}_x \left(W^s_{\mathrm{loc}}(\xi, x), W^s_{\mathrm{loc}}(\eta, x)\right) \geqslant k + 1;$$

(b) *for m-almost every $\mathscr{x} = (\xi, x)$ and for $m(\,\cdot \mid \mathcal{F}^-(\xi, x))$-almost every η*

$$\mathrm{inter}_x \left(W^s_{\mathrm{loc}}(\xi, x), W^s_{\mathrm{loc}}(\eta, x)\right) \leqslant k.$$

Proof. The relation defined on $\mathcal{X}$ by $(\xi, x) \simeq_k (\eta, y)$ if $x = y$ and $W^s_{\mathrm{loc}}(\xi, x)$ and $W^s_{\mathrm{loc}}(\eta, y)$ have order of contact at least $k + 1$ at x is an equivalence relation which defines a partition $\mathcal{Q}_k$ of $\mathcal{X}$. We shall see below that $\mathcal{Q}_k$ is a measurable partition. Since $F\colon \mathcal{X} \to \mathcal{X}$ acts by diffeomorphisms on the fibers X of $\mathcal{X}$, we get that $F(\mathcal{Q}_k(x)) = \mathcal{Q}_k(F(x))$ for almost every $x \in \mathcal{X}$. Then, the proof of [22, Lemma 9.9] applies verbatim to show that if

$$m\left(\left\{x \,;\; m(\mathcal{Q}_k(x)|\mathcal{F}^-(x)) > 0\right\}\right) > 0, \tag{9.2}$$

then

$$m\left(\left\{x \,;\; m(\mathcal{Q}_k(x)|\mathcal{F}^-(x)) = 1\right\}\right) = 1. \tag{9.3}$$

This is exactly the desired statement. (This assertion says more than the mere ergodicity of m, which only implies that $m(\{x,\ m(\mathcal{Q}_k(x)|\mathcal{F}^-(x)) > 0\}) = 1$.)

It remains to explain why $\mathcal{Q}_k$ is a measurable partition. For this, we have to express the atoms of $\mathcal{Q}_k$ as the fibers of a measurable map to a Lebesgue space. As for the measurability of the intersection multiplicity, we consider an exhaustion of $\mathcal{X}$ by countably many Pesin sets; then, it is sufficient to work in restriction to some compact set $\mathcal{K} \subset \mathcal{X}$ on which local stable manifolds have uniform size and vary continuously. Taking a finite cover of X by good charts (see § 7.4.2), and restricting $\mathcal{K}$ again to keep only those local stable manifolds which are graphs over some fixed direction, we can also assume that $\pi_X(\mathcal{K})$ is contained in the image of a chart $\Phi_{x_0}\colon U_{x_0} \to V_{x_0} \subset X$ and there is an orthonormal basis (e_1, e_2) such that for every $y \in \mathcal{K}$ the local stable manifold $\pi_X(W^s_{\mathrm{loc}}(y))$ is a graph $\{ze_1 + \psi^s_y(z)e_2\}$ in this chart, for some holomorphic function ψ^s_y on $\mathbb{D}(r)$. Now the map from $\mathcal{K}$ to $\mathbf{C}^2 \times \mathbf{C}^k$ defined by

$$x \longmapsto \left(\Phi_{x_0}^{-1}(\pi_X(x)), (\psi^s_x)'(0), \ldots, (\psi^s_x)^{(k)}(0)\right) \tag{9.4}$$

is continuous. Since the fibers of this map are precisely the (intersection with $\mathcal{K}$ of the) atoms of $\mathcal{Q}_k$, we are done. □

The previous lemma is stated on $\mathcal{X}$ because its proof relies on the ergodic properties of F. However, since stable manifolds depend only on the future, it admits the following more elementary formulation on X:

Corollary 9.5. *Let $k \geqslant 1$ be an integer. Exactly one of the following assertions holds:*

(a) *for μ-almost every $x \in X$ and $(\nu^{\mathbf{N}})^2$-almost every (ω, ω'),*

$$\mathrm{inter}_x\left(W^s_{\mathrm{loc}}(\omega, x), W^s_{\mathrm{loc}}(\omega', x)\right) \geqslant k + 1;$$

(b) *or for μ-almost every $x \in X$ and $(\nu^{\mathbf{N}})^2$-almost every (ω, ω'),*

$$\mathrm{inter}_x\left(W^s_{\mathrm{loc}}(\omega, x), W^s_{\mathrm{loc}}(\omega', x)\right) \leqslant k.$$

Combined with results from the previous sections, this alternative leads to the existence of a finite order of contact k_0 between generic stable manifolds $W^s_{\mathrm{loc}}(\omega, x)$ and $W^s_{\mathrm{loc}}(\omega', x)$:

Lemma 9.6. *There exists a unique integer $k_0 \geqslant 1$ such that for μ-almost every $x \in X$ and $(\nu^{\mathbf{N}})^2$-almost every pair (ω, ω'), $\mathrm{inter}_x(W^s(\omega, x), W^s(\omega', x)) = k_0$.*

Proof. Fix a small $\varepsilon > 0$ and consider a compact set $\mathcal{R}_\varepsilon \subset \mathcal{X}_+$ with $m_+(\mathcal{R}_\varepsilon) \geqslant 1 - \varepsilon$, along which local stable manifolds have size at least $r(\varepsilon)$ and vary continuously. Since by Theorem 8.2 for m_+-a.e. x, the only Nevanlinna current associated to $W^s(x)$ is T^s_ω, we can further assume that this property holds for every $x \in \mathcal{R}_\varepsilon$. Let $A \subset X$ be a subset of full μ-measure on which the alternative of Corollary 9.5 holds for every $k \geqslant 1$. In $\mathcal{X}_+$, consider the measurable partition into fibers of the form $\Omega \times \{x\}$; it corresponds to the partition $\mathcal{F}^-$ in Lemma 9.4. Then, the associated conditional measures $m_+(\cdot \mid \Omega \times \{x\})$ are naturally identified with $\nu^{\mathbf{N}}$. Fix $x \in A$ such that $m_+(\mathcal{R}_\varepsilon | \Omega \times \{x\}) > 0$. Since (X, ν) is non-elementary, Theorems 5.8 and 6.12 provide pairs (ω_1, ω_2) in $(\pi_\Omega(\mathcal{R}_\varepsilon))^2$ for which the currents $T^s_{\omega_1}$ and $T^s_{\omega_2}$ are not cohomologous. By Theorem 8.2 these currents describe respectively the asymptotic distribution of $W^s(\omega_1, x)$ and $W^s(\omega_2, x)$ so we infer that $W^s(\omega_1, x) \neq W^s(\omega_2, x)$ and by the analytic continuation principle it follows that $W^s_{\mathrm{loc}}(\omega_1, x) \neq W^s_{\mathrm{loc}}(\omega_2, x)$. Let $k_1 < \infty$ be the intersection multiplicity of these manifolds at x. Since the intersection multiplicity is upper semi-continuous, we infer that for $\omega'_j \in \mathcal{R}_\varepsilon$ close to ω_j, $j = 1, 2$, $\mathrm{inter}_x(W^s_{\mathrm{loc}}(\omega'_1, x), W^s_{\mathrm{loc}}(\omega'_2, x)) \leqslant k_1$. Thus for $k = k_1$ we are in case (b) of the alternative of Corollary 9.5. Applying then Corollary 9.5 successively for $k = 1, \ldots, k_1$, there is a first integer k_0 for which case (b) holds, and since (a) holds for $k_0 - 1$, we conclude that generically $\mathrm{inter}_x\,(W^s_{\mathrm{loc}}(\omega, x), W^s_{\mathrm{loc}}(\omega', x)) = k_0$. □

9.3. **Transversal perturbations.** The key ingredient in the proof of Theorem 9.1 is the following basic geometric lemma, which is a quantitative refinement of [7, Lemma 6.4].

Lemma 9.7. *Let k be a positive integer. If r and ε are positive real numbers, then there are two positive real numbers $\delta = \delta(k, r, c)$ and $\alpha = \alpha(k, r, c)$ with the following property. Let M_1 and M_2 be two complex analytic curves in $\mathbb{D}(r) \times \mathbb{D}(r) \subset \mathbf{C}^2$ such that*

(i) *M_1 and M_2 are graphs $\{(z, f_j(z))\ ;\ w \in \mathbb{D}_r\}$ of holomorphic functions $f_j \colon \mathbb{D}(r) \to \mathbb{D}(r)$;*
(ii) *$M_1 \cap M_2 = \{(0,0)\}$, and $\mathrm{inter}_{(0,0)}(M_1, M_2) = k$;*
(iii) *the k-th derivative satisfies $\left|(f_1 - f_2)^{(k)}(0)\right| \geqslant c$.*

If $M_3 \subset \mathbb{D}(r) \times \mathbb{D}(r)$ is a complex curve that does not intersect M_1 but is δ-close to M_1 in the C^1-topology , then M_2 and M_3 have exactly k transverse intersection points in $\mathbb{D}(\alpha r) \times \mathbb{D}(\alpha r)$ (i.e. with multiplicity 1).

Proof. Without loss of generality we may assume that $\delta < 1$.

Step 1.– We claim that there exists $\alpha_1 = \alpha_1(k, r, c)$ such that for every $\alpha \leqslant \alpha_1$ and every $z \in \mathbb{D}(\alpha r)$ the following estimates hold:

$$\frac{1}{2}\frac{\left|(f_1 - f_2)^{(k)}(0)\right|}{k!}\,|z|^k \leqslant |f_1(z) - f_2(z)| \leqslant \frac{3}{2}\frac{\left|(f_1 - f_2)^{(k)}(0)\right|}{k!}\,|z|^k \tag{9.5}$$

$$\frac{1}{2}\frac{\left|(f_1 - f_2)^{(k)}(0)\right|}{(k-1)!}\,|z|^{k-1} \leqslant \left|f'_1(z) - f'_2(z)\right| \leqslant \frac{3}{2}\frac{\left|(f_1 - f_2)^{(k)}(0)\right|}{(k-1)!}\,|z|^{k-1}\,. \tag{9.6}$$

Indeed put $g = f_1 - f_2 = \sum_{m \geqslant k} g_m z^m$. Assumptions (i) and (iii) give$|g(z)| \leqslant 2r$ on $\mathbb{D}(r)$, and $g^{(k)}(0) \neq 0$. By the Cauchy estimates, $|g_n| \leqslant 2r^{1-n}$ for all $n \geqslant 0$. Then on $\mathbb{D}(\alpha r)$ we get

$$\left| g(z) - \frac{g^{(k)}(0)}{k!} z^k \right| \leqslant 2r \left(\frac{|z|}{r}\right)^{k+1} \left(1 - \frac{|z|}{r}\right)^{-1} \leqslant 2r^{1-k} \frac{\alpha}{1-\alpha}\,|z|^k\,.$$

There exists $\alpha_1(k, r, c)$ such that as soon as $\alpha \leqslant \alpha_1$, the right hand side of this inequality is smaller than $c\,|z|^k/2$; hence Estimate (9.5) follows. The same argument applies for (9.6) because

$$\left| g'(z) - \frac{g^{(k)}(0)}{(k-1)!} z^{k-1} \right| \leqslant 4(k+1) \left(\frac{|z|}{r} \right)^k \left(1 - \frac{|z|}{r} \right)^{-2} \leqslant 4(k+1) r^{1-k} \frac{\alpha}{(1-\alpha)^2} |z|^{k-1} .$$

Step 2.– For every $\alpha \leqslant \alpha_1$, if $\delta < c(\alpha r)^k/2k!$, M_2 and M_3 have exactly k intersection points, counted with multiplicities, in $\mathbb{D}(\alpha r) \times \mathbb{D}(\alpha r)$.

Indeed, the intersection points of M_3 and M_2 correspond to the solutions of the equation $f_3 = f_2$. To locate its roots, note that on the circle $\partial \mathbb{D}(\alpha r)$, the Inequality (9.5) implies

$$|f_1 - f_2| \geqslant \frac{1}{2} \frac{c}{k!} (\alpha r)^k. \tag{9.7}$$

Since $|f_1 - f_3| < \delta$, the choice $\delta < c(\alpha r)^k/2k!$ is tailored to assure that the hypothesis of the Rouché theorem is satisfied in $\mathbb{D}(\alpha r)$; so, counted with multiplicities, there are k solutions to the equation $f_3 = f_2$ on that disk. Furthermore by the Schwarz lemma $|f_2| < \alpha r$ on $\mathbb{D}(\alpha r)$ so the corresponding intersection points between M_2 and M_3 are contained in $\mathbb{D}(\alpha r) \times \mathbb{D}(\alpha r)$.

If $k = 1$ the proof is already complete at this stage, so from now on we assume $k \geqslant 2$.

Step 3.– Set $\delta_0 = |f_3(0)|$, and note that $\delta_0 \leqslant \delta$. Then for every $\alpha \leqslant 1/2$, in $\mathbb{D}(\alpha r)$ we have

$$\delta_0^{\frac{1+\alpha}{1-\alpha}} \leqslant |f_1(z) - f_3(z)| \leqslant \delta_0^{\frac{1-\alpha}{1+\alpha}} \tag{9.8}$$

$$\left| f_1'(z) - f_3'(z) \right| \leqslant \frac{1}{\alpha r} \delta_0^{\frac{1-2\alpha}{1+2\alpha}} . \tag{9.9}$$

For this, recall the Harnack inequality: for any negative harmonic function in $\mathbb{D}$

$$\frac{1-|\zeta|}{1+|\zeta|} \leqslant \frac{u(\zeta)}{u(0)} \leqslant \frac{1+|\zeta|}{1-|\zeta|}. \tag{9.10}$$

Since $f_1 - f_3$ does not vanish and $|f_1 - f_3| \leqslant \delta < 1$ in $\mathbb{D}(r)$, the function $\log |f_1 - f_3|$ is harmonic and negative there. Thus for $\alpha \leqslant 1/2$, the Harnack inequality can be applied to $\zeta \mapsto (f_1 - f_3)(r\zeta)$ in $\mathbb{D}$: this gives (9.8). Likewise, we infer that

$$\delta_0^{\frac{1+2\alpha}{1-2\alpha}} \leqslant |f_1(z) - f_3(z)| \leqslant \delta_0^{\frac{1-2\alpha}{1+2\alpha}} \tag{9.11}$$

in $\mathbb{D}(2\alpha r)$, and (9.9) follows from the Cauchy estimate $\|g'\|_{\mathbb{D}(\alpha r)} \leqslant (\alpha r)^{-1} \|g\|_{\mathbb{D}(2\alpha r)}$.

Step 4.– We now conclude the proof. Fix $\alpha = \alpha(k, r, c)$ such that $\alpha \leqslant \alpha_1$ and

$$\beta(\alpha) := \frac{1-2\alpha}{1+2\alpha} - \frac{k-1}{k} \times \frac{1+\alpha}{1-\alpha} > 0. \tag{9.12}$$

(This will be our final choice for α.) Fix $\delta < c(\alpha r)^k/2k!$ and consider a solution z_0 of the equation $f_2(z) = f_3(z)$ in $\mathbb{D}(\alpha r)$ provided by Step 2. The transversality of M_2 and M_3 at $(z_0, f_2(z_0))$ is equivalent to $f_3'(z_0) \neq f_2'(z_0)$, so we only need

$$\left| (f_3 - f_1)'(z_0) \right| < \left| (f_2 - f_1)'(z_0) \right| . \tag{9.13}$$

Since $(f_1 - f_3)(z_0) = (f_1 - f_2)(z_0)$, combining the right hand side of Inequality (9.5) and the left hand side of Inequality 9.8, we get that

$$\frac{3}{2} \frac{\left| (f_1 - f_2)^{(k)}(0) \right|}{k!} |z_0|^k \geqslant \delta_0^{\frac{1+\alpha}{1-\alpha}}, \tag{9.14}$$

thus

$$|z_0| \geqslant \delta_0^{\frac{1}{k}\frac{1+\alpha}{1-\alpha}} \left(\frac{2k!}{3}\right)^{\frac{1}{k}} \left|(f_1 - f_2)^{(k)}(0)\right|^{-\frac{1}{k}} \tag{9.15}$$

Hence by (9.6) we get that

$$\begin{aligned} \left|(f_2 - f_1)'(z_0)\right| &\geqslant \frac{1}{2(k-1)!} \left(\frac{2k!}{3}\right)^{\frac{k-1}{k}} \delta_0^{\frac{k-1}{k}\frac{1+\alpha}{1-\alpha}} \left|(f_1 - f_2)^{(k)}(0)\right|^{\frac{1}{k}} \\ &\geqslant \frac{1}{2(k-1)!} \left(\frac{2k!}{3}\right)^{\frac{k-1}{k}} \delta_0^{\frac{k-1}{k}\frac{1+\alpha}{1-\alpha}} c^{\frac{1}{k}}. \end{aligned} \tag{9.16}$$

On the other hand by Estimate (9.9)

$$\left|(f_3 - f_1)'(z_0)\right| \leqslant \frac{1}{\alpha r} \delta_0^{\frac{1-2\alpha}{1+2\alpha}} \tag{9.17}$$

Since $\delta_0 \leqslant \delta$, we only need to impose one more constraint on δ (together with $\delta < c(\alpha r)^k/2k!$), namely

$$\delta^{\beta(\alpha)} < \frac{1}{2(k-1)!} \left(\frac{2k!}{3}\right)^{\frac{k-1}{k}} c^{\frac{1}{k}} r\alpha, \tag{9.18}$$

to get the desired inequality $|(f_3 - f_1)'(z_0)| < |(f_2 - f_1)'(z_0)|$. □

Let Δ_1 and Δ_2 be two disks of size r at $x \in X$, which are tangent at x; let $e_1 \in T_xX$ be a unit vector in $T_x\Delta_1 = T_x\Delta_2$ and e_2 a unit vector orthogonal to e_1 for κ_0. Then, in the chart Φ_x, Δ_1 and Δ_2 are graphs $\{ze_1 + \psi_i(z)e_2\}$ of holomorphic functions $\psi_i \colon \mathbb{D}(r) \to \mathbb{D}(r)$, $i = 1, 2$, such that $\psi_i(0) = 0$ and $\psi_i'(0) = 0$. If $\mathrm{inter}_x(\Delta_1, \Delta_2) = k$, then for $j = 1, \ldots, k-1$ one has $\psi_1^{(j)}(0) = \psi_2^{(j)}(0)$ and $\psi_1^{(k)}(0) \neq \psi_2^{(k)}(0)$. We define the k**-osculation** of Δ_1 and Δ_2 at x to be

$$\mathrm{osc}_{k,x,r}(\Delta_1, \Delta_2) = \left|\psi_1^{(k)}(0) - \psi_2^{(k)}(0)\right|. \tag{9.19}$$

If $s \leqslant r$ and we consider Δ_1 and Δ_2 as disks of size s, then $\mathrm{osc}_{k,x,s}(\Delta_1, \Delta_2) = \mathrm{osc}_{k,x,r}(\Delta_1, \Delta_2)$. Thus, $\mathrm{osc}_{k,x,r}(\Delta_1, \Delta_2)$ does not depend on r, so we may denote this osculation number by $\mathrm{osc}_{k,x}(\Delta_1, \Delta_2)$. With this terminology, Lemma 9.7 directly implies the following corollary.

