HYPERBOLICITY FOR LARGE AUTOMORPHISM GROUPS OF PROJECTIVE SURFACES

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ABSTRACT. We study the hyperbolicity properties of the action of a non-elementary automorphism group on a compact complex surface, with an emphasis on K3 and Enriques surfaces. A first result is that when such a group contains parabolic elements, Zariski diffuse invariant measures automatically have non-zero Lyapunov exponents. In combination with our previous work, this leads to simple criteria for a uniform expansion property on the whole surface, for groups with and without parabolic elements. This, in turn, has consequences on the dynamics: description of orbit closures, equidistribution, ergodicity properties, etc. Along the way, we provide a reference discussion on uniform expansion of non-linear discrete group actions on compact (real) manifolds and the construction of Margulis functions under optimal moment conditions.

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1. INTRODUCTION

Let X be a compact complex surface and denote by $\mathsf{Aut}(X)$ its group of automorphisms, i.e. of holomorphic diffeomorphisms. Let Γ be a subgroup of $\mathsf{Aut}(X)$. We say that Γ is **non-elementary** if the subgroup $\Gamma^* \leqslant \mathsf{GL}(H^*(X, \mathbf{C}))$ induced by the action of Γ on the De Rham cohomology of X contains a non-abelian free group; the existence of a non-elementary subgroup of $\mathsf{Aut}(X)$ implies that X is projective (see [18]). In a series of articles [21, 22, 20] we have explored the dynamics of such a non-elementary group Γ on X, notably by means of random walk techniques. In this paper, we study the hyperbolicity properties of such random actions and their consequences.

1.1. **Wehler examples.** To understand the motivation behind our general results, it is interesting to start with the Wehler family $\mathcal{W}$ of surfaces of degree $(2, 2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, which has been a recurring example in our work (see e.g. [21, §3]). This family $\mathcal{W}$ depends on 26 parameters and is naturally parameterized by $\mathbb{P}^{26}(\mathbf{C})$; we shall denote by $\mathcal{W}_0 \subset \mathcal{W}$ the Zariski open subset of smooth Wehler surfaces which do not contain any fiber of the three coordinate projections $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Note that $\mathsf{Aut}(\mathbb{P}^1)^3$ acts on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as well as on $\mathcal{W}$ and $\mathcal{W}_0$. For $X \in \mathcal{W}_0$, the three natural projections $X \to \mathbb{P}^1 \times \mathbb{P}^1$ are ramified covers of degree 2; their deck transformations yield three holomorphic involutions σ_1, σ_2, and σ_3; the group Γ generated by these involutions is non-elementary and isomorphic to $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$.

Since every $X \in \mathcal{W}_0$ is a K3 surface, there is a canonical $\mathsf{Aut}(X)$-invariant volume form vol_X on $X(\mathbf{C})$; furthermore, when X is defined over $\mathbf{R}$ there is a canonical area form $\mathsf{vol}_{X(\mathbf{R})}$ on $X(\mathbf{R})$ which is invariant under the action of $\mathsf{Aut}(X_\mathbf{R})$ (see Example 1.5 below). Slightly abusing notation, we respectively denote by vol_X and $\mathsf{vol}_{X(\mathbf{R})}$ the associated measures on X and $X(\mathbf{R})$, normalized to have mass 1.

Our first main result is a complete description of orbit closures for most parameters $X \in \mathcal{W}_0$. Recall that a 2-dimensional real submanifold $Y \subset X$ is *totally real* if for every $x \in Y$, T_xY spans T_xX as a complex vector space.

Theorem 1.1. *There exists a dense and Zariski open subset $\mathcal{W}_{\rm exp} \subset \mathcal{W}_0(\mathbf{C})$ such that for every $X \in \mathcal{W}_{\rm exp}$, the action of $\Gamma = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ on X satisfies the following properties. There exists a Γ-invariant finite set $F \subset X$ and a Γ-invariant totally real analytic surface $Y \subset X$ (with possibly finitely many singular points) such that for* every $x \in X$,

(a) *either $x \in F$ (and its orbit is finite);*
(b) *or $\overline{\Gamma(x)}$ is a union of connected components of Y;*
(c) *or $\overline{\Gamma(x)} = X$.*

As a general convention in the paper, by "dense" we mean dense for the Euclidean topology; when working with the Zariski topology, we specify "Zariski dense". Note also that a Zariski open subset of a variety W is dense if and only if it intersects every component of W. In this statement both F and Y may be empty, depending on X. For instance, [22, Thm A] says that F is empty for a very general Wehler surface $X \in$

$\mathcal{W}_0(\mathbf{C})$, i.e. for X in the complement of countably many proper Zariski closed subsets. A typical situation for case (b) is that X is defined over $\mathbf{R}$ and $Y = X(\mathbf{R})$.

Remark 1.2. To check that an orbit $\Gamma(x)$ is dense, it suffices to find a point $x' = (x_1', x_2', x_3')$ in $\Gamma(x)$ such that (1) the fiber $X_{x_3'}$ of the third projection containing x' is a smooth curve and (2) $\sigma_1 \circ \sigma_2$ acts on this genus 1 curve $X_{x_3'}$ as a translation with dense orbits. Then, the closure of $\Gamma(x)$ is infinite and is not contained in a totally real surface, so case (c) occurs. Now, for x_3 outside a countable union of real analytic curves, $\sigma_1 \circ \sigma_2$ has dense orbits along X_{x_3}. Thus, it is easy to produce examples of dense orbits. On the other hand, given a specific Wehler surface X, it is a priori hard to decide whether there is a Γ invariant real anaytic surface $Y \subset X$.

If we restrict to real parameters in $\mathcal{W}$, we also have a fairly complete understanding of the asymptotic distribution of random orbits. By this we mean the following. Let ν be the probability measure on Γ defined by $\nu = \frac{1}{3}(\delta_{\sigma_1} + \delta_{\sigma_2} + \delta_{\sigma_3})$. For any x in $X(\mathbf{R})$, and for any sequence (g_i) of automorphisms $g_i \in \Gamma$ chosen independently with distribution ν, consider the trajectory $(g_n \cdots g_0(x))_{n\geqslant 0}$. Let $X'(\mathbf{R})$ be a union of connected components of $X(\mathbf{R})$. We say that these random trajectories, starting at x, are **equidistributed** in $X'(\mathbf{R})$ if for $\nu^{\mathbf{N}}$-almost every (g_i), the empirical measures $\frac{1}{n}\sum_{k=0}^{n-1} \delta_{g_k\cdots g_0(x)}$ converge to the normalized volume form induced by $\mathsf{vol}_{X(\mathbf{R})}$ on $X'(\mathbf{R})$ as $n \to \infty$. The appearance of $X'(\mathbf{R})$ is due to the fact that Γ may not act transitively on the components of $X(\mathbf{R})$.

Theorem 1.3. *There exists a dense and Zariski open subset $\mathcal{W}_{\exp}(\mathbf{R}) \subset \mathcal{W}_0(\mathbf{R})$ such that for every $X \in \mathcal{W}_{\exp}(\mathbf{R})$, there exists a Γ-invariant finite set $F \subset X(\mathbf{R})$ such that for every $x \in X(\mathbf{R})$:*

(a) *either $x \in F$;*
(b) *or the random trajectories starting at x are equidistributed in a union of connected components of $X(\mathbf{R})$.*

An interesting point in Theorems 1.1 and 1.3 is that their conclusions hold for *every* $x \in X$. Let us explain how these theorems fall within the progression of [21, 22, 20] and what the last missing ingredient was until the present paper.

First, the existence of the maximal finite invariant set F follows from [22, Thm C]. One key point here is that Γ contains *parabolic elements*, that is automorphisms whose action on $H^*(X; \mathbf{C})$ is virtually unipotent and of infinite order (see Section 6).

Now, the scheme of proof of Theorem 1.3 is as follows. The random walk on Γ induced by ν gives rise to a random dynamical system on X. We refer to [42, 13] for general references on this topic, and to Sections 4 and 7 of [21] for our holomorphic context. In particular, we shall use the notions of stationary and invariant measures μ, of fibered entropy $h_\mu(X, \nu)$, etc. Fix $x \in X \backslash F$. By Breiman's ergodic theorem, for almost every sequence $(g_n)_{n\geqslant 0}$ with respect to the measure $\nu^{\mathbf{N}}$, every cluster value of the sequence of empirical measures $\frac{1}{n}\sum_{k=0}^{n-1} \delta_{g_k\cdots g_0(x)}$ is a ν-stationary measure (see [5, §2.2]). We proved in [21] that every ν-stationary measure is Γ-invariant([1]), and in [20] we showed that any invariant measure is either supported on F, or of the form $\mathsf{vol}_{X'(\mathbf{R})}$, for some union of components $X'(\mathbf{R})$ of $X(\mathbf{R})$. Therefore, any cluster value of $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{g_k\cdots g_0(x)}$ is a convex combination of point masses on F and $\mathsf{vol}_{X'(\mathbf{R})}$. Thus the last step is to show that if $\Gamma(x)$ is infinite, the limiting empirical measures give no mass to F.

For Theorem 1.1 the situation is similar: most of the work was done in [20, §8], except that there we could not exclude that the accumulation locus of an infinite orbit could be contained in a finite invariant set. Note that since we are talking about orbit closures and not asymptotic distribution, the full classification of stationary measures, which is much harder and not yet complete in the complex surface $X(\mathbf{C})$, is not required here.

These difficulties were already addressed for homogeneous random dynamical systems in [4, 31] and in the context of non-linear actions on real surfaces in [45, 24]. The key is to show that if X belongs to the dense Zariski open set $\mathcal{W}_{\rm exp}$ of Theorem 1.1 (resp. $\mathcal{W}_{\rm exp}(\mathbf{R})$ of Theorem 1.3), the maximal finite invariant set F is *repelling* for the random dynamics. Since we do not know the set F, nor its cardinality (examples of Wehler surfaces with large finite invariant sets were recently constructed in [35]), we make a large detour and prove a uniform hyperbolicity property for the dynamics on *the whole of* X, which is interesting in its own right: this is the *uniform expansion* property that we present in detail in § 1.3. Establishing this property relies on ergodic-theoretic arguments, the first of which is an automatic hyperbolicity property that we describe in the next paragraph.

1.2. **Hyperbolicity of invariant measures.** It is a fundamental (and widely open) problem in conservative dynamics to show the typicality of non-zero Lyapunov exponents on a set of positive Lebesgue measure. In deterministic dynamics, a recent breakthrough is the work of Berger and Turaev [6]. Adding some randomness makes such a hyperbolicity result easier to obtain: see [8] for random perturbation of the standard map, and [2, 47] for random conservative diffeomorphisms on closed real surfaces. The results of Barrientos and Malicet [2] and of Obata and Poletti [47] are perturbative in nature, so they do not give explicit examples. In our context, the rigidity properties of holomorphic diffeomorphisms will enable us to exhibit explicit criteria ensuring such a non-uniform hyperbolicity.

In [20] we have classified invariant measures for non-elementary groups containing parabolic elements. We say that a measure μ on X is **Zariski diffuse** if it gives zero mass to proper Zariski closed subsets. If μ is Γ-invariant and ergodic for some $\Gamma \subset \mathsf{Aut}(X)$, this is equivalent to its support $\mathrm{Supp}(\mu)$ being Zariski dense. Roughly speaking, our classification of invariant measures says that every Zariski diffuse, ergodic, invariant probability measure is given by an analytic 4-form on X or by an analytic 2-form on some invariant, real analytic subset $Y \subset X$ of dimension 2. Here we

[1]Here, we use the fact that on a typical Wehler surface there is no Γ-invariant curve, see [22, Lem. 2.3].

proceed to a finer study of the dynamical properties of these invariant measures. For this, we fix a probability measure ν on $\mathsf{Aut}(X)$ satisfying the moment condition

$$\int \left(\log \|f\|_{C^1(X)} + \log \left\|f^{-1}\right\|_{C^1(X)}\right) d\nu(f) < +\infty; \tag{M}$$

(see Sections 2.1 and 4.2 and Remark 4.2 for discussions of stronger moment conditions), and we view any invariant measure μ as a ν-stationary measure, that is,

$$\int f_\star \mu \, d\nu(f) = \mu. \tag{1.1}$$

Then by (M), the Lyapunov exponents of μ are well defined: for $\nu^{\mathbf{N}}$-almost every sequence (g_i), and μ-almost every $x \in X$, $\frac{1}{n}\log \|D_x(g_{n-1} \circ \cdots \circ g_0)\|$ converges towards the upper Lyapunov exponent $\lambda^+(x) \in \mathbf{R}$; by ergodicity of μ, $\lambda^+(x)$ is almost surely equal to some constant $\lambda^+(\mu)$. Similarly, one defines the lower Lyapunov exponent $\lambda^-(\mu)$, and μ is said to be hyperbolic if $\lambda^+(\mu) > 0 > \lambda^-(\mu)$.

We denote by $\Gamma_\nu \subset \mathsf{Aut}(X)$ the closed subgroup generated by $\mathrm{Supp}(\nu)$ ([2]).

Theorem 1.4. *Let X be a compact complex surface and Γ be a non-elementary subgroup of $\mathsf{Aut}(X)$ containing parabolic elements. Let μ be a Zariski diffuse ergodic Γ-invariant probability measure on X. Let ν be any probability measure on $\mathsf{Aut}(X)$ satisfying $\Gamma_\nu = \Gamma$ and the moment condition* (M).

Then, viewed as a ν-stationary measure, μ is hyperbolic and its fiber entropy $h_\mu(X, \nu)$ is positive.

A variant of this result will also be obtained when Γ_ν contains a Kummer example instead of a parabolic element (see Theorem 7.4).

Example 1.5. When X is a torus or a K3 surface, the canonical bundle K_X is trivial and, up to multiplication by a complex number of modulus 1, there is a unique section Ω_X of K_X that satisfies $\int_X \Omega_X \wedge \overline{\Omega_X} = 1$. The volume form $\mathsf{vol}_X := \Omega_X \wedge \overline{\Omega_X}$ is $\mathsf{Aut}(X)$-invariant. Likewise, every Enriques surface S inherits such an invariant volume form vol_S from its universal cover X (a 2-to-1 cover by a K3 surface). Under the assumptions of Theorem 1.4, vol_X is Γ-ergodic, thus we conclude that *it is hyperbolic*. Other examples are provided by some rational surfaces (see the discussion on Coble surfaces in [18]).

In these situations the 2-form Ω_X also induces a natural measure vol_Y on any totally real surface $Y \subset X$ (see [20, Rmk 2.3]). For instance, if X is projective and defined over $\mathbf{R}$, Γ is contained in $\mathsf{Aut}(X_{\mathbf{R}})$, and Y is a Γ-invariant connected component of $X(\mathbf{R})$, Theorem 1.4 asserts that vol_Y is hyperbolic.

1.3. **Uniform expansion.** Fix a Riemannian metric on X. We say that the measure ν on $\mathsf{Aut}(X)$ is **uniformly expanding** if there exists $c > 0$ and an integer n_0 such that for

[2]Note that $\mathsf{Aut}(X)$ is discrete unless X is a torus, see [21, §3] so in most cases $\Gamma_\nu = \langle \mathrm{Supp}(\nu) \rangle$.

every $x \in X$ and *every* $v \in T_xX\backslash\{0\}$,

$$\int_{\mathsf{Aut}(X)} \log \frac{\|D_x f(v)\|}{\|v\|} d\nu^{(n_0)}(f) \geqslant c; \tag{1.2}$$

here $\nu^{(n)}$ denotes the $n^{\rm th}$ convolution power of ν. This notion is taken from [24, 26, 45, 52] (see also [31, 48] for the linear context) and has a number of strong ergodic and topological consequences on the action of Γ_ν. So far, uniform expansion has been verified only in the context of homogeneous dynamics, or for certain perturbative situations, or with the help of numerical methods. The geometric analysis of stationary measures developed in [21] together with Theorem 1.4 will be used to obtain the following result.

Theorem 1.6. *Let X be a compact complex surface which is not rational. Let ν be a probability measure on* $\mathsf{Aut}(X)$. *Assume that:* (i) *ν satisfies the moment condition* (M) *and* (ii) *the group $\Gamma = \Gamma_\nu$ is non-elementary and contains parabolic elements.*

Then ν is uniformly expanding if and only if the following two conditions hold:

(1) *every finite Γ-orbit is uniformly expanding;*
(2) *there is no Γ-invariant algebraic curve.*

Here, by definition, a finite orbit F of Γ is said to be **uniformly expanding** if condition (1.1) holds for every $x \in F$. This is the repulsion property alluded to at the end of § 1.1.

Checking Condition (2) of Theorem 1.6 is not hard in practice and boils down to cohomological computations (see § 6.3). Therefore, most of the complexity in applying this theorem to practical situations comes from the analysis of finite orbits. The simplest instance is when there are no finite orbits at all:

Corollary 1.7. *Under the assumptions of Theorem 1.6, if there is no proper algebraic Γ_ν-invariant subset, then ν is uniformly expanding.*

By [22, Thm A] the automorphism group of a very general Wehler surface has no proper Zariski closed invariant set. Since uniform expansion is an open property in the C^1 topology, it holds on an open and dense set – in the Euclidean topology – of Wehler examples. In the next few paragraphs we explain why it actually holds on a dense Zariski open set, which is the main point in Theorems 1.1 and 1.3.

