ORBITS OF AUTOMORPHISM GROUPS OF AFFINE SURFACES OVER p-ADIC FIELDS

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ABSTRACT. We study orbit closures and stationary measures for groups of automorphisms of p-adic affine surfaces.

CONTENTS

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1. INTRODUCTION

1.1. **Affine varieties over local fields.** If $V \subset \mathbb{A}^N$ is an affine variety defined over a ring R, we denote by $\mathsf{Aut}(V_R)$ the group of automorphisms of V defined over R. Specifically, $f\colon V \to V$ is an element of $\mathsf{Aut}(V_R)$ if it is an automorphism and both f and f^{-1} are defined by formulas with coefficients in R. If R is the valuation ring of a local field (see Section 2.1), we shall endow $\mathbb{A}^N(R) = R^N$ with the product topology, and then $V(R)$ with the induced topology, which is independent of the embedding of V into $\mathbb{A}^N$. The group $\mathsf{Aut}(V_R)$ determines a subgroup of $\mathsf{Homeo}(V(R))$, the group of homeomorphisms of $V(R)$.

1.2. **Stationary measures.** Let T be a topological space. Let μ be a probability measure on $\mathsf{Homeo}(T)$. A stationary measure, with respect to μ, is a probability measure ν on T such that $\int \nu(f^{-1}(B))\, d\mu(f) = \nu(B)$ for every Borel subset $B \subset T$. Let Γ be a subgroup of $\mathsf{Homeo}(T)$. By definition, a stationary measure for Γ is a stationary measure with respect to a probability measure μ on $\mathsf{Homeo}(T)$ whose support generates Γ. Given a compact Γ-invariant set $K \subset T$, and a probability measure μ on Γ, there is always at least one μ-stationary measure ν whose support is contained in K.

1.3. **Orbit closures.** An automorphism of an affine surface X is said to be loxodromic when its (first) dynamical degree is greater than 1: we refer to Section 3 for a description of the different types of automorphisms. Our goal is to prove the following theorem.

Theorem A.– *Let X be a normal affine surface defined over $\mathbf{Z}_p$, the ring of p-adic integers. Let Γ be a subgroup of $\mathsf{Aut}(X_{\mathbf{Z}_p})$ containing a loxodromic element and a non loxodromic element of infinite order. Then,*

(1) *$X(\mathbf{Z}_p)$ contains at most finitely many finite Γ-orbits;*
(2) *if a Γ-orbit is infinite, its closure is a clopen subset of $X(\mathbf{Z}_p)$;*
(3) *the decomposition of the compact set $X(\mathbf{Z}_p)$ into closures of Γ-orbits is a partition into (at most) countably many clopen subsets and (at most) finitely many finite orbits;*
(4) *this partition is finite if and only if every finite orbit of Γ in $X(\mathbf{Z}_p)$ is an isolated subset of $X(\mathbf{Z}_p)$.*

Moreover, if μ is a probability measure, whose support generates Γ, then each orbit closure O is the support of a Γ-invariant probability measure, and this invariant measure is the unique stationary measure supported in O.

Parts of Theorem A hold when $\mathbf{Z}_p$ is replaced by the valuation ring of an arbitrary p-adic local field, see in particular Theorem D in Section 6. The important example of Markov surfaces is studied in Section 8 (see Theorem E in Section 8.2).

Theorem A is motivated by the results of William Goldman concerning the ergodic theory of the mapping class group action on character varieties [22], those of Jean Bourgain, Alex Gamburd, and Peter Sarnak concerning Markov surfaces over finite fields [10], and the classification of stationary measures recently obtained by the first author with Christophe Dupont and Florestan Martin-Baillon for Markov surfaces [14].

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2. From automorphisms to flows

In this section, we recall an important theorem proven by Jason Bell; the version that we shall use is due to Bjorn Poonen (see [7, 8, 33]). Then, following Sections 2 and 9 of [18], we explain how this theorem can be applied to study the dynamics of groups of automorphisms (the reader should also consult [1, §3.1], [3, Prop. 22–23], and [6, Thm. 3.3]). In Subsections 2.5 and 2.6, we prove a general finiteness result on orbit closures (see Theorem B). In the rest of the paper, we will see how this general result may be used to reach our Theorem A.

2.1. **Local fields of characteristic** 0. A non-archimedian local field $\mathbf{K}$ of characteristic zero, which will be idiomatically called a *p-adic local field*, is the same thing as a finite extension of $\mathbf{Q}_p$ for some prime p, with the topology induced by the (unique) extension of the p-adic absolute value $|\cdot|_p$. We denote the absolute value of $\mathbf{K}$ by $|\cdot|$; the normalization is $|p| = 1/p$. The valuation ring is $\mathfrak{o}_{\mathbf{K}} = \{x \in \mathbf{K}\ ;\ |x| \leq 1\}$; its group of units is $\{x \in \mathbf{K}\ ;\ |x| = 1\}$. The valuation group is the image $|\mathbf{K}^\times| \subset \mathbf{R}_+^\times$ of the absolute value, and it is cyclic, equal to $|\pi|^{\mathbf{Z}}$ for some element $\pi \in \mathbf{K}^\times$ with $|\pi| < 1$. Such a π is called a uniformizer, and it is unique up to multiplication by a unit of $\mathfrak{o}_{\mathbf{K}}$. The ring $\mathfrak{o}_{\mathbf{K}}$ contains a unique maximal ideal, namely $\mathfrak{m}_{\mathbf{K}} = \{x \in \mathfrak{o}_{\mathbf{K}}\ ;\ |x| < 1\} = \pi\mathfrak{o}_{\mathbf{K}}$. The residue field is $\mathbf{k} = \mathfrak{o}_{\mathbf{K}}/\mathfrak{m}_{\mathbf{K}}$, and it is a finite extension of $\mathbf{F}_p$.

We consider $\mathfrak{o}_{\mathbf{K}}$ as the closed unit disk of $\mathbf{K}$ and we refer to ${\mathfrak{o}_{\mathbf{K}}}^m$ as the (closed) polydisk of dimension m.

2.2. **Analytic diffeomorphisms of the polydisk.** Set $\mathcal{U} = {\mathfrak{o}_{\mathbf{K}}}^m$. An analytic function φ on the polydisk $\mathcal{U}$ is an analytic function in the sense of Tate, that is, if we denote by $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_m)$ the coordinates, and by $\mathbf{x}^I = \mathbf{x}_1^{i_1} \cdots \mathbf{x}_m^{i_m}$ the monomial associated with a multi-index $I = (i_1, \ldots, i_m)$, then φ is defined by a power series

$$\varphi(\mathbf{x}) = \sum_I a_I \mathbf{x}^I \tag{2.1}$$

with coefficients $a_I \in \mathbf{K}$ such that $|a_I|$ goes to 0 as I goes to infinity (i.e. the length $|I| = i_1 + \cdots + i_m$ goes to $+\infty$). The set of analytic functions with coefficients in a subring $R \subset \mathbf{K}$ will be denoted by $R\langle \mathbf{x} \rangle$. We will be particularly interested in the cases $R = \mathfrak{o}_{\mathbf{K}}$ and $R = \mathbf{Z}_p \subset \mathfrak{o}_{\mathbf{K}}$. Similarly, an analytic map $\mathcal{U} \to \mathbf{K}^n$ is a map defined by n analytic functions $(\varphi_1, \ldots, \varphi_n)$. This notion is usually called Tate analyticity ([1]).

An analytic endomorphism of $\mathcal{U}$ is just an analytic map $g\colon \mathcal{U} \to \mathcal{U}$ that can be written as $g(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}))$ for some analytic functions $g_i \in \mathfrak{o}_{\mathbf{K}}\langle \mathbf{x} \rangle$. Endomorphisms form an $\mathfrak{o}_{\mathbf{K}}$-module $\mathsf{End}^{an}(\mathcal{U})$, and a monoid for the composition. Invertible elements (with respect to the composition) form a group, the group $\mathsf{Diff}^{an}(\mathcal{U})$ of analytic diffeomorphisms of the polydisk. One can also define analytic vectors fields and analytic flows. By definition, a flow parametrized by $\mathfrak{o}_{\mathbf{K}}$ is an analytic map $\Phi\colon \mathfrak{o}_{\mathbf{K}} \times \mathcal{U} \to \mathcal{U}$, $(\mathbf{t}, \mathbf{x}) \mapsto \Phi^{\mathbf{t}}(\mathbf{x})$, such that $\Phi^{\mathbf{s}+\mathbf{t}}(\mathbf{x}) = \Phi^{\mathbf{s}}(\Phi^{\mathbf{t}}(\mathbf{x}))$. The vector field associated with such a flow is $\Theta(\mathbf{x}) = \partial_{\mathbf{t}}(\Phi^{\mathbf{t}}(\mathbf{x}))_{\mathbf{t}=0}$. An integral curve of this vector field is the same as an orbit $t \in \mathfrak{o}_{\mathbf{K}} \mapsto \Phi^t(z) \in {\mathfrak{o}_{\mathbf{K}}}^m$, for fixed $z \in \mathcal{U}$.

Let $V \subset \mathbb{A}^N$ be an affine variety of dimension m. Let U be a subset of V. If there is an analytic map $\varphi\colon \mathcal{U} \to \mathbb{A}^N(\mathbf{K})$ such that $\varphi(\mathcal{U}) = U$ and φ is a diffeomorphism from $\mathcal{U}$ to U, we say that U is a polydisk of V (see [18, Prop. 3.3, 3.4]).

2.3. **Reduction and the Bell-Poonen theorem.** If ℓ is a positive integer and $\varphi = \sum_I a_I \mathbf{x}^I$ is an element of $\mathfrak{o}_{\mathbf{K}}\langle \mathbf{x} \rangle$, its reduction modulo $\mathfrak{m}_{\mathbf{K}}^{\ell}$ is the formal power series $\overline{\varphi}$ with coefficients in $\mathfrak{o}_{\mathbf{K}}/\mathfrak{m}_{\mathbf{K}}^{\ell}$ obtained by reduction of the coefficients a_I modulo $\mathfrak{m}_{\mathbf{K}}^{\ell}$. Thus, $\overline{\varphi} = \sum_I \overline{a_I} \mathbf{x}^I$. Since $|a_I|$ goes to 0 as $|I|$ goes to

[1]Note that it requires a global power series expansion on the polydisk $\mathcal{U}$. As such, it is not a direct analogue of the classical definition of complex analytic functions, which requires only a local analytic expansion around each point.

infinity, this power series is a polynomial in $(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ with coefficients in the ring $\mathfrak{o}_{\mathbf{K}}/\mathfrak{m}_{\mathbf{K}}^{\ell}$. Recall that $\mathfrak{m}_{\mathbf{K}} = \pi\mathfrak{o}_{\mathbf{K}}$ for some uniformizer π. Let e be the integer such that $|\pi|^e = |p|$. One writes $\varphi \equiv \psi \pmod{p^c}$ if there is an integer $\ell \geq ec$ such that $\varphi \equiv \psi \pmod{\mathfrak{m}_{\mathbf{K}}^{\ell}}$.

Theorem 2.1 (Bell-Poonen)**.** *Let f be an analytic endomorphism of the polydisk $\mathcal{U} = {\mathfrak{o}_{\mathbf{K}}}^m$ with $f \equiv \mathrm{id} \pmod{p^c}$ for some real number $c > 1/(p-1)$. Then, f is an analytic diffeomorphism of $\mathcal{U}$ and there exists a unique analytic flow $\Phi\colon \mathfrak{o}_{\mathbf{K}} \times \mathcal{U} \to \mathcal{U}$ such that $\Phi^n(\mathbf{x}) = f^n(\mathbf{x})$ for every $n \in \mathbf{Z}$; moreover $\Phi^t(\mathbf{x}) \equiv \mathbf{x} \pmod{p^c}$ for all $t \in \mathfrak{o}_{\mathbf{K}}$.*

Thus, if the congruence $f \equiv \mathrm{id} \pmod{p^c}$ holds, the dynamics of f on $\mathcal{U}$ are “embedded” into the dynamics of a flow Φ. For instance, *if $\mathbf{K} = \mathbf{Q}_p$, then $\mathfrak{o}_{\mathbf{K}} = \mathbf{Z}_p$ and $\mathbf{Z}$ is dense in $\mathbf{Z}_p$, so that the closure of an f-orbit $f^{\mathbf{Z}}(x)$ for the p-adic topology equals the trajectory $\Phi^{\mathbf{Z}_p}(x)$.* As a continuous image of a compact set $\mathbf{Z}_p \times \{x\}$, the trajectory $\Phi^{\mathbf{Z}_p}(x)$ is closed in $\mathcal{U}$.

We provide a proof of the Bell-Poonen theorem in the appendix.

Remark 2.2. As the residue field $\mathbf{k}$ of $\mathfrak{o}_{\mathbf{K}}$ is finite, the following property is *not* sufficient for the congruence $f \equiv \mathrm{id} \pmod{\mathfrak{m}_{\mathbf{K}}^{\ell}}$: for all $x \in \mathcal{U}$, we have $f(x) - x \in (\mathfrak{m}_{\mathbf{K}}^{\ell})^m$. For instance, let $q = |\mathbf{k}|$ be the residue field order, and set $f(\mathbf{x}_1) = \mathbf{x}_1 + (\mathbf{x}_1 - \mathbf{x}_1^q)^{\ell}$ a univariable polynomial. Then we have $f \not\equiv \mathrm{id} \pmod{\mathfrak{m}_{\mathbf{K}}^{\ell}}$ yet $f(x_1) \equiv x_1 \pmod{\mathfrak{m}_{\mathbf{K}}^{\ell}}$ for all $x_1 \in \mathfrak{o}_{\mathbf{K}}$. This issue does not arise if $\mathbf{K}$ is extended to its algebraic closure; one can also avoid it by a finite extension of $\mathbf{K}$ whose residue field is sufficiently large (relative to f and ℓ).

2.4. **Automorphisms.** Let V be an affine variety, defined over $\mathfrak{o}_{\mathbf{K}}$, where $\mathfrak{o}_{\mathbf{K}}$ is the valuation ring of $\mathbf{K}$, as described above.

2.4.1. *Reduction.* Reduction mod $\mathfrak{m}_{\mathbf{K}}^{\ell}$ provides a homomorphism

$$f \in \mathsf{Aut}(V_{\mathfrak{o}_{\mathbf{K}}}) \mapsto \overline{f} \in \mathsf{Aut}(V_{\mathfrak{o}_{\mathbf{K}}/\mathfrak{m}_{\mathbf{K}}^{\ell}}). \tag{2.2}$$

Example 2.3. Consider the affine plane $\mathbb{A}^2$ and the group $\mathsf{Aut}(\mathbb{A}^2_{\mathfrak{o}_{\mathbf{K}}})$. This group contains all affine transformations $f(\mathbf{x}_1, \mathbf{x}_2) = L(\mathbf{x}_1, \mathbf{x}_2) + T$ with $L \in \mathsf{GL}_2(\mathfrak{o}_{\mathbf{K}})$ and $T \in \mathbb{A}^2(\mathfrak{o}_{\mathbf{K}})$, as well as all elementary automorphisms $g(\mathbf{x}_1, \mathbf{x}_2) = (a\mathbf{x}_1 + b(\mathbf{x}_2), c\mathbf{x}_2 + d)$, where $a, c, d \in \mathfrak{o}_{\mathbf{K}}$, $b \in \mathfrak{o}_{\mathbf{K}}[\mathbf{x}_2]$, and $|a| = |c| = 1$. Any affine or elementary automorphism of $\mathbb{A}^2_{\mathbf{k}}$ is the reduction modulo $\mathfrak{m}_{\mathbf{K}}$ of such an affine or elementary automorphism of $\mathbb{A}^2_{\mathfrak{o}_{\mathbf{K}}}$. By the theorem of Jung and van der Kulk (see [27], Thm. 2, and references therein), the reduction homomorphism $\mathsf{Aut}(\mathbb{A}^2_{\mathfrak{o}_{\mathbf{K}}}) \to \mathsf{Aut}(\mathbb{A}^2_{\mathbf{k}})$ is onto.

Example 2.4. Consider a Markoff cubic surface M, defined by the equation

$$\mathbf{x}_1^2+\mathbf{x}_2^2+\mathbf{x}_3^2+\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3 = A\mathbf{x}_1+B\mathbf{x}_2+C\mathbf{x}_3+D \tag{2.3}$$

for some parameters A, B, C, D in $\mathfrak{o}_\mathbf{K}$.A theorem of El'Huti shows that the homomorphism $\mathsf{Aut}(M_{\mathfrak{o}_\mathbf{K}}) \to \mathsf{Aut}(M_\mathbf{k})$ is onto (see [20, Thm. 2]).

Let us embed $V_{\mathfrak{o}_\mathbf{K}}$ into an affine space $\mathbb{A}^N_{\mathfrak{o}_\mathbf{K}}$. As explained in the introduction, we endow $\mathbb{A}^N(\mathbf{K}) = \mathbf{K}^N$ with the product topology, and $V(\mathbf{K})$ (resp. $V(\mathfrak{o}_\mathbf{K})$) with the induced topology. This topology does not depend on the choice of an embedding into an affine space. With this topology, $V(\mathfrak{o}_\mathbf{K})$ is compact.