Corollary 9.8. *Let k be a positive integer, and r and c be positive real numbers. Then, there are two positive real numbers δ and α, depending on (k, r, c), satisfying the following property. Let Δ_1 and Δ_2 be two holomorphic disks of size r through x, such that $\mathrm{inter}_x(\Delta_1, \Delta_2) = k$ and $\mathrm{osc}_{k,x}(\Delta_1, \Delta_2)) \geqslant c$. Let Δ_3 be a holomorphic disk of size r such that Δ_3 is δ-close to Δ_1 in the C^1-topology but $\Delta_3 \cap \Delta_1 = \varnothing$. Then Δ_3 intersects Δ_2 transversely in exactly k points in $B_X(x, \alpha r)$.*

The following lemma follows directly from the first step of the proof of Lemma 9.7.

Lemma 9.9. *Let k be a positive integer, and r and c be positive real numbers. Then there exists a constant β depending only on (r, k, c) such that if Δ_1 and Δ_2 are two holomorphic disks of size r through x, such that $k = \mathrm{inter}_x(\Delta_1, \Delta_2)$ and $\mathrm{osc}_{k,x}(\Delta_1, \Delta_2)) \geqslant c$, then x is the only point of intersection between Δ_1 and Δ_2 in the ball $B_X(x, \beta r)$.*

9.4. **Proof of Theorem 9.1.** Before starting the proof, we record the following two facts from elementary measure theory:

Lemma 9.10. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\delta \in (0,1)$.*

(1) *If φ is a measurable function with values in $[0,1]$ and such that $\int \varphi \, d\mathbb{P} \geqslant 1-\delta$, then*

$$\mathbb{P}\left(\left\{x \; ; \; \varphi(x) \geqslant 1-\sqrt{\delta}\right\}\right) \geqslant 1-\sqrt{\delta}.$$

(2) *If A_j is a sequence of measurable subsets such that $\mathbb{P}(A_j) \geqslant 1-\delta$ for every j, then $\mathbb{P}(\limsup A_j) \geqslant 1-\delta$.*

Let us now prove Theorem 9.1. If the integer k_0 of Lemma 9.6 is equal to 1, then Pesin stable manifolds corresponding to different itineraries at a μ-generic point $x \in X$ are generically transverse; hence, we are in case (a) of the theorem –note that the conclusion is actually stronger than mere non-randomness. So, we now assume $k_0 > 1$ and we prove that μ is almost surely invariant and that its entropy is equal to zero.

Step 1.– First, we construct a subset $\mathcal{G}_\varepsilon$ of "good points" in $\mathcal{X}$.

As described in Section 7.1.2, the atoms of $\mathcal{F}^-$ are the sets $\mathcal{F}^-(\mathcal{x}) = \Sigma^u_{\mathrm{loc}}(\xi) \times \{x\}$ and the measures $m(\,\cdot\,|\mathcal{F}^-(\mathcal{x}))$ can be naturally identified to $\nu^{\mathbf{N}}$ under the natural projections $\mathcal{F}^-(\mathcal{x}) \xrightarrow{\sim} \Sigma^u_{\mathrm{loc}}(\xi) \xrightarrow{\sim} \Omega$. For notational simplicity we denote these measures by $m^{\mathcal{F}^-}_{\mathcal{x}}$.

For a small $\varepsilon > 0$, let $\mathcal{R}_\varepsilon \subset \mathcal{X}$ be a compact subset with $m(\mathcal{R}_\varepsilon) > 1-\varepsilon$, along which local stable manifolds have size at least $2r(\varepsilon)$ and vary continuously. Since $\int m^{\mathcal{F}^-}_{\mathcal{x}}(\mathcal{R}_\varepsilon)\, dm(\mathcal{x}) \geqslant 1-\varepsilon$, by Lemma 9.10 (1) we can select a compact subset $\mathcal{R}'_\varepsilon \subset \mathcal{R}_\varepsilon$ with $m(\mathcal{R}'_\varepsilon) \geqslant 1-\sqrt{\varepsilon}$ such that for every $\mathcal{x} \in \mathcal{R}'_\varepsilon$ one has $m^{\mathcal{F}^-}_{\mathcal{x}}(\mathcal{R}_\varepsilon) \geqslant 1-\sqrt{\varepsilon}$.

By assumption, $\mathrm{inter}_x(W^s_{\mathrm{loc}}(y_1), W^s_{\mathrm{loc}}(y_2)) = k_0$ for m-almost every $\mathcal{x} = (\xi, x) \in \mathcal{R}'_\varepsilon$ and for $(m^{\mathcal{F}^-}_{\mathcal{x}} \otimes m^{\mathcal{F}^-}_{\mathcal{x}})$-almost every pair of points $(y_1, y_2) \in (\mathcal{F}^-(\mathcal{x}) \cap \mathcal{R}_\varepsilon)^2$. Then there exists $\mathcal{R}''_\varepsilon \subset \mathcal{R}'_\varepsilon$ of measure at least $1-2\sqrt{\varepsilon}$ and a constant $c(\varepsilon) > 0$ such that

$$\mathrm{osc}_{k_0, x, r(\varepsilon)}(W^s_{\mathrm{loc}}(y_1), W^s_{\mathrm{loc}}(y_2)) \geqslant c(\varepsilon) \tag{9.20}$$

for every $\mathcal{x} = (\xi, x) \in \mathcal{R}''_\varepsilon$ and all pairs (y_1, y_2) in a subset $A_{\varepsilon, \mathcal{x}} \subset (\mathcal{F}^-_{\mathcal{x}} \cap \mathcal{R}_\varepsilon)^2$ depending measurably on $\mathcal{x}$ and of measure

$$(m^{\mathcal{F}^-}_{\mathcal{x}} \otimes m^{\mathcal{F}^-}_{\mathcal{x}})(A_{\varepsilon,\mathcal{x}}) \geqslant 1-4\sqrt{\varepsilon} \tag{9.21}$$

(we just used $(m^{\mathcal{F}^-}_{\mathcal{x}} \otimes m^{\mathcal{F}^-}_{\mathcal{x}})((\mathcal{F}^-_{\mathcal{x}} \cap \mathcal{R}_\varepsilon)^2) \geqslant (1-\sqrt{\varepsilon})^2 > 1-4\sqrt{\varepsilon}$). Finally, Fubini's theorem and Lemma 9.10 (1) provide a set $\mathcal{G}_\varepsilon \subset \mathcal{R}''_\varepsilon$ such that

(a) $m(\mathcal{G}_\varepsilon) \geqslant 1-2\varepsilon^{1/4}$

(b) for every $\mathcal{x} \in \mathcal{G}_\varepsilon$, $W^s_{\mathrm{loc}}(\mathcal{x})$ has size $2r(\varepsilon)$;

(c) for every $\mathcal{x} \in \mathcal{G}_\varepsilon$, there exists a measurable set $\mathcal{G}_{\varepsilon,\mathcal{x}} \subset \mathcal{F}^-_{\mathcal{x}}$ with $m^{\mathcal{F}^-}_{\mathcal{x}}(\mathcal{G}_{\varepsilon,\mathcal{x}}) \geqslant 1-2\varepsilon^{1/4}$ such that for every y in $\mathcal{G}_{\varepsilon,\mathcal{x}}$, $W^s_{\mathrm{loc}}(y)$ has size $\geqslant r(\varepsilon)$ and, viewed as a subset of X,
   - it is tangent to $W^s_{\mathrm{loc}}(\mathcal{x})$ to order k_0 at x,
   - $\mathrm{osc}_{k_0,x,r(\varepsilon)}(W^s_{\mathrm{loc}}(\mathcal{x}), W^s_{\mathrm{loc}}(y)) \geqslant c(\varepsilon)$.

Note that $\mathcal{x} \notin \mathcal{R}_{\varepsilon,\mathcal{x}}$: indeed, when the local stable manifolds vary continuously, one can think of $A_{\varepsilon,\mathcal{x}}$ as the complement of a small neighborhood of the diagonal in $\Omega \times \Omega$.

Step 2.– To make the argument more transparent, we first show that the fiber entropy vanishes.

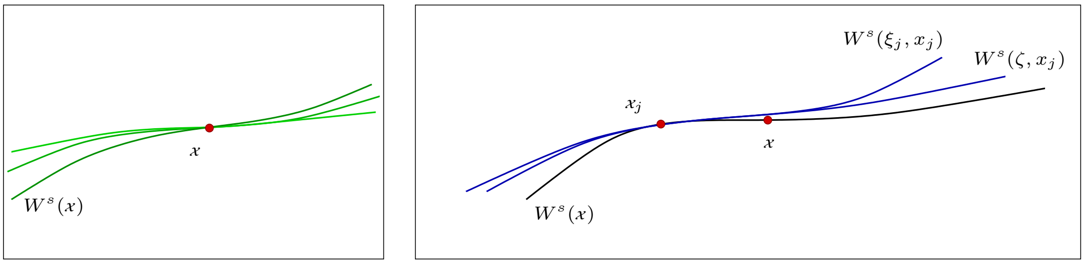

FIGURE 1. On the left, a generic point x with the local stable manifolds $W^s_{\rm loc}(\xi_i, x)$ for distinct $(\xi_i)_{i\geqslant 0}$ (see Step 1). On the right, the choice of the sequence (ζ, x_j) gives a family of local stable manifolds (see Step 2).

Let η^s be a Pesin partition subordinate to local stable manifolds in $\mathcal{X}$. By Corollary 7.16 it is enough to show that for m-almost every x, $m(\cdot|\eta^s(\mathrm{x}))$ is atomic (hence concentrated at x). Assume by contradiction that this is not the case. Therefore for $\varepsilon > 0$ small enough there exists $\mathrm{x} = (\xi, x) \in \mathcal{G}_\varepsilon$ such that $m(\cdot|\eta^s(\mathrm{x}))_{|\eta^s(\mathrm{x})\cap\mathcal{G}_\varepsilon}$ is non-atomic, and there exists an infinite sequence of points $\mathrm{x}_j = (\xi, x_j)$ in $\mathcal{G}_\varepsilon \cap \eta^s(\mathrm{x})$ converging to x. Then with $\mathcal{G}_{\varepsilon,\star}$ as in Property (c) of the definition of $\mathcal{G}_\varepsilon$, we have $m^{\mathcal{F}^-}_{\mathrm{x}_j}(\mathcal{G}_{\varepsilon,x_j}) \geqslant 1 - 2\varepsilon^{1/4}$ for every j.

Identifying all $\mathcal{F}^-(\mathrm{x}_j)$ with $\Sigma^u_{\rm loc}(\xi)$, by Lemma 9.10 (2) we can find $\zeta \in \Sigma^u_{\rm loc}(\xi)$ such that (ζ, x_j) belongs to $\mathcal{G}_{\varepsilon,(\zeta,x_j)}$ for infinitely many j's. Along this subsequence the local stable manifolds $W^s_{\rm loc}(\zeta, x_j)$ form a sequence of disks of uniform size $r = 2r(\varepsilon)$ at x_j. Two such local stable manifolds are either pairwise disjoint or coincide along an open subset because they are associated to the same itinerary ζ.

Let us now use the notation from Corollary 9.8 and Lemma 9.9. We know that $W^s_{r(\varepsilon)}(\zeta, x_j)$ is tangent to $W^s_{r(\varepsilon)}(\xi, x)$ at x_j to order k_0, with $\mathrm{osc}_{k_0,x_j,r(\varepsilon)}(W^s_{r(\varepsilon)}(x), W^s_{r(\varepsilon)}(\zeta, x_j)) \geqslant c(\varepsilon)$; so, by Lemma 9.9, $W^s_{r(\varepsilon)}(\zeta, x_j)$ and $W^s_{r(\varepsilon)}(\zeta, x_{j'})$ are disjoint as soon as $\mathrm{dist}_X(x_j, x_{j'}) < \beta r(\varepsilon)$. Finally, if j and j' are large enough, then $\mathrm{dist}_X(x_j, x_{j'}) < \alpha r(\varepsilon)$ and the C^1 distance between $W^s_{r(\varepsilon)}(\zeta, x_j)$ and $W^s_{r(\varepsilon)}(\zeta, x_{j'})$ is smaller than δ; thus, Corollary 9.8 asserts that $W^s_{r(\varepsilon)}(\zeta, x_j)$ and $W^s_{r(\varepsilon)}(\zeta, x_{j'})$ cannot both be tangent to $W^s_{r(\varepsilon)}(\xi, x)$. This is a contradiction, and we conclude that the fiber entropy of m vanishes.

Step 3.– We now prove the almost sure invariance.

As in [22, Eq. (11.1)] we consider a measurable partition $\mathcal{P}$ of $\mathcal{X}$ with the property that for m-almost every (ξ, x),

$$\Sigma^s_{\rm loc}(\xi) \times W^s_{r(\xi,x)}(\xi, x) \subset \mathcal{P}(\xi, x) \subset \Sigma^s_{\rm loc}(\xi) \times W^s(\xi, x). \tag{9.22}$$

The existence of such a partition is guaranteed, for instance, by Lemma 7.13. By [22, Prop 11.1]([5]), to show that μ is almost surely invariant it is enough to prove that:

$$\text{for } m \text{ almost every } \xi,\ m(\,\cdot\,|\mathcal{P}(\xi, x)) \text{ is concentrated on } \Sigma^s_{\rm loc}(\xi) \times \{x\}\,. \tag{9.23}$$

[5]Brown and Rodriguez-Hertz make it clear that this result holds for an arbitrary smooth random dynamical system on a compact manifold.

By contradiction, assume that (9.23) fails. By contraction along the stable leaves, it follows that almost surely $\Sigma^s_{\mathrm{loc}}(\xi) \times \{x\}$ is contained in

$$\operatorname{Supp}\left(m(\cdot|\mathcal{P}(\xi,x))_{|\mathcal{P}(\xi,x)\setminus\Sigma^s_{\mathrm{loc}}(\xi)\times\{x\}}\right) \tag{9.24}$$

(this is identical to the argument of Corollary 7.16). In particular for small ε we can find $\mathscr{x} = (\xi, x) \in \mathcal{G}_\varepsilon$ and a sequence of points $\mathscr{x}_j = (\xi_j, x_j) \in \mathcal{G}_\varepsilon$ such that $\mathscr{x}_j$ belongs to $\mathcal{P}(\mathscr{x}) \cap \mathcal{G}_\varepsilon$, $x_j \neq x$ and (x_j) converges to x in X. We can also assume that the x_j are all distinct. By definition of $\mathcal{G}_\varepsilon$, $m^{\mathcal{F}^-}_{\mathscr{x}_j}\left(\mathcal{G}_{\varepsilon,x_j}\right) \geqslant 1 - 2\varepsilon^{1/4}$ for every j. For $(\xi,\zeta) \in \Sigma^2$, set

$$[\xi,\zeta] = \Sigma^u_{\mathrm{loc}}(\xi) \cap \Sigma^s_{\mathrm{loc}}(\zeta); \tag{9.25}$$

that is, $[\xi,\zeta]$ is the itinerary with the same past as ξ and the same future as ζ. As above, identifying the atoms of the partition $\mathcal{F}^-$ with Ω, Lemma 9.10 (2) provides an infinite subsequence (j_ℓ) and for every ℓ an itinerary $\zeta_{j_\ell} \in \Sigma^u_{\mathrm{loc}}(\xi_{j_\ell})$ such that $y_{j_\ell} := (\zeta_{j_\ell}, x_{j_\ell})$ belongs to $\mathcal{G}_{\varepsilon,x_{j_\ell}}$ and all the ζ_{j_ℓ} have the same future, that is ζ_{j_ℓ} is of the form $[\xi_{j_\ell},\zeta]$ for a fixed ζ. By definition,

$$\operatorname{inter}_{x_{j_\ell}}(W^s_{\mathrm{loc}}(\mathscr{x}_{j_\ell}), W^s_{\mathrm{loc}}(y_{j_\ell})) = k_0 \tag{9.26}$$

$$\operatorname{osc}_{k_0,x_{j_\ell},r(\varepsilon)}(W^s_{\mathrm{loc}}(\mathscr{x}_{j_\ell}), W^s_{\mathrm{loc}}(y_{j_\ell})) \geqslant c(\varepsilon). \tag{9.27}$$

In addition the disks $\pi_X(W^s_{\mathrm{loc}}(y_{j_\ell}))$ are pairwise disjoint or locally coincide because the x_{j_ℓ} are distinct and the ζ_{j_ℓ} have the same future. Moreover, since $\mathscr{x}_{j_\ell}$ belongs to $\mathcal{P}(\mathscr{x})$, $W^s(\mathscr{x}_{j_\ell})$ coincides with $W^s(\mathscr{x})$. Therefore, the $\pi_X(W^s_{\mathrm{loc}}(y_{j_\ell}))$ form a sequence of disjoint disks of size $2r(\varepsilon)$ at x_j, all tangent to $\pi_X(W^s_{\mathrm{loc}}(\mathscr{x}))$ to order k_0, with osculation bounded from below by $c(\varepsilon)$. Since this sequence of disks is continuous and (x_j) converges towards x, Lemma 9.9 and Corollary 9.8 provide a contradiction, exactly as in Step 2. This completes the proof of the theorem. □

10. Stiffness

Here we study Furstenberg's stiffness property for automorphisms of compact Kähler surfaces, thereby proving Theorem A. Our first results in §10.3 deal with elementary subgroups of $\mathsf{Aut}(X)$. The argument relies on the classification of such elementary groups together with general group-theoretic criteria for stiffness; these criteria are recalled in § 10.1 and 10.2. Theorem 10.10 concerns the much more interesting case of non-elementary subgroups; its proof combines all results of the previous sections with the work of Brown and Rodriguez-Hertz [22].

10.1. **Stiffness.** Following Furstenberg [65], a random dynamical system (X,ν) is **stiff** if any ν-stationary measure is almost surely invariant; equivalently, every ergodic stationary measure is almost surely invariant. This property can conveniently be expressed in terms of ν-harmonic functions on Γ. Indeed if $\xi\colon X \to \mathbf{R}$ is a continuous function and μ is ν-stationary, then $\Gamma \ni g \mapsto \int_X \xi(gx)\, d\mu(x)$ is a bounded, continuous, right ν-harmonic function on Γ; thus proving that μ is invariant amounts to proving that such harmonic functions are constant. Stiffness can also be defined for group actions: a group Γ **acts stiffly** on X if and only if (X,ν) is stiff for every probability measure ν on Γ whose support generates Γ; in this definition, the measures ν can also be restricted to specific families, for instance symmetric finitely supported measures, or measures satisfying some moment condition. There are some general criteria ensuring stiffness directly from the properties of Γ. A first case is when G is a topological group acting continuously on X and $\Gamma \subset G$ is relatively compact. Then Γ acts stiffly on X: this follows from the maximum principle for harmonic functions on $\overline{\Gamma}$ (see also [65, Thm 3.5]). Another important case for us is that of Abelian and nilpotent groups.

Theorem 10.1. *Let G be a locally compact, second countable, topological group. Let ν be a probability measure on G. If G is nilpotent of class $\leqslant 2$, then any measurable, ν-harmonic, and bounded function $\varphi\colon G \to \mathbf{R}$ is constant; thus, every measurable action of such a group is stiff.*

This is due to Dynkin-Malyutov and to Guivarc'h; we refer to [72] for a proof ([6]). The case of Abelian groups is the famous Blackwell-Choquet-Deny theorem. We shall apply Theorem 10.1 to subgroups $A \subset \mathsf{Aut}(X)$; what we implicitly do is first replace A by its closure in $\mathsf{Aut}(X)$ to get a locally compact group, and then apply the theorem to this group.