First, for a given finite Γ-orbit F, if ν is symmetric and satisfies a slightly stronger moment condition (M_+), Theorem 8.14 provides a checkable necessary and sufficient condition for F to be uniformly expanding: it is equivalent to the tangent action of Γ being proximal and strongly irreducible. It follows that when ν is symmetric and X is not rational *the uniform expansion property depends only on* Γ, *and not on* ν (Corollary 8.15). Anticipating on these results, for a non-elementary subgroup $\Gamma \subset \mathsf{Aut}(X)$ on a non-rational surface X, we can say that **the action is uniformly expanding** if this property holds for some (hence any) symmetric probability measure ν satisfying (M_+) and generating Γ.

In § 9.1 we show that uniform expansion can be checked algorithmically. The starting point is the fact that if X is not a torus and Condition (2) of Theorem 1.6 holds, then by [22, Thm C], there are only finitely many finite orbits. The difficulty is that there is no a priori bound on their number so far, even in the Wehler family. Fortunately, we prove that the number of non-expanding finite orbits can be controlled:

Proposition 1.8. *Let X be a smooth projective surface and Γ be a non-elementary subgroup of $\mathsf{Aut}(X)$ containing parabolic elements and without invariant algebraic curve. Then there is a computable number $N(X, \Gamma)$ such that any finite orbit of length greater than $N(X, \Gamma)$ is uniformly expanding. Moreover, in the Wehler family, the number $N(X, \langle\sigma_1, \sigma_2, \sigma_3\rangle)$ is uniformly bounded.*

See Theorem 9.1 for details on what we mean by computable. To conclude from Condition (2) that uniform expansion holds on a Zariski open subset of $\mathcal{W}$, we use the fact from [22] that on a Zariski open subset $\mathcal{W}_N \subset \mathcal{W}$, all finite orbits have length greater than N (see Theorem 9.3 below). We do not know the value of N for the Wehler family but we do not expect it to be large([3]). In particular the equations defining $\mathcal{W}_{\rm exp}$ and $\mathcal{W}_{\rm exp}(\mathbf{R})$ could in principle be written down explicitly.

1.4. **Ergodicity.** Given an action of a general non-elementary group Γ on a compact complex surface X, one may ask the following two basic questions: does there exist a dense orbit? Is the action ergodic with respect to Lebesgue measure? (The latter makes sense even when there is no invariant volume form.) If Γ contains a parabolic element, by [20] the answer to both questions is 'yes', but without parabolic elements, the answer is unknown. A natural obstruction to the existence of a dense orbit could be the presence of a non-trivial Fatou component for Γ. No example of such a Fatou component is known so far; note that examples do exist for algebraic actions on *affine* surfaces (see [15, §4.1] or [49, Thm E]).

As a matter of fact, the failure of ergodicity is associated to a lack of expansion: indeed a theorem of Dolgopyat and Krikorian [26, §10] asserts that a conservative uniformly expanding action on a (real) surface must be ergodic. It is not difficult to extend their argument to the complex setting (see Theorem 10.2). In Theorem 8.9 we state a general criterion (i.e. without parabolic elements) for uniform expansion which shows that under the conditions (1) and (2) of Theorem 1.6, the failure of uniform expansion is due to the existence of a Γ-invariant measure with exceptional properties (see Theorems 8.9 and A.1). We expect it to be an extremely rare phenomenon. Incidentally, this shows that the question of ergodicity for general non-elementary groups (i.e. without parabolic elements) ultimately boils down to the classification of Γ-invariant measures.

Another consequence of our results, together with [26], is that a generic real Wehler example is **stably ergodic** among C^2 volume preserving actions, that is, if X belongs to the open set $\mathcal{W}_{\rm exp}(\mathbf{R})$ of Theorem 1.3 and σ_1', σ_2', σ_3' are C^2 volume preserving diffeomorphisms sufficiently close to of σ_1, σ_2, σ_3 in the C^1 topology, then $\Gamma' := \langle\sigma_1', \sigma_2', \sigma_3'\rangle$ is ergodic for $\mathsf{vol}_{X(\mathbf{R})}$.

[3]We show that N depends on the cardinality of a set of special "non-twisting" fibers of elliptic fibrations with automorphisms, which seems to be quite scarce, see [28, Rmk 7.7.14].

In an opposite direction, the examples from [20, §9] of $\mathsf{Aut}(X_{\mathbf{R}})$-invariant domains with boundary in $X(\mathbf{R})$ (which admit an invariant curve) provide explicit counterexamples to uniform expansion. For another example, start with the round sphere $x^2 + y^2 + z^2 = 3$ in the affine space, and view it as a singular Wehler surface. The three involutions act by changing the signs of the coordinates and the points $(\epsilon_1, \epsilon_2, \epsilon_3)$ with $\epsilon_i = \pm 1$ form an orbit of size 8. Now, choose a smooth Wehler surface X containing these 8 points and tangent to the sphere at each of them; this imposes 16 linear conditions on the coefficients of the equation defining X, thus such examples exist. For such a surface, the 8 points form a finite, non-expanding orbit of $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ (the action of the stabilizer of $(1, 1, 1)$ on the tangent space is identical to that of the round sphere so it factorizes through a finite group).

1.5. **Organization of the paper.** The first part of this paper (Sections 2 to 5) is devoted to a general study of the notion of uniform expansion on compact (real) manifolds. Much of this material is inspired from other sources; the novelty here is that we strive for optimal moment conditions. We see several reasons for this. First, it is an important trend in random dynamics to look for optimal conditions in the measure rigidity results (an explicit motivation of [31] is to extend the results of Benoist and Quint to measures with finite first moment). Next, when a random dynamical system is generated by a probability measure ν with finite support $S = \mathrm{Supp}(\nu)$ and one considers a finite index subgroup Γ_0 of $\Gamma := \langle S \rangle$, then the support of the measure induced by ν on Γ_0 is infinite (albeit with exponential moments). Also, when looking for random dynamical systems with atypical features, the first examples are usually given by probability measures with only weak moment conditions. Finally, we expect fine moment estimates to be important in the process of trying to improve Theorem 4.5 to include the case of some singular subvarieties $Y \subset X$ (possibly in the spirit of [3, Thm B']).

In Section 2 we give several equivalent definitions of uniform expansion: this is inspired by Liu [45] and Chung [24]. In Section 3 we show that uniform expansion is preserved when restricting to a finite index subgroup or taking a finite extension (Proposition 3.3); this is useful when dealing with invariant sets made of finitely many connected components. Section 4 deals with the construction of **Margulis functions**. In a nutshell, a Margulis function near a finite uniformly expanding invariant set F is a function $u : M \backslash F \to \mathbf{R}_+$ that tends to infinity at F and decreases on average along orbits. The existence of such a function guarantees that empirical measures of random orbits do not accumulate at F. These functions have played an important role in random dynamics since the work of Eskin and Margulis [32]. Here, thanks to the work by Bénard and De Saxcé [3], we construct such Margulis functions under optimal moment conditions (Theorem 4.1); note that the usual average decay property $\int u(f(x))d\nu(f) \leqslant au(x) + b$, $a < 1$, is then replaced by $\int u(f(x))d\nu(f) \leqslant u(x) - \gamma$, $\gamma > 0$. This repulsion property does not hold if F is an invariant submanifold (see Example 4.6). However in the holomorphic context, Margulis functions can be constructed for invariant totally real manifolds of maximal dimension (Theorem 4.5): a typical situation is that of $X(\mathbf{R}) \subset X$ for real projective manifolds. In Section 5, we elaborate on an ergodic-theoretic criterion for uniform expansion borrowed from [24].

In the second part of the paper (Sections 6 to 10), we consider groups of automorphisms of projective surfaces. Theorem 1.4 is established in Section 7. In Section 8 we prove a general version of Theorem 1.6 and study uniform expansion along periodic orbits; this makes essential use of the results of the first part. The focus in § 9.1 is on finding algorithmically checkable conditions for uniform expansion along finite orbits (Theorem 9.1); this leads to a precise description of the locus of uniform expansion in the Wehler family (Theorem 9.3). In § 9.2, we construct uniformly expanding actions by perturbing Kummer examples in the Wehler family; in particular this work for "thin" subgroups of $\mathsf{Aut}(X)$ containing no parabolic element. In Section 10 we study orbit closures and equidistribution by proving general versions of Theorems 1.1 and 1.3; we also explain the adaptation to the complex setting of the ergodicity theorem of Dolgopyat and Krikorian [26].

The paper ends with an appendix on the rigidity of zero entropy measures.

1.6. **Notes and comments.** Theorem 1.4 was included in the first preprint version of [21]. We were informed of ongoing projects by Aaron Brown, Alex Eskin, Simion Filip and Federico Rodriguez Hertz, as well as Megan Roda, on the classification of stationary measures for uniformly expanding actions. This should fit nicely with our work; indeed, parts of this article are written so as to to be easily combined with such a classification (see e.g. Theorem 10.5),

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Part 1. **Uniform expansion for discrete group actions on manifolds**

2. GENERALITIES

In this section, M denotes a compact manifold. The group $\mathsf{Diff}^1(M)$ of C^1 diffeomorphisms of M endowed with the C^1 topology is a polish space (see [50]) and we consider a Borel probability measure ν on it. We fix a Riemannian metric on M. We denote by $\|\cdot\|$ the norm induced by the metric on the tangent bundle TM, and by T^1M the unit tangent bundle.

2.1. **Moment conditions.** If f is a C^1-diffeomorphism of M, we denote by $f_\star$ its action on TM. Note that if $v \in TM$ is a tangent vector based at x (that is, $v \in T_xM$), then $f_\star v = D_xf(v)$ is based at $f(x)$. By definition, $\|f\|_{C^1(X)}$ is the supremum of $v \mapsto \|f_\star v\|$ on T^1M. For $f \in \mathsf{Diff}^1(M)$ we put

$$\mathrm{L}(f) = \log \|f\|_{C^1(X)} + \log \left\|f^{-1}\right\|_{C^1(X)}; \tag{2.1}$$

this quantity is subadditive: $\mathrm{L}(f\circ g) \leqslant \mathrm{L}(f)+\mathrm{L}(g)$. For $p \geqslant 1$ we consider the moment conditions

$$\int \mathrm{L}(f)^p \, d\nu(f) < +\infty, \tag{M_p}$$

$$\exists p > 1, \ (\mathrm{M}_p) \text{ holds}, \tag{M_+}$$

$$\exists t > 0, \ \int \|f\|^t_{C^1(X)} + \left\|f^{-1}\right\|^t_{C^1(X)} d\nu(f) < +\infty. \tag{$\mathrm{M}_{\mathrm{exp}}$}$$

When $p = 1$, (M_p) coïncides with the moment condition (M) from the introduction. For $p > 1$, (M_p) implies (M_+) which implies (M). The subadditivity of L and the convexity inequality $(r^{-1}\sum_{i=1}^r L_i)^p \leqslant r^{-1}\sum_{i=1}^r L_i^p$ imply

$$\int \mathrm{L}(f)^p \, d\nu^{(r)}(f) \leqslant r^p \int \mathrm{L}(f)^p \, d\nu(f) \tag{2.2}$$

for $p \in [1, +\infty[$ and $r \in \mathbf{N}^*$, where $\nu^{(r)}$ denotes the r^{th} convolution power of ν.

2.2. **Notation for random compositions.** Set $\Omega = \mathsf{Diff}^1(M)^{\mathbf{N}}$; its elements are sequences $\omega = (f_n)_{n\geqslant 0}$ of diffeomorphisms. We use the probabilistic notation $\mathbb{E}(\cdot)$ and $\mathbb{P}(\cdot)$ for the expectation and probability with respect to $\nu^{\mathbf{N}}$ on the probability space Ω. We let $(\mathcal{F}_n)_{n\geqslant 1}$ be the increasing sequence of σ-algebras in Ω generated by cylinders of length n, so that an event is $\mathcal{F}_n$-measurable if it depends only on the first n terms $f_0, \ldots, f_{n-1}$ of $\omega = (f_n)_{n\geqslant 0}$. For $\omega = (f_n)_{n\geqslant 0} \in \Omega$ we put $f^0_\omega = \mathrm{id}$ and

$$f^n_\omega = f_{n-1} \circ \cdots \circ f_0 \tag{2.3}$$

for $n \geqslant 1$; in particular $f^1_\omega = f_0$. For x in M and $v \in T_xM\backslash\{0\}$ we set

$$x_{\omega,n} = f^n_\omega(x) \quad \text{and} \quad v_{\omega,n} = \frac{(f^n_\omega)_\star(v)}{\|(f^n_\omega)_\star(v)\|} \in T^1_{x_{\omega,n}}M. \tag{2.4}$$

For any sequence of integers $0 = k_0 < k_1 < \cdots < k_p = n$ the chain rule gives

$$\log \|(f^n_\omega)_\star v\| = \sum_{j=0}^{p-1} \log \left\| \left(f^{k_{j+1}-k_j}_{\sigma^{k_j}\omega} \right)_\star v_{\omega,k_j} \right\|. \tag{2.5}$$

If x is a point of M, we denote by δ_x the Dirac mass at x. If μ is a probability measure on M, $\nu \star \mu$ is the measure defined by $(\nu \star \mu)(B) = \int (f_*\mu)(B) \, d\nu(f)$ for every Borel subset B of M.

2.3. **Equivalent conditions for uniform expansion.** Recall that the probability measure ν on $\mathsf{Diff}^1(M)$ is **uniformly expanding** if there exists a real number $c > 0$ and an integer $n_0 \geqslant 1$ such that

$$\text{for every } v \in T^1X, \quad \int \log \|f_\star(v)\| \, d\nu^{(n_0)}(f) \geqslant c. \tag{2.6}$$

Then, the cocycle relation for $\log \frac{\|f_\star(v)\|}{\|v\|}$ implies that

$$\int \log \|f_\star(v)\| \, d\nu^{(kn_0)}(f) \geqslant kc \tag{2.7}$$

for every $k \geqslant 1$. Thus, ν is uniformly expanding if and only if $\nu^{(n)}$ is uniformly expanding for some (and hence for all) n. It follows that the uniform expansion property does not depend on our choice of a Riemannian metric on M.

Remark 2.1. If ν is uniformly expanding and the submanifold $N \subset M$ is invariant under every diffeomorphism in the support of ν, then ν induces a uniformly expanding measure on $\mathsf{Diff}^1(N)$.

Lemma 2.2. *Let ν be a probability measure on Γ satisfying* (M). *It is uniformly expanding if and only if*

$$\forall v \in T^1M,\ \exists n = n(v) \ \textit{such that} \ \int \log \|f_\star v\| \, d\nu^{(n)}(f) > 0. \tag{2.8}$$

This is Lemma 4.3.1 of [45], but Liu assumes that the support of ν is compact; thus we briefly reproduce his proof, assuming only (M).

