INVARIANT MEASURES FOR LARGE AUTOMORPHISM GROUPS OF PROJECTIVE SURFACES

SERGE CANTAT AND ROMAIN DUJARDIN

ABSTRACT. We classify invariant measures for non-elementary groups of automorphisms, on any compact Kähler surface X, under the assumption that the group contains a so-called "parabolic automorphism". We also provide finiteness results for the number of invariant, ergodic, probability measures with a Zariski dense support.

CONTENTS

1. INTRODUCTION

1.1. **Non-elementary groups and parabolic automorphisms.** Let Γ be a group of automorphisms of a compact Kähler surface X. We say that Γ is **non-elementary** if its image Γ^* in $\mathsf{GL}(H^2(X;\mathbf{Z}))$, induced by its action on the cohomology of X, contains a non-abelian free group. We refer to [11] for a description of such groups of automorphisms. In particular, it is shown in [11] that the existence of a non-elementary subgroup in $\mathsf{Aut}(X)$ implies that X is a projective surface.

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By definition, an element g of $\mathsf{Aut}(X)$ is a **parabolic automorphism** if $\|(g^n)^*\|$ grows quadratically with the number n of iterates, where $\|\cdot\|$ is any operator norm on $\mathsf{End}(H^2(X;\mathbf{C}))$. Any parabolic automorphism g preserves some genus 1 fibration, and a group $\Gamma \subset \mathsf{Aut}(X)$ containing a parabolic automorphism is non-elementary if and only if it contains two parabolic automorphisms preserving distinct fibrations [11, 10].

1.2. **Classification.** In this article, we classify probability measures on X which are invariant by a non-elementary group containing a parabolic automorphisms. Our first theorem was already announced in [11, 10]:

Theorem A. *Let X be a compact Kähler surface. Let Γ be a non-elementary subgroup of $\mathsf{Aut}(X)$ containing a parabolic element. Let μ be a Γ-invariant ergodic probability measure on X. Then, μ satisfies exactly one of the following properties.*

(a) *μ is the average on a finite orbit of Γ;*
(b) *μ is nonatomic and supported on a Γ-invariant curve $D \subset X$;*
(c) *There is a Γ-invariant proper algebraic subset Z of X, and a Γ-invariant, totally real analytic surface Σ of $X \setminus Z$ such that* (1) *$\mu(\overline{\Sigma}) = 1$ and $\mu(Z) = 0$;* (2) *Σ has finitely many irreducible components;* (3) *the singular locus of Σ is locally finite; and* (4) *μ is absolutely continuous with respect to the Lebesgue measure on Σ, and* (5) *its density (with respect to any real analytic area form on the regular part of Σ) is real analytic;*
(d) *There is a Γ-invariant proper algebraic subset Z of X such that* (1) *$\mu(Z) = 0$,* (2) *the support of μ is equal to X;* (3) *μ is absolutely continuous with respect to the Lebesgue measure on X; and* (4) *the density of μ with respect to any real analytic volume form on X is real analytic on $X \setminus Z$.*

Remark 1.1.

(1) Recall that an analytic surface Σ in an open $U \subset X$ is **totally real** if for every smooth point x of Σ, the (real) tangent space $T_x\Sigma$ contains a basis of the complex tangent space T_xX; equivalently $T_x\Sigma$ and its image $\mathsf{j}_X(T_x\Sigma)$ by the complex structure satisfy $T_x\Sigma \oplus_{\mathbf{R}} \mathsf{j}_X(T_x\Sigma) = T_xX$.
(2) Thus, each of the four cases (a), (b), (c), and (d) is characterized by a property of the support $\mathrm{Supp}(\mu)$: being finite, Zariski dense in a curve, totally real, or equal to X.
(3) Given any non-elementary group $\Gamma \subset \mathsf{Aut}(X)$, there is a unique maximal Γ-invariant curve $D_\Gamma \subset X$ (see § 4.1 below). The invariant algebraic set Z is independent of μ and admits an explicit description (see Propositions 4.9 and 4.15). It is made of components of D_Γ together with a residual finite set.

Corollary B. *Let X be a compact Kähler surface. Let Γ be a non-elementary subgroup of $\mathsf{Aut}(X)$ that contains a parabolic element and does not preserve any proper algebraic subset of X. If μ is a Γ-invariant and ergodic probability measure on X, then μ is*

(a) *either a measure with real-analytic density on a compact, smooth, totally real, and real analytic surface Σ of X;*
(b) *or a measure with real-analytic density on X.*

In the totally real case (c) of Theorem A, it is natural to inquire about the structure of $\overline{\Sigma}$ on the whole surface X, including Z. Under a mild geometric condition (AC) we are indeed able to show that $\overline{\Sigma}$ admits a semi-analytic extension across Z (see Theorem A' in Section 6); this means that $\overline{\Sigma}$ is defined locally by finitely many analytic inequalities (see §5.1). Since it requires some additional concepts, the condition (AC) will be described only in § 6.3: it concerns the action of Γ on the singular fibers of the elliptic fibration invariant by a parabolic element of Γ. This condition is satisfied in many interesting cases (for instance if D_Γ is empty) and can be checked on concrete examples.

In Sections 8 and 9, we provide examples showing that the geometric conclusions of Theorems A and A' are, in a sense, optimal. More precisely it is shown that in case (c),

– Σ is **not** necessarily contained in the real part of X, for some real structure on X, in other words, it is not necessarily contained in the fixed point set of an anti-holomorphic involution of X (see Corollary 8.2);
– Σ can have a non-empty boundary (see §9.2).

1.3. **Finitely many invariant measures.** Theorem A is a key ingredient of the finiteness results for the number of periodic orbits in [10]. It also leads to the following alternative which is reminiscent of, but independent from, [10, Thm B].

Theorem C. *Let X be a compact Kähler surface, and let Γ be a non-elementary subgroup of $\mathsf{Aut}(X)$ containing a parabolic element. Then there are only finitely many ergodic Γ-invariant probability measures giving no mass to proper Zariski closed subsets, unless (X, Γ) is a Kummer group.*

We refer to [10] for the definition of Kummer groups; roughly speaking it means that the dynamics of Γ on X comes from the dynamics of a group of automorphisms on some torus $\mathbf{C}^2/\Lambda$. This result will be established in Section 7. We also show in § A.3 that a Kummer group can indeed admit infinitely many ergodic invariant measures of totally real type. Together with [10, Thm. C] (finitely many finite orbits) we thus obtain:

Corollary D. *Let X be a compact Kähler surface which is not a torus. Let Γ be a subgroup of $\mathsf{Aut}(X)$ which contains a parabolic element and does not preserve any algebraic curve. Then there are at most finitely many Γ-invariant, ergodic probability measures on X.*

1.4. **Notes.** A weak form of Theorem A is proven in [5] in the special case of K3 surfaces[1]. The results of [5] do not describe the support of the measure or the smoothness of its density, and are not sufficient to derive the global structure of Σ given by Corollary B, nor the finiteness result of Theorem C.

1.5. **Structure of the paper.** In Section 2 we start by briefly describing a few basic examples which are useful to be kept in mind; more advanced examples are given in Sections 8 and 9. In Section 3 we collect some preliminary results on genus 1 fibrations, their singular fibers, and the dynamics of automorphisms preserving such fibrations. We hope that this could prove useful beyond this paper (see also Duistermaat's monograph [21] for a thorough treatment with a different focus). The core of the paper extends from Sections 4 to 7. A basic dichotomy is whether X is birationally equivalent to a torus, or not. The torus case relies on elementary tools from homogeneous dynamics, and the details are given in Appendix A. The proof of Theorem A occupies Sections 4 and 5. Theorems A' on the semi-analyticity of $\overline{\Sigma}$ and Theorem C are largely intertwined, and rest on a careful analysis of the action of the parabolic elements of Γ near the singular fibers of the associated elliptic fibrations. The details are given in Sections 6 and 7.

2. Two examples

2.1. **K3 surfaces.** Let X be any K3 surface. There is a holomorphic 2-form Ω_X on X that does not vanish and satisfies $\int_X \Omega_X \wedge \overline{\Omega_X} = 1$; this form is unique up to multiplication by a complex number of modulus 1. Thus, the volume form $\mathsf{vol}_X := \Omega_X \wedge \overline{\Omega_X}$ is $\mathsf{Aut}(X)$-invariant. If X comes with a real structure for which $X(\mathbf{R})$ is non-empty, then $X(\mathbf{R})$ is orientable and some multiple of Ω restricts to a positive area form on $X(\mathbf{R})$ (see [29, §VIII.4] and §1 of [18]). This area form is multiplied by ± 1 by elements of $\mathsf{Aut}(X_{\mathbf{R}})$; in particular, the measure induced by this form is invariant (see also Remark 2.3 below). We refer to [17, 19] for the topology of $X(\mathbf{R})$: it can be a sphere, the union of a sphere and a surface of genus 2, a torus, etc. Here are two explicit examples.

Example 2.1. (See [11, 14]).– Take three copies of $\mathbb{P}^1$, with respective coordinates $z_i = [x_i : y_i]$, $i = 1, 2, 3$. Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a Wehler surface, i.e. a smooth surface of degree $(2, 2, 2)$; assume that X *is very general in the family of such surfaces*. In particular, X is smooth and it is a K3 surface. Fix an index $k \in \{1, 2, 3\}$, let $i < j$ be the two indices such that $\{i, j, k\} = \{1, 2, 3\}$, and let $\pi_{ij} \colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the projection

[1]Also, one of the statements in [5] is slightly erroneous. In case (c), one first shows that μ gives mass to some *germs of real analytic surfaces*, and one has to glue these germs together to construct the surface Σ; to do so there is a monodromy problem which is overlooked in [5]. To say it differently, the results of [5] only imply that Σ is the analytic continuation of such a germ, which could a priori be dense in X. Overcoming this problem occupies a significant part of the present paper.

that forgets the k-th coordinate; this projection is a 2-to-1 cover, and we denote by σ_k the involution that permutes the points in the fibers of π_{ij}. Then, $\mathsf{Aut}(X)$ is generated by the three involutions σ_k and is non-elementary (see [4, 11, 31]); the composition $\sigma_i \circ \sigma_j$ is a parabolic automorphism preserving the genus 1 fibration $\pi_k(z_1, z_2, z_3) = z_k$; and the composition $\sigma_1 \circ \sigma_2 \circ \sigma_3$ is a loxodromic automorphism –with topological entropy $\log(9 + 4\sqrt{5}) > 0$. It is shown in [10, Thm A] that for a very general X, there is no $\mathsf{Aut}(X)$-invariant proper algebraic subset. If X is defined by a polynomial equation with real coefficients, then $\mathsf{Aut}(X)$ preserves the real structure $X_{\mathbf{R}}$ because the three involutions do. In particular, the real part $X(\mathbf{R})$ is $\mathsf{Aut}(X)$-invariant.

For future reference let us note that the canonical invariant 2-form admits a simple explicit expression: consider affine coordinates $x_i \in \mathbf{C}$ corresponding to each of the three $\mathbb{P}^1$ factors (with $z_i = [x_i : 1]$); then, X is defined by a polynomial equation $P(x_1, x_2, x_3) = 0$, and at every point in X, one of the partial derivatives of P does not vanish because X is smooth; then, up to some constant factor,

$$\Omega_X = \frac{dx_1 \wedge dx_2}{\partial_{x_3} P} = \frac{dx_2 \wedge dx_3}{\partial_{x_1} P} = \frac{dx_3 \wedge dx_1}{\partial_{x_2} P}. \tag{2.1}$$

Example 2.2. (See [11, §3.2]).– Fix five lengths $(\ell_0, \ell_2, \ldots, \ell_4) \in (\mathbf{R}_+^*)^5$ such that there is at least one pentagon $\mathsf{P} = (a_0, \ldots, a_4)$ in $\mathbf{R}^2$, the sides of which satisfy $(a_i, a_{i+1}) = \ell_i$ (for i taken modulo 5); here, by a **pentagon**, we just mean an ordered set of five points a_i in $\mathbf{R}^2$. Assume that the family of such pentagons does not contain any flat pentagon (for instance this imposes $\ell_0 + \ell_1 \neq \ell_2 + \ell_3 + \ell_4$). Consider the set of all such pentagons modulo affine positive isometries of $\mathbf{R}^2$; thus, each pentagon can now be put in a normal position, with $a_0 = (0, 0)$ and $a_1 = (\ell_0, 0)$. This set can be identified with a real algebraic surface $X(\mathbf{R})$ that depends on $(\ell_0, \ldots, \ell_4)$. There are five natural involutions acting algebraically on this surface: given one of the vertices a_i of a pentagon $\mathsf{P} \in X(\mathbf{R})$, consider the two circles with centers a_{i-1} and a_{i+1} and respective radii ℓ_{i-1} and ℓ_i, where indices are taken modulo 5; these circles intersect in two points a_i and a_i'; thus we get an involution σ_i of $X(\mathbf{R})$, mapping P to the pentagon $\sigma_i(\mathsf{P})$ with the same vertices except for a_i that is replaced by a_i'. Our hypotheses imply that $X(\mathbf{R})$ is the real part of some real K3 surface $X_{\mathbf{R}}$ and $\sigma_i \in \mathsf{Aut}(X_{\mathbf{R}})$. Again, the composition $\sigma_i \circ \sigma_{i+1}$ is a parabolic automorphism of X when the lengths are chosen generically.

Similar examples of large groups of automorphisms preserving a volume form on X (resp. a smooth measure on $X(\mathbf{R})$) can be constructed on some abelian surfaces and on most Enriques surfaces (see [11] for instance).

Remark 2.3. If Σ is a totally real surface (of class C^1, say) in an abelian or K3 surface X, which is invariant under a group $\Gamma \subset \mathsf{Aut}(X)$, then the canonical 2-form Ω_X induces a Γ-invariant measure on Σ. Indeed for every $x \in X$, the tangent space T_xX contains two $\mathbf{C}$ linearly independent vectors, thus $\Omega_X|_\Sigma$ induces a complex valued 2-form on Σ that does not vanish. Thus locally we can define an area form Ω_Σ by $\Omega_{\Sigma,x} = \xi(x)\Omega_{X,x}$ where ξ is a function with values in the unit circle, and whenever Σ is orientable or not, this induces (by taking the associated density $|\Omega_\Sigma|$ in the non-orientable case) the desired measure on Σ.

If X is an Enriques surface, the universal cover $q\colon X' \to X$ is an étale 2-to-1 cover by a K3 surface. If $\Sigma \subset X$ is a totally real surface, then its pre-image $\Sigma' = q^{-1}(\Sigma)$ is also totally real, and the automorphism of the covering q is an element of $\mathsf{Aut}(X'; \Sigma')$. Thus, applying the above construction in X' and pushing forward to X, we get an invariant measure on Σ as well.

Finally, if X is a blow-up of an abelian, K3, or Enriques surface, the same construction applies, except that the density of the associated volume form may vanish along the exceptional divisor of the blow-up.

2.2. **Rational surfaces.** The family of Coble surfaces (see [9, 11]) and the examples described by Blanc in [3] give rational surfaces X such that $\mathsf{Aut}(X)$ is non-elementary and contains parabolic elements (see [11], §3.4, and Example 6.8 below). They are constructing by blowing up a finite number of points in $\mathbb{P}^2$: the 10 double points of some rational sextic S for Coble surfaces; a finite number of points on a cubic curve C for Blanc surfaces. The strict transform S' and C' of these curves are preserved by $\mathsf{Aut}(X)$. Denote by K_X the canonical bundle of the surface.

(1) In the Coble case, there is a meromorphic section Ω of $K_X^{\otimes 2}$ that does not vanish and has a simple pole along S'.
(2) In Blanc's example, there is a meromorphic section Ω of K_X that does not vanish and has a simple pole along C'.

In both cases, Ω induces a natural measure on X: for Blanc surfaces, it is given by the form $\Omega \wedge \overline{\Omega}$; for Coble surfaces, it is given by $\Omega^{1/2} \wedge \overline{\Omega}^{1/2}$. In the Coble case, the total mass of this measure is finite, while in Blanc's example it is infinite. Moreover, if $\Gamma \subset \mathsf{Aut}(X)$ is any subgroup generated by parabolic elements, then Γ preserves this measure.

2.3. **Subgroups.** In each of the previous examples, one can replace $\mathsf{Aut}(X)$ by a **thin** subgroup, i.e. a subgroup $\Gamma \subset \mathsf{Aut}(X)$ of infinite index but with

$$\mathrm{Zar}(\mathsf{Aut}(X)^*)/\mathrm{Zar}(\Gamma^*) < +\infty; \tag{2.2}$$

here, $\mathrm{Zar}(\Gamma^*)$ is the Zariski closure in $\mathsf{GL}(H^2(X; \mathbf{R}))$. For instance, pick finitely many parabolic automorphisms g_i in $\mathsf{Aut}(X)$, and consider the group Γ generated by high powers g_i^m of the g_i. If the g_i do not preserve the same fibrations (see below), Γ is non-elementary; and if one chooses the g_i correctly, Γ is thin. In case of Wehler surfaces, it suffices to take $g_1 = \sigma_2 \circ \sigma_3$ and $g_2 = \sigma_3 \circ \sigma_1$ and $m = 3$.

3. THE DYNAMICS OF HALPHEN TWISTS

3.1. **Parabolic automorphisms and Halphen twists.** An automorphism f of a compact Kähler surface X is said **parabolic** if

$$\int_X (f^n)^* \kappa \wedge \kappa \asymp n^2 \tag{3.1}$$

for some (hence any) Kähler form κ on X; equivalently, some power $(f^m)^*$ of $f^* \in \mathsf{GL}(H^2(X; \mathbf{Z}))$ is unipotent and the maximal size of its Jordan blocks is equal to 3; equivalently, f^* acts on $H^{1,1}(X; \mathbf{R})$ as a parabolic isometry with respect to the intersection form given by the cup product (see [8]).

Theorem 3.1. *Let $f\colon X \to X$ be a parabolic automorphism of a compact Kähler surface.*

(1) *there exists a genus 1 fibration $\pi\colon X \to B$ and an automorphism f_B of the Riemann surface B such that $\pi \circ f = f_B \circ \pi$;*
(2) *if E is any (scheme theoretic) fiber of π, and F is a member of the linear system $|E|$, then F is a fiber of π;*
(3) *the foliation determined by the fibration π is the unique f-invariant complex analytic (smooth or singular) foliation on X;*
(4) *f_B has finite order, unless X is a compact torus.*

The existence of the invariant fibration is proven in [26] when X is a rational surface, and is easily obtained for other types of surfaces by the Riemann-Roch theorem (see [4, Proposition 1.4], and [8] for a survey). The second assertion is not specific to invariant fibrations, but is good to be kept in mind; phrased differently, it says that the space of sections $H^0(X; \mathcal{O}(E))$ is 2-dimensional, and the divisor of zeroes $\mathrm{div}(s)_0$ of any section $s \in H^0(X; \mathcal{O}(E))$ is a fiber of π. The uniqueness of the invariant foliation and the last assertion are proven in [12] (see Remark 3.10 below for the Kähler case).

When $f_B = \mathrm{id}_B$ we say that f is a **Halphen twist**. From the last item of the theorem, we see that if X is not a torus, for every parabolic f there exists $k \geqslant 1$ such that f^k is a Halphen twist. Beware that *this terminology may differ from other references.*

Remark 3.2. Consider a fiber E of π, as in Assertion (2) of Theorem 3.1. Its class $[E] \in H^{1,1}(X;\mathbf{R})$ generates a ray $\mathbf{R}_+[E]$ with the following property: if D is an effective divisor with $[D] \in \mathbf{R}_+[E]$, then D is mapped to a point by π. Thus, $\mathbf{R}_+[E]$ characterizes π as the unique fibration contracting these curves. In particular, an automrophism h of X permutes the fibers of π if and only if h^* preserves $\mathbf{R}_+[E] \subset H^{1,1}(X;\mathbf{R})$.

3.2. **Complex and real analytic structures on fibers.** Let $\pi\colon X \to B$ be a genus 1 fibration on a compact Kähler surface X (in this section, we do not assume that π is invariant by a parabolic automorphism). Our goal is to describe a foliation which is associated to π and the choice of a section of π (see [13, §2.1] for further details).

Denote by $\mathrm{Crit}(\pi) \subset B$ the set of critical values of π, and by B° the complement of this finite set. On $\pi^{-1}(B^\circ)$, π is a proper submersion and each fiber $X_w := \pi^{-1}(w)$ is a curve of genus 1. Let $U \subset B^\circ$ be an open subset, endowed with

- a holomorphic section $\sigma\colon U \to X$ of π above U,
- a continuous choice of basis of $H_1(X_w;\mathbf{Z})$, for $w \in U$.

If we declare that $\sigma(w)$ is the neutral element of X_w, then X_w becomes an elliptic curve for each $w \in U$. There is, therefore, a unique holomorphic function τ from U to the upper half-plane $\mathbb{H}_+ \subset \mathbf{C}$ such that

- for every $w \in U$, $X_w = \mathbf{C}/\mathrm{Lat}(w)$ where

$$\mathrm{Lat}(w) = \mathbf{Z} \oplus \mathbf{Z}\tau(w) \simeq H_1(X_w;\mathbf{Z}) \tag{3.2}$$

- the basis $(1,\tau(w))$ of $\mathrm{Lat}(w)$ corresponds to the chosen basis of $H_1(X_w;\mathbf{Z})$.

Indeed, $X_U := \pi^{-1}(U)$ is holomorphically equivalent to the quotient of $U \times \mathbf{C}$ by the action of $\mathbf{Z}^2$ defined by $(p,q)\cdot(w,z) = (w, z+p+q\tau(w))$ for $(p,q) \in \mathbf{Z}^2$ and $(w,z) \in U \times \mathbf{C}$.

In the real-analytic category all one-dimensional complex tori are equivalent to $\mathbf{R}^2/\mathbf{Z}^2$ as real Lie groups. Concretely, there is a unique isomorphism $\Psi_w\colon X_w \to \mathbf{R}^2/\mathbf{Z}^2$ which maps the basis $(1,\tau(w))$ of $\mathrm{Lat}(w)$ to the canonical basis $((1,0),(0,1))$ of $\mathbf{Z}^2$; in coordinates, if $\tau(w) = \tau_1(w) + \mathrm{i}\tau_2(w)$ and $z = x + \mathrm{i}y$, then

$$\Psi_w(z) = \left(x - \frac{\tau_1(w)}{\tau_2(w)} y,\ \frac{1}{\tau_2(w)} y \right). \tag{3.3}$$

The real analytic diffeomorphism $\Psi\colon \pi^{-1}(U) \to U \times \mathbf{R}^2/\mathbf{Z}^2$ defined by

$$\Psi(w,z) = (w, \Psi_w(z)) \tag{3.4}$$

is the unique homeomorphism such that (1) $\pi_1 \circ \Psi = \pi$, where π_1 is the first projection; (2) Ψ maps the basis of $H_1(X_w;\mathbf{Z})$ to the canonical basis of $\mathbf{Z}^2$; and (3) Ψ is an isomorphism of Lie groups in each fiber. In particular, $\Psi(w,\sigma(w)) = (w,(0,0))$.

In the following remarks, $\pi_2\colon U \times \mathbf{R}^2/\mathbf{Z}^2 \to \mathbf{R}^2/\mathbf{Z}^2$ denotes the second projection.

Remark 3.3. For any $(a,b) \in \mathbf{R}^2/\mathbf{Z}^2$, the holomorphic map $w \in U \mapsto a + b\tau(w) \in \mathbf{C}/\mathrm{Lat}(w)$ determines a local section of π above U; this section coincides with

$$\Psi^{-1}\{(w,(x,y))\ ;\ (x,y) = (a,b)\}. \tag{3.5}$$

So, if we consider the real analytic foliation of $U \times \mathbf{R}^2/\mathbf{Z}^2$ whose leaves are the fibers of π_2, and if we pull-back this foliation by Ψ, we get a real analytic foliation $\mathcal{F}_U$ of X_U with holomorphic leaves, which will be referred to as the (local) **Betti foliation**. Now, consider the holomorphic map $[k]_U\colon \pi^{-1}(U) \to \pi^{-1}(U)$ given by multiplication by some integer $k \geqslant 2$ along the fibers of π; by definition, it fixes $\sigma(U)$ pointwise. Then, $\mathcal{F}_U$ is invariant under the action of $[k]_U$. A leaf is pre-periodic if and only if it contains a torsion point of X_w, for some and then for any $w \in U$. The union of these preperiodic leaves is dense and each leaf of $\mathcal{F}_U$ is a limit of such leaves.

Remark 3.4. Suppose that X is projective and σ is the restriction of an algebraic multi-section of degree ℓ. This means that there is an irreducible curve C in X intersecting the general fiber of π in ℓ points such that the graph of σ is contained in C. Set $k = \ell + 1$. Then the multiplication map $[k]_U$ extends as a rational transformation $[k]_B\colon X \dashrightarrow X$. Indeed, if X_w is a general fiber and x is a point of X_w, there is a unique point y such that $(\ell+1)x - y$ is linearly equivalent to the divisor $C \cap X_w$: by definition, $[k]_B(x) = y$.

If C is a section, i.e. $\ell = 1$, $\mathcal{F}_U$ extends globally to a foliation $\mathcal{F}$ of $\pi^{-1}(B^\circ)$; we shall also refer to $\mathcal{F}$ as the (global) Betti foliation. The leaves of $\mathcal{F}$ corresponding to torsion points are, in fact, algebraic curves in X, since they correspond to the curves defined by $[2]_B^{m+q}(x) = [2]_B^m(x)$ for some $m \geqslant 0$ and $q \geqslant 1$. On the other hand, the local projections $\pi_2 \circ \Psi\colon \pi^{-1}(U) \to \mathbf{R}^2/\mathbf{Z}^2$ are not canonically defined; if γ is a loop in B°, with base point $w_0 \in U$, then the analytic continuation of $\pi_2 \circ \Psi$ along the loop is $M(\gamma) \circ \pi_2 \circ \Psi$, where $M(\gamma) \in \mathsf{SL}_2(\mathbf{Z})$ is given by the monodromy of the fibration (the determinant of $M(\gamma)$ is 1 because the orientation of the fibers is preserved). In other words, the monodromy of the fibration is induced by the holonomy of the Betti foliation. The section σ provides a fixed point $\sigma(w_0)$ of the holonomy; the curves of pre-periodic points of $[2]_B$ correspond simultaneously to finite orbits of the holonomy group and to torsion points of the fiber X_{w_0}.

On the other hand, if C is a multisection of degree $\ell \geqslant 2$, the Betti foliation $\mathcal{F}_U$ does not extend to B°: instead we obtain a web of degree at most ℓ, which is locally the superposition of the local Betti foliations associated to the ℓ choices of local sections whose graphs are contained in C.

Remark 3.5. (see [13]). The form $\pi_2^*(dx \wedge dy)$ is a smooth closed form. Its pull back to X_U is the local **Betti form** $\omega_{\mathrm{B}} = \Psi^*\pi_2^*(dx \wedge dy)$: (1) ω_{B} is a closed semi-positive $(1,1)$-form; (2) it vanishes along the leaves of $\mathcal{F}_U$ (its kernel coïncides with $T\mathcal{F}_U$); and (3) for $w \in B^\circ$, $\omega_{\mathrm{B}|X_w}$ is the unique translation invariant form of type $(1,1)$ such that

$\int_{X_w} \omega_{\mathrm{B},} = 1$. These properties characterize $\omega_{\mathrm{B},}$. If there is a global section, these forms patch together to define a global real analytic Betti form ω_{B} on X_{B° (the monodromy group is contained in $\mathsf{SL}_2(\mathbf{Z})$, so it preserves $dx \wedge dy$).

Note that the Betti form and Betti foliation depend on the choice of a section, but not on the choice of a basis of $H_1(X_{w_0}; \mathbf{Z})$.

3.3. **Singular fibers.** Our goal in this section is to collect some facts concerning the geometry of a genus 1 fibration $\pi\colon X \to B$ around one of its singular fibers. Furthermore, if f is a Halphen twist preserving π, we describe how its dynamical properties degenerate at a singular fiber, and how they are affected by the stabilization process, which reduces a singular fiber to a canonical model (see below). Of particular interest to us is the set of points $w \in B$ such that the orbits of f in X_w are finite, or dense, or have a closure of dimension 1 (cf. Section 3.4). A first instance of stabilization is when π is not relatively minimal, that is when there is an exceptional curve of the first kind E contained in a fiber of π. There are finitely many such curves, so f permutes them, and some positive iterate f^m fixes each of them. Thus, one can contract E in an f^m-equivariant way, to end up with a birational morphism $\varepsilon\colon X \to X'$, a fibration $\pi'\colon X' \to B$ such that $\pi' \circ \varepsilon = \pi$, and an automorphism f' of X' such that $\varepsilon \circ f^m = f' \circ \varepsilon$. The dynamical properties of f' are the same as the ones of f: for example, the parameters $w \in B^\circ$ such that each orbit of f in X_w is dense coïncide with the parameters for which the orbits of f' in X'_w satisfy the same property.