10.2. **Subgroups and hitting measures.** A basic tool is the **hitting measure** on a subgroup, which we briefly introduce now (see [12, Chap. 5] for details). Let G be a locally compact second countable topological group. A notion of length can be defined in this context as follows: given a neighborhood U of the unit element, for any $g \in G$, $\mathrm{length}_U(g)$ is the least integer $n \geqslant 1$ such that $g \in U^n$. By definition a probability measure ν on G has a finite first moment (resp. a finite exponential moment) if $\int \mathrm{length}_U(g)\, d\nu(g) < \infty$ (resp. if $\int \exp(\alpha\, \mathrm{length}_U(g))\, d\nu(g) < \infty$ for some $\alpha > 0$). This condition does not depend on the choice of U.

Let ν be a probability measure on G, and consider the left random walk on G governed by ν. Given a subgroup $H \subset G$, for $\omega = (g_i) \in G^{\mathbf{N}}$, define the hitting time

$$T(\omega) = T_H(\omega) := \min\{n \geqslant 1\ ;\ g_n \cdots g_1 \in H\}\,. \tag{10.1}$$

If T is almost surely finite we say that H is **recurrent** and the distribution of $g_{T(\omega)} \cdots g_1$ is by definition the **hitting measure** of ν on H, which will be denoted ν_H. The key property of ν_H is that if $\varphi : G \to \mathbf{R}$ is a ν-harmonic function, then $\varphi|_H$ is also ν_H-harmonic. Therefore, if μ is a ν-stationary measure, then it is also ν_H-stationary. Conversely, any bounded ν_H-harmonic function h on H admits a unique extension $\widetilde{h}$ to a bounded ν-harmonic function on G; this extension is defined by the formula

$$\widetilde{h}(x) = \mathbb{E}_x(h(g_{T_{x,H}(\omega)} \cdots g_1 x)) = \int h(g_{T_{x,H}(\omega)} \cdots g_1 x)\, d\nu^{\mathbf{N}}(\omega) \tag{10.2}$$

where the stopping time $T_{x,H}$ is defined by $T_{x,H}(\omega) = \min\{n \geqslant 0\ ;\ g_n \cdots g_1 x \in H\}$. The uniqueness comes from Doob's optional stopping theorem, which asserts that if $(M_t)_{t\geqslant 0}$ is a bounded martingale and T is a stopping time which is almost surely finite then $\mathbb{E}(M_T) = \mathbb{E}(M_0)$. Thus, any bounded ν-harmonic function h on G satisfies Formula (10.2).

If $[G : H] < \infty$ then H is recurrent and its stopping time admits an exponential moment. It follows that ν_H has a finite first (resp. exponential) moment if and only if ν does.

Likewise, assume that H is a normal subgroup of G with G/H isomorphic to $\mathbf{Z}$, and that ν is symmetric with a finite first moment. Then, the projection $\overline{\nu}$ of ν on G/H is symmetric with a finite first moment, so the random walk governed by $\overline{\nu}$ on $G/H \simeq \mathbf{Z}$ is recurrent (see the Chung-Fuchs Theorem in [55, §5.4] or [39]) and H is recurrent.

[6]The proof in [102] is not correct, because Lemma 2.5 there is false. But the proof works perfectly, and is quite short, if the support of ν is countable or if the nilpotency class of the group is $\leqslant 2$.

Lemma 10.2. *Let ν be a probability measure on $\mathsf{Aut}(X)$ and Γ' be a closed subgroup which is recurrent for the random walk induced by ν. Let ν' be the induced measure on Γ'. If (X, ν') is stiff then (X, ν) is stiff as well. This holds in particular if:*

(i) *either $[\Gamma_\nu : \Gamma'] < \infty$*
(ii) *or Γ' is a normal subgroup of Γ_ν with Γ_ν/Γ' isomorphic to $\mathbf{Z}$, and ν is symmetric with a finite first moment.*

Proof. Let μ be a ν-stationary measure on X. Then μ is ν'-stationary, hence by stiffness it is Γ'-invariant. Therefore for every Borel set $B \subset X$, the function $\Gamma \ni g \mapsto \mu(g^{-1}B)$ is a bounded ν-harmonic function which is constant on Γ' so by the uniqueness of harmonic extension it is constant, and ν is Γ-invariant. □

10.3. **Elementary groups.** Recall that $\mathsf{Aut}(X)$ is a topological group for the topology of uniform convergence and is in fact a complex Lie group (with possibly infinitely many connected components). Let $\mathsf{Aut}(X)^\circ$ be the connected component of the identity in $\mathsf{Aut}(X)$ and

$$\mathsf{Aut}(X)^\# = \mathsf{Aut}(X)/\mathsf{Aut}(X)^\circ. \tag{10.3}$$

Let $\rho : \mathsf{Aut}(X) \to \mathsf{GL}(H^*(X; \mathbf{Z}))$ be the natural homomorphism; its image is $\mathsf{Aut}(X)^* = \rho(\mathsf{Aut}(X))$ (see § 2.1.1); is kernel contains $\mathsf{Aut}(X)^\circ$ and a theorem of Lieberman [90] shows that $\mathsf{Aut}(X)^\circ$ has finite index in $\ker(\rho)$. If Γ is a subgroup of $\mathsf{Aut}(X)$, we set $\Gamma^* = \rho(\Gamma)$.

Theorem 10.3. *Let X be a compact Kähler surface. Let ν be a symmetric probability measure on $\mathsf{Aut}(X)$ satisfying the moment condition* (4.1)*. If Γ_ν is elementary and Γ_ν^* is infinite, then (X, ν) is stiff.*

Note that stiffness can fail when Γ_ν^* is finite: see Example 10.4 below. The proof relies on the classification of elementary subgroups of $\mathsf{Aut}(X)$ (see [28, Thm 3.2], [59]): if Γ_ν is elementary and Γ_ν^* is infinite there exists a finite index subgroup $A^* \subset \Gamma_\nu^*$ which is

(a) either cyclic and generated by a loxodromic map;
(b) or a free Abelian group of parabolic transformations possessing a common isotropic line; in that case, there is a genus 1 fibration $\tau \colon X \to S$, onto a compact Riemann surface S, such that Γ_ν permutes the fibers of τ.

Denote by $\rho_{\Gamma_\nu} : \Gamma_\nu \to \Gamma_\nu^*$ the restriction of ρ to Γ_ν. We distinguish two cases.

Proof when the kernel of ρ_{Γ_ν} is finite. Let A be the pre-image of A^* in Γ; it fits into an exact sequence $1 \to F \to A \to A^* \to 0$ with F finite, so a classical group theoretic lemma (see Corollary 4.8 in [35]) asserts that A contains a finite index, free Abelian subgroup A_0, such that $\rho_{\Gamma_\nu}(A_0)$ has finite index in A^*. Since A_0 is Abelian, Theorem 10.1 shows that the action of (A_0, ν_{A_0}) on X is stiff. The index of A_0 in Γ being finite, Lemma 10.2 concludes the proof. □

Proof when the kernel of ρ_{Γ_ν} is infinite. In case (a), X is a torus $\mathbf{C}^2/\Lambda$ and $\ker(\rho_{\Gamma_\nu})$ is a group of translations of X (see Proposition 3.18). Let $A \subset \Gamma_\nu$ be the pre-image of A^*; setting $K = \ker(\rho_{\Gamma_\nu})$, we obtain an exact sequence $0 \to K \to A \to A^* \to 0$, with $A \subset \Gamma_\nu$ of finite index, $A^* \simeq \mathbf{Z}$ generated by a loxodromic element, and $K \subset X$ an infinite group of translations. Since ν is symmetric, the measure ν_A is also symmetric; since ν_A satisfies the moment condition (4.1), its projection on A^* has a first moment (note that if f is loxodromic, then $\log(\|(f^*)^n\|) \asymp |n|$).

Since K is Abelian, its action on X is stiff; thus, as in Lemma 10.2.(ii), the action of A on X is stiff. Since A has finite index in Γ, the action of Γ on X is stiff too by Lemma 10.2.(i).

In case (b), we apply Proposition 2.19. So, either X is a torus, or the action of Γ_ν on the base S of its invariant fibration $\tau\colon X \to S$ has finite order. In the latter case, a finite index subgroup Γ_0 of Γ preserves each fiber of τ; then, Γ_0 contains a subgroup of index dividing 12 acting by translations on these fibers. This shows that Γ is virtually Abelian; in particular, Γ is stiff. The last case is when the image of Γ in $\mathsf{Aut}(S)$ is infinite and X is a torus $\mathbf{C}^2/\Lambda_X$. Then, $S = \mathbf{C}/\Lambda_S$ is an elliptic curve and τ is induced by a linear projection $\mathbf{C}^2 \to \mathbf{C}$, say the projection $(x, y) \mapsto x$. Lifting Γ to $\mathbf{C}^2$, and replacing Γ by a finite index subgroup if necesssary, its action is by affine transformations of the form

$$\tilde{f}\colon (x, y) \mapsto (x + a, y + mx + b) \tag{10.4}$$

with m in $\mathbf{C}^*$, and (a, b) in $\mathbf{C}^2$. This implies that Γ is a nilpotent group of length $\leqslant 2$; by Theorem 10.1 it also acts stiffly and we are done. □

Example 10.4. If $X = \mathbb{P}^2(\mathbf{C})$, its group of automorphism is $\mathsf{PGL}_3(\mathbf{C})$ and for most choices of ν there is a unique stationary measure, which is not invariant; the dynamics is proximal, and this is opposite to stiffness (see [65]). If $X = \mathbb{P}^1(\mathbf{C}) \times C$, for some algebraic curve C, then $\mathsf{Aut}(X)$ contains $\mathsf{PGL}_2(\mathbf{C}) \times \mathsf{Aut}(C)$; if ν is a probability measure on $\mathsf{PGL}_2(\mathbf{C}) \times \{\mathrm{id}_C\}$, then in most cases the stationary measures are again non invariant.

Proposition 10.5. *Let X be a complex projective surface, and Γ be a subgroup of $\mathsf{Aut}(X)$ such that Γ^* is finite. If Γ preserves a probability measure, whose support is Zariski dense in X, then the action of Γ on X is stiff.*

The main examples we have in mind is when the invariant measure is given by a volume form, or by an area form on the real part $X(\mathbf{R})$ for some real structure on X, with $X(\mathbf{R}) \neq \varnothing$.

Proof. Replacing Γ by a finite index subgroup we may assume that $\Gamma \subset \mathsf{Aut}(X)^\circ$. Denote by μ the invariant measure. Let G be the closure (for the euclidean topology) of Γ in the Lie group $\mathsf{Aut}(X)^\circ$; then G is a real Lie group preserving μ.

Let $\alpha_X\colon X \to A_X$ be the Albanese morphism of X. There is a homomorphism of complex Lie groups $\tau\colon \mathsf{Aut}(X)^\circ \to \mathsf{Aut}(A_X)^\circ$ such that $\alpha_X \circ f = \tau(f) \circ \alpha_X$ for every f in $\mathsf{Aut}(X)^\circ$.

Pick a very ample line bundle L on X, denote by $\mathbb{P}^N(\mathbf{C})$ the projective space $\mathbb{P}(H^0(X, L)^\vee)$, where $N + 1 = h^0(X, L)$, and by $\Psi_L\colon X \to \mathbb{P}^N(\mathbf{C})$ the Kodaira-Iitaka embedding of X given by L. By hypothesis, $(\Psi_L)_*\mu$ is not supported by a hyperplane of $\mathbb{P}^N(\mathbf{C})$.

Step 1.— Suppose $\tau(G) = 1$. Since $\mathrm{Pic}^0(X)$ and A_X are dual to each other, G acts trivially on $\mathrm{Pic}^0(X)$ and L is G-invariant, that is $g^*L = L$ for every $g \in G$. Thus there is a homomorphism $\beta\colon G \to \mathsf{PGL}_{N+1}(\mathbf{C})$ such that $\Psi_L \circ g = \beta(g) \circ \Psi_L$ for every $g \in L$. If G is not compact, there is a sequence of elements $g_n \in G$ going to infinity in $\mathsf{PGL}_{N+1}(\mathbf{C})$: in the KAK decomposition $g_n = k_n a_n k'_n$, the diagonal part a_n goes to ∞. Then, any probability measure on $\mathbb{P}^N(\mathbf{C})$ which is invariant under all g_n is supported in a proper projective subspace of $\mathbb{P}^N(\mathbf{C})$, and this contradicts our preliminary remark. So, G is compact in that case.

Step 2.— Now, assume that $\tau(G)$ is infinite. Identifying $\mathsf{Aut}(A_X)^\circ$ with A_X, $\tau(\mathsf{Aut}(X)^\circ)$ is a complex algebraic subgroup of the torus A_X, of positive dimension since it contains $\tau(G)$. If the kernel of τ is finite, then $\mathsf{Aut}(X)^\circ$ is compact and virtually Abelian; thus, we may assume $\dim(\ker(\tau)) \geqslant 1$. In particular the fibers of α_X have positive dimension, $\dim(\alpha_X(X)) \leqslant 1$

and $\alpha_X(X)$ is a curve, which is elliptic because it is invariant under the action of $\tau(\mathsf{Aut}(X)^\circ)$. Then, the universal property of the Albanese morphism implies $\alpha_X(X) = A_X$. In particular, α_X is a submersion, for its critical values form a proper, $\tau(\mathsf{Aut}(X)^\circ)$-invariant subset of A_X. Thus, X is a $\mathbb{P}^1(\mathbf{C})$-bundle over A_X because the fibers of α_X are smooth, are invariant under the action of $\ker(\tau)$, and can not be elliptic since otherwise X would be a torus. From [93, Thm 3] (see also [92, 100] for instance), there are two cases:

(1) either $X = A_X \times \mathbb{P}^1(\mathbf{C})$, $\mathsf{Aut}(X) = \mathsf{Aut}(A_X) \times \mathsf{PGL}_2(\mathbf{C})$ and we deduce as in the first step that G is a compact group;
(2) or $\mathsf{Aut}(X)^\circ$ is Abelian.

In both cases stiffness follows, and we are done. □

Remark 10.6. Pushing the analysis further, it can be shown that, under the assumptions Proposition 10.5, Γ is relatively compact. Indeed in the last considered case, if Γ is not bounded it can be deduced from [93, Thm 3] that there are elements with wandering dynamics: all orbits in some Zariski open subset converge towards a section of α_X. This contradicts the invariance of μ.

10.4. **Invariant algebraic curves II.** Let us start with an example.

Example 10.7 (See also [32])**.** Consider an elliptic curve $E = \mathbf{C}/\Lambda$ and the Abelian surface $A = E \times E$. The group $\mathsf{GL}_2(\mathbf{Z})$ determines a non-elementary group of automorphisms of $E \times E$ of the form $(x, y) \mapsto (ax + by, cx + dy)$. The involution $\eta = -\operatorname{id}$ generates a central subgroup of $\mathsf{GL}_2(\mathbf{Z})$, hence $\mathsf{PGL}_2(\mathbf{Z})$ acts on the (singular) Kummer surface A/η. Each singularity gives rise to a smooth $\mathbb{P}^1(\mathbf{C})$ in the minimal resolution X of A/η, the group $\{B \in \mathsf{PGL}_2(\mathbf{Z})\ ;\ B \equiv \operatorname{id} \mod 2\}$ preserves each of these 16 rational curves, and its action on these curves is given by the usual linear projective action of $\mathsf{PGL}_2(\mathbf{Z})$ on $\mathbb{P}^1(\mathbf{C})$. In particular, it is proximal and strongly irreducible so it admits a unique, non-invariant, stationary measure.

The next result shows that when ν is symmetric, every non-invariant stationary measure is similar to the previous example.

Proposition 10.8. *Let (X, ν) be a random holomorphic dynamical system, with ν symmetric. Let μ be an ergodic ν-stationary measure giving positive mass to some proper Zariski closed subset of X. Then μ is supported on a Γ_ν-invariant proper Zariski closed subset and*

(a) *either μ is invariant;*
(b) *or the Zariski closure of $\mathrm{Supp}(\mu)$ is a finite, disjoint union of smooth rational curves C_i, the stabilizer of C_i in Γ induces a strongly irreducible and proximal subgroup of $\mathsf{Aut}(C_i) \simeq \mathsf{PGL}_2(\mathbf{C})$, and $\mu(C_i)^{-1}\mu|_{C_i}$ is the unique stationary measure of this group of Möbius transformations.*

Moreover, if (X, ν) is non-elementary, the curves C_i have negative self-intersection and can be contracted on cyclic quotient singularities.

Note that no moment assumption is assumed here. Before giving the proof, let us briefly discuss the question of stiffness for Möbius actions on $\mathbb{P}^1(\mathbf{C})$. Let ν be a symmetric measure on $\mathsf{PGL}_2(\mathbf{C})$. As already said, by Furstenberg's theory, if Γ_ν is strongly irreducible and unbounded it admits a unique stationary measure, and this measure is not invariant. Otherwise, any ν-stationary measure is invariant because

– either Γ_ν is relatively compact and stiffness follows from [65, Thm. 3.5];
– or Γ_ν admits an invariant set made of two points, then Γ_ν is virtually Abelian and stiffness follows from Theorem 10.1;
– or Γ_ν is conjugate to a subgroup of the affine group $\mathsf{Aff}(\mathbf{C})$ with no fixed point.

In the latter case after conjugating Γ_ν to a subgroup of $\mathsf{Aff}(\mathbf{C})$ we can write any $g \in \Gamma_\nu$ as $g(z) = a(g)z + b(g)$. If $a(g) \equiv 1$ then Γ_ν is Abelian and we are done. Otherwise Γ_ν is merely solvable and we apply the following lemma which follows from a result of Bougerol and Picard (see [20, Thm. 2.4]).

Lemma 10.9. *Let ν be a symmetric probability measure on $\mathsf{Aff}(\mathbf{C})$. If no point of $\mathbf{C}$ is fixed by ν-almost every g, then the only ν-stationary probability on $\mathbb{P}^1(\mathbf{C})$ is the point mass at ∞.*

Proof. Assume by contradiction that there exists a stationary measure μ such that $\mu(\mathbf{C}) = 1$ and $\mu(\{\infty\}) = 0$. If Γ_ν is abelian, it is made of translations because it has no fixed point in $\mathbf{C}$; on the other hand if Γ_ν is not abelian, its derived subgroup contains a non-trivial translation. Thus, in any case Γ_ν contains a non-trivial translation, and we infer that Γ_ν does not preserve any finite measure on $\mathbf{C}$. In particular μ is not invariant.

Let now r_n be the right random walk associated to ν on $\mathsf{Aff}(\mathbf{C})$. Put $\nu^\infty = \sum_{k=0}^\infty 2^{-k+1}\nu^{*k}$. A classical martingale convergence argument (see [19, Lem. II.2.1]) provides a measurable set Ω_0 with $\nu^{\mathbf{N}}(\Omega_0) = 1$ such that, for all $\omega \in \Omega_0$,

– $r_n(\omega)_*\mu$ converges toward a probability measure μ_ω and $\mu = \int \mu_\omega d\nu^{\mathbf{N}}(\omega)$;
– for ν^∞-almost every γ, $r_n(\omega)_*\gamma_*\mu$ converges towards the same limit μ_ω.

