LINEAR SOLVABLE GROUPS AND METABELIAN GROUPS

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ABSTRACT. In these notes, I prove a theorem of Margulis and Soifer: the proof follows arguments from Philip Hall that date back to the 50's, as explained to me by Adrien Le Boudec and Vincent Guirardel. Then, I go on to prove a second result of Hall, namely that finitely generated metabelian groups are residually finite. I also describe examples of solvable groups with uncountably many distinct maximal subgroups, due to Yves de Cornulier. The main references are an article of Margulis and Soifer [4], the book by Lennox and Robinson [3], papers by Hall [1, 2].([1])

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Part 1.– Maximal Subgroups

1. Maximal subgroups of linear solvable groups

Theorem A.– *Let* $\mathbf{k}$ *be a field and* m *be a positive integer. Let* G *be a finitely generated and solvable subgroup of* $\mathsf{GL}_m(\mathbf{k})$. *If* P *is a maximal subgroup of* G*, then* G/P *is finite.*

This result is the second part of a theorem of Margulis and Soifer which says that a finitely generated subgroup of $\mathsf{GL}_m(\mathbf{k})$ satisfies exactly one of the following assertions:

- either G contains a non-abelian free group and then G contains uncountably many maximal subgroups, in particular it contains maximal subgroups of infinite index;
- or G is virtually solvable, and then all its maximal subgroups have finite index, so that in particular G contains at most countably many maximal subgroups.

Our goal is to give a simple proof of Theorem A. The proof given by Margulis and Soifer relies on a deep result of Roseblade on polycyclic groups; here, we reduce the proof to a simpler theorem concerning metabelian groups.

2. Maximal subgroups of metabelian groups

Theorem B.– *Let* G *be a finitely generated metabelian group. If* $P \subset G$ *is a maximal subgroup, then* P *has finite index in* G.

Proof. Let $A = [G, G]$ be the derived subgroup and $Q = G/A$ be the abelianization of G; both A and Q are abelian, and Q is finitely generated. We have an exact sequence

$$1 \to A \to G \to Q \to 1$$

and we can write

$$Q = \mathbf{Z}^r \oplus \bigoplus_{i=1}^{s} \mathbf{Z}/a_i\mathbf{Z}$$

for some prime powers $a_i \geq 1$. Since the maximal subgroups of Q have finite index in Q, we can now assume that the projection from P to Q is onto. Thus, every $g \in G$ can be written as a product $g = pa$ for some $a \in A$ and $p \in P$. This implies that $A \cap P$ is a normal subgroup of G, because if $x \in A \cap P$ and $g = pa \in G$, then $gxg^{-1} = pxp^{-1}$ (A being abelian) is both in A (A being normal) and in P (x and p being in P). After taking the quotient by $A \cap P$, we can assume that $A \cap P = \{1_G\}$. This does not change Q and this implies that the projection from P to Q is an isomorphism. Thus, we can now identify P to Q and write G as a semi-direct product $G = A \rtimes P$.

Consider A as a $\mathbf{Z}[P]$-module, P acting on A by conjugation. This module is simple, because if B were a non-trivial proper sub-module of A, we could adjoin B to P to get a proper subgroup $B \rtimes P$ of G, contradicting the maximality of P. The key point is given by the following lemma.

Lemma 2.1. *A simple* $\mathbf{Z}[Q]$*-module is automatically finite.*

Indeed, if we take this lemma for granted, then A is finite, and P has finite index in $G = A \rtimes P$. □

Proof of Lemma 2.1. First, note that $\mathbf{Z}[Q]$ is the commutative, finitely generated ring

$$\mathbf{Z}[Q] = \mathbf{Z}[t_i, t_i^{-1}; 1 \leq i \leq r+s]/I$$

where I is the ideal generated by the polynomial functions $t_i^{a_i} - 1$ for $i = r+1$ to $r+s$. Let A be a simple $\mathbf{Z}[Q]$-module and pick a non-trivial element y in A. Then $A = \mathbf{Z}[Q] \cdot y = \mathbf{Z}[Q]/J$ where J is the ideal of $\mathbf{Z}[Q]$ defined by

$$J = \{f \in \mathbf{Z}[Q]\ ;\ \ f \cdot y = 0\}.$$

Since A is simple, J is a maximal ideal in $\mathbf{Z}[Q]$, and since $\mathbf{Z}[Q]$ is finitely generated, A is a field which is finitely generated as a ring. This implies that A is finite (see Section 8 below). □

Corollary 2.2. *If G is an extension of an abelian group by a virtually abelian group, and $P \subset G$ is a maximal subgroup, then G/P is finite.*

Indeed, we have the following lemma.