Proof. We have to show that (2.8) implies (2.6). Since $|\log \|f_\star v\|| \leqslant \mathrm{L}(f)$ for every $v \in T^1X$, the dominated convergence theorem implies that, for every n,

$$v \mapsto \int \log \|f_\star(v)\| \, d\nu^{(n)}(f) \tag{2.9}$$

is continuous. Thus by compactness, there exists a finite open cover $V_1, \ldots, V_p$ of T^1M, positive real numbers c_i, and integers n_i such that

$$\int \log \|f_\star(v)\| \, d\nu^{(n_i)}(f) \geqslant c_i \tag{2.10}$$

for every $v \in V_i$. Set $c_0 = \min(c_i)$ and $n_0 = \max(n_i)$. For $v \in T^1X$ and $\omega \in \Omega$, define the stopping time $\tau_1(v, \omega)$ to be the first integer $n \geqslant 1$ such that $\int \log \|f_\star v\| \, d\nu^{(n)}(f) \geqslant c_0$, and then define inductively

$$\tau_{k+1}(v, \omega) = \tau_k(v, \omega) + \tau_1(v_{\omega,k}, \sigma^k(\omega)). \tag{2.11}$$

By construction, τ_1 depends on v (hence on x) but not on ω, while τ_k depends on both v and ω when $k \geqslant 2$; in addition $\tau_k(v, \omega) \leqslant kn_0$ for all $k \geqslant 1$. For $n \geqslant 1$, define $K_n(v, \omega)$, or $K(n)$ for short, by $K_n(v, \omega) = \max\{k;\ \tau_k \leqslant n\}$. Then $K(n) \geqslant n/n_0$ and

$n - K(n) \leqslant n_0 - 1$. With the convention $\tau_0 = 0$, the chain rule (2.5) gives

$$\begin{aligned}\mathbb{E}\left(\log \|(f_\omega^n)_\star v\|\right) &= \mathbb{E}\left(\sum_{j=0}^{K(n)-1} \log \left\| \left(f_{\sigma^{\tau_j}\omega}^{\tau_{j+1}-\tau_j}\right)_\star v_{\omega,\tau_j} \right\|\right) + \mathbb{E}\left(\left(f_{\sigma^{\tau_{K(n)}}\omega}^{n-\tau_{K(n)}}\right)_\star v_{\omega,\tau_{K(n)}}\right) \\ &\geqslant \frac{n}{n_0} c_0 - \max_{1\leqslant q\leqslant n_0} \mathbb{E}\left(L(f_\omega^q)\right) \qquad (2.12) \\ &\geqslant \frac{n}{n_0} c_0 - n_0 \int L(f) d\nu(f). \qquad (2.13)\end{aligned}$$

Thus, for $n \geqslant \frac{n_0}{2} + \frac{n_0^2}{c_0} \int L(f)d\nu(f)$, we have $\mathbb{E}\left(\log \|(f_\omega^n)_\star v\|\right) \geqslant \frac{c_0}{2} > 0$ independently of v, as was to be shown. □

Lemma 2.3. *Under the moment condition* (M$_+$)*, ν is uniformly expanding if and only if*

$$\forall v \in T^1X, \exists c > 0 \ \textit{such that} \quad \mathbb{P}\left(\frac{1}{n}\log \|(f_\omega^n)_\star v\| \geqslant c\right) \underset{n\to\infty}{\longrightarrow} 1. \qquad (2.14)$$

Under the moment condition (M)*, Property* (2.14) *implies uniform expansion.*

Proof. Let us first show that (2.14) implies (2.8) under the assumption (M). Fix $v \in T^1X$, set $\Omega_n = \left\{\omega \in \Omega \; ; \; \frac{1}{n}\log\|(f_\omega^n)_\star v\| \geqslant c\right\}$, and split $\mathbb{E}\left(\frac{1}{n}\log\|(f_\omega^n)_\star v\|\right)$ into the sum of an integral over Ω_n and an integral over $\Omega_n^\complement$. The first one is larger than $c\mathbb{P}(\Omega_n)$, and $\mathbb{P}(\Omega_n)$ tends to 1 as n goes to $+\infty$. The second one satisfies

$$\left|\mathbb{E}\left(\frac{1}{n}\log\|(f_\omega^n)_\star v\| \mathbf{1}_{\Omega_n^\complement}\right)\right| \leqslant \mathbb{E}\left(\frac{1}{n}\mathrm{L}(f_\omega^n)\mathbf{1}_{\Omega_n^\complement}\right). \qquad (2.15)$$

The moment condition and Kingman's subadditive ergodic theorem show that $\frac{1}{n}\mathrm{L}(f_\omega^n)$ is uniformly integrable and converges almost surely to some finite constant; since $\mathbb{P}(\Omega_n^\complement)$ converges to 0, we conclude that $\mathbb{E}\left(\frac{1}{n}\log\|(f_\omega^n)_\star v\|\right) \geqslant c/2$ for large n.

For the converse implication we use a martingale convergence argument, as in [45, Lem. 4.3.5] and [24, Prop. 2.2]([4]). Choose $p > 1$ such that (M$_p$) holds. For convenience, let us first replace ν by $\nu^{(n_0)}$, where n_0 is given by the expansion property (2.6). Define (for some fixed unit vector v)

$$X_k = \log\left\|(f_{\sigma^k\omega}^1)_\star v_{\omega,k}\right\| - \int \log\|f_\star(v_{\omega,k})\| d\nu(f). \qquad (2.16)$$

These increments X_k are uniformly bounded in L^p because

$$\mathbb{E}\left(\left|\log\left\|(f_{\sigma^k\omega}^1)_\star v_{\omega,k}\right\|\right|^p\right)^{1/p} \leqslant \mathbb{E}\left(L(f_{\sigma^k\omega}^1)^p\right)^{1/p} = \left(\int L(f)^p d\nu(f)\right)^{1/p} \qquad (2.17)$$

[4]Chung only assumes the moment condition (M) however it seems to us that a stronger assumption is needed for the control of the martingale differences.

and the second term in (2.16) is pointwise bounded by

$$\left|\int \log \|f_\star v_{\omega,k}\| d\nu(f)\right| \leqslant \int L(f) d\nu(f) \leqslant \left(\int L(f)^p d\nu(f)\right)^{1/p}. \tag{2.18}$$

Thus, the sums $S_n = \sum_{k=0}^{n-1} X_k$ are all in L^p. Since $\mathbb{E}(X_n \mid \mathcal{F}_n) = 0$ and S_n is $\mathcal{F}_{n-1}$-measurable, (S_n) is a martingale relative to the filtration $(\mathcal{F}_{n-1})$. It follows from Theorem 2.22 in [38, §2.7] that $\frac{1}{n}S_n$ converges to 0 in probability and in L^p. Now, the chain rule gives

$$\frac{1}{n}S_n(\omega) = \frac{1}{n}\log \|(f_\omega^n)_\star v\| - \frac{1}{n}\int \log \|f_\star v\| d\nu^{(n)}(f), \tag{2.19}$$

and (2.7) asserts that $\int \log \|f_\star v\| d\nu^{(n)}(f) \geqslant cn$, so we conclude that for any $c' < c$

$$\mathbb{P}\left(\frac{1}{n}\log \|(f_\omega^n)_\star v\| \geqslant c'\right) \underset{n\to\infty}{\longrightarrow} 1, \tag{2.20}$$

as desired. Recall however that we are working with $\nu^{(n_0)}$: coming back to ν this means that (2.20) holds along the subsequence (nn_0). We then write $n = kn_0 + r$, with $0 \leqslant r \leqslant n_0 - 1$, so that

$$(f_\omega^n)_\star v = (f^r_{\sigma^{kn_0}\omega})_\star (f_\omega^{kn_0})_\star v \tag{2.21}$$

and what we have to show is that applying $f^r_{\sigma^{kn_0}\omega}$ does not affect the linear growth of $\log \left\|(f_\omega^{kn_0})_\star v\right\|$. But the inequality (2.2), applied with $p = 1$, gives

$$\mathbb{P}\left(\exists 0 \leqslant r \leqslant n_0 - 1,\ \left|\log \left\|(f^r_{\sigma^{kn_0}})_\star\right\|\right| \geqslant \varepsilon k\right) \leqslant \sum_{r=0}^{n_0-1} \nu^{(r)}\left(L(f) \geqslant \varepsilon k\right) \leqslant \frac{Cn_0^2}{\varepsilon k}, \tag{2.22}$$

and we are done. □

Remark 2.4. In the first part of the proof, the implication (2.14)⇒(2.8) is true for a given v, while the converse implication requires uniform expansion on the whole of X.

Remark 2.5. This proof shows that if ν satisfies (M_2), then the convergence in probability in (2.14) can be replaced by an almost sure convergence. (Indeed by Theorem 3 of [34, p. 243], $\frac{1}{n}S_n$ converges almost surely to 0 when the X_k are uniformly L^2.)

Remark 2.6. So far, we have not really used that we are dealing with diffeomorphisms: the results from this section hold for a semigroup action, by replacing $\mathrm{L}(f)$ by $\|f\|_{C^1(X)}$.

3. Inducing on a finite index subgroup

3.1. **Hitting times and hitting measures (see** [5, Chap. 5]**).** Let ν be a Borel probability measure on $\mathsf{Diff}^1(M)$ and let G be the closed subsemigroup of $\mathsf{Diff}^1(M)$ generated by ν. Let $H \subset G$ be a closed finite index subsemigroup; this means that there is a continuous and transitive action $G \times F \to F$ on some finite set F such that H is the stabilizer of some element $x_0 \in F$; the index of H is $[G : H] = |F|$ and F is the quotient space. For instance, H can be the stabilizer of a point x in a finite G-orbit.

The hitting time T_H of H for the random walk induced by ν (starting from the neutral element) is

$$T_H(\omega) = \min\{n \geqslant 1,\ f_\omega^n \in H\}. \tag{3.1}$$

Lemmas 5.4 and 5.5 in [5] show that T_H is almost surely finite, admits an exponential moment, and satisfies $\mathbb{E}(T_H) = [G : H]$. By definition the **hitting measure** (or induced measure) ν_H is the probability measure on H describing the distribution of $f_\omega^{T_H(\omega)}$.

Define the k**-th hitting time** $T_{H,k}$ of H by $T_{H,1} = T_H$ and the induction

$$T_{H,k+1}(\omega) = \min\{n \geqslant T_{H,k} + 1\ ;\ f_\omega^n \in H\}. \tag{3.2}$$

The convolution $\nu_H^{(k)}$ describes the distribution of $f_\omega^{T_{H,k}(\omega)}$. If H is a finite index semi-group and $g \in H$, hg belongs to H if and only if h belongs to H. Thus, $T_{H,k+1}(\omega) - T_{H,k}(\omega) = T_{H,1}(\sigma^{T_{H,k}(\omega)}(\omega))$ and the Markov property implies that the random variables $(T_{H,k+1} - T_{H,k})$ are independent and identically distributed: each of them is distributed as T_H. Since their expectation equals $[G : H]$, the law of large numbers gives

$$\lim_{k\to+\infty} \frac{1}{k} T_{H,k}(\omega) = [G : H] \tag{3.3}$$

$\nu^{\mathbf{N}}$-almost surely.

Theorem 3.1. *The hitting measure on a finite index subgroup satisfies the following properties*

(1) if ν_H satisfies (M_p) *for some $p \geqslant 1$, then so does ν;*
(2) if ν satisfies (M_p)*, then ν_H satisfies $(M_{p'})$ for any $1 < p' < p$;*
(3) ν satisfies (M)*, or* (M_+)*, or* $(\mathrm{M}_{\exp})$ *if and only if ν_H does.*

Moreover, ν_H generates H as a semigroup, which means that H is the smallest closed subsemigroup of G containing the support of ν_H.

This result still holds if we substitute any subbaditive function to $\log \|f\|_{C^1(X)}$ in the definition of L (see Equation 2.1), with exactly the same proof.

Proof. Consider the finite quotient F of G by H and denote the action of G on F by left translations by ($u \mapsto au$, $a \in G$); by definition H is the stabilizer of some $x_0 \in F$. Set $K = |F| = [G : H]$.

For each $u \in F$, choose a sequence of measurable subsets $A_1(u)$, $A_2(u)$, …, $A_k(u)$ in G, with $k = k(u) \leqslant K$ such that $\nu(A_i(u)) > 0$ for each i and, for all sequences $a_i \in A_i(u)$, $(a_k \cdots a_1)u = x_0$ while $(a_j \cdots a_1)u \neq x_0$ if $j < k$. Since F is finite, there is a real number $\varepsilon > 0$ such that $\nu(A_1(u)) \cdots \nu(A_{k(u)}(u)) \geqslant \varepsilon$ for all u. Shrinking the $A_i(u)$ if necessary, we may assume that $\mathrm{L}(g) \leqslant C$ for some $C > 0$ and all g in $\bigcup_{u,i} A_i(u)$.

We split the integral of $\mathrm{L}(f)^p$ as a finite sum $\int \mathrm{L}(f)^p d\nu(f) = \sum_{u\in F} \int_{\{fx_0=u\}} \mathrm{L}(f)^p d\nu(f)$. If

$$(a_{k(u)}, \ldots, a_1) \in A_{k(u)}(u) \times \cdots \times A_1(u), \tag{3.4}$$

then $\mathrm{L}(f) \leqslant \mathrm{L}(a_{k(u)} \cdots a_1 f) + KC$ because L is subadditive; thus,

$$\int_{\{fx_0=u\}} \mathrm{L}(f)^p d\nu(f) \leqslant \int_{\{fx_0=u\}} (\mathrm{L}(a_{k(u)} \cdots a_1 f) + KC)^p d\nu(f). \tag{3.5}$$

By construction, the product $a_{k(u)} \cdots a_1 f$ is a first return in H. Thus, integrating over the $A_i(u)$, the distribution of $a_{k(u)} \cdots a_1 f$ contributes positively to ν_H, and we get

$$\varepsilon \int_{\{fx_0=u\}} \mathrm{L}(f)^p d\nu(f) \leqslant \int_H (\mathrm{L}(g) + KC)^p d\nu_H(g). \tag{3.6}$$

Assertion (1) follows from this estimate.

For assertion (2), we must bound $\int \mathrm{L}(f)^{p'} d\nu_H(f) = \mathbb{E}(\mathrm{L}(f_\omega^{T_H(\omega)})^{p'})$. By subadditivity of L and convexity of $s \mapsto s^p$, we have

$$\mathrm{L}\left(f_\omega^{T_H(\omega)}\right)^p \leqslant T_H(\omega)^{p-1} \sum_{i=0}^{T_H(\omega)-1} \mathrm{L}(f_i)^p. \tag{3.7}$$

Raising this inequality to power p'/p gives

$$\mathrm{L}\left(f_\omega^{T_H(\omega)}\right)^{p'} \leqslant T_H(\omega)^{p'(1-1/p)} \left(\sum_{i=0}^{T_H(\omega)-1} \mathrm{L}(f_i)^p\right)^{p'/p}. \tag{3.8}$$

On the other hand, Lemma 5.4 of [5] says that $\mathbb{E}(\sum_{i=1}^{T_H(\omega)} \varphi \circ \sigma^i) = \mathbb{E}(T_H)\mathbb{E}(\varphi)$ for any integrable function φ. This shows that $\sum_{i=0}^{T_H(\omega)-1} \mathrm{L}(f_i)^p$ is integrable, and in particular its p'/p power is in $\mathrm{L}^{p/p'}(\Omega; \nu^{\mathbf{N}})$. Since the hitting time T_H admits moments of all orders, we can apply the Hölder inequality with parameters $r = p/p' > 1$ and q such that $1/q + 1/r = 1$: it shows that $\mathrm{L}\left(f_\omega^{T_H(\omega)}\right)^{p'}$ is integrable, as desired.

Assertion (3) follows from Assertions (1) and (2) and Corollary 5.6 in [5].

For the last assertion, fix an element h of H and an open neighborhood U of h in H. Since H is of finite index in G, it is open and closed, so U is also a neighborhood of h in G. By assumption the support of ν generates a dense sub-semigroup of G, so the random walk induced by ν starting at the neutral element visits U, thus ν_H generates H as a semigroup. □

3.2. Uniform expansion of the induced measure.

Proposition 3.2. *Let ν be a probability measure on* $\mathsf{Diff}^1(M)$ *satisfying* (M). *Assume that ν is uniformly expanding and let n_0 be as in* (2.6). *Then, the measure induced by $\nu^{(n_0)}$ on H is uniformly expanding.*

In fact, Proposition 3.3 below shows that, under condition (M_+), ν is uniformly expanding if and only if ν_H is. The proof of Proposition 3.2 is based on a simple martingale argument, while Proposition 3.3 relies on the criterion of Lemma 2.3.

Proof. We use ideas from [45, §4.3] and [24, Prop. 2.2]. To ease notation we rename $\nu^{(n_0)}$ into ν so that (2.6) holds with $n_0 = 1$ and some $c > 0$; as above, we denote by ν_H the measure induced by ν (i.e. by $\nu^{(n_0)}$) on H. Fix $v \in T^1X$, and define a sequence of random variables $(Y_k)_{k\geqslant 0}$ by

$$Y_k(\omega) = \log \left\|(f^1_{\sigma^k\omega})_\star v_{\omega,k}\right\| - c. \tag{3.9}$$

Then for all $k \geqslant 1$, $\mathbb{E}(Y_k \mid \mathcal{F}_k) \geqslant 0$, so that the sequence $(S_n)_{n\geqslant 1}$ defined by $S_n = \sum_{k=0}^{n-1} Y_k$ is a submartingale relative to the filtration $(\mathcal{F}_n)$: $\mathbb{E}(S_{n+1} \mid \mathcal{F}_n) \geqslant S_n$. The moment condition (M) implies that $\mathbb{E}(|S_{n+1} - S_n| \mid \mathcal{F}_n) = \mathbb{E}(|Y_n| \mid \mathcal{F}_n)$ is uniformly bounded. Since the hitting time T_H is integrable, we can apply the optional stopping theorem [30, Thm. 4.7.5], which implies that $\mathbb{E}(S_{T_H}) \geqslant \mathbb{E}(S_1) \geqslant 0$. Unwinding the definitions and applying the chain rule, we see that

$$\mathbb{E}(S_{T_H}) = \int \log \|f_\star v\| \, d\nu_H(f) - c[G : H], \tag{3.10}$$

where we use $\mathbb{E}(T_H) = [G : H]$. Therefore $\int \log \|f_\star v\| \, d\nu_H(f) \geqslant c[G : H] > 0$, and ν_H is uniformly expanding. □

Proposition 3.3. *Let ν be a probability measure on $\mathsf{Diff}^1(M)$ satisfying* (M$_+$). *Let ν_H be the measure induced on a closed finite index subsemigroup. Then ν is uniformly expanding if and only if ν_H is uniformly expanding.*

Proof. Let us show that if ν_H is uniformly expanding then ν is uniformly expanding. The converse implication is similar and is left to the reader (in this direction, Proposition 3.2 will actually be sufficient for our purposes). Fix $v \in T^1M$. In view of Lemma 2.3, we have to show that for some $c > 0$,

$$\mathbb{P}\left(\frac{1}{n}\log \|(f^n_\omega)_\star v\| \geqslant c\right) \xrightarrow[n\to\infty]{} 1. \tag{3.11}$$

Consider the sequence of hitting times $(T_{H,k})$ defined in § 3.1 and denote it by (T_k) for simplicity (hence $T_1 = T_H$). By Theorem 3.1, ν_H satisfies (M$_+$), so we can apply Lemma 2.3 to get a real number $c > 0$ such that

$$\mathbb{P}\left(\frac{1}{k}\log \left\|\left(f^{T_k(\omega)}\right)_\star v\right\| \geqslant c\right) \xrightarrow[k\to\infty]{} 1. \tag{3.12}$$

Let $\gamma = [G : H]$ and fix a positive real number $\varepsilon < \frac{c}{\gamma}$. Let also $\varepsilon_1 \ll \varepsilon$ which will be specified later. For $K \geqslant 1$, set

$$\Omega_1(K) = \left\{\omega \ ; \ \forall k \geqslant K, \quad \left|\frac{T_k(\omega)}{k} - \gamma\right| < \varepsilon_1\right\}. \tag{3.13}$$

If $K > \gamma/(2\varepsilon_1)$ and $\omega \in \Omega_1(K)$, then for $n \geqslant K\gamma + 1$ we get

$$T_{\lfloor n/(\gamma+\varepsilon_1)\rfloor}(\omega) - \frac{4\varepsilon_1}{\gamma}n \leqslant n \leqslant T_{\lfloor n/(\gamma+\varepsilon_1)\rfloor}(\omega) + \frac{4\varepsilon_1}{\gamma}n. \tag{3.14}$$

Now, define $\Omega_2(n)$ to be the set of sequences ω such that the inequality involved in (3.12) is satisfied at time $T_{\lfloor n/(\gamma+\varepsilon_1)\rfloor}(\omega)$; in other words, $\omega \in \Omega_2(n)$ if and only if

$$\log \left\| \left(f^{T_{\lfloor n/(\gamma+\varepsilon_1)\rfloor}(\omega)}\right)_* v \right\| \geqslant c\lfloor n/(\gamma+\varepsilon_1)\rfloor. \tag{3.15}$$

Then $\mathbb{P}(\Omega_1(\lfloor\sqrt{n}\rfloor)\cap\Omega_2(n))$ converges to 1 as n goes to $+\infty$ and as soon as $n \geqslant \gamma\sqrt{n}+1$, any $\omega \in \Omega_1(\lfloor\sqrt{n}\rfloor) \cap \Omega_2(n)$ satisfies (3.14) and (3.15).