The local geometry of π around a critical value $s \in \mathrm{Crit}(\pi)$ was described by Kodaira. The reader is referred to [1] for details, in particular Sections III.10, V.9, and V.10 there. From the above discussion, we may *assume that X is relatively minimal*. We fix $s \in \mathrm{Crit}(\pi)$ and further assume that X_s is not a multiple fiber; the adaptation to the case of a multiple fiber will be described in § 3.3.3

3.3.1. *Local sections.* A first observation is that when X_s is not a multiple fiber there exists a local section of the fibration π around s. More precisely, for every component C of multiplicity 1 of $\pi^{-1}(s)$, any small disk transverse to C is the graph of a section σ; and by Kodaira's classification, such a component always exists (see [1, §V.7]). Let us fix a small open disk $V \subset B$ around s, such that $\overline{V} \cap \mathrm{Crit}(\pi) = \{s\}$, together with such a local section $\sigma\colon V \to X$. Set $X_V = \pi^{-1}(V)$ and let $X_V^\sharp$ be the complement in X_V of the irreducible components of X_s that do not intersect $\sigma(V)$ (resp. of the singular point of X_s if X_s is irreducible); this set depends the chosen section. In other words, we keep from X_s the smooth locus of the unique component intersecting $\sigma(V)$; we shall denote by $X_s^\sharp \subset X_V^\sharp$ this residual curve.

3.3.2. *Type I_b.* The main example of singular fibers are those of type I_b, with $b \in \mathbf{N}^*$ (type I_0 corresponds to the smooth case). For $b = 1$, X_s is a rational curve with a unique normal crossing singularity; when $b \geqslant 2$, X_s is a cycle of b smooth rational curves of self-intersection -2. So, $X_s^\sharp$ is biholomorphic to $\mathbf{C}^\times = \mathbb{P}^1(\mathbf{C})\setminus\{0, \infty\}$. Shrinking V if necessary, we can identify (V, s) with a disk $(\mathbb{D}_R, 0)$ of radius $R < 1$, and $X_V^\sharp$ with the quotient of $\mathbb{D}_R \times \mathbf{C}$ by the family of lattices $\mathrm{Lat}(w) = \mathbf{Z} \oplus \mathbf{Z}\tau(w)$ given by

$$\tau(w) = \frac{b}{2\mathsf{i}\pi}\log(w), \quad \text{for } w \in \mathbb{D}_R \tag{3.6}$$

(note that $\log(w)$ is not well-defined but $\mathrm{Lat}(w)$ is). For $w = 0$, the lattice $\mathrm{Lat}(w)$ degenerates to $\mathrm{Lat}(0) = \mathbf{Z} \subset \mathbf{C}$. If γ is a loop making one positive turn around s, the monodromy $M(\gamma)$ maps the basis $(1, \tau(w_0))$ to $(1, \tau(w_0) + b)$.

Let $t\colon V \to \mathbf{C}$ be a holomorphic function. The transformation $g\colon V \times \mathbf{C} \to V \times \mathbf{C}$ defined by $g(w, z) = (w, z + t(w))$ induces, by taking the quotient, a holomorphic diffeomorphism of $X_V^\sharp$. By [1, Prop. III.(8.5)], it extends to a diffeomorphism of X_V that preserves π. Conversely, if f is a holomorphic diffeomorphism of X_V that preserves each fiber of π, some positive iterate f^m of f preserves each component of X_s, and then f^m maps σ to another section $f^m \circ \sigma$ intersecting $X_s^\sharp$. Lifting to the universal cover, we see that there is a holomorphic function $t\colon V \to \mathbf{C}$ such that f is induced by $(w, z) \mapsto (w, z + t(w))$; the function $t(w) = f(\sigma(w)) - \sigma(w)$ can be viewed as a section of the Jacobian fibration associated to π (see [1, §V.9] for details).

Consider the map

$$(w, z) \in \mathbb{D}_R \times \mathbf{C} \mapsto (w, v) := (w, \exp(2\mathsf{i}\pi z)) \in \mathbb{D}_R \times \mathbf{C}^\times. \tag{3.7}$$

It is the quotient map for the action of $\mathbf{Z} \subset \mathrm{Lat}(w)$ on $\mathbf{C}$ by integral translations. If $w \neq 0$, the vertical fiber $\{w\} \times \mathbf{C}^\times$ is mapped in X_V to the elliptic curve $\mathbf{C}^\times/w^{b\mathbf{Z}} \simeq X_w$ (because $\exp(2\mathsf{i}\pi\tau(w)) = w^b$). The fiber $\{0\} \times \mathbf{C}^\times$ is mapped injectively onto the central fiber $X_s^\sharp$ of $X_V^\sharp$.

Consider the Betti foliation $\mathcal{F}$ defined in $\pi^{-1}(V\setminus\{s\})$ by the choice of the section σ. In $\mathbb{D}_R^* \times \mathbf{C}$, the leaves of $\mathcal{F}$ correspond to the curves $(w, c + d\tau(w))$, for $(c, d) \in \mathbf{R}^2$. They are mapped in $\mathbb{D}_R^* \times \mathbf{C}^\times$ to the curves $\gamma_{c,d}(w) = (w, \exp(2\mathsf{i}\pi c)w^{bd})$; here, $|\exp(2\mathsf{i}\pi c)| = 1$ because $c \in \mathbf{R}$ and w^{bd} is multivalued as soon as $bd \notin \mathbf{Z}$. Let us describe the local dynamics of $\mathcal{F}$ around X_s. To simplify the exposition, we contract the components of X_s that do not intersect the neutral section $\sigma(V)$ onto a point q; this gives a new surface $\overline{X}_V$. The central fiber of $\overline{X}_V$ is irreducible and q is its unique singularity; when $b \geqslant 2$, q is also a singular point of $\overline{X}_V$. By construction, $X_V^\sharp$ is biholomorphically equivalent to $\overline{X}_V\setminus\{q\}$. When $d = 0$, the leaf defined by the curve $\gamma_{c,0}$ extends to a local holomorphic section of π, given by $(w, v) = (w, \exp(2\mathsf{i}\pi c))$; the union of these curves is, locally, an $\mathcal{F}$-invariant real 3-manifold which intersects the central fiber $X_s^\sharp \simeq \mathbf{C}^\times$ along the unit circle $\{v \in \mathbf{C}^\times \; ; \; |v| = 1\}$. When d is rational, $\gamma_{c,d}(w) = (w, \exp(2\mathsf{i}\pi c)w^{bd})$ extends to a local multisection of π; when σ is the restriction of a global section of π to the disk $V \subset B$, this local multisection $\gamma_{c,d}$ extends to an algebraic curve of X (a pre-periodic curve for $[2]_B$, see Remarks 3.3 and 3.4). Finally, when $d \in \mathbf{R}\backslash\mathbf{Q}$, $\gamma_{c,d}$ is a transcendental and multivalued curve; in $\overline{X}_V$, the singularity q is the unique limit point of this curve on the central fiber.

Remark 3.6. In $\mathbb{D}_R^* \times \mathbf{C}^\times$,

$$\tilde{\alpha}_V = \log(|v|)\frac{dw}{w} - \log(|w|)\frac{dv}{v} \tag{3.8}$$

is a real analytic $(1,0)$-form that vanishes along the curves $\gamma_{c,d}$. Being invariant under the transformation $(w, v) \mapsto (w, w^b v)$, it induces by taking the quotient a $(1,0)$-form α_V on $\pi^{-1}(V\backslash\{s\})$, the kernel of which coincides with the tangent space of the Betti foliation. To get the Betti form ω_{B} defined by π and σ, one needs to multiply $\alpha_V \wedge \overline{\alpha_V}$ by a factor $\varphi(w)$ to ensure $\int_{X_w} \varphi(w)\alpha_V \wedge \overline{\alpha_V} = 1$ for every w in $V\backslash\{s\}$. The result is

$$\omega_{\mathrm{B}} = \frac{\mathsf{i}}{2\pi} \cdot \frac{b}{2(\log|w|)^3}\alpha_V \wedge \overline{\alpha_V}. \tag{3.9}$$

3.3.3. *Multiple fibers (see* [1, §III.9 and V.10]*).* Let us assume in this paragraph that X_s is a multiple fiber; it is necessarily of type mI_b for some $b \geqslant 0$. Let us do a local base change under the map $p\colon \zeta \mapsto \zeta^m = w$; in other words, we consider the surface X'_V given locally above $V \simeq \mathbb{D}_R$ by $X'_V = \{\, (\zeta, x) \in \mathbb{D}_R \times X_V \; ; \; \pi(x) = \zeta^m\}$, together with the projection $\pi'\colon X'_V \to \mathbb{D}_R$ defined by $\pi'(\zeta, x) = \zeta$. Then, the map $P\colon (\zeta, x) \in X'_V \mapsto x \in X_V$ satisfies $\pi \circ P = p \circ \pi'$. The surface X'_V may be singular, so we let X''_V be the normalization of X'_V and $X_V^{(m)}$ be the minimal resolution of X''_V; there is a natural fibration $\pi^{(m)}\colon X_V^{(m)} \to \mathbb{D}_R$ and a natural map $P^{(m)}\colon X_V^{(m)} \to X_V$ such that $\pi \circ P^{(m)} = p \circ \pi^{(m)}$. Now, it turns out that $\pi^{(m)}$ has no multiple fiber and that its central fiber (the unique possible singular fiber above V) is of type I_b.

If f is a holomorphic diffeomorphism of X_V such that $\pi \circ f = \pi$, then f can be lifted to a holomorphic diffeomorphism $f^{(m)}$ of $X_V^{(m)}$ such that $P^{(m)} \circ f^{(m)} = f$. First, one lifts f to X'_V by $(w, z) \mapsto (w, f(z))$ and then to the normalization and its minimal resolution. Conversely, one recovers X_V by taking the quotient of $X^{(m)}$ by the action of a finite group $\mathbf{Z}/m\mathbf{Z}$ that commutes to $f^{(m)}$. Thus, to study the local dynamics around multiple fibers, one only needs to study the case of fibers of type I_b (including smooth fibers), and take a quotient by such a finite group.

3.3.4. *Unstable fibers (see* [1, §III.10 and V.10]*).* Let us now assume that X_s is not multiple and not of type I_b; it is an unstable fiber of type II, III, IV, or I_b^*, II^*, III^*, IV^*. As in the previous paragraph, a local base change can be performed to end up with a local stable fibration; its central fiber will be smooth, except for the types I_b^*, $b \geqslant 1$ which lead to a central fiber of type I_b. To do so, one first blows up the central fiber to ensure that its singularities are nodes, which gives rise to a new surface $\overline{X}_V$; then, one does a base change to construct a new surface $\overline{Y}_V$ (as above with $X^{(m)}$); a priori, the induced fibration on $\overline{Y}_V$ is not relatively minimal anymore, so one contracts curves in the central fiber to construct a surface Y_V with a relatively minimal fibration. Finally, X_V can be recovered from Y_V by taking a finite quotient, however only up to bimeromorphic equivalence (see [1, §III.10 and V.10]).

After taking some positive iterate f^m, so that f^m fixes each irreducible component of the singular fiber, the holomorphic diffeomorphism can be lifted to a holomorphic diffeomorphism of Y_V; indeed, f^m induces a meromorphic map of Y_V, and this map is a local diffeomorphism by [1, Prop. III(8.5)]. Thus, to study the dynamical properties of f we can focus locally on regular fibrations and singular fibrations of type I_b, and then take a quotient by a finite group. Moreover, this finite group acts by multiplication by a root of unity on the base $V \simeq \mathbb{D}_R$.

3.4. **The dynamics of Halphen twists: twisting property.** We pursue the study of a Halphen twist $f\colon X \to X$ and of its invariant fibration $\pi\colon X \to B$. Let $U \subset B^\circ$ be an open disk, endowed with a section $\sigma\colon U \to X$ of π and a continuous choice of basis of $H_1(X_w; \mathbf{Z})$, for $w \in U$. As in Section 3.2, there is a holomorphic function $\tau\colon U \to \mathbb{H}_+$ such that the fibers X_w can be idendified to $\mathbf{C}/\mathrm{Lat}(w)$, where $\mathrm{Lat}(w) = \mathbf{Z} + \mathbf{Z}\tau(w)$. Along each fiber $X_w = \mathbf{C}/\mathrm{Lat}(w)$, f can be expressed in the coordinate $z \in \mathbf{C}$ as $\xi z + t(w)$, where $t\colon U \to \mathbf{C}$ is holomorphic. Here, ξ is a root of unity (of order dividing 12) which is determined by the action of f on $H_1(X_w; \mathbf{Z}) \simeq \mathbf{Z}^2$ and, as such, is locally constant; if $\xi \neq 1$, f has finite order on each fiber X_w, $w \in U$, so f is periodic, and this contradicts the parabolicity assumption. Thus, $f(z) = z + t(w)$ along X_w, and *f acts by translation along each fiber X_w, $w \in U$, of π.* Now, conjugating f locally by the diffeomorphism $\Psi\colon \pi^{-1}(U) \to U \times \mathbf{R}^2/\mathbf{Z}^2$ introduced in Section 3.2, f becomes

$$f_\Psi(w,(x,y)) = (w,(x,y) + T(w)) \tag{3.10}$$

for some real analytic map $T\colon U \to \mathbf{R}^2$. The following lemma says that "$t(w)$ *varies independently from* $\tau(w)$".

Lemma 3.7. *The analytic map $w \in U \mapsto T(w) \in \mathbf{R}^2/\mathbf{Z}^2$ is not constant.*

This is shown in [5], proof of Proposition 2.2, but the proof makes use of a global multisection of π, which exists if and only if X is projective[2]. For completeness, we present an argument which is more straightforward and applies to all Kähler surfaces.

[2]The proof in [5] is written for K3 surfaces but extends to other projective surfaces

Remark 3.8. If σ is replaced by another local section σ', the diffeomorphism Ψ is replaced by $\Psi' = \Phi \circ \Psi$, with $\Phi(w,(x,y)) = (w,(x,y)+S(w))$ for some analytic map $S\colon U \to \mathbf{R}^2/\mathbf{Z}^2$. Then, f_Ψ is changed into $\Phi \circ f_\Psi \circ \Phi^{-1}$ and T *is unchanged*. Likewise, if the basis of $H_1(X_w;\mathbf{Z})$ is changed, T is mapped to $A \circ T$ for some $A \in \mathsf{GL}_2(\mathbf{Z})$. Thus, the property that T is locally constant does not depend on the choices we made. Moreover, T is constant on U if and only if f preserves the Betti foliation $\mathcal{F}_U$ associated to σ; thus, if this property holds above U for some choice of section, then it holds above any disk $U' \subset B^\circ$ and for any choice of local section.

Proof of Lemma 3.7. We fix a Kähler form κ on X, and compute the norm of the tangent map $\|Df\|$ with respect to κ. Assume that T is constant on U. Then at each point $(w,(x,y))$ of $U \times \mathbf{R}^2/\mathbf{Z}^2$, the tangent map $(Df_\Psi)_{(w,(x,y))}$ is the identity; this implies that $\|Df^n\|$ is uniformly bounded above U, independently of n. By Remark 3.8, this property propagates over B°: if $K \subset B^\circ$ is any compact subset, there is a constant $C(K)$ such that $\|Df^n_x\| \leqslant C(K)$ for all $n \geqslant 0$ and all $x \in \pi^{-1}(K)$.

Let us now study the behavior of f and Df near a singular fiber X_s of type I_b, for some $s \in \mathrm{Crit}(\pi)$. Let V be a small disk around s, endowed with a section $\sigma\colon V \to X$, as in § 3.3.2. We identify V to $\mathbb{D}_R$ and $X_V^\sharp$ to the quotient of $\mathbb{D}_R \times \mathbf{C}$ by the family of lattices $\mathrm{Lat}(w) = \mathbf{Z} + \mathbf{Z}\tau(w)$, with $\tau(w)$ as in Equation (3.6). Then, we change f into a positive iterate to assume that it fixes each irreducible component of the central fiber X_s. From Remark 3.8 we know that the local Betti foliation $\mathcal{F}$ defined above $V\setminus\{s\}$ is f-invariant. As shown in Section 3.3.2, the leaves of $\mathcal{F}$ which extend to local sections of $X_V^\sharp$ foliate a unique local 3-manifold (that intersects $X_s^\sharp$ on a circle). Since f is a diffeomorphism preserving $\mathcal{F}$ and π, it preserves this 3-manifold. In the local coordinates $(w,v) \in \mathbb{D}_R \times \mathbf{C}^\times$ introduced in Section 3.3.2, this means that f acts by $(w,v) \mapsto (w,h(w)v)$ for some holomorphic map $w \in \mathbb{D}_R \to h(w) \in \mathbf{C}^\times$ taking its values in the unit circle. This shows that h is a constant of modulus 1, and that the iterates of f are locally contained, above V, in a compact group of local diffeomorphisms. Thus, $\|Df^n\|$ is also uniformly bounded around every singular fiber of type I_b.

As explained in §3.3.4, we can contract curves and perform a stable reduction to get an automorphism g of a compact Kähler surface Y preserving a fibration $\pi'\colon Y \to B'$, a finite map $\eta\colon B' \to B$, and a meromorphic, dominant map $\varepsilon\colon Y \dashrightarrow X$ such that $\pi \circ \varepsilon = \eta \circ \pi'$ and $\varepsilon \circ g = f \circ \varepsilon$. The fact that T is constant on some open set is an intrinsic property, so it also holds for g. Since all singular fibers of π' are now of type I_b, the previous argument shows that $\|Dg^n\|$ is uniformly bounded on Y; in particular, $\|(g^n)^*\|_{H^2(Y;\mathbf{R})}$ is uniformly bounded. Let us show that this contradicts the parabolic behavior of f. Write ε as a composition $\varphi \circ \psi^{-1}$ where $\varphi\colon Z \to X$ is regular and $\psi\colon Z \to Y$ is a bimeromorphic morphism. Set $h = \psi^{-1} \circ g \circ \psi$, which is a bimeromorphic map of Z. By [20, Prop. 1.15], $\|(h^n)^*\|_{H^{1,1}(Z;\mathbf{R})}$ is bounded. Then,

$$\int_Z ((h^n)^\star \varphi^\star \kappa) \wedge (\varphi^\star \kappa) = \int_Z \varphi^\star \left((f^n)^\star \kappa \wedge \kappa \right) = \deg(\varphi) \int_X (f^n)^\star \kappa \wedge \kappa. \tag{3.11}$$

The left hand side is bounded because $\|(h^n)^*\|_{H^{1,1}(Z;\mathbf{R})}$ is bounded. This contradicts the parabolic behavior of f and concludes the proof. □

Lemma 3.9. *The differential $DT_w \colon T_wU \to \mathbf{R}^2$ of T has rank 2 everywhere, except for finitely many points $w_j \in U$ at which $DT_{w_j} = 0$. The analytic map $T \colon U \to \mathbf{R}^2$ is an open mapping.*

If $DT_w = 0$, we shall say that f **does not twist** along the fiber X_w; this notion is intrinsic: it does not depend on the choice of the local diffeomorphism Ψ (the argument is the same as that of Remark 3.8). We define the set of non-twisting points by

$$\mathrm{NT}_f = \{w \in B \; ; \; DT_w = 0\}. \tag{3.12}$$

Proof of Lemma 3.9. Since f is holomorphic, its differential commutes with the complex structure j_X of X. This implies that $DT_w \circ \mathrm{j} = j(w) \circ DT_w$ where j is the complex structure on $U \subset B$ and $j(w)$ is a complex structure on $\mathbf{R}^2/\mathbf{Z}^2$ that depends on w (it is the conjugate of the complex structure of $\mathbf{C}/\mathrm{Lat}(w)$ by the linear map $D\Psi_w$). As a consequence, the rank of DT_w is equal to 2 or 0. Assume that the real analytic set $\{w \in U \; ; \; DT_w = 0\}$ contains some connected real analytic curve $C \subset U$. Along C, T is a constant T_0; then, the image $f_\Psi(\{(w,(0,0)) \; ; \; w \in U\})$ of the zero section intersects the horizontal disk $\{(w,(x,y)) \; ; \; (x,y) = T_0\}$ on a real analytic curve. Now, let us come back to the complex surface X. The image $f(\sigma(U))$ of the zero section and, by Remark 3.3, the set $\Psi^{-1}\{(w,(x,y)) \; ; \; (x,y) = T_0\}$ are two connected complex analytic curves. Since they intersect along a non-discrete subset, they coincide; this implies that T is constant, in contradiction with Lemma 3.7. Thus, the set of points at which DT_w has rank < 2 is locally finite. The last assertion follows easily from the first. □

Remark 3.10. Lemma 3.9 shows that *every f-invariant holomorphic foliation $\mathcal{G}$ on X is given by the invariant fibration π.* Indeed, if $T(w) \in \mathbf{Q}^2/\mathbf{Z}^2$ and $DT_w \neq 0$, then a positive iterate f^m fixes X_w pointwise and, at every point $x \in X_w$, $\mathrm{Ker}(D\pi_x) \subset T_xX$ is the unique line invariant by the tangent map Df^m_x; thus $\mathcal{G}$ must be everywhere tangent to X_w and X_w must be a leaf of $\mathcal{G}$. Since T is open, $T^{-1}(\mathbf{Q}^2/\mathbf{Z}^2)$ is dense in U, so $\mathcal{G}$ coincides with the fibration above U, hence everywhere on X. This proves Assertion (3) of Theorem 3.1 for compact Kähler surfaces ([12] considered only projective surfaces).

Lemma 3.9 does not say that NT_f is finite: it could cluster at a critical value of π. To exclude this possibility, let us first reformulate the twisting property. Consider an open set $U \subset B$ endowed with a section σ and the induced local Betti foliation $\mathcal{F}$.

Its image $f_*(\mathcal{F})$ is generically transverse to $\mathcal{F}$; these foliations are tangent at $x \in X_U$ if, and only if $DT_{\pi(x)} = 0$ if, and only if they are tangent along the fiber $X_{\pi(x)}$. So, $\pi^{-1}(\mathrm{NT}_f) = \mathrm{Tang}(\mathcal{F}, f_*(\mathcal{F}))$.

Lemma 3.11. *Let $s \in \mathrm{Crit}(\pi)$ be the projection of a fiber of type I_b, $b \geqslant 1$. Let V be an open disk containing s, with a section $\sigma\colon V \to X$ of π; let $\mathcal{F}$ be the Betti foliation determined by σ above $V \setminus \{s\}$. If V is small enough, $f_*(\mathcal{F})$ is everywhere transverse to $\mathcal{F}$ above $V \setminus \{s\}$.*

Proof. We make use of Remark 3.6, and work in $\mathbb{D}_R \times \mathbf{C}^\times$. The automorphism f is given by $f(w, v) = (w, h(w)v)$ for some holomorphic function $h(w) = \exp(2\mathrm{i}\pi t(w))$ that does not vanish. In these coordinates, the neutral section σ is given by $\sigma(w) = (w, 1)$, and its image under f is the curve $f \circ \sigma\colon w \mapsto (w, \exp(2\mathrm{i}\pi t(w)))$. The points w above which $\mathcal{F}$ is tangent to $f_*\mathcal{F}$ are the points where the image of $f \circ \sigma$ is tangent to $\mathcal{F}$; they are determined by the equality $\tilde{\alpha}_V((f \circ \sigma)'(w)) = 0$, with $\tilde{\alpha}_V$ as in Equation (3.8). This leads to the constraint

$$-\mathrm{i}\log(|w|)wt'(w) = \mathsf{Im}(t(w)); \tag{3.13}$$

writing $t(w) = \alpha w^k + \mathrm{h.o.t.}$, for some k in $\mathbf{N}$, we get $-\mathrm{i}\alpha k \log(|w|)w^k \simeq \mathsf{Im}(\alpha w^k)$. As a consequence there is no solution close to the origin, and the proof is complete. □

According to Section 3.3, the local dynamics of f near an unstable or multiple fiber is covered by the dynamics of a parabolic automorphism near a curve of type I_b. So, the next result is a corollary of the previous lemmas. (Note that the existence of a section in Assertion (2) implies that X is projective and that π does not have multiple fibers[3].)

Proposition 3.12. *Let f be a parabolic automorphism of a compact Kähler surface X, acting trivially on the base of its invariant fibration $\pi\colon X \to B$.*

(1) *The set of fibers $\pi^{-1}(w)$ along which f does not twist is finite, i.e. NT_f is finite.*
(2) *If π admits a global section $\sigma\colon B \to X$ and $\mathcal{F}$ is the associated Betti foliation on $\pi^{-1}(B^\circ)$, then $\mathrm{Tang}(\mathcal{F}, f_*(\mathcal{F}))$ is a finite union of fibers.*

3.5. **The dynamics of Halphen twists: orbit closure.** We keep the notation from Section 3.4: we let $U \subset B^\circ$ be a disk, on which a continuous choice of basis for $H_1(X_w; \mathbf{Z})$ and a section of π have been chosen, so that $H_1(X_w; \mathbf{Z}) \simeq \mathbf{Z}^2$ and $X_w \simeq \mathbf{R}^2/\mathbf{Z}^2$.

[3]Indeed, if $D \subset X$ is the graph of a section and F is a fiber, then $F \cdot D = 1$, so F can not be multiple; and if F is reducible, D intersects F along a component of multiplicity 1. Moreover, if $a \in \mathbf{Z}_+$ is large enough, then $aF + D$ is big and nef.

3.5.1. A proper and closed subgroup of X_w is finite or 1-dimensional.

A closed 1-dimensional subgroup $L \subset X_w$ is characterized by two data: a slope (p,q), given by a primitive vector in $\mathbf{Z}^2$, and the number $k \geqslant 1$ of connected components of L. The connected component of the identity $L^0 \subset L$ is the kernel of the homomorphism $X_w \simeq \mathbf{R}^2/\mathbf{Z}^2 \to \mathbf{R}/\mathbf{Z}$ defined by

$$(x,y) \mapsto qx - py, \tag{3.14}$$

and L is the preimage of the unique cyclic subgroup of order k in $\mathbf{R}/\mathbf{Z}$. Equivalently, L is the kernel of $(x,y) \mapsto k(qx-py)$. We denote this subgroup by $L_w(k,(p,q))$. The integer k is intrinsically defined, but the slope depends on the basis of $H_1(X_w;\mathbf{Z})$.

Notation 3.13. For $z \in X$, we denote by $L_f(z)$ the closure of the orbit of z, and by $L_f^0(z)$ the connected component of z in $L_f(z)$. For w in B, we denote by f_w the restriction of f to the fiber X_w. If $\pi(z) \notin \mathrm{Crit}(\pi)$, $f_{\pi(z)}$ is a translation in $X_{\pi(z)}$; thus $L_f(z)$ is either finite, or a translate of a 1-dimensional subgroup of $X_{\pi(z)}$, or equal to $X_{\pi(z)}$. By definition a translate of a connected, closed, 1-dimensional subgroup of $X_{\pi(z)}$ will be called a **circle**.

3.5.2. Define

$$\begin{aligned} \mathrm{Tor}(U) &= \{w \in U \; ; \; f_w \colon X_w \to X_w \text{ is a periodic translation}\} \\ &= \{w \in U \; ; \; t(w) \text{ has finite order in } X_w = \mathbf{C}/\mathrm{Lat}(w)\} \\ &= \{w \in U \; ; \; T(w) \in \mathbf{Q}^2/\mathbf{Z}^2\}. \end{aligned} \tag{3.15}$$

This set is intrinsically defined (it does not depend on the section or on the basis of $H_1(X_w;\mathbf{Z})$). By Lemma 3.9, $\mathrm{Tor}(B^\circ)$ is a countable and dense subset of B°. A point w belongs to $\mathrm{Tor}(B^\circ)$ if and only if the orbit of every $z \in X_w$ is a finite subset of X_w.