Lemma 2.3. *Let G be a finitely generated group and let H be a finite index subgroup of G. If every maximal subgroup of H has finite index in H, then the same property holds for G.*

Proof. First, recall that G is finitely generated if and only if H is. Now, suppose that every maximal subgroup of H has finite index. Let P be a maximal subgroup of G, assume that its index is infinite, and set $P_H = P \cap H$. Since H is finitely generated, P_H is contained in a maximal subgroup M; by assumption, M/P_H is infinite. The subgroup $M' = \cap_{p \in P} pMp^{-1}$ has finite index in G, it contains P_H, and again, M'/P_H is infinite. Thus, the subgroup of G generated by P and M' contains P strictly and is a proper subgroup of G, contradicting the maximality of P. □

3. Nilpotent by abelian groups; application

Remark 3.1. Let G be a group and P be a maximal subgroup of G. If $N \subset G$ is a normal subgroup, then

- either $N \subset P$, in which case P/N is maximal in G/N;
- or N and P generate G, which implies that
$$G = NP = \{np\ ;\ n \in N,\ p \in P\}$$
because N is normal.

Theorem C.– *Let G be a finitely generated group that contains a normal, nilpotent subgroup N such that G/N is virtually abelian. Then, every maximal subgroup of G has finite index.*

Lemma 3.2. *If N is a nilpotent group and $H \subset N$ is a subgroup such that $[N,N]H = N$, then $H = N$.*

Note that $[N,N]H = N$ means that H projects *onto* the abelianization of N.

Proof. We argue by induction on the nilpotent lengh of N. If N is abelian, the lemma is obvious. Then, consider the quotient of N by its center Z. Again, the group $H/Z \subset N/Z$ maps onto the abelianization of N/Z, thus $H/Z = N/Z$, by the induction hypothesis. In other words, $HZ = N$. This implies that H is normal in N, and that N/H is a nilpotent group with trivial abelianization. Thus $N = H$. □

Proof. Let P be a maximal subgroup of G. The derived subgroup $[N,N]$ is characteristic in N, hence normal in G. From Remark 3.1, there are a priori two (mutually exclusive) cases: either P contains $[N,N]$ or $G = [N,N]P$.

Let us exclude the second case. Indeed, we would have $([N,N]P) \cap N = N$ and, by Lemma 3.2, $P \cap N = N$. But then, $N \subset P$ and we are in the first case: $[N,N] \subset P$.

Taking the quotient by $[N,N]$, the projection $\overline{P} = P/[N,N]$ of P becomes a maximal subgroup of $\overline{G} = G/[N,N]$. Doing so, we are reduced to the case when $\overline{N}$ is a normal and abelian subgroup of $\overline{G}$ with $\overline{G}/\overline{N}$ virtually abelian. In this case, the conclusion follows from Corollary 2.2. □

Proof of Theorem A. Let G be a (finitely generated) virtually solvable subgroup of $\mathsf{GL}_m(\mathbf{k})$, for some field $\mathbf{k}$. Assume that $\mathbf{k}$ is algebraically closed, for simplicity. Let $\mathsf{Zar}(G)$ be the Zariski closure of G in $\mathsf{GL}_m(\mathbf{k})$ and let $\mathsf{Zar}(G)^\circ$ the connected component of the identity in $\mathsf{Zar}(G)$. Set $G^\circ := G \cap \mathsf{Zar}(G)^\circ$ has finite index in G. Moreover, by Borel fixed point theorem, $\mathsf{Zar}(G)^\circ$ is conjugate to a group of upper triangular matrices. Thus, its derived subgroup is nilpotent. This shows that G° is nilpotent by abelian, and we can apply Theorem D to get Theorem A. □

4. VARIATION

The following is a variation on the first part of the proof of Theorem B.

Theorem D.– *Let G be a group containing (a) a non-trivial, normal, and abelian subgroup $A \subset G$ and (b) a maximal subgroup $P \subset G$ that does not contain any non-trivial normal subgroup of G. Then,*

(1) $G = A \rtimes P$;

(2) *if $B \subset A$ is normal in G, then $B = \{1_G\}$ or $B = A$;*

(3) *the centralizer of A in G coincides with A (it intersects P trivially);*

(4) *every non-trivial normal subgroup of G contains A;*

(5) *A is characteristically simple, which means that it does not contain any non-trivial proper subgroup that would be invariant under* $\mathsf{Aut}(A)$.

Proof. (1).– The subgroup A is not contained in P, hence $G = AP$, as explained above. Moreover, $A \cap P$ is normal in G (as in the proof of Theorem B), and the hypothesis made on P implies $A \cap P = \{1_G\}$. Thus, $G = A \rtimes P$.