Now consider the set $\Omega_3(n) \subset \Omega_1(\lfloor\sqrt{n}\rfloor)\cap\Omega_2(n)$ made of those ω such that $\mathrm{L}\left(f^{n-T_k}_{\sigma^{T_k}\omega}\right) < \varepsilon n$ for $k = \lfloor n/(\gamma+\varepsilon_1)\rfloor$. Then

$$\mathbb{P}(\Omega_3(n)^{\complement}) \leqslant \mathbb{P}\left(\mathrm{L}\left(f^{n-T_k}_{\sigma^{T_k}\omega}\right) \geqslant \varepsilon n\right) \leqslant \mathbb{P}\left(\max_{0\leqslant q\leqslant \frac{4\varepsilon_1}{\gamma}n} \sum_{i=0}^{q-1} L_i \geqslant \varepsilon n\right) \tag{3.16}$$

where $(L_i)_{i\geqslant 0}$ is a sequence of independent random variables, each of which being distributed as $\mathrm{L}(g)$ for $d\nu(g)$. Since the L_i are non-negative, $\mathbb{P}(\Omega_3(n)^{\complement}) \leqslant \mathbb{P}\left(\sum_{i=0}^{4\varepsilon_1 n/\gamma} L_i \geqslant \varepsilon n\right)$. If ε_1 is chosen such that $\frac{4\varepsilon_1}{\gamma}\mathbb{E}(L_1) < \varepsilon$, the law of large numbers entails that

$$\mathbb{P}\left(\sum_{i=0}^{4\varepsilon_1 n/\gamma} L_i \geqslant \varepsilon n\right) \underset{n\to\infty}{\longrightarrow} 0, \tag{3.17}$$

thus $\mathbb{P}\left(\Omega_3(n)\right)$ tends to 1 as $n \to \infty$.

Then, for $\omega \in \Omega_3(n)$, the estimates (3.14), (3.15) and $\mathrm{L}\left(f^{n-T_k}_{\sigma^{T_k}\omega}\right) < \varepsilon n$ imply that $\frac{1}{n}\log\|(f^n_\omega)_* v\| \geqslant \frac{c}{\gamma} - \varepsilon$ and the conclusion follows. □

4. Margulis functions

In this section we develop some tools for the proof of the equidistribution Theorem 1.3. Under appropriate assumptions, we show that the measures $\nu^n * \delta_x$ and $\frac{1}{n}\sum_{k=1}^n \delta_{f^k_\omega(x)}$ do not cluster at a Γ-periodic orbit, except when $\Gamma(x)$ is itself finite. The basic tool is the construction of a proper function, defined on the complement of such a periodic orbit, which "essentially decreases" along random trajectories. After [32] it is often referred to as a **Margulis function**, even if this strategy has a long history in the Markov chain literature (see [46]). Our presentation is greatly influenced by [4] and [3].

4.1. A general recurrence criterion.

For concreteness, instead of general Markov chains, we consider the setting of group actions.

Theorem 4.1 (Bénard-De Saxcé [3]). *Let U be a locally compact topological space. Let Γ be a group of homeomorphisms of U, and ν be a probability measure on Γ. Assume that there exists a function $u : U \to \mathbf{R}_+$ satisfying the assumptions:*

$$\exists A > 0,\ \exists\gamma > 0,\ \forall x \in U,\ u(x) \geqslant A \Rightarrow \int u(f(x))d\nu(f) \leqslant u(x) - \gamma \tag{4.1}$$

$$\exists B > 0, \exists\eta > 0, \forall x \in U,\ \int |u(f(x)) - u(x)|^{1+\eta}\, d\nu(f) \leqslant B. \tag{4.2}$$

Then for every $\varepsilon > 0$ there exists $R > 0$ such that for all x in U,

(1) there exists $n_x \geqslant 0$, such that $(\nu^n * \delta_x)(\{u \geqslant R\}) \leqslant \varepsilon$ for all $n \geqslant n_x$;

(2) for $\nu^{\mathbf{N}}$-almost every ω,

$$\limsup_{n\to\infty} \frac{1}{n} \# \left\{k \in \{1, \ldots, n\} \ ; \ u\left(f_\omega^k(x)\right) \geqslant R\right\} \leqslant \varepsilon.$$

Furthermore the integer n_x in (1) depends only on $u(x)$.

In the most interesting cases, u will be a proper function on U. Then, Equation (4.1) expresses that, on average, the random dynamics does not send points too far off at infinity, and Equation (4.2) can be understood as a moment condition in the u-variable; then, the Conclusions (1) and (2) correspond to "non-escape of mass" and "quantitative recurrence" properties.

For the proof, see Proposition 1.2 in [3], and the comments following it. More precisely, we refer to [3, Prop. 2.5] for the conclusion (1), including the uniformity statement on n_x, and to [3, Prop. 2.7] for (2).

The original decay property for the Margulis function u in [32, 4] is

$$\exists 0 < a < 1,\ \exists b > 0,\ \forall x \in U,\ \int u(f(x)) d\nu(f) \leqslant a u(x) + b \tag{4.3}$$

instead of (4.1). One easily checks that if u satisfies (4.1) and the following strong integrability property

$$\exists B > 0, \forall x \in U,\ \int \exp\left(|u(f(x)) - u(x)|\right) d\nu(f) \leqslant B, \tag{4.4}$$

then $e^{\delta u}$ satisfies (4.3) for small $\delta > 0$. Under this assumption, Theorem 4.1 was established in [4].

4.2. **Finite orbits of C^2 actions.** Let ν be a probability measure on the group of C^2 diffeomorphisms of a compact Riemannian manifold M of dimension d. As in § 2.1 we consider the moment conditions

$$\int \left(\log \|f\|_{C^2} + \log \left\|f^{-1}\right\|_{C^2}\right)^p d\nu(f) < +\infty, \tag{M$_{2,p}$}$$

$$\exists p > 1,\ (\mathrm{M}_{2,p}) \text{ holds}, \tag{M$_{2,+}$}$$

Remark 4.2. For a holomorphic action on a compact complex manifold X, these conditions are equivalent to their respective C^1 analogues (M$_p$) and (M$_+$), because a uniform control on the first derivatives provides a uniform control on the higher derivatives as well. Here is an outline of the argument. Cover X by finitely many charts Ω_i. Then there exists $c > 0$ such that for every $x \in X$ and every $f \in \mathsf{Aut}(X)$, the balls $B(x, r)$ and $f(B(x, r))$ are contained in a single chart, as soon as $r \leqslant c\|f\|_{C^1}^{-1}$. Then the Cauchy estimates imply that $\left\|f|_{B(x,r/2)}\right\|_{C^2} \leqslant (C/r)\left\|f|_{B(x,r)}\right\|_{C^1} \leqslant C\|f\|_{C^1}^2$ and the result follows.

Before stating our next result, recall that the notion of uniform expansion along a finite orbit was defined in Section 1.3.

Theorem 4.3. *Let Γ be a group of C^2 diffeomorphisms of a compact Riemannian manifold M, and ν be a measure on Γ satisfying the moment condition ($M_{2,+}$). Let F be a finite orbit of Γ such that ν is uniformly expanding on F. Then for every $x \in M\backslash F$, for every $\varepsilon > 0$ there exists a compact set $K \Subset M\backslash F$ such that:*

(1) $(\nu^n * \delta_x)(K) \geqslant 1 - \varepsilon$ for $n \geqslant n_x$, and

(2) for $\nu^{\mathbf{N}}$-almost every ω,

$$\limsup_{n\to\infty} \frac{1}{n} \# \left\{ k \in \{1, \dots, n\} \,,\; f_\omega^k(x) \in K \right\} \geqslant 1 - \varepsilon.$$

Furthermore the integer n_x in (1) is locally uniform in $M\backslash F$.

This result seems to be new: it appears under stronger (exponential) moment assumptions in e.g. [45, 24]. Note that such a result is not expected to hold under the first moment condition ($\mathrm{M}_{2,1}$), as explained in Examples 1 and 2 of Section 2 in [3].

Proof. First, the proof of Proposition 3.3 in [3] shows that if the conclusions (1) and (2) hold for $\nu^{(n_0)}$, then they hold for ν. So we can replace ν by $\nu^{(n_0)}$ and hence assume that the uniform expansion property (1.1) holds (on F) for $n_0 = 1$.

Let $d(\cdot, \cdot)$ be the Riemannian distance on M. According to Theorem 4.1, we only need to show that $u \colon x \mapsto -\log d(x, F)$ is a proper function $M\backslash F \to \mathbf{R}_+$ satisfying Properties (4.1) and (4.2).

Preliminaries.– We set $N(f) = \|f\|_{C^2} + \|f^{-1}\|_{C^2}$ and note that $N(f) \geqslant \mathrm{Lip}(f) + \mathrm{Lip}(f^{-1})$ for every $f \in \Gamma$. In particular, for every $x \in X$

$$\frac{1}{N(f)} \leqslant \frac{d(f(x), F)}{d(x, F)} \leqslant N(f). \tag{4.5}$$

For $R > 0$, set $\Gamma(R) = \{f \in \Gamma \;;\; N(f) \leqslant R\}$. We choose $\eta > 0$ such that the moment condition ($\mathrm{M}_{2,p}$) is satisfied with $p = 1 + \eta$. Then,

$$I_\eta := \int_\Gamma (\log(N(f)))^{1+\eta} \, d\nu(f) \tag{4.6}$$

is a finite positive number. In what follows, we choose $R > 1$ such that

$$\frac{2I_\eta}{(\log(R))^\eta} < \frac{c}{4} \tag{4.7}$$

where c is the expansion factor in Equation (1.1) (along the finite orbit F).

Take $s > 0$ such that

- s is smaller than the injectivity radius of M at y, for every $y \in F$;
- the balls $B(y; s)$, for y in F, are pairwise disjoint;
- $C_0 R^2 s < c/4$, where c is the expansion factor as above, and C_0 is the constant appearing below in the Taylor expansion (Equation (4.9)).

Then, define V and V' by

$$V = \bigcup_{y \in F} B(y; s), \quad V' = \bigcup_{y \in F} B(y; s/R). \tag{4.8}$$

By (4.5) we have $f(V') \subset V$ for every $f \in \Gamma(R)$.

If x belongs to V, we denote by $\pi(x)$ the unique point of F at distance $\leqslant s$ from x, and we denote by w_x the unique vector in $T_{\pi(x)}M$ such that $\exp_{\pi(x)}(w_x) = x$ and $\|w_x\| = d(x, \pi(x))$.

First estimate.– For f in $\Gamma(R)$ and $x \in V'$, Taylor's second order formula yields

$$|d(f(x), f(\pi(x))) - \|f_\star(w_x)\|| \leqslant C_0 N(f) d(x, \pi(x))^2, \tag{4.9}$$

for some uniform constant C_0, that does not depend on f. This gives

$$\left| \frac{d(f(x), F)}{d(x, F)} - \frac{\|f_\star(w_x)\|}{\|w_x\|} \right| \leqslant C_0 N(f) d(x, F). \tag{4.10}$$

Now, using the Lipschitz estimate (4.5) and the fact that $|\log(a) - \log(b)| \leqslant N\,|a - b|$ when $a, b \in [N^{-1}, \infty[$, we obtain

$$\left| \log\left(\frac{d(f(x), F)}{d(x, F)}\right) - \log\left(\frac{\|f_\star(w_x)\|}{\|w_x\|}\right) \right| \leqslant C_0 N(f)^2 d(x, F). \tag{4.11}$$

By the definition of $\Gamma(R)$ and the requirements on s, we get

$$\int_{f \in \Gamma(R)} \left| \log\left(\frac{d(f(x), F)}{d(x, F)}\right) - \log\left(\frac{\|f_\star(w_x)\|}{\|w_x\|}\right) \right| d\nu(f) \leqslant C_0 R^2 d(x, F) \leqslant \frac{c}{4}, \tag{4.12}$$

because $d(x, F) \leqslant s$.

Second estimate.– Now, for any f in Γ we also have

$$\left| \log\left(\frac{d(f(x), F)}{d(x, F)}\right) - \log\left(\frac{\|f_\star(w_x)\|}{\|w_x\|}\right) \right| \leqslant 2 \log(N(f)) \tag{4.13}$$

hence Markov's inequality and our choice of R give

$$\int_{f \in \Gamma(R)^\complement} \left| \log\left(\frac{d(f(x), F)}{d(x, F)}\right) - \log\left(\frac{\|f_\star(w_x)\|}{\|w_x\|}\right) \right| d\nu(f) \leqslant \frac{2}{\log(R)^\eta} I_\eta \leqslant \frac{c}{4}. \tag{4.14}$$

Conclusion.– Summing the integrals over f in $\Gamma(R)$ and $\Gamma(R)^\complement$, we obtain

$$\int_{f \in \Gamma} \left| \log\left(\frac{d(f(x), F)}{d(x, F)}\right) - \log\left(\frac{\|f_\star(w_x)\|}{\|w_x\|}\right) \right| d\nu(f) \leqslant \frac{c}{2}. \tag{4.15}$$

Since w_x is a vector tangent to M at $\pi(x) \in F$, the uniform expansion along F yields

$$\int \log\left(\frac{\|f_\star(w_x)\|}{\|w_x\|}\right) d\nu(f) \geqslant c \tag{4.16}$$

and then (4.15) implies that

$$\int -\log d(f(x), F) d\nu(f) \leqslant -\log(d(x, F)) - c/2. \tag{4.17}$$

In other words, $u\colon x \mapsto -\log(d(x,F))$ satisfies Property (4.1) (with $A = -\log(s)$). Property (4.2) is obtained from (4.5) and the moment condition. Thus, as announced above, u satisfies the assumptions of Theorem 4.1, and we are done. □

The local uniformity of n_x in Theorem 4.3 has the following interesting consequence.

Proposition 4.4. *Under the assumptions of Theorem 4.3, any stationary Radon measure on $M \backslash F$ has finite mass.*

Proof. Let μ be such a stationary measure. Fix $\varepsilon > 0$, say $\varepsilon = 1/2$ and let K be as in Theorem 4.3. The stationarity of μ implies that for every $n \geqslant 0$,

$$(\nu^{(n)} \times \mu)\left(\{(g,x),\ gx \in K\}\right) = \mu(K), \tag{4.18}$$

hence for every borel set $B \subset M\backslash F$,

$$\int_B \nu^{(n)}\left(\{g,\ gx \in K\}\right) d\mu(x) \leqslant \int_X \nu^{(n)}\left(\{g,\ gx \in K\}\right) d\mu(x) = \mu(K). \tag{4.19}$$

Now if B is an arbitrary compact subset of $M\backslash F$, the uniformity statement in Theorem 4.3 implies that there exists $n = n_B$ such that for every $x \in B$,

$$\nu^{(n_B)}\left(\{g,\ gx \in K\}\right) \geqslant \frac{1}{2}. \tag{4.20}$$

Plugging this into (4.19), we obtain $\frac{1}{2}\mu(B) \leqslant \mu(K)$. Since B is arbitrary, this implies that $\mu(M\backslash F) \leqslant 2\mu(K)$ and we are done. □

4.3. **Totally real invariant manifolds.** We now consider a situation which is specific to the complex setting.