For any integer $\ell \geq 1$, denote by $\mathrm{red}_\ell \colon V(\mathfrak{o}_\mathbf{K}) \to V(\mathfrak{o}_\mathbf{K}/\mathfrak{m}_\mathbf{K}^\ell)$ the reduction map modulo $\mathfrak{m}_\mathbf{K}^\ell$. The group $\mathsf{Aut}(V_{\mathfrak{o}_\mathbf{K}})$ acts by permutation on $V(\mathfrak{o}_\mathbf{K}/\mathfrak{m}_\mathbf{K}^\ell)$, which yields a homomorphism $\mathsf{Aut}(V_{\mathfrak{o}_\mathbf{K}}) \to \mathsf{Bij}(V(\mathfrak{o}_\mathbf{K}/\mathfrak{m}_\mathbf{K}^\ell))$. Fix a point $z \in V(\mathfrak{o}_\mathbf{K}/\mathfrak{m}_\mathbf{K}^\ell)$. Its preimage $\mathrm{red}_\ell^{-1}(z)$ in $V(\mathfrak{o}_\mathbf{K})$ is an open and closed subset of $V(\mathfrak{o}_\mathbf{K})$. By the Hensel lemma, if ℓ is large enough and if z is a smooth point of $V(\mathbf{k})$, the clopen set $\mathrm{red}_\ell^{-1}(z)$ is a polydisk $\simeq {\mathfrak{o}_\mathbf{K}}^m$; for details, see [6, Prop. 2.2] and [18, Prop. 3.4]. Then, the stabilizer of $z \in V(\mathfrak{o}_\mathbf{K}/\mathfrak{m}_\mathbf{K}^\ell)$ in $\mathsf{Aut}(V_{\mathfrak{o}_\mathbf{K}})$ coincides with the stablizer of $\mathrm{red}_\ell^{-1}(z)$ and is isomorphic to a subgroup of $\mathsf{Diff}^{an}({\mathfrak{o}_\mathbf{K}}^m)$.

2.4.2. *Good polydisks (see* [18, §3.2]*).* Suppose, for simplicity, that V is smooth. Fix an integer $\ell_1 \geq 1$ such that each preimage $\mathrm{red}_{\ell_1}^{-1}(z)$, for $z \in V(\mathfrak{o}_\mathbf{K}/\mathfrak{m}_\mathbf{K}^{\ell_1})$, is diffeomorphic to a polydisk. By the finiteness of $\mathfrak{o}_\mathbf{K}/\mathfrak{m}_\mathbf{K}^{\ell_1}$, this provides a finite cover of $V(\mathfrak{o}_\mathbf{K})$ by polydisks $\mathcal{U}_i$, with i between 1 and $N_1 := |V(\mathfrak{o}_\mathbf{K}/\mathfrak{m}_\mathbf{K}^{\ell_1})|$. In what follows, we identify each $\mathcal{U}_i$ with the polydisk ${\mathfrak{o}_\mathbf{K}}^m$ via some analytic diffeomorphism.

Let Γ be a subgroup of $\mathsf{Aut}(V_{\mathfrak{o}_\mathbf{K}})$. Each of these polydisks is invariant under the action of the kernel Γ_1 of the representation $\Gamma \to \mathsf{Bij}(V(\mathfrak{o}_\mathbf{K}/\mathfrak{m}_\mathbf{K}^{\ell_1}))$ and, by restriction to each $\mathcal{U}_i$, Γ_1 determines N_1 subgroups

$$\Gamma_1(i) \subset \mathsf{Diff}^{an}({\mathfrak{o}_\mathbf{K}}^m), \tag{2.4}$$

where ${\mathfrak{o}_\mathbf{K}}^m \simeq \mathcal{U}_i$. Now, choose ℓ_2 such that $p^4 \in \mathfrak{m}_\mathbf{K}^{\ell_2}$. Each $\Gamma_1(i)$ acts, after reduction modulo $\mathfrak{m}_\mathbf{K}^{\ell_2}$, by polynomial automorphisms on $(\mathfrak{o}_\mathbf{K}/\mathfrak{m}_\mathbf{K}^{\ell_2})^m$. Denote by $\Gamma_0(i) \subset \Gamma_1$ the kernel of the action on the tangent space of $(\mathfrak{o}_\mathbf{K}/\mathfrak{m}_\mathbf{K}^{\ell_2})^m$; by this we mean that in $\mathcal{U}_i$, every element f of $\Gamma_0(i)$ satisfies

$$f(x) = x \ (\mathrm{mod}\ \mathfrak{m}_\mathbf{K}^{\ell_2}) \ \text{ and } \ Df_x = \mathrm{id} \ (\mathrm{mod}\ \mathfrak{m}_\mathbf{K}^{\ell_2}) \tag{2.5}$$

for every $x \in (\mathfrak{o}_\mathbf{K}/\mathfrak{m}_\mathbf{K}^{\ell_2})^m$. The intersection of all $\Gamma_0(i)$'s is a normal subgroup Γ_0 of Γ.

Now, pick an element f in Γ_0 and a point x in $V(\mathfrak{o}_{\mathbf{K}})$. Then x is in some $\mathcal{U}_i \simeq \mathfrak{o}_{\mathbf{K}}{}^m$. In this polydisk $\mathfrak{o}_{\mathbf{K}}{}^m$, if we conjugate f by the translation t_x that maps the origin o to x, we can write f as a series

$$f(\mathbf{x}) = A_0 + A_1(\mathbf{x}) + A_2(\mathbf{x}) + \cdots + A_n(\mathbf{x}) + \cdots \tag{2.6}$$

where

(1) each $A_n\colon \mathbf{K}^m \to \mathbf{K}^m$ is a homogeneous polynomial mapping of degree n with coefficients in $\mathfrak{o}_{\mathbf{K}}$;
(2) $A_0 = 0 \pmod{p^4}$ and $A_1(\mathbf{x}) = \mathbf{x} \pmod{p^4}$.

If we conjugate f by the homothety $h_p\colon \mathbf{x} \mapsto p\mathbf{x}$, then $h_p^{-2} \circ f \circ h_p^2$ satisfies the hypothesis of Theorem 2.1. Since the origin corresponds to x before conjugating by t_x, and h_p^2 maps $\mathfrak{o}_{\mathbf{K}}{}^m$ to $(p^2\mathfrak{o}_{\mathbf{K}})^m$, we see that x is contained in a (smaller) polydisk $\mathcal{U}(x) \subset \mathcal{U}_i$ that is invariant by Γ_0 and where each element of Γ_0 satisfies the hypotheses of Theorem 2.1.

Theorem 2.5. *Let $\mathbf{K}$ be a non-archimedian local field of characteristic 0 and $\mathfrak{o}_{\mathbf{K}}$ be its valuation ring. Let V be a smooth affine variety defined over $\mathfrak{o}_{\mathbf{K}}$. Let Γ be a subgroup of $\mathsf{Aut}(V_{\mathfrak{o}_{\mathbf{K}}})$. There is a finite index subgroup Γ_0 of Γ with the following property. Given any $x \in V(\mathfrak{o}_{\mathbf{K}})$, there is a compact open neighborhood $\mathcal{U}(x)$ of x such that*

(1) *$\mathcal{U}(x)$ is diffeomorphic to the polydisk $\mathfrak{o}_{\mathbf{K}}{}^m$;*
(2) *$\mathcal{U}(x)$ is Γ_0-invariant;*
(3) *for any $f \in \Gamma_0$, there is a unique analytic flow $\Phi_f\colon \mathfrak{o}_{\mathbf{K}} \times \mathcal{U}(x) \to \mathcal{U}(x)$ such that $f^n_{|\mathcal{U}(x)} = \Phi_f^n$ for every $n \in \mathbf{Z}$.*

We shall say that the orbit $\{\Phi^t(x)\,;\, t \in \mathfrak{o}_{\mathbf{K}}\}$ is the **analytic trajectory** of x determined by f.

Remark 2.6. When V is singular, the same result holds locally on the complement of the singularities. If the singularities of V are quotient singularities, one can get a finite cover as in Theorem 2.5, except that some of the $\mathcal{U}(x)$ will be the quotient $\mathfrak{o}_{\mathbf{K}}{}^m/G_x$ of the polydisk by a finite group. Then the third assertion is better stated on $\mathfrak{o}_{\mathbf{K}}{}^m$ rather than that on the quotient.

Remark 2.7. If V is defined over $\mathfrak{o}_{\mathbf{K}}$ and $W \subset V(\mathbf{K})$ is a Γ-invariant compact open subset, then the same statement holds on W.

Remark 2.8. If f is an element of Γ_0, y is in $\mathcal{U}(x)$ and $f^k(y) = y$ for some $k \geq 1$, then $\Phi_f^{kn}(y) = y$ and, by the principle of isolated zeroes (see [35, Sec.

VI.2]), $\Phi_f^t(y) = y$ for every $t \in \mathfrak{o}_{\mathbf{K}}$; taking $t = 1$ shows that $f(y) = y$. Thus, in $V(\mathfrak{o}_{\mathbf{K}})$, every periodic point of f is fixed. In particular, Γ_0 is torsion free as soon as $V(\mathfrak{o}_{\mathbf{K}})$ is Zariski dense in V. See [4] for a more general result.

2.5. **The distribution $L_\Gamma(x)$ and its dimension $s_\Gamma(x)$.** Let x be a smooth point of $V(\mathfrak{o}_{\mathbf{K}})$, $\mathcal{U} \subset V(\mathfrak{o}_{\mathbf{K}})$ be a polydisk containing x, and f be an element of Γ stabilizing $\mathcal{U}$. If $f = \Phi_f^1$ for some flow $\Phi_f \colon \mathfrak{o}_{\mathbf{K}} \times \mathcal{U} \to \mathcal{U}$, we define

$$\Theta_f(x) = \left(\frac{\partial \Phi_f^t(x)}{\partial t}\right)_{|t=0}. \tag{2.7}$$

Changing f to f^k for some $k \in \mathbf{Z} \setminus \{0\}$ changes $\Theta_f(x)$ to $k\Theta_f(x)$. Thus, in the tangent space T_xV, viewed as a vector space over $\mathbf{Q}_p$, we have $\mathbf{Q}_p\Theta_f(x) = \mathbf{Q}_p\Theta_{f^k}(x)$ for every $k \neq 0$. This space is a line if and only if $\Theta_f(x) \neq 0$, if and only if $f(x) \neq x$ (see Remark 2.8). We denote this line by

$$[\Theta_f(x)] \in \mathbb{P}(T_xV) \tag{2.8}$$

where $\mathbb{P}(T_xV) = (T_xV \setminus \{0\})/\mathbf{Q}_p^\times$ (note that T_xV is viewed as a vector space over $\mathbf{Q}_p$, rather than over $\mathbf{K}$). Now, suppose $\mathcal{U}$ is stabilized by a finite index subgroup $\Gamma_0 \subset \Gamma$ such that every element of Γ_0 is in a flow. The subspace

$$L_{\Gamma_0}(x) = \mathrm{vect}_{\mathbf{Q}_p}(\Theta_f(x); f \in \Gamma_0) \subset T_xV \tag{2.9}$$

is equal to $L_{\Gamma_1}(x)$ if Γ_1 has finite index in Γ_0. Thus, it depends only on Γ and we denote it by $L_\Gamma(x)$. By definition, $L_\Gamma(x)$ is the **tangent space to the action** of Γ at x. Its dimension, as a $\mathbf{Q}_p$-vector space, is denoted by $s_\Gamma(x)$.

Proposition 2.9. *Let $\mathcal{U} \subset V(\mathfrak{o}_{\mathbf{K}})$ be a Γ_0-invariant polydisk, for some finite index subgproup Γ_0 of Γ. The function $s_\Gamma \colon \mathcal{U} \to \mathbf{Z}_+$ is semi-continuous in the analytic topology: this means that the sets $\{x \in \mathcal{U}\,;\, s_\Gamma(x) \leq k\}$ are $\mathbf{Q}_p$-analytic subsets (in the sense of Tate).*

Note that in this statement the polydisk $\mathcal{U} \simeq {\mathfrak{o}_{\mathbf{K}}}^m$ is seen as a polydisk $\mathbf{Z}_p^{km}$ of dimension km over $\mathbf{Z}_p$, with $\mathfrak{o}_{\mathbf{K}} \simeq \mathbf{Z}_p^k$.

Proof. Let $f_1, \ldots, f_{k+1}$ be elements of Γ_0. Then, the set $\{x \in \mathcal{U}\,;\, s_\Gamma(x) \leq k\}$ is contained in the $\mathbf{Q}_p$-analytic set $\{x \in \mathcal{U}\,;\, \Theta_{f_1}(x) \wedge \cdots \wedge \Theta_{f_{k+1}}(x) = 0\}$, and is equal to the intersection of these sets as we vary the f_i. □

2.6. **A general finiteness result.**

Theorem B.– *Let V be a smooth affine variety defined over $\mathfrak{o}_{\mathbf{K}}$. Let Γ be a subgroup of* $\mathsf{Aut}(V_{\mathfrak{o}_{\mathbf{K}}})$*. Let $B \subset V(\mathfrak{o}_{\mathbf{K}})$ be a Γ-invariant compact subset. If $s_\Gamma(x) = [\mathbf{K} : \mathbf{Q}_p]\dim(V)$ for every $x \in B$, then the orbit closures of Γ in B form a partition of B into finitely many compact open subsets.*

Proof. Let x be a point of $V(\mathfrak{o}_{\mathbf{K}})$. Let $\mathcal{U} \subset V(\mathfrak{o}_{\mathbf{K}})$ be a polydisk containing x which is stabilized by a finite index subgroup Γ_0 of Γ, as in Theorem 2.5.

Set $N = [\mathbf{K} : \mathbf{Q}_p]\dim(V)$. If $s_\Gamma(x) = N$, the orbit closure $C_0(x) := \overline{\Gamma_0(x)}$ contains an open neighborhood of x. Indeed, we can choose elements $f_1, \ldots,$ f_N in Γ_0 such that $(\Theta_{f_i}(x))$, $1 \leq i \leq N$, form a basis of T_xV as a $\mathbf{Q}_p$-vector space. Then, if we denote by Φ_i the flow associated with f_i, the map

$$(t_1, \ldots, t_N) \in \mathbf{Z}_p^N \mapsto \Phi_{f_N}^{t_N} \circ \cdots \circ \Phi_{f_1}^{t_1}(x) \in V(\mathfrak{o}_{\mathbf{K}}) \tag{2.10}$$

is a local diffeomorphism. On the other hand, given any N-tuple $(t_1, \ldots, t_N)$ and any $\varepsilon > 0$, there are integers $(n_1, \ldots, n_N)$ such that $|t_i - n_i|_p \leq \varepsilon$ for each i. Thus, there is a neighborhood of x in which $\Gamma_0(x)$ is ε-dense. Since this holds for every $\varepsilon > 0$, $C_0(x)$ contains a neighborhood of x.

Set $C(x) = \overline{\Gamma(x)}$. Since Γ_0 has finite index in Γ, $C(x)$ is made of finitely many images $g(C_0(x))$, with $g \in \Gamma/\Gamma_0$. In particular, $C(x)$ is compact and open. As $C(x)$ is open, the sets $C(x)$, for $x \in V(\mathfrak{o}_{\mathbf{K}})$, are disjoint: if two orbit closures intersect, they coincide. This proves that the orbit closures form a partition of $V(\mathfrak{o}_{\mathbf{K}})$ into clopen subsets, and since $V(\mathfrak{o}_{\mathbf{K}})$ is compact, this partition contains only finitely many atoms. □

Remark 2.10. The same argument shows the following. *If $s_\Gamma(x) = [\mathbf{K} : \mathbf{Q}_p]\dim(V)$ for every x in the complement of some closed analytic subset $Z \subset V$, then the orbit closures form a partition of $V(\mathfrak{o}_{\mathbf{K}}) \setminus Z(\mathfrak{o}_{\mathbf{K}})$ into (at most) countably many compact open subsets; this partition is locally finite.*

3. AUTOMORPHISMS OF AFFINE SURFACES

In this section and the next one, we collect a few facts concerning automorphisms of affine surfaces: the leitmotif is to explain how general results concerning birational transformations of surfaces specialize — and become more precise — for automorphisms of affine surfaces. See, in particular, Theorem C in Section 3.3. Here, we denote by X_0 an affine surface and by X a projective completion of X_0. This notation is specific to this section (in the rest of the paper, X is an affine surface).

