ANALYTIC ACTIONS OF MAPPING CLASS GROUPS ON SURFACES.

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ABSTRACT. Let Σ be a closed orientable surface of genus $g \geq 3$. Let Γ be any finite index subgroup of the mapping class group of Σ. We prove that there is no faithful analytic action of Γ on compact surfaces with non zero Euler characteristic.

1. INTRODUCTION

Let Σ_g be a closed oriented surface of genus g. If k is a non negative integer or ∞, $\mathsf{Diff}^k(\Sigma_g)$ will denote the group of diffeomorphisms of Σ_g of class $\mathcal{C}^k$. By definition, when $k=0$, this group coïncides with the group $\mathsf{Homeo}(\Sigma_g)$ of homeomorphisms of the surface Σ_g. The same notation, but with $k=\omega$, will denote the group of real analytic diffeomorphisms of Σ_g. Note that we made an implicit choice of a $\mathcal{C}^k$-structure on the surface. If we change the $\mathcal{C}^k$-structure, then the subgroup $\mathsf{Diff}^k(\Sigma_g)$ of $\mathsf{Homeo}(\Sigma_g)$ changes by a conjugacy.

For all k in $\{0,1,...,\infty\}$, $\mathsf{Diff}^k_0(\Sigma_g)$ will stand for the group of $\mathcal{C}^k$-diffeomor--phisms which are isotopic to the identity. This coincides with the connected component of the identity in $\mathsf{Diff}^k(\Sigma_g)$. The *modular group*, or *mapping class group*, of Σ_g will be denoted $\mathsf{MCG}(\Sigma_g)$. Whatever the choice of k in $\{0,1,...,\infty\}$, $\mathsf{MCG}(\Sigma_g)$ is isomorphic to the quotient of $\mathsf{Diff}^k(\Sigma_g)$ by its normal subgroup $\mathsf{Diff}^k_0(\Sigma_g)$. This definition provides an exact sequence

$$\{Id_{\Sigma_g}\} \to \mathsf{Diff}^k_0(\Sigma_g) \to \mathsf{Diff}^k(\Sigma_g) \to \mathsf{MCG}(\Sigma_g) \to \{1\}$$

where the second and third morphisms

$$\iota : \mathsf{Diff}^k_0(\Sigma_g) \to \mathsf{Diff}^k(\Sigma_g), \quad \pi : \mathsf{Diff}^k(\Sigma_g) \to \mathsf{MCG}(\Sigma_g)$$

are respectively the inclusion and the canonical projection.

A natural question, which appears as one of Thurston's question in Kirby's list of problems [11], is whether this exact sequence splits, i.e. whether there is a morphism $s : \mathsf{MCG}(\Sigma_g) \to \mathsf{Diff}^k(\Sigma_g)$, called a section of π, such that $\pi \circ s$ is the identity. When the genus g is 1, such a section indeed exists: Σ_1 is the torus $\mathbf{R}^2/\mathbf{Z}^2$ and the group $\mathsf{GL}\,(2,\mathbf{Z})$ acts faithfully on the torus by analytic

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diffeomorphisms; the restriction of π to $\mathsf{GL}(2,\mathbf{Z})$ is a bijection onto $\mathsf{MCG}(\Sigma_1)$ and the inverse mapping provides the desired section (for all k).

For higher genus, there is no section: This was first proved by Morita for $k \geq 2$ in [15, 16], and then by Markovic for $k = 0$ in [13].

Markovic-Morita Theorem. *If Σ_g is a closed orientable surface of genus $g > 5$, the canonical projection*

$$\pi : \mathsf{Homeo}(\Sigma_g) \to \mathsf{MCG}(\Sigma_g)$$

does not have any section.

It is conjectured that the same result holds as soon as the genus g is at least 2 ; a recent preprint by Franks and Handel extends this result to $g \geq 3$ if we consider sections into the group $\mathsf{Diff}^1(\Sigma_g)$ [7]. The first goal of this paper is to provide a simple proof of a slightly more precise result for sections into the group of analytic diffeomorphisms of Σ_g.

Theorem A. *Let Σ_g be a closed orientable surface of genus $g \geq 2$. Let Γ be a finite index subgroup of $\mathsf{MCG}(\Sigma_g)$. Then, there is no homomorphism $s : \Gamma \to \mathsf{Diff}^\omega(\Sigma_g)$ such that $\pi \circ s$ is the identity of Γ.*

Morita's proof of the previous theorem implies the same statement for sections into $\mathsf{Diff}^2(\Sigma_g)$ as soon as $g \geq 5$. Markovic's arguments use finite order elements in $\mathsf{MCG}(\Sigma_g)$; since there is a torsion free, finite index subgroup in $\mathsf{MCG}(\Sigma_g)$, it is not possible to adapt easily his ideas to get a proof of theorem A. Nevertheless, it seems reasonable to expect that the same result holds for sections into $\mathsf{Homeo}(\Sigma_g)$ with $g \geq 2$.

The second result that we shall prove is much stronger.

