DENSE ORBITS

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There is an interesting problem, the goal of which is to classify complex projective varieties (resp. compact Kähler manifolds) with an automorphism $f\colon X \to X$ acting minimally. This means that every orbit of f is dense in X; the problem can be ask for the Zariski topology or the euclidean topology (this is the topology that should be used for the compact Kähler case).

Example.– Consider a torus $A = \mathbf{C}^g/\Lambda_A$, where Λ_A is a (cocompact) lattice in $\mathbf{C}^g$, and then take a totally irrational translation $f(z) = z + t$. Then all orbits of f are dense.

Example.– Consider an elliptic curve $E = \mathbf{C}/\Lambda_E$ and the surface $S = E \times E$. On S, consider the automorphism $f(x,y) = (x+t, x+y)$, for some t such that all orbits of $x \mapsto x+t$ are dense in E for the euclidean topology. Then, a result of Furstenberg shows that all orbits of f are dense in S for the euclidean topology.

In fact, all examples known so far are automorphisms of (compact) tori $\mathbf{C}^g/\Lambda$, and a natural question is

Question.– Let $\mathbf{k}$ be an algebraically closed field. Let X be a projective variety over $\mathbf{k}$. If there is an automorphism f of X such that every orbit of f in $X(\mathbf{k})$ is Zariski dense, does it follow that X is an abelian variety?

This question has been asked by many since around 2000 and is still open (in 2024). An intersting dual problem is to look for affine varieties with an automorphism f, all of whose orbits are Zariski dense. Here is one such example.

Theorem A.–
(1) *An automorphism of $\mathbb{A}^2_{\mathbf{C}}$ always has an orbit which is not Zariski dense.*
(2) *All orbits of the automorphism of $\mathbb{A}^3_{\mathbf{C}}$ defined by*

$$f(x,y,z) = (y + x^2 + z, x, z+1)$$

are Zariski dense in $\mathbb{A}^3_{\mathbf{C}}$.

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Proof.

(1).– If h is conjugate to an affine transformation $h(x,y) = L(x,y) + (s,t)$, then h preserves an affine line. If h is conjugate to an elementary automorphism

$$h(x,y) = (ax + p(y), dy + e)$$

with $p \in \mathbf{C}[y]$, a, d, e in $\mathbf{C}$ and $ad \neq 0$, then. Then if $d \neq 1$ the line $y = -e/(d-1)$ is invariant. And if $d = 1$ one can conjugate by $(x,y) \mapsto (x + y^m, y+1)$ to get

$$h'(x,y) = (ax + ay^m + p(y+1) - (y+1+e)^m, y+1)$$

to decrease the degree of p and reduce the study to affine maps. Then, if h is conjugate to a composition of Hénon maps, a result of Bedford, Lyubich and Smillie shows that h has infinitely many periodic points. By the work of Friedland and Milnor, this exhausts all possibilities.

(2).– Since f acts by translation $z \mapsto z+1$ on the last variable, all orbits of f are infinite. We have to exclude the existence of an invariant curve and the existence of an invariant surface.

(2.a).– Suppose C is an invariant curve. Let D be an irreducible component of C and $k \geq 1$ be an integer such that $f^k(D) = D$. The projection $\pi_D \colon D \subset \mathbb{A}^3 \to \mathbb{A}^1$ on the z-axis must be onto because if z_0 is in $\pi(D)$ then so is $z_0 + k$. Moreover, π_D has no ramification point, because if z_0 is a critical value of π_D, so is $z_0 + k$. Thus, D is a rational, affine curve with an automorphism of infinite order and, as such, is isomorphic to $\mathbf{C}$ or $\mathbf{C}^*$.

Let us consider the first case. Then D that can be parametrized by $(a(z), b(z), z)$ for some polynomial functions a and b of z. Assume that the degree d of a with respect to z and the degree e of b satisfy $e < 2d$. Then,

$$f(a,b,z) = (b + a^2 + z, a, z+1)$$

is a curve parametrized by $a_1 = b(z) + a^2(z) + z$, of degree $d_1 = 2d$, $b_1(z) = a(z)$, of degree $e_1 = d$, and $c_1(z) = z+1$. Thus, we still have $2d_1 > e_1$. Repeating this argument k times, we obtain a contradiction because the degree of the parametrization of $f^k(D)$ is too large, in contradiction with f_D being an homography. If $e \geq 2d$, then $d < 2e$ and we can apply the same argument with f^{-1} in place of f.

(2.b).– Suppose S is an f-invariant surface, with equation $P(x,y,z) = 0$. Then, isolate the term of highest degree in P with respect to the lexicographical order on multidegrees (a,c,b) of monomials $x^a y^b z^c$ (note that we first put a, then c, and then b in (a,c,b)). Write $P(y+x^2+z,x,z+1) = \alpha P(x,y,z)$ for some $\alpha \neq 0$, and derive a contradiction from that. □

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