3.1. **Elliptic, parabolic, loxodromic.** Let X be a projective surface, over an algebraically closed field $\mathbf{K}$. Let $\mathsf{Bir}(X)$ be its group of birational transformations. Let H be a polarization of X; the degree of a rational map $f\colon X \dashrightarrow X$ with respect to H is the intersection number

$$\deg_H(f) = (H \cdot f^*H). \tag{3.1}$$

If $\varphi\colon Y \dashrightarrow X$ is a birational map and H' is a polarization of Y, then

$$c^{-1}\deg_H(f) \leq \deg_{H'}(\varphi^{-1}\circ f\circ\varphi) \leq c\deg_H(f) \tag{3.2}$$

for some constant $c \geq 1$ and all $f \in \mathsf{Bir}(X)$; in what follows, the degree is simply denoted by $\deg(\cdot)$. Each element f of $\mathsf{Bir}(X)$ satisfies exactly one of the following properties (see [13, Thm. 4.6], [16, Thm. 4.6], [15, Sec. 3.2]).

(a) The sequence $(\deg(f^n))_{n\geq 0}$ is bounded, and then there is a smooth projective surface Y, a birational map $\varphi\colon Y \dashrightarrow X$, and a positive integer k such that $f_Y := \varphi^{-1}\circ f\circ\varphi$ is in $\mathsf{Aut}(Y)$ and f_Y^k is in the connected component $\mathsf{Aut}(Y)^0$ of id_Y in $\mathsf{Aut}(Y)$.

(b) The sequence $(\deg(f^n))$ grows linearly with $n \in \mathbf{N}$, and then there is a smooth projective surface Y, a birational map $\varphi\colon Y \dashrightarrow X$, a curve B, and a genus 0 fibration $\pi\colon Y \to B$ such that $f_Y := \varphi^{-1}\circ f\circ\varphi$ permutes the fibers of π. This fibration π is uniquely determined by f_Y (up to post-composition by an automorphism of B). Such a birational transformation is never conjugate to an automorphism of a *projective* surface Y by a birational map $Y \dashrightarrow X$.

(b') The sequence $(\deg(f^n))$ grows quadratically with $n \in \mathbf{N}$, and then there is a smooth projective surface Y, a birational map $\varphi\colon Y \dashrightarrow X$, a curve B, and a genus 1 fibration $\pi\colon Y \to B$, such that $f_Y := \varphi^{-1}\circ f\circ\varphi$ permutes the fibers of π. Moreover, on a relatively minimal model of the invariant fibration, f becomes an automorphism.

(c) The sequence $(\deg(f^n))$ grows exponentially with $n \in \mathbf{N}$. In that case, f does not preserve any pencil of curves.

In cases (b) and (b'), *genus k fibration* means that π is a fibration with connected fibers and with smooth general fibers of genus k; and f_Y *permutes the fibers of* π means that there is an automorphism f_B of B such that $\pi\circ f_Y = f_B\circ\pi$.

In case (a), we say that f is **elliptic**; in cases (b) and (b') that f is **parabolic**; and in case (c) that f is **loxodromic**. Thus, there are two types of parabolic transformations: in case (b), f is called a **Jonquières twist**; in case (b') it is a **Halphen twist**. In all cases, the limit

$$\lambda_1(f) = \lim_{n\to+\infty} \deg(f^n)^{1/n} \tag{3.3}$$

exists and is, by definition, the dynamical degree of f; we have $\lambda_1(f) \geq 1$, and the inequality is strict if and only if f is loxodromic.

Example 3.1. A non-loxodromic automorphism of the affine plane is conjugate to an elementary map $(\mathbf{x},\mathbf{y}) \mapsto (a\mathbf{x}+b, c\mathbf{y}+P(\mathbf{x}))$ and, as such, is elliptic. For any polynomial function $P(\mathbf{x})$ of degree ≥ 2, the Hénon map $(\mathbf{x},\mathbf{y}) \mapsto (\mathbf{y}+P(\mathbf{x}),\mathbf{x})$ is loxodromic, with dynamical degree $\deg(P)$.

Proposition 3.2. *Let X_0 be an affine surface and Γ be a subgroup of $\mathsf{Aut}(X_0)$. Assume there is a projective surface Y and a birational map $\varphi_0 \colon X_0 \dashrightarrow Y$ such that the group $\Gamma_Y := \varphi_0\Gamma\varphi_0^{-1}$ is contained in $\mathsf{Aut}(Y)$. Then, Γ is contained in an algebraic subgroup of $\mathsf{Aut}(X_0)$; in particular, every element of Γ is elliptic and Γ is elementary.*

Proof. Let $\mathrm{Exc}(\varphi_0)$ be the union of the irreducible curves $E \subset X_0$ which are contracted by φ_0 (i.e. the strict transform of E is a point). Let E be such a curve and let $q \in Y$ be the point to which E is contracted. If f is in Γ and $f_Y = \varphi_0 \circ f \circ \varphi_0^{-1}$, then $f(E)$ is a curve contracted by φ_0 onto the point $f_Y(q)$. Thus, Γ preserves $\mathrm{Exc}(\varphi_0)$. Similarly, the indeterminacy locus $\mathrm{Ind}(\varphi_0)$ is a finite Γ-invariant subset of X_0.

Now, consider a projective completion X of X_0, the extension $\varphi \colon X \dashrightarrow Y$ of φ_0, and a resolution of the indeterminacies of φ: this is given by a projective surface Z and birational morphisms $\varepsilon \colon Z \to X$, $\eta \colon Z \to Y$ such that $\varphi = \eta \circ \varepsilon^{-1}$.

There is an effective and ample divisor H supported on $X \setminus X_0$ (see [23, Thm. 1]). Set $H_Z = \varepsilon^* H$ and $H_Y = \eta_*(H_Z)$. If D is an irreducible component of H_Y, and if $f_Y = \varphi \circ f \circ \varphi^{-1}$ is an element of Γ_Y, then $f_Y(D)$ is also a component of H_Y. Indeed, if $f_Y(D)$ is not contained in $\eta_*(H_Z)$, its strict transform in Z under η^{-1} is a curve D_Z that intersects $\varepsilon^{-1}(X_0)$; then $\varepsilon(D_Z)$ gives a curve in X_0 or an indeterminacy point of φ_0. In both cases $\varepsilon(D_Z)$ is mapped by f^{-1} into $X \setminus X_0$, and this contradicts the fact that f is in $\mathsf{Aut}(X_0)$. Thus, Γ_Y preserves the support of H_Y, and a finite index subgroup of Γ_Y preserves individually each component of H_Y.

Since H is ample, H_Y is big and nef. Hence, Γ_Y preserves a big and nef divisor. This implies that Γ_Y is contained in an algebraic subgroup of $\mathsf{Aut}(Y)$. Consequently, Γ is contained in an algebraic subgroup of $\mathsf{Aut}(X_0)$.([2]) □

3.2. **Non-elementary groups.** There is a natural hyperbolic space $\mathbb{H}_X$ associated with X on which $\mathsf{Bir}(X)$ acts by isometries. The notions of elliptic, parabolic, or loxodromic birational transformations coincide with the three possible kinds of isometries (see [11, §3.5] and [13, Thm. 4.6]). Similarly, there is a notion of non-elementary group of isometries, which can be described without any reference to $\mathbb{H}_X$: a subgroup Γ of $\mathsf{Bir}(X)$ is non-elementary if it contains a non-abelian free group, the element of which (except the identity) are all loxodromic. When Γ contains a parabolic element f, Γ is non-elementary if and only if it does not preserve the f-invariant fibration (see [11, Prop. 6.12] and [13, §6]).

3.3. **Automorphisms of affine surfaces, characterizations of $\mathbb{G}_m \times \mathbb{G}_m$.** Let X_0 be an affine surface. Let X be a projective completion of X_0. We obtain an embedding $\mathsf{Aut}(X_0) \subset \mathsf{Bir}(X)$, and the above classification can be applied to the elements of $\mathsf{Aut}(X_0)$. The following results are proven in [2, Prop. 4.4.20] and [1, Cor. 10.11].

Theorem 3.3. *Let X_0 be an affine surface and g be a loxodromic automorphism of X_0. Then, g does not preserve any curve $C \subset X_0$.*

Theorem 3.4. *Let X_0 be an affine surface. Assume that* (i) $\mathsf{Aut}(X_0)$ *contains a loxodromic automorphism and* (ii) *there is a non-constant and non-vanishing function $\xi\colon X_0 \to \mathbf{K}^\times$. Then, X_0 is isomorphic to $\mathbb{G}_m \times \mathbb{G}_m$ over $\overline{\mathbf{K}}$.*

Our next result concerns parabolic automorphisms $f\colon X_0 \to X_0$. With the notation from Section 3.1, in cases (b) and (b'), we shall denote by $\pi_0\colon X_0 \dashrightarrow B$ the rational map $\pi_0 := (\pi_Y \circ \varphi^{-1})_{|X_0}$; its fibers are permuted by f. If X_0 and f are defined over a field $\mathbf{K}$, then π_0 is defined on a finite extension of $\mathbf{K}$.

Theorem C.– *Let X_0 be a normal, affine surface defined over a field* $\mathbf{K}$ *of characteristic* 0*. Let f be a parabolic automorphism of X_0, let $\pi_0\colon X_0 \dashrightarrow B$*

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[2] Let us recall the proof of this last step (see [12, Sec. 2] for references). The group $\mathsf{Aut}(Y)$ acts on the Néron-Severi group $\mathrm{NS}(Y)$, the connected component of the identity $\mathsf{Aut}(Y)^0$ is an algebraic group, and if $[A]$ is a class of positive self-intersection, then $\mathsf{Aut}(Y)^0$ has finite index in the stabilizer of $[A]$ in $\mathsf{Aut}(Y)$. On the other hand, Γ_Y preserves a big and nef class, and the self-intersection of such a class is automatically positive. Thus, $\mathsf{Aut}(Y)^0 \cap \Gamma_Y$ has finite index in Γ_Y, which shows that Γ_Y is contained in a finite extension of $\mathsf{Aut}(Y)^0$.

be its invariant fibration, and let $f_B \in \mathsf{Aut}(B)$ *be the automorphism induced by* f *and* π_0. *Then,*

(1) f *is a Jonquières twist,*
(2) π_0 *does not have base points in* $X_0(\overline{\mathbf{K}})$*,*
(3) *if* f_B *has infinite order, then there is a non-constant and non-vanishing function* $\xi\colon X_0 \to \mathbf{K}^\times$.

Moreover, if f_B *has infinite order then* X_0 *is smooth.*

As we shall see below, Jonquières twists exist on some regular and some singular affine surfaces, for instance, on singular Markov surfaces (see §5.3).

Preliminary remarks. To prove Theorem C, we assume that $\mathbf{K}$ is algebraically closed. Indeed, if ξ is a non-vanishing function defined over some Galois extension $\mathbf{L}$ of $\mathbf{K}$, then the product of its Galois conjugate provides a non-vanishing function defined over $\mathbf{K}$.

In the proof, we shall freely use some known results on rational fibrations of surfaces. In particular, as explained in [5], if Z is a smooth projective surface and $\pi\colon Z \to B$ is a fibration whose general fiber is a smooth rational curve, then there are no multiple fibers, and every singular fiber $F = \sum_i n_i C_i$ is made of smooth rational curves C_i of negative self intersection, with at least one having self intersection -1. In particular, by blowing down such -1 curves and iterating this process, one obtains a relatively minimal surface W with a ruling $\pi_W : W \to B$.

Proof of Theorem C. Let X be a projective completion of X_0. Denote by ∂X the complement of X_0 in X; we may and do assume that X is smooth near ∂X and that ∂X is a normal crossing divisor. By a theorem of Goodman [23] (see p. 166, Cor. of its Thm. 1), ∂X is connected.

• Let us prove Assertion (1) by contradiction.

Otherwise, f is a Halphen twist and according to Case (b') of Section 3.1, f is birationally conjugate to an automorphism f_Y of some projective surface Y. This contradicts Proposition 3.2, and f is therefore a Jonquières twist.

• The rational map π_0 extends to a rational map $\pi\colon X \dashrightarrow B$, where B is a smooth projective curve. Let Z be a smooth projective surface and $\psi\colon Z \to X$ be a birational morphism on which $\pi_Z := \pi \circ \psi$ is regular. The surface Z is obtained from X by a sequence of blow-ups, and we choose Z to minimize this number of blow-ups. Doing so, f lifts to a birational transformation f_Z of Z

that preserves the Zariski open subset

$$\mathcal{U}_Z = \psi^{-1}(X_0)$$

and acts on it as an automorphism. The total transform of ∂X in Z will be denoted by ∂Z.

Remark 3.5. If π_0 (or more precisely the pencil defined by π_0) had base points in X_0, then ψ^{-1} would have indeterminacy points in X_0. We denote by $I_Z \subset \mathcal{U}_Z$ the total transform of the base points of π_0 by ψ. If non-empty, I_Z is a curve that intersects the general fiber of π_Z and is disjoint from ∂Z.

Remark 3.6. If X_0 has a singularity, then the singular locus of X_0 is a finite, f-invariant set, and its preimage in Z is an invariant curve in $\mathcal{U}_Z$ (this curve can share some components with I_Z).

The general fiber of π_Z is a smooth rational curve. Thus, as explained in the preliminary remarks, there is a ruled surface W, with ruling $\pi_W\colon W \to B$, and a birational morphism $\eta\colon Z \to W$ such that $\pi_Z = \pi_W \circ \eta$. The fibers of π_W are smooth rational curves of self-intersection 0, and the fibers of π_Z are obtained from them by a finite number of blow-ups. Thus, if a fiber of π_Z is not smooth, it is a tree of smooth rational curves of negative self-intersection.

Let E be an irreducible component of ∂Z. If π_Z maps E to a point, then E is an irreducible component of a fiber and, as such, it is a smooth rational curve. Such a curve $E \subset \partial Z$ will be called a **vertical boundary component**. If C is an irreducible component of a fiber that intersects $\mathcal{U}_Z$, we say that C is an **internal component**.

• Now, we modify Z as follows. If there is a vertical boundary component of self-intersection -1, then we contract it. This gives a new surface Z', with a birational morphism $Z \to Z'$ which is an isomorphism from $\mathcal{U}_Z$ to its image, denoted by $\mathcal{U}_{Z'}$; and f_Z induces a birational transformation $f_{Z'}$ of Z' that preserves $\mathcal{U}_{Z'}$, acting as an automorphism on it. The fibration π_Z induces a fibration $\pi'\colon Z' \to B$. By definition, the complement of $\mathcal{U}_{Z'}$ is the boundary $\partial Z'$.

We repeat this process until we obtain a surface Y, with a fibration $\pi_Y : Y \to B$, and a birational morphism $\eta'\colon Z \to Y$ such that

(a) η' contracts only vertical boundary components and is an isomorphism from $\mathcal{U}_Z$ to its image, which we denote by $\mathcal{U}_Y$;
(b) $\pi_Z = \pi_Y \circ \eta'$ and π_Y does not have any vertical boundary component with self intersection (-1);

(c) η' conjugates f_Z to a birational transformation f_Y of Y that preserves the fibration π_Y (permuting its fibers) and preserves the open set $\mathcal{U}_Y$, acting regularly on it.

Again, we denote by ∂Y the complement of $\mathcal{U}_Y$. We can moreover assume (changing W if necessary) that there is a birational morphism $\eta'' \colon Y \to W$ such that $\eta = \eta'' \circ \eta'$. Then,

(d) a vertical boundary component $E \subset \partial Y$ is either a smooth rational curve of self-intersection 0, in which case it coincides with a fiber of π_Y, or a smooth rational curve with self-intersection ≤ -2, in which case the fiber $F_b = \pi_Y^{-1}(b)$ containing E is a tree of rational curves that contains at least one internal component with self-intersection -1;

(e) f_Y permutes the internal components of the fibers (none of them is contracted by f_Y).

• We say that a fiber $F_b = \pi_Y^{-1}(b)$ of π_Y is **mobile** if the orbit of b under f_B is infinite. Let us prove that

(f) mobile fibers are smooth and irreducible, and do not contain indeterminacy points of f_Y or f_Y^{-1}; they do not contain vertical boundary components.

(g) If a fiber F_b contains an internal component, then it does not contain any indeterminacy points (for any iterate of f_Y).

To prove (f), let F_b be a mobile fiber. Since the f_B-orbit of b is infinite, there is a positive integer n such that $F_{f_B^n(b)}$ is a smooth, irreducible, internal curve that avoids indeterminacy points of f_Y. The proper transform of $F_{f_B^n(b)}$ by f_Y^{-n} is an internal component C of F_b, and is the only one. Thus, by (d), either F_b coincides with it, and is smooth and irreducible, or $C^2 = -1$ and the other irreducible components E_i of F_b are contained in ∂Y. In the latter case, f_Y^n should contract the E_i. But, writing a resolution of the indeterminacies of f_Y^n along F_b, one sees that the first curve f_Y^n should contract is C, since by (d) the E_i have self-intersection ≤ -2. This is a contradiction, because C is internal. Hence, F_b is smooth, irreducible, and is equal to the internal curve C.

This result can be applied to $F_{f_B^n(b)}$ for every $n \in \mathbf{Z}$. If F_b contained an indeterminacy point of f_Y, $F_{f_B(b)}$ would contain a boundary component or would not be smooth, and we would obtain a contradiction. Therefore, F_b does not contain any indeterminacy point of f_Y (resp. of f^{-1}).