(2).– Let B be a proper subgroup of A and a normal subgroup of G. By (1), applied to B, we see that either $B = \{ 1_G\}$ or $G = B \rtimes P$, which implies $B = A$ since $G = A \rtimes P$.

(3).– Let C_G be the centralizer of A in G. Let $C_P = C_G \cap P$ be the centralizer of A in P. Then C_P is centralized by A and normalized by P, so that C_P is normal in G. Since $C_P \subset P$, the hypothesis on P implies $C_P = \{1_G\}$. Since $G = A \rtimes P$, this implies that $C_G = A$.

(4).– Let N be a non-trivial normal subgroup of G. Then $N \cap A$ is normal in G, and by (2) either N contains A or $N \cap A = \{1_G\}$. Since both A and N are normal, we have $[A,N] \subset A \cap N$. Thus, if N does not contain A, N centralizes A, in contradiction with (3).

(5).– If A contained a non-trivial characteristic subgroup B, B would be normal in G, in contradiction with (4). □

Remark 4.1. Conversely, suppose that $G = A \rtimes P$ with A abelian, normal, non-trivial, and minimal for these properties. Then P is a maximal subgroup of G.

Remark 4.2. Let P be a maximal subgroup of a group G. Let M be the normal core of P:

$$M = \bigcap_{g \in G} gPg^{-1}.$$

Then, taking quotient by M, P/M is a maximal subgroup of G/M, it does not contain any non-trivial normal subgroup of G/M, and Theorem C can be applied to $P/M \subset G/M$. For instance, if G is solvable, and if A/M is the last derived subgroup of G/M, we see that $G/M = (A/M) \rtimes (P/M)$.

Corollary 4.3. *If G, A, and P are as in Theorem C, and if G is residually finite, then G is finite.*

Proof. Fix $x \in A$, $x \neq 1_G$. Since G is residually finite, there is a normal, finite index subgroup $N \subset G$ that does not contain x. By Assertion (4) of Theorem C we deduce that $N = \{1_G\}$, which means that G is finite. □

Remark 4.4. If we use the theorem of P. Hall that says that a finitely generated metabelian group is residually finite, then we recover Theorem B from the previous corollary. But this theorem of Hall is harder to prove than Theorem B.

5. Examples of Hall and Cornulier

5.1. **Hall's example.** Consider the $\mathbf{Q}$-vector space

$$V = \bigoplus_{i \in \mathbf{Z}} \mathbf{Q}\mathbf{e}_i;$$

the $\mathbf{e}_i$ form a basis of V and the elements of V are sums $v = \sum_i a_i \mathbf{e}_i$ with finite support (i.e. only finitely many of the a_i are $\neq 0$). Consider the following linear automorphisms of V. Firstly, the shift $s \colon V \to V$ defined by

$$s(\mathbf{e}_i) = \mathbf{e}_{i+1}.$$

Secondly, the diagonal transformation $m\colon V \to V$ defined by

$$m_\lambda(\mathbf{e}_i) = \lambda_i \mathbf{e}_i$$

where $(\lambda_i)_{i\in\mathbf{Z}}$ is a sequence of non-zero rational numbers. The group $P = \langle s, m_\lambda \rangle \subset \mathsf{GL}(V)$ acts on V, and V can be considered as a P-module or, equivalently, as a $\mathbf{Z}[P]$-module. We also obtain a semi-direct product

$$G = P \ltimes V.$$

(1) The group G is solvable.

Proof. Elements of G can be written as affine transformations of V of type $f(v) = \ell(v) + u$ where the linear part ℓ is a composition $\ell = s^k m$ for some integer k and some diagonal map m, the coefficients of which are products of λ_i's and their inverses. When one derives G once, one kills the shift part s^k. Then, all elements of the second derived subgroup $[[G,G],[G,G]]$ are translations, so that this group is abelian. □

(2) Suppose every non-zero rational numbers α appears in (λ_i). Then, G is generated by s, m, and $\mathbf{e}_0$.

Proof. If one $\mathbf{e}_0$ by s^j, apply m, and shift back, one gets $\lambda_j \mathbf{e}_0$. If one shift again, one obtain $\lambda_j \mathbf{e}_i$ for every pair (i, j). By assumption, the $\lambda_j \mathbf{e}_i$ generate V as a group. This concludes the proof. □

(3) Suppose every finite sequence of non-zero rational numbers $(\alpha_j)_{j=1}^r$ appears in (λ_i), meaning there is an integer i_0 such that $\alpha_j = \lambda_{i_0+j}$ for all $j \le r$. Then, the $\mathbf{Z}[P]$-module V is simple and P is a maximal subgroup of G of infinite index.