3.5.3. The next level of complexity is when $t(w)$ belongs to a 1-dimensional subgroup of X_w. Write $f_\Psi(w,(x,y)) = (w,(x,y)+T(w))$. For (α,β) in $\mathbf{Q}^2/\mathbf{Z}^2$ and $(p,q) \in \mathbf{Z}^2 \backslash \{(0,0)\}$, we set

$$\mathrm{R}_{p,q}^{\alpha,\beta}(U) = \{w \in U \; ; \; T(w) \in (\alpha,\beta) + \mathbf{R}\cdot(p,q)\}. \tag{3.16}$$

This set depends only on the slope (p,q) and on $q\alpha - p\beta \in \mathbf{R}/\mathbf{Z}$, but not really on (α,β) itself. Using holomorphic coordinates, this real analytic curve $\mathrm{R}_{p,q}^{\alpha,\beta}(U)$ is alternatively expressed as

$$\mathrm{R}_{p,q}^{\alpha,\beta}(U) = \{w \; ; \; \text{the complex numbers } t(w) - \sigma_{(\alpha,\beta)}(w) \text{ and } p + q\tau(w) \text{ are } \mathbf{R}\text{-collinear}\},$$

where $\sigma_{(\alpha,\beta)}$ is the local holomorphic section of π defined by

$$\sigma_{(\alpha,\beta)}(w) = \Psi^{-1}(w,(\alpha,\beta)) = \alpha + \beta\tau(w) \mod \mathrm{Lat}(w); \tag{3.17}$$

here, as in Equation (3.4) , Ψ_w maps the basis $(1, \tau(w))$ of $\mathrm{Lat}(w) \subset \mathbf{C}$ onto the basis $((1,0),(0,1))$ of $\mathbf{Z}^2 \subset \mathbf{R}^2$. This equation of $R^{\alpha,\beta}_{p,q}$ can be written locally as

$$\mathsf{Im}\left(\frac{t(w) - \sigma_{(\alpha,\beta)}(w)}{p + q\tau(w)}\right) = 0. \tag{3.18}$$

This curve may have singularities, as does the curve defined by $\mathsf{Im}(w^k) = 0$ in the unit $\mathbb{D}$ when $k \geqslant 2$; this happens precisely when $w \in \mathrm{NT}_f$.

Let $\mathrm{R}^k_{p,q}(U) \subset U$ be the closure of the set of points w for which $L_f(z)$ has k connected components of slope (p,q) for all $z \in X_w$; it is the (finite) union of the $\mathrm{R}^{\alpha,\beta}_{p,q}(U)$, for all (α,β) in $\mathbf{Q}^2/\mathbf{Z}^2$ such that $q\alpha - p\beta$ has order k in $\mathbf{R}/\mathbf{Z}$. Thus, $\mathrm{R}^k_{p,q}(U)$ is an analytic curve in U. For $w \in \mathrm{R}^k_{p,q}(U) \setminus \mathrm{Tor}(U)$ and $z \in X_w$, $L_f(z)$ is a translate of $L_w(k,(p,q))$.

When performing an analytic continuation of $\mathrm{R}^k_{p,q}(U)$ around a critical value of π, the continuation may hit U again along a component of $\mathrm{R}^k_{p',q'}(U)$ for some new slope (p',q'); the vector (p',q') is in the orbit of (p,q) under the action of the monodromy group of the fibration. Since the orbit of (p,q) is typically infinite, the analytic continuation could a priori intersect U on infinitely many distinct $\mathrm{R}^k_{p',q'}(U)$.

Finally, for each integer $k \geqslant 1$, we set

$$\mathrm{R}^k(U) = \bigcup_{(p,q)\in\mathbf{Z}^2 \text{ primitive}} \mathrm{R}^k_{p,q}(U), \quad \text{and} \quad \mathrm{R}(U) = \bigcup_{k\geqslant 1} \mathrm{R}^k(U). \tag{3.19}$$

These sets are intrinsically defined. Intuitively $\mathrm{R}^k(U)$ should be thought as the set of $w \in U$ such that for $z \in X_w$, $L_f(z)$ is a union of k circles; formally, this is not a correct characterization of $\mathrm{R}^k(U)$ because $\mathrm{R}^k(U)$ contains points of $\mathrm{Tor}(U)$.

3.5.4. Summarizing this discussion, and keeping the same notation, we obtain:

Lemma 3.14. *Let f be a parabolic automorphism of a compact Kähler surface acting trivially on the base of its invariant fibration $\pi\colon X \to B$. Let $U \subset B^\circ$ be an open disk.*

(1) *The set $\mathrm{Tor}(B^\circ)$ is a dense and countable subset of B, and $w \in \mathrm{Tor}(B^\circ)$ if and only if the translation f_w is periodic, if and only if $L_f(z)$ is finite for every $z \in X_w$.*
(2) *For each slope (p,q), the set $\mathrm{R}^k_{p,q}(U)$ is either empty, or a (possibly singular) real analytic curve; the set $\mathrm{R}^k(U)$ is the union of these curves. Moreover, $w \in \mathrm{R}(B^\circ) \setminus \mathrm{Tor}(B^\circ)$ if and only if $L^0_f(z)$ is a circle for each z in X_w if and only if the closure of each orbit of f_w is a union of circles embedded in X_w;*
(3) *If $w \in B^\circ \setminus \mathrm{R}(B^\circ)$, then each orbit of f_w is dense in X_w.*

In the second case, every f_w-invariant and ergodic probability measure is the Lebesgue measure on $L_f(z)$ for some $z \in X_w$; in the third case, the only f-invariant probability measure on X_w is the Lebesgue measure.

3.6. **Additional notes on the curves** $\mathrm{R}^{\alpha,\beta}_{p,q}(U)$**.** Fix a finite set of real analytic arcs (slits) $\gamma_j \subset B$ intersecting transversally, and containing the critical values of π, such that the complement of $\bigcup \gamma_j$ in B is topologically a disk. Denote this disk by U. Then, choose a section σ of π and a continuous family of basis for $H_1(X_w; \mathbf{Z})$ above U. Each $\mathrm{R}^{\alpha,\beta}_{p,q}(U)$ is a real analytic curve in U; but when crossing an arc γ_j, the analytic continuation of $\mathrm{R}^{\alpha,\beta}_{p,q}(U)$ must be glued to another $\mathrm{R}^{\alpha',\beta'}_{p',q'}(U)$.

In §6.2, we will have to understand the local structure of these curves near critical values of π and, as before, the important case is that of singular fibers of type I_b. Here, we content ourselves with a few simple remarks, which help understand some of the subtleties of the problem of the semi-analytic extension of the curves σ_g in Section 6.

Choose a local coordinate w as in § 3.3.2 and, to fix the ideas, suppose $(\alpha, \beta) = (0, 0)$. From Equations (3.18) and (3.6), $R^{0,0}_{p,q}$ is determined by the equation

$$\mathsf{Im}\left(\frac{t(w)}{p + q\frac{b}{2\mathrm{i}\pi}\log(w)}\right) = 0, \tag{3.20}$$

where $\log(w)$ is viewed as a multivalued function. The multivaluedness of the logarithm takes care of the monodromy in the sense that winding around the origin changes $\log(w)$ into $\log(w) + 2\mathrm{i}\pi$, which in this equation corresponds to the monodromy action $(p, q) \mapsto (p + qb, q)$. The function $t(w)$ is a well-defined local holomorphic function, which may or may not vanish at the origin (see §3.3.2).

If $q = 0$, Equation (3.20) reduces to $\mathsf{Im}(t(w)) = 0$ which is an analytic subset of the disk (including the origin if $\mathsf{Im}(t(0)) = 0$).

Now, we focus on the case $q \neq 0$, and we assume that $t(w) = t_0 w^k$. Write $w = e^{-s}$, where s belongs to some right half plane, and write $t_0 = e^{ks_0}$. Then, Equation (3.20) becomes

$$\mathsf{Im}\left(\frac{e^{-k(s-s_0)}}{p + q\frac{b}{2\mathrm{i}\pi}(-s)}\right) = 0. \tag{3.21}$$

Writing $s = x + iy$ and $s_0 = x_0 + \mathrm{i}y_0$, and making a few elementary manipulations, this equation reduces to

$$x = \left(y + \frac{2\pi p}{bq}\right)\tan k(y - y_0), \tag{3.22}$$

which after a vertical translation gives $x = (y - y_1)\tan(ky)$ for some $y_1 \in \mathbf{R}$. There are two different regimes:

(1) if $k = 0$ the curve is a line, which descends to a logarithmic spiral in the w-coordinate;
(2) if $k > 0$, one distinguishes two cases, depending on whether y_1 is of the form $\frac{1}{k}(\frac{\pi}{2} + j\pi)$, $j \in \mathbf{Z}$, or not (see Figure 1). In both cases, these curves have infinitely

many branches asymptotic to the horizontal lines $y = \frac{1}{k}(\frac{\pi}{2}+j\pi)$, $j \in \mathbf{Z}$, as $x \to +\infty$. In the w-plane, these branches have well-defined tangent directions at the origin, so they extend as C^1 curves at 0, but they are not semi-analytic.

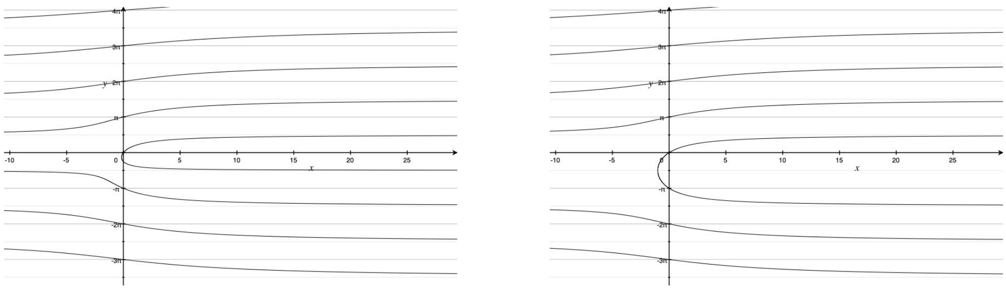


FIGURE 1. The curves $x = \left(y + \frac{\pi}{4}\right) \tan y$ (left) and $x = \left(y + \frac{\pi}{2}\right) \tan y$ (right).

4. Proof of Theorem A: preliminaries, first steps, and corollaries

Let X be a smooth, compact, Kähler surface. Fix a subgroup Γ of $\mathsf{Aut}(X)$ such that:

(i) Γ is non-elementary,
(ii) Γ contains a parabolic element.

Then, as shown in [11], X is a smooth complex projective surface. Our goal in this section is to prove Theorem A under the stronger assumption:

(ii') Γ contains a Halphen twist g (see § 3.1); equivalently Γ contains a parabolic automorphism acting with finite order on the base of its invariant fibration.

As explained in Theorem 3.1, this hypothesis automatically follows from (ii) when X is not an Abelian surface. For the easier case of Abelian surfaces, a direct proof of Theorem A is given in the Appendix.

Notation 4.1. If h is a Halphen twist, we denote by $\pi_h \colon X \to B_h$ its invariant fibration, and by $\mathrm{Sing}(\pi_h)$ the union of its singular and multiple fibers. For $U \subset B_h$ (resp. $w \in B_h$), we set $X^h_U = \pi_h^{-1}(U)$ (resp. $X^h_w = \pi_h^{-1}(w)$). Similarly, we make use of the notation $\mathrm{R}_h(U)$ and $\mathrm{Tor}_h(U)$, with the index h. We denote by $\mathrm{Hal}(\Gamma)$ the set of Halphen twists $h \in \Gamma$ preserving each irreducible component of each fiber of π_h. This set is invariant under conjugation: if $f \in \Gamma$ and $h \in \mathrm{Hal}(\Gamma)$, then $f^{-1} \circ h \circ f$ is an element of $\mathrm{Hal}(\Gamma)$ and its invariant fibration is given by $\pi_g \circ f$.

Remark 4.2. The foliation defined by π_h is uniquely determined by h, but π_h itself is not canonically defined: post-composition by an automorphism of B_h would give another projection defining the same fibration. Thus, the notation means that a projection π_h was chosen for every fibration invariant by an element $h \in \mathrm{Hal}(\Gamma)$, the choice being the same for two twists preserving the same fibration. Then $\pi_h \circ f$ equals $\pi_{f^{-1}\circ h\circ f}$ up to post composition by an automorphism of the base.

4.1. **Invariant curves,** Tang **and** STang**.** According to [11, Lemma 2.13] and [10, Section 3], there is a unique reduced, effective, and Γ-invariant divisor D_Γ in X such that:

(1) the Γ-periodic irreducible curves $C \subset X$ are exactly the irreducible components C_i of D_Γ;
(2) the intersection form is negative definite on

$$V(\mathrm{D}_\Gamma) := \mathrm{Vect}([C_i], i = 1, \ldots, k) \subset H^2(X; \mathbf{R}). \tag{4.1}$$

Remark 4.3. If $g \in \mathrm{Hal}(\Gamma)$, the divisor D_Γ is made of irreducible components of fibers of π_g, but D_Γ *does not contain any complete fiber*: indeed the intersection form is negative definite on $V(\mathrm{D}_\Gamma) \subset H^2(X; \mathbf{R})$, while the self-intersection of a fiber is 0.

For $(g, h) \in \mathrm{Hal}(\Gamma)^2$, we let $\mathrm{Tang}(\pi_g, \pi_h)$ be the set of points $x \in X$ such that $(d\pi_g \wedge d\pi_h)(x) = 0$, and define

$$\mathrm{Tang}_\Gamma = \bigcap_{(g,h)\in \mathrm{Hal}(\Gamma)^2} \mathrm{Tang}(\pi_g, \pi_h) \tag{4.2}$$

In plain words, $x \notin \mathrm{Tang}_\Gamma$ if one can find g and h in $\mathrm{Hal}(\Gamma)$ such that π_g and π_h are transverse projections in a neighborhood of x (this is compatible with our convention for π_g, see Remark 4.2). Note that by definition if F is a multiple component of a fiber of π_g then $d\pi_g \wedge d\pi_h = 0$ along F for all $h \in \mathrm{Hal}(\Gamma)$ (see §5.2.2 for more on this). Let Tang^1_Γ (resp. Tang^0_Γ) be the union of the 1-dimensional (resp. 0-dimensional) components of Tang_Γ. We also put

$$\mathrm{STang}(\pi_g, \pi_h) = \mathrm{Tang}(\pi_g, \pi_h) \cup \mathrm{Sing}(\pi_g) \cup \mathrm{Sing}(\pi_h) \tag{4.3}$$

and

$$\mathrm{STang}_\Gamma = \bigcap_{(g,h)\in \mathrm{Hal}(\Gamma)^2} \mathrm{STang}(\pi_g, \pi_h), \tag{4.4}$$

and define STang^1_Γ and STang^0_Γ similarly.

Lemma 4.4. *The reduced divisors given by* D_Γ,Tang^1_Γ *and* STang^1_Γ *are all equal. The* 0*-dimensional parts* Tang^0_Γ *and* STang^0_Γ *are finite* Γ*-invariant sets.*

Proof. By definition, $\mathrm{Tang}(\Gamma)$ and STang_Γ are Γ-invariant algebraic sets, and STang^1_Γ contains Tang^1_Γ. So, $\mathrm{Tang}^1_\Gamma \subset \mathrm{STang}^1_\Gamma \subset \mathrm{D}_\Gamma$. Now, fix $g \in \mathrm{Hal}(\Gamma)$. If C is a g-periodic irreducible curve, Lemma 3.14 entails that $\pi_g(C)$ must be a point. So, all fibrations π_g must be tangent along any component of D_Γ, and it follows that $\mathrm{D}_\Gamma = \mathrm{Tang}^1_\Gamma$. The second assertion is straightforward. ☐

Proposition 4.5 (See [10], Proposition 3.9)**.** *There is a normal projective surface X_0 and a birational morphism $\eta\colon X \to X_0$ such that*

(1) *η contracts D_Γ on a finite number of points;*
(2) *Γ induces a subgroup of $\mathsf{Aut}(X_0)$, i.e. there is an injective homomorphism $f \in \Gamma \mapsto f_0 \in \mathsf{Aut}(X_0)$ such that $\eta \circ f = f_0 \circ \eta$ for all $f \in \Gamma$.*

Example 4.6. The Coble surfaces and the surfaces constructed by Blanc (see §2.2) provide examples of pairs (X, Γ) such that X is rational and Γ preserves a smooth rational curve or a smooth curve of genus 1, respectively.

4.2. **Analytic subsets of positive mass, $d_{\mathbf{R}}(\mu)$, and $d_{\mathbf{C}}(\mu)$.** Let μ be a Γ-invariant and ergodic probability measure. Denote by $d_{\mathbf{R}}(\mu)$ the minimum of the dimensions $k \in \{0, 1, 2, 3, 4\}$ such that there exist an open set $U \subset X$, for the euclidean topology, and a real analytic submanifold $W \subset U$ of real dimension k with $\mu(W) > 0$.

Remark 4.7. Pick such a local analytic submanifold $W \subset U$ with $\mu(W) > 0$ and $\dim_{\mathbf{R}}(W) = d_{\mathbf{R}}(\mu)$. By ergodicity, μ gives full mass to $\bigcup_{g\in\Gamma} g(W)$. If $W \cap g(W)$ has empty relative interior in W and $g(W)$, then $g(W) \cap W$ is contained in at most countably many analytic submanifolds of lower dimension, so $\mu(g(W) \cap W) = 0$.

Likewise, let $d_{\mathbf{C}}(\mu)$ be the minimal dimension $k \in \{0, 1, 2\}$ such that $\mu(Z) > 0$ for some irreducible local complex analytic subset $Z \subset X$ of (complex) dimension k.

Lemma 4.8. *Assume that Γ satisfies (i) and (ii'), and let μ be an ergodic Γ-invariant measure.*

(1) *If $d_{\mathbf{C}}(\mu) = 0$, μ is supported on a finite orbit of Γ.*
(2) *If $d_{\mathbf{C}}(\mu) = 1$, μ is supported on D_Γ.*

So, we see that $d_{\mathbf{C}}(\mu) = 2$ is equivalent to: *μ gives no mass to algebraic sets* (or more precisely *to proper algebraic subsets*).

Proof. The zero-dimensional case is left to the reader. So, suppose that μ has no atom and let $Z \subset X$ be a local, 1-dimensional, complex analytic set such that $\mu(Z) > 0$. If Z is not contained in D_Γ, one can find $g \in \mathrm{Hal}(\Gamma)$ and a holomorphic disk $\Delta \subset Z$ of positive measure which is transverse to π_g. By Lemma 3.9, for any $k \neq \ell$, $g^k(\Delta) \cap g^\ell(\Delta)$ is a finite set; since μ is atomless and g-invariant, this implies $\mu(\bigcup_{n\geqslant 0} g^n(\Delta)) = \sum_{n\geqslant 0} \mu(\Delta) = \infty$. This contradiction completes the proof. □

4.3. **The smooth case.** For $g \in \mathrm{Hal}(\Gamma)$, the (marginal) measure $\mu_g := (\pi_g)_*\mu$ is a probability measure on B_g. Let $\mathrm{R}_g = \mathrm{R}(B_g^\circ)$ (cf. Equation (3.19)).

Proposition 4.9. *Let μ be an invariant and Γ-ergodic measure which gives no mass to proper algebraic subsets (i.e. $d_{\mathbf{C}}(\mu) = 2$)). Assume that there exists an element $g \in \mathrm{Hal}(\Gamma)$ such that*

$$\mu_g(B_g^\circ \setminus \mathrm{R}_g) > 0. \tag{4.5}$$

Then μ is absolutely continuous with respect to the Lebesgue measure on X and its support is equal to X. Moreover, the density of μ with respect to any real analytic volume form on X is a real analytic function in the complement of STang_Γ.

The proof occupies §§ 4.3.1 to 4.3.4 below.

4.3.1. *Preliminaries: special subgroups.*

Definition 4.10. A pair (g, h) of Halphen twists is **special** if

(1) the group $\langle g, h\rangle$ is a non-abelian free group on two generators (in particular the fibrations π_g and π_h are distinct);
(2) an element $f \in \langle g, h\rangle$ is a power of g if and only if it fixes the class of the fibration π_g in the Néron-Severi group $\mathrm{NS}(X; \mathbf{Z})$, if and only if it permutes the fibers of π_g, if and only if it maps some smooth fiber of π_g to a fiber of π_g;
(3) Property (2) holds also for h in place of g.

In Assertion (2), the important part is that an $f \in \langle g, h\rangle$ that permutes the fibers of π_g is an element of $g^{\mathbf{Z}}$. Indeed, the following remark shows that the last three properties in Assertion (2) are always equivalent.

Remark 4.11. Consider an arbitrary pair (g, h) of Halphen twists, and let E be a scheme theoretic fiber of π_g (that is, if w is a local coordinate near $w_0 = \pi_g(E)$, the equation of E is $\pi_g(\xi) = w$). If $f \in \Gamma$ maps E in a fiber E' of π_h, then the class $[f(E)]$ must be proportional to $[E']$, because the self-intersection of $f(E)$ is zero and in the vector space generated by the classes of the components of E', all isotropic vectors are proportional to $[E']$. Thus the classes of the fibers of $\pi_g \circ f$ and π_h generate the same ray in $H^{1,1}(X; \mathbf{R})$; by Remark 3.2, $\pi_g \circ f^{-1}$ and π_h are equal, up to post composition by an isomorphism $B_h \to B_g$. In particular, if g and h preserve distinct fibrations, no fiber of π_g is entirely contracted by π_h.

Lemma 4.12. *If g and h are Halphen twists associated to distinct fibrations, then for large enough n, the pair (g^n, h^n) is special.*

Proof. Consider the action of Γ on $H^{1,1}(X; \mathbf{R})$. It preserves the isotropic cone

$$\{u \in H^{1,1}(X; \mathbf{R}) \ ; \ \langle u | u\rangle = 0\}, \tag{4.6}$$

where $\langle \cdot | \cdot \rangle$ is the intersection form. Projectively, this cone is a sphere $\mathbb{S}$ (which can be considered as the boundary of a hyperbolic space, see [8, 11]). Now, g^* is a parabolic transformation on $\mathbb{S}$, fixing a unique point u_g. Let us fix a small neighborhood U_g of u_g; if x is a point of $\mathbb{S}\backslash\{u_g\}$ there is small neighborhood $V(x)$ of x in $\mathbb{S}$ and a positive integer $m_g(V(x))$ such that $(g^n)^*(V(x)) \subset U_g$ for every $n \in \mathbf{Z}$ with $|n| \geqslant n_g(V(x))$. A similar property is satisfied by h. The points u_g and u_h are determined by the classes of the fibers of the invariant fibrations π_g and π_h. We choose U_g and U_h small and disjoint, and n_0 such that for all $n \in \mathbf{Z}$ with $|n| \geqslant n_0$

$$(g^n)^*(U_h) \subset U_g\backslash\{u_g\} \text{ and } (h^n)^*(U_g) \subset U_h\backslash\{u_h\}\,. \tag{4.7}$$

In the following we fix such an n. Then, the first assertion follows from the ping-pong lemma (see [16], Chapter I).

Let us show that if $f \in \langle g^n, h^n\rangle$, then $f^*u_g = u_g$ if and only if $f \in \langle g\rangle$. We assume that $f^*u_g = u_g$, and we want to show that f is an iterate of g. Write f as a word in g^n and h^n: $f = g^{nk_1} \circ h^{n\ell_1} \circ \cdots \circ h^{n\ell_s}$, the k_i and ℓ_i being non-zero integers, except possibly for k_1 and ℓ_s which may vanish. The proof is by induction on $|f| = |\{i\ ;\ k_i \neq 0\}| + |\{i\ ;\ \ell_i \neq 0\}|$. The result is obvious when $|f| = 0$ or 1.

The point u_g corresponds to an isotropic line in $H^{1,1}(X;\mathbf{R})$, and this line intersects $H^2(X;\mathbf{Z})$ on a discrete set $\mathbf{Z}[D]$, for some integral class $[D]$ on the boundary of the ample cone.Since f^* preserves $H^2(X;\mathbf{Z})$ and the ray $\mathbf{R}_+[D]$, it fixes $[D]$. Thus, f^* cannot be loxodromic (see [8]).

If f starts with h and ends with g (i.e. $\ell_s k_1 \neq 0$), then f^* maps U_h strictly inside itself and $(f^*)^{-1}$ maps U_g strictly inside itself. This implies that f^* is loxodromic, a contradiction. Therefore f starts and ends with the same letter. If this letter were h, then u_g would in fact be mapped into U_h by f; thus, f starts and ends by a power of g. Conjugating f by a power of g, we reduce its length without changing the property $f^*u_g = u_g$. Thus, by induction, f is a power of g. □

4.3.2. *Preliminaries: disintegration.* If g is any Halphen twist, we may disintegrate μ with respect to π_g: there is a measurable family of probability measures $\mu_{g,w}$ on the fibers X^g_w such that

$$\int_X \xi(x)\, d\mu(x) = \int_{B_g}\int_{X^g_w} \xi(z)\, d\mu_{g,w}(z)\, d\mu_g(w) \tag{4.8}$$

for every Borel function $\xi\colon X \to \mathbf{R}$. The measures $\mu_{g,w}$ are unique, in the following sense: if $\mu'_{g,w}$ is another family of probability measures satisfying Equation (4.8), then $\mu'_{g,w} = \mu_{g,w}$ for μ_g-almost every w. Thus, the measures $\mu_{g,w}$ are invariant under the action of $g_{|X^g_w}$.

4.3.3. *Proof of Proposition 4.9: special case.* In this paragraph we assume that there exists $h \in \mathrm{Hal}(\Gamma)$ such that the pair (g,h) is special in the sense of Definition 4.10, and let $\Gamma_1 = \langle g, h\rangle$. Note that Γ_1 satisfies assumptions (i) and (ii') from p.20. Let μ be as in the statement of Proposition 4.9. Let us show that the conclusions of the proposition hold under the *additional assumption that that μ is Γ_1-ergodic* (also in this first part of the proof, the analyticity of the density will only be established outside $\mathrm{STang}(\pi_g, \pi_h)$).

Let $\mathcal{B}_g$ be a Borel subset of B_g° which is disjoint from R_g and satisfies $\mu_g(\mathcal{B}_g) > 0$. According to Lemma 3.14, the dynamics of g on X_w^g is uniquely ergodic for every $w \in \mathcal{B}_g$. Thus, we get:

Step 1.– *If $w \in \mathcal{B}_g$, then $g_{|X_w^g}$ is uniquely ergodic and $\mu_{g,w}$ is equal to the Haar measure $\lambda_{g,w}$ on the fiber $X_w^g \simeq \mathbf{C}/\mathrm{Lat}_g(w)$.* Here, $\mathrm{Lat}_g(w) = \mathbf{Z} \oplus \mathbf{Z}\tau_g(w)$ and $\lambda_{g,w} = (\mathrm{Im}(\tau_g(w)))^{-1} \mathsf{i} dz \wedge d\bar{z}$, as in Section 3.

Step 2.– *We have $d_{\mathbf{C}}(\mu) = 2$ and $\mu_g(\mathrm{R}_g) = 0$.*

The first assertion follows directly from Lemma 4.8 and Lemma 4.4.

For the second one, we argue by contradiction, assuming that there is an analytic arc $\gamma \subset \mathrm{R}_g$ such that $\mu_g(\gamma) > 0$. Since $d_{\mathbf{C}}(\mu) > 1$, we can shorten γ to ensure that it does not contain any critical value of π_g. Set $W_\gamma = \pi_g^{-1}(\gamma)$. Then $\mu(W_\gamma) > 0$, so $d_{\mathbf{R}}(\mu) \leqslant 3$. By ergodicity of μ, $\bigcup_{f \in \Gamma_1} f(W_\gamma)$ is a subset of full measure.

Pick $f \in \Gamma_1$. If it permutes the fibers of π_g, then $f \in g^{\mathbf{Z}}$ because (g, h) is special, and thus $f(W_\gamma) = W_\gamma$; thus $\mu(f(W_\gamma) \cap \pi_g^{-1}(\mathcal{B}_g)) = \mu(\varnothing) = 0$ in that case. Now, suppose f does not permute the fibers of π_g. Note that if W_γ intersects an irreducible curve $C \subset X$ on some non-empty open subset of C, then C must be a fiber of π_g, because its projection in B_g is locally contained in γ. Thus, if $w \in B_g^\circ$ and $f(W_\gamma) \cap X_w^g$ contains a non-empty open subset of X_w^g then f^{-1} maps X_w^g into a fiber of π_g above γ, and this contradicts the fact that (g, h) is special. We deduce that $f(W_\gamma) \cap X_w^g$ is contained in a countable union of real analytic submanifolds of dimension 1 in X_w^g, for every $w \in B_g^\circ$. In particular if $w \in \mathcal{B}_g$, $\lambda_{g,w}(f(W_\gamma) \cap X_w^g) = 0$, and we conclude that $\mu(f(W_\gamma) \cap \pi_g^{-1}(\mathcal{B}_g)) = 0$ in that case too. Since $\mu\big(\bigcup_{f \in \Gamma_1} f(W_\gamma)\big) = 1$ and simultaneously $\mu(\pi_g^{-1}(\mathcal{B}_g)) > 0$ we obtain a contradiction, which concludes the proof of the second step.