Theorem 4.5. *Let X be a compact complex manifold of dimension d. Let Γ be a group of holomorphic diffeomorphisms of X, endowed with a probability measure ν satisfying* (M$_+$). *Let $Y \subset X$ be a Γ-invariant, real analytic, totally real submanifold of maximal (real) dimension d, such that ν is uniformly expanding on Y. Then for any $x \in X\backslash Y$ and any $\varepsilon > 0$, there exists a compact subset $K \Subset X\backslash Y$ such that the conclusions (1) and (2) of Theorem 4.3 hold.*

By "uniformly expanding along Y" we mean that the restriction of Γ to Y is uniformly expanding viewed as an action on Y, or equivalently that the uniform expansion condition (1.1) holds in X for every $x \in Y$; the equivalence between the two conditions comes from the fact that for every $x \in Y$, the complex span of T_xY is T_xX. When Y is singular, we require that (1.1) holds in X along $\mathrm{Sing}(Y)$.

Note also that this statement is specific to totally real submanifolds and holomorphic actions. In other words, there is no analogue of Theorem 4.1 when F is replaced by an arbitrary submanifold: see Example 4.6 below.

Proof. We suppose Y smooth and show that there exists $n \geqslant 1$ such that

$$x \mapsto -\log d(x,Y) \tag{4.21}$$

defines a Margulis function (i.e. satisfies (4.1) and (4.2)) for $\nu^{(n)}$. Then, as explained before, [3] shows that (4.1) and (4.2) are automatically satisfied with $n = 1$. As in Theorem 4.3, Property (4.2) follows from the invariance of Y and the bilipschitz property; so we focus on (4.1).

For every $x \in Y$ there exists a local chart in which the equation of Y becomes $\mathsf{Im}(z) = 0$, where $\mathsf{Im}(z) = \mathsf{Im}(z_1, \ldots, z_d) = (\mathsf{Im}(z_1), \ldots, \mathsf{Im}(z_d))$ (see [1, Prop. 1.3.8 and 1.3.11]). We fix a finite family $\phi_i : U_i \to \mathbf{C}^d$ of such charts, covering a neighborhood of Y. The charts being bilipschitz, there exists an absolute constant D such that if $x \in U_i$,

$$|\log d(\phi_i(x), \phi_i(Y)) - \log d(x, Y)| \leqslant D.$$

Then from (2.7), replacing ν by $\nu^{(n)}$ we may assume that the uniform expansion holds for $n = 1$ and the expansion constant c is bigger than $10D$. We will work in local charts to show that $-\log d(\cdot, Y)$ is a Margulis function.

Let d_{U_i} denote the Euclidean distance in the i-th chart (pulled back by ϕ_i). In U_i, write $\phi_i(x) = z = (z_1, \ldots, z_d)$ and $\phi_i(Y) = \{\mathsf{Im}(z) = 0\}$. Let $\pi(\phi_i(x)) = (\mathsf{Re}(z_1), \ldots, \mathsf{Re}(z_d))$ be the projection of $\phi_i(x)$ on Y, so that

$$d_{U_i}(x, Y) = \|\phi_i(x) - \pi(\phi_i(x))\| = \|(\mathsf{Im}(z_1), \ldots, \mathsf{Im}(z_d))\| = \|\mathsf{Im}(\phi_i(x))\|. \quad (4.22)$$

As before let $\Gamma(R) = \{f \in \Gamma\ ;\ N(f) \leqslant R\}$, where $N(f) = \|f\|_{C^2} + \|f^{-1}\|_{C^2}$, and fix $f \in \Gamma(R)$. If x is sufficiently close to Y, then so does $f(x)$, hence $f(x)$ belongs to some chart U_j and working in this chart we get $d_{U_j}(f(x), Y) = \|\mathsf{Im}(\phi_j(f(x)))\|$. Applying Taylor's formula to the coordinate expression $\tilde{f}$ of f, we obtain

$$\begin{aligned}\phi_j(f(x)) &= \tilde{f}(\phi_i(x)) \\ &= \tilde{f}(\pi(\phi_i(x))) + d\tilde{f}_{\pi(\phi_i(x))}(\phi_i(x) - \pi(\phi_i(x))) + O\left(\|\phi_i(x) - \pi(\phi_i(x))\|^2\right).\end{aligned}$$

Now, observe that the vector $d\tilde{f}_{\pi(\phi_i(x))}(\phi_i(x) - \pi(\phi_i(x)))$ is purely imaginary because $\phi_i(x) - \pi(\phi_i(x))$ is purely imaginary and $d\tilde{f}_{\pi(\phi_i(x))}$ is real, since it preserves Y. Thus, taking imaginary parts and using (4.22) yields

$$\left| \frac{d_{U_j}(f(x), Y)}{d_{U_i}(x, Y)} - \left\| df_{\pi(x)}(v_x) \right\| \right| \leqslant CRd_{U_i}(x, Y), \quad (4.23)$$

where $v_x = \phi_i^\star \left(\frac{\phi_i(x) - \pi(\phi_i(x))}{\|\phi_i(x) - \pi(\phi_i(x))\|} \right)$, $\pi(x) = \phi_i^{-1}\pi(\phi_i(x))$, and the constant C depends only on the charts. Arguing as in (4.11), plugging in the bilipschitz estimate for the distance to Y, and increasing C if necessary we get

$$\left| \log \frac{d(f(x), Y)}{d(x, Y)} - \log \left\| df_{\pi(x)}(v_x) \right\| \right| \leqslant CR^2 d(x, Y) + 2D. \quad (4.24)$$

Finally, using the moment condition to deal with the contribution of $\Gamma \backslash \Gamma(R)$ as in Theorem 4.1, we obtain

$$\int_\Gamma \left| \log \frac{d(f(x), Y)}{d(x, Y)} - \log \left\| df_{\pi(x)}(v_x) \right\| \right| d\nu(f) \leqslant CR^2 d(x, Y) + 2D + \frac{C}{(\log R)^\eta}, \quad (4.25)$$

and we conclude that $\log d(\cdot, Y)$ is a Margulis function by first fixing a large R and then choosing x sufficiently close to Y, as in Theorem 4.1. $\square$

Example 4.6. *There exists a group $\Gamma = \langle f, g\rangle$ of diffeomorphisms of the 3-torus $\mathbf{R}^3/\mathbf{Z}^3$ and a finitely supported measure ν on Γ with $\langle \mathrm{Supp}(\nu)\rangle = \Gamma$ such that:*

- *Γ preserves $Y := \mathbf{R}^2/\mathbf{Z}^2 \times \{0\}$;*
- *there exists a neighborhood U of Y on which the dynamics of (Γ, ν) is uniformly expanding;*
- *for every $x \in U$ and almost every trajectory ω, $f^n_\omega(x)$ converges to Y.*

Proof. Let $0 < \gamma < 1$ and ψ be a diffeomorphism of the circle $\mathbf{R}/\mathbf{Z}$ such that

(i) ψ fixes 0, $\psi(z) = \gamma z$ on $[-1/8, 1/8]$, and $\psi([-1/4, 1/4]) = [-1/4, 1/4]$;
(ii) in the interval $[-1/4, 1/4]$, the only fixed points of ψ are $-1/4$, 0 and $1/4$.

Then there is a diffeomorphism $\varphi\colon]-1/4, 1/4[\to \mathbf{R}$ that conjugates $\psi|_{]-1/4,1/4[}$ to $t \mapsto \gamma t$.

Pick A and B in $\mathsf{SL}(2, \mathbf{Z})$ and (c_1, c_2) in $\mathbf{Z}^2$ such that

(iii) A and B generate a non-elementary subgroup;
(iv) γ is not an eigenvalue of B and $(c_1, c_2) \neq (0, 0)$.

Define two diffeomorphisms g and h of $\mathbf{R}^3/\mathbf{Z}^3$ by

$$g(x, y, z) = (A(x, y) + (c_1z, c_2z), \psi(z)) \text{ and } h(x, y, z) = (B(x, y), \psi(z)). \tag{4.26}$$

Let ν be a probability measure supported on $\{g, h, g^{-1}, h^{-1}\}$ such that $0 < \nu(g^{-1}) < \nu(g)$ and $0 < \nu(h^{-1}) < \nu(h)$.

First, let us prove that there exists $\Omega_0 \subset \Omega$ of full $\nu^{\mathbf{N}}$-measure such that $f^n_\omega(p)$ converges towards Y for every $p = (x, y, z) \in \mathbf{R}^2/\mathbf{Z}^2\times]-1/4, 1/4[$, and $\omega \in \Omega_0$. Indeed, writing $f^n_\omega(p) = (x_n, y_n, z_n)$, we have:

(1) $z_n \in]-1/4, 1/4[$ because ψ preserves $]-1/4, 1/4[$;
(2) $\varphi(z_n) = \gamma^{\sum_{i=1}^n \varepsilon_i}\varphi(z)$, where (ε_n) is a sequence of independent random variables with $\mathbb{P}(\varepsilon = 1) = \nu(g) + \nu(h)$ and $\mathbb{P}(\varepsilon = -1) = \nu(g^{-1}) + \nu(h^{-1})$. Since, $\nu(g^{-1}) + \nu(h^{-1}) < \nu(g) + \nu(h)$, $\varphi(z_n)$ converges almost surely to 0.

Now, we show that the dynamics of (Γ, ν) is uniformly expanding in $\mathbf{R}^2/\mathbf{Z}^2\times]-1/4, 1/4[$. Indeed, if $p \in \mathbf{R}^2/\mathbf{Z}^2\times]-1/4, 1/4[$ and $\omega \in \Omega_0$, there is $n(\omega)$ such that $f^n_\omega(p) \in \mathbf{R}^2/\mathbf{Z}^2\times]-1/8, 1/8[$ for $n \geqslant n(\omega)$. Now, in $\mathbf{R}^2/\mathbf{Z}^2\times]-1/8, 1/8[$ the dynamics is linear, and the tangent action is generated by

$$\tilde{g} = \begin{pmatrix} A & \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ 0 & \gamma \end{pmatrix} \text{ and } \tilde{h} = \begin{pmatrix} B & 0 \\ 0 & \gamma \end{pmatrix}. \tag{4.27}$$

We claim that the linear action of $(\tilde{\Gamma}, \tilde{\nu})$ on $\mathbf{R}^3$ is uniformly expanding, where $\tilde{\Gamma} = \langle \tilde{g}, \tilde{h}\rangle$ and $\tilde{\nu}$ is the measure naturally corresponding to ν. Indeed, the action is uniformly expanding on $\mathbf{R}^2 \times \{0\}$ and if it were not uniformly expanding on $\mathbf{R}^3$, by Furstenberg-Kifer [36], there would exist a $\tilde{\Gamma}$-invariant line transverse to $\mathbf{R}^2 \times \{0\}$ along which the

Lyapunov exponent would be non-positive. But the hypothesis (iv) guarantees that such a line does not exist. From this, we deduce that there exists $c > 0$ (any constant smaller than the Lyapunov exponent of the random product generated by A and B will do) such that for every $p \in \mathbf{R}^2/\mathbf{Z}^2 \times]-1/4, 1/4[$, every unit tangent vector v at p and almost every ω, $\frac{1}{n} \log \|(f_\omega^n)_\star v\| \geqslant c$ if n is large enough. Applying Lemma 2.3 finishes the proof. □

5. An ergodic-theoretic criterion for expansion

5.1. **Construction of stationary measures.** Let M be a compact manifold endowed with a Riemannian metric; let T^1M denote its unit tangent bundle and $\pi \colon T^1M \to M$ be the canonical projection. As in Section 2, if f is a diffeomorphism of M, we denote by $f_\star$ its action on TM. Let ν be a probability measure on $\mathsf{Diff}^1(M)$ satisfying the moment condition (M). We apply a classical strategy to get the following theorem (see e.g. [24, Prop. 3.17], and [39, Lem. 3.3]).

Theorem 5.1. *Assume that there exists an increasing sequence $(n_k) \in \mathbf{N}^{\mathbf{N}}$ and a sequence of unit tangent vectors $(u_k) \in (T^1M)^{\mathbf{N}}$ such that*

$$\lim_{k\to\infty} \frac{1}{n_k} \int \log \|f_\star u_k\| d\nu^{(n_k)}(f) = \chi_0. \tag{5.1}$$

Then, there exists a real number $\chi \geqslant \chi_0$, an ergodic ν-stationary probability measure $\hat{\mu}$ on T^1M, and a ν-almost surely invariant sub-bundle $V \subset TM$ such that the top Lyapunov exponent of the projected measure $\mu := \pi_\star \hat{\mu}$ in restriction to V is equal to χ.

Likewise, there exists a real number $\chi' \leqslant \chi_0$ that satisfies the same property for some pair $(\hat{\mu}', V')$ and the projection $\mu' := \pi_* \hat{\mu}'$.

Note that if $\hat{\mu}$ is a probability measure on T^1M that is ν-stationary for the tangent action, then its projection μ on M is ν-stationary as well; and if $\hat{\mu}$ is ergodic, so is μ. When $\chi > 0$, one typically obtains $V = TM$.

Proof (see [24, 39]*).* Consider the sequence of measures $\hat{\mu}_k$ on T^1M defined by

$$\hat{\mu}_k = \frac{1}{n_k} \sum_{j=0}^{n_k-1} \nu^{(j)} \star \delta_{u_k} = \frac{1}{n_k} \sum_{j=0}^{n_k-1} \int \frac{f_\star u_k}{\|f_\star u_k\|} d\nu^{(j)}(f), \tag{5.2}$$

where $\nu^{(j)} \star \delta_{u_k}$ denotes the convolution for the action of $\mathsf{Diff}(M)$ on the unit tangent bundle. Since T^1M is compact and the $\hat{\mu}_k$ are probability measures, we can extract a subsequence (still denoted by $\hat{\mu}_k$ for simplicity) that converges weakly towards a probability measure $\hat{\mu}_\infty$ on T^1M. By construction, this measure is ν-stationary.

The function $\mathrm{Dil}(f, u) := \log \|f_\star u\|$ is continuous on $\mathsf{Diff}^1(M) \times T^1M$. And by our moment assumption, so is the function $u \in T^1M \mapsto \int \mathrm{Dil}(f, u) d\nu(f)$. For $u \in T^1M$ the chain rule gives

$$\frac{1}{n}\int \log \|(f_\omega^n)_\star u\| d\nu^{\mathbf{N}}(\omega) = \frac{1}{n}\sum_{j=0}^{n-1}\int \mathrm{Dil}\left(f_j, \frac{(f_\omega^j)_\star u}{\|(f_\omega^j)_\star u\|}\right) d\nu^{\mathbf{N}}(\omega)$$

$$= \int_{g\in\mathsf{Diff}(M)}\left(\frac{1}{n}\sum_{j=0}^{n-1}\int \mathrm{Dil}\left(g, \frac{h_\star u}{\|h_\star u\|}\right) d\nu^{(j)}(h)\right) d\nu(g) \tag{5.3}$$

If we apply this equation to $n = n_k$ and $u = u_k$ the term between parentheses in the last integral is equal to $\int \mathrm{Dil}\,(g, u)\, d\hat{\mu}_k(u)$, so, letting k go to $+\infty$, we conclude that

$$\lim_{k\to\infty}\frac{1}{n_k}\int \log \|f_\star u_k\| d\nu^{(n_k)}(f) = \chi_0 = \int_{\mathsf{Diff}^1(M)}\int_{T^1M} \mathrm{Dil}(g, u) d\hat{\mu}_\infty(u) d\nu(g) \tag{5.4}$$

Thus, there exists $\chi \geqslant \chi_0$ (resp. $\chi \leqslant \chi_0$) and an ergodic component $\hat{\mu}$ of $\hat{\mu}_\infty$ such that

$$\int_{\mathsf{Diff}^1(M)}\int_{T^1M} \mathrm{Dil}(g, u) d\hat{\mu}(u) d\nu(g) = \chi. \tag{5.5}$$

As observed above, $\mu = \pi_\star\hat{\mu}$ is an ergodic ν-stationary probability measure. Denote by $\hat{\mu}_x$ the conditional measures obtained by disintegration of $\hat{\mu}$ with respect to the fibers of π, that is, $\hat{\mu} = \int \hat{\mu}_x d\mu(x)$. For μ-almost every x, let $V(x)$ be the linear span of $\mathrm{Supp}(\hat{\mu}(x))$. Since $\mathrm{Supp}(\hat{\mu})$ is ν-almost invariant and $f_\star$ acts linearly along the fibers of TM, we infer that V is a ν-almost invariant measurable sub-bundle. The Furstenberg formula asserts that the top Lyapunov exponent of μ in restriction to V is equal to χ. For completeness let us recall the argument: the ergodic theorem shows that for $(\nu^{\mathbf{N}} \times \mu)$-almost every (ω, x) and $\hat{\mu}_x$-almost every $u \in T_x^1M$,

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1} \mathrm{Dil}\left(f_j, (f_\omega^j)_\star u\right) = \int_{\mathsf{Diff}^1(M)^{\mathbf{N}}}\int_{T^1M} \mathrm{Dil}(f_\omega^1, u)\, d\hat{\mu}(u) d\nu^{\mathbf{N}}(\omega) = \chi \tag{5.6}$$

where as usual $\omega = (f_0, f_1, \ldots)$, $f_\omega^1 = f_0$, and $f_\omega^j = f_{j-1}\circ\cdots\circ f_0$. On the other hand the Oseledets theorem asserts that for $(\nu^{\mathbf{N}} \times \mu)$-almost every (ω, x), there exists a proper subspace $W(\omega, x) \subset V(x)$ such that for $u \notin W(\omega, x)$, $\frac{1}{n}\log\|(f_\omega^n)_\star u\|$ converges to the top Lyapunov exponent $\chi^+(\mu, V)$ of μ in restriction to V. Thus by (5.6), $\chi^+(\mu, V) = \chi$, and the proof is complete. □

5.2. **Application: Chung's criterion.** The following theorem, taken from [24, Prop 3.17], plays an important role in this paper; a variant of this result appears in [12]. It is stated in [24] for C^2 actions on surfaces but it holds in greater generality. The proof follows directly from the second assertion of Theorem 5.1.