Theorem B. *Let Σ_g be a closed orientable surface of genus $g \geq 3$ and Γ be any finite index subgroup of $\mathsf{MCG}(\Sigma_g)$. Then, there is no faithful analytic action of Γ on a closed surface of non zero Euler characteristic.*

The hypothesis $g \geq 3$ and the hypothesis on the Euler characteristic are technical; the same result should hold for $g = 2$, and for analytic actions on the torus. Unfortunately, our proof does not work in this wider context. Once again, theorem B should also hold for actions by homeomorphisms.

In order to prove theorem B, we first study commuting groups of germs of analytic diffeomorphisms near a fixed point. The main result, which is summarized in theorem 3.1 below, is already an interesting and independant statement that may be useful for other purposes.

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2. SECTIONS INTO THE GROUP OF ANALYTIC DIFFEOMORPHISMS

In this section, we prove theorem A.

2.1. Centralizer of an analytic diffeomorphism.

Let f be an analytic diffeomorphism of a closed surface S. The centralizer of f in $\mathsf{Diff}^{\omega}(S)$ is the subgroup of all analytic diffeomorphisms g which commute with f. The following result is proved in [4].

Theorem 2.1. *Let S be a closed surface. If f is an analytic diffeomorphism of S with positive topological entropy, then f generates a finite index subgroup in its centralizer.*

Let us sketch the proof of a weaker statement which is sufficient for our purpose. Let A be the centralizer of f. We shall prove that *A contains a finite index abelian subgroup* (see [17] for a similar argument).

Sketch of proof. Since the topological entropy of f is positive, f has a periodic saddle point q, the stable and unstable manifolds of which intersect each other (see [9]). As a consequence, $W^u(q)$ and $W^s(q)$ are Zariski-dense (if an analytic function vanishes along $W^u(q)$, the function vanishes identically).

The group A permutes the isolated fixed points of f^k, where k is the period of q, so that a finite index subgroup A' in A fixes q and stabilizes $W^s(q)$ and $W^u(q)$. If g is an element of A', the restriction of g to $W^s(q)$ determines g, because $W^s(q)$ is Zariski-dense, and commutes to f.

If λ denotes the derivative of f^k at q along $W^s(q)$, then $|\lambda| < 1$, and there exists an analytic parametrization $\xi : R \to S$ of $W^s(q)$ such that $f^k \circ \xi(t) = \xi(\lambda t)$. If g is an element of A', its restriction to $W^s(q)$ commutes to $t \mapsto \lambda t$ and is therefore linear in the variable t. This implies that the restriction of A' to $W^s(q)$, and therefore A' itself, are abelian groups. □

2.2. Action on the fundamental group and entropy.

We now describe a result due to Bowen and Katok which provides a criterium in order to prove that a homeomorphism has positive entropy.

Let M be a compact manifold, and x be a base point on M. Let f be a homeomorphism of M which fixes the base point x. Then f induces an automorphism f_* of the fundamental group $\pi_1(M,x)$. Since M is compact, we can choose a finite generating set $\{a_1,...,a_k\}$ for $\pi_1(M,x)$. From this we get a length function on

$\pi_1(M,x)$: the distance $L(b)$ from a loop b to the trivial loop is the smallest integer l such that b is a product of at most l generators a_i. The asymptotic stretching factor $\lambda(f_*)$ is then defined by

$$\lambda(f_*) = \limsup_{n\to+\infty} \left(\max\{L(f_*^n(a_i)) \,|\, i = 1,...,k\}\right)^{1/n}.$$

Theorem 2.2 (Bowen, Katok, see [3] or [10])**.** *The topological entropy of a homeomorphism f of a compact manifold M is not less than the logarithm of the asymptotic stretching factor of $f_* : \pi_1(M) \to \pi_1(M)$.*

Together with theorem 2.1, we get the following result.

Corollary 2.3. *If f is an analytic diffeomorphism of a closed surface with $\lambda(f_*) > 1$, the centralizer of f is virtually cyclic.*

A weaker result which depends only on what is fully proved in the previous section asserts that *the centralizer of f is almost abelian when $\lambda(f_*) > 1$.*

2.3. **Proof of theorem A.** Let Γ be a finite index subgroup of $\mathsf{MCG}(\Sigma_g)$, with $g \geq 2$. Assume that $s : \Gamma \to \mathsf{Diff}^k(\Sigma_g)$ is a section of the projection π.

Let a_1, b_1, a_2, b_2, ... be the loops on Σ_g which are described in figure 1. Let t_c denote the Dehn twist along the curve c, with $c \in \{a_1, b_1, a_2, b_2 ..., \}$. Since s is a section of π, we get

- $s(t_{a_1})$ and $s(t_{b_1})$ commute with $s(t_{a_2})$ and $s(t_{b_2})$;
- $s(t_{a_1})$ and $s(t_{b_2})$ generate a non abelian free group ;
- $(s(t_{a_1} \circ t_{b_1}))_* = t_{a_1} \circ t_{b_1}$ has a positive asymptotic stretching factor.

Together with the previous corollary (even in its weak formulation), this shows that the image of s can not be contained in $\mathsf{Diff}^{\omega}(\Sigma_g)$. Theorem A is proved.

3. Commuting germs of analytic diffeomorphisms

The main goal of this section is to prove the following result, which concerns the group $\widehat{\mathsf{Diff}}_{Id}(\mathbf{C}^2, 0)$ of formal diffeomorphisms at the origin in $\mathbf{C}^2$ which are tangent to the identity.