Step 3.– From Step 2 we can suppose $\mu_g(\mathcal{B}_g) = 1$. Let us show that $\mu_h := (\pi_h)_*\mu$ *is absolutely continous with respect to the Lebesgue measure. In particular, h satisfies $\mu_h(B_h^\circ \setminus \mathrm{R}_h) = 1$ too.*

Let $\Delta \subset B_h$ be a Borel set of Lebesgue measure 0. Remark 4.11 shows that if X_w^g is a smooth fiber of π_g, then X_w^g is not contracted by π_h, so $\pi_{h|X_w^g} \colon X_w^g \to B_h$ is a finite ramified cover, and $\lambda_{g,w}((\pi_{h|X_w^g})^{-1}(\Delta)) = 0$. This shows that

$$\lambda_{g,w}(\pi_h^{-1}(\Delta) \cap X_w^g) = 0 \tag{4.9}$$

for μ_g almost every point $w \in B_g$. Since $\mu_g(\mathcal{B}_g) = 1$ and $\mu_{g,w}$ coincides with the Haar measure $\lambda_{g,w}$ for μ_g almost every $w \in \mathcal{B}_g$, Equation (4.8) implies $\mu(\pi_h^{-1}(\Delta)) = 0$; thus $\mu_h(\Delta) = 0$, as required.

Step 4.– *The support of* μ *is* X.

Indeed, μ_h being absolutely continuous with respect to the Lebesgue measure of B_h, $\mu_h(\mathrm{R}_h) = 0$. Thus, symetrically, μ_g is absolutely continuous with respect to the Lebesgue measure of B_g, and from Equation (4.8) we deduce that μ is absolutely continuous with respect to the Lebesgue measure on X. If U is an open subset of B_g, X_U^g intersects every smooth fiber of π_h on a set of positive Haar-measure; thus $\mu(X_U^g) = \mu_g(U)$ is positive, and we infer that the support of μ_g is equal to B_g. Since $\mu_{g,w}$ is μ_g-almost surely the Haar measure $\lambda_{g,w}$, we conclude that the support of μ is equal to X.

Step 5.– *The density is analytic outside* $\mathrm{STang}(\pi_g, \pi_h)$.

Let x_0 be a point of $X \backslash \mathrm{STang}(\pi_g, \pi_h)$ and V be a small neighborhood of x_0 such that

- π_g and π_h are everywhere transverse on V,
- $U_g := \pi_g(V)$ and $U_h := \pi_h(V)$ are small disks in B_g and B_h, respectively,
- π_g (resp. π_h) is a proper submersion above U_g (resp. U_h).

In a chart Ψ_g mapping $X_{U_g}^g$ to $U_g \times \mathbf{R}^2/\mathbf{Z}^2$, Equation (4.8) implies that $(\Psi_g)_*(\mu_{|X_{U_g}^g})$ is invariant under the action of *all* vertical translations; the same property holds with respect to π_h. Coming back to X, these translations act analytically and locally transitively on V: for every y in V there is a pair of such translations such that their composition maps x_0 to y (V is not invariant under such translations). Following the proof of [22, Proposition 1], we deduce that μ has a real analytic density on V, with respect to the analytic structure of X. Indeed, embed V in $\mathbf{R}^4$ and denote by vol a volume form on $\mathbf{R}^4$. In these coordinates, $\mu = \xi \, \mathsf{vol}$, for some non-negative integrable function ξ. Changing $x_0 \in V$ if necessary, we suppose that x_0 is a Lebesgue density point for μ; this means that $\mu(\varepsilon K)\mathsf{vol}(\varepsilon K)^{-1}$ converges to $\xi(x_0)$ when $\varepsilon \to 0$, for every ellipsoid K centered at x_0. If x_0 is mapped to y by a diffeomorphism φ_y preserving μ, then the boundedness of the distortion of φ_y shows that y is also a Lebesgue density point of μ, with density $\xi(y) = (\mathrm{Jac}(\varphi_y)(x_0))^{-1}\xi(x_0)$. Now, choosing φ_y as a composition of two translations, we can assume that $y \mapsto \mathrm{Jac}(\varphi_y)(x_0)$ is an analytic function; thus, the density ξ is real-analytic in a neighborhood of x_0. □

4.3.4. *Proof of Proposition 4.9: general case.* Let μ and g be as in the statement of the proposition. Fix $h \in \mathrm{Hal}(\Gamma)$ such that $d\pi_g \wedge d\pi_h \not\equiv 0$. Then by Lemma 4.12, we find $n \geqslant 1$ such that (g^n, h^n) is special; set $\Gamma_1 = \langle g^n, h^n \rangle$. Since $\mathrm{R}_{g^n} = \mathrm{R}_g$, the assumption (4.5) holds with g^n instead of g. However μ is not necessarily Γ_1-ergodic, so the results of § 4.3.3 cannot be applied directly. To get around this difficulty, we use the Γ_1-ergodic decomposition of μ (see [30]): $\mu = \int_X \beta_x \, d\mu(x)$ for an essentially unique, Γ-invariant, Borel map $\beta : x \mapsto \beta_x$ such that β_x is μ-almost surely a Γ_1-ergodic probability measure.

Set $\Omega_j = \{x \in X \ ; \ d_{\mathbf{C}}(\beta_x) = j\}$, for $j = 0, 1$. By ergodicity, $\beta_x(D_{\Gamma_1}) = 1$ for every $x \in \Omega_1$. Since $\mu(D_{\Gamma_1}) = 0$, we deduce that $\mu(\Omega_1) = 0$. For $x \in \Omega_0$, β_x gives full mass to the union of the (fixed) countable set $\pi_g^{-1}(\mathrm{Tor}(B_g^\circ)) \cap \pi_h^{-1}(\mathrm{Tor}(B_h^\circ))$ and the Zariski closed set $\mathrm{Sing}(\pi_g) \cup \mathrm{Sing}(\pi_h)$; since $d_{\mathbf{C}}(\mu) = 2$, we also get $\mu(\Omega_0) = 0$. Thus $\dim_{\mathbf{C}}(\beta_x) = 2$ for μ-almost every x.

So, for x in a subset $\Omega \subset X$ with $\mu(\Omega) = 1$, the results of § 4.3.3 apply to β_x. There are two possibilities:

– either $\beta_x(\pi_g^{-1}(B_g^\circ \setminus \mathrm{R}_g)) > 0$, β_x is absolutely continuous, and its support is X;
– or $\beta_x(\pi_g^{-1}(\mathrm{R}_g)) = 1$.

Denote by Ω_{ac} (for "absolutely continuous") and by Ω_{si} (for "singular") the set of points such that the first or second alternative holds, respectively. Both are Borel subsets and $\mu(\Omega_{\mathrm{ac}} \cup \Omega_{\mathrm{si}}) = 1$. Assumption (4.5) implies that $\mu(\Omega_{\mathrm{ac}}) > 0$. If $\mu(\Omega_{\mathrm{si}}) > 0$, then $\mu(\pi_g^{-1}(\mathrm{R}_g)) > 0$, and since μ is Γ-ergodic, we infer that $\mu(\Gamma \cdot \pi_g^{-1}(\mathrm{R}_g)) = 1$. But the Lebesgue measure of $\Gamma \cdot \pi_g^{-1}(\mathrm{R}_g)$ is zero, so $\int_{x \in \Omega_{\mathrm{ac}}} \beta_x(\Gamma \cdot \pi_g^{-1}(\mathrm{R}_g)) \, d\mu(x) = 0$ and we deduce that $\mu(\Omega_{\mathrm{ac}}) = 0$. This contradiction shows that $\mu(\Omega_{\mathrm{si}}) = 0$ and $\mu(\Omega_{\mathrm{ac}}) = 1$.

Finally, if x and y are in Ω_{ac}, then β_x and β_y are Γ_1-ergodic measures of full support, the densities of which are analytic on the complement of proper analytic sets. So $\beta_x = \beta_y$, and this implies that $\beta_x = \mu$ almost surely. In particular, μ is Γ_1-ergodic and satisfies the conclusions of § 4.3.3.

Remark 4.13. This argument shows that if μ is a Γ-ergodic probability measure, μ gives no mass to algebraic subsets, and $\mu_g(B_g^\circ \setminus \mathrm{R}_g) > 0$ for some $g \in \mathrm{Hal}(\Gamma)$, then $\mu_{g'}(B_{g'}^\circ \setminus \mathrm{R}_{g'}) = 1$ for *any* $g' \in \mathrm{Hal}(\Gamma)$: this follows from the absolute continuity of μ.

To conclude the proof of Proposition 4.9, it remains to show that the density of μ is analytic outside STang_Γ. For every $x \in X \setminus \mathrm{STang}_\Gamma$ there exists a pair $(g', h') \in \mathrm{Hal}(\Gamma)^2$ such that $x \notin \mathrm{STang}(\pi_g, \pi_h)$, and by Lemma 4.12 we can assume that this pair is special. By the previous remark, the results of § 4.3.3 apply to (g', h'). So, μ is smooth near x, and we are done. □

4.4. **Dimension $\leqslant 1$.**

Lemma 4.14. *If $d_{\mathbf{R}}(\mu) \leqslant 1$ then μ is either supported on a finite orbit, or on* D_Γ.

Proof. By Lemma 4.8, we may assume that $d_{\mathbf{C}}(\mu) = 2$, in particular μ_g is atomless, and we seek a contradiction. Pick $g \in \mathrm{Hal}(\Gamma)$. By assumption there is a (local) real analytic curve $W \subset X$ such that $\mu(W) > 0$ and W is Zariski dense in X. Shrinking it, we can assume that W is an analytic path, transverse to the fibration π_g, that intersects each fiber above $\pi_g(W)$ in a unique point. By Proposition 4.9, $\pi_g(W)$ is contained in R_g. Let us disintegrate μ with respect to π_g. Since μ_g is atomless, Lemma 3.14 shows that, for μ_g-almost every w, the measures which are invariant under g_w are atomless; thus, the conditional measure $\mu_{g,w}$ is almost surely atomless and $\mu_{g,w}(W \cap X_w^g) = 0$. This contradicts $\mu(W) > 0$. □

4.5. **The totally real case.** We are now reduced to the case where $d_{\mathbf{R}}(\mu) \geqslant 2$ and μ_g is supported on R_g for every $g \in \mathrm{Hal}(\Gamma)$. The properties of μ are summarized in the following proposition, the proof of which will be given in Section 5.

Proposition 4.15. *Let μ be an ergodic Γ-invariant measure. Assume that μ gives no mass to algebraic subsets and that $\mu_g(\mathrm{R}_g) > 0$ for some $g \in \mathrm{Hal}(\Gamma)$. Then $d_{\mathbf{R}}(\mu) = 2$, and there exists a totally real analytic subset Σ of $X \backslash \mathrm{STang}_\Gamma$ of pure dimension 2 such that:*

(1) *$\mu(\Sigma) = 1$ and $\mathrm{Supp}(\mu) = \overline{\Sigma}$;*
(2) *Σ has finitely many irreducible components;*
(3) *the singular locus of Σ is locally finite;*
(4) *on the regular part of Σ, μ has a real analytic density with respect to any real analytic area form;*

Example 4.16. The main example is when the projective surface X is defined over $\mathbf{R}$, and μ is a measure supported on $X(\mathbf{R})$ giving no mass to algebraic subsets. This occurs for instance when $\Gamma \subset \mathsf{Aut}(X_{\mathbf{R}})$ preserves an area form on $X(\mathbf{R})$ (see Section 2). An example of a different kind is given in Section 8.

A noteworthy consequence of Propositions 4.15 and 4.9 is:

Corollary 4.17. *The dimension $d_{\mathbf{R}}(\mu)$ cannot be equal to 3.*

4.6. **Proof of the main theorem, and consequences.**

Proof of Theorem A. If μ gives positive mass to a proper algebraic subset of X, then according to Lemma 4.8, either Assertion (a) or (b) of Theorem A is satisfied. Otherwise, we know from Lemma 4.14 and Corollary 4.17 that $d_{\mathbf{R}}(\mu)$ is either equal to 2 or 4, and exactly one of Proposition 4.15 or Proposition 4.9 applies. If $d_{\mathbf{R}}(\mu) = 2$, Proposition 4.15 shows that the conclusions of case (c) of the theorem hold. The remaining case (d) is covered by Proposition 4.9. In both cases the exceptional set Z is equal to STang_Γ. □

Proof of Corollary B. When Γ does not preserve any proper algebraic subset, then we must be in one of the cases (c) (if $d_{\mathbf{R}}(\mu) = 2$) or (d) (if $d_{\mathbf{R}}(\mu) = 4$) of Theorem A.

Furthermore STang_Γ is empty. Thus if $d_{\mathbf{R}}(\mu) = 4$ we infer that μ has an analytic positive density on X. If $d_{\mathbf{R}}(\mu) = 2$, note that the analytic surface Σ is smooth everywhere, since otherwise its singular locus would be a finite invariant subset (see Assertion (4) of Proposition 4.15). □

Let us point out the following immediate consequence of Theorem A.

Corollary 4.18. *If X is a real projective surface, $X(\mathbf{R})$ is nonempty, $\Gamma \subset \mathsf{Aut}(X_{\mathbf{R}})$, and μ is a Γ-invariant ergodic probability measure on $X(\mathbf{R})$, then*

(a) *either μ is supported by a Γ-invariant proper real algebraic subset of X,*
(b) *or there is a Γ-invariant proper real algebraic subset Z of X such that μ is supported by a union of connected components of $X(\mathbf{R})\backslash Z(\mathbf{R})$ and μ has a real analytic density on that set.*

In particular if Γ does not preserve any proper real algebraic subset of X, μ is given by a real analytic area form on $X(\mathbf{R})$, restricted to a Γ-invariant union of connected components of $X(\mathbf{R})$.

Let X be (a blow up of) an abelian, K3, or Enriques surface, and let vol_X be the natural probability measure on X (see 2.1). Likewise, if Σ is a totally real submanifold in X, let vol_Σ be the measure induced by the normalized 2-form Ω_X (see Remark 2.3). By positivity of the density of μ and ergodicity, we obtain:

Corollary 4.19. *If in Theorem A, X is a blow-up of an abelian, K3, or Enriques surface, then in case (c) μ is a multiple of vol_Σ, and in case (d) it coincides with vol_X.*

In case (c), this implies in particular that Σ has finite area. Similarly, case (d) does not appear in Blanc's example (see § 2.2).

5. Proof of Proposition 4.15

In this section, we prove Proposition 4.15. Thus, unless otherwise specified, we study ergodic invariant measures with $d_{\mathbf{C}}(\mu) = 2$ and $\mu_g(\gamma) > 0$ for some $g \in \mathrm{Hal}(\Gamma)$ and some local, smooth, analytic arc $\gamma \subset \mathrm{R}_g$. The main step of the proof is to show that γ extends to an analytic curve σ in B_g°. From Proposition 4.9, we already know that the support of μ_g is contained in the closure of R_g; however the analytic continuation of γ and the support of μ_g still could a priori be a complicated subset in B_g (see Remark 5.8). The main point is to exclude such a phenomenon.

After some preliminaries in §§ 5.1 and 5.2, the proof of Proposition 4.15 is carried out in §§ 5.3 to 5.7.

5.1. **Vocabulary of analytic and semi-analytic geometry.** Let us recall briefly a few basic facts from analytic and semi-analytic geometry. A **semi-analytic** set E in a (σ-compact) real-analytic manifold M is a subset such that every $x \in M$ admits an open neighborhood U in which $E \cap U$ is defined by finitely many inequalities of the form $f \geqslant 0$ or $f > 0$, where $f\colon U \to \mathbf{R}$ is analytic. The class of semi-analytic sets is stable under many operations such as taking a finite union or intersection, the closure, the boundary, the connected components, or the preimage under an analytic map. However the image of a semi-analytic set by a proper analytic map needs not be semi-analytic: adding such projections, one obtains the class of **subanalytic sets**. The main fact that we will need from subanalytic geometry is: *any subanalytic set of dimension* $\leqslant 1$ *is semi-analytic*. In this way we will only have to deal with semi-analytic sets (see [2, 28] for more details on these facts).

A point x in E is **regular** if E is an analytic submanifold in the neighborhood of x, otherwise E is **singular** at x. The **dimension** of E is the maximal dimension of E at its regular points. We say that E is of pure dimension if its dimension is the same at all regular points.

A delicate point in semi-analytic and real-analytic geometry is that the notion of irreducible component is not well behaved (we will discuss only the analytic case). To get around this problem, the following notion was introduced by Cartan [15] and Whitney and Bruhat [32]: a subset E of a real analytic manifold M is **C-analytic** (or global analytic) if it the common zero set of a finite number of analytic functions defined on the whole of M. Equivalently, there is a coherent analytic sheaf whose zero set is E (see [32, Prop. 10]). This class is stable by union and intersection. Every C-analytic set E admits a unique locally finite decomposition $E = \bigcup_i E_i$ into irreducible components; here, the irreducibility means that E_i is not the union of two distinct C-analytic sets (beware that it might be reducible as an analytic set, see [32, §11.a]). If $E \subset M$ is a smooth analytic submanifold, or if E is locally a finite union of smooth plaques of the same dimension, then E is C-analytic. Indeed, in this case it is easy to see that for every $x \in E$ there exists an open neighborhood $U \ni x$ and a finite family of analytic functions f_i on U such that for every open subset $V \subset U$, the intersection of the zero sets of the f_i coincides with E in V: this implies that E is the zero set of a coherent analytic sheaf, hence it is C-analytic. Another useful fact is that every 1-dimensional analytic set is C-analytic; more generally, if E is analytic, the canonical ideal sheaf of analytic functions vanishing on E is coherent outside a codimension 2 subset of E, (see [25]).

A semi-analytic subset $\Sigma \subset X$ of pure dimension 2 is **totally real** if at every regular point $x \in \Sigma$, $T_x\Sigma$ is a totally real subspace of T_xX, that is, $\mathrm{j}_x(T_x\Sigma) \oplus_{\mathbf{R}} T_x\Sigma = T_xX$, where j_x is the complex structure (multiplication by $\sqrt{-1}$).

5.2. **Preliminaries and conventions.**

5.2.1. *Choice of Halphen twists.* Recall from Remark 4.13 that under the assumptions of Proposition 4.15, $\mu_g(\mathrm{R}_g) = 1$ for any $g \in \mathrm{Hal}(\Gamma)$. We fix a pair of elements $(g, h) \in \mathrm{Hal}(\Gamma)^2$ associated to different fibrations. This pair will be kept fixed until Subsection 5.7 . Recall from Notation 4.1, that we denote by X^g_w the fiber $\pi_g^{-1}(w)$, and similarly for h.

5.2.2. *A decomposition of the tangency locus.* By definition, the tangency locus $\mathrm{Tang}(\pi_g, \pi_h)$ is the locus where the map $(\pi_g, \pi_h)\colon X \to B_g \times B_h$ is not a local diffeomorphism; $\mathrm{Tang}(\pi_g, \pi_h)$ is a curve that contains all multiple components of fibers of π_g and π_h, as well as the curves along which the foliations determined by these fibrations are tangent. To be more precise, we split $\mathrm{Tang}(\pi_g, \pi_h)$ into four parts,

$$\mathrm{Tang}^{\mathrm{ff}}(\pi_g, \pi_h) \cup \mathrm{Tang}^{\mathrm{ft}}(\pi_g, \pi_h) \cup \mathrm{Tang}^{\mathrm{tf}}(\pi_g, \pi_h) \cup \mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h) \tag{5.1}$$

which are defined as follows. An irreducible component C of $\mathrm{Tang}(\pi_g, \pi_h)$ is

- contained in $\mathrm{Tang}^{\mathrm{ff}}(\pi_g, \pi_h)$ if and only if C is contained in a fiber of π_g and in a fiber of π_h (in that case, C is both g and h-invariant, and its self intersection is negative, see § 4.1);
- contained in $\mathrm{Tang}^{\mathrm{ft}}(\pi_g, \pi_h)$ if and only if C is a multiple component of a fiber of π_g but is generically transverse to π_h;
- contained in $\mathrm{Tang}^{\mathrm{tf}}(\pi_g, \pi_h)$ if and only if C is a multiple component of a fiber of π_h but is generically transverse to π_g;
- contained in $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h)$ if it is generically transverse to both fibrations.

The superscripts f, t stand for fiber and transverse.

Lemma 5.1. *If* $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h)$ *is empty, the fibrations* π_g *and* π_h *are both isotrivial.*

The isotriviality of π_g means that the j-invariant of the fibers X^g_w is constant on B°_g. In this case, the discussion of § 3.3.2 shows that no fiber of π_g is of type I_b or I^*_b, $b \geqslant 1$, and that after a finite base change, π_g becomes birationnally equivalent to a trivial fibration.

Proof. If $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h)$ is empty, the foliation associated to π_h is transverse to π_g on $\pi_g^{-1}(B^\circ_g)$ (though a multiple fiber of π_h may intersect every fiber of π_g). If $\beta\colon [0, 1] \to B^\circ_g$ is a smooth path, the holonomy of this foliation determines a holomorphic diffeomorphism $\mathrm{hol}_\beta\colon X^g_{\beta(0)} \to X^g_{\beta(1)}$. Thus, all fibers of π_g are isomorphic, and likewise for π_h. □

5.3. **Geometry of g-orbits.** Let us fix a Kähler metric on X, given by a Kähler form κ, as well as a Kähler form κ_g on B_g (hence also on B_h, see § 5.2.1). Lengths, areas, and diameters will be computed with respect to these metrics.

According to the Notation 3.13, a circle in an elliptic curve is a translate of a 1-dimensional, closed, and connected subgroup. Being invariant under translation, this notion is well defined on X^g_w, for every $w \in B^\circ_g$. If $w \in \mathrm{R}^k_g$ and $z \in X^g_w$, the closure of its g-orbit is a union of k circles; the circle containing z is denoted $L^0_g(z)$. In the next lemma we use the notation introduced in Section 3.5; the norm of a slope (p, q) is, by definition $\|(p, q)\| = (p^2 + q^2)^{1/2}$. Since circles are homotopically non-trivial and long circles become asymptotically dense, we get:

Lemma 5.2. *Let $U \Subset B^\circ_g$ be a disk, endowed with a continuous choice of basis for $H_1(X^g_w; \mathbf{Z})$ and a local section of π_g.*

(1) *There is a real number $\ell(U)$ such that the length of every circle of every fiber X^g_w, for $w \in U$, is bounded from below by $\ell(U)$.*
(2) *For every $\varepsilon > 0$, there is a real number $D > 0$ such that for all (p, q) with $\|(p, q)\| > D$, and all $w \in U$, every circle of slope (p, q) is ε-dense in X^g_w. In particular, for every $\varepsilon > 0$, there is a real number $D > 0$ such that if $\|(p, q)\| > D$, and $w \in \mathrm{R}^k_{p,q}(U) \setminus \mathrm{Tor}(U)$ for some k then the circle $L^0_g(z)$ is ε-dense in X^g_w for every $z \in X^g_w$.*
(3) *For every $\varepsilon > 0$, there is an integer $k_0 > 0$ such that for every $k \geqslant k_0$, every $w \in \mathrm{R}^k_{p,q}(U) \setminus \mathrm{Tor}(U)$, and every $z \in X^g_w$, the orbit closure $L_g(z)$ is ε-dense in X^g_w.*

As a consequence, if K is a compact subset of B°_g, there is a real number $\ell(K)$ such that the length of every circle of every fiber X^g_w, for $w \in K$, is bounded from below by $\ell(K)$. Another consequence is:

Lemma 5.3. *For every $\delta_1 > 0$ there exists $\delta_2 > 0$ such that if $w \in B_g$ is δ_2-far from $\mathrm{Crit}(\pi_g)$, any circle in X^g_w escapes the δ_1-neighborhood of $\mathrm{Tang}(\pi_g, \pi_h)$.*

Let η denote half of the injectivity radius of the metric κ_g. For every $w \in B_g$, the (riemannian) exponential map is a diffeomorphism from the disk of radius η in $T_w B_g$ to some open subset of B_g. By definition, the diameter of an interval $J \subset T_w B_g$ is its length with respect to $\kappa_{g,w}$; its radius is half its diameter.

Now, let $I \subset B_g$ be a smooth real analytic arc, and let w be a point of I. The tangent direction to I at w determines an orthogonal decomposition of $T_w B_g$ into a direct sum $T_w I \oplus_{\mathbf{R}} (T_w I)^\perp$. Let r be a positive number $\leqslant \eta$. We say that I **is of size** (at least) r at w if its preimage in $T_w B_g$ by the exponential map contains the graph $\{s + \psi(s) \; ; s \in J\}$ of a function $\psi\colon J \subset T_w I \to (T_w I)^\perp$ such that:

(i) ψ is defined on an interval J of radius r around 0 in $T_w I$;
(ii) $\psi(0) = 0$, its first derivative ψ' is bounded by 1, and its second derivative ψ'' is bounded by $1/r$.

Note that since ψ' is bounded by 1, ψ takes its values in an interval of length $\leqslant r$ in $(T_w I)^\perp$.

This definition is scale invariant. Similar notions can be defined on B_h, or on X, or along the fibers X^g_w. When talking about the size of an arc, it should be clear from the context whether we are working in X, B_g, or B_h.

Since circles on a torus are geodesics for the flat metrics, we get:

Lemma 5.4. *For every $\delta > 0$ there exists $r_1 = r_1(\delta) > 0$ such that if $w \in B_g$ is such that $\mathsf{dist}(w, \mathrm{Crit}(\pi_g)) > \delta$, then any circle of X^g_w is of size at least r_1 at any of its points.*

The ramification points of the restriction of π_g to the leaves of the foliation induced by π_h are located in $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h) \cup \mathrm{Tang}^{\mathrm{ft}}(\pi_g, \pi_h) \cup \mathrm{Tang}^{\mathrm{ff}}(\pi_g, \pi_h)$. This implies:

Corollary 5.5. *For every $\delta > 0$ there exists $r_2 = r_2(\delta) > 0$ such that if $\xi \in X$ is δ-far from $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h) \cup \mathrm{Tang}^{\mathrm{ft}}(\pi_g, \pi_h) \cup \mathrm{Tang}^{\mathrm{ff}}(\pi_g, \pi_h)$, then the projection under π_g of any circle in $X^h_{\pi_h(\xi)}$ has size at least r_2 at $\pi_g(\xi)$.*

Of course a similar result holds by swapping g and h.

5.4. **Local structure of μ.** Recall that we work under the hypotheses of Proposition 4.15. By assumption, $\mu_g(\mathrm{R}_g)$ is positive and $\mu(\mathrm{Tang}(\pi_g, \pi_h)) = 0$, because the tangency locus $\mathrm{Tang}(\pi_g, \pi_h)$ is a proper algebraic subset of X. And by Remark 4.13, $\mu_g(\mathrm{R}_g) = \mu_h(\mathrm{R}_h) = 1$.

Pick an analytic arc $\gamma \in \mathrm{R}_g$ such that $\mu_g(\gamma) > 0$; shrinking γ if necessary, we choose an open subset U as in Section 3.5, and parameters (α, β) and (p, q) such that γ is a smooth, real analytic subset of $R^{\alpha,\beta}_{p,q}(g; U)$ diffeomorphic to an interval. Then

$$0 < \mu(\pi_g^{-1}(\gamma)) = \mu(\pi_g^{-1}(\gamma) \cap \pi_h^{-1}(\mathrm{R}_h)) \leqslant 1; \tag{5.2}$$

consequently, there is an open subset U' of B°_h and a smooth analytic arc γ' in some $R^{a',b'}_{p',q'}(h; U')$ such that $\mu(\pi_g^{-1}(\gamma) \cap \pi_h^{-1}(\gamma')) > 0$. On the complement of $\mathrm{Tang}(\pi_g, \pi_h)$, the intersection $\pi_g^{-1}(\gamma) \cap \pi_h^{-1}(\gamma')$ is transverse; thus reducing γ, U, γ' and U' again if necessary, we may assume that (π_g, π_h) is a local diffeomorphism from $X^g_U \cap X^h_{U'}$ to $U \times U'$; in particular the intersection $\pi_g^{-1}(\gamma) \cap \pi_h^{-1}(\gamma')$ is transverse. Then, the set

$$\pi_g^{-1}(\gamma) \cap \pi_h^{-1}(\gamma') = S_1 \sqcup \cdots \sqcup S_l \tag{5.3}$$

is a disjoint union of small "squares" – diffeomorphic to $\gamma \times \gamma'$ – which, for $w \in \gamma$, intersect the fiber X^g_w along pieces of circles (of the same slope). In what follows, we denote by S any one of the squares S_j such that $\mu(S_j) > 0$; hence

$$S \subset \pi_g^{-1}(\gamma) \cap \pi_h^{-1}(\gamma'), \quad \mu(S) > 0, \tag{5.4}$$

and $(\pi_g, \pi_h)\colon S \to \gamma \times \gamma'$ is a diffeomorphism. Thus, $d_{\mathbf{R}}(\mu) \leqslant 2$, and Lemma 4.14 implies

Lemma 5.6. $d_{\mathbf{R}}(\mu) = 2$.