Since $\mu = \int \mu_\omega d\nu^{\mathbf{N}}(\omega)$, we have $\mu_\omega(\mathbf{C}) = 1$ almost surely. Now, assume that for some $\omega \in \Omega_0$, $r_n(\omega)$ does not go to ∞ in $\mathsf{PGL}_2(\mathbf{C})$. Extracting a convergent subsequence $r_{n_j}(\omega) \to r$, we infer that $\gamma_*\mu = \gamma'_*\mu = (r^{-1})_*\mu_\omega$ for $(\nu^\infty \times \nu^\infty)$-almost-every (γ, γ'); hence μ is Γ_ν-invariant, a contradiction. Thus $r_n(\omega)$ goes to ∞ in $\mathsf{PGL}_2(\mathbf{C})$ for almost every ω.

Suppose that $(a(r_n(\omega)), b(r_n(\omega)))$ is unbounded in $\mathbf{C}^2$ for a subset $\Omega'_0 \subset \Omega_0$ of positive measure. Set

$$\tilde{r}_n(\omega) = \frac{1}{\max(|a(r_n(\omega))|\,, |b(r_n(\omega))|)} r_n(\omega) \tag{10.5}$$

and extract a subsequence n_j so that $\tilde{r}_{n_j}(\omega) \to \ell(\omega)$, where $\ell(\omega)$ is an affine endomorphism of $\mathbf{C}$. If $\ell(\omega)(z) \neq 0$ then $r_{n_j}(\omega)(z) \to \infty$. Since $r_{n_j}(\omega)_*\mu \to \mu_\omega$ and $\mu_\omega(\mathbf{C}) = 1$, we deduce that $\mu(\ell(\omega)^{-1}\{0\}) = 1$. This is a contradiction because μ is not concentrated at a single point. Thus, $(a(r_n(\omega)), b(r_n(\omega)))$ is almost surely bounded. Since $r_n(\omega)$ goes to ∞ in $\mathsf{PGL}_2(\mathbf{C})$, $a(r_n(\omega))$ goes to 0 almost surely, in contradiction with the symmetry of ν. This concludes the proof. $\square$

Proof of Proposition 10.8. If μ has an atom then, by ergodicity, μ is supported on a finite orbit and it is invariant. So we now assume that μ is atomless. By ergodicity, μ gives full mass to a Γ_ν-invariant curve D; let $C_1, \ldots, C_n$ be its irreducible components. Let Γ' be the finite index subgroup of Γ_ν stabilizing each C_i and ν' be the hitting measure induced by ν on Γ'; it is symmetric, μ is ν'-stationary, and so are its restrictions $\mu|_{C_i}$, for each C_i.

If the genus of (the normalization of) C_1 is positive, then $\Gamma'|_{C_1} \subset \mathsf{Aut}(C_1)$ is virtually Abelian, hence $\mu|_{C_1}$ is Γ'-invariant. Since μ is ergodic, Γ_ν permutes transitively the C_i, and arguing as in Lemma 10.2, we see that μ is ν-invariant as well. Now, assume that the normalization $\hat{C}_1$ is isomorphic to $\mathbb{P}^1(\mathbf{C})$. If C_1 is not smooth, or if it intersects another Γ_ν-periodic curve, then the image of Γ' in $\mathsf{Aut}(\hat{C}_1) \simeq \mathsf{PGL}_2(\mathbf{C})$ is not strongly irreducible, and the discussion preceding this proof shows that μ is Γ'-invariant. Again, this implies that μ is Γ_ν-invariant. The same holds if Γ' is a bounded subgroup of $\mathsf{Aut}(\hat{C}_1)$. The only possibility left is that C_1 is smooth, disjoint from the other periodic curves, and Γ' induces a strongly irreducible subgroup of $\mathsf{Aut}(C_1)$. Since Γ_ν permutes transitively the C_i, conjugating the dynamics of the groups $\Gamma'|_{C_i}$, the same property holds for each C_i.

If Γ_ν is non-elementary, Lemma 2.14 shows that $C_i^2 = -m$ for some $m > 0$, which does not depend on i because Γ_ν permutes the C_i transitively. Then, the C_i being disjoint, one can contract them simultaneously, each of the contractions leading to a quotient singularity $(\mathbf{C}^2, 0)/\langle\eta\rangle$ with $\eta(x, y) = (\alpha x, \alpha y)$ for some root of unity α of order m (see [6, §III.5]). □

10.5. **Non-elementary groups: real dynamics.** We now consider general non-elementary actions. As explained in the introduction, so far our results are restricted to subgroups of $\mathsf{Aut}(X)$ preserving a totally real surface Y. We further assume that there exists a Γ_ν-invariant volume form on Y; this is automatically the case if X is an Abelian, a K3, or an Enriques surface (see Lemma 11.3). Note that, *a posteriori*, the results of §11 and 12 suggest that measures supported on a totally real surface and invariant under a non-elementary subgroup of $\mathsf{Aut}(X)$ tend to be absolutely continuous, unless they are supported by a curve or a finite set. We saw in Example 10.7 that stiffness can fail in presence of invariant rational curves along which the dynamics is that of a proximal and strongly irreducible random product of Möbius transformations. The next theorem shows that for actions preserving a totally real surface, this obstruction to stiffness is the only one.

Theorem 10.10. *Let (X, ν) be a non-elementary random holomorphic dynamical system satisfying the moment condition* (4.1). *Assume that $Y \subset X$ is a Γ_ν-invariant totally real 2-dimensional smooth submanifold such that the action of Γ on Y preserves a probability measure vol_Y equivalent to the Riemannian volume on Y. Then, every ergodic stationary measure μ on Y is:*

(a) *either almost surely invariant,*
(b) *or supported on a Γ_ν-invariant algebraic curve.*

In particular if there is no Γ_ν-invariant curve then (Y, ν) is stiff. Moreover, if the fiber entropy of μ is positive, then μ is the restriction of vol_Y to a subset of positive volume.

Recall from Lemma 2.14 that Γ_ν-invariant curves can be contracted. For the induced random dynamical system on the resulting singular surface, stiffness holds unconditionally. If furthermore ν is symmetric then the result can be made more precise by applying Proposition 10.8.

Proof of Theorem 10.10. We split the proof in two steps.

Step 1.– Let μ be an ergodic stationary measure supported on Y. We assume that μ is not invariant, and we want to prove that it is supported on a Γ_ν-invariant curve. Since the action is volume preserving, its Lyapunov exponents satisfy $\lambda^- + \lambda^+ = 0$ (see Lemma 7.6). The invariance principle (Theorem 7.4) shows that μ is hyperbolic: indeed μ is almost surely invariant when $\lambda^- \geqslant 0$. We can therefore apply Theorem 3.4 of [22] to obtain the following trichotomy:

(1) either μ has finite support, so it is invariant;

(2) or the distribution of Oseledets stable directions is non-random;
(3) or μ is almost surely invariant and absolutely continuous with respect to vol_Y: even more, it is the restriction of vol_Y to a subset of positive volume.

Since μ is not invariant, we are in case (2). Theorem 9.1 then implies that μ is supported on an invariant algebraic curve. This concludes the proof of the first assertions in Theorem 10.10, including the stiffness property when Γ has no periodic curve.

Step 2.– It remains to prove the last assertion. Let then μ be an ergodic stationary measure with $h_\mu(X, \nu) > 0$. In the above trichotomy, (1) is now excluded. To exclude the alternative (2), by Theorem 9.1, it suffices to show that μ is not supported on an invariant curve. By Proposition 7.12 (i.e. the fibered Margulis-Ruelle inequality), μ is hyperbolic. If μ is supported on an algebraic curve, the proof of Corollary 8.3 leads to the following alternative: either μ is atomic or the Lyapunov exponent along that curve is negative. In the latter case μ is proximal along that curve and its stable conditionals are points. In both cases the fiber entropy would vanish, in contradiction with our hypothesis, so μ is not supported on an algebraic curve, as desired. □

We conclude this section with a variant of Theorem 10.10 for singular volume forms; it may be applied to Blanc's examples (see § 3.4).

Theorem 10.11. *Let (X, ν) be a non-elementary random holomorphic dynamical system satisfying the moment condition* (4.1)*, and preserving a totally real 2-dimensional submanifold $Y \subset X$. Assume that there exists a meromorphic 2-form η which is almost invariant under every $f \in \Gamma_\nu$ (i.e. $f^*\eta = \mathrm{Jac}_\eta(f)\eta$ with $|\mathrm{Jac}_\eta(f)| = 1$). Then every ν-stationary measure supported on Y is either supported on a Γ_ν-invariant algebraic curve or almost surely invariant.*

Proof. The proof is identical to that of Theorem 10.10, except that we use Proposition 7.8 instead of Lemma 7.6. Indeed by ergodicity if μ is not supported on an invariant algebraic curve it gives zero mass to the set of zeros and poles of Ω so, by Proposition 7.8, we have $\lambda^+ + \lambda^- = 0$. □

11. Subgroups with parabolic elements

We say that $\Gamma \subset \mathsf{Aut}(X)$ is **twisting** if it contains a parabolic automorphism (this terminology is justified below). This section investigates the dynamics of (X, ν) when Γ_ν is non-elementary and twisting. Under this assumption invariant measures can be classified (Theorem 11.4): they are either hyperbolic or carried by some proper algebraic subset (Theorem 11.7).

Remark 11.1. In many examples for which $\mathsf{Aut}(X)$ contains a non-elementary group, $\mathsf{Aut}(X)$ contains also a parabolic automorphism (see the examples in §§3.1–3.4). So, if we are interested in random dynamical systems for which Γ_ν has finite index in $\mathsf{Aut}(X)$, the twisting assumption is quite natural. Also, if $\mathsf{Aut}(X)$ is both twisting and non-elementary, then there are thin subgroups $\Gamma \subset \mathsf{Aut}(X)$ with the same property: one can take two parabolics automorphisms g and h generating a non-elementary group, and set $\Gamma = \langle g^m, h^n \rangle$ for large integers m and n.

11.1. Dynamics of parabolic automorphisms.

Recall from §2.4 that if h is a parabolic automorphism of a compact Kähler surface X, it preserves a unique genus 1 fibration, given by the fibers of a rational map $\pi_h \colon X \to B$. In particular there is an automorphism h_B of B such that

$$\pi \circ h = h_B \circ \pi. \tag{11.1}$$

Moreover, if X is not a torus there exists an integer $m > 0$ such that h^m preserves every fiber of π and acts by translation on every smooth fiber (Proposition 2.19). As shown in Lemma 11.2, h behaves like a "complex Dehn twist", acting by translations along the fibers of π, with a shearing property in the transversal direction. This twisting property justifies the vocabulary introduced for "twisting groups". When X is rational, the invariant fibration comes from a Halphen pencil of $\mathbb{P}^2_{\mathbf{C}}$ (see [29]); this is why parabolic automorphisms are also called **Halphen twists**.

Let h be a parabolic automorphism with $h_B = \mathrm{id}_B$. The critical values of π form a finite subset $\mathrm{Crit}(\pi) \subset B$; we denote its complement by B°. Each fiber $X_w := \pi^{-1}(w)$, $w \in B^\circ$, is a smooth curve of genus 1, isomorphic to $\mathbf{C}/L(w)$ for some lattice $L(w) = \mathbf{Z} \oplus \mathbf{Z}\tau(w)$; and h induces a translation $h_w(z) = z + t(w)$ of X_w, for some $t(w) \in \mathbf{C}/L(w)$. The points w for which h_w is periodic are characterized by the relation $t(w) \in \mathbf{Q} \oplus \mathbf{Q}\tau(w)$. If

$$t(w) - (a + b\tau(w)) \in \mathbf{R} \cdot (p + q\tau(w)) \tag{11.2}$$

for some $(a, b) \in \mathbf{Q}^2$ and $(p, q) \in \mathbf{Z}^2$, the closure of $\mathbf{Z}t(w)$ in $\mathbf{C}/L(w)$ is an Abelian Lie group of dimension 1, isomorphic to $\mathbf{Z}/k\mathbf{Z} \times \mathbf{R}/\mathbf{Z}$ for some $k > 0$; then, the closure of each orbit of h_w is a union of k circles. Locally in B° this occurs along a countable union of analytic curves (R_j). Otherwise, the orbits of h_w are dense in X_w, and the unique h_w invariant probability measure is the Haar measure on X_w.

Now, assume that $Y \subset X$ is a real analytic subset of X of real codimension 2, and that h preserves Y; for instance h may preserve a real structure on X, and Y be a connected component of $X(\mathbf{R})$. Then, $\pi(Y) \subset B$ is (locally) contained in the curves R_j. The smooth fibers $\pi_{|Y}^{-1}(w)$, for $w \in \pi(Y) \backslash \mathrm{Crit}(\pi)$, are unions of circles along which the orbits of h_w are either dense (for most $w \in \pi(Y)$) or finite (for countably many $w \in \pi(Y)$).

Lemma 11.2. *Assume that h_B is the identity. Let $U \subset B^\circ$ be a simply connected open subset. There is a countable union of analytic curves $R_j \subset U$, such that*

(1) *h acts by translation on each fiber $X_w = \pi^{-1}(w)$, $w \in U$;*
(2) *for $w \in U \backslash \cup_j R_j$, the action of h in the fiber X_w is a totally irrational translation (it is uniquely ergodic, and its orbits are dense in X_w);*
(3) *for w in some countable subset of U, the orbits of h_w are finite;*
(4) *if the orbits of h_w are neither dense nor finite, then $w \in \cup_j R_j$ and the closure of each orbit of h_w is dense in a finite union of circles;*
(5) *there is a finite subset $\mathrm{Flat}(h) \subset U$ such that for $x \notin \pi^{-1}(\mathrm{Flat}(h))$*

$$\lim_{n \to \pm\infty} \| D_x h^n \| \to +\infty$$

locally uniformly in x; more precisely for every $v \in T_x X \backslash T_x X_{\pi(x)}$, $\| Dh^n_x(v) \|$ grows linearly while $\frac{1}{n} \pi_*(D_x h^n(v))$ converges to 0.

Moreover, if h preserves a 2-dimensional real analytic subset $Y \subset X$, then

(6) *π induces on Y a singular fibration whose generic leaves are union of (one or two) circles, and there exists an integer $m \in \{1, 2\}$ such that h^m preserves these circles and is uniquely ergodic along these circles except countably many of them.*

This lemma is proven in [26, 32]; Property(5) is the above mentioned twisting property of h.

11.2. **Classification of invariant measures.** In this paragraph, we review the classification of invariant ergodic probability measures for twisting non-elementary groups of automorphisms; we refer to [26, 32] for details and examples. If X is a real K3 or Abelian surface and $X(\mathbf{R}) \neq \varnothing$ there is a unique section of the canonical bundle of X which, when restricted to $X(\mathbf{R})$, induces a positive area form of total area 1; we denote this area form by $\mathsf{vol}_{X(\mathbf{R})}$. The associated probability measure is invariant under the action of $\mathsf{Aut}(X_{\mathbf{R}})$, the subgroup of $\mathsf{Aut}(X)$ preserving the real structure. In fact, such a smooth invariant probability measure exists on any totally real invariant surface (see [32, §5]):

Lemma 11.3. *Let X be an Abelian surface, or a K3 surface, or an Enriques surface with universal cover X'. Let $Y \subset X$ be a (real) surface of class C^1. Let $\mathsf{Aut}(X;Y)$ be the subgroup of $\mathsf{Aut}(X)$ preserving Y. If Y is totally real, Ω_X (resp. $\Omega_{X'}$) induces a smooth $\mathsf{Aut}(X;Y)$-invariant probability measure vol_Y on Y.*

Note that there indeed exists examples of subgroups preserving a totally real surface $Y \subset X$ which is not a real form of X (see [32, §6]). The classification of invariant measures then reads as follows.

Theorem 11.4. *Let X be a compact Kähler surface. Let Γ be a twisting non-elementary subgroup of $\mathsf{Aut}(X)$. Let μ be a Γ-invariant ergodic probablity measure on X. Then, μ satisfies one and only one of the following properties.*

(a) *μ is the average on a finite orbit of Γ;*
(b) *μ is supported by a Γ-invariant curve $D \subset X$;*
(c) *there is a Γ-invariant proper algebraic subset Z of X, and a Γ-invariant, totally real, real analytic submanifold Y of $X \setminus Z$ such that* (1) *$\mu(Z) = 0$,* (2) *the support of μ is a union of finitely many connected components of Y,* (3) *μ is absolutely continuous with respect to the Lebesgue measure on Y, and* (4) *the density of μ with respect to any real analytic area form on Y is real analytic;*
(d) *there is a Γ-invariant proper algebraic subset Z of X such that* (1) *$\mu(Z) = 0$,* (2) *the support of μ is equal to X,* (3) *μ is absolutely continuous with respect to the Lebesgue measure on X, and* (4) *the density of μ with respect to any real analytic volume form on X is real analytic on $X \setminus Z$.*

If X is not a rational surface, then in case (c) *(resp.* (d)*) we can further conclude that the invariant measure is locally proportional to vol_Y (resp. equal to vol_X).*

The reason why we say that μ is proportional to vol_Y (and not equal to it) in the last sentence is because μ may be equal to zero on some components of $Y \setminus Z$. This theorem is a combination of Theorem 1.1 and § 5.3 of [32]. Let us also point out the following corollary of the proof.

Corollary 11.5. *Let $\Gamma \leqslant \mathsf{Aut}(X)$ be as in Theorem 11.4. Assume furthermore that X and Γ are defined over $\mathbf{R}$ and Γ does not preserve any proper Zariski closed subset of X. Then any Γ-invariant ergodic measure supported on $X(\mathbf{R})$ is supported by a union $X(\mathbf{R})' = \cup_j X(\mathbf{R})_j$ of connected components $X(\mathbf{R})_j$ of $X(\mathbf{R})$, and is locally given by positive real analytic 2-forms on $X(\mathbf{R})'$. If X is not rational, μ is equal to the restriction of $\mathsf{vol}_{X(\mathbf{R})}$ to $X(\mathbf{R})'$, up to some normalizing factor.*

Using this classification we can now sharpen the conclusion of Theorem 10.10 in the presence of parabolic automorphisms. When $Y = X(\mathbf{R})$, the statement can also be combined with Corollary 11.5 to get an even more precise result.

Corollary 11.6. *Let (X, ν) be a random holomorphic dynamical system on a compact Kähler surface, satisfying* (4.1) *and such that Γ_ν is twisting and non-elementary. Let $Y \subset X$ be a Γ_ν-invariant, smooth, totally real surface such that, on Y, Γ_ν preserves a probability measure vol_Y equivalent to the Riemannian volume.*

Then up to a positive multiplicative factor, every ergodic stationary measure μ supported on Y is :

- *either the counting measure on a finite orbit;*
- *or supported on a Γ_ν-invariant algebraic curve;*
- *or the restriction of vol_Y to a Γ_ν-invariant open subset of Y whose boundary is piecewise smooth.*

In the last alternative, the boundary is obtained by intersecting an algebraic curve $D \subset X$ with Y; it may have a finite number of singularities.