Proof. To prove that V is simple, pick a non-zero vector $v = \sum_{i=i_-}^{i_+} a_i \mathbf{e}_i$ in V. Shifting by s, multiplying by m, and shifting back again, one constructs a new vector $v' = \sum_{i=i_-}^{i_+} b_i \mathbf{e}_i$ with coefficients $b_i = -a_i$ for $i < i_+$ and $b_{i_+} = 1 - a_{i_+}$. Then, $v + v' = \mathbf{e}_{i_+}$. Shifting by s^{-i_+}, one gets $\mathbf{e}_0$, hence V by Assertion (2). □

Putting everythin together, we obtain

Hall's Example.– *If every finite sequence of non-zero rational numbers appears in (λ_i), the group G is finitely generated, solvable (of length* 3*), and $P \subset G$ is a maximal subgroup of infinite index.*

Remark 5.1. We could also start with a vector space over $\mathbf{F}_q$ (q a prime power) instead of $\mathbf{Q}$ in this construction. There are uncountably many possible choices for the sequence $(\lambda_i)_{i\in\mathbf{Z}}$ (see below § 5.2.4).

5.2. **Cornulier's examples.** The goal of this section is to construct a group with uncountably many maximal subgroups of infinite indices. The idea is to vary the choice of the sequence $(\lambda_i)_{i\in\mathbf{Z}}$.

5.2.1. Consider the wreath product $L = \mathbf{Z}/2\mathbf{Z} \wr \mathbf{Z}$, i.e. the lamplighter group. It is the semi-direct product of $W \rtimes \mathbf{Z}$ where W is the additive group of finitely supported sequences $\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$; elements of W are written as sequences $(u_i)_{i\in\mathbf{Z}}$ with $u_i = 0$ or 1, and all $u_i = 0$ except finitely many of them. The group L is generated by two elements:

- the element u, all of whose coefficients are 0 except $u_0 = 1$;
- the shift s that maps a sequence (u_i) to (u_{i+1}).

This group L is solvable, its derived group being the additive group W.

5.2.2. Fix a prime p and a sequence $\lambda = (\lambda_i)_{i\in\mathbf{Z}}$ be a sequence of non-zero elements of $\mathbf{F}_p$. Consider the $\mathbf{F}_p$-vector space with basis $(\mathbf{e}_i)_{i\in\mathbf{Z}}$,

$$M = \bigoplus_{i\in\mathbf{Z}} \mathbf{F}_p \mathbf{e}_i,$$

and the representation of L in $\mathsf{GL}(M)$ given by

$$s(\mathbf{e}_i) = \mathbf{e}_{i+1} \quad \forall i \in \mathbf{Z}$$
$$u(\mathbf{e}_i) = \lambda_i \mathbf{e}_i \quad \forall i \in \mathbf{Z}$$

(i.e. u is mapped to m_λ with the notation from the previous section). This endows M with the structure of an L-module (or a $\mathbf{Z}[L]$-module) , denoted by M_λ in what follows. The argument given in the previous section shows that

(1) *If λ contains every finite sequences of elements in $\mathbf{F}_p^\times$, then M_λ is a simple $\mathbf{Z}[L]$-module and the subgroup L of $L \ltimes M_\lambda$ is maximal, of infinite index.*

5.2.3. Define a new group G by

$$G = L \ltimes \mathbf{F}_p[L]$$

where L acts on $\mathbf{F}_p[L]$ by left multiplication. This group is finitely generated, by s and u in L and $1_L \in \mathbf{F}_p[L]$, and is solvable of length three.

The important feature of G is that, for each sequence $\lambda \in (\mathbf{F}_p^\times)^{\mathbf{Z}}$, the group $G_\lambda = L \ltimes M_\lambda$ is a quotient of G. This group G_λ contains a maximal subgroup

$\overline{P}_\lambda$ of infinite index, namely the image of L in G_λ. We shall denote by P_λ the preimage of $\overline{P}_\lambda$ in G. We obtain

(2) *The subgroups $P_\lambda \subset G$ are maximal subgroups of G of infinite index.*