To prove (g), we can now assume that F_b is not mobile, hence that it is fixed. We argue as before: in a minimal resolution of the indeterminacies of f_Y along

F_b, the only curve one could be contracted would be the strict transform of an internal component, and this is not allowed.

Remark 3.7. As in Remark 3.5, suppose that π_0 has a base point. Let I_Y be the image of I_Z in Y. The curve I_Y intersects the general fiber of π_Y, and hence all of them. Since I_Y is contained in $\mathcal{U}_Y$, every fiber contains an internal component. By Property (g), f_Y has no indeterminacy points, which contradicts f_Y being a Jonquières twist (see Item (b) in Section 3.1). Thus, *if f is an automorphism of an affine surface X_0 and is a Jonquières twist, then its invariant fibration π_0 is regular on X_0 (it does not have base points).* This proves Assertion (2) of the theorem.

• Up to now, we have not assumed f_B to be of infinite order. We now make this hypothesis and identify B and f_B:

(h) B is a rational curve, f_B is conjugate to a translation $\mathbf{b} \mapsto \mathbf{b}+1$ or to a scalar multiplication $\mathbf{b} \mapsto \alpha\mathbf{b}$, where α is not a root of 1. In particular, f_B has either 1 or 2 fixed points, and the orbit of a point that is not fixed is infinite.

Indeed, if B is not a rational curve, then it is a curve of genus 1 and every orbit of f_B is infinite. Thus, Y does not contain any vertical boundary component (by (f)), and f_Y is regular (again by (f)). On the other hand, a Jonquières twist is never a regular automorphism (see Item (b) in Section 3.1). Property (h) follows from this contradiction.

As a consequence, with Property (f) we get

(i) a fiber of π_Y which is not above a fixed point of f_B is mobile; it is smooth and irreducible, and it does not contain any indeterminacy point (of f^n, for any $n \in \mathbf{Z}$).

• Since every fiber of π_0 intersects ∂X, there is a component D of ∂Z such that π_Z maps D onto B. We continue to denote by D its image in Y. Since f_Y preserves the fibration, it cannot contract D: f_Y permutes the components of ∂Y which are mapped onto B by π_Y. Changing f_Y into some positive iterate f_Y^k, we may assume that these components are fixed by f_Y. Doing so, f_Y determines an automorphism f_D of D, and the restriction of π_Y to D is a dominant morphism $\pi_D \colon D \to B$ such that $\pi_D \circ f_D = f_B \circ \pi_D$.

(j) The ramification points of the morphism $\pi_D \colon D \to B$ are mapped to fixed points of f_B. If E is a reduced curve that is a component of a fiber (with some multiplicity ≥ 1), then E is transverse to D.

Indeed, suppose that D is tangent to the fibration at some point q. If $\pi_Y(q)$ is not a fixed point of f_B, then f_Y^n is regular on a neighborhood of q for all $n \in \mathbf{Z}$ (by Property (i)), and the orbit of q is infinite. Thus, D should be tangent to the fibration on an infinite set, a contradiction. Thus, $\pi_Y(q)$ is a fixed point of f_B. The same argument applies to ramification points of π_D.

If $f_B(\mathbf{b}) = \mathbf{b}+1$, then f_D is also a translation, and π_D has a unique ramification point. This is a contradiction since a map from $\mathbb{P}^1$ to $\mathbb{P}^1$ which is not étale must have at least 2 ramification points. Thus, π_D is an isomorphism onto B (because $B \simeq \mathbb{P}^1$), and D is everywhere transverse to the fibration.

If $f_B(\mathbf{b}) = \alpha\mathbf{b}$, then f_D is also a scalar multiplication, and π_D has 2 ramification points, one above 0 and one above ∞. Note that one of the fibers F_0 or F_∞ is contracted by f_Y, because f_Y is a Jonquières twist (see, again, Item (b) in Section 3.1). On the other hand, if a fiber F is contracted by a Jonquières twist, then any invariant irreducible curve that is generically transverse to the fibration must intersect F transversally (see Section 3, and in particular Lemma 3.28 in [37]). This concludes the proof of Property (j).

• If $f_B(\mathbf{b}) = \mathbf{b}+1$ the fiber F_∞ is the only fiber contracted by f_Y. From Properties (d) and (g), this fiber is a smooth boundary component. Thus, by Property (f), Y is ruled. Doing elementary transformations along F_∞, we do not modify $\mathcal{U}_Y$ and we can construct a birational map $\varepsilon\colon Y \to \mathbb{P}^1 \times \mathbb{P}^1$ that transforms π_Y to the first projection, maps D to the section $\mathbb{P}^1 \times \{\infty\}$, and transforms F_∞ to $\{\infty\} \times \mathbb{P}^1$. Then, using affine coordinates $(\mathbf{x}, \mathbf{y})$ on $\mathbb{P}^1 \times \mathbb{P}^1$, ε conjugates f_Y (i.e. f_Y^k) to a map of type

$$(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}+1, a(\mathbf{x})\mathbf{y} + b(\mathbf{x}))$$

where $a(\mathbf{x})$ is some regular function of $\mathbf{x} \in \mathbb{A}^1$ that does not vanish, since otherwise a mobile fiber would be contracted. Thus, a is a constant, and this contradicts the fact that f_Y is a Jonquières twists (the degree of the iterates would be bounded). So, the case $f_B(\mathbf{b}) = \mathbf{b}+1$ does not appear.

• If $f_B(\mathbf{b}) = \alpha\mathbf{b}$, we apply the same strategy. At least one fiber among F_0 and F_∞ is contracted, say F_∞; it is a smooth irreducible component of ∂Y.

Assume first that F_0 does not contain any internal component. Then it is smooth and contained in ∂Y. Doing elementary transformations centered on F_∞, we get a map $\varepsilon\colon Y \to \mathbb{P}^1 \times \mathbb{P}^1$ that maps $\mathcal{U}_Y$ into $(\mathbb{P}^1 \setminus \{0, \infty\}) \times (\mathbb{P}^1 \setminus \{\infty\}))$ and conjugates f_Y^k to a map of type

$$(\mathbf{x}, \mathbf{y}) \mapsto (\alpha\mathbf{x}, a(\mathbf{x})\mathbf{y} + b(\mathbf{x}))$$

where a is non-constant and does not vanish on $\mathbb{P}^1 \setminus \{0,\infty\}$. Thus, $a \circ \varepsilon$ is the desired non-constant, non-vanishing function we were looking for.

Assume now that F_0 contains an internal component. Then f_Y acts regularly on $Y \setminus F_\infty$. Let us contract the (-1)-components of F_0 iteratively to reach a ruled surface W; this process may modify $\mathcal{U}_Y$. With elementary transformations based on F_∞, we construct a birational map $\varepsilon\colon W \to \mathbb{P}^1 \times \mathbb{P}^1$ as above. It maps D to $\mathbb{P}^1 \times \{\infty\}$ and conjugates f_Y to a map of type

$$(\mathbf{x},\mathbf{y}) \mapsto (\alpha\mathbf{x}, a(\mathbf{x})\mathbf{y} + b(\mathbf{x}))$$

where a is not constant and does not vanish if $\mathbf{x} \neq 0$. Thus, $a(\mathbf{x}) = \beta\mathbf{x}^d$ for some $d \neq 0$ and $\beta \neq 0$. Since $d \neq 0$, the fiber $\{0\} \times \mathbb{P}^1$ is contracted, hence so is F_0. Thus, in fact, F_0 is smooth and contained in ∂Y, contradicting our hypothesis.

Remark 3.8. We have excluded the case $f_B(\mathbf{b}) = \mathbf{b} + 1$ and we have proven that F_0 and F_∞ do not contain any internal components. Therefore, every point of $\mathcal{U}_Y$ has an infinite orbit (since its projection under π_Y also has an infinite orbit). In particular, X_0 is smooth, since otherwise a singularity would create a fiber of π_Y with a finite orbit intersecting $\mathcal{U}_Y$. This remark proves the last sentence of Theorem C.

• To conclude, we need to go back to X_0. In the last step, we have constructed a regular function $\xi := a \circ \varepsilon$ on $\mathcal{U}_Y$ that is non-constant and does not vanish. But $\mathcal{U}_Y$ is isomorphic to $\eta'(\psi^{-1}(X_0))$, and $\psi \circ (\eta')^{-1} \colon Y \dashrightarrow Z \to X$ is obtained by a sequence of blow-ups (of the base points of π_0). By construction, $\xi \circ \eta' \circ \psi^{-1}$ is regular and does not vanish outside the indeterminacy points of $\psi \circ (\eta')^{-1}$. Thus, it extends to a regular, non-vanishing function on X_0 (because X_0 is normal). This concludes the proof of Theorem C. □

Corollary 3.9. *Let X_0 be an affine surface. Assume $\Gamma \subset \mathsf{Aut}(X_0)$ is non-elementary and contains a parabolic element acting as an automorphism of infinite order on the base of its invariant fibration. Then, X_0 is isomorphic to $\mathbb{G}_m \times \mathbb{G}_m$ over $\overline{\mathbf{K}}$.*

Indeed, Theorem C shows that there is a non-constant and non-vanishing regular function on X_0. Since Γ contains a loxodromic element, Theorem 3.4 shows that X_0 is isomorphic to $\mathbb{G}_m \times \mathbb{G}_m$.

Corollary 3.10. *Let X_0 be an affine surface that is not isomorphic to $\mathbb{G}_m \times \mathbb{G}_m$ over $\overline{\mathbf{K}}$. Assume that $\mathsf{Aut}(X_0)$ contains a loxodromic automorphism. If f is a parabolic element of $\mathsf{Aut}(X_0)$ then*

(1) *f preserves a regular fibration $\pi_0 \colon X_0 \to B$ with base and fibers of genus 0;*
(2) *a positive iterate f^k of f preserves each fiber of this fibration;*
(3) *the Zariski closure of every orbit of f has dimension 0 (if the orbit is finite) or 1 (if the orbit is infinite, in which case its closure is a finite number of fibers of π_0).*

Indeed, π_0 does not have base points in X_0 by Assertion (2) in Theorem C (or Remark 3.7).

3.4. **Finite orbits of parabolic automorphisms.** For a subring R of $\mathbf{K}$, denote by $\mathrm{Per}_f(R)$ the set of periodic points of f contained in $X_0(R)$.

Theorem 3.11. *Let X_0 be an affine surface defined over the valuation ring $\mathfrak{o}_{\mathbf{K}}$ of a p-adic field $\mathbf{K}$. Let f be a parabolic automorphism of X_0 defined over $\mathfrak{o}_{\mathbf{K}}$. Then, there is an integer $m \geq 1$ and a Zariski closed subset P_f of X_0 such that* $\mathrm{Per}_f(\mathfrak{o}_{\mathbf{K}}) = P_f(\mathfrak{o}_{\mathbf{K}}) = \{x \in X_0(\mathfrak{o}_{\mathbf{K}})\,;\, f^m(x) = x\}$.

Of course, P_f and m depend on f and on $\mathbf{K}$. The same result does not hold on the algebraic closure of $\mathbf{K}$: see Example 5.1.

Lemma 3.12. *Let f be an automorphism of an irreducible, affine curve $F \subset \mathbb{A}^N$. If $x \in F$ is f-periodic, then either f has finite order in $\mathsf{Aut}(F)$ or $f(x) = x$.*

Proof. Let $\overline{F}$ be the completion of F in $\mathbb{P}^N$, and let s be the number of points of $\overline{F}$ at infinity. Let r be the size of the orbit of x. If x is not fixed, we have $r \geq 2$ and $s \geq 1$ so $r+s \geq 3$. If we set $m = (r+s)!$, then f^m fixes at least 3 points of $\overline{F}$. Thus, $f^m = \mathrm{id}_F$. □

Proof of Theorem 3.11. Let $\pi_0 \colon X_0 \to B$ be the f-invariant fibration, and let f_B be the automorphism of B induced by f.

If f_B has infinite order, then f_B has at most two periodic points, which are fixed. Denote them by $\{b, b'\}$. Then, the finite orbits of f are contained in the fibers $\pi_0^{-1}(b)$ and $\pi_0^{-1}(b')$, and the theorem follows from the previous lemma, applied to the restriction of f to $\pi_0^{-1}(b)$ and $\pi_0^{-1}(b')$.

Now, assume that f_B is an element of order $1 \leq \ell < \infty$ in $\mathsf{Aut}(B)$. Then, changing f to f^ℓ, we are reduced to the case $f_B = \mathrm{id}_B$. Let k be a positive integer such that f^k fixes every irreducible component of every fiber of π_0.

From the previous lemma, the problem is to show that the number of points $b \in B(\mathfrak{o}_{\mathbf{K}})$ such that $f_{|F_b}$ has finite order is finite. For this, we can reduce the study to regular fibers. But, in some completion X of X_0, every regular fiber has a neighborhood U isomorphic to $U_0 \times \mathbb{P}^1$ (for some $U_0 \subset B$, see [5, Thm. III.4]), on which f acts by

$$f(b,z) = (b, A_b(z)) \tag{3.4}$$

where $A\colon b \in B \mapsto A_b \in \mathsf{PGL}_2(\overline{\mathbf{K}})$ is a non-constant rational map. Both π_0 and A are defined on a finite extension $\mathbf{L}$ of $\mathbf{K}$. So, to conclude, we only need to show that the set

$$\{b \in B(\mathfrak{o}_{\mathbf{L}})\ ;\ A_b \text{ has finite order in } \mathsf{PGL}_2(\mathbf{L})\} \tag{3.5}$$

is finite. To see this, we lift A_b to an element A'_b in $\mathsf{GL}_2(\overline{\mathbf{L}})$ and set

$$r(b) = \frac{\mathsf{Tr}(A'_b)^2}{\mathsf{det}(A'_b)} - 2; \tag{3.6}$$

then the order of A_b is finite if and only if $r(b) = \alpha + \alpha^{-1}$ for some root of unity $\alpha \in \overline{\mathbf{L}}$; more precisely, A_b is conjugate to $z \mapsto \alpha z$ and the order of α is the order of A_b. When b is in $B(\mathfrak{o}_{\mathbf{L}})$, $r(b)$ is in $\mathbf{L}$ and α is a root of unity in $\overline{\mathbf{L}}$ that satisfies a quadratic equation over $\mathbf{L}$. The structure of the multiplicative group of finite extensions of $\mathbf{Q}_p$ (see [32], p.140, or Section 5.1 below) implies that the order of such a root of unity is uniformly bounded, independently of $r(b) \in \mathbf{L}$. This concludes the proof. □

Remark 3.13. If $f_B = \mathrm{id}_B$ and, in some invariant bidisk $\mathcal{U} \simeq \mathfrak{o}_{\mathbf{K}}{}^2$, $f = \Phi^1$ for some flow $\Phi^t\colon \mathfrak{o}_{\mathbf{K}} \times \mathcal{U} \to \mathcal{U}$, then the vector field $\Theta_f = \partial_t \Phi^t_{|t=0}$ is everywhere tangent to the fibers of π_0. Thus, we may extend the distribution of lines $x \mapsto \mathbf{K}\Theta_f(x)$ globally as the algebraic distribution of tangent lines $x \mapsto \mathrm{Ker}(d\pi_{0,x})$. Note that Θ_f vanishes identically on each fiber F_b on which f induces a finite order automorphism.

4. Elliptic automorphisms

4.1. **Bounded degrees.** Let f be an automorphism of an affine variety V, both defined over a field $\mathbf{K}$ of characteristic 0.

If $(\deg(f^n))$ is bounded, then f is contained in a linear algebraic group G acting algebraically on V (see [17, Prop. 3.2], with B as a point). The Zariski closure of $f^{\mathbf{Z}}$ in G is an abelian algebraic subgroup A of G; we call it the *algebraic completion of* $f^{\mathbf{Z}}$. Let A° denote the (Zariski) connected component of the identity in A, and let k be the index of A° in A; then, f^k is contained in A°, and its iterates form a Zariski dense subset of A°. In what follows, we assume that $\dim(A^\circ) \geq 1$; equivalently, the order of f in $\mathsf{Aut}(X_0)$ is infinite.

On the algebraic closure $\overline{\mathbf{K}}$ of $\mathbf{K}$, A° is isomorphic to $\mathbb{G}_a^r \times \mathbb{G}_m^s$ for some integers $r, s \geq 0$ (here we use $\mathrm{char}(\mathbf{K}) = 0$). Since the cyclic group $(f^k)^{\mathbf{Z}}$ is Zariski dense in A°, we have $r \leq 1$. The algebraic subgroups of $\mathbb{G}_m^s$ form a countable family of subgroups, so the generic point of X_0 has a trivial stabilizer in $\mathbb{G}_m^s \subset A^\circ$. Since the only nontrivial algebraic subgroup of $\mathbb{G}_a$ is $\mathbb{G}_a$ itself, the general point of X_0 also has a trivial stabilizer in $\mathbb{G}_a \subset A^\circ$. Thus $r + s \leq \dim(V)$.