Theorem 3.1. *Let F and G be two subgroups of $\widehat{\mathsf{Diff}}_{Id}(\mathbf{C}^2, 0)$ such that*

- *neither F nor G is a solvable group ;*
- *F and G commute: $f \circ g = g \circ f$ for all f in F and g in G.*

Then, there exist Ψ_F and Ψ_G, two quotients of formal power series in two variables, such that

- *$d\Psi_F \wedge d\Psi_G$ does not vanish identically;*
- *Ψ_F is F-invariant, i.e. $\Psi_F \circ f = \Psi_F$ for all f in F, and Ψ_G is G-invariant;*

- *there is an injective morphism* ε_F *(resp.* ε_G*) from* F *to* $\widehat{\mathsf{Diff}}_{Id}(\mathbf{C},0)$ *or* $\mathsf{PSL}_2(\mathbf{C})$ *such that*

$$\Psi_G \circ f = \varepsilon_F(f) \circ \Psi_G, \quad \forall f \in F$$

(resp. $\Psi_F \circ g = \varepsilon_G(g) \circ \Psi_F, \quad \forall g \in G$*).*

3.1. **Formal vector fields and the exponential mapping.** By $\widehat{O}(\mathbf{C}^n)$ we denote the ring of formal power series in n complex variables. The field of fractions of formal power series is denoted by $\widehat{\mathcal{M}}(\mathbf{C}^n)$.

Let $\widehat{\chi}_0(\mathbf{C}^n,0)$ be the Lie algebra of formal vector fields at the origin of $\mathbf{C}^n$ with vanishing first jet. If X is an element of $\widehat{\chi}_0(\mathbf{C}^n,0)$, the flow $\phi(X,t)$ of X is a formal power series; this series is polynomial with respect to the time variable t:

$$\phi(X,t) = \sum_I a_I(t) x^I$$

where I descibes the set of multi-indices, x^I are the corresponding monomials and each $a_I : \mathbf{C} \to \mathbf{C}^n$ is a polynomial application in the variable t. In particular, $\phi(X,t)$ is a well defined germ of formal diffeomorphism fixing the origin. By definition, the exponential mapping is the map

$$\exp : \widehat{\chi}_0(\mathbf{C}^n,0) \to \widehat{\mathsf{Diff}}_{Id}(\mathbf{C}^n,0), \quad \exp(X) = \phi(X,1).$$

It follows from the fact that groups of k-jets of elements of $\widehat{\mathsf{Diff}}_{Id}(\mathbf{C}^n,0)$ are nilpotent groups that exp is a bijection. In other words, $\widehat{\chi}_0(\mathbf{C}^n,0)$ plays the role of the Lie algebra for the group $\widehat{\mathsf{Diff}}_{Id}(\mathbf{C}^n,0)$, and the exponential mapping coincides with the formal flow at time 1.

If f is an element of $\widehat{\mathsf{Diff}}_{Id}(\mathbf{C}^n,0)$, L_f will denote the unique element of $\widehat{\chi}_0(\mathbf{C}^n,0)$ such that

$$\exp(L_f) = f.$$

Two formal germs of diffeomorphisms f and g commute if and only if the corresponding vector fields L_f and L_g commute (i.e. if their Lie bracket vanishes identically).

3.2. **Linear part and Jordan decomposition.** Let us briefly describe the Jordan decomposition in $\widehat{\mathsf{Diff}}(\mathbf{C}^n,0)$ (see [1], §23). This will not be used until section 3.7.

If Δ is a diagonalizable matrix with eigenvalues α_i, $i = 1,\ldots n$, a resonnance for Δ is a relation of type

$$\alpha_i = \prod \alpha_j^{m_j},$$

where m_j are positive integers, and $\sum m_j \geq 2$. Let x_i, $i = 1, \ldots n$, be coordinates in which Δ is diagonal. A resonnant monomial is a monomial $M = \prod x_j^{m_j}$, $\sum m_j \geq 2$, which satisfies an equation of type

$$M \circ \Delta = \alpha_i M$$

for at least one eigenvalue α_i.

If f is a formal germ of diffeomorphism, we can write f uniquely as the composition $s \circ u$ of two formal germs of diffeomorphisms such that

- s and u commute: $s \circ u = u \circ s$;
- s is diagonalizable: there is a formal change of coordinates ϕ such that $\Delta = \phi \circ s \circ \phi^{-1}$ is a linear diagonal mapping;
- the linear part of s and u coincide respectively with the diagonalizable part and the unipotent part in the Jordan decomposition of $D_0 f$;
- since u commutes with s, the higher order monomial terms of $\phi \circ u \circ \phi^{-1}$ are resonnant with respect to Δ.

The diagonalizable term s is called the semi-simple part of f and u is called the unipotent part of f. The unipotent part u is the flow at time 1 of a unique formal vector field which vanishes at the origin and has a nilpotent first jet.

By uniqueness of the decomposition, if g commutes to f, then g commutes at the same time to the semi-simple part and to the unipotent part of f. Similarly, if Φ is an f-invariant meromorphic function, then Φ is both s and u-invariant (see [2], chapter I.4).