Proof. We just have to repeat the proof of Theorem 10.10, by incorporating the classification given in Theorem 11.4. Note that Y is automatically real analytic in this case. □

11.3. **Hyperbolicity of the invariant volume.** It is a fundamental (and mostly open) problem in conservative dynamics to show the typicality of non-zero Lyapunov exponents on a set of positive Lebesgue measure. In deterministic dynamics, a recent breakthrough is the work of Berger and Turaev [13]. Adding some randomness makes such a hyperbolicity result easier to obtain: see [15] for random perturbation of the standard map, and [5, 99] for random conservative diffeomorphisms on (closed real) surfaces. The results of Barrientos and Malicet or Obata and Poletti [5, 99] are perturbative in nature and do not give explicit examples. Here the high rigidity of complex algebraic automorphisms will be sufficient to show that twisting, non-elementary, random dynamical systems (X, ν) automatically satisfy some non-uniform hyperbolicity with respect to the volume.

Theorem 11.7. *Let X be a compact Kähler surface, and let Γ be a non-elementary, twisting subgroup of $\mathsf{Aut}(X)$. Let μ be an ergodic Γ-invariant measure giving no mass to proper Zariski closed subsets of X ([7]). Then for every probability measure ν on $\mathsf{Aut}(X)$ satisfying the moment condition* (4.1) *and such that $\Gamma_\nu = \Gamma$, μ is hyperbolic and the fiber entropy $h_\mu(X, \nu)$ is positive.*

The same argument leads to a variant of this result when Γ_ν contains a Kummer example. Before stating our next result, let us recall the definition of classical Kummer examples (see also Example 10.7, and [33, §1.3] for a more general definition). Let $A = \mathbf{C}^2/\Lambda$ be a complex torus and let η be the involution given by $\eta(z_1, z_2) = (-z_1, -z_2)$, which has 16 fixed points. Then $A/\langle\eta\rangle$ is a surface with 16 singular points, and resolving these singularities (each of them requires a single blow-up) yields a so-called **Kummer surface** X: a K3 surface with 16 disjoint nodal curves. Let f_A be a loxodromic automorphism of A which is induced by a linear transformation of $\mathbf{C}^2$ preserving Λ; then f_A commutes to η and descends to an automorphism f of X; such automorphisms will be referred to as **classical Kummer examples**. Of course, they

[7]Hence by Theorem 11.4, μ is equivalent to vol_X or vol_Y for some real analytic invariant surface with boundary.

preserve the canonical volume vol_X. Notice that the Kummer surface X also supports automorphisms which are not coming from automorphisms of A (see [79] and [49] for instance).

Theorem 11.8. *Let (X, ν) be a non-elementary random dynamical system on a Kummer K3 surface satisfying* (4.1) *and such that Γ_ν contains a classical Kummer example. Then any ergodic Γ_ν-invariant measure giving no mass to proper Zariski closed subsets of X is hyperbolic and has positive fiber entropy.*

In this statement we do not assume that Γ_ν contains a parabolic element. In Theorem 12.5 below, we classify invariant probability measures which are supported on an invariant, real analytic, and totally real surface Y, when Γ_ν contains a Kummer example.

Theorems 11.7 and 11.8 will be proven in §11.5.

11.4. **Ledrappier's invariance principle and invariant measures on $\mathbb{P}T\mathcal{X}$.** This paragraph contains preliminary results for the proof of Theorems 11.7 and 11.8. Our presentation is inspired by the exposition of [5]. It is similar in spirit to that of [99], which relies on the "pinching and twisting" formalism of Avila and Viana (see [110] for an introduction[8]). Most of this discussion is valid for a random holomorphic dynamical system on an arbitrary complex surface (not necessarily compact), satisfying (4.1).

Let μ be an ergodic ν-stationary measure. We introduce the projectivized tangent bundles $\mathbb{P}T\mathcal{X}_+ = \Omega \times \mathbb{P}TX$ and $\mathbb{P}T\mathcal{X} = \Sigma \times \mathbb{P}TX$. The tangent bundles TX and $\mathbb{P}TX$ admit measurable trivializations over a set of full measure. Consider any probability measure $\hat\mu$ on $\mathbb{P}TX$ that is stationary under the random dynamical system induced by (X, ν) on $\mathbb{P}TX$ and whose projection on X coincides with μ, i.e. $\pi_*\hat\mu = \mu$ where $\pi\colon \mathbb{P}TX \to X$ is the natural projection. Such measures always exist. Indeed the set of probability measures on $\mathbb{P}TX$ projecting to μ is compact and convex, and it is non-empty since it contains the measures $\int \delta_{[v(x)]} d\mu(x)$ for any measurable section $x \mapsto [v(x)]$ of $\mathbb{P}TX$; thus, the operator $\int \mathbb{P}(Df)\, d\nu(f)$ has a fixed point in that set. The stationarity of $\hat\mu$ is equivalent to the invariance of $\nu^{\mathbf{N}} \times \hat\mu$ under the transformation $\widehat{F}_+ \colon \Omega \times \mathbb{P}TX \to \Omega \times \mathbb{P}TX$ defined by

$$\widehat{F}_+(\omega, x, [v]) = (\sigma(\omega), f^1_\omega(x), \mathbb{P}(D_x f^1_\omega)[v]) \tag{11.3}$$

for any non-zero tangent vector $v \in T_xX$. We denote by $\hat\mu_x$ the family of probability measures – on the fibers $\mathbb{P}T_xX$ of π – given by the disintegration of $\hat\mu$ with respect to π; the conditional measures of $\nu^{\mathbf{N}} \times \hat\mu$ with respect to the projection $\mathbb{P}T\mathcal{X} \to X$ are given by $\hat\mu_{\omega,x} = \nu^{\mathbf{N}} \times \hat\mu_x$.

Remark 11.9. Even when μ is Γ_ν-invariant, this construction only provides a stationary measure on $\mathbb{P}TX$. This is exactly what happens for twisting non-elementary subgroups: indeed we will show in §11.5 that projectively invariant measures do not exist in this case.

The tangent action of our random dynamical system gives rise to a stationary product of matrices in $\mathsf{GL}(2, \mathbf{C})$. To see this, fix a measurable trivialization $P\colon TX \to X \times \mathbf{C}^2$, given by linear isomorphisms $P_x\colon T_xX \to \mathbf{C}^2$, which conjugates the action of DF_+ to that of a linear cocycle $A : \mathcal{X}_+ \times \mathbf{C}^2 \to \mathcal{X}_+ \times \mathbf{C}^2$ over $(\mathcal{X}_+, F_+, \nu^{\mathbf{N}} \times \mu)$. In this context, Ledrappier establishes in [86] the following "invariance principle".

[8]Beware that the word "twisting" has a different meaning there.

Theorem 11.10. *If $\lambda^-(\mu) = \lambda^+(\mu)$, then for any stationary measure $\hat\mu$ on $\mathbb{P}TX$ projecting to μ, we have $\mathbb{P}(D_xf)_*\hat\mu_x = \hat\mu_{f(x)}$ for μ-almost every x and ν-almost every f.*

From this point the second main ingredient of the proof is a classification of such projectively invariant measures; this is where we follow [5]. To explain this result a bit of notation is required. Let V and W be hermitian vector spaces of dimension 2. Endow the projective lines $\mathbb{P}(V)$ and $\mathbb{P}(W)$ with their respective Fubini-Study metrics. If $g\colon V \to W$ is a linear isomorphism, set

$$[\![g]\!] = \|\mathbb{P}(g)\|_{C^1} \tag{11.4}$$

where $\mathbb{P}(g)\colon \mathbb{P}(V) \to \mathbb{P}(W)$ is the projective linear map induced by g and $\|\cdot\|_{C^1}$ is the maximum of the norms of $D_z\mathbb{P}(g)\colon T_z\mathbb{P}(V) \to T_{\mathbb{P}gz)}\mathbb{P}(W)$ with respect to the Fubini-Study metrics. Let us fix two isometric isomorphisms $\iota_V\colon V \to \mathbf{C}^2$ and $\iota_W\colon W \to \mathbf{C}^2$ to the standard hermitian space $\mathbf{C}^2$. If we denote by $\iota_W \circ g \circ \iota_V^{-1} = k_1ak_2$ the KAK decomposition of $\iota_W \circ g \circ \iota_V^{-1}$ in $\mathsf{PSL}(2,\mathbf{C})$, we get $[\![g]\!] = \|a\|^2 = \left\|\iota_W \circ g \circ \iota_V^{-1}\right\|^2$where $\|\cdot\|$ is the matrix norm in $\mathsf{PSL}_2(\mathbf{C}) = \mathsf{SL}_2(\mathbf{C})/\langle\pm\,\mathrm{id}\rangle$ associated to the Hermitian norm in $\mathbf{C}^2$. In particular:

(a) $[\![g]\!] = 1$ if and only if $\mathbb{P}(g)$ is an isometry from $\mathbb{P}(V)$ to $\mathbb{P}(W)$;
(b) for a sequence (g_n) of linear maps $V \to W$, $[\![g_n]\!]$ tends to $+\infty$ as n goes to $+\infty$ if and only if $\mathbb{P}(\iota_W \circ g \circ \iota_V^{-1})$ tends to ∞ in $\mathsf{PSL}_2(\mathbf{C})$.

We are now ready to state the classification of projectively invariant measures.

Theorem 11.11. *Let (X,ν) be a random dynamical system on a complex surface and let μ be an ergodic stationary measure. Let $\hat\mu$ be a stationary measure on $\mathbb{P}TX$ such that $\pi_*\hat\mu = \mu$ and $(\mathbb{P}D_xf)_*\hat\mu_x = \hat\mu_{f(x)}$ for μ-almost every x and ν-almost every f. Then, exactly one of the following two properties is satisfied:*

(1) *For $(\nu^{\mathbf{N}} \times \mu)$-almost every $x = (\omega, x)$, the sequence $[\![D_xf_\omega^n]\!]$ is unbounded and then:*
  (1.a) *either there exists a measurable Γ_ν-invariant family of lines $E(x) \subset T_xX$ such that $\hat\mu_x = \delta_{[E(x)]}$ for μ-almost every x;*
  (1.b) *or there exists a measurable Γ_ν-invariant family of pairs of lines $E_1(x), E_2(x) \subset T_xX$ and positive numbers λ_1, λ_2 with $\lambda_1 + \lambda_2 = 1$ such that $\hat\mu_x = \lambda_1\delta_{[E_1(x)]} + \lambda_2\delta_{[E_2(x)]}$ for μ-almost every x;*

(2) *The projectivized tangent action of Γ_ν is reducible to a compact group, that is there exists a measurable trivialization of the tangent bundle $(P_x : T_xX \to \mathbf{C}^2)_{x\in X}$, such that for every $f \in \Gamma_\nu$ and every x, $\mathbb{P}\left(P_{f(x)} \circ D_xf \circ P_x^{-1}\right)$ belongs to the unitary group $\mathsf{PU}_2(\mathbf{C})$.*

In assertion (1.b), the pair is not naturally ordered, i.e. there is no natural distinction of E_1 and E_2, the random dynamical system may a priori permute these lines. The proof is obtained by adapting the arguments of [5] to the complex case. Details are given in Appendix C.

11.5. **Proofs of Theorems 11.7 and 11.8.**

11.5.1. *Proof of Theorem 11.7.* By Theorem 11.4, μ is either equivalent to the Lebesgue measure on X, or to the 2-dimensional Lebesgue measure on some components of an invariant totally real surface $Y \subset X$. Let us assume, by contradiction, that μ is not hyperbolic. Hence its Lyapunov exponents vanish, and by Theorem 11.10 and Theorem 11.11, there is a measurable set $X' \subset X$ with $\mu(X') = 1$ such that one of the following properties is satisfied along X':

(a) there is a measurable Γ_ν-invariant line field $E(x)$;

(b) there exists a measurable Γ_ν-invariant splitting $E(x) \oplus E'(x) = T_xX$ of the tangent bundle; here, the invariance should be taken in the following weak sense: an element f of Γ_ν maps $E(x)$ to $E(f(x))$ or $E'(f(x))$;

(c) there exists a measurable trivialization $P_x \colon T_xX \to \mathbf{C}^2$ such that in the corresponding coordinates the projectivized differential $\mathbb{P}(Df_x)$, $f \in \Gamma_\nu$, takes its values in $\mathsf{PU}_2(\mathbf{C})$.

Fix a small $\varepsilon > 0$. By Lusin's theorem, there is a compact set K_ε with $\mu(K_\varepsilon) > 1 - \varepsilon$ such that the data $x \mapsto E(x)$, resp. $x \mapsto (E(x), E'(x))$ or $x \mapsto P_x$ in the respective cases (a,b,c) are continuous on K_ε. In particular, in case (c), the norms of P_x and P_x^{-1} are bounded by some uniform constant $C(\varepsilon)$ on K_ε; hence, if $g \in \Gamma_\nu$ and x and $g(x)$ belong to K_ε, $[\![Dg_x]\!]$ is bounded by $C(\varepsilon)^2$.

Fix a pair of Halphen twists g and $h \in \Gamma_\nu$ with distinct invariant fibrations $\pi_g \colon X \to B_g$ and $\pi_h \colon X \to B_h$ respectively (see Lemma 2.20). In a first stage assume that X is not a torus: then by Proposition 2.19 we may assume that g and h preserve every fiber of their respective invariant fibrations (see Section 11.1).

First assume that μ is absolutely continuous with respect to the Lebesgue measure on X, with a positive real analytic density on the complement of some invariant, proper, Zariski closed subset. Since the invariant fibration is holomorphic, the disintegration μ_b of μ is absolutely continuous on almost every fiber $\pi_h^{-1}(b)$. Thus, there exists a fiber $\pi_h^{-1}(b)$ such that (1) the Haar measure of $K_\varepsilon \cap \pi_h^{-1}(b)$ is positive, (2) $b \notin \mathrm{Flat}(h))$ and (3) the dynamics of h in $\pi_h^{-1}(b)$ is uniquely ergodic (see Lemma 11.2). Then we can pick $x \in \pi_h^{-1}(b)$ such that $(h^k(x))_{k \geqslant 0}$ visits K_ε infinitely many times. The fifth assertion of Lemma 11.2 rules out case (c) because the twisting property implies that the projectivized derivative $[\![Dh_x^n]\!]$ tends to infinity, while it should be bounded along the sequence of times n for which $h^n(x) \in K_\varepsilon$. Case (b) is also excluded: under the action of h^n, tangent vectors projectively converge to the tangent space of the fibers, so the only possible invariant subspace is $\ker(D\pi_h)$. Thus we are in case (a) and moreover $E(x) = \ker D_x\pi_h$ for μ-almost every x. But then, using g instead of h and the fact that μ does not charge the algebraic curve along which the fibrations π_g and π_h are tangent, we get a contradiction. This shows that alternative (a) does not hold either, and this contradiction proves that μ is hyperbolic.

If μ is supported by a 2-dimensional real analytic subset $Y \subset X$, the same proof applies, except that we disintegrate μ along the singular foliation of Y by circles induced by π_h and use the fact that a generic leaf is a circle along which h is uniquely ergodic (see Lemma 11.2.(6)).

If X is a torus, then its tangent bundle is trivial and the differential of an automorphism is constant. In an appropriate basis, the differential of a Halphen twist h is of the form

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \text{ with } \alpha \neq 0. \tag{11.5}$$

Thus we are in case (a) with $E(x) = \ker D_x\pi_h$ for μ-almost every x. Using another twist g transverse to h we get a contradiction as before.

Since μ is invariant then the invariant measure m on $\mathcal{X}$ is equal to $\nu^{\mathbf{Z}} \times \mu$. In both cases $\mu \ll \mathsf{vol}_X$ and $\mu \ll \mathsf{vol}_Y$. The absolute continuity of the foliation by local Pesin unstable manifolds implies that the unstable conditionals of m cannot be atomic (see the classical argument showing that an absolutely continuous invariant measure has the SRB property, as in [87]). Thus positivity of the entropy follows from Corollary 7.16, and the proof of Theorem 11.7 is complete. □

11.5.2. *Proof of Theorem 11.8.* The proof is similar to that of Theorem 11.7 so we only sketch it. Assume by contradiction that μ is not hyperbolic; since X is a K3 surface, Corollary 7.7 shows that the sum of the Lyapunov exponents of μ vanishes; thus, each of them is equal to 0, and one of the alternatives of Theorem 11.11 holds, referred to as (a), (b), (c) on page 85. By assumption, Γ_ν contains a map f which is uniformly hyperbolic in some Zariski open set U, which is thus of full μ-measure. We denote by $x \mapsto E^u_f(x) \oplus E^s_f(x)$ the associated splitting of $TX|_U$. Since f is uniformly expanding/contracting on $E^{u/s}_f$, alternative (c) is not possible.

If alternative (a) holds, then $E(x)$ being f-invariant on a set of full measure, it must coincide with E^u_f or E^s_f, say with E^u_f. By continuity any $g \in \Gamma_\nu$ preserves E^u_f pointwise. On the other hand, E^u_f is everywhere tangent to an f-invariant (singular) holomorphic foliation $\mathcal{F}^u$, induced by a linear foliation on the torus A given by the Kummer structure. Every leaf of that foliation, except for a finite number of them, is biholomorphically equivalent to $\mathbf{C}$, and the Ahlfors-Nevanlinna currents of these entire curves are all equal to the unique closed positive current T^+_f that satisfies $\mathbf{M}(T^+_f) = 1$ and $f^*T^+_f = \lambda(f)T^+_f$ with $\lambda(f) > 1$. Now, pick any element g of Γ_ν. Since g preserves the line field E^u_f, g preserves $\mathcal{F}^u$ as well, hence also the ray $\mathbf{R}_+[T^+_f]$, contradicting the non-elementary assumption.

Finally, if alternative (b) holds, any $g \in \Gamma_\nu$ preserves $\{E^u_f(x), E^s_f(x)\}$ on a set of full measure so by the continuity of the hyperbolic splitting it must either preserve or swap these directions. Passing to an index 2 subgroup both directions are preserved, and we are back to case (a). □

12. Measure rigidity

In view of the results of Sections 10 and 11, it is natural to wonder whether a classification of invariant measures is possible without assuming the existence of parabolic elements in Γ. The results in this section belong to a thread of measure rigidity results starting with Rudolph's theorem [106] on Furstenberg's $\times 2 \times 3$ conjecture. If μ is a probability measure on X, we denote by $\mathsf{Aut}_\mu(X)$ the group of automorphisms of X preserving μ.

Theorem 12.1. *Let f be an automorphism of a compact Kähler surface X, preserving a totally real and real analytic surface $Y \subset X$. Let μ be an ergodic f-invariant measure on Y with positive entropy. Then*

(a) *either μ is absolutely continuous with respect to the Lebesgue measure on Y;*
(b) *or $\mathsf{Aut}_\mu(X)$ is virtually cyclic.*

If in addition the Lyapunov exponents of f with respect to μ satisfy $\lambda^s(f,\mu) + \lambda^u(f,\mu) \neq 0$, then case (a) does not occur, so $\mathsf{Aut}_\mu(X)$ is virtually cyclic.

This result, and its proof, may be viewed as a counterpart, in our setting, to Theorems 5.1 and 5.3 of [22]; again the possibility of invariant line fields is ruled out by using the complex structure. As before the typical case to keep in mind is when X is a projective surface defined over $\mathbf{R}$ and $Y = X(\mathbf{R})$. Observe that by ergodicity, if f preserves a smooth volume vol_Y, then in case (a) μ will be the restriction of vol_Y to an $\mathsf{Aut}_\mu(X)$-invariant Borel set of positive volume.

Proof of Theorem 12.1. Since it admits a measure of positive entropy, f is a loxodromic transformation. By the Ruelle-Margulis inequality μ is hyperbolic with respect to f and it does not charge any point, nor any piecewise smooth curve: indeed, the entropy of a homeomorphism of the circle or the interval is equal to zero.