The map $p_\lambda \colon \mathbf{F}_p[L] \mapsto M_\lambda$ defined by $p_\lambda(w) = w(\mathbf{e}_0)$, or in full by

$$p_\lambda(\sum_{h\in L} \alpha_h h) = \sum_{h\in L} \alpha_h h(\mathbf{e}_0)$$

(where h is the linear transformation associated to $h \in L$ in $\mathsf{GL}(M_\lambda)$), is a homomorphism of abelian groups (resp. of $\mathbf{F}_p$-vector spaces). Its kernel I_λ is

$$I_\lambda = \left\{ \sum_{h\in L} \alpha_h h \,;\, \sum_{h\in L} \alpha_h h(\mathbf{e}_0) = 0 \right\}.$$

This is a ideal for the action of L on $\mathbf{F}_p[L]$ by left multiplication. And this ideal determines λ, because $\alpha 1_L - s^k u s^{-k}$ is in I_λ if and only if $\alpha = \lambda$. Thus,

(3) *The intersection $P_\lambda \cap \mathbf{F}_p[L] = I_\lambda$ determines λ, so $P_\lambda \subset G$ determines λ.*

5.2.4. There are uncountably many sequences λ containing all possible finite sequences of elements of $\mathbf{F}_p^\times$. To see this, note that finite sequences of elements of a countable set (as $\mathbf{F}_p^\times$ or $\mathbf{Q}^\times$) form a countable set. So we can list these sequences and then concatenate them in an infinite, one-sided, sequence. For instance, with $\mathbf{F}_2^\times$, one can take

$$(0;1;0,0;0,1;1,0;1,1;0,0,0;0,0,1;0,1,0;1,0,0;0,1,1;1,0,1;\ldots).$$

Then, one can put any sequence $(\lambda_i)_{i\le -1}$ on the negative side. So, putting everything together, we obtain

Cornulier's Example.– *Let p be a prime, let L be the lamplighter group $L = \mathbf{Z}/2\mathbf{Z} \wr \mathbf{Z}$, and set $G = L \ltimes \mathbf{F}_p[L]$. Then G is a finitely generated, solvable group that contains uncountably many maximal subgroups P_λ.*

Part 2.– Subgroups of finite index

6. Residual finiteness I

Theorem E.– *Let G be finitely generated group. If G is virtually metabelian, then it is residually finite.*

This theorem is also due to Philip Hall. It holds more generally for finitely generated groups which are abelian by nilpotent. Note that if $H \subset G$ is a finite index subgroup, then G is residually finite if and only if H is. Thus, Theorem E holds if and only if it holds for finitely generated metabelian groups.

To prove this theorem, we shall use the Artin-Rees lemma from commutative algebra.

Lemma 6.1 (Artin-Rees)**.** *Let R be a noetherian ring and I an ideal of R. Let M be a finitely generated R-module, and let N be a submodule of M. Then, there is an integer $k \geq 0$ such that*

$$I^n M \cap N = I^{n-k}(I^k M \cap N)$$

for every $n \geq k$.

The proof of this lemma will be given below in Section 9. Note that, with $n = k+1$, we get the existence of an integer $n \geq 1$ such that

$$I^n M \cap N \subset IN.$$

Proof of Theorem E. As explained above, we can assume that G is metabelian. Let A be the derived subgroup of G, and let Q be the quotient G/A.

Fix $x \in G$, $x \neq 1_G$. Since G is finitely generated, there is a normal subgroup K of G which is maximal among normal subgroups that do not contain x. Our goal is to show that G/K is finite. Thus, in what follows, we can suppose that $K = 1$, or more precisely that x is contained in every non-trivial normal subgroup of G. In other words, x is contained in the normal core M of G, i.e. in the intersection of all non-trivial normal subgroups of G. In particular,

- A contains M and x.
- viewed as a $\mathbf{Z}[Q]$-module (for the action of G and thus Q by conjugation), M is a simple submodule of A.

From Lemma 2.1, we deduce that M is finite and, consequently, that its centralizer $C \subset G$ has finite index in G. Since A is abelian, C contains A. Since G/C is finite, C is finitely generated, and C/A is a finitely generated abelian group.

In what follows, we view A as a module under the action of C by conjugation, or more precisely as a $\mathbf{Z}[C/A]$-module.

Now, consider the ideal $I_C \subset \mathbf{Z}[C/A]$ generated by all the elements $1-c$, with $c \in C$. Set

$$\begin{aligned} I_C^* &= \{a \in A \; ; \; (1-c)\cdot a = 0, \text{ for all } \; c \in C/A\} \\ &= \{a \in A \; ; \; a - cac^{-1} = 0, \text{ for all } \; c \in C/A\} \\ &= \text{centralizer of } C \text{ in } A. \end{aligned}$$