4.2. **The Lie algebra of A° and the Bell-Poonen theorem.** Taking the derivative of the action of A° on V, the Lie algebra $\mathsf{Lie}(A^\circ)$ determines a subalgebra of the Lie algebra $\Gamma(V, TV)$ of regular vector fields on V. We identify $\mathsf{Lie}(A^\circ)$ with its image in $\Gamma(V, TV)$.

Assume, now, that $\mathbf{K}$ is a p-adic local field, and that V and f are defined over its valuation ring $\mathfrak{o}_{\mathbf{K}}$. As in the Bell-Poonen theorem, suppose there exists an open set $\mathcal{U} \simeq \mathfrak{o}_{\mathbf{K}}{}^2 \subset V(\mathfrak{o}_{\mathbf{K}})$ preserved by f on which $f = \Phi^1$ for some analytic flow $\Phi \colon \mathfrak{o}_{\mathbf{K}} \times \mathcal{U} \to \mathcal{U}$. Then, Φ^t corresponds to a 1-parameter subgroup of A°. In particular, there is an algebraic vector field $a_f \in \mathsf{Lie}(A^\circ)$ such that

$$\Theta_f(y) = \left(\frac{\partial \Phi^t(y)}{\partial t}\right)_{t=0} = a_f(y) \tag{4.1}$$

for $y \in \mathcal{U}$. Thus, Θ_f extends from $\mathcal{U}$ to V as a globally defined algebraic vector field.

For each y in $\mathcal{U}$ (resp. V), we denote by $L_f(y)$ the line $\mathbf{K} a_f(y) \subset T_y V$, with a small abuse of notation because this distribution of lines is not well-defined at the points y where $a_f(y) = 0$. (Here, in comparison with §2.5, the lines are defined over $\mathbf{K}$ instead of $\mathbf{Q}_p$.) We have shown the following.

Lemma 4.1. *If f is an elliptic automorphism, the distribution of tangent lines*

$$L_f \colon y \mapsto L_f(y) := \mathbf{K}\Theta_f(y) = \mathbf{K} a_f(y)$$

depends algebraically on x. In other words, it defines a rational section of the projectivized tangent bundle $\mathbb{P}(TV)$.

Let x be a point of $V(\mathfrak{o}_{\mathbf{K}})$ and set $\gamma(t) = \Phi^t(x)$. Let g be an element of $\mathsf{Aut}(V_{\mathfrak{o}_{\mathbf{K}}})$. We say that g **preserves locally the analytic trajectory** of x determined by f if g maps $\gamma(\mathfrak{m}_{\mathbf{K}}^{\ell})$ into $\gamma(\mathfrak{o}_{\mathbf{K}})$ for some $\ell \geq 1$. There we obtain

$$g(\gamma(t)) = \gamma(\varphi(t)) \tag{4.2}$$

for some germ of analytic function φ around the origin in $\mathfrak{o}_{\mathbf{K}}$. Taking derivatives, we get

$$Dg_{\gamma(t)}\dot{\gamma}(t) = \varphi'(t)\dot{\gamma}(\varphi(t)) \tag{4.3}$$

where Dg is the differential, $\dot{\gamma}$ is the velocity vector of the curve γ, and φ' is the derivative of the function φ. Thus, along the curve γ, g preserves the distribution of tangent lines to the orbits of Φ^t. But this distribution of lines $x \mapsto \mathbf{K}\Theta_f(x)$ depends algebraically on x. This gives the following lemma.

Lemma 4.2. *Suppose $g \in \mathsf{Aut}(V_{\mathfrak{o}_{\mathbf{K}}})$ preserves the analytic trajectory of x, and that this trajectory is Zariski dense in V. Then, the distribution of lines*

$$L_f\colon\, y \in V \mapsto L_f(y) := \mathbf{K}a_f(y)$$

is g-invariant.

4.3. **Finite orbits.**

Lemma 4.3. *Let f be an elliptic automorphism of an affine variety V, let A° be the connected algebraic group associated with f, and let P_f be the algebraic set defined by $P_f = \{x \in V\ ;\ a(x) = 0\ \ \forall a \in \mathsf{Lie}(A^\circ)\}$. Then $q \in V(\overline{\mathbf{K}})$ has a finite f-orbit if and only if $q \in P_f(\overline{\mathbf{K}})$; there is an integer ℓ such that the period of every point $q \in P_f(\overline{\mathbf{K}})$ divides ℓ.*

Note that this lemma works for all finite orbits in $V(\overline{\mathbf{K}})$, in contrast with Theorem 3.11.

Proof. We may assume that the order of f is infinite, since otherwise $P_f = V$ and we can set ℓ to be the order of f. A point $q \in V(\overline{\mathbf{K}})$ has a finite orbit if and only if its stabilizer has finite index in $f^{\mathbf{Z}}$. Since such a subgroup intersects A° on a Zariski dense subgroup, q has a finite orbit if and only if $A^\circ(q) = q$, if and only if $q \in P_f(\overline{\mathbf{K}})$. We can then take ℓ to be the index of A° in A. □

Example 4.4. If f is the automorphism of the plane defined by $f(\mathbf{x},\mathbf{y}) = (\mathbf{x}+\mathbf{y}^2+\mathbf{y}^3, -\mathbf{y})$, then f is elliptic. Indeed, $f^{2n}(\mathbf{x},\mathbf{y}) = (\mathbf{x}+2n\mathbf{y}^2, \mathbf{y})$ has degree 2, and $f^{2n+1} = (\mathbf{x}+(2n+1)\mathbf{y}^2+\mathbf{y}^3, -\mathbf{y})$ has degree 3. This reveals that $P_f = (\mathbf{y}^2 = 0) = (\mathbf{y} = 0)$. We can also explicitly compute the algebraic completion A of f: the identity component A° acts by $\Phi^t(\mathbf{x},\mathbf{y}) = (\mathbf{x}+t\mathbf{y}^2, \mathbf{y})$; its second component is obtained by the f-shift $f\circ\Phi^t(\mathbf{x},\mathbf{y}) = (\mathbf{x}+(t+1)\mathbf{y}^2+\mathbf{y}^3,-\mathbf{y})$. One can further generalize this to elementary transformations of the type $(\mathbf{x},\mathbf{y})\mapsto(\alpha\mathbf{x}+q(\mathbf{y}),\beta\mathbf{y})$, where α and β are roots of unity.

4.4. **Structure of A°.** Let us apply Sections 4.1 and 4.2 when f is an elliptic automorphism of an affine surface X. We obtain $A^\circ = \mathbb{G}_a^r\times\mathbb{G}_m^s$ with $r\le 1$ and $r+s\le 2$. More precisely, over $\overline{\mathbf{K}}$, exactly one of the following situations occurs:

(1) $r=0$ and $s=2$, and $A^\circ=\mathbb{G}_m\times\mathbb{G}_m$ has a unique open orbit, which is isomorphic to A°;
(2) $r=1$ and $s=1$, and $A^\circ=\mathbb{G}_a\times\mathbb{G}_m$ has a unique open orbit, which is isomorphic to A°;
(3) $r=0$ and $s=1$, or $r=1$ and $s=0$, then $A^\circ=\mathbb{G}_m$ or $\mathbb{G}_a$, respectively. The general orbits of A° have dimension 1, and they define a rational fibration $\pi_0\colon X\dashrightarrow B$ onto a curve B such that $\pi_0\circ h=\pi_0$ for every $h\in A^\circ$.

If f has a Zariski dense orbit, we are in one of the first two cases.

Example 4.5. If f is the automorphism of the plane defined by $f(\mathbf{x},\mathbf{y}) = (\mathbf{x}+1,\alpha\mathbf{y})$, where $\alpha\in\mathbf{K}^\times$ is not a root of unity, then the orbit of (x,y) is Zariski dense in $\mathbb{A}^2$ if and only if $y\neq 0$. If $f(\mathbf{x},\mathbf{y})=(\alpha\mathbf{x},\beta\mathbf{y})$ and α and β are elements of $\mathbf{K}^\times$ that are multiplicatively independent, then the orbit of (x,y) is Zariski dense if, and only if, $xy\neq 0$.

4.5. **The Fixed Locus.** If a reductive algebraic group G acts on an affine variety V, then any two disjoint, invariant, Zariski closed subsets can be separated by a regular invariant function on V (see [31, Cor. 1.2]).

Lemma 4.6. *Let $\mathbb{G}_m\times V\to V$ be a nontrivial algebraic action on an affine variety V. For each point $x\in V$,*

(1) *the Zariski closure of the $\mathbb{G}_m$-orbit $\mathbb{G}_m(x)$ contains at most one fixed point;*
(2) *if x is fixed by $\mathbb{G}_m$, there is a $\mathbb{G}_m$-orbit which is 1-dimensional and whose Zariski closure contains x.*

Proof. (1) Fix $x\in V$. If the Zariski closure of $\mathbb{G}_m(x)$ has more than one $\mathbb{G}_m$-fixed point, then these points must be separated by a regular invariant function [31, Cor. 1.2]. Such an invariant function must be constant on the Zariski closure, a contradiction.

(2) We proceed by induction on $\dim V$. If $\dim V = 1$, then as $\mathbb{G}_m$ acts nontrivially on V, there is a dense orbit in V, and the closure of this orbit contains x.

Now suppose $\dim V \geq 2$. If every 1-dimensional $\mathbb{G}_m$-orbit closure contains the fixed point x, then we are done. Otherwise, suppose there exists $y \in V$, not fixed by $\mathbb{G}_m$, such that x is not contained in the Zariski closure of $\mathbb{G}_m(y)$.

Consider the GIT quotient $V/\!/\mathbb{G}_m$ (see, e.g., [30, Def. 5.8]) and the natural projection $\pi\colon V \to V/\!/\mathbb{G}_m$. Observe that (a) the fiber $\pi^{-1}(\pi(x))$ contains a unique fixed point, namely x, and (b) the fiber $\pi^{-1}(\pi(x))$ has dimension $< \dim V$, as π separates the fiber from $\mathbb{G}_m(y)$. On the other hand, the general orbit of $\mathbb{G}_m$ has dimension 1, so all fibers of π have positive dimensions. By choosing $V' \subset \pi^{-1}(\pi(x))$ to be an irreduible component containing x, we have $0 < \dim V' < \dim V$. Since $\mathbb{G}_m$ is connected, V' is invariant, and since π separates fixed points, the action of $\mathbb{G}_m$ on V' is nontrivial. Thus, we can conclude by induction. □

Using this, we can study the dimension of the fixed locus P_f, for f an elliptic automorphism of an affine surface X. Let A be the algebraic completion of $f^{\mathbf{Z}}$. By section 4.4, we know that A° is either $\mathbb{G}_m$, $\mathbb{G}_a$, $\mathbb{G}_a \times \mathbb{G}_m$, or $\mathbb{G}_m \times \mathbb{G}_m$. We have observed, in Example 4.4, that if $\dim A^\circ = 1$ then P_f may be a curve.

Proposition 4.7. *If* $\dim A^\circ = 2$*, then* P_f *has at most one point.*

Proof. • First, suppose $A^\circ \simeq \mathbb{G}_a \times \mathbb{G}_m$. Denote simply by $\mathbb{G}_m$ the subgroup $\{0\} \times \mathbb{G}_m \subset A^\circ$, and consider the quotient $\pi\colon X \to B := X/\!/\mathbb{G}_m$. By [31, Thm. 1.1], B is affine. As $\mathbb{G}_m$ acts nontrivially on X, the general fibers of π have dimension 1 and $\dim B \leq 1$.

If $\dim B = 0$, then B is a point because X is irreducible; since P_f is pointwise fixed by $\mathbb{G}_m$, and π separates fixed points, there is at most one point in P_f.

Suppose $\dim B = 1$. The action of A° on X induces an algebraic action of $\mathbb{G}_a$ on B. If A° has a fixed point q, then $\pi(q)$ is fixed by $\mathbb{G}_a$, and thus $\mathbb{G}_a$ acts trivially on B ([3]). So, we can assume that $\mathbb{G}_a$ acts trivially on B. Then $\mathbb{G}_a \times \mathbb{G}_m$ acts faithfully on the generic fiber of π (i.e., $\pi^{-1}(\eta_B)$ where η_B is the generic

[3] If $\mathbb{G}_a$ acted non trivially on an irreducible affine curve B, it would have no fixed point. Indeed, $\mathbb{G}_a$ acts on the normalization $\tilde{B}$ of B, and since the action is nontrivial, $\tilde{B}$ must be a smooth, affine, rational curve. In other words, $\tilde{B}$ is isomorphic to $\mathbb{P}^1 \setminus F$, for some finite subset F, with $|F| \geq 1$. But then, since $\mathbb{G}_a$ acts nontrivially on it, $\tilde{B} \simeq \mathbb{A}^1$. The action of $\mathbb{G}_a$ has no fixed point, and $\tilde{B} \simeq B$.

point of B; as the fiber is dense in X, we have the faithfulness). This leads to a contradiction, as the group $\mathbb{G}_a \times \mathbb{G}_m$ cannot act faithfully on a curve.

• Next, suppose $A^\circ \simeq \mathbb{G}_m \times \mathbb{G}_m$. As remarked in item (1) of §4.4, we have a unique open orbit $A^\circ(x)$ isomorphic to A° in X, which must be Zariski dense as well. Thus, any A°-invariant regular function on X must be constant, equal to the value taken at x. Therefore the GIT quotient $X/\!/A^\circ$ must be a point. Since P_f coincides with the set of fixed points of A°, and the quotient $X \to X/\!/A^\circ$ separates fixed points (by [31, Cor. 1.2] again), we conclude that $|P_f| \leq 1$. □

Example 4.8. The above proof shows that, if $A^\circ = \mathbb{G}_a \times \mathbb{G}_m$ then A° does not have a fixed point unless $\dim(X/\!/\mathbb{G}_m) = 0$. To get an example, set $h^s = \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)$ for $s \in \mathbb{G}_a$, and let $\mathbb{G}_a \times \mathbb{G}_m$ act on $X = \mathbb{A}^2$ by $(\mathbf{s}, \mathbf{t}) \cdot (\mathbf{x}, \mathbf{y}) = \mathbf{t}h^{\mathbf{s}}(\mathbf{x}, \mathbf{y}) = (\mathbf{t}(\mathbf{x} + \mathbf{sy}), \mathbf{ty})$. The point $(0,0)$ is fixed and is the only fixed point of A° in X.

4.6. **Elliptic automorphisms with Zariski dense orbits.**

Theorem 4.9. *Let X be a normal affine surface defined over the valuation ring $\mathfrak{o}_{\mathbf{K}}$ of a p-adic local field $\mathbf{K}$. Let f be an elliptic automorphism of X, defined over $\mathfrak{o}_{\mathbf{K}}$. Let $\mathcal{U} \subset X(\mathfrak{o}_{\mathbf{K}})$ be an f-invariant bidisk, on which $f = \Phi^1$ for some analytic flow. If there is a loxodromic automorphism g in $\mathsf{Aut}(X)$ defined over $\mathfrak{o}_{\mathbf{K}}$ and a point $x \in \mathcal{U}$ such that (i) the analytic trajectory $t \mapsto \Phi^t(x)$ is Zariski dense in X and (ii) g preserves locally this analytic trajectory, then X is isomorphic to $\mathbb{G}_m \times \mathbb{G}_m$ over $\overline{\mathbf{K}}$.*

The proof is related to the classification of foliations on complex surfaces invariant by an infinite group of birational transformations (see [15, Thm. 1.1]).

Proof. Let γ be the curve $t \mapsto \Phi^t(x)$. By assumption, g is loxodromic and the image of γ is Zariski dense. Hence, the algebraic completion A of $f^{\mathbf{Z}}$ has $\dim(A) = 2$ and A° is isomorphic to $\mathbb{G}_a \times \mathbb{G}_m$ or $\mathbb{G}_m^2$ over $\overline{\mathbf{K}}$. Let k be the index of $A^\circ \subset A$.

Step 1.– By Lemma 4.2, L_f is g-invariant, hence there is a rational function ξ on X such that $g_* a_f = \xi \cdot a_f$. This function ξ is regular on the complement of the zero set

$$Z(a_f) = \{x \in X \; ; \; a_f(x) = 0\}; \tag{4.4}$$

but Proposition 4.7 shows that this set is finite because it is pointwise fixed by f^k, hence by A°. Since X is normal, ξ extends as a regular function on X. Moreover, if $\{\xi = 0\}$ were non-empty, it would be a curve, and the differential dg_x would have rank ≤ 1 along a curve. However, since g is an automorphism, we see that ξ does not vanish at all. By Theorem 3.4, we may assume that ξ is a constant. This implies that g conjugates f^k to another element of A°. Since $(f^k)^{\mathbf{Z}}$ is Zariski dense in A°, the action of g by conjugacy on $\mathsf{Aut}(X)$ preserves A°. In other words, there is an algebraic automorphism ψ_g of the algebraic group A° such that $ghg^{-1} = \psi_g(h)$ for every $h \in A^\circ$.