3.3. **First integral of foliations.** If X is a non zero element of $\widehat{\chi}_0(\mathbf{C}^n, 0)$, then X defines a formal germ of dimension 1 foliation $\mathcal{F}_X$ at the origin. Two elements X and Y of $\widehat{\chi}_0(\mathbf{C}^n, 0)$ define the same foliation if and only if X is *parallel* to Y, which means that there exists a formal meromorphic function r such that $X = rY$. If f is an element of $\widehat{\mathsf{Diff}}_{Id}(\mathbf{C}^n, 0)$, we shall denote by $\mathcal{F}_f$ the foliation which is determined by L_f.

A (formal) first integral of X, or of $\mathcal{F}_X$, is a formal power series $\Psi \in \widehat{\mathcal{O}}(\mathbf{C}^n)$ such that its Lie derivative $X \cdot \Psi$ vanishes identically. A meromorphic first integral is an element of $\widehat{\mathcal{M}}(\mathbf{C}^n)$ which satisfies the same property. The set of first integrals forms a ring, and the set of meromorphic first integrals forms a field. The existence of a non constant meromorphic first integral is not granted : there are examples of holomorphic germs of foliations without any non constant formal meromorphic first integral.

Recall from section 3.1 that the flow $\phi(X, t)$ is polynomial with respect to the time variable t. As a consequence,

- Ψ is a formal (meromorphic) first integral of X if and only if Ψ is invariant by the flow of $\phi(X,t)$;
- if f is an element of $\widehat{\mathsf{Diff}}_{Id}(\mathbf{C}^n,0)\setminus\{Id\}$, then Ψ is a first integral of $\mathcal{F}_f$ if and only if Ψ is f-invariant.

Let us now assume that n is 2. The following results are proved in [14] and [5].

Theorem 3.2. *Let X be an element of $\widehat{\chi}_0(\mathbf{C}^2,0)$. If X admits a non constant formal first integral, then*

- *there exists a first integral Ψ which is not a (non trivial) power of another first integral ;*
- *for any choice of such a Ψ, the ring of formal first integral coincides with the ring $\mathbf{C}\{\{\Psi\}\}$ of formal power series in Ψ ;*
- *Ψ is unique up to post composition with a germ of formal diffeomorphism near the origin in $\mathbf{C}$.*

A purely meromorphic first integral Ψ is a quotient of two formal power series that does not coincide with a formal power series or the inverse of a formal power series. The following statement is a consequence of Luroth's theorem ([5], section 5.V, page 137).

Theorem 3.3. *Let X be an element of $\widehat{\chi}_0(\mathbf{C}^2,0)$ without non constant formal first integral. If X admits a purely meromorphic first integral, then*

- *there exists a purely meromorphic first integral Ψ such that the field of meromorphic first integrals coincides with the field $\mathbf{C}(\Psi)$ of rational functions in Ψ;*
- *this first integral Ψ is unique up to composition by a homographic transformation $M\in\mathsf{PGL}_2(\mathbf{C})$.*

If X has a non constant first integral, any generator Ψ of the ring of first integral will be called a *minimal first integral.* If X does not possess any non constant first integral but admits a purely meromorphic first integral then any generator Ψ of the field of meromorphic first integrals will be called a *minimal first integral.*

If G is a group of formal diffeomorphisms which preserves the foliation $\mathcal{F}$, then G acts on the set of first integrals of $\mathcal{F}$; if g is an element of G and Ψ a first integral, then $\Psi\circ g^{-1}$ is of the form $\varepsilon(g)\Psi$, where $\varepsilon(g)$ is an element of $\widehat{\mathsf{Diff}}(\mathbf{C}^1,0)$ or $\mathsf{PGL}_2(\mathbf{C})$ according to the type of the first integral Ψ (power series or purely meromorphic).

3.4. **Proof of theorem 3.1, step 1.** In order to prove theorem 3.1, we now assume that F and G are two commuting non solvable subgroups of $\widehat{\mathsf{Diff}}_{Id}(\mathbf{C}^2,0)$.

Let g_0 be an element of $G\setminus\{Id\}$. In this first step, we assume that all vector fields L_g, g in G, are parallel to L_{g_0}, by which we mean that there exists r_g in $\widehat{\mathcal{M}}(\mathbf{C}^2)$ such that

$$L_g = r_g L_{g_0}.$$

Since F and G commute, $f_* L_g = L_g$ for any pair (f,g) in $F\times G$. It follows that

$$r_g L_{g_0} = L_g = f_* L_g = (r_g\circ f^{-1}) f_* L_{g_0} = (r_g\circ f^{-1}) L_{g_0},$$

and that r_g is f-invariant for all pairs (f,g) in $F\times G$.

Since G is not abelian, there is at least one element g_1 in G for which r_{g_1} is not constant. This implies that r_{g_1} is a non constant first integral of L_f for all f in F, and therefore that all L_f are parallel. As a consequence, the vector fields L_f, f in F, define a unique formal foliation $\mathcal{F}_F$.