Then, I_C^* is a submodule of A. The ring $\mathbf{Z}[C/A]$ is noetherian[1]. By the lemma of Artin and Rees, there is a positive integer m such that

$$(I_C^m \cdot A) \cap I_C^* \subset I_C \cdot I_C^* = 0,$$

since by definition $I_C \cdot I_C^* = 0$. This means that $I_C^m A$ does not intersect the centralizer of C in A; in particular, it does not intersect M. But, using multiplicative notations instead of additive ones, $I_C \cdot A$ is the subgroup of A generated by the commutators $[a,c]$, and then by recursion we see that $I_C^m \cdot A$ is the subgroup of A generated by the elements of type

$$[[\ldots[[a,c_1],c_2],\ldots],c_m],$$

with a in A and the c_i in C. Since C/A is abelian, $[C,C]$ is contained in A, and it follows that the subgroup of G generated by commutators

$$[[\ldots[[c_0,c_1],c_2],\ldots],c_{m+1}]$$

of elements of C does not intersect M. But M being contained in any non-trivial normal subgroup of G, this implies that C is nilpotent, of length $\leq m+2$.

To conclude, we rely on the next section, which proves that any nilpotent, finitely generated group is residually finite. □

7. Residual finiteness II

Theorem F.– *If G is a finitely generated nilpotent group, then G is residually finite.*

Lemma 7.1. *If G is nilpotent and finitely generated, then its derived group is finitely generated too.*

[1] As in the proof of Lemma 2.1, if C/A is isomorphic to $\mathbf{Z}^r \oplus \bigoplus_i \mathbf{Z}/a_i\mathbf{Z}$, then $\mathbf{Z}[C/A]$ is the commutative, finitely generated ring $\mathbf{Z}[t_i, t_i^{-1}; 1 \leq i \leq r+s]/I$ where I is the ideal generated by the polynomial functions $t_i^{a_i} - 1$ for $i = r+1$ to $r+s$.

Of course, this is specific to nilpotent groups. For instance, the derived subgroup of a non-abelian free group is note finitely generated, and the derived subgroup of the solvable group $2^{\mathbf{Z}} \ltimes \mathbf{Z}[1/2]$ is not finitely generated either.

Proof of Lemma 7.1. In fact, every element in the lower central series $\gamma_m G$ is finitely generated. Recall that $\gamma_1 G = G$ and $\gamma_{m+1} G = [G, \gamma_m G]$, so that $\gamma_2 G$ is the derived subgroup. Then, $\gamma_1 G/\gamma_2 G$ is finitely generated, because $G = \gamma_1 G$ is.

The key remark is that $\gamma_2 G/\gamma_3 G$ is finitely generated too. More precisely, if g_1, ..., g_{n_1} is a system of generators of $\gamma_1 G$, then the commutators $[g_i, g_j]$ generate $\gamma_2 G/\gamma_3 G$. To see this, note that for any triple of elements g, h, k in G we have

$$\begin{aligned}[gh,k] &= ghkh^{-1}g^{-1}k^{-1} \\ &= g[h,k][k,g^{-1}]g^{-1} \\ &= (g[h,k]g^{-1}g)([k,g^{-1}]g^{-1}).\end{aligned}$$

Thus, $\gamma_2 G$ is generated by the conjugates of the commutators $[g_i, g_j]$. Modulo $\gamma_3 G = [G, \gamma_2 G]$, the $[g_i, g_j]$ become central, thus $\gamma_2 G/\gamma_3 G$ is generated by the $[g_i, g_j]$. Denote these commutators by $g_{2,i}$ with $i = 1, \ldots, n_2$ with $n_2 \leq n_1(n_1 - 1)/2$.

Then, the same argument shows that the commutators $[g_i, g_{2,j}]$ generate $\gamma_3 G/\gamma_4 G$, and so on. Since $\gamma_m G = \{1_G\}$ for some $m \geq 1$, we see that $\gamma_{m-1} G$ is finitely generated, and then that each $\gamma_j G$ is finitely generated. □

Proof of Theorem F. Let x be an element of G, $x \neq 1_G$. We are looking for a finite index subgroup of G that does not contain x. The abelianization of G is a finitely generated abelian group, and such a group is residually finite, so we can assume that $x \in [G, G]$. By recursion on the derived length of G, and the fact that the derived subgroup is also finitely generated, we know that $[G, G]$ is residually finite; thus, we can take the quotient by a characteristic subgroup of $[G, G]$ of finite index and assume that $[G, G]$ is finite.

Consider the centralizer C of the finite group $[G, G]$. Its index in G is finite, hence C is also finitely generated. And $[C, C]$ is finite and central in C, so C is nilpotent of length ≤ 2. If $x \notin C$ we are done. So, we can assume $x \in C$.