Theorem 5.2 (Chung). *Let M be a compact manifold. Let ν be a probability measure on $\mathsf{Diff}^1(M)$ that satisfies* (M). *If ν is not uniformly expanding there exists an ergodic ν-stationary measure μ on M and a μ-measurable subbundle $W \subset TM$ such that*

(a) $0 < \dim(W) \leqslant \dim(M)$*;*

(b) W is ν-almost surely invariant;
(c) in restriction to W, the top Lyapunov exponent of μ is non-positive.
Conversely, if such a pair (μ, W) exists, then ν is not uniformly expanding.

When M is a surface and ν is supported by the group of diffeomorphisms preserving some fixed area form the Lyapunov exponents of any ergodic stationary measure μ satisfy $\lambda^+(\mu) + \lambda^-(\mu) = 0$. Thus, in Chung's theorem, either $\lambda^-(\mu) = \lambda^+(\mu) = 0$ and we can take $W = TM$ or $\lambda^-(\mu) < 0 < \lambda^+(\mu)$ and W coïncides with the stable line field provided by the Oseledets theorem; thus, μ is not hyperbolic or it is hyperbolic and its stable line field is non-random.

Part 2. **Non-elementary actions on complex surfaces**

From now on we denote by X a compact complex surface, endowed with a group Γ of holomorphic diffeomorphisms. Recall from [21, 18] that if Γ is non-elementary, then X is necessarily projective and $\Gamma \subset \mathrm{Aut}(X)$.

6. Preliminaries

In this section we briefly recall some results from [20] (see also [14, 17]).

6.1. **Parabolic automorphisms and their dynamics (see** [20, §3]**).** Let h be a parabolic automorphism of a compact projective surface X (most of this discussion is valid for a compact Kähler surface). Then, h preserves a genus 1 fibration $\pi_h \colon X \to B$, and every h-invariant holomorphic (singular) foliation – in particular any invariant fibration – coïncides with π. Let h_B denote the automorphism of B such that

$$\pi \circ h = h_B \circ \pi. \tag{6.1}$$

If X is not a torus there is a positive integer m such that h^m preserves every fiber of π, i.e. $h_B^m = \mathrm{id}_B$. When $h_B = \mathrm{id}_B$ we say that h is a **Halphen twist**. The set of Halphen twists in a given subgroup $\Gamma \subset \mathsf{Aut}(X)$ is denoted by $\mathrm{Hal}(\Gamma)$.

Remark 6.1. If Γ is non-elementary and contains a Halphen twist (resp. a parabolic automorphism) h, then the conjugacy class of h in Γ contains Halphen twists (resp. parabolic automorphisms) associated with infinitely many distinct invariant fibrations (see [22, §3.1]).

Suppose now that h is a Halphen twist. Then, h acts by translation on every smooth fiber of π (see [20, §3]). To be more precise, denote by $\mathrm{Crit}(\pi) \subset B$ the finite set of critical values of π and set $B^\circ = B \setminus \mathrm{Crit}(\pi)$. Fix some simply connected open subset $U \subset B^\circ$, endowed with a section σ of π and a continuous choice of basis for $H_1(X_w, \mathbf{Z})$. Each fiber $X_w := \pi^{-1}(w)$, $w \in U$, is an elliptic curve with zero $\sigma(w)$, and one can find a holomorphic function τ on U, with values in the upper half plane, such that X_w is isomorphic to $\mathbf{C}/\mathrm{Lat}(w)$ for $\mathrm{Lat}(w) = \mathbf{Z} \oplus \mathbf{Z}\tau(w)$. On X_w, h is a translation $h_w(z) = z + t(w)$, for some holomorphic function $w \in U \mapsto t(w) \in \mathbf{C}/\mathrm{Lat}(w)$. Moreover, Lemma 6.2(4) says that h behaves like a "complex Dehn twist", with a shearing property in the direction which is transversal to the fibers; thus shearing (or twisting) occurs along X_w whenever t and τ are "transverse" at w (see § 9.1 for more details on the non-twisting locus).

The points w for which h_w is periodic are characterized by the relation $t(w) \in \mathbf{Q} \oplus \mathbf{Q}\tau(w)$. If

$$t(w) - (\alpha + \beta\tau(w)) \in \mathbf{R} \cdot (p + q\tau(w)) \tag{6.2}$$

for some $(\alpha, \beta) \in \mathbf{Q}^2$ and $(p, q) \in \mathbf{Z}^2$, the closure of $\mathbf{Z}t(w)$ in $\mathbf{C}/\mathrm{Lat}(w)$ is an abelian Lie group of dimension 1, isomorphic to $\mathbf{Z}/k\mathbf{Z} \times \mathbf{R}/\mathbf{Z}$ for some $k > 0$; then, the closure of each orbit of h_w is a union of k circles. This occurs along a countable union of analytic curves $\mathrm{R}^{\alpha,\beta}_{p,q} \subset U$. Otherwise, the orbits of h_w are dense in X_w, and the unique h_w invariant probability measure is the Haar measure on X_w.

The following lemma summarizes this discussion.

Lemma 6.2. *Let h be a Halphen twist with invariant fibration $\pi : X \to B$. Then,*

(1) *h acts by translation on each fiber $X_w = \pi^{-1}(w)$, $w \in B^\circ$;*
(2) *for w in a dense countable subset of B°, the orbits of h_w are finite;*
(3) *there is a dense, countable union of analytic curves R_j in B°, such that:*
   (a) *for $w \notin \bigcup_j \mathrm{R}_j$, the action of h in the fiber X_w is a totally irrational translation (it is uniquely ergodic, and its orbits are dense in X_w);*
   (b) *for $w \in \bigcup_j \mathrm{R}_j$ the orbits of h_w are either finite or dense in a finite union of circles;*
(4) *there is a finite subset NT_h such that for $x \notin \pi^{-1}(\mathrm{NT}_h)$*

$$\lim_{n\to\pm\infty} \|D_x h^n\| \to +\infty$$

locally uniformly in x; more precisely for every $v \in T_xX \setminus T_xX_{\pi(x)}$, $\|D_xh^n(v)\|$ grows linearly while $\frac{1}{n}\pi_*(D_xh^n(v))$ converges to 0.

If moreover h preserves a totally real 2-dimensional real analytic subset $Y \subset X$, then:

(5) *the generic fibers of $\pi|_Y$ are union of circles, there exists an integer m such that h^m preserves each of these circles, and h^m is uniquely ergodic along each of these circles, except for countably many fibers.*

Property (4) is the above mentioned twisting property of h. Property (5) occurs, for instance, when X and h are defined over $\mathbf{R}$ and $Y = X(\mathbf{R})$ is the real part of X. There are also examples of subgroups $\Gamma \subset \mathsf{Aut}(X)$ preserving a totally real surface $Y \subset X$ which is not the real part of X for any real structure, see [20, §9].

6.2. **Classification of invariant measures.** Recall from Example 1.5 that if X is a torus, a K3 surface, or an Enriques surface it admits a canonical $\mathsf{Aut}(X)$-invariant volume form vol_X. The associated probability measure will also be denoted by vol_X. Such an area form exists also on any totally real surface, by virtue of the following lemma.

Lemma 6.3 (see [20, Remark 2.3])**.** *Let X be an Abelian surface, or a K3 surface, or an Enriques surface with universal cover $\tilde{X}$. Let $Y \subset X$ be a totally real surface of class*

C^1, and $\mathsf{Aut}(X;Y)$ be the subgroup of $\mathsf{Aut}(X)$ preserving Y. If Y is totally real, the canonical holomorphic 2-form Ω_X (resp. $\Omega_{\tilde{X}}$) induces a smooth $\mathsf{Aut}(X;Y)$-invariant probability measure vol_Y on Y.

Theorem 6.4 (see [20, Thm A]). *Let X be a projective surface. Let Γ be a non-elementary subgroup of $\mathsf{Aut}(X)$ containing a parabolic element. Let μ be a Γ-invariant ergodic probability measure on X. Then, μ satisfies exactly one of the following properties.*

(a) *μ is the average on a finite orbit of Γ;*
(b) *μ is non-atomic and supported on a Γ-invariant algebraic curve $D \subset X$;*
(c) *there is a Γ-invariant proper algebraic subset Z of X, and a Γ-invariant, totally real analytic surface Y of $X\backslash Z$ such that* (1) *$\mu(\overline{Y}) = 1$ and $\mu(Z) = 0$;* (2) *Y has finitely many irreducible components;* (3) *the singular locus of Y is locally finite in $X\backslash Z$;* (4) *μ is absolutely continuous with respect to the Lebesgue measure on Y; and* (5) *its density (with respect to any real analytic area form on the regular part of Y) is real analytic;*
(d) *there is a Γ-invariant proper algebraic subset Z of X such that* (1) *$\mu(Z) = 0$,* (2) *the support of μ is equal to X;* (3) *μ is absolutely continuous with respect to the Lebesgue measure on X; and* (4) *the density of μ with respect to any real analytic volume form on X is real analytic on $X\backslash Z$.*

If X is not a rational surface, then in case (c) (resp. (d)) we can further conclude that the invariant measure is proportional to vol_Y (resp. vol_X).

6.3. **Invariant curves.** By [21, Lem. 2.12], any action of a non-elementary group Γ on a projective surface X admits a maximal invariant curve D_Γ, which can be easily detected from the action of Γ on $H^2(X, \mathbf{Z})$ since it corresponds to an invariant class. Bounds on the degrees of such invariant curves in terms of the action are given in [22, §3]. If in addition Γ contains a parabolic element, D_Γ is the set of common components of the singular fibers of all elliptic fibrations associated to parabolic elements in Γ (see [20, §4.1]).

7. HYPERBOLICITY OF INVARIANT MEASURES

Here, X is a compact Kähler surface. We fix a Kähler form κ_0 on X; norms of tangent vectors and differentials will be computed with respect to it.

7.1. **Ledrappier's invariance principle and invariant measures on $\mathbb{P}T\mathcal{X}$.** In this paragraph we collect some preliminary results for the proof of Theorems 1.4 and 7.4. The reader should also consult [2] and [47] for comparison; [47] relies on the "pinching and twisting" formalism of Avila and Viana (see [51] for an introduction[5]). Most of this discussion is valid for a random holomorphic dynamical system on an arbitrary complex surface (not necessarily compact) satisfying (M).

[5] Beware that the word "twisting" has a different meaning there.

We denote by $\mathbb{P}TX$ the projectivized tangent bundle of X; if f is an automorphism of X, we denote by $\mathbb{P}(Df)$ the induced action on $\mathbb{P}TX$.

Let ν be a probability measure on $\mathsf{Aut}(X)$ that satisfies the moment condition (M). We endow $\Omega := \mathsf{Aut}(X)^{\mathbf{N}}$ (resp. $\Sigma := \mathsf{Aut}(X)^{\mathbf{Z}}$) with the probability measure $\nu^{\mathbf{N}}$ (resp. $\nu^{\mathbf{Z}}$), and set $\mathcal{X}_+ = \Omega \times X$ (resp. $\mathcal{X} = \Omega \times X$); σ will denote the shift (on Ω or Σ). For $\omega = (f_i)_{i\geqslant 0} \in \Omega$, we keep the notation f_ω^n from § 2.2. Then, we define $F_+\colon \mathcal{X}_+ \to \mathcal{X}_+$ by $F_+(\omega, x) = (\sigma(\omega), f_\omega^1(x))$; $F\colon \mathcal{X} \to \mathcal{X}$ is defined by the same formula. For further standard notations, we refer to [21, §7].

Let μ be an ergodic ν-stationary measure on X. We introduce the projectivized tangent bundles $\mathbb{P}T\mathcal{X}_+ = \Omega \times \mathbb{P}TX$ and $\mathbb{P}T\mathcal{X} = \Sigma \times \mathbb{P}TX$. The bundles TX and $\mathbb{P}TX$ admit measurable trivializations over a set of full measure. Consider any probability measure $\hat\mu$ on $\mathbb{P}TX$ that is stationary under the random dynamical system induced by (X, ν) on $\mathbb{P}TX$ and whose projection on X coincides with μ, i.e. $\pi_\star\hat\mu = \mu$ where $\pi\colon \mathbb{P}TX \to X$ is the natural projection. Such measures always exist: indeed, the set of probability measures on $\mathbb{P}TX$ projecting to μ is compact and convex, and it is non-empty since it contains the measures $\int \delta_{[v(x)]} d\mu(x)$ for any measurable section $x \mapsto [v(x)]$ of $\mathbb{P}TX$. Thus, the operator $\int \mathbb{P}(Df)\, d\nu(f)$ has a fixed point on that set. The stationarity of $\hat\mu$ is equivalent to the invariance of $\nu^{\mathbf{N}} \times \hat\mu$ under the transformation $\widehat{F}_+\colon \Omega \times \mathbb{P}TX \to \Omega \times \mathbb{P}TX$ defined by

$$\widehat{F}_+(\omega, x, [v]) = (\sigma(\omega), f_\omega^1(x), \mathbb{P}(D_x f_\omega^1)[v]) \tag{7.1}$$

for any non-zero tangent vector $v \in T_xX$. We denote by $\hat\mu_x$ the family of probability measures on the fibers $\mathbb{P}T_xX$ of π given by the disintegration of $\hat\mu$ with respect to π. The conditional measures of $\nu^{\mathbf{N}} \times \hat\mu$ with respect to the projection $\mathbb{P}T\mathcal{X}_+ \to X$ are given by $\hat\mu_{\omega,x} = \nu^{\mathbf{N}} \times \hat\mu_x$.

Remark 7.1. Even when μ is Γ_ν-invariant, this construction only provides a stationary measure on $\mathbb{P}TX$. This is exactly what happens for non-elementary subgroups with a parabolic automorphism: indeed, we show in § 7.2 that projectively invariant measures do not exist in this case.

The tangent action of our random dynamical system gives rise to a stationary product of matrices in $\mathsf{GL}(2, \mathbf{C})$. To see this, fix a measurable trivialization $P\colon TX \to X \times \mathbf{C}^2$, given by linear isomorphisms $P_x\colon T_xX \to \mathbf{C}^2$. It conjugates the action of DF_+ to that of a linear cocycle $A : \mathcal{X}_+ \times \mathbf{C}^2 \to \mathcal{X}_+ \times \mathbf{C}^2$ over $(\mathcal{X}_+, F_+, \nu^{\mathbf{N}} \times \mu)$. In this context, Ledrappier establishes in [43] the following "invariance principle".

Theorem 7.2. *If $\lambda^-(\mu) = \lambda^+(\mu)$, then for any stationary measure $\hat\mu$ on $\mathbb{P}TX$ projecting to μ, we have $\mathbb{P}(D_xf)_\star\hat\mu_x = \hat\mu_{f(x)}$ for μ-almost every x and ν-almost every f.*

The second ingredient in the proof of Theorem 1.4 is a description of such projectively invariant measures; this is where we follow [2]. To explain this result a bit of notation is required. Let V and W be hermitian vector spaces of dimension 2; we fix two isometric isomorphisms $\iota_V\colon V \to \mathbf{C}^2$ and $\iota_W\colon W \to \mathbf{C}^2$ to the standard hermitian space $\mathbf{C}^2$, and we endow the projective lines $\mathbb{P}(V)$ and $\mathbb{P}(W)$ with their respective

Fubini-Study metrics. If $g\colon V \to W$ is a linear isomorphism, we set

$$[\![g]\!] = \|\mathbb{P}(g)\|_{C^1} \tag{7.2}$$

where $\mathbb{P}(g)\colon \mathbb{P}(V) \to \mathbb{P}(W)$ is the projective linear map induced by g and $\|\cdot\|_{C^1}$ is the maximum of the norms of $D_z\mathbb{P}(g)\colon T_z\mathbb{P}(V) \to T_{\mathbb{P}(g)(z)}\mathbb{P}(W)$ with respect to the Fubini-Study metrics. If $\iota_W \circ g \circ \iota_V^{-1} = k_1 a k_2$ is the KAK decomposition of $\iota_W \circ g \circ \iota_V^{-1}$ in $\mathsf{GL}(2, \mathbf{C})$, we get

$$[\![g]\!] = \frac{\|a\|^2}{|\det(a)|} = \frac{\left\|\iota_W \circ g \circ \iota_V^{-1}\right\|^2}{\left|\det(\iota_W \circ g \circ \iota_V^{-1})\right|} \tag{7.3}$$

where $\|\cdot\|$ is the matrix norm in $\mathsf{GL}_2(\mathbf{C})$ associated to the Hermitian norm of $\mathbf{C}^2$. In particular,

(a) $[\![g]\!] = 1$ if and only if $\mathbb{P}(g)$ is an isometry from $\mathbb{P}(V)$ to $\mathbb{P}(W)$;
(b) for a sequence (g_n) of linear maps $V \to W$, $[\![g_n]\!]$ tends to $+\infty$ with n if and only if $\mathbb{P}(\iota_W \circ g \circ \iota_V^{-1})$ diverges to infinity in $\mathsf{PGL}_2(\mathbf{C})$.