In other words, *the vector fields* L_g, $g\in G$, *are all parallel one to another if and only if the vector fields* L_f, $f\in F$, *are.*

The foliation $\mathcal{F}_F$ admits r_{g_1} as a non constant first integral. Let Ψ_F be a minimal first integral of the foliation $\mathcal{F}_F$ (see §3.3). Since G commutes to F, G preserves the foliation, and each $g\in G$ sends Ψ_F to another minimal first integral. According to section 3.3, two distinct cases may arise:

(i) Ψ_F is purely meromorphic. In this case, there exists a morphism $\varepsilon_G : G\to \mathsf{PSL}(2,\mathbf{C})$ such that

$$\Psi_F\circ g = \varepsilon_G(g)\circ \Psi_F, \quad \forall g\in G;$$

(ii) Ψ_F is a formal power series. In this case, there exists a morphism $\varepsilon_G : G\to \widehat{\mathsf{Diff}}_{Id}(\mathbf{C},0)$ such that

$$\Psi_F\circ g = \varepsilon_G(g)\circ \Psi_F, \quad \forall g\in G.$$

Of course, a similar result holds if we permute F and G. This provides a minimal first integral Ψ_G for the foliation $\mathcal{F}_G$ and a morphism ε_F such that

$$\Psi_G\circ f = \varepsilon_F(f)\circ \Psi_G, \quad \forall f\in F.$$

3.5. **Proof of theorem 3.1, step 2.** Still assuming that all vector fields L_g are parallel, we now prove that both ε_G and ε_F are injective mophisms.

Let us assume that there is an element g in $G\setminus\{Id\}$ which is contained in the kernel of ε_G ; by definition of ε_G, Ψ_F is g-invariant, and therefore

$$\Psi_F\circ g^n = \Psi_F, \quad \forall n\in\mathbf{Z}.$$

This implies that Ψ_F is invariant under the flow of L_g, and that $\mathcal{F}_F$ coincides with $\mathcal{F}_G$ (see section 3.3). In particular, Ψ_F is both F and G-invariant, and the foliations $\mathcal{F}_F$ and $\mathcal{F}_G$ coincide. To simplify the notation, we may now denote by $\mathcal{F}$ this foliation and Ψ the chosen first integral.

Let $\tilde{\mathbf{C}}^2$ be the surface obtained by blowing-up the origin of $\mathbf{C}^2$, let E be the exceptional divisor, and $\mathbf{c} : \tilde{\mathbf{C}}^2 \to (\mathbf{C}^2, 0)$ be the blowing down of E. The (formal) foliation $\mathbf{c}^*\mathcal{F}$ has a finite number of singularities along E. If $\mathcal{F}$ is not dicrital, then E is a leaf of $\mathbf{c}^*\mathcal{F}$, and if $\mathcal{F}$ is dicritical, then $\mathbf{c}^*\mathcal{F}$ is transverse to E in the complement of a finite set.

Let p be a generic point of E. There are formal coordinates (x,t) at p such that p corresponds to the origin $(0,0)$ and

$$\Psi \circ \mathbf{c} = x.$$

If $\mathcal{F}$ is not dicritical, then x vanishes along E.

Since F and G are tangent to the identity, we can lift F and G to groups of formal germs of diffeomorphisms $\tilde{F}$ and $\tilde{G}$ in $\widehat{\mathsf{Diff}}(\tilde{\mathbf{C}}^2, p)$. Since Ψ is both F and G-invariant, elements of $\tilde{F}$ and $\tilde{G}$ may be written

$$\tilde{f}(x,t) = (x, f_2(x,t)), \quad \tilde{g}(x,t) = (x, g_2(x,t))$$

in local coordinates (x,t). In other words, $\tilde{f}$ and $\tilde{g}$ correspond to formal vector fields of type

$$\tilde{L}_f = A_f(x,t)\frac{\partial}{\partial t}, \quad \tilde{L}_g = B_g(x,t)\frac{\partial}{\partial t}$$

where A_f and B_f are formal power series. Since F commutes to G, we get

$$\frac{\partial}{\partial t}\left(\frac{A_f}{B_g}\right) = 0$$

for any pair of elements (f,g) in $F \times G$. As a consequence, if g_1 is a fixed element of $G \setminus \{Id\}$, we can write the series A_f in the form

$$A_f(x,t) = a_f(x) B_{g_1}(x,t),$$

where a_f is a formal meromorphic function in one variable. This implies that the group $\tilde{F}$, and therefore F itself, is abelian. This contradiction shows that ε_G is indeed injective.

3.6. **Proof of theorem 3.1, step 3.** In order to conclude the proof of theorem 3.1, we may now assume that the formal vector fields L_g, g in G, are not all parallel; we may therefore fix two elements h and k of G such that L_h and L_k are not parallel. If g is an element of G, there is a unique pair (H_g, K_g) in $\widehat{\mathcal{M}}(\mathbf{C}^2, 0)$ such that

$$L_g = H_g L_h + K_g L_k.$$

Since F commutes to G, we obtain

$$H_g \circ f = H_g, \quad K_g \circ f = K_g$$

for all (f,g) in $F \times G$. If one of the formal power series H_g or K_g is not constant, then, as in step 1, all formal vector fields L_f are parallel, and therefore all formal vector fields L_g are also parallel, a contradiction. This implies that all formal power series H_g and K_g, g in G, ar in fact constant.

From this we deduce that the Lie algebra generated by the formal vector fields L_g has dimension 2, and, as such, is solvable. This implies that G itself is a solvable group; this contradiction completes the proof of theorem 3.1.