We first assume that X is projective; non-projective surfaces will be studied at the end of the proof. For μ-almost every $x \in X$, the stable manifold $W^s(f, x)$ is an entire curve in X which is either transcendental or contained in a periodic rational curve (see [28, Thm. 6.2]). Since f has only finitely many invariant algebraic curves (see [28, Prop. 4.1]) and μ gives no mass to curves, $W^s(f, x)$ is μ-almost surely transcendental; then, the only Ahlfors-Nevanlinna current associated to $W^s(f, x)$ is T_f^+; similarly, the Ahlfors-Nevanlinna currents of the unstable manifolds give T_f^-. (This is the analogue in deterministic dynamics of Theorem 8.2.) Fix $g \in \mathsf{Aut}_\mu(X)$ and set $\Gamma := \langle f, g\rangle$. Our first goal is to prove the following:

Alternative: *either Γ^* is virtually cyclic and preserves $\{\mathbb{P}[T_f^+], \mathbb{P}[T_f^-]\} \subset \partial\mathbb{H}_X$; or μ is absolutely continuous with respect to the Lebesgue measure on Y.*

Let $Y' \subset Y$ be the union of the connected components of Y of positive μ-measure. The measure μ does not charge any analytic subset of Y of dimension $\leqslant 1$; thus, by analytic continuation, any $h \in \Gamma$ preserves Y'. So, without loss of generality we can replace Y by Y'.

We divide the argument into several cases according to the existence or non-existence of certain Γ-invariant line fields. In the first two cases we will conclude that Γ is elementary. In the third case, μ will be absolutely continuous with respect to the Lebesgue measure on Y; then by the Pesin formula its Lyapunov exponents satisfy $\lambda^u(f, \mu) = -\lambda^s(f, \mu) = h_\mu(f)$ so when $\lambda^u(f, \mu) + \lambda^s(f, \mu) \neq 0$, Case 3 is actually impossible.

Case 1.– *There exists a Γ-invariant measurable line field.* Specifically, we mean a measurable field of complex lines $x \mapsto E(x) \in \mathbb{P}(T_xX)$, defined on a set of full μ-measure, such that $D_xh(E(x)) = E(h(x))$ for every $h \in \Gamma$ and almost every $x \in X$; since μ is supported on the totally real surface Y, the field of real lines $E(x) \cap T_xY \subset T_xY$ is also invariant, and determines $E(x)$. Now, μ being ergodic and hyperbolic for f, the Oseledets theorem shows that either $E(x) = E^s_f(x)$ μ-almost everywhere or $E(x) = E^u_f(x)$ μ-almost everywhere. Changing f into f^{-1} if necessary, we may assume that $E(x) = E^s_f(x)$.

Consider the automorphism $h = g^{-1}fg \in \mathsf{Aut}_\mu(X)$. Since h is conjugate to f, μ is also ergodic and hyperbolic for h. Thus, either $E^s_h(x) = E^s_f(x)$ for μ-almost every x or $E^u_h(x) = E^s_f(x)$ for μ-almost every x.

Lemma 12.2. *If there is a measurable set A of positive measure along which $E^s_h(x) = E^s_f(x)$ (resp. $E^u_h(x) = E^s_f(x)$), then $W^s(f, x) = W^s(h, x)$ for almost every x in A (resp. $W^u(h, x) = W^s(f, x)$).*

Let us postpone the proof of this lemma and conclude the argument. Suppose first that $E^s_h(x) = E^s_f(x)$ on a subset A with $\mu(A) > 0$. Then $T_f^+ = T_h^+$ because for μ-almost every x, the unique Ahlfors-Nevanlinna current associated to the (complex) stable manifold $W^s(f, x)$ (resp. $W^s(h, x)$) is T_f^+ (resp. T_h^+). Since $T_h^+ = \mathbf{M}(g^*T_f^+)^{-1}g^*T_f^+$, we see that g, and therefore Γ itself, preserve the line $\mathbf{R}[T_f^+] \subset H^{1,1}(X)$. Since $[T_f^+]^2 = 0$, Γ fixes a point $\mathbb{P}[T_f^+]$ of the boundary $\partial\mathbb{H}_X$, so it is elementary. Since in addition Γ contains a loxodromic element, Theorem 3.2 of [28] shows that Γ^* is virtually cyclic.

Now, suppose that $E^u_h(x) = E^s_f(x)$ on A. Then, $T^-_h = T^+_f$ and the group generated by f and h is elementary. Since it contains a loxodromic element [28, Thm 3.2] says that $\langle f^*, h^*\rangle$ is virtually cyclic and fixes also $\mathbb{P}[T^-_f] \in \partial\mathbb{H}_X$. This implies that g, hence Γ, preserves the pair of boundary points $\{\mathbb{P}[T^+_f], \mathbb{P}[T^-_f]\} \subset \partial\mathbb{H}_X$. Thus, in both cases Γ^* is virtually cyclic and preserves $\{\mathbb{P}[T^+_f], \mathbb{P}[T^-_f]\} \subset \partial\mathbb{H}_X$.

Proof of Lemma 12.2. The argument is similar to that of Theorem 9.1, in a simplified setting, so we only sketch it. For μ-almost every x, $W^s(f,x)$ and $W^s(h,x)$ are tangent at x. Assume by contradiction that there exists a measurable subset A' of A of positive measure such that $W^s(f,x) \neq W^s(h,x)$ for every $x \in A'$. Then for small $\varepsilon > 0$ there exists two positive constants $r = r(\varepsilon)$ and $c = c(\varepsilon)$, an integer $k \geqslant 2$, and a measurable subset $\mathcal{G}_\varepsilon \subset A'$ such that $\mu(\mathcal{G}_\varepsilon) > 0$ and

- $W^s_{\rm loc}(f,x)$ and $W^s_{\rm loc}(h,x)$ are well defined and of size r for every $x \in \mathcal{G}_\varepsilon$,
- $W^s_{\rm loc}(f,x)$ and $W^s_{\rm loc}(h,x)$ depend continuously on x on $\mathcal{G}_\varepsilon \subset X$,
- $\mathrm{inter}_x(W^s_{\rm loc}(f,x), W^u_{\rm loc}(f,x)) = k$ for every $x \in \mathcal{G}_\varepsilon$,
- and $\mathrm{osc}_{(k,x,r)}(W^s_r(f,x), W^s_r(h,x)) \geqslant c$ for every $x \in \mathcal{G}_\varepsilon$.

Indeed, to get the first and second properties, one intersects A' with a large Pesin set $\mathcal{R}_\varepsilon$. On $A' \cap \mathcal{R}_\varepsilon$ the multiplicity of intersection $x \mapsto \mathrm{inter}_x(W^s_{\rm loc}(f,x), W^u_{\rm loc}(f,x))$ is semi-continuous, so we can find $k \geqslant 2$ and a subset $\mathcal{R}'_\varepsilon \subset (A' \cap \mathcal{R}_\varepsilon)$ of positive measure such that

$$\mathrm{inter}_x(W^s_{\rm loc}(f,x), W^u_{\rm loc}(f,x)) = k \tag{12.1}$$

for every $x \in \mathcal{R}'_\varepsilon$. Thus, the k-th osculation number is well defined, and the last property holds on a subset $\mathcal{G}_\varepsilon \subset \mathcal{R}'_\varepsilon$ of positive measure if c is small.

Let η^s be a Pesin partition subordinate to the local stable manifolds of f. Since $h_\mu(f) > 0$ the conditional measures $\mu(\cdot|\eta^s)$ are non-atomic. Thus there exists $x \in \mathcal{G}_\varepsilon$ such that x is an accumulation point of $\mathrm{Supp}\left(\mu(\cdot|\eta^s(x))|_{\mathcal{G}_\varepsilon\cap\eta^s(x)}\right)$. Fix a neighborhood N of x such that $W^s_r(f,x) \cap W^s_r(h,x) \cap N = \{x\}$, and then pick a sequence (x_j) of points in $\mathcal{G}_\varepsilon \cap \eta^s(x) \cap N$ converging to x. The local stable manifolds $W^s_r(h,x_j)$ form a sequence of disks of size r at x_j, each of them tangent to $W^s_r(f,x)$ (at x_j), and all of them disjoint from $W^s_r(h,x)$ (because x_j does not belong to $W^s_r(h,x)$). This contradicts Corollary 9.8, and the proof is complete. □

Case 2.– *There is a pair of distinct measurable line fields $\{E_1(x), E_2(x)\}$ invariant under Γ.* Again by the Oseledets theorem applied to f, necessarily $\{E_1(x), E_2(x)\} = \{E^s_f(x), E^u_f(x)\}$. For μ-almost every x, $g(\{E^s_f(x), E^u_f(x)\}) = \{E^s_f(g(x)), E^u_f(g(x))\}$. As before, consider $h = g^{-1}fg \in \mathsf{Aut}_\mu(X)$. Since h is conjugate to f, it is hyperbolic and ergodic with respect to μ, and $\{E^s_f(x), E^u_f(x)\} = \{E^s_h(x), E^u_h(x)\}$ for almost every x. Replacing h by h^{-1} if necessary, there exists a set A of positive measure for which $E^s_h(x) = E^s_f(x)$, and we conclude as in Case 1.

Case 3.– *There is no Γ-invariant line field or pair of line fields.* In other words, Cases 1 or 2 are now excluded. This part of the argument is identical to the proof of [22, Thm 5.1.a].

First, we claim that there exists $g_1, g_2 \in \Gamma$ and a subset A of positive measure such that $D_xg_1(E^s_f(x)) \notin \{E^s_f(g_1(x)), E^u_f(g_1(x))\}$ and $D_xg_2(E^u_f(x)) \notin \{E^s_f(g_2(x)), E^u_f(g_2(x)\}$ for every x in A. Indeed since we are not in Case 2 (possibly switching E^u_f and E^s_f) there exists $g_1 \in \Gamma$ and a set A of positive measure such that for $x \in A$, $D_xg_1(E^s_f(x)) \not\subset E^s_f(g_1(x)) \cup E^u_f(g_1(x))$.

Since we are not in Case 1, there exists $g \in \Gamma$ and a set B of positive measure such that for $x \in B$, $D_xg(E^u_f(x)) \neq E^u_f(g(x))$. If $D_xg(E^s_f(x)) \in \{E^s_f(g(x)), E^u_f(g(x))\}$ on a subset B' of B of positive measure, then choose $k > 0$ and $\ell > 0$ such that $\mu(f^\ell(A) \cap B') > 0$ and $\mu(f^k(g(f^\ell(A))) \cap A) > 0$ and define $g_2 = g_1 f^k g f^\ell$; otherwise, set $g_2 = gf^\ell$ with ℓ such that $\mu(f^\ell(A) \cap B) > 0$. Then change A into $A = A \cap f^{-\ell}(B')$ (resp. $A \cap f^{-\ell}(B)$).

Denote by Δ the simplex $\{(a,b,c,d) \in (\mathbf{R}^*_+)^4\ ;\ a+b+c+d = 1\}$. For $\alpha = (a,b,c,d)$ in Δ, let ν_α be the probability measure $\nu_\alpha = a\delta_f + b\delta_{f^{-1}} + c\delta_g + d\delta_{g^{-1}}$. Then μ is ν_α-stationary and since μ is f-ergodic and $\nu_\alpha(\{f\}) > 0$, it is also ergodic as a ν_α-stationary measure (see [12, §2.1.3]). Since we are not in Cases 1 or 2 and μ is hyperbolic for f, Theorems 11.10 and 11.11 imply that the Lyapunov exponents of μ, viewed as a ν_α-stationary measure, satisfy $\lambda^-_\alpha(\mu) < \lambda^+_\alpha(\mu)$.

Lemma 12.3. *There exists a choice of $\alpha \in \Delta$ such that μ is a hyperbolic ν_α-stationary measure, i.e. $\lambda^-_\alpha(\mu) < 0 < \lambda^+_\alpha(\mu)$*

Proof. This is automatic when f and g are volume preserving because $\lambda^-_\alpha(\mu) = -\lambda^+_\alpha(\mu)$ in that case. For completeness, let us copy the proof given in [22, §13.2.4]. The assumptions of Case 3 and the strict inequality $\lambda^-(\mu) < \lambda^+(\mu)$ imply that

$$\alpha \in \Delta \mapsto (\lambda^-_\alpha(\mu), \lambda^+_\alpha(\mu)) \in \mathbf{R}^2 \tag{12.2}$$

is continuous (see [22, Prop. 13.7] or [110, Chap. 9]). Since $\lambda^-_\alpha(\mu) < \lambda^+_\alpha(\mu)$ for every $\alpha \in \Delta$, one of λ^-_α and λ^+_α is non zero. Furthermore, μ being invariant, the involution $(a,b,c,d) \mapsto (b,a,d,c)$ interchanges the Lyapunov exponents. It follows that $P = \{\alpha \in \Delta, \lambda^+_\alpha > 0\}$ and $N = \{\alpha \in \Delta, \lambda^-_\alpha < 0\}$ are non-empty open subsets of Δ such that $P \cup N = \Delta$. The connectedness of Δ implies $P \cap N \neq \varnothing$, as was to be shown. □

Fix $\alpha \in \Delta$ such that μ is hyperbolic as a ν_α-stationary measure. The assumptions of Case 3 imply that the stable directions depend on the itinerary so the main result of [22] shows that μ is fiberwise SRB (on the surface Y), that is, the unstable conditionals of the measures μ_x (here $\mu_x = \mu$) are given by the Lebesgue measure (in some natural affine parametrizations of the unstable manifolds by the real line $\mathbf{R}$). Since μ is invariant, we can revert the stable and unstable directions by applying the argument to F^{-1}, and we conclude that the stable conditionals are given by the Lebesgue measure as well. The absolute continuity property of the stable and unstable laminations then implies that μ is absolutely continuous with respect to the Lebesgue measure on Y.

Conclusion.– Let us assume that μ is not absolutely continuous with respect to the Lebesgue measure on Y. The above alternative holds for all subgroups $\Gamma = \langle f, g\rangle$, with $g \in \mathsf{Aut}_\mu(X)$ arbitrary. Therefore, if X is projective, we deduce that $\mathsf{Aut}_\mu(X)^*$ preserves $\{\mathbb{P}[T^+_f], \mathbb{P}[T^-_f]\} \subset \partial\mathbb{H}_X$, which implies that $\mathsf{Aut}_\mu(X)^*$ is virtually cyclic. By Lemma 3.20, $\mathsf{Aut}_\mu(X)^*$ is also virtually cyclic when X is not projective. So the only remaining issue is to prove that $\mathsf{Aut}_\mu(X)$ itself is virtually cyclic. If this is not the case, then $\mathsf{Aut}(X)^\circ$ is infinite, X must be a torus $\mathbf{C}^2/\Lambda$ (see Proposition 3.18), and $\mathsf{Aut}_\mu(X) \cap \mathsf{Aut}(X)^\circ$ is a normal subgroup of $\mathsf{Aut}_\mu(X)$ containing infinitely many translations. This group is a closed subgroup of the compact Lie group $\mathsf{Aut}(X)^\circ = \mathbf{C}^2/\Lambda$; thus, the connected component of the identity of $\mathsf{Aut}_\mu(X) \cap \mathsf{Aut}(X)^\circ$ is a (real) torus $H \subset \mathbf{C}^2/\Lambda$ of positive dimension. This torus is invariant under the action of f by conjugacy. Since $X = \mathbf{C}^2/\Lambda$, f is a complex linear Anosov diffeomorphism of X, and it follows that $\dim_{\mathbf{R}}(H) \geqslant 2$. Being H-invariant, μ is then absolutely continuous with respect to the Lebesgue measure of Y; this contradiction completes the proof. □

Remark 12.4. Theorem 12.1 can be extended to the case of singular analytic subsets Y, after minor adjustments of the proof, because μ cannot charge its singular locus.

It is natural to expect that the positive entropy assumption in Theorem 12.1 could be replaced by a much weaker assumption, namely, "*μ gives no mass to proper Zariski closed subsets*". In full generality this seems to exceed the scope of techniques of this paper, however we are able to deal with a special case.

Theorem 12.5. *Let f be a Kummer example on a compact Kähler surface X. Let μ be an atomless, f-invariant, and ergodic probability measure that is supported on a totally real, real analytic surface $Y \subset X$. If $g \in \mathsf{Aut}(X)$ preserves μ, then:*

(a) *either μ is absolutely continuous with respect to vol_Y;*
(b) *or $\langle f, g\rangle$ is virtually isomorphic to $\mathbf{Z}$.*

Thus, as in the case of subgroups containing parabolic transformations, the stiffness Theorem 10.10 takes a particularly strong form when $\mathrm{Supp}(\nu)$ contains a Kummer example.

Proof. Let us start with a preliminary remark. Assume that $\mu(C) > 0$ for some irreducible curve $C \subset X$; since μ does not charge any point the support of $\mu_{|C}$ is Zariski dense in C, and C is an f-periodic curve. But f being a Kummer example, such a curve is a rational curve $C \simeq \mathbb{P}^1(\mathbf{C})$ (obtained by blowing-up a periodic point of a linear Anosov map on a torus), on which f has a north-south dynamics; thus, all f-invariant probability measures on C are atomic, and we get a contradiction. This means that the assumption "*μ has no atom*" is equivalent to the assumption "*μ gives no mass to proper Zariski closed subsets of X*".

Now, we follow step by step the proof of Theorem 12.1, only insisting on the points requiring modification. Since μ does not charge any curve, we can contract all f-periodic curves, and lift (f, μ) to $(\tilde f, \tilde\mu)$, where $\tilde f$ is a linear Anosov diffeomorphism of some compact torus $\mathbf{C}^2/\Lambda$ and $\tilde\mu$ is an $\tilde f$-invariant probability measure (see [37] for details on Kummer examples). We deduce that $\tilde\mu$ is hyperbolic for $\tilde f$ and then, coming back to X, that μ is hyperbolic for f. Case 3 of the proof of Theorem 12.1 only requires hyperbolicity of μ so it carries over in this case without modification. In Cases 1 and 2 we have to show that if $\Gamma = \langle f, g\rangle$ preserves a measurable line field or a pair of measurable line fields then Γ^* is elementary. In either case we consider $h = gfg^{-1}$ and up to possibly replacing E^u_f by E^s_f and h by h^{-1}, we have that $E^s_f(x) = E^s_h(x)$ on a set of positive measure. But now f and h are Kummer examples so their respective stable foliations $\mathcal{F}^s_f$ and $\mathcal{F}^s_h$ are (singular) holomorphic foliations. From the previous reasoning $\mathcal{F}^s_f$ and $\mathcal{F}^s_h$ are tangent on a set of positive μ-measure, so, since μ gives no mass to subvarieties we infer that $\mathcal{F}^s_f = \mathcal{F}^s_h$ and we conclude exactly as in Theorem 11.8. □

Unlike most results in this paper, Theorem 12.1 can be extended to a rigidity theorem for polynomial automorphisms of $\mathbf{R}^2$ with essentially the same proof.