If f is an automorphism of X and x is a point of X, then κ_0 endows T_xX and $T_{f(x)}X$ with hermitian structures, and we can apply this discussion to $D_xf\colon T_xX \to T_{f(x)}X$. We are now ready to state the classification of projectively invariant measures.

Theorem 7.3. *Let (X, ν) be a random dynamical system on a complex surface and let μ be an ergodic stationary measure. Let $\hat\mu$ be a stationary measure on $\mathbb{P}TX$ such that $\pi_*\hat\mu = \mu$ and $\mathbb{P}(D_xf)_*\hat\mu_x = \hat\mu_{f(x)}$ for μ-almost every x and ν-almost every f. Then, exactly one of the following two properties is satisfied:*

(1) *For $(\nu^{\mathbf{N}} \times \mu)$-almost every (ω, x), the sequence $[\![D_xf_\omega^n]\!]$ is unbounded and then:*
  (1.a) *either there exists a measurable Γ_ν-invariant family of lines $E(x) \subset T_xX$ such that $\hat\mu_x = \delta_{[E(x)]}$ for μ-almost every x;*
  (1.b) *or there exists a measurable Γ_ν-invariant family of pairs of lines $E_1(x), E_2(x) \subset T_xX$ and positive numbers λ_1, λ_2 with $\lambda_1 + \lambda_2 = 1$ such that $\hat\mu_x = \lambda_1\delta_{[E_1(x)]} + \lambda_2\delta_{[E_2(x)]}$ for μ-almost every x.*

(2) *The projectivized tangent action of Γ_ν is reducible to a compact group, that is there exists a measurable trivialization of the tangent bundle $(P_x : T_xX \to \mathbf{C}^2)_{x\in X}$, such that for almost every $f \in \Gamma_\nu$ and every x, $\mathbb{P}\left(P_{f(x)} \circ D_xf \circ P_x^{-1}\right)$ belongs to the unitary group $\mathsf{PU}_2(\mathbf{C})$.*

In assertion (1.b), the pair is not ordered: there is no natural distinction of E_1 and E_2, the elements of Γ_ν may a priori permute these lines. The proof can be obtained by adapting the arguments of [2] to the complex case; full details are given in § 7.4 of [19]. We provide a shorter proof, suggested by one of the referees, that relies on results of Furstenberg and Zimmer. Yet another approach, suggested by another referee, would be to view $\mathbb{P}^1(\mathbf{C})$ as the boundary of $\mathbb{H}^3$ and use the notion of the conformal barycenter of Douady-Earle [27].

Proof. Consider a probability space $(Y, \mathcal{A}, m)$ together with an ergodic measure preserving transformation $T\colon Y \to Y$ and a measurable cocycle

$$\eta\colon (Y, \mathcal{A}) \to (\mathsf{GL}_2(\mathbf{C}), \mathcal{B}(\mathsf{GL}_2(\mathbf{C})). \tag{7.4}$$

Suppose we are given a measurable map $\hat{m}\colon Y \to \mathrm{Prob}(\mathbb{P}^1(\mathbf{C}))$ from Y to the space of probability measures on $\mathbb{P}^1(\mathbf{C})$ (equipped with its Borel σ-algebra) and that this map is η-equivariant, i.e.

$$\hat{m}_{Ty} = \eta(y)_* \hat{m}_y \tag{7.5}$$

for almost every y in Y. The ergodicity of m and Theorem 3.2.6 of [53] imply that, on a subset of full measure in Y, $\hat{m}$ takes values in a unique $\mathsf{GL}_2(\mathbf{C})$-orbit in $\mathrm{Prob}(\mathbb{P}^1(\mathbf{C}))$. Let $\lambda \in \mathrm{Prob}(\mathbb{P}^1(\mathbf{C}))$ be a point in this orbit. The equivariance of $\hat{m}\colon Y \to \mathsf{GL}_2(\mathbf{C})_\star\lambda$ means that $\hat{m}$ is cohomologous to a cocycle taking values in the stabilizer of λ in $\mathsf{GL}_2(\mathbf{C})$, that is, there exists a measurable map $Y \ni y \mapsto A_y \in \mathsf{GL}_2(\mathbf{C})$ such that $A_{Ty}^{-1}\eta_y A_y \in \mathrm{Stab}(\lambda)$. Indeed $\hat{m}_y = (A_y)_\star\lambda$ for some A_y. Then, according to Lemma 3.2.1 of [53] and its Corollary 3.2.2, there are only three possibilities. Either λ is a Dirac mass, or λ is an average of two Dirac masses, or the stabilizer of λ is compact.

In our situation, we take Y to be $\mathcal{X}^+ = \Omega \times X$, T is F_+, m is $\nu^{\mathbf{N}} \otimes \mu$, and $\hat{m}$ is the family of disintegrations $(\omega, x) \mapsto \hat{\mu}_x$, which is a Borel map since it is the disintegration of a Borel measure on $X \times \mathbb{P}^1$ relative to the projection $X \times \mathbb{P}^1 \to X$. Applying the above results, we get the desired conclusion, except for one point, which is the fact that in the conclusion of the theorem, the trivialization P in Assertion (2) (resp. the lines in Assertion (1)) depends only on x, and not on (ω, x), Theorem 7.3. a fact that will be used later. Since the lines in Assertion (1) are directly determined by $\hat{\mu}_x$ it is obvious that they depend uniquely on x.

Assume that we are in the situation where $\mathrm{Stab}(\lambda)$ is a compact subgroup. To show that P can be chosen to depend only on x we argue as follows. First, observe that $\mathrm{Stab}(\lambda) \subset P_0^{-1}\mathsf{U}_2(\mathbf{C})P_0$ for some $P_0 \in \mathsf{GL}_2(\mathbf{C})$, so with notation as above we obtain that $D_x f$ is of the form $A_{F_+(\omega,x)}P_0UP_0^{-1}A_{(\omega,x)}^{-1}$, for some $U \in \mathsf{U}_2$. We need to show that $A_{(\omega,x)}$ can be chosen to depend on x only. Let $E = \{(x, A) \in X \times \mathsf{GL}_2(\mathbf{C})\ ;\ A_\star\lambda = \hat{\mu}_x\}$. Since $x \mapsto \mu_x$ is Borel, E is a Borel subset of $X \times \mathsf{GL}_2(\mathbf{C})$ whose fibers in $\mathsf{GL}_2(\mathbf{C})$ are empty or compact ([6]). From the Borel selection theorem (see [9, Thm 6.9.6] or [53, Appendix A]), there exists a Borel map $x \mapsto A'_x$ such that for every $x \in X$, (x, A'_x) belongs to E; replacing $A_{(\omega,x)}$ by A'_x concludes the proof. □

7.2. **Proof of Theorem 1.4.** By Theorem 6.4, μ is either equivalent to the Lebesgue measure on X, or to the 2-dimensional Lebesgue measure on some components of an invariant totally real surface $Y \subset X$.

7.2.1. *Proof of the hyperbolicity of* μ. Let us assume, by way of contradiction, that μ is not hyperbolic. Hence its Lyapunov exponents vanish, and by Theorem 7.2 and Theorem 7.3, there is a measurable set $X' \subset X$ with $\mu(X') = 1$ such that one of the following properties is satisfied along X':

[6]On the null set where the fibers are empty, we may replace the fiber by some fixed matrix, say $A = \mathrm{id}$.

(a) there is a measurable Γ_ν-invariant line field $E(x)$;
(b) there exists a measurable Γ_ν-invariant splitting $E(x) \oplus E'(x) = T_xX$ of the tangent bundle; here, the invariance should be taken in the following weak sense: an element f of Γ_ν maps $E(x)$ to $E(f(x))$ or $E'(f(x))$;
(c) there exists a measurable trivialization $P_x \colon T_xX \to \mathbf{C}^2$ such that in the corresponding coordinates the projectivized differential $\mathbb{P}(Df_x)$ takes its values in $\mathsf{PU}_2(\mathbf{C})$ for all $f \in \Gamma_\nu$ and μ-almost all $x \in X'$.

Fix a small $\varepsilon > 0$. By Lusin's theorem, there is a compact set K_ε with $\mu(K_\varepsilon) > 1-\varepsilon$ such that the data $x \mapsto E(x)$, or $x \mapsto (E(x), E'(x))$ or $x \mapsto P_x$ in the respective cases (a), (b), and (c) are continuous on K_ε. In particular, in case (c), the norms of P_x and P_x^{-1} are bounded by some uniform constant $C(\varepsilon)$ on K_ε; hence, if $g \in \Gamma_\nu$ and x and $g(x)$ belong to K_ε, $[\![Dg_x]\!]$ is bounded by $C(\varepsilon)^2$.

Fix a pair of parabolic elements g and $h \in \Gamma_\nu$ with distinct invariant fibrations $\pi_g \colon X \to B_g$ and $\pi_h \colon X \to B_h$ respectively (see Remark 6.1). These two fibrations are tangent along some curve $\mathrm{Tang}(\pi_g, \pi_h)$ in X.

• In a first stage we assume that X is not a torus. According to Section 6.1, there is an integer $N > 0$ such that g^N and h^N preserve every fiber of their respective invariant fibrations. From now on, we replace g by g^N and h by h^N. Also, Γ_ν is discrete, so we may also assume $\nu(g)\nu(h) > 0$ (see footnote (2) page 5).

First assume that μ is absolutely continuous with respect to the Lebesgue measure on X, with a positive real analytic density on the complement of some invariant, proper, Zariski closed subset. We apply Lemma 6.2 to h and remark that $(\pi_h)_*\mu$ can not charge the union of the curves R_j. Then, we disintegrate μ with respect to π_h to obtain conditional measures μ_b, for $b \in B_h$; since π_h is holomorphic, the measures μ_b are absolutely continuous with respect to the Haar measure on almost every fiber $\pi_h^{-1}(b)$. By Lemma 6.2, there exists a fiber $\pi_h^{-1}(b)$ such that (1) the Haar measure of $K_\varepsilon \cap \pi_h^{-1}(b)$ is positive, (2) $b \notin \mathrm{NT}_h$ and (3) the dynamics of h in $\pi_h^{-1}(b)$ is uniquely ergodic. These properties hold for $b = \pi_h(z)$, for μ-almost all z in K_ε. Then we can pick $x \in \pi_h^{-1}(b)$ such that $(h^k(x))_{k\geqslant 0}$ visits K_ε infinitely many times ([7]). The fifth assertion of Lemma 6.2 rules out case (c) because the twisting property implies that the projectivized derivative $[\![Dh^n_x]\!]$ tends to infinity, while it should be bounded by $C(\varepsilon)^2$ when $h^n(x) \in K_\varepsilon$. Case (b) is also excluded: under the action of h^n, tangent vectors projectively converge to the tangent space of the fibers, so the only possible invariant subspace of dimension 1 is $\ker(D\pi_h)$. Thus we are in case (a) and moreover $E(x) = \ker D_x\pi_h$ for μ-almost every x. But then, using g instead of h and the fact that μ does not charge the curve $\mathrm{Tang}(\pi_g, \pi_h)$, we get a contradiction. This shows that the last alternative (a) does not hold either, and this contradiction proves that μ is hyperbolic.

If μ is supported by a 2-dimensional real analytic subset $Y \subset X$, the same proof applies, except that we disintegrate μ along the singular foliation of Y by circles induced

[7] Note that we use the invariance of μ here, not mere ν-stationarity.

by π_h and we use the fact that a generic leaf is a circle along which h is uniquely ergodic (see Lemma 6.2.(4)).

• If X is a torus its tangent bundle is trivial and the differential of an automorphism is constant. In an appropriate basis, the differential of a Halphen twist h is of the form

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \text{ with } \alpha \neq 0. \tag{7.6}$$

Thus we are in case (a) with $E(x) = \ker D_x\pi_h$ for μ-almost every x. Using another twist g transverse to h we get a contradiction as before.

7.2.2. *Proof of the positivity of the fiber entropy.* This follows from classical arguments. Since μ is invariant the measure $m := \nu^{\mathbf{Z}} \times \mu$ on $\mathcal{X}$ is F-invariant. In both cases $\mu \ll \mathsf{vol}_X$ and $\mu \ll \mathsf{vol}_Y$, respectively. The absolute continuity of the foliation by local Pesin unstable manifolds implies that the unstable conditionals of m cannot be atomic, see e.g. [44, Theorem B, Section 3]. Since the unstable conditionals of a zero entropy stationary measure are automatically atomic (see [21, Cor. 7.14]), we conclude that μ has positive fiber entropy.

This concludes the proof of Theorem 1.4 □

7.3. **A variant of Theorem 1.4.** Let us first recall the definition of classical Kummer examples (see [22, §4] for a thorough treatment)). Let $A = \mathbf{C}^2/\Lambda$ be a complex torus and let η be the involution given by $\eta(z_1, z_2) = (-z_1, -z_2)$; it has 16 fixed points. Then $A/\langle\eta\rangle$ is a surface with 16 singular points, and resolving these singularities (each of them requires a single blow-up) yields a Kummer surface X. Let f_A be a loxodromic automorphism of A which is induced by a linear transformation of $\mathbf{C}^2$ preserving Λ; then f_A commutes to η and goes down to an automorphism f of X; such automorphisms will be referred to as loxodromic, classical, Kummer examples. They preserve the canonical volume vol_X. The Kummer surface X also supports automorphisms which do not come from automorphisms of A (see [41] and [25] for instance).

In the following statement we do not assume that Γ_ν contains a parabolic element.

Theorem 7.4. *Let (X, ν) be a non-elementary random dynamical system on a Kummer K3 surface satisfying* (M) *and such that Γ_ν contains a loxodromic classical Kummer example. Then any ergodic Γ_ν-invariant measure giving no mass to proper Zariski closed subsets of X is hyperbolic.*

Proof. The proof is similar to that of Theorem 1.4 so we only sketch it. Assume by contradiction that μ is not hyperbolic. Since X is a K3 surface, the invariance of the volume shows that the sum of the Lyapunov exponents of μ vanishes (see [21, §7.3]), thus both are equal to 0, and one of the alternatives of Theorem 7.3 holds, referred to as (a), (b), (c) as in Section 7.2, page 31.

By assumption, Γ_ν contains a loxodromic, classical Kummer example f associated to a linear automorphism f_A of a torus A. This automorphism f is uniformly hyperbolic in some dense Zariski open subset U, which is thus of full μ-measure: its complement is given by the sixteen rational curves coming from the resolution of the singularities of A/η. We denote by $x \mapsto E^u_f(x) \oplus E^s_f(x)$ the associated splitting of $TX|_U$. The line field E^u_f (resp. E^s_f) is everywhere tangent to an f-invariant (singular) holomorphic foliation $\mathcal{F}^u$ (resp. $\mathcal{F}^s$) coming from the f_A invariant linear unstable (resp. stable) foliation on A. Since f is uniformly expanding/contracting on $E^{u/s}_f$, Alternative (c) is not possible.

If Alternative (a) holds, then $E(x)$ being f-invariant on a set of full measure, it must coincide with E^u_f or E^s_f, say with E^u_f. By continuity any $g \in \Gamma_\nu$ preserves E^u_f pointwise on $\mathrm{Supp}(\mu)$. Since in addition μ is Zariski diffuse, g preserves E^u_f everywhere on X, so it preserves also the unstable holomorphic foliation $\mathcal{F}^u$. From this, we shall contradict the fact that Γ_ν is non-elementary. We use a dynamical argument, based on basic constructions which are surveyed in [17]; one can also derive a contradiction from [23].

Every leaf of $\mathcal{F}^u$, except a finite number of them, is parametrized by an injective entire holomorphic curve $\varphi\colon \mathbf{C} \to X$, the image of which is Zariski dense. Fix a Kähler form κ on X and consider the positive currents defined by

$$\alpha \mapsto \left(\int_0^R \int_{\mathbb{D}(0;t)} \varphi^* \kappa \frac{dt}{t} \right)^{-1} \int_0^R \int_{\mathbb{D}(0;t)} \varphi^* \alpha \frac{dt}{t} \tag{7.7}$$

for any smooth $(1,1)$-form α. As R goes to $+\infty$, it is known that this sequence of currents converges to a closed positive current T^+_f that does not depend on the parametrization φ of the leaf, nor on the leaf itself (provided it is Zariski dense). This current is uniquely determined by $\mathcal{F}^u$ and the normalization $\langle T^+_f | \kappa \rangle = 1$. Dynamically, it is the unique closed positive current T^+_f that satisfies $\langle T^+_f | \kappa \rangle = 1$ and $f^* T^+_f = \lambda(f) T^+_f$ for some $\lambda(f) > 1$. Its cohomology class $[T^+_f]$ is a non-zero element of $H^{1,1}(X; \mathbf{R})$ of self-intersection 0.