3.7. **A corollary.**

Corollary 3.4. *Let G and H be two groups. If neither G nor H is solvable, there is no injective morphism of the group $\mathbf{Z} \times G \times H$ into the group of formal germs of diffeomorphisms $\widehat{\mathsf{Diff}}(\mathbf{C}^2, 0)$.*

We shall use the following lemmas.

Lemma 3.5. *Let f be an element of $\widehat{\mathsf{Diff}}(\mathbf{C}^1, 0)$. If f is not periodic, there exists $p \in \mathbf{Z}$ such that the centralizer of f is isomorphic to $\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$. In particular, if f commutes to a subgroup of $\widehat{\mathsf{Diff}}(\mathbf{C}^1, 0)$ which is not solvable, then f is periodic.*

Proof. If $f'(0)$ is not a root of unity, f is formally linearizable. Up to a change of formal coordinate

$$f(z) = f'(0)z,$$

and in this new coordinate, the centralizer of f coïncides with the group of homotheties. This group is abelian.

If $f'(0)$ is a root of unity, we decompose f into the composition of its semi-simple part s and its unipotent part u. Up to conjugacy, $s(z) = f'(0)z$. If f is not periodic, the centralizer of f is contained in the centralizer of its unipotent part, and this group is of type $\mathbf{Z} \times \mathbf{Z}/pZ$ (see [6], or [12], chapter 1). □

Lemma 3.6. *Let f be an element of $\widehat{\mathsf{Diff}}(\mathbf{C}^2, 0)$. Let Φ and Ψ be two formal power series at the origin of $\mathbf{C}^2$ such that $d\Phi \wedge d\Psi$ is not identically 0. If f preserves both Φ and Ψ, then f is periodic.*

Proof. If f is tangent to the identity, there exists a unique formal vector field $X \in \widehat{\chi}_0(\mathbf{C}^2, 0)$ for which $f = \exp(X)$. Since f preserves Φ and Ψ, both Lie derivatives $X \cdot \Phi$ and $X \cdot \Psi$ vanish identically (see section 3.3). Since $d\Phi \wedge d\Psi \neq 0$, X vanishes identically and f is the identity.

If f is unipotent, i.e. its semi-simple is trivial, the same proof applies (with X a formal vector field, the linear part of which is nilpotent).

If $f = s \circ u$ is the decomposition of f into its semi-simple and unipotent parts, then Φ and Ψ are both s and u-invariant (see section 3.2). From $d\Phi \wedge d\Psi \neq 0$, we deduce that s is periodic. If k is the period of s, then f^k is unipotent and preserves Φ and Ψ, so that f^k is the identity. □

Proof of the corollary. Let us assume that there is a faithful representation of $\mathbf{Z} \times G \times H$ into $\widehat{\mathsf{Diff}}(\mathbf{C}^2, 0)$. We identify G and H to their images, and we fix a generator f for the image of $\mathbf{Z}$.

• Let us show that there are two non solvable subgroups $G_1 \subset G$ and $H_1 \subset H$ with trivial linear part at the origin.

If k is an element of $\widehat{\mathsf{Diff}}(\mathbf{C}^2, 0)$, we shall denote its linear part, or 1-jet, at the origine by $J^1(k)$.

First we assume the existence of an element g in G, the linear part of which has two distinct eigenvalues α and β. We then choose formal coordinates at p so that the semi-simple part of g is a diagonal linear transformation (see section 3.2). Let h be an element of H. Since h commutes to g, its linear part $J^1(h)$ is also diagonal. The morphism $h \mapsto J^1(h)$ has an abelian image, and its kernel is therefore a non nonsolvable subgroup H_1 of H. Higher order terms of elements of H_1 are resonnant with respect to the eigenvalues α and β. Since elements of G commute with elements of H_1, their semi-simple parts commute with resonnant monomials, and are therefore diagonal. This implies that a non solvable subgroup G_1 of G has trivial linear part.

Now assume that there is an element h of H, the linear part of which has two distinct eigenvalues. Permuting G and H, the same argument shows that there are two subgroups G_1 and H_1 in G and H which are tangent to the identity and are not solvable.

If all elements of G and of H have a unique eigenvalue, the groups of linear parts $J^1(G)$ and $J^1(H)$ are solvable, and the same conclusion holds.

• We can now apply theorem 3.1 to $G_1 \times H_1$. Since f commutes to G_1 there exists a formal diffeomorphism η_f (resp. an element η_f of $\mathsf{PSL}\,(2, \mathbf{C})$, depending on the type of the first integral Ψ_{G_1}) such that

$$\Psi_{G_1} \circ f = \eta_f \circ \Psi_{G_1}.$$

The formal diffeomorphism (resp. Möbius transformation) η_f commutes to the non solvable subgroup $\varepsilon_{H_1}(H_1)$. This implies that η_f has finite order, i.e. that Ψ_{G_1} is f^k-invariant for some $k > 0$ (lemma 3.5). The same argument shows that

Ψ_{H_1} is also f^l-invariant for some $l > 0$. From this follows that f^{kl} preserves both Ψ_{G_1} and Ψ_{H_1}, and that f has finite order, because $d\Psi_{G_1} \wedge d\Psi_{H_1}$ is not identically 0 (lemma 3.6). This contradicts the starting assumption, namely that $\mathbf{Z} \times G \times H$ embeds into $\widehat{\mathsf{Diff}}(\mathbf{C}^2, 0)$. □

4. ANALYTIC ACTIONS OF MAPPING CLASS GROUPS

We now prove theorem B. This requires a few technical results on groups of diffeomorphisms of the circle.