Step 2.– If A° were isomorphic to $\mathbb{G}_a \times \mathbb{G}_m$, the action of g on it by conjugacy would preserve the factor $\mathbb{G}_a$, and then g would preserve the rational fibration of X given by the orbits of $\mathbb{G}_m$. This contradicts to the fact that g is loxodromic (see Theorem 3.3). Thus, A° is isomorphic to $\mathbb{G}_m^2$ over $\overline{\mathbf{K}}$, and ψ_g is a monomial automorphism of $\mathbb{G}_m \times \mathbb{G}_m$. From this, $X_{\overline{\mathbf{K}}}$ contains a copy of $\mathbb{G}_m \times \mathbb{G}_m$, represented by the unique open orbit of A°, which is g-invariant because $gA^\circ g^{-1} = A^\circ$, and g acts on it via a loxodromic monomial transformation. This implies that X is isomoprhic to $\mathbb{G}_m \times \mathbb{G}_m$. Indeed, let Y be a projective completion of X. It is a projective completion of $\mathbb{G}_m \times \mathbb{G}_m$, and g induces a birational transformation of Y acting as a regular automorphism on $\mathbb{G}_m \times \mathbb{G}_m$. Let $(\mathbf{x}_1, \mathbf{x}_2)$ be the affine coordinates on $\mathbb{G}_m \times \mathbb{G}_m$. There are elements α, β in $\overline{\mathbf{K}}^\times$, and a matrix

$$M_g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.5}$$

of $\mathsf{GL}_2(\mathbf{Z})$ such that $g(\mathbf{x}_1, \mathbf{x}_2) = (\alpha \mathbf{x}_1^a \mathbf{x}_2^b, \beta \mathbf{x}_1^c \mathbf{x}_2^d)$. Since g is loxodromic, its dynamical degree is > 1, and M_g is conjugate in $\mathsf{GL}_2(\mathbf{R})$ to a diagonal matrix with eigenvalues $\pm\lambda_1(g)$ and $\pm\lambda_1(g)^{-1}$, with eigenvectors in $\mathbf{R}^2 \setminus \mathbf{Q}^2$. This implies that $Y \setminus (\mathbb{G}_m \times \mathbb{G}_m)$ is entirely contracted by some positive (resp. negative) iterate of g on an indeterminacy point of g^{-1} (resp. of g). Since g induces an automorphism of X, this implies that $X = \mathbb{G}_m \times \mathbb{G}_m$. □

5. The multiplicative group $\mathbb{G}_m^2$

5.1. **Multiplicative groups.** Let d be a positive integer, and let V be the mutiplicative group $\mathbb{G}_m^d$, viewed as an affine variety. The automorphism group $\mathsf{Aut}(V)$ is $\mathsf{GL}_d(\mathbf{Z}) \ltimes V$, where V acts on itself by tranlation and $\mathsf{GL}_d(\mathbf{Z})$ by monomial transformations: the action of a matrix $(a_{i,j}) \in \mathsf{GL}_d(\mathbf{Z})$ is given by

$$(v_1, \ldots, v_d) \mapsto (v_1^{a_{1,1}} \cdots v_d^{a_{d,d}}, \ldots, v_1^{a_{d,1}} \cdots v_d^{a_{d,d}}). \tag{5.1}$$

Let p be a prime and $\mathbf{K}$ be a p-adic local field, with uniformizer π, valuation ring $\mathfrak{o}_{\mathbf{K}}$, and group of principal units $U_{\mathbf{K}} = 1 + \mathfrak{m}_{\mathbf{K}}$. Let $\mathbf{k}$ be the residue field of $\mathbf{K}$; it is isomorphic to $\mathbf{F}_q$ where $q = p^f$. According to [32, Chap. II.5], the multiplicative group $\mathbf{K}^\times$ is isomorphic, as an abelian topological group, to the product

$$\pi^{\mathbf{Z}} \times \mu_{q-1} \times \mathbf{Z}/p^a\mathbf{Z} \times \mathbf{Z}_p^m \tag{5.2}$$

for some $m \geq 1$ and some $a \geq 0$, with *both m and a bounded in terms of* $[\mathbf{K} : \mathbf{Q}_p]$. The torsion group $\mathbf{Z}/p^a\mathbf{Z}$ comes from the roots of unity contained in $U_{\mathbf{K}}$. The reduction homomorphism $\mathfrak{o}_{\mathbf{K}} \to \mathbf{k}$ provides a bijection from $\mu_{q-1} = \{\zeta \in \mathbf{K}\,;\, \zeta^{q-1} = 1\}$ to $\mathbf{k}^\times$. Thus,

$$V(\mathfrak{o}_{\mathbf{K}}) = F \times (\mathbf{Z}_p^m)^d, \tag{5.3}$$

where F is the finite group $(\mu_{q-1} \times \mathbf{Z}/p^a\mathbf{Z})^d$.

The action of $\mathsf{GL}_d(\mathbf{Z})$ on V preserves $V(\mathfrak{o}_{\mathbf{K}})$. This gives an action of the subgroup $\mathsf{GL}_d(\mathbf{Z}) \ltimes V(\mathfrak{o}_{\mathbf{K}})$ of $\mathsf{Aut}(V)$ on $V(\mathfrak{o}_{\mathbf{K}})$. A finite index subgroup G of $\mathsf{GL}_d(\mathbf{Z}) \ltimes V(\mathfrak{o}_{\mathbf{K}})$ acts trivially on the finite part $F = V(\mathfrak{o}_{\mathbf{K}})/(\mathbf{Z}_p^m)^d$ and by affine transformations on $(\mathbf{Z}_p^m)^d$; more precisely, each element of G acts on $(\mathbf{Z}_p^m)^d$ by

$$(w_1, \ldots, w_d) \mapsto (s_1 + \sum_{j=1}^d a_{1,j}w_j, \ldots, s_d + \sum_{j=1}^d a_{d,j}w_j) \tag{5.4}$$

for some $(a_{i,j}) \in \mathsf{GL}_d(\mathbf{Z})$ and $(s_j)_{j=1}^d \in (\mathbf{Z}_p^m)^d$.

Example 5.1. Consider the monomial action of $\mathsf{GL}_d(\mathbf{Z})$ on $\mathbb{G}_m(\overline{\mathbf{Q}_p})^d$. A point has a finite orbit if and only if it is a torsion point, if and only if its coordinates $(\xi_1, \ldots, \xi_d)$ are roots of unity. Thus, finite orbits in $\mathbb{G}_m(\overline{\mathbf{Z}_p})^d$ are Zariski dense, but only finitely many are contained in $\mathbb{G}_m(\mathbf{Z}_p)^d$.

5.2. **Dimension** 2. We keep the same notation, but assume $d = 2$. Let Γ be a subgroup of $\mathsf{Aut}(V_{\mathfrak{o}_{\mathbf{K}}})$. The intersection $\Gamma_0 = \Gamma \cap G$ has a finite index in Γ. Let H_Γ be the image of Γ in $\mathsf{GL}_2(\mathbf{Z})$, and let $\overline{H}_\Gamma$ be the closure of H_Γ in $\mathsf{GL}_2(\mathbf{Z}_p)$; by construction, $\overline{H}_\Gamma$ is contained in $\mathsf{SL}_2^{\pm}(\mathbf{Z}_p)$, the group of matrices with determinant ± 1.

Lemma 5.2. *If* $\dim(V) = 2$ *and* Γ *is a non-elementary subgroup of* $\mathsf{Aut}(V_{\mathfrak{o}_{\mathbf{K}}})$, *then* $\overline{H}_\Gamma$ *is an open subgroup of* $\mathsf{SL}_2^{\pm}(\mathbf{Z}_p)$.

Proof. The group Γ contains a non-abelian free group, hence so do Γ_0 and H_Γ. This implies that $\overline{H}_\Gamma$ is open in $\mathsf{SL}_2^{\pm}(\mathbf{Z}_p)$ (see [29, Window 9, Thm. 2]).([4]) □

[4]Let us sketch a proof of this last fact. Pick two elements A and B in $\mathsf{SL}_2(\mathbf{Z}_p)$ generating a non-abelian free group. Let H be the closure of $\langle A, B\rangle$ in $\mathsf{SL}_2(\mathbf{Z}_p)$. Taking iterates, we can assume that $A = \mathrm{id}$ and $B = \mathrm{id}$ modulo p^2. Then, $A^{\mathbf{Z}}$ and $B^{\mathbf{Z}}$ are contained in 1-parameter

So, we assume that $\overline{H}_\Gamma$ is an open subgroup of $\mathsf{SL}_2^{\pm}(\mathbf{Z}_p)$. There are two cases, in each of them we prove a refined version of Theorem A:

5.2.1. Assume that Γ_0 is contained in $\mathsf{SL}_2^{\pm}(\mathbf{Z}_p)$. Its action on $(\mathbf{Z}_p^m)^2$, as given in Equation (5.4), coincides with its diagonal, linear action on $(\mathbf{Z}_p^2)^m$.

The action of Γ_0 on $\mathbf{Z}_p^2$ fixes the origin $o = (0,0)$, and if $u \in \mathbf{Z}_p^2 \setminus \{o\}$ its orbit closure is open. So, if $m = 1$, Γ_0 has one fixed point and infinitely many open orbit closures. If $m \geq 2$, the function $\mathsf{det}(v_1, v_2)$ is invariant, hence no orbit closure contains an open set. (If $m \geq 3$, any m-tuple of vectors (v_i) satisfies linear relations of length 3, such as $b_1v_1 + b_2v_2 + b_3v_3 = 0$ for some non-zero vector $(b_1, b_2, b_3) \in \mathbf{Z}_p^3$. This shows also that no orbit closure is open). Thus, one sees that

(1) the general orbit closure of Γ on $V(\mathfrak{o}_\mathbf{K})$ is open if and only if $m = 1$, that is, if and only if $\mathbf{K} = \mathbf{Q}_p$;
(2) Γ has a finite number of finite orbits (corresponding to the elements of $F \times \{o\}$);
(3) there are infinitely many distinct orbit closures.

5.2.2. Suppose that Γ_0 is not conjugate to a subgroup of $\mathsf{SL}_2^{\pm}(\mathbf{Z}_p)$. Since $\overline{H}_\Gamma$ contains an open set, the closure of Γ_0 in the affine group $\mathsf{SL}_2^{\pm}(\mathbf{Z}_p) \ltimes \mathbf{Z}_p^2$ contains an open neighborhood of the origin in $\mathbf{Z}_p^2$. This implies that every orbit closure is open when $m = 1$ or 2. On the other hand, when $m \geq 3$, the interior of every orbit closure is empty because the function $\mathsf{det}(v_2 - v_1, v_3 - v_1)$ is invariant.

5.3. **An example: the Cayley cubic.** Consider the surface $V = \mathbb{G}_m \times \mathbb{G}_m$ with the monomial action of $\mathsf{GL}_2(\mathbf{Z})$: each element M of $\mathsf{GL}_2(\mathbf{Z})$ determines an automorphism f_M of V. The involution $\eta := f_{-\mathrm{id}}$ is a central element, and the quotient $V/\langle\eta\rangle$ is the Cayley cubic. It is an affine cubic with four isolated singularities (the maximal number for a cubic). More precisely, the map $(v_1, v_2) \in V \mapsto -(v_1 + 1/v_1, v_2 + 1/v_2, v_1v_2 + 1/(v_1v_2)) \in \mathbb{A}^3$ identifies $V/\langle\eta\rangle$ to the affine surface of equation $\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2 + \mathbf{xyz} = 4$ (it is a member of the Markov family described below). Since $-\mathrm{id}$ is central in $\mathsf{GL}_2(\mathbf{Z})$, $\mathsf{PGL}_2(\mathbf{Z})$ acts by automorphisms on $V/\langle\eta\rangle$. On V and $V/\langle\eta\rangle$, f_M is loxodromic if and only $\mathsf{Tr}(M)^2 > 4$, and f_M is a Jonquières twist if and only if M has infinite

subgroups parametrized by $\mathbf{Z}_p$. Taking derivatives and using that $\langle A, B\rangle$ is free, one sees that the Lie algebra of H is equal to $\mathfrak{sl}_2$; hence H contains an open subgroup of $\mathsf{SL}_2(\mathbf{Z}_p)$.

order and $\mathsf{Tr}(M)^2 = 4$. If f_M is elliptic, then f_M has finite order. In particular, there are Jonquières twists on singular affine surfaces, a phenomenon that appears on every singular Markov surface (see §8.3).

6. Decomposition into orbit closures

In this section we prove Theorem A, except for the statement concerning stationary measures. So, let X be an affine surface defined over $\mathbf{Z}_p$, and let Γ be a subgroup of $\mathsf{Aut}(X_{\mathbf{Z}_p})$ such that

(a) Γ contains a loxodromic element g;
(b) Γ contains a non loxodromic element f of infinite order.

From the results of Section 5, we can assume that X is not isomorphic to $\mathbb{G}_m \times \mathbb{G}_m$ over $\overline{\mathbf{K}}$. Thus, by Corollary 3.10, if f is parabolic, it is a Jonquières twist acting by a finite order automorphism on the base of its invariant fibration $\pi\colon X \to B$. Changing f into a positive iterate, we assume that it preserves each fiber of π, i.e. $f_B = \mathrm{id}_B$.

We denote by Γ_0 a normal, finite index subgroup of Γ that satisfies the conclusion of Theorem 2.5. Then, Γ_0 is torsion free (see Remark 2.8). As in Section 2.5, we denote by $s_\Gamma(x)$ the dimension of $L_\Gamma(x)$ for $x \in X(\mathbf{Z}_p)$.

Step 1.– *There are only finitely many finite orbits in $X(\mathbf{Z}_p)$.*

This finiteness result holds over any p-adic local field:

Theorem D.– *Let $\mathbf{K}$ be a p-adic local field, with valuation ring $\mathfrak{o}_\mathbf{K}$. Let X be an affine surface defined over $\mathfrak{o}_\mathbf{K}$, and let Γ be a non-elementary subgroup of $\mathsf{Aut}(X_{\mathfrak{o}_\mathbf{K}})$ containing a non-loxodromic element of infinite order. Then, Γ has at most finitely many finite orbits in $X(\mathfrak{o}_\mathbf{K})$.*

Proof. Let $\mathrm{Per}_\Gamma(\mathfrak{o}_\mathbf{K}) \subset X(\mathfrak{o}_\mathbf{K})$ be the set of points with a finite Γ-orbit. Such a point is periodic under the action of f and all its conjugates hfh^{-1}, for $h \in \Gamma$. Thus,

$$\mathrm{Per}_\Gamma(\mathfrak{o}_\mathbf{K}) \subset \bigcap_{h\in\Gamma} P_{hfh^{-1}}(\mathfrak{o}_\mathbf{K}) \tag{6.1}$$

where $P_{hfh^{-1}} = h(P_f)$ and P_f is defined in Lemma 4.3 when f is elliptic or in Theorem 3.11 when f is parabolic. Thus, $\cap_{h\in\Gamma} P_{hfh^{-1}}$ is an intersection of Zariski closed subsets of dimension ≤ 1, and, as such, it is Zariski closed of dimension ≤ 1. Since it is Γ-invariant, it must be finite, because the loxodromic automorphism element g does not preserve any curve (Theorem 3.3). Thus, $\mathrm{Per}_\Gamma(\mathfrak{o}_\mathbf{K})$ is finite and coincides with the set of $\mathfrak{o}_\mathbf{K}$-points in $\cap_{h\in\Gamma}P_{hfh^{-1}}$. $\square$

Step 2.– *For $x \in X(\mathbf{Z}_p)$, we have $s_\Gamma(x) = 2$ if the orbit of x is infinite and $s_\Gamma(x) = 0$ otherwise.*

Indeed, consider a point x of $X(\mathbf{Z}_p)$ and a neighborhood $\mathcal{U}(x) \simeq \mathbf{Z}_p^2$ of x, as well as the subgroup $\Gamma_0 \subset \Gamma$, as in Theorem 2.5. If the orbit of x is finite, it is fixed by the flow $\Phi_h\colon \mathbf{Z}_p \times \mathcal{U}(x) \to \mathcal{U}(x)$ associated with any $h \in \Gamma_0$. Hence, $s_{\Gamma_0}(x) = s_\Gamma(x) = 0$. Conversely, if $s_\Gamma(x) = 0$, then the vector field Θ_h vanishes at x for every $h \in \Gamma_0$, which implies that $h(x) = \Phi_h^1(x) = x$. Therefore, Γ_0 fixes x and the Γ-orbit has cardinality $|\Gamma(x)| \leq [\Gamma : \Gamma_0]$.