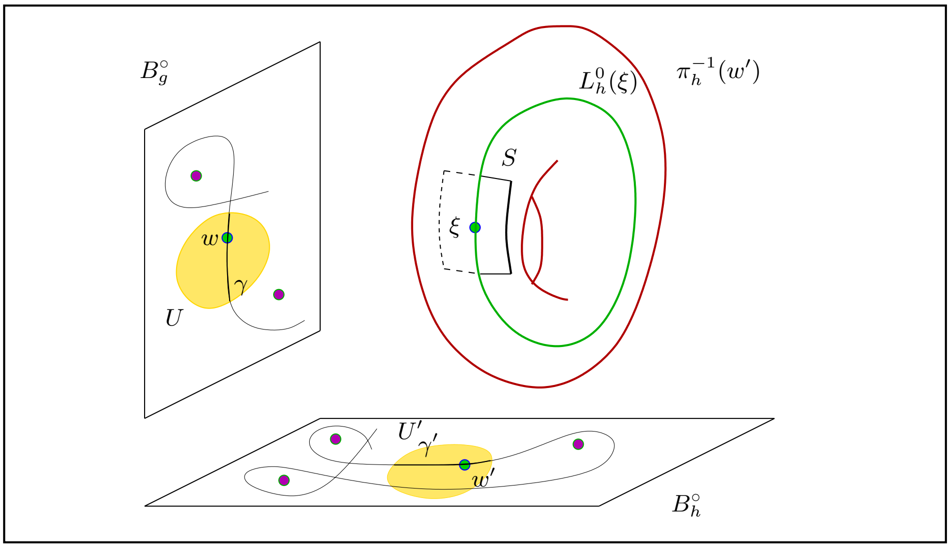


FIGURE 2. This figure represents (one of) the square(s) S. The intersection of S with $X_{w'}^h$ is contained in the orbit closure $L_h^0(z)$. The points ξ, w, w' are in green; the magenta points are critical values of π_g and π_h. The pre-image of γ in $X_{w'}^h$ is locally contained in $L_h^0(\xi)$.

We now initiate the study of the analytic continuation of γ. We say that an arc $J \subset B_g$ is *evenly charged* by μ_g if $\mu_g(J') > 0$ for every non-empty, open interval $J' \subset J$. Likewise, we say that a surface S in X is evenly charged by μ if $\mu(S') > 0$ for any non-empty relatively open subset $S' \subset S$.

For $w' \in \gamma' \setminus \mathrm{Tor}(B_h^\circ)$ and $\xi' \in X_{w'}^h$, $L_h(\xi')$ is a finite union of circles of $X_{w'}^h$, and $L_h^0(\xi')$ is invariant under some iterate of h. For μ_h-almost every w', the conditional measure $\mu_{h,w'}$ is invariant under $h_{w'}$; as such, it is supported on a union of such orbit closures. The set $S \cap X_{w'}^h$ is an interval, which is a piece of the circle $L_h^0(\xi')$, for $\xi' \in S \cap X_{w'}^h$. The set[4]

$$L_h(S) = \overline{\bigcup_{\xi' \in S} L_h(\xi')} \text{ (resp. } L_h^0(S) = \overline{\bigcup_{\xi' \in S} L_h^0(\xi')}\text{),} \tag{5.5}$$

is a finite union of real analytic annuli – each of which a circle bundle above γ'– which is h-invariant (resp. is an h^k-invariant annulus, for some $k > 0$). For $\xi' \in S$ such that $w' := \pi_h(\xi') \notin \mathrm{Tor}_h(B_h^\circ)$ the restriction of the conditional measure $\mu_{h,w'}$ to $L_h^0(\xi')$ is the Haar measure. Projecting under the real analytic map π_g, we get:

Lemma 5.7. *With the above notation, $\pi_g(L_h^0(S))$ is a semi-analytic curve containing γ, which is evenly charged by μ_g. In particular any (semi-)analytic continuation of γ contains $\pi_g(L_h^0(S))$.*

[4]We must take the closure in Equation 5.5 because $L_h^0(z')$ is reduced to $\{z'\}$ when $\pi_h(z') \in \mathrm{Tor}(B_h^\circ)$.

Remark 5.8. Pick ξ' in S and set $w' = \pi_g(\xi')$. If $L_h^0(\xi')$ is disjoint from the ramification points of $\pi_g|_{X^h_{w'}} : X^h_{w'} \to B_g$, then $\pi_g(L_h^0(S))$ is an analytic loop in B_g, and this loop provides a complete description of the analytic continuation of γ. On the other hand, if $L_h^0(\xi')$ hits a ramification point of $\pi_g|_{X^h_{w'}}$, its image $\pi_g(L_h^0(\xi'))$ should be thought of as a segment which is strictly contained in the analytic continuation of γ, and whose endpoints are contained in the projections of the ramification points. As ξ' moves in S, the endpoints and the length of $\pi_g(L_h^0(\xi'))$ may vary within a curve contained in R_g and after successively saturating as in (5.5), μ_g might fill up a complicated, possibly dense, curve in B_g.

Lemma 5.9. *The restriction of μ to S is absolutely continuous with respect to the 2-dimensional Lebesgue measure on S and its density with respect to any real analytic area form on S is given by a positive real analytic function.*

The argument is the same as in Step 5 in the proof of Proposition 4.9. Now, from Section 5.3, we get the following a priori bound on the analytic continuation of γ.

Lemma 5.10. *For every $\delta > 0$ there exists $r = r(\delta) > 0$ such that if $\gamma \in \mathrm{R}_g$ is an (analytic) arc with $\mu_g(\gamma) > 0$ and w is a point of γ such that $\mathsf{dist}(w, \mathrm{Crit}(\pi_g)) > \delta$, then γ admits an analytic continuation to an analytic arc of size r at w, which is evenly charged by μ_g.*

Proof. We may assume that $w \in \gamma \setminus \mathrm{Tor}(B_g)$. With notation as above, pick $\xi \in X^g_w \cap S$. By Lemma 5.3, there exists $\xi' \in L_g^0(\xi)$ such that $\mathsf{dist}(\xi', \mathrm{Tang}(\pi_g, \pi_h)) > \delta_2 = \delta_2(\delta)$. By g-invariance of μ, ξ' plays the same role as ξ. Now by Corollary 5.5, $\pi_g(L_h^0(\xi'))$ has size $r_2(\delta_2)$ at $\pi_g(\xi') = w$ and applying Lemma 5.7 concludes the proof. ☐

Remark 5.11. It may happen that the local equation for some $\mathrm{R}^{\alpha,\beta}_{p,q}(B_g^\circ)$ is of type $\mathsf{Im}(w^k) = 0$, with w in a small disk $\mathbb{D}_\varepsilon \subset B_g^\circ$ (see the definition of NT_g in § 3.4). At the origin of such a disk, the curve is singular, with several branches going through the origin. Lemma 5.10 says that if $\varepsilon \ll r$ and γ is a smooth analytic arc in one of these branches, then it can be continued to an evenly charged analytic arc accross the origin. So, if one of the branches is charged by μ, its symmetric with respect to the origin is charged too.

5.5. **Analytic continuation of γ: the isotrivial case.** From this point the proof splits in two separate arguments according to the isotrivial or non-isotrivial nature of the fibrations (more precisely, according to the emptyness, or not, of $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h)$).

Lemma 5.12. *If π_g is isotrivial there is an analytic subset $\tilde{\sigma}$ of B_g such that:*

(1) *$\tilde{\sigma}$ is of pure dimension 1 and extends γ;*

(2) $\mu_g(\tilde\sigma) > 1/2$ *and* $\mathrm{Supp}(\mu_g|_{\tilde\sigma})$ *is a semi-analytic set* $\sigma \subset \tilde\sigma$*; it is a finite union of immersed real analytic loops in* B_g° *and immersed real analytic arcs with endpoints in* $\mathrm{Crit}(\pi_g)$.

Proof. As observed in § 5.2.2, π_g becomes birationally equivalent to a trivial fibration after a finite base change $B' \to B_g$. In particular its monodromy is finite and the curves $R^k_{p,q}$ define global analytic subsets of B'. Coming back to $\pi_g \colon X \to B_g$, the local curves $R^k_{p,q}(U)$ extend as (singular) global analytic subsets of B_g. Since $\mu_g(\mathrm{R}_g) = 1$ we can find a finite number of smooth analytic arcs $\gamma_1, \ldots, \gamma_\ell$ contained in R_g such that $\mu_g(\gamma_1 \cup \cdots \cup \gamma_\ell) > 1/2$. Since every γ_j is contained in some $R^k_{p,q}(U)$, it is contained in a global analytic subset $\tilde\sigma_j$ of B_g. We put $\tilde\sigma = \tilde\sigma_1 \cup \cdots \cup \tilde\sigma_\ell$. By Lemma 5.10, every irreducible component of $\tilde\sigma \cap B_g^\circ$ of positive mass is evenly charged by μ_g. To conclude, we define σ to be the closure of the union of the components of $\tilde\sigma \cap B_g^\circ$ charged by μ. □

5.6. **Analytic continuation of γ: the general case.** By Lemma 5.1, if $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h) = \emptyset$ then π_g and π_h are isotrivial; in that case, Lemma 5.12 applies. Now, we assume $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h) \neq \emptyset$, and our goal is to establish the following lemma.

Lemma 5.13. *If* $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h) \neq \emptyset$*, then there exists a unique analytic curve* σ_g *in* B_g° *such that*

(1) *if* μ *is any ergodic* Γ*-invariant probability measure such that* $\mu_g(\mathrm{R}_g) > 0$*, then* $\mu_g(\sigma_g) = 1$*;*
(2) *if* $\gamma \subset \sigma_g$ *is any arc, then* $\mu(\gamma) > 0$ *for at least one such measure.*

This result is both stronger and weaker than Lemma 5.12: indeed its conclusion holds for *all invariant measures with* $\mu_g(R_g) > 0$ (this fact will be important for Theorem C); on the other hand it gives no information on the structure of the analytic continuation σ_g near $\mathrm{Crit}(\pi_g)$ (this issue will be investigated in the next section).

The curve σ_g is **defined** by this lemma. Note that, at this stage of the proof, it could contain for instance a sequence of small topological circles converging to a critical value of π_g. We shall exclude such a possibility later.

5.6.1. *An elementary lemma.* Let $M_k : \mathbf{C} \to \mathbf{C}$ be the monomial $M_k(z) = z^k$.

Lemma 5.14. *Let* r *be a positive real number. For any* $0 < \varepsilon \ll r$ *there is a constant* $C_r(\varepsilon) > 0$ *with the following property. If* $z_0 \in \mathbf{C}$ *satisfies* $|z_0| \leqslant \varepsilon$*, and if* γ *is an analytic arc of size* $\geqslant r$ *at* z_0 *with* $0 \notin \gamma$*, then* $M_k(\gamma)$ *contains a point at which the curvature is* $\geqslant C_r(\varepsilon)\varepsilon^{-k}$.

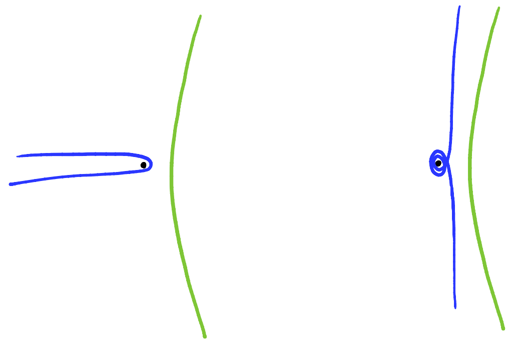


FIGURE 3. A (green) curve of bounded geometry passing near the origin and its image (in blue) by $z \mapsto z^2$ on the left, and $z \mapsto z^5$ on the right.

Proof. We prove it via an explicit computation; see Figure 5.6.1 for a graphical explanation. We may assume that $r = 1$, so that $0 < \varepsilon \ll 1$, and that γ is of size 1 (we truncate γ if necessary). Let z_1 be the point closest to 0 on γ. Applying a rotation we may assume that $z_1 = \eta \in \mathbf{R}_+$ for some $\eta \leqslant \varepsilon$. Since z_1 is closest to the origin, γ is a graph over the y-axis: it is of the form $x = \varphi(y)$, with $\varphi(y) = \eta + \alpha y^2 + O(y^3)$. Since γ has size 1 at z_0 and $\varepsilon \ll 1$, it has size $\geqslant 1/2$ at z_1, in particular $|\alpha| \leqslant 2$. In polar coordinates (ρ, θ), the equation of γ is of the form

$$\rho = \psi(\theta) = \eta\left(1 + \left(\frac{1}{2} + \alpha\eta\right)\theta^2 + O(\theta^3)\right) \tag{5.6}$$

for θ small. Thus the polar equation of $M_k(\gamma)$ near $\theta = 0$ is

$$\rho = \psi(\theta/k)^k = \eta^k\left(1 + \frac{1}{k}\left(\frac{1}{2} + \alpha\eta\right)\theta^2 + O(\theta^3)\right), \tag{5.7}$$

and finally the value of the curvature at $\theta = 0$ is

$$\left.\frac{\rho^2 + 2(\rho')^2 - \rho\rho''}{(\rho^2 + (\rho')^2)^{3/2}}\right|_{\theta=0} = \eta^{-k}\frac{k - 1 - 2\alpha\eta}{k} \geqslant \eta^{-k} \geqslant \varepsilon^{-k}. \tag{5.8}$$

This completes the proof. □

5.6.2. *Extension of γ in B_g°: proof of Lemma 5.13.* Let μ be an arbitrary Γ-invariant and ergodic measure, giving no mass to algebraic subsets, and assume that $\mu_g(\mathrm{R}_g) > 0$. Then, by Remark 4.13, μ_g gives full mass to R_g.

Fix a small open disk $U \Subset B_g^\circ$ in which we have fixed a section of π_g and a continuous choice of basis for $H^1(X_w^g, \mathbf{Z})$. In U, R_g is the union of the analytic curves $\mathrm{R}_{p,q}^k(U)$ (see §3.5). To prove the lemma, we seek a uniform bound on $\max(k, \|(p,q)\|)$ for the indices $(k, (p,q))$ such that $\mu_g(\mathrm{R}_{p,q}^k(U)) > 0$ for at least one Γ-invariant ergodic measure μ.

Note that even for a single μ, μ_g could charge infinitely many of the $\mathrm{R}^k_{p,q}(U)$; due to the monodromy of the fibration, the analytic continuation of an arc γ of positive mass could come back infinitely many times in U, each time along a new $\mathrm{R}^k_{p,q}(U)$.

Suppose that there is a sequence of curves $\gamma_n = \mathrm{R}^{k_n}_{p_n,q_n}(U)$ with $\max(k_n, \|(p_n, q_n)\|) \to \infty$ such that $\mu_{n,g}(\gamma_n) > 0$ for some Γ-invariant measures μ_n (that depend a priori on n). The γ_n are evenly charged (by μ_n), and by Lemma 5.10 they have uniformly bounded geometry (the constant $r(\delta)$ in Lemma 5.10 does not depend on μ_n). In particular the accumulation locus of (γ_n) is uncountable.

Over U, there exists $\varepsilon = \varepsilon(k, p, q)$, with $\varepsilon(k, p, q) \to 0$ as $\max(k, \|(p, q)\|) \to \infty$ such that for every $w \in U$, any translate of $L_{g,w}(k, (p, q))$ is ε-dense in X^g_w (see § 3.5 and Lemma 5.2). Hence, there is a sequence $(\varepsilon_n) \in (\mathbf{R}^*_+)^{\mathbf{N}}$ converging to 0 such that $L_g(\xi)$ is ε_n-dense in X^g_w for any $w \in \gamma_n \backslash \mathrm{Tor}_g(U)$ and $\xi \in X^g_w$.

Since $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h)$ is non-empty, it intersects every X^g_w along some non-empty finite subset, which by definition is not persistently contained in $\mathrm{Sing}(\pi_h)$. Furthermore $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h)$ is generically transverse to π_g. So we can pick w_0 in the accumulation set of (γ_n) together with $\xi_{\mathrm{t}} \in \mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h) \cap X^g_{w_0}$ such that $\pi_h(\xi_{\mathrm{t}})$ is 2δ-far from $\mathrm{Crit}(\pi_h)$ for some $\delta > 0$, and ξ_{t} can be locally holomorphically followed as a point $\xi_{\mathrm{t}}(w)$ in $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h) \cap X^g_w$ for w near w_0.

Pick a sequence $w_n \in \gamma_n \backslash \mathrm{Tor}_g(U)$ converging to w_0, and consider the corresponding sequence $\xi_{\mathrm{t}}(w_n)$. For each n there is a finite union of annuli $A_n \subset X^g_{\gamma_n}$ such that π_g maps A_n onto γ_n, $\pi_g \colon A_n \to \gamma_n$ is a fiber bundle whose fibers are unions of k_n circles of type $L^0_g(\xi)$, and μ charges A_n evenly. We can choose $\xi'_n \in A_n$ such that $0 < \mathsf{dist}(\xi'_n, \xi_{\mathrm{t}}(w_n) < 2\varepsilon_n$. In $X^g_{w_n}$ the curve $L^0_g(\xi'_n)$ is a circle, with uniformly bounded geometry. Near $\xi_{\mathrm{t}}(w_n)$ the restriction of π_h to $X^g_{w_n}$ is locally conjugate to $z \mapsto z^k$ where $k \geqslant 2$ is the order of tangency between π_g and π_h along $\mathrm{Tang}^{\mathrm{tt}}(\pi_h, \pi_h)$ at $\xi_{\mathrm{t}}(w_n)$. The local change of coordinates that transforms π_h into $z \mapsto z^k$ depends on the fiber, hence on w_n, but the first and second derivatives of these changes of coordinates are uniformly controlled, independently of n. Thus, Lemma 5.14 shows that the curvature of the projections $\pi_h(L^0_g(\xi'_n))$ goes to $+\infty$ as $n \to \infty$. But $\pi_h(L^0_g(\xi'_n))$ is a piece of R_h which is charged by μ_h and is δ-far from $\mathrm{Crit}(\pi_h)$ for large n; so by Lemma 5.10, its curvature must be uniformly bounded (with respect to n). This is a contradiction, and the proof of Lemma 5.13 is complete. □

Remark 5.15. The proof shows that under the assumptions of Lemma 5.13, if γ is an arc of a curve $\mathrm{R}^k_{p,q}$ such that $\mu_g(\gamma) > 0$ for some invariant probability measure μ, then $\max(k, \|(p, q)\|)$ is uniformly bounded.

5.7. **Conclusion of the proof of Proposition 4.15.**

Proof of Proposition 4.15. Let g be as in the statement of the proposition. Recall the definition of STang_Γ from (4.4):

$$\mathrm{STang}_\Gamma = \bigcap_{(h_1,h_2)\in \mathrm{Hal}(\Gamma)^2} \left(\mathrm{Sing}(\pi_{h_1}) \cup \mathrm{Sing}(\pi_{h_2}) \cup \mathrm{Tang}(\pi_{h_1}, \pi_{h_2})\right) \tag{5.9}$$

By the Noether property, this infinite intersection can be written as a finite intersection. Thus, we can choose a finite set of Halphen twists $(g_i)_{1\leqslant i\leqslant s}$ with $g_1 = g$ and such that if $x \notin \mathrm{STang}_\Gamma$, there exist $i \neq j$ such that the fibers of π_{g_i} and π_{g_j} at x are smooth fibers and are transverse at x.

For $x \notin \mathrm{STang}_\Gamma$, there are $k < \ell$ and a (Zariski) neighborhood $V \ni x$ disjoint from $\mathrm{Sing}(\pi_{g_k}) \cap \mathrm{Sing}(\pi_{g_\ell})$ such that π_{g_k} and π_{g_ℓ} are transverse in V. Let us show that in V, $\mathrm{Supp}(\mu)$ is a real analytic set satisfying the conclusions of the proposition. There are two possibilities:

- (A1) either there exists $k \neq \ell$ such that $\mathrm{Tang}^{\mathrm{tt}}(\pi_{g_k}, \pi_{g_\ell}) = \varnothing$; then by Lemma 5.1 π_{g_k} and π_{g_ℓ} are isotrivial;
- (A2) or for every $k \neq \ell$, $\mathrm{Tang}^{\mathrm{tt}}(\pi_{g_k}, \pi_{g_\ell}) \neq \varnothing$.

• Let us first complete the proof in case (A1). Fix $k \neq \ell$ given by Condition (A1) and apply Lemma 5.12: there exist analytic curves $\tilde\sigma_{g_k}$ in B_{g_k} and $\tilde\sigma_{g_\ell}$ in B_{g_ℓ} such that $\mu_{g_k}(\tilde\sigma_k) > 1/2$ and $\mu_{g_\ell}(\tilde\sigma_\ell) > 1/2$. Then $\tilde\Sigma_{kl} := \pi_{g_k}^{-1}(\tilde\sigma_{g_k}) \cap \pi_{g_\ell}^{-1}(\tilde\sigma_{g_\ell})$ is a real analytic subset of X such that $\mu(\tilde\Sigma_{kl}) > 0$. Observe that $\tilde\Sigma_{kl}$ is C-analytic in X (see § 5.1 for this notion): indeed the curves $\tilde\sigma_{g_k}$ and $\tilde\sigma_{g_k}$, being of dimension 1, are defined by global equations in B_{g_k} and B_{g_ℓ}, hence so does $\tilde\Sigma_{kl} = \pi_{g_k}^{-1}(\tilde\sigma_{g_k}) \cap \pi_{g_\ell}^{-1}(\tilde\sigma_{g_\ell})$ on X. Let $\tilde\Sigma_0$ be an irreducible component of $\tilde\Sigma_{kl}$ such that $\mu(\tilde\Sigma_0) > 0$. If f is an element of Γ, $\mu(f(\tilde\Sigma_0)) = \mu(\tilde\Sigma_0)$, and $\mu(f(\tilde\Sigma_0) \cap \tilde\Sigma_0) = 0$, unless $f(\tilde\Sigma_0) \cap \tilde\Sigma_0$ contains a non-empty, relatively open subset. In the latter case $f(\tilde\Sigma_0) = \tilde\Sigma_0$. Thus, a finite index subgroup Γ_0 of Γ fixes $\tilde\Sigma_0$. Define

$$\tilde\Sigma = \bigcup_{f\in\Gamma} f(\tilde\Sigma_0) = \bigcup_{f\in\Gamma/\Gamma_0} f(\tilde\Sigma_0) \tag{5.10}$$

This is a C-analytic subset of X (hence of $X\setminus\mathrm{STang}_\Gamma$) such that $\mu(\tilde\Sigma) = 1$. We finally define Σ to be the union of the irreducible components of $\tilde\Sigma\setminus\mathrm{STang}_\Gamma$ that are charged by μ. Using the Γ-invariance of $X\setminus\mathrm{STang}_\Gamma$ and repeating the previous argument shows that Σ has finitely many irreducible components which are permuted by Γ (Assertion (2) of the proposition), and by construction Σ (hence $\overline{\Sigma}$) is semi-analytic in X. By definition $\mu(\Sigma) = 1$.

Now choose any irreducible component Σ_1 of Σ and any $x \in \Sigma_1$. Around x, there are two transverse projections π_{g_k} and π_{g_ℓ}. The results of §§ 5.3 and 5.4 imply that Σ_1 is locally a finite union of "squares" of the form $\pi_g^{-1}(\gamma) \cap \pi_h^{-1}(\gamma')$ (where γ and γ' are smooth analytic curves), along which μ has a positive real analytic density. In particular $\mu(\Sigma_1) > 0$ and Σ_1 is evenly charged by μ. Thus, Assertions (1) and (4) of the proposition are established.

It remains to prove Assertion (3). For this, note that if non-empty, the 1-dimensional part of $\mathrm{Sing}(\Sigma)$ is Γ-invariant. Suppose we can find an arc β in $\mathrm{Sing}(\Sigma)$. If $\pi_g(\beta)$ were not a point, the closure of the g-orbit of β would contain a 2-dimensional annulus that would be contained in $\mathrm{Sing}(\Sigma)$: contradiction. Thus $\pi_g(\beta)$ is a point, and so is $\pi_{g'}(\beta)$ for any $g' \in \mathrm{Hal}(\Gamma)$. This contradicts $\beta \subset X \backslash \mathrm{STang}^1_\Gamma$. Thus, $\mathrm{Sing}(\Sigma)$ is a discrete subset of $X \backslash \mathrm{STang}^1_\Gamma$.

• In case (A2), we fix $k \neq \ell$ and apply Lemma 5.13 to π_{g_k} and π_{g_ℓ}: it yields analytic curves σ_{g_k} in $B^\circ_{g_k}$ and σ_{g_ℓ} in $B^\circ_{g_\ell}$ such that $\mu_{g_k}(\sigma_k) = 1$ and $\mu_{g_\ell}(\sigma_\ell) = 1$. Set $\tilde{\Sigma}_{kl} := \pi_{g_k}^{-1}(\sigma_{g_k}) \cap \pi_{g_\ell}^{-1}(\sigma_{g_\ell})$; it is a 2-dimensional totally real C-analytic subset of $X \backslash (\mathrm{Sing}(\pi_{g_k}) \cup \mathrm{Sing}(\pi_{g_\ell}))$. We then further restrict it to $X_{k\ell} := X \backslash \mathrm{STang}(\pi_{g_k}, \pi_{g_\ell}))$ and define $\Sigma_{k\ell}$ to be the union of irreducible components of $\tilde{\Sigma}_{k\ell} \cap X_{k\ell}$ of positive μ-mass. Note that $\mu(\Sigma_{k\ell}) = 1$. From the analysis of § 5.4, we know that any irreducible component of $\Sigma_{k\ell}$ is evenly charged by μ. So, for any other pair (k', ℓ'), the equality $\mu(\Sigma_{k'\ell'} \cap \Sigma_{k\ell}) = 1$ implies that the analytic sets $\Sigma_{k\ell}$ and $\Sigma_{k'\ell'}$ coincide on $X_{k\ell} \cap X_{k'\ell'}$. Thus the $\Sigma_{k\ell}$ patch together to form a real analytic subset Σ of $\bigcup_{k\neq\ell} X_{k\ell} = X \backslash \mathrm{STang}_\Gamma$. Since it is locally a finite union of 2-dimensional real analytic (and totally real) plaques, we infer that Σ is C-analytic in $X \backslash \mathrm{STang}_\Gamma$. Using the Γ-invariance of $X \backslash \mathrm{STang}_\Gamma$, we see that any component Σ_0 of positive mass is invariant under a finite index subgroup of Γ (see Equation (5.10) and the lines preceding it); hence, Σ has finitely many irreducible components. The remaining properties of Σ are obtained exactly as in case (A1), and the proof is complete. ☐

Remark 5.16. The proof shows that if the curve σ_g constructed in Lemma 5.13 is semi-analytic in B_g, then the surface Σ is semi-analytic in X. This holds automatically in case (A1).

6. SEMI-ANALYTICITY OF $\overline{\Sigma}$ AND COMPLEMENTS

In this section we continue the investigation of case (c) of Theorem A by studying the semi-analyticity of $\overline{\Sigma}$. This leads to Theorem A' in § 6.3, and also prepares the ground for Theorem C. We keep the choice of Halphen twists g, h from § 5.2.1. By Remark 5.16 the semi-analyticity of Σ is already established in the isotrivial case, so:

throughout this section we assume that $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h) \neq \varnothing$.

By Remark 5.16, we only need to show that the curve σ_g from Lemma 5.13 admits a semi-analytic continuation to B_g. So, the work takes place near the singular fibers of π_g.

6.1. **Preparation and strategy.** Recall that locally in B_g°, σ_g is a union of smooth branches with uniformly bounded geometry. Recall also from § 5.1 that analytic curves have well-defined irreducible components.

Lemma 6.1. *Assume that* $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h) \neq \emptyset$. *Any irreducible component of* σ_g *is either an analytic loop in* B_g° *or an immersed line converging to* $\mathrm{Crit}(\pi_g)$ *at its two endpoints.*

Here by analytic loop we mean an analytic immersion of the circle, with possible self-intersections. And by an immersed line we mean an analytic immersion of the real line $\mathbf{R} \to B_g^\circ$.