4.1. Preliminaries on diffeomorphisms of the circle.

Let $\mathsf{Homeo}_+(\mathbb{S}^1)$ be the group of orientation preserving homeomorphisms of the circle.

Proposition 4.1. *Let G be a finitely generated subgroup of $\mathsf{Homeo}_+(\mathbb{S}^1)$. If all elements of G are periodic, then G is finite. In particular G is finite as soon as all G-orbits are finite.*

Proof. Since all elements of G are periodic, G does not contain any free non abelian group. Margulis-Tits' alternative for $\mathsf{Homeo}_+(\mathbb{S}^1)$ shows that G preserves a probability measure μ (see [8]). In particular, the rotation number $\rho : G \to \mathbf{R}/\mathbf{Z}$ is a morphism, with values in $\mathbf{Q}/\mathbf{Z}$; its image is finite because G is finitely generated. Elements of the kernel of ρ are periodic, and have a fixed point. Since any periodic, orientation preserving, homeomorphism of the circle with a fixed point is the identity, G is finite. □

Proposition 4.2. *Let G be an infinite, finitely generated subgroup of $\mathsf{Diff}^\omega(\mathbb{S}^1)$. Let H be a finitely generated subgroup of $\mathsf{Diff}^\omega(\mathbb{S}^1)$. If (i) all elements of G have a rational rotation number and (ii) G and H commute, then*

- *either H is finite,*
- *or $G \times H$ has a finite orbit.*

Proof. From proposition 4.1, we know that there exists an element g in G which is not periodic; since its rotation number is rational, we can change g in g^k, $k > 0$, and assume that the set of fixed points $Fix(g)$ is non empty. This set is finite because g is an analytic diffeomorphism.

The group H commutes to g, and therefore permutes its fixed points. As a consequence, there is a finite index subgroup H_1 in H which fixes $Fix(g)$ pointwise. If H_1 is finite, so is H. Otherwise, the set of fixed points of H_1 is finite. But this set is G-invariant, because G and H_1 commute. This provides a finite $(G \times H_1)$-orbit, and therefore a finite $G \times H$ orbit. □

4.2. **Notation and strategy of the proof.** In what follows, Σ_g is the orientable closed surface of genus g, with $g \geq 3$, and Γ is a finite index subgroup of $\mathsf{MCG}(\Sigma_g)$. We shall consider the following set of non separating closed curves on Σ_g and

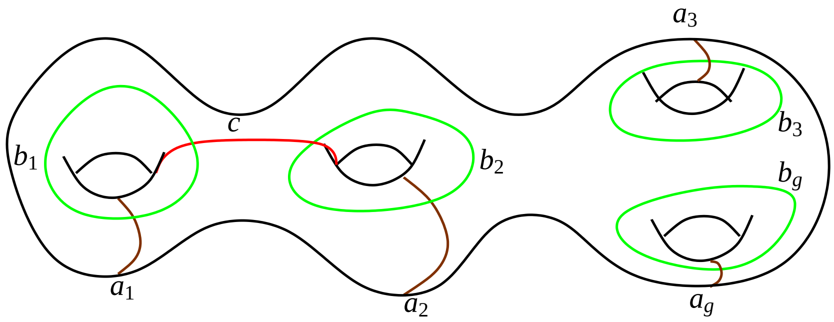

FIGURE 1. Curves on Σ_g.

denote by F_0, G_0, and H_0 the subgroups of $\mathsf{MCG}(\Sigma_g)$ generated by the following Dehn twits.

- F_0 is generated by the twist along a_1, and the twist along b_1.
- G_0 is generated by the twist along a_2, and the twist along b_2.
- H_0 is generated by the twists along a_k, and the twist along b_k, $3 \leq k \leq g$.

Since Γ has finite index in $\mathsf{MCG}(\Sigma_g)$, Γ intersects F_0 (resp. G_0, H_0) on a finite index subgroup F (resp. G, H) of F_0 (resp. G_0, H_0). The groups F and G are non abelian free groups, while H is a commutative product of $g-2$ non abelian free groups. The groups F, G and H commute to each other, so that the product $F \times G \times H$ embeds into Γ.

Let us now assume that Γ acts faithfully by analytic diffeomorphisms on a compact surface S, so that we can identify Γ to a subgroup of $\mathsf{Diff}^{\omega}(S)$. In the following sections, we shall study the action of $F \times G \times H$ on S in order to find common fixed points and then get a contradiction from corollary 3.4.

4.3. **Isolated fixed points.** Let f be an element of $F \setminus \{Id\}$. Since the Euler characteristic of S is different from 0, the set of fixed points $Fix(f)$ is a *non empty* real analytic subset of S. Isolated fixed points of f correspond to the 0-dimensional part of $Fix(f)$ (a finite set), while non isolated fixed points form a finite union of analytic curves. Since the Euler characteristic of the surface S is not zero, the set $Fix(f)$ is non empty.