Now, consider the set $S = \{y \in X(\mathbf{Z}_p)\ ;\ s_\Gamma(y) \leq 1\}$. Let us prove that S is the set of $\mathbf{Z}_p$-points of a Zariski closed, Γ-invariant set. For this, we consider the distribution L_f: by definition, if f is elliptic, then L_f is the distribution of lines determined by an algebraic vector field a_f (see Section 4.2), and if f is parabolic, L_f is the distribution of lines tangent to the f-invariant fibration $\pi\colon X \to B$ (see Remark 3.13). Thus, given any $h \in \Gamma$, $L_{hfh^{-1}} = h_*L_f$ is a globally defined algebraic distribution of lines in the tangent space TX. This implies that the tangency locus

$$T_\Gamma = \{y \in X\ ;\ \dim \mathrm{Vect}(L_{hfh^{-1}}(x); h \in \Gamma) \leq 1\} \tag{6.2}$$

is a Γ-invariant algebraic subset ([5]). Since Γ contains loxodromic elements, T_Γ coincides with X or is a finite set.

Lemma 6.1. *The algebraic set T_Γ is a finite subset of X.*

Proof. We must show that T_Γ does not coincide with X. Otherwise, the distribution of lines L_f is invariant under the action of Γ. This invariant line field is not tangent to a fibration, because Γ contains loxodromic elements, and such automorphisms do not preserve any fibration. Thus, f is elliptic, and

- L_f is determined by an algebraic vector field a_f on X, as in Section 4.2;
- the algebraic group A in which $f^\mathbf{Z}$ is Zariski dense has dimension 2, since otherwise its orbits would determine an invariant fibration tangent to L_f, but g does not preserve any fibration;
- A° is isomorphic to $\mathbb{G}_a \times \mathbb{G}_m$ or to $\mathbb{G}_m \times \mathbb{G}_m$, has an open orbit, and almost all orbits of f are Zariski dense in X.

[5] Indeed, the tangency locus between two algebraic distributions of lines is an algebraic subset of X, and an intersection of algebraic subsets is algebraic.

We shall give two arguments to reach a contradiction. The first one is based on the classification of birational symmetries of foliations; the second one is a simple variation on Theorem 4.9.

First argument.– Let $\overline{X}$ be a completion of X, smooth at infinity. Since Γ contains loxodromic elements and the pair (X,Γ) is defined over a field of characteristic 0, we can apply [15, Cor. 1.3] to the triple $(\overline{X},\Gamma,L_f)$. This gives four possibilities. In the first case $\overline{X}$ is birationally equivalent to an abelian surface A. Since the group $\mathsf{Bir}(A)$ coincides with $\mathsf{Aut}(A)$, Proposition 3.2 provides a contradiction. The same argument applies when $\overline{X}$ is birationally equivalent to the quotient of such an abelian surface by a finite group action. The third case leads to $X=\mathbb{G}_m\times\mathbb{G}_m$, which is excluded by hypothesis. And the fourth one leads to X being the quotient of $\mathbb{G}_m\times\mathbb{G}_m$ by $(u,v)\mapsto(1/u,1/v)$: this is excluded too, because this surfaces does not have infinite order elliptic elements.

Second argument.– If g is an element of Γ, then g_*a_f is a new vector field and is proportional to a_f. This means that there is a rational function ξ_g on X such that $g_*a_f=\xi_g\cdot a_f$. This rational function may have poles on $\{a_f=0\}$, but according to Proposition 4.7, this set is finite; thus, ξ_g is actually a regular function on X. Moreover, ξ_g cannot vanish because g is an automorphism. Thus, by Theorem 3.4, ξ_g is a constant. This implies that Γ normalizes the algebraic group A° associated with the elliptic element f. The automorphisms of $\mathbb{G}_a\times\mathbb{G}_m$ preserve the fibration onto $\mathbb{G}_m$, hence if $A^\circ\simeq\mathbb{G}_a\times\mathbb{G}_m$, there would be a Γ-invariant fibration, a contradiction. Thus $A^\circ\simeq\mathbb{G}_m\times\mathbb{G}_m$, and as in the proof of Theorem 4.9, we see that X is isomorphic to $\mathbb{G}_m\times\mathbb{G}_m$ over $\overline{\mathbf{K}}$, a contradiction. □

By noetherianity, there is a finite number of conjugates $f_i=h_ifh_i^{-1}$, $1\leq i\leq q$, such that T_Γ coincides with $\{y\in X\ ;\ \dim\mathrm{Vect}(L_{f_i}(y);1\leq i\leq q)\leq 1\}$. Let x be a point with an infinite orbit. As T_Γ is finite, one can find a point x' in the orbit of x that is not on T_Γ. For such a point, $\Theta_{f_i}(x')$ generate $T_{x'}X$, and thus $s_\Gamma(x')=2$. But the set $\{s_\Gamma=2\}$ is Γ-invariant, thus $s_\Gamma(x)=2$. This concludes Step 2.

Step 3.– To conclude, we apply Theorem B and Remark 2.10 from Section 2.6. These results show that in the complement of the finite set $\mathrm{Per}_\Gamma(\mathbf{Z}_p)$, every orbit closure is open. Being open, the orbit closures intersect if and only if they coincide, so they form a partition of $X(\mathfrak{o}_{\mathbf{K}})$, as desired.

7. STATIONARY MEASURES

The goal of this section is to prove Proposition 7.2. This proposition completes the proof of Theorem A.

7.1. **Compact groups.** Let G be a compact topological group, and let ω_G be its Haar measure, normalized so that $\omega_G(G) = 1$. Let S be a closed subgroup of G, and let $q\colon G \to G/S$ be the quotient map. The group G acts by left translations on G/S, and the push-forward measure $\omega_{G/S} = q_*\omega_G$ is a G-invariant probability measure on G/S. Now, let μ be a probability measure on G whose support generates a dense subgroup of G. If ν is a μ-stationary measure on G/S, then it is automatically G-invariant (this follows from the maximum principle for continuous functions on G, see also [21, Thm. 3.5]), and the uniqueness of the Haar measure implies that ν coincides with $\omega_{G/S}$.

Remark 7.1. Let μ be a probability measure on G such that (i) the subgroup generated by the support $\mathrm{Supp}(\mu)$ is dense in G, and (ii) $\mathrm{Supp}(\mu)$ is not contained in a coset of a closed normal subgroup of G. Then, the convolutions $\mu^{\star n}$ converge towards ω_G as n goes to $+\infty$ (see [36, Main Thm. 3.3.5]). Both (i) and (ii) are necessary, but (ii) can be dropped if one is interested only in the classification of stationary measures.

7.2. **Isometry groups.** Let $\mathbf{K}$ be a non-archimedean local field. As above, we endow the affine space $\mathbb{A}^N_{\mathbf{K}}$ with the distance given by the sup-norm:

$$\mathsf{dist}(x,y) = \max_{i=1,\ldots,N} |x_i - y_i|. \tag{7.1}$$

Let $V \subset \mathbb{A}^N$ be a subvariety defined over $\mathfrak{o}_{\mathbf{K}}$. If f is an element of $\mathsf{Aut}(V_{\mathfrak{o}_{\mathbf{K}}})$, then f is the restriction to V of an endomorphim $\tilde{f}$ of $\mathbb{A}^N$ defined over $\mathfrak{o}_{\mathbf{K}}$. As explained in [18, §2.1.2], f induces a 1-Lipschitz homeomorphism of $V(\mathfrak{o}_{\mathbf{K}})$ with respect to $\mathsf{dist}(\cdot,\cdot)$. Since f^{-1} is also 1-Lipschitz, f acts by isometry on $V(\mathfrak{o}_{\mathbf{K}})$. This provides a homomorphism

$$\mathsf{Aut}(V_{\mathfrak{o}_{\mathbf{K}}}) \to \mathsf{Isom}(V(\mathfrak{o}_{\mathbf{K}})). \tag{7.2}$$

Moreover, with the topology induced by uniform convergence, $\mathsf{Isom}(V(\mathfrak{o}_{\mathbf{K}}))$ is a compact group.

Proposition 7.2. *Let $\mathbf{K}$ be a non-archimedean local field. Let V be an affine variety defined over $\mathfrak{o}_{\mathbf{K}}$. Let Γ be a subgroup of $\mathsf{Aut}(V_{\mathfrak{o}_{\mathbf{K}}})$ and let G be the closure of Γ in $\mathsf{Isom}(V(\mathfrak{o}_{\mathbf{K}}))$. Let μ be a probability measure on G whose support generates a dense subgroup of G. Then, every orbit closure $\overline{\Gamma(x)} \subset$*

$V(\mathfrak{o}_{\mathbf{K}})$ supports a unique μ-stationary probability measure. This measure is the unique G-invariant probability measure on $\overline{\Gamma(x)}$.

Proof. The orbit closure $\overline{\Gamma(x)}$ is equal to $G(x)$ and can be identified (as a topological G-space) to G/S_x, where $S_x \subset G$ is the stabilizer of x in G. Let ω_x be the push-forward of ω_G on $G(x)$ by the map $g \mapsto g(x)$. Then, the result follows from Section 7.1. □

As an example, consider a countable group $\Gamma \subset \mathsf{Aut}(V_{\mathfrak{o}_{\mathbf{K}}})$ and a probability measure $\mu = \sum_{f\in\Gamma}\mu(f)\delta_f$ such that $\{f \in \Gamma\,;\, \mu(f) \neq 0\}$ generates Γ([6]). In the example of Markov surfaces, μ will be symmetric, with finite support.

8. THE MARKOV SURFACES

In this section, we study the family of surfaces $S_{A,B,C,D} \subset \mathbb{A}^3$ defined by the equation

$$\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2 + \mathbf{xyz} = A\mathbf{x} + B\mathbf{y} + C\mathbf{z} + D. \tag{8.1}$$

Here, we suppose that the parameters A, B, C, and D are in the valuation ring $\mathfrak{o}_{\mathbf{K}}$ of a p-adic local field $\mathbf{K}$. For simplicity, we denote $S_{A,B,C,D}$ by S. The group Γ that we shall study is the one generated by the Vieta involutions $s_1(\mathbf{x},\mathbf{y},\mathbf{z}) = (-\mathbf{x}+A-\mathbf{yz},\mathbf{y},\mathbf{z})$, $s_2(\mathbf{x},\mathbf{y},\mathbf{z}) = (\mathbf{x},-\mathbf{y}+B-\mathbf{zx},\mathbf{z})$, and $s_3(\mathbf{x},\mathbf{y},\mathbf{z}) = (\mathbf{x},\mathbf{y},-\mathbf{z}+C-\mathbf{xy})$. It is a finite index subgroup of $\mathsf{Aut}(S)$, defined over the ring $\mathbf{Z}[A,B,C,D]$, and hence on $\mathfrak{o}_{\mathbf{K}}$.

The surface S, if smooth, supports a regular 2-form Ω, defined locally by

$$\Omega = \frac{d\mathbf{x}\wedge d\mathbf{y}}{2\mathbf{z}-C+\mathbf{xy}} = \frac{d\mathbf{y}\wedge d\mathbf{z}}{2\mathbf{x}-A+\mathbf{yz}} = \frac{d\mathbf{z}\wedge d\mathbf{x}}{2\mathbf{y}-B+\mathbf{zx}}. \tag{8.2}$$

By p-adic integration, this gives a measure on $S(\mathfrak{o}_{\mathbf{K}})$ (see Igusa's book [25, Chap. 7.4], or [34]). If U is a non-empty open subset of $S(\mathfrak{o}_{\mathbf{K}})$ (with respect to the p-adic topology), we shall refer to the measure

$$\frac{1}{\int_U \Omega}\Omega_{|U} \tag{8.3}$$

as the **normalized symplectic measure** on U.

[6]The reason for choosing μ as a probability measure on G (rather than on Γ) in Proposition 7.2 is because otherwise we should introduce a σ-algebra on $\mathsf{Aut}(V_{\mathfrak{o}_{\mathbf{K}}})$ for which the map $f \in \mathsf{Aut}(V_{\mathfrak{o}_{\mathbf{K}}}) \mapsto f_{|V(\mathfrak{o}_{\mathbf{K}})} \in \mathsf{Isom}(V(\mathfrak{o}_{\mathbf{K}}))$ is measurable.

8.1. **Parabolic automorphisms and bounded orbits.** The composition $f = s_1 \circ s_2$ is an automorphism that preserves the fibration defined by $\pi_{\mathbf{z}} \colon (\mathbf{x}, \mathbf{y}, \mathbf{z}) \mapsto \mathbf{z}$. If we cut S by a horizontal plane $\{\mathbf{z} = z\}$, for some $z \in \mathbf{K}$, we get a conic C_z, with the equation

$$\mathbf{x}^2 + \mathbf{y}^2 + z\mathbf{x}\mathbf{y} = A\mathbf{x} + B\mathbf{y} + (Cz + D - z^2). \tag{8.4}$$

The action of f on this conic section is given by the restriction of an affine map $A_z(\mathbf{x}, \mathbf{y}) = M_z(\mathbf{x}, \mathbf{y}) + T_z$ where

$$M_z = \begin{pmatrix} z^2 - 1 & z \\ -z & -1 \end{pmatrix} \quad \text{and} \quad T_z = \begin{pmatrix} A - Bz \\ B \end{pmatrix}. \tag{8.5}$$

The determinant of M_z is 1, its trace is $z^2 - 2$, so its eigenvalues are α and $1/\alpha$ with $\alpha + 1/\alpha = z^2 - 2$. Extracting a root, we get $2\alpha(z) = (z^2 - 2) \pm z\sqrt{z^2 - 4}$. Thus, over a quadratic extension of $\mathbf{K}$, A_z is conjugate to the linear map $(\mathbf{x}, \mathbf{y}) \mapsto (\alpha(z)\mathbf{x}, \mathbf{y}/\alpha(z))$. This implies that $\deg(f^n)$ grows like $2n$ as n goes to $+\infty$, so f is a Jonquières twist. Also, if $|z| > 1$, we see that one of the two eigenvalues, say $\alpha(z)$, satisfies $|\alpha(z)| > 1$.

The next lemma follows from a direct computation.

Lemma 8.1. *If $z^2 \neq 4$, the affine map A_z has a fixed point in $\mathbf{K}^2$; this fixed point is on C_z (equivalently on S) if and only if $z = 0$, in which case $A_z \circ A_z = \mathrm{id}$.*

If $z = 2\varepsilon$, with $\varepsilon = \pm 1$, then A_z has either no fixed point in $\mathbf{K}^2$, or a line of fixed point defined by the equation $\varepsilon x = y + B/2$; this last case occurs if and only if $\varepsilon A + B = 0$.

Thus, if (x, y, z) is a point of $S(\mathbf{K})$ with $|z| > 1$, the eigenvalues of M_z satisfy $|\alpha(z)| > 1 > |1/\alpha(z)|$, and the unique fixed point of A_f in $\mathbf{K}$ is outside C_z. This implies that the orbit of (x, y, z) under the action of $f^{\mathbf{Z}}$ is not bounded.

Proposition 8.2. *A point $(x, y, z) \in S(\mathbf{K})$ has a bounded Γ-orbit if and only if $(x, y, z) \in S(\mathfrak{o}_{\mathbf{K}})$. Thus, $S(\mathfrak{o}_{\mathbf{K}})$ is the unique maximal, compact, invariant subset of $S(\mathbf{K})$.*

Proof. The compact set $S(\mathfrak{o}_{\mathbf{K}})$ is Γ-invariant. If (x, y, z) is not in $S(\mathfrak{o}_{\mathbf{K}})$, then one of the coordinates has an absolute value > 1. If $|z| > 1$, then we know that $f^{\mathbf{Z}}(x, y, z)$ is not bounded. Otherwise, we may apply $s_3 \circ s_1$ or $s_2 \circ s_3$ to conclude in the same way. □

8.2. **General parameters.** The finite orbits of Γ in $S_{A,B,C,D}(\overline{\mathbf{K}})$ have been classified in [28] (see also [14, §3.1]). For a general parameter (A,B,C,D), every Γ-orbit in $S = S_{A,B,C,D}$ is infinite.

Theorem E.– *Assume that A, B, C, and D are in $\mathbf{Z}_p$, and that $S(\mathbf{Z}_p)$ does not contain any finite orbit. Let μ be a probability measure on $\{s_1,s_2,s_3\}$ with $\mu(s_1)\mu(s_2)\mu(s_3) > 0$. Let ν be an ergodic μ-stationary probability measure on $S(\mathbf{Q}_p)$. Then,*

(1) ν *is supported on an orbit closure $\overline{\Gamma(x)}$, for some $x \in S(\mathbf{Z}_p)$;*
(2) *this orbit closure is a clopen subset;*
(3) ν *is equal to the normalized symplectic measure on $\overline{\Gamma(x)}$.*

Moreover, there are only finitely many orbit closures in $S(\mathbf{Z}_p)$, and the convex set of μ-stationary measures on $S(\mathbf{Q}_p)$ is a finite dimensional simplex.

Proof. According to [14, Pro. 6.1, Rem. 6.2], ν is supported on a compact invariant subset. Thus, it is supported in $S(\mathfrak{o}_\mathbf{K})$, and by Theorem A, ergodicity, and Proposition 7.2, it is the unique invariant measure on an orbit closure $\overline{\Gamma(x)}$. But $\Omega_{\overline{\Gamma(x)}}$ is such an invariant measure, thus $\nu = \Omega_{\overline{\Gamma(x)}}$. □

8.3. **Examples.** Assume that $S(\mathfrak{o}_\mathbf{K})$ contains a finite orbit F, and that F is not an isolated subset in $S(\mathfrak{o}_\mathbf{K})$. Since the action of Γ is isometric, there are infinitely orbit closures accumulating to F, and this implies that $S(\mathfrak{o}_\mathbf{K})$ supports infinitely many ergodic stationary measures.