Proof. Let σ be a component of σ_g. If σ is compactly contained in B_g° then by Lemmas 5.10 and 5.13 it is an analytic curve in B_g and we are in the first situation. Otherwise, there is a semi-infinite branch σ^+ of σ; since σ is analytic outside the finite set $\mathrm{Crit}(\pi_g)$ and the accumulation set of σ^+ is connected, we deduce that the accumulation set of σ^+ is reduced to a singleton $\{c_0\} \subset \mathrm{Crit}(\pi_g)$. Then the second branch must accumulate $\mathrm{Crit}(\pi_g)$ as well (otherwise σ would be an analytic loop) and we are done. □

From now on we study the structure of σ_g locally near a fixed $s \in \mathrm{Crit}(\pi_g)$. We already know that σ_g is locally a union of smooth branches of the form $\mathrm{R}_{p,q}^{\alpha,\beta}$, with $(\alpha, \beta, p, q) \in \mathbf{Q}^2/\mathbf{Z}^2 \times \mathbf{Z}^2$. The study of σ_g will employ two types of arguments:

- first, we only use the dynamics of g and analyze the curves $\mathrm{R}_{p,q}^{\alpha,\beta}$ near s, as started in Section 3. The rest of the group Γ is not taken into account. This allows us to make some operations (base change, blowing down (-1)-curves in fibers of π_g, etc) which are not Γ-equivariant but preserve the curves $\mathrm{R}_{p,q}^{\alpha,\beta}$. This analysis, which is also crucial for Theorem C, is developed in §6.2.
- then in §6.3 we take into account the whole action of Γ on the singular fibers; doing so, we have to work on X, without simplifying the fibration π_g. This is where condition (AC) enters into play.

6.2. **Geometry of the curves** $\mathrm{R}_{p,q}^{\alpha,\beta}$**.** As explained above, in this paragraph, the only information that we retain from the dynamics of Γ is the existence of the curve σ_g, its geometric properties, and the fact that it is locally a union of smooth branches of the form $\mathrm{R}_{p,q}^{\alpha,\beta}$. After base change and some birational modification, as described in §§ 3.3.3 and 3.3.4, which we simply refer to as the "reduction" of the singular fiber, we only have to consider a central fiber X_s^g of type I_0 (that is, a regular fiber) or I_b with $b \geqslant 1$.

6.2.1. *Case 1: type* I_0. This corresponds to the stable reduction of fibers of type ${}_mI_0$, II, III, IV, I_0^*, II^*, III^*, IV^* and their blow-ups.

Lemma 6.2. *If the reduction of X_s^g is of type I_0, then σ_g admits a semi-analytic extension at s. In particular it admits only finitely many irreducible components in a neighborhood of s.*

Proof. We may, and do assume that X_s^g is of type I_0 (but the results obtained so far for σ_g hold only outside s).

Fix a pair of disks $V \Subset V'$ centered at s, with a local section of π_g and a fixed basis for $H_1(X_{V'}^g, \mathbf{Z})$. Recall the real-analytic map $T\colon V' \to \mathbf{R}^2$ introduced in §3.2 and the definition (3.16) of $\mathrm{R}_{p,q}^{\alpha,\beta}(V)$ in terms of T, which says that locally $\mathrm{R}_{p,q}^{\alpha,\beta}(V)$ is the preimage of a straight line under T, in particular every branch of $\mathrm{R}_{p,q}^{\alpha,\beta}(V)\setminus\{s\}$ admits a semi-analytic extension at s. In V, σ_g is a union of branches of $\mathrm{R}_{p,q}^{\alpha,\beta}(V)\setminus\{s\}$. We need to show that σ_g includes only finitely many such branches. For this, observe that by Lemma 5.13, σ_g is analytic in $V'\setminus V$. So if we can show that any irreducible component of σ_g in V reaches ∂V, the finiteness follows. This relies on a topological argument that avoids explicit computations and will be used again below to prove Lemma 6.3.

By Lemma 3.9 we may assume that either T is a diffeomorphism from V' to V' or that s is the unique critical point of T in V'. Suppose that some branch γ of $\mathrm{R}_{p,q}^{\alpha,\beta}(V)\setminus\{s\}$ is completely contained in V (see Figure 4 below): it is either an analytic loop in $V\setminus\{s\}$ or an immersed arc clustering at s at its two ends. We claim that γ contains a critical point of T, which is a contradiction. Indeed, parameterize γ by a smooth immersion $\varphi : (0,1) \to V$, which extends continuously to $[0,1]$, with $\varphi(0) = \varphi(1) = c$ (with $c = s$ if γ is not a loop). Then, $T \circ \varphi$ is a smooth map from $(0,1)$ to the line $(\alpha,\beta) + \mathbf{R}(p,q)$,

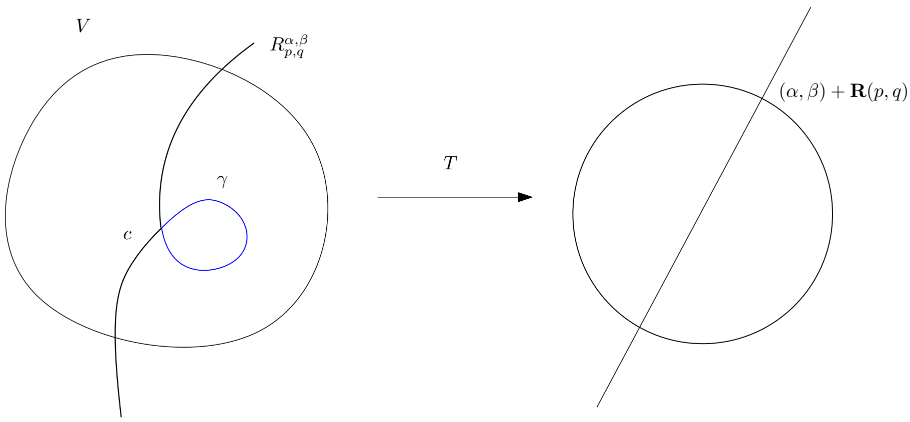


FIGURE 4.

it is continuous up to the boundary, and $T \circ \varphi(0) = T \circ \varphi(1)$; by Rolle's Theorem it admits a critical point in $(0,1)$, as claimed. $\square$

6.2.2. *Case 2: type I_b, $b \geqslant 1$.* This corresponds to the stable reduction of fibers of type I_b, ${}_mI_b$, I_b^* (with $b \geqslant 1$), and their blow-ups. As before we fix a pair of disks $V \Subset V'$ centered at s. The real analytic map T is only locally defined in $V \setminus \{s\}$, but its critical points form a well defined subset of $V' \setminus \{s\}$ and, by Proposition 3.12, we can assume this set to be empty. We resume the computations from §3.6. From Equations (3.18) and (3.6), there is a coordinate w around s in which $s = 0$ and the equation of $R_{p,q}^{\alpha,\beta}$ is

$$\mathsf{Im}\left(\frac{t(w) - \left(\alpha + \beta\frac{b}{2\mathrm{i}\pi}\log w\right)}{p + q\frac{b}{2\mathrm{i}\pi}\log w}\right) = 0; \tag{6.1}$$

here $\log w$ is viewed as a multivalued function and $t(w)$ is a well-defined holomorphic function near the origin (see § 3.3.2). As already observed, the curve $R_{p,q}^{\alpha,\beta}$ depends on $(\alpha, \beta) \in \mathbf{Q}^2$ only through $q\alpha - p\beta \in \mathbf{Q}/\mathbf{Z}$; we denote by k the order of this torsion point $q\alpha - p\beta \in \mathbf{Q}/\mathbf{Z}$.

Lemma 6.3. *If $R_{p,q}^{\alpha,\beta}$ admits a compact component in $V \setminus \{s\}$, then $q = 0$ and this component winds non-trivially around s.*

Note that the notion of slope (p, q) depends a priori on our choice of local coordinates, but the property $q = 0$ does not: indeed the action of the monodromy around the loop is given by $(p, q) \mapsto (p + bq, q)$ (geometrically, circles of slope $(1, 0)$ correspond to the "vanishing cycle").

Proof. Let C be such a component. It can be parametrized by a degree 1 immersion $\varphi : \mathbb{S}^1 \to C$ (not necessarily injective, for C may have self-crossings). If C does not wind around the origin, there is a continuous determination of $T \circ \varphi$ on $\mathbb{S}^1$. In this way, $T \circ \varphi$ is a smooth map from $\mathbb{S}^1$ to a line of slope (p, q) in $\mathbf{R}^2$, so it admits a critical point. This contradicts our choice of V'. So, the winding number ℓ_C around the origin is not zero. Consider a small open disk $U \subset V \setminus \{s\}$ containing a point w_0 of C. If we follow the local determination of T along C by turning counterclockwise around the origin, T is composed with the monodromy $(u, v) \mapsto (u + \ell_C bv, v)$. Thus, in U, C locally satisfies an equation of the form $kT(x, y) \in \mathbf{R}(p, q)$ as well as $kT(x, y) \in \mathbf{R}(p + n\ell_C bq, q)$ for every $n \in \mathbf{Z}$. Since C has only finitely many components in U, this implies $q = 0$. □

Lemma 6.4. *Assume that the reduction of X_s^g is of type I_b, $b \geqslant 1$. Let C be a branch of σ_g in $V \setminus \{s\}$ with $q = 0$ along some open subset of C. Then C admits a semi-analytic extension at s.*

Proof. Since (p, q) is primitive, $p = 1$ and in some small disk $U \subset V \setminus \{s\}$, C is of the form $\mathrm{R}_{1,0}^{\alpha,\beta}$. We can choose (α, β) of the form $(0, \beta)$, and the local equation of C

becomes

$$\mathsf{Im}\left(t(w) - \beta\frac{b}{2\mathrm{i}\pi}\log w\right) = 0. \tag{6.2}$$

If $\beta = 0$ this defines an analytic subset of V. This set contains the origin if and only if the imaginary part ot $t(w)$ vanishes at $w = 0$, in which case C coincides with a branch of $\{\mathsf{Im}\,(t(w)) = 0\} \setminus \{0\}$.

If $\beta \neq 0$, we write $w = e^{-s}$ as in §3.6, where $s = x + iy$ ranges in some right half plane $x \geqslant x_0$. The Equation (6.2) rewrites as

$$\mathsf{Im}(t(e^{-s})) - \frac{\beta b}{2\pi}x = 0, \quad \text{that is,} \quad \tilde{t}(x,y) - \frac{\beta b}{2\pi}x = 0, \tag{6.3}$$

where $\tilde{t}(x,y)$ is (2π)-periodic in y and admits a finite limit $\mathsf{Im}(t(0))$ as $x \to +\infty$; this defines an analytic curve $\tilde{C}$ in the s-plane. In particular $\tilde{t}$ is uniformly bounded along $\tilde{C}$, hence $\tilde{C}$ is contained in a vertical strip $\{x_0 \leqslant x \leqslant x_1\}$. The branch C is the projection under $s \mapsto e^{-s}$ of a connected component $\tilde{C}_0$ of $\tilde{C}$. It is contained in the annulus $\{\exp(-x_1) \leqslant |w| \leqslant \exp(-x_0)\}$ and it is an analytic subset of this annulus because it is contained in σ_g. According to Lemma 6.3, C must then be a loop that winds around the critical value s of π_g. □

Remark 6.5. If $\mathsf{Im}(t(0)) \neq 0$ and $\beta \neq 0$ is small, Equation (6.3) defines a small loop around the origin. A priori, at this stage of the proof, σ_g might have arbitrarily many small components of this type, converging to s. We shall rule out this phenomenon in Proposition 6.7.

Lemma 6.6. *If the reduction of X_s^g is of type I_b, $b \geqslant 1$ and if C is a branch of σ_g in $V \setminus \{s\}$ such that $q \neq 0$ along some open subset of C, then one end of C converges to s and the other one escapes V. There are at most finitely many such branches in σ_g.*

Proof. The last statement follows from the first because every such branch reaches $V' \setminus V$ and σ_g is analytic in $V' \setminus V$.

Since $q \neq 0$, we can choose (α, β) of the form $(\alpha, 0)$ for some $\alpha \in \mathbf{Q}$, and the equation of C becomes

$$\mathsf{Im}\left(\frac{t(w) - \alpha}{p + q\frac{b}{2\mathrm{i}\pi}\log w}\right) = 0 \tag{6.4}$$

for some (primitive) slope (p, q). The denominator of this expression does not vanish because $|w| < 1$. Again, we shall write $w = e^{-s}$, $s = x + \mathrm{i}y$, with $x > x_0 \geqslant 0$.

By Lemma 6.3, C contains a branch that accumulates towards s. If the other branch escapes V, we are done. So, by Lemma 6.1, all we have to do is to show that the second branch does not converge towards s. We argue by contradiction, and parameterize C

by an analytic immersion $u \in \mathbf{R} \mapsto \varphi(u) \in V\backslash\{s\}$ such that $\varphi(u)$ goes to $s = 0$ as u goes to $+\infty$ and $-\infty$. Along that curve, the function $u \mapsto t(\varphi(u))$ converges towards $t(0)$ as u goes to $+\infty$ and to $-\infty$. Writing $w = \rho\exp(2\mathrm{i}\pi\theta)$, Equation (3.6) yields $\frac{1}{b}\tau(w) = \theta - \frac{\mathrm{i}}{2\pi}\log(\rho)$; hence, the real part τ_1 of τ remains bounded while its imaginary part τ_2 goes to $+\infty$ along both ends of C. Now, let us consider the function T on the curve C. We start with a local definition of T in a small open subset $U \subset V\backslash\{s\}$ that intersects C; locally, $T|_C$ takes values in a line $L \subset \mathbf{R}^2$ of slope (p, q). Now, since $\mathbf{R}$ is simply connected, the function $T|_C$ can be analytically continued along C; since its values are locally contained in L, they are indefinitely contained in that line. Using Equation (3.3), we can write $T = (t_1 - \frac{\tau_1}{\tau_2}t_2, \frac{1}{\tau_2}t_2)$ in U, and this local equality propagates along C by analytic continuation. Thus, $T|_C$ converges to $(t_1(0), 0)$ at both ends. By Rolle's theorem, we deduce that the derivative of $T|_C$ vanishes at least once and, by Lemma 3.9, T has a critical point in $V\backslash\{s\}$. This contradiction concludes the proof. □

6.2.3. *A general finiteness result.* The above results give a rather complete account of the geometry of the branches of σ_g near a critical value of π_g. However, for fibers of type I_b, $b \geqslant 1$, our results do not yet imply that σ_g is semi-analytic at s: this is already apparent for the case of curves $\mathrm{R}^{\alpha,\beta}_{p,q}$ with $q \neq 0$ in the toy calculations of § 3.6. Still, we have the following finiteness result. It does not rely on the choice of a particular invariant measure, so it is stronger than the finiteness of the number of components of Σ in Proposition 4.15. This will play a key role in Theorem C.

Proposition 6.7. *If* $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h) \neq \emptyset$*, then* σ_g *has finitely many irreducible components.*

Proof. Since σ_g is analytic in B_g°, only finitely many components of σ_g intersect any given compact $K \Subset B_g^\circ$. So, we can work locally near some fixed $s \in \mathrm{Crit}(\pi_g)$. By Lemmas 6.2 and 6.6, the only case to deal with is that of fibers reducing to type I_b and branches of type $\mathrm{R}^k_{1,0}$ (i.e. with $q = 0$). According to (the proof of) Lemma 6.4, we have to rule out the existence of an infinite sequence of small loops of the form $\mathrm{R}^{0,\beta_n}_{1,0}$, with $\beta_n \to 0$, winding around the origin s and converging to it (see Remark 6.5). For this we come back to the analysis of the local structure of invariant measures from § 5.4. If an arc $\gamma \subset \mathrm{R}^{0,\beta}_{1,0}$ is locally contained in σ_g in some open set $U \subset V\backslash\{s\}$, there exists an ergodic invariant measure μ such that for $w \in \gamma\backslash\mathrm{Tor}_g(U)$, its conditional $\mu_{g,w}$ on X_g^w puts positive weight on some circle $L_g^0(\xi)$ of slope $(1, 0)$ through ξ (see § 5.4). Then, the g-invariance shows that $\mu_{g,w}$ puts positive mass on each of the k components of $L_g(\xi) \simeq L_w(k, (1, 0))$, where k is the order of β in $\mathbf{R}/\mathbf{Z}$ (see the definition of $L_w(k, (p, q))$ in § 3.5.1 and the discussion around Equation (5.5)). Now, since β_n tends to 0, its order k_n tends to infinity. Near s, this means that when σ_g contains a component of type $\mathrm{R}_{1,0}^{0,\beta_n}$, it contains also a component of $\mathrm{R}_{1,0}^{0,j\beta_n}$ for all $j = 1, \ldots, \lfloor \beta_n^{-1} \rfloor$. Fix $\delta > 0$ small and consider the level sets $\mathrm{R}_{1,0}^{0,j\beta_n}$ for $\delta \leqslant j\beta_n \leqslant 2\delta$. Then, from Lemma 6.4 (see Remark 6.5), this creates an accumulation of components of σ_g away from $\mathrm{Crit}(\pi_g)$, which is a contradiction. □

6.3. **Condition (AC) and conclusion.** If X_s^g is a singular fiber of π_g with reduction of type I_b, $b \geqslant 1$, we define the **active components** of X_s^g to be the set of its irreducible components which are not contracted during the reduction process. More precisely, if X_s^g is not relatively minimal, there exists a unique birational morphism $\varepsilon : X \to X'$, with X' smooth, so that $\pi_g \circ \varepsilon^{-1}$ is relatively minimal: no fiber contains a (-1)-curve. By definition, the components of the exceptional divisor of ε are not active. After this contraction, X_s^g becomes a fiber $\varepsilon(X_s^g)$ of type ${}_mI_b$, for some $m \geqslant 1$, or I_b^*. In the first case the active components of X_s^g are the components which are not contracted by ε; equivalently, they are the components of the proper transform of $\varepsilon((X')_s^g)$. In the I_b^* case, we retain only the proper transform of the $b+1$ components of multiplicity 2 in $\varepsilon((X')_s^g)$.

The Active Components condition reads as follows:

(AC) there exists $g \in \mathrm{Hal}(\Gamma)$ such that every fiber of π_g reducing to type I_b, $b \geqslant 1$, contains an active component which is not in D_Γ.

Theorem A'. *If in Theorem A we further assume the non-degeneracy condition (AC), then in the totally real case (c) we can add the conclusion: (5) $\overline{\Sigma}$ is a semi-analytic subset of X.*

Example 6.8. If D_Γ is empty, then (AC) is satisfied. This is the case for general Wehler surfaces and general Enriques surfaces if one takes $\Gamma = \mathsf{Aut}(X)$ (see [10] and the references therein). If X is minimal and π_g does not contain singular fibers of type I_b^*, then (AC) is satisfied.

Let us show that the examples obtained by Blanc's construction for three points (see § 2.2) satisfy condition (AC). One starts with a smooth cubic $C \subset \mathbb{P}^2(\mathbf{C})$ and three general points p, q, r on C. Then, X is the blow-up of $\mathbb{P}^2(\mathbf{C})$ at p, q, r, and at the 12 points $a(p)$, $b(p)$, $c(p)$, $d(p)$, $a(q)$, …, $d(q)$, $a(r)$, …, $d(r)$ of C such that the tangent to C at one of these points intersects C in p, q, or r, respectively. The Jonquières involution s_p (resp. s_q, s_r) that preserves the pencil of lines through p (resp. q, r) and fixes C pointwise lifts to an automorphism of X. According to [3], the subgroup of $\mathsf{Aut}(X)$ generated by s_p, s_q, s_r is a free product $\mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z}$. From the formulas given in [3, Lem. 17] for the action of $s_\star$ on $\mathrm{NS}(X)$, it can be deduced that the composition $g = s_p \circ s_q$ is a parabolic automorphism, preserving the pencil of plane quartic curves passing through p and q with multiplicity 2 and through the eight points $a(p)$, …, $d(q)$

with multiplicity 1. The union of C and the line $L_{p,q}$ through p and q belongs to this pencil; this gives a reducible fiber of $\pi_g\colon X \to \mathbb{P}^1(\mathbf{C})$ which, after blowing down the strict transform of $L_{p,q}$, is of type I_0. It is easy to see that the unique effective divisor which is invariant by s_p, s_q and s_r is the proper transform of C, i.e. $D_{\langle s_p,s_q,s_r\rangle} = C$. Since condition (AC) deals only with fibers of type I_b, $b \geqslant 1$, we conclude that it holds for $\langle s_p, s_q, s_r\rangle$.

Example 6.9. General Coble surfaces with $\Gamma = \mathsf{Aut}(X)$ *do not satisfy condition* (AC). More precisely, if g is a Halphen twist, then π_g has 10 singular fibers with

- 8 singular fibers of type I_1, each of them made of a single, active component;
- 1 fiber made of two rational curves (intersecting transversally in two points), the first is a (-1)-curve E_g (contracted by the reduction process), the second is a (-2)-curve S which reduces to a I_1 fiber, so it is active;
- one multiple fiber M_g of type $2I_0$.

The curve S does not depend on g, and $D_\Gamma = S$, so we see that condition (AC) is violated. On the other hand, if we blow-down the (-2)-curve S onto a point, we get a singular surface on which Theorem A' applies.

The relevance of condition (AC) comes from the following lemma. We resume the context of Lemma 6.6.

Lemma 6.10. *Let $s \in \mathrm{Crit}(\pi_g)$ be such that X^g_s reduces to type I_b, $b \geqslant 1$. Let C be a branch of σ_g accumulating s, of type $\mathrm{R}^k_{p,q}$ with $q \neq 0$. Let (w_n) be a sequence of points of $C\setminus \mathrm{Tor}(B_g^\circ)$ converging toward s. Pick an arbitrary sequence $\xi_n \in X^g_{w_n}$. If A is any active component of X^g_s, then $L^0_g(\xi_n)$ accumulates A along a non-trivial curve.*

Proof. Since A is not contracted during the reduction process, we can assume that X^g_s is already of type I_b. We now rely on the description of fibers of type I_b given in §§ 3.3.1 and 3.3.2. On a small disk $V \subset B_g$ containing s, we pick a local section of π_g intersecting A at a smooth point of X^g_s and we construct the surface $X^{g,\sharp}_V$, with central fiber $X^{g,\sharp}_s$ corresponding to A. Since $q \neq 0$, the circle $L^0_g(\xi_n)$ is not homotopic to a vanishing cycle, so its length is bounded from below by the injectivity radius of X. More precisely, we can extract a subsequence $L^0_g(\xi_{n_i})$ that converges in $X^{g,\sharp}_V$ towards a logarithmic spiral in the central fiber $X^{g,\sharp}_s \simeq \mathbf{C}^\times$. Here, by a logarithmic spiral we mean a translate of a one parameter subgroup that goes from 0 to ∞ in the complex multiplicative group $\mathbf{C}^\times$. The result follows. □

Proof of Theorem A'. By assumption Γ satisfies (AC). We shall revisit the choice of the Halphen twists g and h from §5.2.1. First, we choose g with the required property in (AC); an extra condition will also be imposed on h (see below).

From the first lines of §6.1, we may assume $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h) \neq \varnothing$, and we have to show that σ_g admits a semi-analytic continuation to B_g. Fix $s \in \mathrm{Crit}(\pi_g)$ and small disks $V \Subset V'$ centered at s, as in § 6.2. If the singular fiber X^g_s reduces to type I_0 then σ_g is semi-analytic at s by Lemma 6.2. If it reduces to type I_b, $b \geqslant 1$, any branch of σ_g with $q = 0$ is semi-analytic by Lemma 6.4. By Proposition 6.7, σ_g has finitely many components near s; thus, we only need to show that any given branch of σ_g with $q \neq 0$ is semi-analytic.

Fix a branch γ converging to s, of type $\mathrm{R}^{\alpha,\beta}_{p,q}$ for some $q \neq 0$. Since Tang_Γ contains D_Γ, we may assume that there is an active component $A \subset X^g_s$ which is not contained in any fiber of h. Now, fix an invariant, ergodic probability measure μ such that μ_g evenly charges γ. We will argue as in Lemma 5.10, except that the uniform geometric estimates from §5.3 are replaced by Lemma 6.10.

The details are as follows. Let r be such that any logarithmic spiral in A contains an arc of size r. Fix $\varepsilon \ll \delta \ll r$. Set $k := \left|\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h) \cap X^g_s\right|$; $k < +\infty$ by definition of $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h)$. Identify V with $\mathbb{D}_R$, and for $0 < R' < R$, consider the set $\pi_g^{-1}(\mathbb{D}_{R'}) \cap \mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h)$. As R' converges towards 0, this open subset of $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h)$ converges towards the finite set $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h) \cap X^g_s$ in the Hausdorff topology. Thus, if R' is small enough, $\pi_g^{-1}(\mathbb{D}_{R'}) \cap \mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h)$ is a union of k subsets, each of diameter $\leqslant \varepsilon$; its projection under π_h is contained in a union of k disks $\Delta_i \subset B_h$, each of diameter less than $O(\varepsilon)$.

Fix a sequence (w_n) in $\gamma \setminus \mathrm{Tor}(B^\circ_g)$ converging towards s. For large n, pick a point $\xi_n \in X^g_{w_n}$ which is contained in a small square above γ on which μ restricts to a smooth measure, as in Section 5.4; the orbit $L^0_g(\xi_n)$ is a circle in $X^g_{w_n}$ and by Lemma 6.10 a subsequence of $L^0_g(\xi_n)$ converges to a logarithmic spiral in A. Changing ξ_n into another point $\xi'_n \in L^0_g(\xi_n)$, we can therefore assume that (a) ξ_n is in the support of μ, (b) $w'_n := \pi_h(\xi'_n)$ is δ-far from $\mathrm{Crit}(\pi_h) \cup \bigcup_{i=1}^k \Delta_i$, and (c) w'_n is not in $\mathrm{Tor}(B_h)$; then, (d) $L^0_h(\xi'_n)$ is a circle in $X^h_{w'_n}$, whose size at every point is bounded from below by a constant that does not depend on n (here we apply Lemma 5.4 to h).

The set $\pi_g^{-1}(\mathbb{D}_R) \cap X^h_{w'_n}$ is an open subset of $X^h_{w'_n}$ that contains ξ'_n. If w'_n is close enough to s and R is small, we may assume that the connected component V'_n of this open set that contains ξ'_n contains a unique point s' of A, that V'_n is a disk, and that $\pi_g|_{V'}\colon V'_n \to V$ is a covering which is ramified at s' only (indeed, property (b) above implies that $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h)$ does not intersect V'). Consider the connected component $I_n \subset V'_n$ of $L^0_h(\xi'_n)$ that contains ξ'_n. The projection $\pi_g(I_n)$ is locally contained in γ around w_n. If I_n did not contain s', then $\pi_g|_{I_n}$ would have no ramification point, so $\pi_g(I_n)$ would be an arc with boundary points in ∂V; being contained in this arc, γ

would not converge to s, a contradiction. Thus I_n contains s' and $\pi_g(I_n)$ is a semi-analytic subset of the disk V that contains s. It is smooth and analytic in $V \setminus \{s\}$, and γ is a component of $\pi_g(I_n) \setminus \{s\}$; therefore $\overline{\gamma}$ is semi-analytic, as desired. □

7. FINITELY MANY INVARIANT MEASURES: PROOF OF THEOREM C

Let as usual X be a compact Kähler surface and Γ be a non-elementary subgroup of $\mathsf{Aut}(X)$ containing a parabolic element. We want to show the following alternative:

- either (X, Γ) is a Kummer example,
- or there are only finitely many Γ-invariant ergodic measures with a Zariski dense support.

It is shown in § A.3 that Kummer groups can indeed admit infinitely many ergodic totally real measures (i.e. with $d_{\mathbf{C}}(\mu) = 2$ and $d_{\mathbf{R}}(\mu) = 2$).

Lemma 7.1. *There is at most one Γ-invariant, ergodic probability measure which is absolutely continuous with respect to the Lebesgue measure.*

Proof. Let μ and μ' be such measures. Fix a real analytic volume form ω on X, for instance $\omega = \kappa \wedge \kappa$ for some Fubini-Study form. From Theorem A, $d\mu(x) = \xi(x)\omega$ (resp. $d\mu'(x) = \xi'(x)\omega$) for some function ξ (resp. ξ') which is positive and real analytic on the complement of some proper real analytic subset $B(\mu)$ (resp. $B(\mu')$). On $X \setminus (B(\mu) \cup B(\mu'))$, the function ξ'/ξ is continuous and Γ-invariant; since μ is ergodic, ξ'/ξ is constant, and $\mu' = \mu$. □

According to Theorem A and Lemma 7.1, we are interested only in measures of type (c): those with a totally real support. To prove the finiteness we revisit the proof of Proposition 4.15 and use an alternative similar to that of §5.7; indeed, exactly one of the following two situations holds (compared with the alternative of §5.7, the quantifiers are switched):

(A1') $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h) = \varnothing$ for every pair g, h in $\mathrm{Hal}(\Gamma)$ such that $\pi_g \neq \pi_h$ (recall the convention of Remark 4.2);

(A2') there exists g, h in $\mathrm{Hal}(\Gamma)$ such that $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h) \neq \varnothing$.