Theorem 12.6. *Let f be a polynomial automorphism of $\mathbf{R}^2$. Let μ be an ergodic f-invariant measure with positive entropy supported on $\mathbf{R}^2$. If $g \in \mathsf{Aut}(\mathbf{R}^2)$ satisfies $g_*\mu = \mu$, then:*

(a) *either f and g are conservative and μ is the restriction of $\mathrm{Leb}_{\mathbf{R}^2}$ to a Borel set of positive measure invariant under f and g;*

(b) *or the group generated by f and g is solvable and virtually cyclic; in particular, there exists $(n,m) \in \mathbf{Z}^2 \backslash \{(0,0)\}$ such that $f^n = g^m$.*

Proof. We briefly explain the modifications required to adapt the proof of Theorem 12.1, and leave the details to the reader. We freely use standard facts from the dynamics of automorphisms of $\mathbf{C}^2$. Let f and g be as in the statement of the theorem, and set $\Gamma = \langle f, g\rangle$.

Since its entropy is positive, f is of Hénon type in the sense of [82]: this means that f is conjugate to a composition of generalized Hénon maps, as in [62], Theorem 2.6. Thus, the support of μ is a compact subset of $\mathbf{C}^2$, because the basins of attraction of the line at infinity for f and f^{-1} cover the complement of a compact set; moreover, as in Theorem 12.1, μ cannot charge any proper Zariski closed subset.

Let γ be an arbitrary element of Γ; then $h := \gamma^{-1} f \gamma$ is also of Hénon type. We run through Cases 1, 2 and 3 as in the proof of Theorem 12.1. Case 3 is treated exactly in the same way as above and implies that μ is absolutely continuous. This in turn implies that the Jacobian of f, a constant $\mathrm{Jac}(f) \in \mathbf{C}^*$ since $f \in \mathsf{Aut}(\mathbf{C}^2)$, is equal to ± 1; and since μ is ergodic for f, it must be the restriction of $\mathrm{Leb}_{\mathbf{R}^2}$ to some Γ-invariant subset. In Cases 1 and 2, arguing as before and keeping the same notation, we arrive at $W^s(h,x) = W^s(f,x)$ or $W^u(f,x)$ on a set of positive measure. For a Hénon type automorphism of $\mathbf{C}^2$, the closure of any stable manifold is equal to the forward Julia set J^+, and J^+ carries a unique positive closed current T^+ of mass 1 relative to the Fubini Study form in $\mathbb{P}^2(\mathbf{C})$ (see [109]). So we infer that $T^+_h = T^+_f$ or $T^+_h = T^-_f$; as a consequence, the Green functions of f and h satisfy $G^+_h = G^+_f$ or $G^+_h = G^-_f$, respectively.

Automorphisms of $\mathbf{C}^2$ act on the Bass-Serre tree of $\mathsf{Aut}(\mathbf{C}^2)$, each automorphism $u \in \mathsf{Aut}(\mathbf{C}^2)$ giving rise to an isometry u_* of the tree. Hénon type automorphisms act as loxodromic isometries; the axis of such an isometry u_* will be denoted $\mathrm{Geo}(u_*)$: it is the unique u_*-invariant geodesic, and u_* acts as a translation along its axis. Theorem 5.4 of [82] shows that $G^+_h = G^+_f$ implies $\mathrm{Geo}(h_*) = \mathrm{Geo}(f_*)$; changing f into f^{-1}, $G^+_h = G^-_f$ gives $\mathrm{Geo}(h_*) = \mathrm{Geo}(f_*^{-1}) = \mathrm{Geo}(f_*)$ because the axis of f_* and f_*^{-1} coincide. Since γ_* maps $\mathrm{Geo}(f_*)$ onto $\mathrm{Geo}(h_*)$, we deduce that Γ preserves the axis of f; so, all elements u of Γ of Hénon type satisfy $\mathrm{Geo}(u_*) = \mathrm{Geo}(f_*)$. From [82, Prop. 4.10], we conclude that Γ is solvable and virtually cyclic. $\square$

Remark 12.7. With the techniques developed in [27], the same result applies to the dynamics of $\mathsf{Out}(\mathbb{F}_2)$ acting on the real part of the character surfaces of the once punctured torus.

APPENDIX A. GENERAL COMPACT COMPLEX SURFACES

Here, we study the concept of non-elementary groups of automorphisms on (non Kähler) compact complex surfaces. We show that the two possible definitions of non-elementary group are equivalent and force the surface to be Kähler.

Let M be a compact manifold. We say that a group Γ of homeomorphisms of M is **cohomologically non-elementary** if its image Γ^* in $\mathsf{GL}(H^*(M;\mathbf{Z}))$ contains a non-Abelian free subgroup, and that Γ is **dynamically non-elementary** if it contains a non-Abelian free group Γ_0 such that the topological entropy of every $f \in \Gamma_0 \backslash \{\mathrm{id}\}$ is positive. When M is a compact Kähler surface and $\Gamma \subset \mathsf{Aut}(M)$, Theorem 3.2 of [28] and the fact that parabolic automorphisms have zero entropy imply that Γ is non-elementary (in the sense of Section 2.3.3) if and only if it is cohomologically non-elementary, if and only if it is dynamically non-elementary.

Lemma A.1. *Let M be a compact manifold, and Γ be a subgroup of $\mathsf{Diff}^\infty(M)$. If Γ is cohomologically non-elementary, then Γ is dynamically non-elementary.*

Proof. We split the proof in two steps, the first one concerning groups of matrices, and the second one concerning topological entropy.

Step 1.- Γ^* *contains a free subgroup Γ_1^*, all of whose non-trivial elements have spectral radius larger than* 1.

The proof uses basic ideas involved in Tits's alternative, here in the simple case of subgroups of $\mathsf{GL}_n(\mathbf{Z})$. Let N be the rank of $H^*_{t.f.}(M;\mathbf{Z})$, where $t.f.$ stands for "torsion free". Fix a basis of this free $\mathbf{Z}$-module. Then Γ^* determines a subgroup of $\mathsf{GL}_N(\mathbf{Z})$. Our assumption implies that the derived subgroup of Γ^* contains a non-Abelian free group Γ_0^* of rank 2.

If all (complex) eigenvalues of all elements of Γ_0^* have modulus $\leqslant 1$, then by Kronecker's lemma all of them are roots of unity. This implies that Γ_0^* contains a finite index nilpotent subgroup (see Proposition 2.2 and Corollary 2.4 of [8]), contradicting the existence of a non-Abelian free subgroup. Thus, there is an element f^* in Γ_0^* with a complex eigenvalue of modulus $\alpha > 1$. Let m be the number of eigenvalues of f^* of modulus α, counted with multiplicities. Consider the linear representation of Γ_0^* on $\bigwedge^m H^*(M;\mathbf{C})$; the action of f^* on this space has a unique dominant eigenvalue, of modulus α^m; the corresponding eigenline determines an attracting fixed point for f^* in the projective space $\mathbb{P}(\bigwedge^m H^*(M;\mathbf{C}))$; the action of f^* on this topological space is proximal.

Let

$$\{0\} = W_0 \subset W_1 \subset \cdots \subset W_k \subset W_{k+1} = \bigwedge^m H^*(M;\mathbf{C}) \tag{A.1}$$

be a Jordan-Hölder sequence for the representation of Γ^*: the subspaces W_i are invariant, and the induced representation of Γ^* on W_{i+1}/W_i is irreducible for all $0 \leqslant i \leqslant k$. Let V be the quotient space W_{i+1}/W_i in which the eigenvalue of f^* of modulus α^m appears. Since Γ_0^* is contained in the derived subgroup of Γ, the linear transformation of V induced by f^* has determinant 1; thus, $\dim(V) \geqslant 2$. Now, we can apply Lemma 3.9 of [8] to (a finite index, Zariski connected subgroup of) $\Gamma_0^*|_V$: changing f is necessary, both $f^*|_V$ and $(f^{-1})^*|_V$ are proximal, and there is an element g^* in Γ^* that maps the attracting fixed points a_f^+ and $a_f^- \in \mathbb{P}(V)$ of $f^*|_V$ and $(f^*|_V)^{-1}$ to two distinct points (i.e. $\{a_f^+, a_f^-\} \cap \{g(a_f^+), g(a_f^-)\} = \varnothing$) ; then, by the ping-pong lemma, large powers of f^* and $g^* \circ f^* \circ (g^*)^{-1}$ generate a non-Abelian free group $\Gamma_1^* \subset \Gamma^*$

such that each element $h^* \in \Gamma_1^* \backslash \{\mathrm{id}\}$ has an attracting fixed point in $\mathbb{P}(V)$. This implies that every element of $\Gamma_1^* \backslash \{\mathrm{id}\}$ has an eigenvalue of modulus > 1 in $H^*(M; \mathbf{C})$.

Step 2.- Since Γ_1^* is free, there is a free subgroup $\Gamma_1 \subset \Gamma$ such that the homomorphism $\Gamma_1 \mapsto \Gamma_1^*$ is an isomorphism. By Yomdin's theorem [115], all elements of $\Gamma_1 \backslash \{\mathrm{id}\}$ have positive entropy, and we are done. □

Theorem A.2. *Let M be a compact complex surface, and Γ be a subgroup of* $\mathsf{Aut}(M)$. *Then, Γ is cohomologically non-elementary if and only if it is dynamically non-elementary. If such a subgroup exists, then M is a projective surface.*

Proof. Indeed it was shown in [24] that every compact complex surface possessing an automorphism of positive entropy is Kähler. Thus, the first assertion follows from Lemma A.1 and Theorem 3.2 of [28], and the second one follows from Theorem E. □

APPENDIX B. STRONG LAMINARITY OF AHLFORS CURRENTS

In this appendix, we sketch the proof of Lemma 8.8, explaining how to adapt arguments of [7, 52, 53], written for $X = \mathbb{P}^2(\mathbf{C})$, to our context.

Proof of Lemma 8.8. Let (Δ_n) be a sequence of unions of disks, as in the definition of injective Ahlfors currents, such that $\frac{1}{\mathbf{M}(\Delta_n)} \{\Delta_n\}$ converges to T. Since X is projective we can choose a finite family of meromorphic fibrations $\varpi_i : X \dashrightarrow \mathbb{P}^1$ such that

- the general fibers of ϖ_i are smooth curves of genus $\geqslant 2$;
- for every $x \in X$, there are at least two of the fibrations ϖ_i, denoted for simplicity by ϖ_1 and ϖ_2, which are well defined in some neighborhood U_x of x (x is not a base point of the corresponding pencils), satisfy $(d\varpi_1 \wedge d\varpi_2)(x) \neq 0$ (the fibrations are transverse), and for which the fibers $\varpi_k^{-1}(\varpi_k(x))$ containing x are smooth.

If we blow-up the base points of ϖ_k, $k = 1, 2$, we obtain a new surface $X' \to X$ on which each ϖ_k lifts to a regular fibration ϖ_k'; the open neighborhood U_x is isomorphic to its preimage in X' so, when working on U_x, we can do as if the two fibrations ϖ_k were local submersions with smooth fibers of genus $\geqslant 2$.

To construct T_r, we follow the proof of [53, Proposition 4.4] (see also [52, Proposition 3.4]). The construction will work as follows: we fix a sequence (r_j) converging to zero, and for every j we extract from $\frac{1}{\mathbf{M}(\Delta_n)} \{\Delta_n\}$ a current T_{n,r_j} made of disks of size $\approx r_j$ which are obtained from Δ_n by only keeping graphs of size r_j over one of the projections ϖ_i.

By a covering argument, it is enough to work locally near a point x, with two projections ϖ_1 and ϖ_2 as above. Let $S \subset \mathbf{C}$ be the unit square $\{x + \mathrm{i}y \,;\ 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\} \simeq [0, 1]^2$. To simplify the exposition, we may assume that

$$\varpi_k(U_x) = S \subset \mathbf{C} \subset \mathbb{P}^1(\mathbf{C}) \quad (\text{for } k = 1, 2). \tag{B.1}$$

Set $r_j = 2^{-j}$ and consider the subdivision $\mathcal{Q}_j$ of $S \simeq [0, 1]^2$ into 4^j squares Q of size r_j. A connected component of $\Delta_n \cap \varpi_k^{-1}(Q)$, for such a small square Q, is called a graph (with respect to ϖ_k) if it lifts to a local section of the fibration $\varpi_k' \colon X' \to \mathbb{P}^1(\mathbf{C})$ above Q. Then, we fix j, intersect Δ_n with $\varpi_k^{-1}(Q)$, and keep only the components of $\varpi_k^{-1}(Q \cap \Delta_n)$, $Q \in \mathcal{Q}_j$

which are graphs with respect to ϖ_k. Such a family of graphs is normal because the fibers of ϖ'_k have genus $\geqslant 2$ (compare to Lemma 3.5 of [52]).

This being done, we can copy the proof of [53, Proposition 4.4]. Letting n go to $+\infty$ and extracting a converging subsequence, we obtain a uniformly laminar current $T_{\mathcal{Q}_j,k} \leqslant T$. Away from the base points of ϖ_k, $T_{\mathcal{Q}_j,k}$ is made of disks of size $\asymp r_j$ which are limits of disks contained in the Δ_n. Combining the two currents $T_{\mathcal{Q}_j,k}$, we get a current $T_{r_j} \leqslant T$ which is uniformly laminar in every cube $\varpi_1^{-1}(Q) \cap \varpi_2^{-1}(Q')$, $Q, Q' \in \mathcal{Q}_j$, and such that

$$\text{(B.2)} \qquad \langle T - T_{r_j}, \varpi_1^* \kappa_{\mathbb{P}^1} + \varpi_1^* \kappa_{\mathbb{P}^1} \rangle \leqslant \langle T - T_{\mathcal{Q}_j,1}, \varpi_1^* \kappa_{\mathbb{P}^1} \rangle + \langle T - T_{\mathcal{Q}_j,2}, \varpi_2^* \kappa_{\mathbb{P}^1} \rangle,$$

where $\kappa_{\mathbb{P}^1}$ is the Fubini-Study form. By definition, T will be strongly approximable if locally $\mathbf{M}(T - T_{r_j}) \leqslant O(r_j^2)$. Using the fact that $\varpi_1^* \kappa_{\mathbb{P}^1} + \varpi_1^* \kappa_{\mathbb{P}^1} \geqslant C\kappa_0$ and the Inequality (B.2), it will be enough to show that $\langle T - T_{\mathcal{Q}_j,k}, \varpi_k^* \kappa_{\mathbb{P}^1} \rangle = O(r_j^2)$ for $k = 1, 2$. This itself reduces to counting (with multiplicity) the number of "good components" of Δ_n for the projections $\varpi_k : \Delta_n \to \mathcal{Q}_j$ that is, the components above the squares Q of Q_j that are kept in the above contruction of $T_{\mathcal{Q}_j,k}$ (the graphs relative to ϖ_k).

The counting argument is identical to [7, §7], except that we apply the Ahlfors theory of covering surfaces to a union of disks, not just one. For notational ease, set $\varpi = \varpi_k$, $r = r_j$ and $\mathcal{Q} = \mathcal{Q}_j$; $\mathcal{Q}$ is a subdivision of $S \simeq [0,1]^2$ by squares of size 2^{-j}. We decompose $\mathcal{Q}$ as a union of four non-overlapping subdivisions $\mathcal{Q}^\ell$, $\ell = 1, 2, 3, 4$; by this we mean that for each ℓ, the squares $Q \in \mathcal{Q}^\ell$ have disjoint closures $\overline{Q}$. Fix such an ℓ and let $q = \#\mathcal{Q}^\ell = 4^{j-1}$. Applying Ahlfors' theorem to each of the disks constituting Δ_n and summing over these disks, we deduce that the number of good components $N(\mathcal{Q}^\ell)$ satisfies ([9])

$$\text{(B.3)} \qquad N(\mathcal{Q}^\ell) \geqslant (q - 4)\,\mathrm{area}_{\mathbb{P}^1}(\Delta_n) - h\,\mathrm{length}_{\mathbb{P}^1}(\partial \Delta_n),$$

where $\mathrm{area}_{\mathbb{P}^1}$ (resp. $\mathrm{length}_{\mathbb{P}^1}$) is the area of the projection $\varpi(\Delta_n)$ (resp. length of $\varpi(\partial\Delta_n)$), counted with multiplicity, and h is a constant that depends only on the geometry of $\mathcal{Q}^\ell$. Dividing by $\mathrm{area}_{\mathbb{P}^1}(\Delta_n)$, using $\mathrm{length}_{\mathbb{P}^1}(\partial\Delta_n) = o(\mathrm{area}_{\mathbb{P}^1}(\Delta_n))$, which is guaranteed by Ahlfors' construction, and letting n go to $+\infty$, we obtain

$$\text{(B.4)} \qquad \langle T_{\mathcal{Q}}|_{\mathcal{Q}^\ell}, \varpi^* \kappa_{\mathbb{P}^1} \rangle \geqslant (q-4) r^2 = \mathrm{area}_{\mathbb{P}^1}\Big(\bigcup\nolimits_{S \in \mathcal{Q}^\ell} S\Big) - 4r^2.$$

Finally, summing from $\ell = 1$ to 4, we see that, relative to $\varpi^*\kappa_{\mathbb{P}^1}$, the mass lost by discarding the bad components of size r in T is of order $O(r^2)$: this is precisely the required estimate.

Let us now justify the geometric intersection statement, following step by step the proof of [53, Thm. 4.2]: let S be a current with continuous normalized potential on X; we have to show that $S \wedge T_r$ increases to $S \wedge T$ as r decreases to 0. Again the result is local so we work near x, use the projections ϖ_1 and ϖ_2, and keep notation as above. Given squares $Q, Q' \in \mathcal{Q}$ and a real number $\lambda < 1$, we denote by λQ the homothetic of Q of factor λ with respect to its center, and by $C(Q, Q')$ the cube $\varpi_1^{-1}(Q) \cap \varpi_2^{-1}(Q')$. Fix $\varepsilon > 0$. We want to show that for $r \leqslant r(\varepsilon)$, the mass of $(T - T_r) \wedge S$ is smaller than ε. The first observation is that there exists $\lambda(\varepsilon) \in (0, 1)$, independent of r, such that translating $\mathcal{Q}$ if necessary, the mass of $T \wedge S$ concentrated in $\bigcup_{Q,Q'} C(Q, Q') \setminus C(\lambda Q, \lambda Q')$ is smaller than $\varepsilon/2$ (see [53, Lem. 4.5]). Fix such a λ. It only remains to estimate the mass of $(T - T_r) \wedge S$ in $\bigcup_{Q,Q'} C(\lambda Q, \lambda Q')$. In such a

[9] The term $(q-4)$ instead of $(q-2)$ in [7] is due to the fact that we are projecting on $\mathbb{P}^1$ and not on $\mathbf{C}$.

cube $C(\lambda Q, \lambda Q')$ the argument presented in [53, pp. 123-124], based on an integration by parts, gives the estimate

$$\int_{C(\lambda Q,\lambda Q')} (T - T_r) \wedge S \leqslant C(\lambda)\mathrm{modc}(u_S, r)\frac{1}{r^2}\mathbf{M}\left((T - T_r)|_{C(Q,Q')}\right), \tag{B.5}$$

where $\mathrm{modc}(u_S, r)$ is the modulus of continuity of the potential u_S of S. To conclude, we sum over all squares Q, Q' and use the estimate $M(T - T_r) = O(r^2)$ to get that

$$\mathbf{M}\left((T - T_r)|_{\bigcup_{Q,Q'} C(\lambda Q,\lambda Q')}\right) \leqslant C\omega(u_S, r). \tag{B.6}$$

This is smaller than $\varepsilon/2$ if $r \leqslant r(\varepsilon)$. ☐

Appendix C. Proof of Theorem 11.11

Let us consider a random dynamical system (X, ν) and μ an ergodic stationary measure, as in Theorem 11.11. We keep the notation from §11.4.