For instance, the origin $o = (0,0,0)$ is a singularity of $S_o = S_{0,0,0,0}$, it is not isolated in $S_o(\mathbf{Z}_p)$, and it is fixed by Γ. On the other hand, if we focus on the subset $S' = \{(x,y,z) \in S_o \; ; \; \mathsf{dist}(o,(x,y,z)) = 1\}$, then the work of Bourgain, Gamburd and Sarnak and of Chen shows that, for p large, the group generated by Γ and the permutations of coordinates act topologically transitively on S': every orbit in S' is dense in S' (see [10, 19]). Thus, S' supports a unique stationary measure.

9. AN OPEN QUESTION

9.1. **Simple examples.** In the following examples, f and g are automorphisms of $\mathbb{A}^2$, both defined over $\mathbf{Z}$, and $\Gamma = \langle f,g \rangle$. In the first set of examples, Γ has only finitely many orbits in $\mathbb{A}^2(\mathbf{Z})$, but the third example has infinitely many orbits, and the orbits of Γ in $\mathbb{A}^2(\mathbf{Z})$ for the example in § 9.1.4 are "sparsed" in terms of height. In each case, we are interested in the number of orbits modulo p as p increases.

9.1.1. This first example can be handled directly, without any reference to Theorem A. Let t be a positive integer. Let $Q(\mathbf{x})$ be a polynomial function of degree $d \geq 2$ with coefficients in $\mathbf{Z}$. Then, we take $g(\mathbf{x},\mathbf{y}) = (\mathbf{y}+Q(\mathbf{x}),\mathbf{x})$ and $f(\mathbf{x},\mathbf{y}) = (\mathbf{x}+t,\mathbf{y})$. Suppose that p does not divide t. Let $z=(x,y)$ be an element of $\mathbb{A}^2(\mathbf{Z}_p)$ and let U be the closure of the Γ-orbit of z. Since $t\mathbf{Z}$ is dense in $\mathbf{Z}_p$, U is the preimage of a subset U_0 of $\mathbb{A}^1(\mathbf{Z}_p)$ under the second projection $\pi(\mathbf{x},\mathbf{y}) = \mathbf{y}$. On the other hand, $g \circ f \circ g^{-1}(\mathbf{x},\mathbf{y}) = (x+Q(\mathbf{y}+t)-Q(\mathbf{y}),\mathbf{y}+t)$ is in Γ. Thus $\pi(U)$ is invariant under the translation $\mathbf{y} \mapsto \mathbf{y}+t$. This implies that $U = \mathbb{A}^2(\mathbf{Z}_p)$, so every Γ-orbit is dense in $\mathbb{A}^2(\mathbf{Z}_p)$. On the other hand, if t and Q are divisible by p, then the action of Γ on $\mathbb{A}^2(\mathbf{Z}/p\mathbf{Z})$ has at least $p(p+1)/2$ orbits, and no orbit is dense in $\mathbb{A}^2(\mathbf{Z}_p)$. This argument also shows that when $t=\pm 1$, then Γ acts transtively on $\mathbb{A}^2(\mathbf{Z})$.

9.1.2. Consider a translation $f(\mathbf{x},\mathbf{y}) = (\mathbf{x}+t,\mathbf{y})$, as in the previous example, and any loxodromic automorphism $g \in \mathsf{Aut}(\mathbb{A}^2_{\mathbf{Z}})$. If p does not divide t, every orbit of Γ modulo p is the preimage of a subset of $\mathbb{A}^1(\mathbf{Z}/p\mathbf{Z})$ under the second projection. The automorphism $h = g \circ f \circ g^{-1}$ acts by translations along the fibers of the function $g_2 := g^*(\mathbf{y})$ and every orbit of h contains p points. Let d be the degree of g_2 with respect to the $\mathbf{y}$ variable. Then, one can easily verify that Γ has at most d orbits modulo p.

9.1.3. Another interesting case is when $f \in \mathsf{GL}_2(\mathbf{Z})$ is a linear diagonalizable automorphism and $g \in \mathsf{Aut}(\mathbb{A}^2_{\mathbf{Z}})$ is any loxodromic automorphism. For instance, one can take $f(\mathbf{x},\mathbf{y}) = (2\mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y})$. Numerical simulations suggest again that the number of orbits of Γ modulo p is uniformly bounded, independently of p. It would be interesting to adapt the techniques of [10] to this example.

9.1.4. Let h_0 and g be the elements of $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{Z}})$ defined by

$$h_0(\mathbf{x},\mathbf{y}) = (2\mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y}),$$
$$g(\mathbf{x},\mathbf{y}) = (\mathbf{y}+\mathbf{x}^2+5,-\mathbf{x}).$$

Let $p < 10^4$ be a prime. Then, our numerical simulations show that if $p \neq 5$, the group $\langle g, h_0 \circ g \circ h_0^{-1}\rangle$ acts transitively on $\mathbb{A}^2(\mathbf{Z}/p\mathbf{Z})$; if $p=5$, the origin is fixed and the group acts transitively on $\mathbb{A}^2(\mathbf{Z}/p\mathbf{Z}) \setminus \{(0,0\}$. On the other hand, each orbit of g alone contains at most $c(g)p\log(p)$ points, for p in this range, with $c(g) = 2$ (for other Hénon maps, $c(g)$ can be larger). We have no explanation yet for these phenomena, and we do not know whether the range $2 \leq p \leq 10^4$ is big enough to derive precise conjectures.

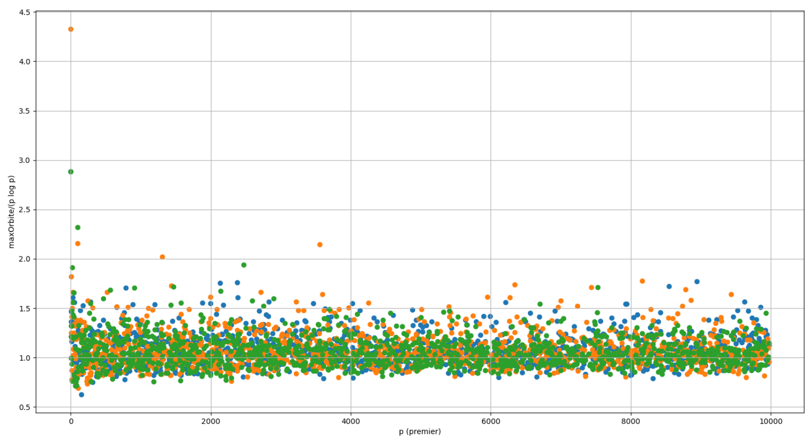

FIGURE 1. This picture represents, for a given p, the maximal length of an orbit of g modulo p divided by $p\log(p)$. Blue points correspond to $g_1(\mathbf{x},\mathbf{y}) = (\mathbf{y}+\mathbf{x}^2+5, -\mathbf{x})$, orange points to $g_2(\mathbf{x},\mathbf{y}) = (-\mathbf{y}, \mathbf{x}+\mathbf{y}^3+2)$, and green points to $g_3 = g_2 \circ h_0 \circ g_1$ where $h_0(\mathbf{x},\mathbf{y}) = (2\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y})$ is as above.

9.2. **Uniform bounds.** Though our numerical simulations concern only small primes (say $p < 10^4$), the examples from the previous section suggest the following question.

Question 9.1. *Let Γ be a non-elementary subgroup of $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{Z}})$. Is the number of orbit closures of Γ in $\mathbb{A}^2(\mathbf{Z}_p)$ bounded by a uniform constant $b(\Gamma)$ that does not depend on p?*

This question is already interesting when Γ contains an element f of infinite order that is not loxodromic (as in Theorem A). This problem is also related to the following one: *under which conditions can we ensure that transitivity modulo p (resp. some p^k) implies topological transitivity over $\mathbf{Z}_p$ (i.e. modulo p^l for all $l \geq 1$)?*

Appendix A. A proof of the Bell-Poonen Theorem

In this appendix, we present an alternative proof of the Bell-Poonen theorem (see Theorem 2.1 above); we refer to [7, 8, 33] for the orginial proofs. The method naturally provides an explicit formula for the vector field Θ_f defined in Equation (2.7): see Equation (A.16) below.

A.1. **Functional Analytic Ingredients.** We use the notations $R = \mathfrak{o}_{\mathbf{K}}$, $\mathcal{U} = R^m$, and $R\langle\mathbf{x}\rangle$ from Section 2.2. On $\mathcal{U}$, we use the ℓ^∞-norm $|(x_1,\ldots,x_m)| = \max_{1\leq i\leq m}|x_i|$, and on the ring $R\langle\mathbf{x}\rangle$, we use the Gauss norm $\|\cdot\|$ defined by

$$\|g\| = \sup_I |a_I| \tag{A.1}$$

for every $g = \sum_I a_I\mathbf{x}^I \in R\langle\mathbf{x}\rangle$. If we view $g \in R\langle\mathbf{x}\rangle$ as a Tate analytic function $g\colon \mathcal{U} \to R$, the norm is an *upper bound* of the function values: i.e., we have $|g(z)| \leq \|g\|$ for all $z \in \mathcal{U}$ [9, Prop. 5.1.4/2]. The function values may not attain the Gauss norm: see Remark 2.2 above. Note that our definition of $f \equiv \mathrm{id} \pmod{p^c}$ in §2.3 asserts $\|f - \mathrm{id}\| \leq p^{-c}$.

By this norm, $R\langle\mathbf{x}\rangle$ is a complete (ultra)metric space [9, Prop. 1.4.1/3]. Although R is not a field, we extend the notion of (faithfully) normed R-modules from [9, Def. 2.1.1/1] and say that $R\langle\mathbf{x}\rangle$ is an R-Banach space with respect to the Gauss norm $\|\cdot\|$. We define the norm on $R\langle\mathbf{x}\rangle^m$ by the supremum of the Gauss norms on each component.

A.2. **Construction of the Flow.** Let A denote the R-algebra of bounded linear endomorphisms $R\langle\mathbf{x}\rangle^m \to R\langle\mathbf{x}\rangle^m$, with norm given by the operator norm $\|\cdot\|_{op}$. This algebra is *normal* in the sense that within the extension of scalars $A \otimes_R \mathbf{K}$, the ball of radius 1 centered at the origin is precisely A.

Let $f\colon \mathcal{U} \to \mathcal{U}$ be a Tate analytic map such that $f \equiv \mathrm{id} \pmod{p^c}$ for some real number $c > 1/(p-1)$. By precomposition, we obtain an operator

$$T_f\colon R\langle\mathbf{x}\rangle^m \to R\langle\mathbf{x}\rangle^m, \quad g \mapsto g\circ f. \tag{A.2}$$

By [18, Lem. 2.1(3)], denoting by Id the identity operator on $R\langle\mathbf{x}\rangle^m$, we have

$$\|T_f - \mathrm{Id}\|_{op} \leq \|f - \mathrm{id}\| \leq p^{-c}. \tag{A.3}$$

In what follows, we denote by exp and log the usual (formal) power series

$$\exp(\mathbf{a}) = \sum_{k=0}^{\infty} \frac{\mathbf{a}^k}{k!}, \tag{A.4}$$

$$\log(1+\mathbf{a}) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\mathbf{a}^k. \tag{A.5}$$

The proof given in Chapter IV of [26], and in particular the Proposition on page 81 of this book, generalizes to normal R-Banach algebras (with unit denoted by 1, or by Id in the case of endomorphisms of $R\langle \mathbf{x}\rangle^m$). This establishes the following properties.

Lemma A.1. *Let A be a normal R-Banach algebra.*

(1) *The series* $\exp(\mathbf{a})$ *converges on the ball* $\{a \in A\ ;\ \|a\| < p^{-1/(p-1)}\}$.
(2) *The series* $\log(1+\mathbf{a})$ *converges on the open unit ball* $\{a \in A\ ;\ \|a\| < 1\}$.
(3) *The exponential function* $\exp(\mathbf{a})$ *defines a* 1*-to-*1 *analytic map from the ball of radius $p^{-1/(p-1)}$ centered at the origin to the ball of radius $p^{-1/(p-1)}$ centered at the identity element* Id*. Its inverse is given by* $\log(\mathbf{a})$.
(4) *In these balls,* $\exp(a+b) = \exp(a)\exp(b)$ *and* $\log(ab) = \log(a)+\log(b)$ *if a and b commute.*

These basic properties imply that

$$(1+a)^n = \exp(n\log(1+a)) \tag{A.6}$$

for any $n \in \mathbf{N}$ and any $a \in A$ with $\|a\| < p^{-1/(p-1)}$. In particular, if $\|a\| < p^{-1/(p-1)}$, the map $n \mapsto (1+a)^n$ extends to a Tate analytic map

$$t \in R \mapsto (1+a)^t := \exp(t\log(1+a)). \tag{A.7}$$

Since $\|T_f - \mathrm{Id}\|_{op} \leq p^{-c}$, we can apply this discussion to $a = (T_f - \mathrm{Id}) \in A$. Doing so, we see that the map $n \mapsto T_f^n$ extends to a Tate analytic map $t \in R \to T_f^t \in A$. Thus, defining

$$\Phi^t = T_f^t(\mathrm{id}) \tag{A.8}$$

we obtain a Tate analytic map $\Phi\colon R \to R\langle \mathbf{x}\rangle^m$ that satisfies

$$\Phi^n = f^n \tag{A.9}$$

for every $n \in \mathbf{N}$. Furthermore, to have $\Phi^t(\mathbf{x}) \equiv \mathbf{x} \pmod{p^c}$, we first need to estimate exp and log:

Lemma A.2. *Let $a \in A$ be an element of a normal R-Banach algebra A. If $\|a\| < p^{-1/(p-1)}$, then $\|\exp(a)-1\| = \|a\|$ and $\|\log(1+a)\| = \|a\|$. Furthermore, for $t \in R$, we have $\|(1+a)^t - 1\| = |t|\cdot\|a\|$.*

Proof. For the estimate on exp, see [24, §5.7, Problem 182], which generalizes to normal R-Banach algebras. The estimate implies that the self-bijection $\mathbf{a} \mapsto \exp(\mathbf{a}) - 1$ on the ball of radius $p^{-1/(p-1)}$ centered at the origin preserves the norm. Its inverse $\mathbf{a} \mapsto \log(1+\mathbf{a})$ must have the same property. The last claim follows from $(1+a)^t = \exp(t\log(1+a))$. □

Based on this, we have

$$\begin{aligned} \|\Phi^t - \mathrm{id}\| &= \|(T_f^t)(\mathrm{id}) - \mathrm{Id}(\mathrm{id})\| \\ &\leq \|T_f^t - \mathrm{Id}\|_{op}\|\mathrm{id}\| \\ &= |t| \cdot \|T_f - \mathrm{Id}\|_{op} \\ &\leq p^{-c}. \end{aligned}$$

This proves the theorem of Bell and Poonen.

A.3. **Formula of the Vector Field.** The proof suggested above gives rise to a formula of the vector field. To see how, first note that, on the operator level,

$$\frac{1}{h}(T_f^h - \mathrm{Id}) \longrightarrow \log(\mathrm{Id} + (T_f - \mathrm{Id}))$$

as $h \to 0$. This follows from the definition of $T_f^h = \exp(h \cdot \log(\mathrm{Id} + (T_f - \mathrm{Id})))$.

Hence if we differentiate $\mathbf{t} \mapsto \Phi^{\mathbf{t}}(\mathbf{x})$ at $\mathbf{t} = 0$, we get

$$\begin{aligned} \Theta_f(\mathbf{x}) &= \lim_{h\to 0} \frac{1}{h}(\Phi^h - \Phi^0)(\mathbf{x}) && \text{(A.10)} \\ &= \lim_{h\to 0} \left(\frac{1}{h}(T_f^h - \mathrm{Id})\mathrm{id}\right)(\mathbf{x}) && \text{(A.11)} \\ &= (\log(\mathrm{Id} + (T_f - \mathrm{Id}))\mathrm{id})(\mathbf{x}). && \text{(A.12)} \end{aligned}$$

This formula can be further understood as

$$\begin{aligned} \Theta_f(\mathbf{x}) &= (\log(\mathrm{Id} + (T_f - \mathrm{Id}))\mathrm{id})(\mathbf{x}) && \text{(A.13)} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}((T_f - \mathrm{Id})^k \mathrm{id})(\mathbf{x}) && \text{(A.14)} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (T_f^j \mathrm{id})(\mathbf{x}) && \text{(A.15)} \end{aligned}$$

which gives

$$\Theta_f(\mathbf{x}) = \sum_{k=1}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{j-1}}{k} \binom{k}{j} f^j(\mathbf{x}). \qquad \text{(A.16)}$$

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