Theorem C is then an immediate consequence of the following two lemmas.

Lemma 7.2. *If Alternative $(A1')$ holds, then (X, Γ) is a Kummer group.*

Lemma 7.3. *If Alternative $(A2')$ holds, then there are only finitely many ergodic, Γ-invariant probability measures with a Zariski dense support.*

Proof of Lemma 7.2. By Proposition 3.12 in [10], we can choose $g, h \in \mathrm{Hal}(\Gamma)$ which are conjugate in Γ and such that every periodic curve for $h \circ g$ is contained in D_Γ. With such a choice, the common components of π_g and π_h coincide with the components of D_Γ, and the foliations $\mathcal{F}_g$ and $\mathcal{F}_h$ induced by π_g and π_h are everywhere transverse, except along $\mathrm{Tang}^{\mathrm{ff}}(\pi_g, \pi_h) = D_\Gamma$.

Remark 7.4. If $X \to X_0$ is the contraction of D_Γ (see Proposition 4.5), the fibrations π_g and π_h define two genus 1 fibrations on X_0, and the associated foliations $\mathcal{F}_g^0$ and $\mathcal{F}_h^0$ are transverse everywhere, except on the finite set $\mathrm{Sing}(X_0)$. Indeed, a smooth point of the surface can not be an isolated point of tangency between two foliations.

If $\beta\colon [0,1] \to B_g^\circ$ is a path joining two points w_0 and w of B_g°, the holonomy of $\mathcal{F}_h$ determines an isomorphism $\mathrm{hol}_h(\beta)\colon X^g_{w_0} \to X^g_w$; thus, π_g is an isotrivial fibration (cf. Lemma 5.1). This construction defines a representation $\mathrm{hol}_h\colon \pi_1(B_g^\circ; w_0) \to \mathsf{Aut}(X^g_{w_0})$, the image of which fixes the finite subsets $X^h_{w'} \cap X^g_{w_0}$ for every $w' \in B_h$. Thus, $\mathrm{hol}_h(\pi_1(B_g^\circ; w_0))$ is a finite subgroup H of automorphisms of the genus 1 curve $E := X^g_{w_0}$. A similar argument applies to π_h in place of π_g; we shall denote by E' the genus 1 curve $X^h_{w_0'}$, for some w_0' in B_h°, and by H' the corresponding holonomy group. (Note that we have $E' \simeq E$ because the two fibrations are conjugate by some automorphism of X.) We also fix a point $\xi_0 \in X$ whose projections are $w_0 = \pi_g(\xi_0)$ and $w_0' = \pi_h(\xi_0)$.

This construction yields a holomorphic map Ψ from $X \backslash D_\Gamma$, or equivalently $X_0 \backslash \mathrm{Sing}(X_0)$, to $E'/H' \times E/H$, which is defined as follows. To a point ξ in $X \backslash D_\Gamma$, we associate the intersection of the leaf of $\mathcal{F}_g$ through ξ (i.e. $X^g_{\pi_g(\xi)}$) with the fiber $X^h_{w_0'}$; this gives a unique point modulo the action of H', hence a point $\psi'(\xi) \in E'/H'$. Doing the same with respect to $\mathcal{F}_h$ and π_g, we get a point $\psi(\xi) \in E/H$, and then we set $\Psi(\xi) = (\psi'(\xi), \psi(\xi))$. Let ξ be a singularity of X_0 and let U be a small neighborhood of ξ. Then $\psi'(X^g_{\pi_g(U)})$ is contained in a small disk $V' \subset E'/H'$; similarly, $\psi(X^h_{\pi_h(U)})$ is contained in a small disk $V \subset E/H$. Thus, Ψ maps $U \backslash \{\xi\}$ in a bidisk $V' \times V$; as a consequence, Ψ extends to U, for the singularities of X_0 are normal (see [10, Prop. 3.9]). Altogether, this defines a finite ramified cover $\Psi\colon X_0 \to E'/H' \times E/H$.

We also define a regular map Φ from $E' \times E$ to X_0. For this, without loss of generality we declare that the neutral element of $X^h_{w_0'}$ is ξ_0 and denote by 0 the neutral element of E'; hence, the pair $(E' \times \{0\}, (0,0))$ is identified to $(X^h_{w_0'}, \xi_0)$ by an isomorphism $\varphi_h\colon (E', 0) \to (X^h_{w_0'}, \xi_0)$. Similarly, we identify $\{0\} \times E$ to $X^g_{w_0}$ via an isomorphism φ_g that maps 0 to ξ_0. We shall denote by $F' \subset E'$ (resp. $F \subset E$) the finite subset $\varphi_h^{-1}(\pi_g^{-1}(\mathrm{Crit}(\pi_g))$ (resp. $\varphi_g^{-1}(\pi_h^{-1}(\mathrm{Crit}(\pi_h))))$; F' corresponds to the intersection of $\varphi_h(E')$ with singular and multiple fibers of π_g. If (u, v) is a point of $E' \times E$ close to $(0,0)$, then the fibers $X^g_{\pi_g(\varphi_h(u))}$ and $X^h_{\pi_h(\varphi_g(v))}$ have a unique intersection point *near* ξ_0. This defines a germ of diffeomorphism $\Phi\colon E' \times E \to X_0$ mapping $(0,0)$ to ξ_0 and preserving the fibrations; it is defined in a a small bidisk $\mathbb{D}' \times \mathbb{D} \subset E' \times E$. Observe that the composition $\Psi \circ \Phi$ coïncides with the natural projection from $E' \times E$ to $E'/H' \times E/H$. Reducing the bidisk if necessary, this map Φ extends uniquely to $\mathbb{D}' \times E$ and provides a local trivialization of the fibration π_g above $\pi_g(\mathbb{D}')$. Similarly, it extends to $E' \times \mathbb{D}$. So Φ is defined in a "cross" of the form $\mathbb{D}' \times E \cup E' \times \mathbb{D}$. By analytic continuation, it extends uniquely to $(E' \backslash F') \times E$ and to $E' \times (E \backslash F)$, that is, to $(E' \times E) \backslash (F' \times F)$. Moreover, $\Psi \circ \Phi \colon E' \times E \to E'/H' \times E/H$ is the quotient map with respect to the action of $H' \times H$. From this, we deduce that the default of injectivity of Φ is given by a subgroup G of $H' \times H$: $\Phi(p) = \Phi(p')$ if and only if $p' - p \in G$. Since $\Psi \colon X_0 \to E'/H' \times E/H$ is a finite map, it follows that for each $(u, v) \in F' \times F$ there is a point $\xi \in X_0$, an open neighborhood U of ξ, and an open neighborhood W of (u, v) such that Φ maps $W \backslash \{(u, v)\}$ into U. Embedding U into some affine space, we see by the Hartogs extension theorem that Φ extends to W. Thus, Φ extends to a holomorphic map $E' \times E \to X_0$ which fits in a sequence

$$E' \times E \xrightarrow{\Phi} X_0 \xrightarrow{\Psi} E'/H' \times E/H, \tag{7.1}$$

such that the composition $\Psi \circ \Phi$ is the natural projection onto the quotient, and the fibers of Φ are orbits of G.

Thus, X_0 is a generalized (singular) Kummer surface (see [10]): it is a quotient of the abelian surface $E' \times E$ by a finite subgroup G of $H' \times H$; the singularities of X_0 correspond to the fixed points of elements of $G \backslash \{\mathrm{id}\}$. Restricting Φ to the complement of these fixed points, we get a regular finite cover onto the regular part of X_0, with G as a group of deck transformations. Denote Λ and Λ' lattices in $\mathbf{C}$ such that $E = \mathbf{C}/\Lambda$ and $E' = \mathbf{C}/\Lambda'$; the universal cover of $E' \times E$ is $\mathbf{C}^2$, with projection $\mathbf{C}^2 \to \mathbf{C}^2/(\Lambda' \times \Lambda)$. If we think of X_0 as an orbifold with quotient singularities, its universal cover is $\mathbf{C}^2$.

From this point, the argument is identical to [10, Thm. 5.15]. Let f be an element of $\mathsf{Aut}(X_0)$ and lift it as a holomorphic diffeomorphism F of $\mathbf{C}^2$. Its differential $DF_{(x,y)}$ at a point $(x, y) \in \mathbf{C}^2$ is an element of $\mathsf{GL}_2(\mathbf{C})$. Let $L(G) \subset \mathsf{GL}_2(\mathbf{C})$ be the linear part of G; it is a finite group, and the class $[DF_{(x,y)}]$ of $DF_{(x,y)}$ in $\mathsf{GL}_2(\mathbf{C})/L(G)$ determines a holomorphic map that is invariant under translations by the lattice $\Lambda' \times \Lambda$; since $\mathsf{GL}_2(\mathbf{C})/L(G)$ is an affine variety, this map is constant. So, all lifts of all elements of $\mathsf{Aut}(X_0)$ to $\mathbf{C}^2$ are affine maps.

Now, consider the full group $\Gamma \subset \mathsf{Aut}(X)$. It preserves D_Γ and induces a subgroup Γ^0 of $\mathsf{Aut}(X_0)$. By the previous paragraph, all elements of Γ^0 come from affine transformations of $E' \times E$. This proves that (X_0, Γ^0) and (X, Γ) are Kummer groups. □

Proof of Lemma 7.3. By Lemma 5.13, there is an analytic curve $\sigma_g \subset B_g^\circ$ (resp. $\sigma_h \subset B_h^\circ$) such that $\mu_g(\sigma_g) = 1$ (resp. $\mu_h(\sigma_h) = 1$) for any ergodic invariant measure μ. By Proposition 6.7, there exists a compact subset $K_g \Subset B_g^\circ$ such that any component of σ_g intersects K_g (and similarly for h). In particular any invariant probability measure gives positive mass to $\pi_g^{-1}(K_g) \cap \pi_g^{-1}(K_h)$. In $\pi_g^{-1}(K_g) \cap \pi_g^{-1}(K_h)$, the regular set of $\pi_g^{-1}(\sigma_g) \cap \pi_g^{-1}(\sigma_h)$, which is semi-analytic, has finitely many connected components (see §5.1 and [2, Cor. 2.7]). By the analyticity of the density in Theorem A.(c), if Σ_0 is such a connected component, there is at most one ergodic invariant probability measure giving positive mass to Σ_0, and the proof is complete. □

8. Invariant analytic surfaces which are not real parts

In this section we construct examples of pairs (X, Γ) such that Property (c) in Theorem A holds and for which:

(1) the support Σ of μ is an analytic real surface;
(2) there is no real structure on X for which Σ is contained in the real part $X(\mathbf{R})$.

8.1. **A family of lattices.** Let t be a positive real number, and set $\tau = \frac{1}{2} + \mathsf{i}t$, where $\mathsf{i} = \sqrt{-1}$. Consider the lattice $\Lambda \subset \mathbf{C}$ defined by

$$\Lambda = \mathbf{Z} \oplus \mathbf{Z}(\frac{1}{2} + \mathsf{i}t) = \mathbf{Z} \oplus \mathbf{Z}\tau. \tag{8.1}$$

Since $\frac{1}{2} - \mathsf{i}t$ belongs to Λ, the complex conjugation $z \mapsto \overline{z}$ induces an anti-holomorphic involution $\sigma_E(z) = \overline{z}$ on the elliptic curve $E = \mathbf{C}/\Lambda$: this gives a real structure on E. The fixed point set of σ_E gives the real part of E; one checks easily that it coincides with the projection of the real axis:

$$E(\mathbf{R}) = \mathbf{R}/\mathbf{Z} = \mathbf{R}/(\Lambda \cap \mathbf{R}). \tag{8.2}$$

8.2. **Abelian and Kummer surfaces.** Now, consider the abelian surface $A := E \times E = \mathbf{C}^2/\Lambda^2$, and the real structure $\sigma_A(x, y) = (\overline{x}, \overline{y}) \mod (\Lambda^2)$. Its fixed point set is $A(\mathbf{R}) = E(\mathbf{R})^2 = \mathbf{R}^2/\mathbf{Z}^2$. The group $\mathsf{GL}_2(\mathbf{Z})$ acts linearly on $\mathbf{C}^2$ by preserving Λ^2, so it also acts on A by "linear automorphisms". Set

$$\mathbf{e}_1 = (1, 0),\ \mathbf{e}_2 = (\tau, 0),\ \mathbf{e}_3 = (0, 1),\ \mathbf{e}_4 = (0, \tau). \tag{8.3}$$

The vectors $\mathbf{e}_1$ and $\mathbf{e}_3$ form a basis of the complex plane $\mathbf{C}^2$, and the four vectors $\mathbf{e}_1$, …, $\mathbf{e}_4$ form a basis of the real vector space $\mathbf{C}^2 \simeq \mathbf{R}^4$. The real planes $\mathrm{Vect}_{\mathbf{R}}(\mathbf{e}_1, \mathbf{e}_3)$ and $\mathrm{Vect}_{\mathbf{R}}(\mathbf{e}_2, \mathbf{e}_4)$ are invariant under the action of $\mathsf{GL}_2(\mathbf{Z})$, and they determine two invariant real tori in A, the first one being equal to $A(\mathbf{R})$. Define

$$\Sigma_A = \mathrm{Vect}_{\mathbf{R}}(\mathbf{e}_2, \mathbf{e}_4)/\Lambda^2 = \mathrm{Vect}_{\mathbf{R}}(\mathbf{e}_2, \mathbf{e}_4)/(\mathbf{Z}\mathbf{e}_2 \oplus \mathbf{Z}\mathbf{e}_4) \simeq \mathbf{R}^2/\mathbf{Z}^2. \tag{8.4}$$

Now, consider the Kummer surface $X_0 = A/\eta$, where η is the holomorphic involution of A defined by $\eta(x, y) = (-x, -y) \mod (\Lambda^2)$. This surface has sixteen singularities, each of which is resolved by a simple blow-up. Let Σ_0 be the projection of Σ_A in X_0. We also let $\Sigma_X \subset X$ be the proper transform of Σ_0 in the smooth K3 surface X obtained by resolving the singularities of X_0.

8.3. **Statements.**

Theorem 8.1. *With notation as above, the following properties are equivalent:*

(a) *there is a real structure s_A on A whose real part contains Σ_A;*
(b) *there is a real structure s_X on X whose real part contains Σ_X;*
(c) *the positive real number t is equal to $\sqrt{3}/2$, $1/2$, or $\sqrt{3}/6$.*

In addition when $t = 1/2$ (resp. $\sqrt{3}/2$ or $\sqrt{3}/6$) the curve E is isomorphic to the quotient of $\mathbf{C}$ by the lattice of Gaussian integers (resp. Eisenstein integers).

The equivalences (a)$\Leftrightarrow$(c) and (b)$\Leftrightarrow$(c) are proven respectively in § 8.4 and 8.5 below.

Corollary 8.2. *There exist examples of abelian and Kummer surfaces X, and non-elementary subgroups $\Gamma \subset \mathsf{Aut}(X)$ such that* (1) *Γ preserves an analytic, totally real surface $\Sigma \subset X$,* (2) *Γ has a dense orbit and an invariant, ergodic, smooth, probability measure with support equal to Σ, and* (3) *there is no real structure on X whose real part contains Σ.*

Indeed, one can just take $\Gamma = \mathsf{GL}_2(\mathbf{Z})$ in the previous examples, and for the invariant probability measure one takes the measure coming from the Lebesgue measure on Σ_A.

8.4. **Proof for the abelian surface.** Let s be a real structure on A, i.e. an anti-holomorphic involution. Then $s \circ \sigma_A$ is holomorphic, so $s = B \circ \sigma_A$ for some automorphism B of A. Now, assume that the fixed point set of s is equal to Σ_A. Since the origin $(0,0)$ of A is fixed by σ_A and s, it is fixed by B too. This means that B is induced by a linear automorphism of the complex vector space $\mathbf{C}^2$, i.e. by an element of $\mathsf{GL}_2(\mathbf{C})$.

Near the origin of A, we can write $s(x,y) = B \circ \sigma_A(x,y) = B(\overline{x}, \overline{y})$, and our assumption implies that the vector $\mathbf{e}_2 = (\tau, 0)$ satisfies $B(\overline{\tau}, 0) = (\tau, 0)$, i.e. $B\mathbf{e}_2 = (\tau/\overline{\tau})\mathbf{e}_2$ because B is $\mathbf{C}$-linear. The same property is satisfied by $\mathbf{e}_4$. Since $(\mathbf{e}_2, \mathbf{e}_4)$ is a basis of the complex plane $\mathbf{C}^2$, B is a homothety: $B = (\tau/\overline{\tau})\mathsf{Id}$. As a consequence, the linear map $M\colon \mathbf{C} \to \mathbf{C}$ defined by $M(z) = (\tau/\overline{\tau})z$ preserves the lattice $\Lambda = \mathbf{Z} \oplus \mathbf{Z}\tau$, with $\tau = 1/2 + \mathrm{i}t$. Thus, we can find a quadruple of integers (a,b,c,d) such that

$$\frac{\tau}{\overline{\tau}} = a + b\tau \quad \text{and} \quad \frac{\tau}{\overline{\tau}}\tau = c + d\tau. \tag{8.5}$$

This implies $a = c = -1$, by looking at the imaginary parts after both equations have been multiplied by $\overline{\tau}$. Then

$$1 = b|\tau|^2 \quad \text{and} \quad \frac{3}{4} - t^2 = d|\tau|^2. \tag{8.6}$$

The first equation plus the relation $|\tau|^2 = 1/4 + t^2 > 1/4$ imply that $1 \leqslant b \leqslant 3$, and more precisely $(b,t) \in \{(1, \sqrt{3}/2), (2, 1/2), (3, \sqrt{3}/6)\}$. Together with the second equation we end up with exactly three possibilities:

(1) $(b,d,t) = (1,0,\sqrt{3}/2)$, the lattice $\mathbf{Z}\oplus\mathbf{Z}\tau$ is the lattice of Eisenstein integers $\mathbf{Z}[(1+\mathsf{i}\sqrt{3})/2]$, and $s(x,y) = (e^{2\mathsf{i}\pi/3}\overline{x}, e^{2\mathsf{i}\pi/3}\overline{x}) = e^{2\mathsf{i}\pi/3}\sigma_A(x,y)$;
(2) $(b,d,t) = (2,1,1/2)$, $\tau = (1+\mathsf{i})/2$, the lattice is the image of the lattice of Gaussian integers $\mathbf{Z}[\mathsf{i}]$ by the similitude $z \mapsto \frac{1-\mathsf{i}}{2}z$, and $s(x,y) = (\mathsf{i}\overline{x}, \mathsf{i}\overline{y}) = \mathsf{i}\sigma_A(x,y)$;
(3) $(b,d,t) = (3,2,\sqrt{3}/6)$ and $\tau = 1/2 + \mathsf{i}\sqrt{3}/6$; modulo the action of $\mathsf{PSL}_2(\mathbf{Z})$ on the upper half plane, τ is equivalent to $2 - 1/\tau = (1+\mathsf{i}\sqrt{3})/2$ so we end up again with the Eisenstein integers, and $s(x,y) = (e^{2\mathsf{i}\pi/6}\overline{x}, e^{2\mathsf{i}\pi/6}\overline{y}) = e^{2\mathsf{i}\pi/6}\sigma_A(x,y)$

This completes the proof of the implication (a)⇒(c), and for the converse implication the explicit formulas for $s(x,y)$ provides the desired real structure.

Note that in each of these cases, s is conjugate to σ_A by $\beta\,\mathrm{id} \in \mathsf{GL}_2(\mathbf{C})$, with respectively $\beta = e^{2\mathsf{i}\pi/6}$, $e^{2\mathsf{i}\pi/8}$, $e^{2\mathsf{i}\pi/12}$. However in the second and third cases, this conjugacy is only satisfied near the origin, because $\beta\,\mathrm{id}$ does not preserve the lattice (i.e. it does not induce an automorphism of A). □

8.5. **Proof for the (smooth) Kummer surface.** Suppose there is a real structure s_X on X whose fixed point set contains Σ_X. Let σ_X be the real structure induced by σ_A on the Kummer surface X. Then, there is an automorphism B_X of X such that $s_X = B_X \circ \sigma_X$.

Consider the origin $(0,0)$ of A, and its blow-up $\varepsilon\colon \hat{A} \to A$. In local coordinates it expresses as $\varepsilon(u,v) = (u,uv) = (x,y) \in A$, with exceptional divisor $D = \{u = 0\}$. The involution η lifts to $\hat{\eta}(u,v) = (-u,v)$; it is the identity on D, it acts transversally as $u \mapsto -u$, and the quotient map $\hat{A} \to X = \hat{A}/\hat{\eta}$ is locally given by $q\colon (u,v) \mapsto (u^2,v)$. Lifting σ_A, we obtain $\hat{\sigma}_A(u,v) = (\overline{u},\overline{v})$. In A, Σ_A is locally parametrized by $(s\tau, s'\tau)$ with s and s' small real numbers; its strict transform is the real analytic surface $\hat{\Sigma}_A$ given by $(u,v) = (s\tau, s'/s)$. So, the intersection of $\hat{\Sigma}_A$ with D is determined by the equation $v \in \mathbf{R}$. In particular, $\hat{\Sigma}_A \cap D$ is fixed by $\hat{\sigma}_A$. The image of D in X is a curve $C \simeq \mathbb{P}^1(\mathbf{C})$ and the image of the subset $\{u = 0,\ v \in \mathbf{R}\}$ is a great circle $S \subset C$. This circle is fixed by σ_X, as we just saw, and by s_X, by definition of Σ_X. Thus, B_X fixes S, hence C itself since S is Zariski dense in C (for the complex algebraic structure on X).

Since B_X fixes C, we can contract C onto a singularity of X_0: B_X and σ_X descend to regular (holomorphic and anti-holomorphic) maps on a neighborhood of the singularity. Since this singularity is the quotient singularity $(\mathbf{C}^2,0)/\eta$, we can lift B_X, σ_X, and s_X to germs of diffeomorphisms B_A, σ_A, and s_A near the origin in $\mathbf{C}^2$. Using the natural, local coordinates given by the projection $\mathbf{C}^2 \to A \to X_0$, the lifts can be written

$$\sigma_A(x,y) = (\overline{x},\overline{y}) \quad \text{and} \quad B_A(x,y) = \Big(\sum_{k,\ell\geqslant 0} a_{k,\ell}x^k y^\ell, \sum_{k,\ell\geqslant 0} b_{k,\ell}x^k y^\ell\Big) \tag{8.7}$$

for some locally convergent power series $\sum_{k,\ell} a_{k,\ell}x^k y^\ell$ and $\sum_{k,\ell} b_{k,\ell}x^k y^\ell$. In these coordinates, Σ_X corresponds to the real plane $(u\tau, v\tau)$ for $(u,v) \in \mathbf{R}^2$, and the equation for the fixed points of s gives $B_A(u\overline{\tau}, v\overline{\tau}) = (u\tau, v\tau)$ for $(u, v) \in \mathbf{R}^2$. This implies that B_A is linear in these coordinates, equal to the homothety of factor $\tau/\overline{\tau}$.

Up to this point, we have worked locally near the singularity of X_0 corresponding to the origin of A, we now globalize the argument. As a consequence of its local form, B_A preserves the horizontal line $\{y = 0\}$, so that B_X preserves the quotient curve

$$(\mathbf{C}/(\Lambda) \times \{0\})/\eta \simeq \mathbb{P}^1(\mathbf{C}) \subset X. \tag{8.8}$$

Furthermore, in the coordinate x given by the projection

$$\mathbf{C} \times \{0\} \to E \times \{0\} = \mathbf{C}/\Lambda \times \{0\} \to \mathbb{P}^1(\mathbf{C}) \tag{8.9}$$

(where the last arrow is a branched cover of degree 2), B_X is covered by the linear map $x \mapsto (\tau/\overline{\tau})x$. Thus the analysis of the previous subsection applies, and shows that $\tau = 1/2 + \mathsf{i}\sqrt{3}/2$, $(1 + \mathsf{i})/2$, or $1/2 + \mathsf{i}\sqrt{3}/6$, and the proof of (b)⇒(c) complete. For the converse implication, it is enough to observe that for each of these three cases, the explicit anti-holomorphic involution on A given in § 8.4 commutes with η, so it descends to the Kummer surface X. □

9. INVARIANT SURFACES WITH BOUNDARY

In this section, we show that in case (c) of Theorem A, the surface Σ may have a non-trivial boundary in X. We provide two examples, one for a Kummer surface, and then a deformation keeping the main features of the first example but on a surface which is not anymore a Kummer surface. We also give examples of invariant curves that do not support any invariant measure.

9.1. **On a Kummer surface.** Consider a Kummer example, with the same construction as in Sections 8.1 and 8.2. Embed the curve $E = \mathbf{C}/\Lambda$ in $\mathbb{P}^2$, in a Weierstrass form. Its equation is

$$y^2 = 4x^3 - g_2 x - g_3 \tag{9.1}$$

with coefficients $g_i \in \mathbf{R}$ depending on the parameter t; the real structure $\sigma_E(z) = \overline{z}$ is the restriction to E of the real structure $[x : y : z] \mapsto [\overline{x} : \overline{y} : \overline{z}]$ on $\mathbb{P}^2$. Since $E(\mathbf{R}) = \mathrm{Fix}(\sigma_E)$ is connected, g_2 is negative. By convention, we fix the origin of the elliptic curve for its group law at the (inflexion) point at infinity. If u and v are two points of E, the line containing u and v intersects E in a third point w. The sum $u + v + w$ is zero for the group law. If $u = (x, y)$ is a point of E, then $-u = (x, -y)$ and the fixed points of this involution $u \mapsto -u$ on $E(\mathbf{R})$ are the two points $(x_0, 0)$ and (∞, ∞) where x_0 is the unique real solution of the equation $4x^3 = g_2 x + g_3$.

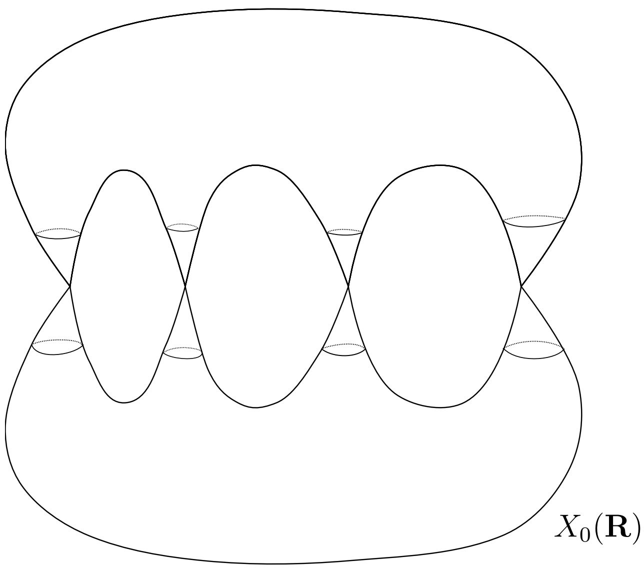
Now, consider the map $\Phi\colon E \times E \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ which is defined as follows: if (u, v) belongs to $E \times E$, with $u = (x_1, y_1)$ and $v = (x_2, y_2)$, and if $w = -(u + v) = (x_3, y_3)$, then $\Phi(u, v) = (x_1, x_2, x_3)$. One easily checks that $\Phi \circ \eta = \Phi$, with $\eta(u, v) = (-u, -v)$, and Φ embeds the Kummer surface $X_0 = (E \times E)/\eta$ into $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as a singular $(2, 2, 2)$-surface (see [7, §8.2]). The singularities of X_0 correspond to the fixed points of η, i.e. to the group $A[2]$ of torsion points of order 2 in $A = E \times E$. This gives 16 points, of which only 4 are real:

$$(x_0, x_0, \infty),\ (x_0, \infty, x_0),\ (\infty, x_0, x_0),\ (\infty, \infty, \infty). \tag{9.2}$$

The real part of X_0 corresponds to the fixed point set of σ_0, i.e. of σ_A viewed on the quotient space X_0. Recall that $\sigma_A(z_1, z_2) = (\overline{z_1}, \overline{z_2})$ if we think of A as $\mathbf{C}^2/\Lambda^2$. The fixed points of σ_0 are of two types: those coming from the fixed points of σ_A, hence from the real part $A(\mathbf{R})$, and those coming from

$$\Delta := \{(z_1, z_2) \in A\ ;\ \sigma_A(z_1, z_2) = \eta(z_1, z_2)\}. \tag{9.3}$$

In the quotient X_0, $A(\mathbf{R})$ projects onto a sphere with four singularities; the projection of Δ is another sphere with the same four singularities. These two spheres are glued along those four points; locally X_0 is the quotient of $\mathbf{C}^2$ by $\{\mathrm{id}, -\mathrm{id}\}$, so up to an analytic change of coordinates, $X_0(\mathbf{R})$ is a quadratic cone isomorphic to $x_1 x_2 = x_3^2$.