Let us first assume that there exists an element f of $F \setminus \{Id\}$ with at least one isolated fixed point p. Let N be the number of isolated fixed points of f. Since G (resp. H) commutes to f, G (resp. H) permutes the set of isolated fixed points of f. From this we deduce that the subgroup G_1 (resp. H_1) of G (resp. H) fixing p has index at most $N!$ in G (resp. H). Both G_1 and H_1 are free groups, and the group $\langle f\rangle \times G_1 \times H_1$ fixes p. Corollary 3.4 provides a contradiction.

4.4. **No isolated fixed point.** Let us now assume that there is no element in F with an isolated fixed point. Let us fix an element f in $F \setminus \{Id\}$, and an irreducible component C of its set of fixed points. If C is not smooth, the set of its singular points is finite, and a finite index subgroup G_1 in G (resp. H_1 in H) fixes all these singular points. We then conclude as in the previous section.

If C is smooth, then C is diffeomorphic to a circle $\mathbb{S}^1$. Let G_1 (resp. H_1) be the finite index subgroup of G (resp. H) which stabilizes the component C of $Fix(f)$. We obtain a morphism $\kappa : G_1 \times H_1 \to \mathsf{Diff}^{\omega}(\mathbb{S}^1)$.

In what follows, we construct a common fixed point $p \in C$ for large subgroups of F, G_1 and H_1 and conclude with corollary 3.4. The basic idea is summarized in the following remark.

Remark 4.3. If $\kappa(G_1 \times H_1)$ has a finite orbit, there exist finite index subgroups $G_2 \leq G_1$ and $H_2 \leq H_1$ and a point $p \in C$ which is fixed by the group $\langle f\rangle \times G_2 \times H_2$. We then get a contradication from corollary 3.4.

4.4.1. *If $\kappa(G_1)$ is finite.* Let us first assume that $\kappa(G_1)$ is a finite group: the kernel of $\kappa : G_1 \to \mathsf{Diff}^{\omega}(\mathbb{S}^1)$ is a finite index subgroup G_2 of G_1 which fixes C pointwise. Since C is an irreducible component of the set of fixed points of G_2, a finite index subgroup F_1 of F stabilizes the curve C and the restriction morphism κ is also defined on F_1 (note that f is contained in F_1).

If $\kappa(H_1)$ is finite, remark 4.3 provides the desired contradiction.

If $\kappa(H_1)$ is infinite but all elements of $\kappa(H_1)$ have a rational rotation number, proposition 4.2 shows that either $\kappa(F_1) \times \kappa(H_1)$ has a finite orbit, or $\kappa(F_1)$ is finite. In the first case, there are finite index subgroups $F_2 \leq F_1$ and $H_2 \leq H_1$ such that $F_2 \times G_2 \times H_2$ has a fixed point p in C. In the second case, there is a finite index subgroup F_2 in F_1 which fixes C pointwise and at least one element h in H_1 with a fixed point p on C (recall all $\kappa(h)$ have a rational rotation number) ; let H_2 be the cyclic group generated by h. In both cases, we get a common fixed point p for $F_2 \times G_2 \times H_2$, and corollary 3.4 provides a contradiction.

The remaining case is when H_1 contains an element h such that the rotation number of $\kappa(h)$ is irrational. In that case, $\kappa(h)$ is conjugate to an irrational rotation by a homeomorphism (see [10], chapters 11 and 12). Since $\kappa(F_1)$ commutes to $\kappa(h)$, the group $\kappa(F_1)$ is abelian ; this implies that the kernel of $\kappa : F_1 \to \mathsf{Diff}^{\omega}(\mathbb{S}^1)$ is a non solvable subgroup F_2 (maybe of infinite index in F_1).

Let now c be the curve on Σ_g shown on figure 1 Let t_c be the Dehn twist along c. The twist t_c commutes to t_{a_2}. Let k be a positive integer such that $t_{a_2}^k$ is in G_2. The curve C is contained in the set of fixed points of $t_{a_2}^k$, and is therefore stabilized by an iterate t_c^l of t_c, $l > 0$. Since t_c commutes to H,

- either t_c^l has a periodic orbit of period $k > 0$ along C. We then get a contradiction since h commutes to t_c and $\kappa(h)$ does not have any periodic orbit.
- or t_c^l has an irrational rotation number along C. In that case $\kappa(H_1)$ is abelian, since it commutes to t_c^l ; as a consequence, a non solvable subgroup H_2 of H_1 fixes C pointwise, and we get a contradiction from corollary 3.4 if we apply it to $F_2 \times G_2 \times H_2$.

4.4.2. *Conclusion.* Since G_1 and H_1 play a symmetric role, we can now assume that both $\kappa(G_1)$ and $\kappa(H_1)$ are infinite. From remark 4.3, we can also assume that all $\kappa(G_1 \times H_1)$-orbits are infinite. Together with proposition 4.2, we may therefore assume that both $\kappa(G_1)$ and $\kappa(H_1)$ contain elements with irrational rotation number. Since G_1 and H_1 commute, this implies that $\kappa(G_1)$ and $\kappa(H_1)$ are abelian subgroups of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$. It follows that the kernel of κ intersects G_1 (resp. H_1) on a non solvable subgroup G_2 (resp. H_2), and we get a contradiction if we apply corollary 3.4 to the group $\langle f \rangle \times G_2 \times H_2$.

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