Thus, we are reduced to prove Theorem F for finitely generated, nilpotent groups G of length ≤ 2 such that $[G, G]$ is finite and central. If such a group has a finite abelianization, then it is finite. Thus, we can restrict our problem to groups with an infinite abelianization and argue by recursion on the rank of the torsion free part of $G/[G,G]$. For this, take a surjective homomorphism from G onto an infinite cyclic group and denote by H its kernel. The abelianization $H/[H,H]$ might be bigger than its image in $G/[G,G]$, but the kernel of the natural homomorphism $H/[H,H] \to G/[G,G]$ is finite, because so is $[G,G]$. Thus, the rank of $H/[H,H]$ is less than the one of $G/[G,G]$ and the induction hypothesis show that H is residually finite, which concludes the proof. □

Remark 7.2. (1) From Theorem F, we deduce that *a finitely generated solvable group S is torsion if and only if it is finite*. This can be proved directly. The case of abelian group follows from the fact that any finitely generated abelian group is isomorphic to a finite sum $\mathbf{Z}^r \oplus \bigoplus_i \mathbf{Z}/a_i\mathbf{Z}$. Thus, the abelianization $S/[S,S]$ of S must be finite. Then, arguing by induction on the solvable length of S, we get that $[S,S]$ is finite and we conclude that S itself is finite.

(2) *If S is a finitely generated solvable group, and if $n \geq 1$, then the subgroup of S generated by the nth power g^n, $g \in S$, is a finite index, normal subgroup of S*. Indeed, the quotient is a finitely generated, nilpotent group in which the order of every element divides n.

Part 3.– Appendices

8. Appendix A

Lemma 8.1. *A field which is finitely generated as a ring is a finite field.*

To prove this lemma, let $\mathbf{K}$ be a field which is finitely generated as a ring. Let $\mathbf{F}$ be its prime field. If the characteristic of $\mathbf{K}$ is 0, $\mathbf{F} = \mathbf{Q}$, if it is positive, equal to some prime p, $\mathbf{F} = \mathbf{F}_p$. Since $\mathbf{K}$ is finitely generated as a field, it is an extension of finite degree of some purely transcendental extension $\mathbf{F}[\mathbf{x}_1, \ldots, \mathbf{x}_m]$, of transcendental degree $m < +\infty$.

If $m = 0$, then either $\mathbf{K}$ is finite, or $\mathbf{K}$ is a number field. In this second case, let us write $\mathbf{K} = \mathbf{Z}[a_1, \ldots, a_n]$ for some $a_j \in \mathbf{K}$. Let $(u_1, \ldots, u_d)$ be a basis of $\mathbf{K}$ as a $\mathbf{Q}$-vector space. The multiplication $m_{a_k} \colon t \in \mathbf{K} \mapsto a_k t$ is a linear endomorphism of $\mathbf{K}$; denote by $c_k(i, j) \in \mathbf{Q}$ the coefficients of its matrix in the basis (u_i). Since $\mathbf{K} = \mathbf{Z}[\varphi_1, \ldots, \varphi_n]$, the matrix of $m_v \colon t \mapsto vt$ has coefficients in the ring $\mathbf{Z}[c_k(i, j); k \leq n, i, j \leq d]$. Taking $v = 1/q$, where q is a prime that does not appear in the prime decompositions of the $c_k(i, j)$, we obtain a contradiction. Thus, $\mathbf{K}$ if finite when $m = 0$.

The same proof applies when $m \geq 1$. Choose a basis (u_i) of $\mathbf{K}$ as a vector space over the field $\mathbf{L} = \mathbf{F}[\mathbf{x}_1, \ldots, \mathbf{x}_m]$. List the coefficients $c_k(i, j) \in \mathbf{L}$ of the matrix of m_{a_k} and pick an irreducible polynomial q in $\mathbf{F}[\mathbf{x}_1]$ of degree larger than the degrees of the irreducible factors in the decompositions of the $c_k(i, j)$. Then, $1/q$ is not in $\mathbf{Z}[a_1, \ldots, a_m]$.

9. Appendix B

Lemma 9.1 (Artin-Rees)**.** *Let A be a noetherian ring. Let M be a finitely generated A-module, and let N be a submodule of M. Let I be an ideal of A. There exists an integer $k \geq 0$ such that*

$$(I^n M) \cap N = I^{n-k}((I^k M) \cap N)$$

for every $n \geq k$.

The goal of this section is to prove this lemma. As a warm up, we start with the Nakayama lemma and Krull's intersection theorem.

Lemma 9.2 (Nakayama)**.** *Let A be a ring, M a finitely generated A-module, and I an ideal of A. The following properties are equivalent:*

- *IM contains M;*

- *there exists an element a of I such that* $(1+a)M = 0$.