The natural action of $\mathsf{GL}_2(\mathbf{Z})$ on $E \times E$ descends to an action of $\mathsf{PGL}_2(\mathbf{Z})$ on X_0, which preserves $X_0(\mathbf{R})$; it also preserves individually each of the two connected components of $X_0(\mathbf{R}) \backslash \mathrm{Sing}(X_0(\mathbf{R}))$. The action of $\mathsf{PGL}_2(\mathbf{Z})$ on these two punctured spheres has dense orbits (and finite orbits too, corresponding to torsion points of A). If we resolve the singularities of $X_0(\mathbf{R})$, the two punctured spheres become two surfaces homeomorphic to a sphere minus four disks; they are glued together along their boundaries to form a closed, orientable surface of genus 3, which is the real part of $X(\mathbf{R})$ for the real structure σ_X. Thus, the generic orbit of $\mathsf{PGL}_2(\mathbf{Z})$ in $X(\mathbf{R})$ is dense in one of these two open subsets of $X(\mathbf{R})$. This gives a first example of an invariant surface with boundary Σ, given by the component $A(\mathbf{R})/\eta$, with an invariant measure μ given by the push forward of the Haar measure from $A(\mathbf{R})$ to Σ.

9.2. **Deformation.** Let us come back to $X_0(\mathbf{R})$. Switching the chart in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ so that the coordinates (x_1, x_2, x_3) are replaced by their inverses $(1/x_1, 1/x_2, 1/x_3)$, the four singularities become

$$(\alpha, \alpha, 0),\ (\alpha, 0, \alpha),\ (0, \alpha, \alpha),\ (0, 0, 0) \tag{9.4}$$

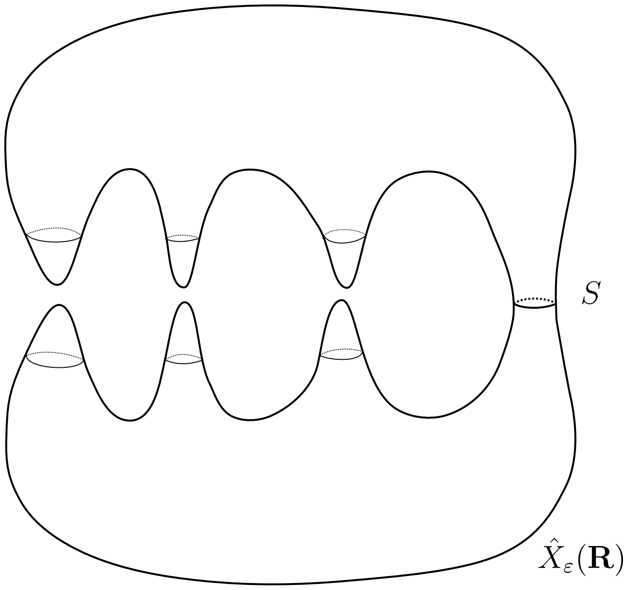
with $\alpha = 1/x_0$. Note that the three vectors $v_1 = (\alpha, \alpha, 0)$, $v_2 = (\alpha, 0, \alpha)$, and $v_3 = (0, \alpha, \alpha)$ are linearly independent (their determinant is $-2\alpha^3$). Thus, given any triple $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{\pm 1\}^3$, there is a real quadratic form $Q(x_1, x_2, x_3)$ such that $\varepsilon_i Q(v_i) > 0$ for each $1 \leqslant i \leqslant 3$. If P denotes the equation of X_0 (after changing the x_i in $1/x_i$ as above) and ε is a small real number, then $P + \varepsilon Q$ is an equation of a new surface X_ε of degree $(2, 2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. At the origin $(0, 0, 0)$, the linear term of the equation $P = 0$ is not changed by the addition of εQ, and the quadratic term is only slightly perturbed if ε is small enough, so X_ε still admits a quadratic singularity, which is non-degenerate of signature $(2, 1)$. At the other three real singularities of X_0, we can choose the sign of $Q(v_i)$ in such a way that $X_\varepsilon(\mathbf{R})$ is locally disconnected (as does a hyperboloid with two sheets).

We claim that, shifting Q a little bit if necessary, all (real or complex) singularities of X_0 disappear in the perturbation X_ε, except the origin. Indeed, by conjugating by a diagonal automorphism $(h, h, h) \in \mathsf{Aut}(\mathbb{P}^1)^3$, such that $h \in \mathsf{PGL}_2(\mathbf{C})$ fixes 0 and α, we can arrange that all singular points of X belong to $\mathbf{C}^3$. Let $(s_j)_{j=0,\ldots,15}$ be the singular points of X_0 (with $s_0 = 0$) and choose Q in such a way that $Q(s_j) \neq 0$ for $j \geqslant 1$. Let $N = \bigcup_j B(s_j, \eta)$, where η is so small that $Q \neq 0$ on $\bigcup_{j \geqslant 1} B(s_j, \eta)$. Take ε to be small and non-zero. Then,X_ε is smooth in $\bigcup_{j \geqslant 1} B(s_j, \eta)$, because its equation $P + \varepsilon Q = 0$ reduces to $P/Q + \varepsilon = 0$ there, and such a hypersurface is smooth for $\varepsilon \neq 0$ small. Finally, smoothness being an open property, X_ε is also smooth in the complement of N.

Remark 9.1. Such a deformation appears naturally in the closely related example of character varieties for the once punctured torus (see [6, 27]).

The result, for a sufficiently small ε and a good choice of Q, is a real surface X_ε of degree $(2, 2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with a real part $X_\varepsilon(\mathbf{R})$ satisfying the following properties.

- The surface $X_\varepsilon(\mathbf{R})$ has a unique singularity, at the origin.
- After a minimal resolution of the singularity, we get a real K3 surface $\hat{X}_\varepsilon$. Indeed, the area form Ω_{X_ε} defined in Example 2.1 lifts to a trivialization of the canonical



bundle of $\hat{X}_\varepsilon$; and $\hat{X}_\varepsilon(\mathbf{C})$ is simply connected (to see this one can use the fibrations $\hat{\pi}_i\colon \hat{X}_\varepsilon(\mathbf{C}) \to X_\varepsilon(\mathbf{C}) \to \mathbb{P}^1(\mathbf{C})$).

- The surface $\hat{X}_\varepsilon(\mathbf{R})$ is homeomorphic to a sphere $\mathbb{S}^2$; the exceptional divisor is a curve $S_\varepsilon \subset \hat{X}_\varepsilon(\mathbf{R})^0$ that separates $\hat{X}_\varepsilon(\mathbf{R})$ in two one-holed spheres with boundary S_ε;
- The three involutions $\sigma_i\colon X_\varepsilon \to X_\varepsilon$ described in Example 2.1 lift to three automorphisms of $\hat{X}_\varepsilon$. They generate a non-elementary subgroup Γ_ε of $\mathsf{Aut}(\hat{X}_\varepsilon)$ that preserves the real structure (the compositions $\sigma_i \circ \sigma_j$ are parabolic automorphisms with respect to distinct fibrations). A subgroup of index 4 preserves simultaneously each component of $\hat{X}_\varepsilon(\mathbf{R})\setminus S$ and the canonical area form of the K3 surface.

This provides *examples with non-trivial invariant open subsets of $X(\mathbf{R})$ for a non-elementary subgroup of* $\mathsf{Aut}(X_\mathbf{R})$, in a case where X is not a Kummer surface (the dynamics of the group Γ_ε is not covered by a linear dynamics on a torus).

9.3. **Invariant curves.** Let us continue with the example given by X_ε.

When $\varepsilon = 0$, X_0 is a (singular) Kummer surface, and its singularities are in 1 to 1 correspondance with the element of $A[2]$. Let $\Gamma(2) \subset \mathsf{GL}_2(\mathbf{Z})$ be the subgroup that fixes $A[2]$ pointwise. The image of $\Gamma(2)$ in $\mathsf{Aut}(X_0)$ preserves the 16 singularities of X_0. This group lifts to a group of automorphisms $\Gamma_0 \subset \mathsf{Aut}(\hat{X}_0)$. Let $S_0 \subset \hat{X}_0$ be the (-2)-curve obtained by the minimal resolution of one of the singularities of X_0. The dynamics of Γ on S_0 coincides, up to conjugacy, with the linear projective action of $\Gamma(2) \subset \mathsf{GL}_2(\mathbf{Z})$ on $\mathbb{P}(\mathbf{C}^2)$. This is a non-elementary subgroup of $\mathsf{Aut}(\mathbb{P}^1) \simeq \mathsf{PGL}_2(\mathbf{C})$; in particular, this action does not preserve any probability measure.

Now, consider the small perturbation $\hat{X}_\varepsilon$ and the group Γ_ε, as constructed in Section 9.2. Then, Γ_ε preserves the (-2)-curve $S_\varepsilon \subset \hat{X}_\varepsilon$, and for $\varepsilon = 0$, we recover the group Γ_0 up to finite index. Since the non-elementary property of $\Gamma_0|_{S_0} \subset \mathsf{PGL}_2(\mathbf{C})$ is invariant under small perturbations or after taking finite index subgroups, we deduce that Γ_ε induces a non-elementary subgroup of S_ε. Thus, we obtain *examples of K3 surfaces $\hat{X}_\varepsilon$ with of a non-elementary subgroup $\Gamma_\varepsilon \subset \mathsf{Aut}(\hat{X}_\varepsilon)$ such that Γ preserves a smooth rational curve $S_\varepsilon \subset \hat{X}_\varepsilon$ but S_ε does not support any Γ_ε-invariant probability measure*; here, the examples are deformations of a Kummer example $(\hat{X}_0, \Gamma_0)$.

Remark 9.2. Similar examples can be constructed on some Coble surfaces Y: $\mathsf{Aut}(Y)$ preserves a rational curve, coming from a plane sextic with ten nodes; and then, taking a double cover of Y ramified along the invariant sextic, one gets K3 surfaces.

Remark 9.3. Consider the above example $(\hat{X}_\varepsilon, S_\varepsilon, \Gamma_\varepsilon)$ and a probability measure ν on Γ_ε whose support is finite and generates Γ_ε. Then, the curve $S_\varepsilon \simeq \mathbb{P}^1(\mathbf{C})$ supports a unique ν-stationary measure μ_ν, because $\Gamma_\varepsilon \subset \mathsf{PGL}_2(\mathbf{C})$ is non-elementary (see [23]). If, as above, everything is defined over $\mathbf{R}$, the support of μ_ν is contained in a circle; but if we apply the same construction with a well chosen, small complex deformation X_ε, the support of μ_ν is supported on a fractal quasi-circle.

Appendix A. Abelian surfaces

In this appendix, we consider the case when all parabolic automorphisms g of Γ induce an automorphism g_B of infinite order on the base of their invariant fibration π_g. In that case, we know from [12, Proposition 3.6] that X is a compact torus, and in fact an abelian surface since Γ is non-elementary. Thus, we assume that

(i) X is an abelian surface, isomorphic to $\mathbf{C}^2/\Lambda$ for some lattice Λ;
(ii) Γ is a non-elementary group of automorphisms of X that contains a parabolic element g;
(iii) every parabolic element g of Γ acts on the base of its invariant fibration $\pi_g \colon X \to B_g$ by an automorphism $g_B \colon B_g \to B_g$ of infinite order.

We provide an argument to complete the proof of Theorem A in that case; the strategy is the same as in Sections 4 and 5, but simpler since the dynamics is linear:

Proposition A.1. *Under the above hypotheses* (i)*,* (ii)*,* (iii)*, if μ is a Γ-invariant and ergodic measure, then either μ is the Haar measure on the abelian surface X, or there are finitely many subtori $S_j \subset X$ of real dimension* 2*, and points $a_j \in X$, $j = 1, \ldots, k$, such that*

(1) *$\bigcup_j (a_j + S_j)$ is Γ-invariant;*
(2) *Γ permutes transitively the subsets $a_j + S_j$, $j = 1, \ldots, k$;*
(3) *μ is supported on $\bigcup_j (a_j + S_j)$ and on each $a_j + S_j$, μ is given by $\frac{1}{k} m_j$ where m_j is the Haar measure on $a_j + S_j$.*

Here, what we call Haar measure on $a_j + S_j$ is the image of the Haar measure on S_j by the translation $s \in S_j \mapsto a_j + s$. With the results of Sections 4, this proposition concludes the proof of Theorem A.

A.1. **Parabolic, affine transformations.** The group Γ acts on X by affine transformations

$$f(x,y) = A_f(x,y) + S_f \mod (\Lambda) \tag{A.1}$$

where the linear part $A_f \in \mathsf{GL}_2(\mathbf{C})$ preserves the lattice $\Lambda \subset \mathbf{C}^2$ and the translation part S_f is an element of $\mathbf{C}^2/\Lambda$. Now, pick a parabolic element $g \in \Gamma$; its linear part is given by

$$A_g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{A.2}$$

after a linear change of coordinates in $\mathbf{C}^2$. In these coordinates, the fibration π_g is induced by the projection $\pi_1 \colon (x,y) \mapsto x$, and a conjugation by a translation reduces g to the form

$$g(x,y) = (x+s, y+x) \mod (\Lambda) \tag{A.3}$$

where s has infinite order in the elliptic curve $B_g = \mathbf{C}/\pi_1(\Lambda)$.

Lemma A.2. *If the orbits of $g_B \colon x \mapsto x+s$ are dense in B_g, then g is uniquely ergodic: the unique g-invariant probability measure on X is the Haar measure.*

This result is due to Furstenberg (see [24, §3.3]). Thus, in this case μ is the Haar measure on X and we are done. So in what follows, we assume that for every $g \in \mathrm{Hal}(\Gamma)$ the orbits of the translation g_B are not dense: they equidistribute along circles $x + T_g$, where T_g is the closure of the group $\mathbf{Z}s \subset B_g$; changing g into a positive iterate we may assume that this closure is isomorphic to $\mathbf{R}/\mathbf{Z}$ (as a real Lie group). We let ℓ_g be the quotient map $\mathbf{C}^2/\Lambda \to B_g/T_g$.

Lemma A.3. *Every fiber of the linear projection ℓ_g is a 3-dimensional g-invariant torus, and g is uniquely ergodic on almost every fiber.*

To prove Lemma A.3, we think of $\mathbf{C}^2$ as a real vector space and fix a basis of Λ. Then $\mathbf{C}^2$ is identified with $\mathbf{R}^4$ and Λ with $\mathbf{Z}^4 \subset \mathbf{R}^4$. The eigenspace of A_g for the eigenvalue 1 is defined over $\mathbf{Z}$ with respect to $\Lambda = \mathbf{Z}^4$. Moreover, A_g acts trivially on the quotient space $\mathbf{R}^4/\mathrm{Fix}(A_g)$. Thus adapting the basis of $\mathbf{Z}^4$ to g, we may assume that

$$g(x_1,x_2,x_3,x_4) = (x_1, x_2+s_2, x_3+ax_1+bx_2, x_4+cx_1+dx_2), \tag{A.4}$$

for some irrational number s_2 and some integers a, b, c, and d. The linear projection ℓ_g is now given by $(x_1,x_2,x_3,x_4) \mapsto x_1$ and Lemma A.3 boils down to the following statement.

Lemma A.4. *If 1, s_1 and s_2 are linearly independent over $\mathbf{Q}$, then g is uniquely ergodic on the level set $\{x_1 = s_1\}$.*

Proof. Let us first observe that $ad - bc \neq 0$. Indeed, otherwise the linear part of g would have a fixed point set of dimension 3, which is impossible because g is holomorphic (see Equation (A.2)). To prove unique ergodicity, we use the following criterion due to Furstenberg (see [24, Prop. 3.10]): let h be a homeomorphism on $\mathbf{R}/\mathbf{Z} \times (\mathbf{R}^2/\mathbf{Z}^2)$ of the form $(x,y) \mapsto (x+u, y+\varphi(x))$, where u is irrational, then h is uniquely ergodic if and only if it is ergodic for the Haar measure. On the fiber $x_1 = s_1$, our map g is of the form

$$(x_2,x_3,x_4) \longmapsto (x_2+s_2, x_3+as_1+bx_2, x_4+cs_1+dx_2), \tag{A.5}$$

so we need to check that it is ergodic for the Haar measure. For this, we pick a measurable invariant subset $A \subset \mathbf{R}^3/\mathbf{Z}^3$, we denote by $\mathbf{1}_A \in L^2(\mathbf{R}^3/\mathbf{Z}^3)$ its indicator function, and we expand it into a Fourier series $\mathbf{1}_A(x_2, x_3, x_4) = \sum_{(k,\ell,m)\in\mathbf{Z}^3} c_{k,\ell,m} e^{2i\pi(kx_2+\ell x_3+mx_4)}$. Then

$$\text{(A.6)}\quad \mathbf{1}_A \circ g(x_2, x_3, x_4) = \sum_{(k,\ell,m)\in\mathbf{Z}^3} c_{k,\ell,m} e^{2i\pi ks_2} e^{2i\pi(\ell a+mc)s_1} e^{2i\pi(k+\ell b+md)x_2} e^{2i\pi\ell x_3} e^{2i\pi mx_4}$$

and, from the g-invariance of $\mathbf{1}_A$ and the uniqueness of the expansion, we get

$$\text{(A.7)}\qquad c_{k,\ell,m} = e^{2i\pi ks_2} e^{2i\pi(\ell a+mc)s_1} c_{k-\ell b-md,\ell,m}$$

for all $(k, \ell, m) \in \mathbf{Z}^3$. For $\ell = m = 0$, the irrationality of s_2 implies that $c_{k,0,0} = 0$ unless $k = 0$. If $\ell b + md \neq 0$, iterating the relation $|c_{k,\ell,m}| = |c_{k-\ell b-md,\ell,m}|$ and using the fact that Fourier coefficients decay to zero at infinity, we infer that $c_{k,\ell,m} = 0$. Finally, if $\ell b + md = 0$ and one of ℓ or m is nonzero, since $ad - bc \neq 0$ we get that $\ell a + mc \neq 0$ and (A.7) gives $c_{k,\ell,m} = e^{2i\pi(ks_2+(\ell a+mc)s_1)} c_{k,\ell,m}$. Since 1, s_1, and s_2 are $\mathbf{Q}$-linearly independent, we derive $c_{k,\ell,m} = 0$. Thus, $\mathbf{1}_A$ is a constant, which means that the Haar measure of A is 0 or 1. □

A.2. **Using distinct parabolic automorphisms.** To complete the proof of Theorem A in the case of tori, one can now follow the same ideas as in Sections 4 and 5. We only need to replace the dimension $\dim_{\mathbf{R}}(\mu)$ by the minimal dimension of a real subtorus $Z \subset X$ such that $\mu(q + Z) > 0$ for some $q \in X$, the invariant fibration π_g by the $\mathbf{R}$-linear projection ℓ_g, and the set $\mathrm{R}_g(B_g^\circ) \subset B_g^\circ$ by

$$\text{(A.8)}\qquad \mathrm{R}(g) = \{y \in B_g/T_g \; ; \; g \text{ is not uniquely ergodic in } \ell_g^{-1}(y)\} \subset B_g/T_g.$$

Lemma A.4 shows that $\mathrm{R}(g)$ is countable.

Lemma A.5. *If there is a parabolic element $g \in \Gamma$ for which $((\ell_g)_*\mu)(\mathrm{R}(g)) < 1$, then μ is the Haar measure on X.*

Proof. The proof is the same as for Proposition 4.9. Pick another parabolic transformation $h \in \Gamma$, such that ℓ_g and ℓ_h are linearly independent; such an h exist because Γ is non-elementary. The main tool is the disintegration of μ with respect to ℓ_g; for y in a subset $\mathcal{Y}_g \subset B_g/T_g$ of positive measure, the conditional measure $\lambda_{g,y}$ is the Haar measure on the 3-dimensional torus $\ell_g^{-1}(y)$. Hence $\dim_{\mathbf{R}}(\mu) \geqslant 3$ and $((\ell_g)_*\mu)(\mathrm{R}(g)) = 0$, as in Step 2 of the proof of Proposition 4.9. As in Steps 3 and 4, we infer that $(\ell_h)_*\mu$ does not charge $\mathrm{R}(h)$ and that $(\ell_h)_*\mu$ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}/\mathbf{Z}$. This, implies that μ itself is invariant by *all* translations along the fibers of ℓ_h, because $\mu = \int_Y \lambda_{h,y} d((\ell_h)_*\mu)(y)$ and $\lambda_{h,y}$ is the Haar measure for almost every y. Permuting the roles of g and h, μ is in fact invariant under all translations. Hence, μ is the Haar measure on X. □

Now, we we are reduced to the case where $(\ell_g)_*\mu(R(g)) = 1$ for every parabolic automorphism g in Γ. Since $R(g)$ is countable, $d_{\mathbf{R}}(\mu) \leqslant 3$ and μ charges some fiber of ℓ_g. Using another parabolic automorphism h, we see that μ gives positive mass to a translate $a_0 + S_0$ of a 2-dimensional torus $S_0 \subset X$ whose projections $\ell_g(a_0 + S_0) = \ell_g(a_0)$ and $\ell_h(a_0)$ are in the countable sets $R(g) \subset B_g/T_g$ and $R(h) \subset B_h/T_h$ respectively. Thus, by ergodicity, we conclude that μ is supported on a finite union of translates of 2-dimensional tori $a_j + S_j \subset X$, $0 \leqslant i \leqslant k-1$ for some $k \geqslant 1$.

A subgroup Γ_0 of index $\leqslant k!$ in Γ preserves $a_0 + S_0$, and $g^{k!}$ and $h^{k!}$ act on $a_0 + S_0 \simeq \mathbf{R}^2/\mathbf{Z}^2$ as two linear parabolic transformations with respect to transverse linear fibrations. So, it follows that $\mu_{|a_0+S_0}$ is proportional to the Haar measure of $a_0 + S_0$, and the proof of Proposition A.1 is complete.

A.3. **No or infinitely many invariant real tori.** Consider a compact complex torus $X = \mathbf{C}^2/\Lambda$ of dimension 2. Let Γ be a subgroup of $\mathsf{Aut}(X)$. As in § A.1, write the elements f of $\mathsf{Aut}(X)$ in the form $f(x, y) = A_f(x, y) + S_f$, and denote by $A_\Gamma \subset \mathsf{GL}_2(\mathbf{C})$ the image of Γ by the homomorphism $f \mapsto A_f$. The group Γ is non-elementary if and only if A_Γ contains a free group, if and only if the Zariski closure of A_Γ in the *real algebraic group* $\mathsf{GL}_2(\mathbf{C})$ is semi-simple.

Now, assume that Γ is non-elementary and preserves at least one ergodic probability measure μ with $\dim_{\mathbf{R}}(\mu) = 2$. Equivalently, after conjugation by a translation, there is a finite index subgroup $\Gamma_0 \subset \Gamma$ that preserves a real, two-dimensional subtorus $\Sigma = \Pi/\Lambda_\Pi$, where $\Pi \subset \mathbf{C}^2$ is a real vector space of dimension 2 and $\Lambda_{\mathbf{R}} := \Pi \cap \Lambda$ is a lattice in Π (the restriction of μ to Π/Λ_Π is proportional to the Haar measure). The goal of this last section is to explain that, in fact, Γ *preserves infinitely many ergodic measures* μ_j *with* $\dim_{\mathbf{R}}(\mu_j) = 2$. Two mechanisms can be used to establish this fact.

The first one relies on the fact that Γ_0 acts on the quotient $Q = X/\Sigma \simeq \mathbf{R}^2/\mathbf{Z}^2$, fixing the origin. Moreover, the action of Γ_0 on $Q = \mathbf{R}^2/\mathbf{Z}^2$ is induced by an *injective homomorphism* $\Gamma_0 \to \mathsf{GL}_2(\mathbf{Z})$ (to see this, note that $\mathbf{C}^2 = \Pi \oplus_{\mathbf{R}} \mathrm{i}\Pi$ and $\mathrm{i}\Pi$ surjects onto Q). This implies that Γ_0 has arbitrarily large finite orbits in Q (coming from torsion points of Q). The preimages of these orbits in X provide surfaces $\Sigma_j \subset X$; they are "parallel" to Σ and have an arbitrarily large number of connected components; they are Γ_0-invariant; and each of them supports a unique invariant, ergodic, probability measure μ_j with $\dim_{\mathbf{R}}(\mu_j) = 2$.

For the second mechanism, we assume that Γ_0 fixes the origin and, changing Γ_0 in a finite index subgroup if necessary, we identify Γ_0 with a subgroup of $\mathsf{SL}_2(\mathbf{C})$. Identify (Π, Λ_Π) to $(\mathbf{R}^2, \mathbf{Z}^2)$ and the restriction $\Gamma_0|_\Pi$ to a subgroup of $\mathsf{GL}_2(\mathbf{Z})$; since Γ_0 is non-elementary, Γ_0 is Zariski dense in $\mathsf{SL}_2(\mathbf{C})$ and $\Gamma_0|_\Pi$ is Zariski dense in $\mathsf{SL}_2(\mathbf{R})$ (resp. in $\mathsf{SL}_2(\mathbf{C})$). In particular, the $\mathbf{Q}$-algebra generated by $\Gamma_0|_\Pi$ is the algebra of 2×2 matrices with rational coefficients. The decomposition $\mathbf{C}^2 = \Pi \oplus_{\mathbf{R}} \mathrm{i}\Pi$ is Γ_0-invariant, and the multiplication by i defines a Γ_0-equivariant map from Π to $\mathrm{i}\Pi$. Thus, Γ_0 preserves each of the real planes $\Pi_\eta = \{(x, y) + \eta\mathrm{i}(x, y)\ ;\ \text{for}\ (x, y) \in \Pi\}$, with $\eta \in \mathbf{R}$. Now, consider the (real) projection q of $\mathbf{C}^2$ onto Π parallel to $\mathrm{i}\Pi$, and set $\Lambda' = q(\Lambda)$. It is a Γ_0-invariant subgroup of Π of rank at most 4, and it contains $\Lambda_\Pi \simeq \mathbf{Z}^2$. Then, one checks easily that

(1) Λ' is commensurable to $\Lambda_\Pi \oplus \alpha\Lambda_\Pi$, for some $\alpha \in \mathbf{R}\backslash\mathbf{Q}$, or to Λ_Π, in which case we set $\alpha = 0$;
(2) Λ is commensurable to $\Lambda_\Pi \oplus K_{\alpha,\beta}(\Lambda_\Pi)$ where $K_{\alpha,\beta}$ is the linear map from Π to $\Pi \oplus \mathrm{i}\Pi$ defined by $K_{\alpha,\beta}(u) = \alpha u + \beta\mathrm{i}u$;
(3) for m in $\mathbf{Z}$, the real plane $\Pi_{m\alpha/\beta}$ is Γ-invariant and intersects Λ on a cocompact lattice $\Lambda_{\Pi_{m\alpha/\beta}}$.

Then, the surfaces $\Sigma_m = \Pi_{m\alpha/\beta}/\Lambda_{\Pi_{m\alpha/\beta}}$ form an infinite family of Γ-invariant tori in X.

Remark A.6. This second argument does not apply in the following case. Let $E = \mathbf{C}/\mathbf{Z}[\mathrm{i}]$, $\Lambda = \mathbf{Z}[\mathrm{i}] \times \mathbf{Z}[\mathrm{i}] \subset \mathbf{C}^2$, and $X = \mathbf{C}^2/\Lambda = E \times E$. The group $\Gamma = \mathsf{SL}_2(\mathbf{Z}) \ltimes \mathbf{R}^2/\mathbf{Z}^2$ is a subgroup of $\mathsf{Aut}(X)$ that preserves the torus Π/Λ_Π for $\Pi = \mathbf{R}^2 \subset \mathbf{C}^2$, but has no fixed point (because Γ contains Π/Λ_Π), and every Γ-invariant surface is a finite union of translates of this torus .

On the other hand, this second argument applies when X and Γ come from a genuine Kummer example, that is, a Kummer example defined on a surface that is not a compact torus. Indeed in that case Γ contains a finite index subgroup with a fixed point.

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SERGE CANTAT, IRMAR, CAMPUS DE BEAULIEU, BÂTIMENTS 22-23 263 AVENUE DU GÉNÉRAL LECLERC, CS 74205 35042 RENNES CÉDEX

Email address: serge.cantat@univ-rennes1.fr

SORBONNE UNIVERSITÉ, CNRS, LABORATOIRE DE PROBABILITÉS, STATISTIQUE ET MODÉLISATION (LPSM), F-75005 PARIS, FRANCE

Email address: romain.dujardin@sorbonne-universite.fr