We say that a sequence of real numbers $(u_n)_{n\geqslant 0}$ **almost converges towards** $+\infty$ if for every $K \in \mathbf{R}$, the set $L_K = \{n \in \mathbf{N}\ ;\ u_n \leqslant K\}$ has an asymptotic lower density

$$\underline{\mathrm{dens}}(L_K) := \liminf_{n\to+\infty}\left(\frac{\sharp(L_K \cap [0, n])}{n + 1}\right) \tag{C.1}$$

which is equal to 0: $\underline{\mathrm{dens}}(L_K) = 0$ for all K.

Lemma C.1. *The set of points $\mathscr{x} = (\omega, x)$ in $\mathcal{X}_+$ such that $[\![D_x f^n_\omega]\!]$ almost converges towards $+\infty$ on $\mathbb{P}(T_xM)$ is F_+-invariant. In particular, by ergodicity,*

(a) *either $[\![D_x f^n_\omega]\!]$ almost converges towards $+\infty$ for $(\nu^{\mathbf{N}} \times \mu)$-almost every (ω, x);*
(b) *or, for $(\nu^{\mathbf{N}} \times \mu)$-almost every (ω, x), there is a sequence (n_i) with positive lower density along which $[\![D_x f^{n_i}_\omega]\!]$ is bounded.*

The proof is straightforward. We are now ready for the proof of Theorem 11.11. Let us first emphasize one delicate issue: in Conclusion (1) of the theorem, it is important that the directions E (resp. E_1 and E_2) only depend on $x \in X$ (and not on $\mathscr{x} = (x, \omega) \in \mathcal{X}_+$). Likewise in Conclusion (2), the trivialization P_x should depend only on x. This justifies the inclusion of a detailed proof of Theorem 11.11, since in the slightly different setting of [5], the authors did not have to check this point carefully.

We fix a measurable trivialization $P\colon TX \to X \times \mathbf{C}^2$, given by linear isometries $P_x\colon T_xX \to \mathbf{C}^2$, where T_xX is endowed with the hermitian form $(\kappa_0)_x$, and $\mathbf{C}^2$ with its standard hermitian form. This trivialization conjugates the action of DF_+ to that of a cocycle $A\colon \mathcal{X}_+ \times \mathbf{C}^2 \to \mathcal{X}_+ \times \mathbf{C}^2$ over F_+. We denote by $A_{\mathscr{x}}\colon \{\mathscr{x}\} \times \mathbf{C}^2 \to \{F_+(\mathscr{x})\} \times \mathbf{C}^2$ the induced linear map; observe that $A_{\mathscr{x}} = A_{(\omega,x)}$ depends only on x and on the first coordinate $f^1_\omega = f_0$ of ω. Using P we transport the measure $\hat{\mu}$ to a measure, still denoted by $\hat{\mu}$, on the product space $X \times \mathbb{P}^1(\mathbf{C})$. By our invariance assumption, its disintegrations $\hat{\mu}_{\mathscr{x}} = \hat{\mu}_x$ satisfy $(\mathbb{P}A_{\mathscr{x}})_*\hat{\mu}_{\mathscr{x}} = \hat{\mu}_{F_+(\mathscr{x})} = \hat{\mu}_{f^1_\omega(x)}$.

The bounded case. – In this paragraph we show that in the essentially bounded case (b) of Lemma C.1, Conclusion (2) of Theorem 11.11 holds. We streamline the argument following the proof of [5, Prop. 4.7] which deals with the more general case of $\mathsf{GL}(d, \mathbf{R})$-cocycles, and is itself a variation on previously known ideas (see e.g. [1, 116]).

Set $G = \mathsf{PGL}(2, \mathbf{C})$, and define the G-extension $\widetilde{F}_+$ of F_+ on $\mathcal{X}_+ \times G$ by

$$\widetilde{F}_+(x, g) = (F_+(x), \mathbb{P}(A_x)g) = ((\sigma(\omega), f^1_\omega(x)), \mathbb{P}(A_{(\omega,x)})g) \tag{C.2}$$

for every $x = (\omega, x)$ in $\mathcal{X}_+$ and g in G; thus $\widetilde{F}_+$ is given by F_+ on $\mathcal{X}_+$ and is the multiplication by $\mathbb{P}(A_x)$ on G. Since $\mathbb{P}(A_{(\omega,x)})$ depends on ω only through its first coordinate, $\widetilde{F}_+$ can be interpreted as the skew product map associated to a random dynamical system on $X\times G$. Denote by $\mathcal{P}$ the convolution operator associated to this random dynamical system; thus $\mathcal{P}$ acts on probability measures on $X \times G$. Let $\mathrm{Prob}_\mu(X \times G)$ the set of probability measures on $X \times G$ projecting to μ under the natural map $X\times G \to X$. Since μ is stationary, $\mathcal{P}$ maps $\mathrm{Prob}_\mu(X\times G)$ to itself.

Recall that by assumption there is a set E of positive measure in $\mathcal{X}_+$, a compact subset K_G of G, and a positive real number ε_0 such that

$$\underline{\mathrm{dens}}\left\{n \ ; \ \mathbb{P}(A^{(n)}_x) \in K_G\right\} \geqslant \varepsilon_0 \tag{C.3}$$

for all x in E.

Lemma C.2. *There exists an ergodic, stationary, Borel probability measure $\widetilde{\mu}_G$ on $X \times G$ with marginal measure μ on X.*

Proof. (See [5, Prop. 4.13] for details). Let $\widetilde{\mu}_G$ be any cluster value of the sequence of probability measures

$$\frac{1}{N}\sum_{i=0}^{N-1} \mathcal{P}^i(\mu \times \delta_{1_G}). \tag{C.4}$$

By the boundedness assumption, $\widetilde{\mu}_G$ has mass $M \geqslant \varepsilon_0$ and is stationary (i.e. $\mathcal{P}$-invariant). Standard arguments show that its projection on the first factor is equal to $M\mu$. We renormalize it to get a probability measure and using the ergodic decomposition and the ergodicity of μ, we may replace it by an ergodic stationary measure in $\mathrm{Prob}_\mu(X \times G)$. □

Denote by $\widetilde{m}_G = \nu^{\mathbf{N}} \times \widetilde{\mu}_G$ the $\widetilde{F}_+$-invariant measure associated to $\widetilde{\mu}_G$. The action of $\widetilde{F}_+$ on $\mathcal{X}_+ \times G$ (resp. of the induced random dynamical system on $X \times G$) commutes to the action of G by right multiplication, i.e. to the diffeomorphisms R_h, $h \in G$, defined by

$$R_h(x, g) = (x, gh). \tag{C.5}$$

Slightly abusing notation we also denote by R_h the analogous map on $X \times G$. The next lemma combines classical arguments due to Furstenberg and Zimmer.

Lemma C.3. *Let $\widetilde{\mu}_G$ be a Borel stationary measure on $X \times G$ with marginal μ on X. Set*

$$H = \{h \in G \ ; \ (R_h)_*\widetilde{\mu}_G = \widetilde{\mu}_G\} = \{h \in G \ ; \ (R_h)_*\widetilde{m}_G = \widetilde{m}_G\}.$$

Then H is a compact subgroup of G and there is a measurable function $Q\colon X \to G$ such that the cocycle $B_x = Q^{-1}_{f^1_\omega(x)} \times \mathbb{P}(A_x) \times Q_x$ takes its values in H for $(\nu^{\mathbf{N}} \times \mu)$-almost every x.

Proof. Clearly, H is a closed subgroup of G. If H were not bounded then, given any compact subset C of G, we could find a sequence (h_n) of elements of H such that the subsets $R_{h_n}(C)$ are pairwise disjoint. Choosing C such that $X \times C$ has positive $\widetilde{\mu}_G$-measure, we would get a contradiction with the finiteness of $\widetilde{\mu}_G$. So H is a compact subgroup of G.

We say that a point (x, g) in $X \times G$ is generic if for ν^N-almost every ω,

$$\frac{1}{N}\sum_{n=0}^{N-1} \varphi\left(\widetilde{F}_+^n(\omega, x, g)\right) \underset{N\to\infty}{\longrightarrow} \int_{\mathcal{X}_+\times G} \varphi \, d\widetilde{m}_G \tag{C.6}$$

for every compactly supported continuous function on $\mathcal{X}_+ \times G$. The Birkhoff ergodic theorem provides a Borel set $\mathcal{E}$ of full $\widetilde{\mu}_G$-measure made of generic points. Now if (x, g_1) and (x, g_2) belong to $\mathcal{E}$, writing $g_2 = g_1 h = R_h(g_1)$ for $h = g_1^{-1} g_2$, we get that h is an element of H.

Given $g \in G$, define $\mathcal{E}_x \subset G$ to be the set of elements $g \in G$ such that (x, g) is generic. Then there exists a measurable section $X \ni x \mapsto Q_x \in G$ such that $Q_x \in \mathcal{E}_x$ for almost all x. By definition of $\mathcal{E}_x$, (ω, x, Q_x) satisfies (C.6) for ν^N-almost every ω. Then for ν-almost every $f_0 = f_\omega^1$, by $\widetilde{F}_+$-invariance of the set of Birkhoff generic points we infer that $(f_\omega^1(x), \mathbb{P}(A_x)Q_x)$ belongs to $\mathcal{E}$. Since $(f_\omega^1(x), Q_{f_\omega^1(x)})$ belongs to $\mathcal{E}$ as well, it follows that $Q^{-1}_{f_\omega^1(x)}\mathbb{P}(A_x)Q_x$ is in H. We conclude that the cocycle $B_x = Q^{-1}_{f_\omega^1(x)} \times \mathbb{P}(A_x) \times Q_x$ takes its values in H for almost all x, as claimed. □

Note that the map $x \mapsto Q_x$ lifts to a measurable map $x \mapsto Q'_x \in \mathsf{GL}_2(\mathbf{C})$. Conjugating H to a subgroup of PU_2 by some element $g_0 \in G$, we can now readily conclude from the two previous lemmas that when $[\![D_x f_\omega^n]\!]$ is essentially bounded, Conclusion (2) of Theorem 11.11 holds (the P_x are obtained by composing the Q'_x with a lift of g_0 to $\mathsf{GL}_2(\mathbf{C})$).

The unbounded case. – Now, we suppose that $[\![D_x f_\omega^n]\!]$ is essentially unbounded (alternative (a) of Lemma C.1), and adapt the results of [5, §4.1] to the complex setting to arrive at one of the Conclusions (1.a) or (1.b) of Theorem 11.11. The main step of the proof is the following lemma.

Lemma C.4. *Let A be a measurable $\mathsf{GL}(2, \mathbf{C})$ cocycle over $(\mathcal{X}_+, F_+, \nu^{\mathbf{N}} \times \mu)$ admitting a projectively invariant family of probability measures $(\hat{\mu}_x)_{x\in X}$ such that almost surely $[\![A_x^{(n)}]\!]$ almost converges to infinity. Then for almost every x, $\hat{\mu}_x$ possesses an atom of mass at least $1/2$; more precisely:*

- *either $\hat{\mu}_x$ has a unique atom $[w(x)]$ of mass $\geqslant 1/2$, that depends measurably on $x \in X$;*
- *or $\hat{\mu}_x$ has a unique pair of atoms of mass $1/2$, and this (unordered) pair depends measurably on $x \in X$.*

For the moment, we take this result for granted and proceed with the proof. By ergodicity, the number of atoms of $\hat{\mu}_x$ and the list of their masses are constant on a set of full measure. A first possibility is that $\hat{\mu}_x$ is almost surely the single point mass $\delta_{[w(x)]}$; this corresponds to (1.a). A second possibility is that $\hat{\mu}_x$ is the sum of two point masses of mass $1/2$; this corresponds to (1.b). In the remaining cases, there is exactly one atom of mass $1/2 \leqslant \alpha < 1$ at a point $[w(x)]$. Changing the trivialization P_x, we can suppose that $[w(x)] = [w] = [1 : 0]$. Then we write $\hat{\mu}_x = \alpha\delta_{[1:0]} + \hat{\mu}'_x$, and apply Lemma C.4 to the family of measures $\hat{\mu}'_x$ (after normalization to get a probability measure). We deduce that almost surely $\hat{\mu}'_x$ admits an atom of mass $\geqslant (1-\alpha)/2$. Two cases may occur:

- $\hat{\mu}'_x$ has a unique atom of mass $\beta \geqslant (1-\alpha)/2$,
- $\hat{\mu}'_x$ has two atoms of mass $(1-\alpha)/2$.

The second one is impossible, because changing the trivialization, we would have $\hat{\mu}_x = \alpha\delta_{[1:0]} + \frac{1-\alpha}{2}(\delta_{[-1:1]} + \delta_{[1:1]})$, and the invariance of the finite set $\{[1:0], [-1:1], [1:1]\}$ would imply that the cocycle $\mathbb{P}(A_x)$ stays in a finite subgroup of $\mathsf{PGL}_2(\mathbf{C})$, contradicting the unboundedness assumption.

If $\hat{\mu}'_x$ has a unique atom of mass $\beta \geqslant (1-\alpha)/2$, we change P_x to put it at $[0:1]$ (the trivialization P_x is not an isometry anymore). We repeat the argument with $\hat{\mu}_x = \alpha\delta_{[1:0]} + \beta\delta_{[0:1]} + \hat{\mu}''_x$. If $\beta = 1-\alpha$, i.e. $\hat{\mu}''_x = 0$, then we are done. Otherwise $\hat{\mu}''_x$ has one or two atoms of mass $\gamma \geqslant (1-\alpha-\beta)/2$, and we change P_x to assume that one of them is $[1:1]$ and the second one –provided it exists– is $[\tau(x):1]$; here, $x \mapsto \tau(x)$ is a complex valued measurable function. Endow the projective line $\mathbb{P}^1(\mathbf{C})$ with the coordinate $[z:1]$; then $\mathbb{P}(A_x)$ is of the form $z \mapsto a(x)z$. Since $\mathbb{P}(A_x)(\{1, \tau(x)\}) = (\{1, \tau(F_+(x))\})$, we infer that:

– either $a(x)1 = 1$ and $\mathbb{P}(A_x)$ is the identity;
– or $a(x)1 = \tau(\pi_X(F_+(x)))$ and $a(x)\tau(x) = 1$ in which case $\tau(\pi_X(F_+(x))) = \tau(x)^{-1}$.

Thus we see that along the orbit of x, $a(F_+^n(x))$ takes at most two values $\tau(\pi_X(F_+^n(x)))^{\pm 1}$, and $[\![A_x^{(n)}]\!]$ is bounded, which is contradictory. This concludes the proof. $\square$

Proof of Lemma C.4. Let r and ε be small positive real numbers. Let $\mathrm{Prob}_{r,\varepsilon}(\mathbb{P}^1(\mathbf{C}))$ be the set of probability measures m on $\mathbb{P}^1(\mathbf{C})$ such that $\sup_{x\in\mathbb{P}^1} m(B(x,r)) \leqslant 1/2 - \varepsilon$, where the ball is with respect to some fixed Fubini-Study metric. This is a compact subset of the space of probability measures on $\mathbb{P}^1$. The set

$$G_{r,\varepsilon} = \left\{\gamma \in \mathsf{PGL}(2,\mathbf{C}),\ \exists m_1, m_2 \in \mathrm{Prob}_{r,\varepsilon}(\mathbb{P}^1(\mathbf{C})),\ \gamma_* m_1 = m_2\right\} \tag{C.7}$$

is a bounded subset of $\mathsf{PGL}(2,\mathbf{C})$. Indeed otherwise there would be an unbounded sequence γ_n together with sequences $(m_{1,n})$ and $(m_{2,n})$ in $\mathrm{Prob}_{r,\varepsilon}(\mathbb{P}^1(\mathbf{C}))$ such that $(\gamma_n)_* m_{1,n} = m_{2,n}$. Denote by $\gamma_n = k_n a_n k'_n$ the KAK decomposition of γ_n in $\mathsf{PGL}(2,\mathbf{C})$, with k_n and k'_n two isometries for the Fubini-Study metric; since γ_n is unbounded, we can extract a subsequence such that the measures $(k'_n)_* m_{1,n}$ and $(k_n^{-1})_* m_{2,n}$ converge in $\mathrm{Prob}_{r,\varepsilon}(\mathbb{P}^1(\mathbf{C}))$ to two measures m_1 and m_2, while the diagonal transformations a_n converge locally uniformly on $\mathbb{P}^1(\mathbf{C})\setminus\{[0:1]\}$ to the constant map $\gamma : \mathbb{P}^1(\mathbf{C})\setminus\{[0:1]\} \mapsto \{[1:0]\}$. Then

$$\gamma_*\left(m_{1|\mathbb{P}^1(\mathbf{C})\setminus\{[0:1]\}}\right) = m_1(\mathbb{P}^1(\mathbf{C})\setminus\{[0:1]\})\delta_a \leqslant m_2; \tag{C.8}$$

since m_1 belongs to $\mathrm{Prob}_{r,\varepsilon}(\mathbb{P}^1(\mathbf{C}))$, $m_1(\mathbb{P}^1(\mathbf{C})\setminus\{[0:1]\}) \geqslant 1/2+\varepsilon$, hence $m_2 \geqslant (1/2+\varepsilon)\delta_a$, in contradiction with $m_2 \in \mathrm{Prob}_{r,\varepsilon}(\mathbb{P}^1(\mathbf{C}))$. This proves that $G_{r,\varepsilon}$ is bounded.

To prove the lemma, let us consider the ergodic dynamical system $\mathbb{P}DF_+$, and the family of conditional probability measures $\hat{\mu}_x$ for the projection $(\omega, x, v) \mapsto x = (\omega, x)$. If there exist $r, \varepsilon > 0$ such that $\hat{\mu}_x$ belongs to $\mathrm{Prob}_{r,\varepsilon}(\mathbb{P}^1(\mathbf{C}))$ for x in some positive measure subset B then, by ergodicity, for almost every $x \in \mathcal{X}_+$ there exists a set of integers $L(x)$ of positive density such that for $n \in L(x)$, $F_+^n(x)$ belongs to B, hence $A_x^{(n)}$ belongs to $G_{r,\varepsilon}$ ([10]). From the above claim we deduce that $[\![A_x^{(n)}]\!]$ is uniformly bounded for $n \in L(x)$, a contradiction. Therefore for every $r, \varepsilon > 0$, the measure of $\left\{x,\ \hat{\mu}_x \in \mathrm{Prob}_{r,\varepsilon}(\mathbb{P}^1(\mathbf{C}))\right\}$ is equal to 0; it follows that for almost every x, $\hat{\mu}_x$ possesses an atom of mass at least $1/2$.

[10]We are slightly abusing here when the Fubini-Study metric depends on x, for instance when P_x is not an isometry; however restricting to subset of large positive measure the metric $(P_x)_*(\kappa_0)_x$ is uniformly comparable to a fixed Fubini-Study metric.

If there is a unique atom of mass $\geqslant 1/2$, this atom determines a measurable map $x \mapsto [w(x)] \in \mathbb{P}T_xX$; since $\hat{\mu}_x$ does not depend on ω, $[w(x)]$ depends only on x, not on ω. If there are generically two atoms of mass $\geqslant 1/2$, then both of them has mass $1/2$, and the pair of points determined by these atoms depends only on x. $\square$

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