Proof. Suppose that IM contains M. Let $u_1, \dots, u_d$ be a finite set of generators of M. Each u_i can be written as a sum $\sum b_{i,j}u_j$ with the $b_{i,j}$ in I. Let B be the $d \times d$ matrix with entries $b_{i,j}$. The transpose of the comatrix ${}^t\mathrm{Com}(\mathrm{id} - B)$ satisfies ${}^t\mathrm{Com}(\mathrm{id} - B)B = \mathtt{det}(\mathrm{id} - B)\mathrm{id}$. And $\mathtt{det}(\mathrm{id} - B)$ is an element of $1+I$; it can be written as $1+a$ for some $a \in I$. From this we get $(1+a)M = 0$.

Conversely, if $(1+a)M = 0$, then the multiplication by a maps M onto M and IM contains M. □

Theorem 9.3 (Krull's intersection theorem)**.** *Let A be a noetherian ring, let I be an ideal of A, and let M be a finitely generated A-module. Then*

$$\bigcap_{n\geq 1} I^n M = \{x \in M \,;\, \exists a \in I, (1-a)x = 0\}.$$

Moreover, such an a exists that does not depend on x.

Proof. Let us derive this theorem from the Artin-Rees and Nakayama lemmas. Set $N = \cap_{n\geq 1} I^n M$. If $(1-a)x = 0$ then $x = a^n x$ for all $n \geq 1$ and $x \in N$. The interesting statement is the converse, which we now prove. First, note that the Artin-Rees lemma, applied to $N \subset M = I$, gives $I^{k+1} \cap N = I(I^k \cap N)$ and therefore $N = IN$. Now, the Nakayama lemma provides an $a \in I$ such that $(1+a)N = 0$, as desired. □

Remark 9.4. Let I be an ideal in a noetherian ring A. Krull's theorem implies that $\cap_n I^n = \{0\}$ if and only if $1+I$ does not contain any zero divisor. So, if the ring A is an integral domain and $I \neq A$, we conclude that $\cap_n I^n = \{0\}$.

We now move to the proof of the Artin-Rees Lemma. Let I be an ideal in a ring A. Let us introduce the blow-up

$$B_I = \bigoplus_n I^n.$$

This A-algebra is isomorphic to the Rees algebra

$$A[I\mathbf{t}] = \bigoplus_{n\geq 0} I^n \mathbf{t}^n \subset A[\mathbf{t}].$$

One says that a decreasing sequence of submodules $M = M_0 \supset M_1 \supset \cdots \subset M_n \supset \cdots$ is an I-filtration if $IM_n \subset M_{n+1}$, and it is a stable one if $IM_n = M_{n+1}$ for sufficiently large n. Given an I-filtration, we set

$$B_I M = \bigoplus_n M_n.$$

It is a graded module over B_I. Now, if the filtration is by finitely generated A-modules, the following are equivalent:

(a) B_IM is finitely generated as a B_I-module;

(b) the filtration is I-stable.

Proof. To see this, suppose (b), that is $IM_n = M_{n+1}$ for $n \geq k$; then B_IM is generated by the first k terms M_k, and since M_0 is finitely generated as an A-module, B_IM is finitely generated as a B_I-module. Conversely, suppose that B_IM is finitely generated as a B_I-module. Pick a finite system of generators $g_{n,i} \in M_n$ for some indices $n \leq n_0$, $i \leq i_0$ (we may assume i_0 does not depend on the degree n). Then, every element f of M_m can be written as a sum

$$f = \sum_{n \leq n_0} \sum_{i \leq i_0} a_{n,i} g_{n,i}$$

with $n \leq m$ and $a_{n,i} \in I^{m-n}$. This implies that f is in $I^{m-n_0}M_{n_0}$, and from this we get the stability $IM_m = M_{m+1}$ for $m \geq n_0$. □

We can now prove the Artin-Rees lemma. Set $M_n = I^nM$, they form an I-filtration, which is stable by construction. By assumption, M is finitely generated and A is noetherian, so each M_n is a finitely generated A-module. Thus, since (b) implies (a), we deduce that $B_IM = \bigoplus_n I^nM$ is finitely generated as a B_I module. But $B_I \simeq A[It]$ is a noetherian ring, because A is. Thus, every B_I-submodule of M is finitely generated. Apply this to B_IN with the filtration $N_n = M_n \cap N$; since (a) implies (b), the filtration N_n is I-stable: there is an integer k such that $I^mN_n = N_{n+m}$ for $n \geq k$, as desired.

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