Now, pick any element g of Γ_ν. Since g preserves $\mathcal{F}^u$, it permutes its leaves and preserves the ray $\mathbf{R}_+[T^+_f]$. Thus, Γ_ν preserves an isotropic line for the intersection form in $H^{1,1}(X; \mathbf{R})$, and this contradicts the non-elementarity assumption (see [21, §2.3]).

Finally, if alternative (b) holds, any $g \in \Gamma_\nu$ preserves $\{E^u_f(x), E^s_f(x)\}$ on a set of full measure so, since μ is Zariski diffuse, it must either preserve or swap these directions. Passing to an index 2 subgroup both directions are preserved, and we again contradict the non-elementary assumption, as in case (a). □

8. Characterization of uniform expansion

In this section we build on the previous results, in conjunction with the measure rigidity results from our previous work [21], to find sufficient conditions for as well as obstructions to uniform expansion for a non-elementary action on a compact complex surface.

8.1. **Proof of Theorem 1.6 and related results.**

8.1.1. *Applying Chung's criterion.*

Definition 8.1. Let ν be a probability measure on $\mathsf{Aut}(X)$. A ν-stationary measure μ on X is said to be **non-expanding** if every ergodic component μ' of μ satisfies:

(i) either both Lyapunov exponents of μ' are non-positive,
(ii) or μ' is hyperbolic and its field of Oseledets stable directions is non-random.

Recall that for a hyperbolic stationary measure μ, for $\nu^{\mathbf{N}} \times \mu$-a.e. (ω, x), the stable Oseledets subspace $E^s(\omega, x) \subset T_xX$ is defined by $v \in E^s(\omega, x)$ if and only if $\limsup_{n\to\infty} \frac{1}{n} \log \|(f_\omega^n)_* v\| < 0$. The field of Oseledets subspaces is said to be non-random if for μ-a.e. x, $\omega \mapsto E^s(\omega, x)$ is constant mod. 0.

Theorem 5.2 asserts that the existence of non-expanding ν-stationary measures is the obstruction to uniform expansion of ν:

Corollary 8.2 (of Theorem 5.2)**.** *Let X be a compact complex surface and ν be a probability measure on the group* $\mathsf{Aut}(X)$*, satisfying the moment condition* (M)*. Then ν is uniformly expanding if and only if non-expanding ν-stationary measures do not exist, hence if and only if every ergodic ν-stationary measure μ on X satisfies one of the following properties:*

- *μ has a positive Lyapunov exponent and its stable distribution depends non-trivially on the itinerary;*
- *the two Lyapunov exponents of μ are strictly positive.*

8.1.2. *Groups with invariant curves.*

Proposition 8.3. *Let X be a compact complex surface. Let Γ be a subgroup of* $\mathsf{Aut}(X)$ *that preserves a complex curve $C \subset X$. If ν is a probability measure on Γ satisfying* (M)*, then ν is not uniformly expanding.*

Remark 8.4. We leave the reader check that the proof adapts to the real case in the following sense: if X, Γ and C are defined over $\mathbf{R}$ and $C(\mathbf{R})$ is of dimension 1 (that is, neither empty nor a finite set), then ν is not uniformly expanding in restriction to $C(\mathbf{R})$.

Lemma 8.5. *Let C be a compact Riemann surface. Then,* $\mathsf{Aut}(C)$ *does not support any uniformly expanding probability measure.*

Proof. Let κ be a Kähler form on C that satisfies $\int_C \kappa = 1$. For every $f \in \mathsf{Aut}(C)$, $\int_C f^*\kappa = 1 = \int_C \|D_xf\|^2\kappa = 1$, so by the Jensen inequality $\int_C \log\|D_xf\|\kappa \leqslant 0$. Now, if ν is any probability measure on $\mathsf{Aut}(C)$, then

$$\int_C \int_{\mathsf{Aut}(C)} \log\|D_xf\| d\nu(f)\kappa \leqslant 0,$$

hence Property (2.6) cannot be satisfied by ν (for any $n_0 \geqslant 1$). □

Note that the same argument applies to conformal diffeomorphisms, in particular for C^1 diffeomorphisms of $\mathbb{S}^1$. Lemma 8.5 and Remark 2.1 imply Proposition 8.3 when C is smooth; we now prove Proposition 8.3 in full generality.

Proof of Proposition 8.3. Arguing by contradiction, we assume that ν is uniformly expanding. Let $\Gamma_1 \leqslant \Gamma$ be the finite index subgroup fixing each component of C, and each of its branches at each of its singular points; let ν_1 be the hitting measure on Γ_1 associated to $\nu^{(n_0)}$, where n_0 is as in Equation (2.6). By Proposition 3.2, ν_1 is uniformly expanding, so by replacing ν by ν_1 and C by one of its components we assume now that C is irreducible and all branches at its singular points are fixed by Γ. To get a contradiction we will construct a stationary measure μ supported on C such that the tangential Lyapunov exponent along TC is non-positive.

By Lemma 8.5 we may assume that the singular set $\mathrm{Sing}(C)$ is non-empty. If the genus of C is $\geqslant 0$, the invariance of $\mathrm{Sing}(C)$ forces $\Gamma|_C$ to be finite, in contradiction with the uniform expansion of ν. Thus, C is a rational curve; let $\pi : \hat{C} \to C$ be its normalization and $\hat{\Gamma} \subset \mathsf{Aut}(\hat{C}) \simeq \mathsf{PGL}_2(\mathbf{C})$ be the group induced by Γ; the measure ν induces a measure $\hat{\nu}$ on $\hat{\Gamma}$. Fix $\hat{p} \in \hat{C}$ such that $p := \pi(\hat{p})$ is singular. The germ of curve given by $\hat{C}$ at $\hat{p}$ determines one of the branches of C at p; our assumptions imply that $\hat{p}$ is fixed by $\hat{\Gamma}$. There are local coordinates $t \in (\mathbf{C}, 0)$ for $(\hat{C}, \hat{p})$ and $(z, w) \in (\mathbf{C}^2, (0, 0))$ for (X, p) in which π is expressed as a Puiseux expansion

$$t \mapsto (\pi_1(t), \pi_2(t)) = (\alpha t^q, \beta t^r) \text{ modulo higher order terms} \tag{8.1}$$

where $1 \leqslant q < r$; if $q = 1$ the branch is smooth at p. In these coordinates, the tangent direction to C at p corresponding to the branch determined by $\hat{p}$ is given by $(1, 0) \in \mathbf{C}^2$. Let $\lambda_{(\hat{C},\hat{p})}$ be the Lyapunov exponent of $\hat{\nu}$ at $\hat{p}$, and $\lambda_{(C,p)}$ be the Lyapunov exponent of ν in the tangent direction of this branch.

Lemma 8.6. *With notation as above $\lambda_{(C,p)} = q\lambda_{(\hat{C},\hat{p})}$. In particular $\lambda_{(C,p)}$ and $\lambda_{(\hat{C},\hat{p})}$ have the same sign.*

Proof. Pick $f \in \Gamma$, write $f(z, w) = (f_1(z, w), f_2(z, w))$ in the local coordinates (z, w), and expand f_1 in power series: $f_1(z, w) = \sum_{i,j} a_{i,j} z^i w^j$. Since the branch determined by $\lambda_{(\hat{C},\hat{p})}$ is f-invariant, we have $D_p f(1, 0) = (a_{1,0}, 0)$ with $a_{1,0} \neq 0$. Thus,

$$f_1(\pi(t)) = \sum_{i,j=0}^{\infty} a_{i,j} \alpha^i \beta^j t^{qi+rj} = a_{1,0}\alpha t^q \mod (t^{q+1}). \tag{8.2}$$

Now, f lifts to an automorphism $\hat{f}$ of $\hat{C}$ fixing $\hat{p}$. Writing $\hat{f}(t) = \lambda t \mod (t^2)$, we get $\pi_1(\hat{f}(t)) = \alpha\lambda^q t^q \mod (t^{q+1})$. Then, the semiconjugacy $f_1(\pi(t)) = \pi_1(\hat{f}(t))$ gives $\lambda^q = a_{1,0}$, and we are done. □

We resume the proof of Proposition 8.3. We fix an affine coordinate s on $\hat{C} \simeq \mathbb{P}^1(\mathbf{C})$ such that $\hat{p} = \infty$. Then, every lift $\hat{g} \in \hat{\Gamma}$ can be written as an affine map $\hat{g}(s) = a_g s + b_g$.

Lemma 8.7. *The functions $\log |a_g|$ and $\log^+ |b_g|$ are ν-integrable and $\mathbb{E}(\log |a_g|) < 0$.*

Proof. For the spherical metric, the derivative of $\hat{g}$ at ∞ in $\hat{C}$ is $1/a_g$. The computations of Lemma 8.6 show that the derivative of g acting on X in the direction of the branch of C at $\pi(\infty)$ is $1/a_g^q$ for some $q \geqslant 1$. So (M) implies that $\mathbb{E}(|\log|a_g||) < \infty$. Since ν is uniformly expanding, this direction is repelling on average: by Lemma 8.6, we get $\mathbb{E}(\log|a_j^{-1}|) > 0$. To estimate $|b_g|$, we note that $\mathrm{dist}_X(\pi(s), p) \asymp |s|^{-q}$ when $s \in \mathbf{C}$ approaches ∞. Changing the affine coordinate s if necessary, we may assume that $\pi(0) \neq p$. We get

$$\frac{1}{|b_g|^q} \asymp \mathrm{dist}_X(\pi(\hat{g}(0)), \pi(\infty)) = \mathrm{dist}_X(g(\pi(0)), g(p)) \leqslant \|g\|_{C^1} \mathrm{dist}_X(\pi(0), p). \quad (8.3)$$

From this and (M) it follows that $\mathbb{E}(\log^+|b_g|) < \infty$. □

The integrability provided by Lemma 8.7 now allows us to construct a stationary measure with full mass in the affine chart $\mathbf{C} \subset C$ with non-positive Lyapunov exponent (relative to the affine metric). This is classical, we briefly recall the argument for completeness (see [11]). For $\omega = (g_n)_{n\geqslant 0}$, write $g_n(s) = a_n s + b_n$, and consider the sequence of right products $r_n(\omega) = g_0 \cdots g_{n-1}$. One easily checks that

$$r_n(\omega)(s) = a_0 \cdots a_{n-1} s + \sum_{j=0}^{n-1} a_0 \cdots a_{j-1} b_j. \quad (8.4)$$

For $\nu^{\mathbf{N}}$-almost every ω, $\frac{1}{n}\log|a_0 \cdots a_{n-1}|$ converges to $\lambda := \mathbb{E}(\log|a_g|) < 0$. Fix $\varepsilon < |\lambda|$. Since $\mathbb{E}(\log^+|b_g|) < \infty$, $\sum_{j=0}^{\infty} \nu\{|b_g| > e^{\varepsilon j}\} < \infty$. By the Borel-Cantelli Lemma, $|b_j| \leqslant e^{\varepsilon j}$ for $\nu^{\mathbf{N}}$-almost every ω and for large j; hence, the series on the right hand side of (8.4) converges. It follows that $r_n(\omega)(s)$ converges almost surely to a limit e_ω that does not depend on $s \in \mathbf{C}$. The distribution of e_ω is the desired stationary measure $\mu_{\mathbf{C}}$. If μ is any stationary measure with $\mu(\mathbf{C}) = 1$, then $r_n(\omega)_\star\mu$ converges to δ_{e_ω} almost surely: this shows that $\mu_{\mathbf{C}}$ is the unique stationary measure with $\mu(\mathbf{C}) = 1$; in particular, $\mu_{\mathbf{C}}$ is ergodic. Since the affine derivative of g is the constant a_g, the Lyapunov exponent of $\mu_{\mathbf{C}}$, relative to the affine metric, is equal to λ.

To conclude the proof, note that $\mu := \pi_\star(\mu_{\mathbf{C}})$ is an ergodic ν-stationary measure on X which has a well-defined Lyapunov exponent, thanks to the moment condition (M). If μ gives positive mass to the singular set of C, then it must be concentrated on a single singular point of C (and likewise $\mu_{\mathbf{C}}$ is a single atom in $\hat{C}$). By Lemma 8.6 the corresponding branch is attracting on average, which contradicts uniform expansion. Therefore μ gives no mass to $\mathrm{Sing}(C)$, and we claim that its Lyapunov exponent $\lambda(\mu)|_{TC}$ in the direction of C equals λ (even if the ratio between the ambient and affine metrics on $\mathbf{C} \subset C$ is unbounded). Indeed, for $\mu \times \nu^{\mathbf{N}}$-almost every (x, ω) and $v \in T_x^1 C$, we can fix a subsequence n_j such that $f_\omega^{n_j}(x)$ is far from the singularities of C (hence from $p = \pi(\infty)$). If j is large, $\frac{1}{n_j}\log\|D_x f_\omega^{n_j}\|$ is both close to λ and to $\lambda(\pi_\star\mu)|_{TC}$. We conclude that $\lambda(\pi_\star\mu)|_{TC} < 0$, which again is contradictory. The proof is complete. □

8.1.3. *Zariski diffuse measures.* From now on we focus on the case of a minimal Kähler surface X of Kodaira dimension zero, that is, a torus, a K3 surface, or an Enriques surface. In this case $\mathsf{Aut}(X)$ preserves a canonical volume form vol_X (see Example 1.5).

From Corollary 8.2, the obstruction to uniform expansion is the existence of a non-expanding stationary measure μ. Moreover, in the first case of Definition 8.1, both exponents must vanish because we are in a volume preserving setting. In this situation, Theorems 7.2 and 7.3 give a precise description of μ.

Theorem 8.8. *Let X be a torus, a K3 surface, or an Enriques surface. Let ν be a probability measure on* $\mathsf{Aut}(X)$ *satisfying* (M) *such that Γ_ν is non-elementary. If μ is a Zariski diffuse ν-stationary measure, the following properties are equivalent*

(a) μ is non-expanding;
(b) the fiber entropy $h_\mu(X, \nu)$ vanishes.

Moreover under these assumptions, μ is invariant and $h_\mu(f) = 0$ for every $f \in \Gamma_\nu$.

Proof. As a preliminary step, observe that almost every ergodic component of μ is Zariski diffuse: this follows from the fact that there are only finitely many invariant curves and countably many isolated periodic points. In addition, by convexity of the entropy (see [40, Prop. 4.3.16]), if $h_\mu(X, \nu) = 0$ then almost every ergodic component of μ has zero fiber entropy as well. Thus for both implications we may further assume that μ is ergodic as a stationary measure.

Let us show that Property (a) implies Property (b). Since there is an invariant volume form, either both Lyapunov exponents of μ vanish or μ is hyperbolic. In the first case, the invariance principle guarantees that μ is Γ_ν-invariant and the fibered version of the Ruelle inequality (see e.g. [21, §7]) implies that its fiber entropy vanishes. If μ is hyperbolic, the invariance of μ and the vanishing of the entropy follow from [21, Thm. 9.1]. Thus, (a) implies (b) together with the invariance of μ.

Consider the converse implication. Again, if μ has zero Lyapunov exponents then it is non-expanding and invariant. Otherwise it is hyperbolic and by applying the whole argument of [13] in the complex case (see Remark 8.10 below for a few comments on this generalization), we infer that if the stable directions of μ depend on the itinerary, its conditionals along Pesin unstable manifolds admit a non-trivial translation invariance; in particular they are non-atomic. It follows that $h_\mu(X, \nu) > 0$ (see also [21, Rmk 9.2]). So under assumption (b) the stable directions are non-random and, as already explained, μ is invariant by [21, Thm. 9.1].

The fact that $h_\mu(f) = 0$ for all $f \in \Gamma_\nu$ will be shown in Theorem A.1. □

8.1.4. *Refined criterion.* The discussion of the previous paragraphs leads to a version of Theorem 1.6 that does not require Γ_ν to contain parabolic elements:

Theorem 8.9. *Let X be a compact Kähler surface which is not rational. Let ν be a probability measure on* $\mathsf{Aut}(X)$ *satisfying* (M) *such that Γ_ν is non-elementary. Then ν is uniformly expanding if and only if the three following conditions hold:*

(1) *every finite Γ_ν-orbit is uniformly expanding;*
(2) *there is no Γ_ν-invariant algebraic curve;*
(3) *there is no Zariski diffuse invariant measure μ with zero fiber entropy.*

Proof. If a compact Kähler surface X is ruled (over a curve of positive genus) or has a positive Kodaira dimension, then $\mathsf{Aut}(X)$ is elementary (in the first case, it preserves the ruling; in the second case, it preserves the Kodaira-Iitaka fibration, acting as a finite group on the base). Thus, the Kodaira dimension of X vanishes. If X is not minimal, the uniqueness of the minimal model shows that there is a $\mathsf{Aut}(X)$-invariant curve, and we know this is incompatible with uniform expansion (Proposition 8.3). Now if $\mathsf{kod}(X) = 0$, X is minimal, and $\mathsf{Aut}(X)$ is non-elementary, then X is a torus, a K3 surface, or an Enriques surface; hence, we can assume that X is such a surface.

If ν is uniformly expanding, Property (1) is obvious, Property (2) follows from Proposition 8.3, and Property (3) follows from Corollary 8.2 and Theorem 8.8.

Conversely, if these properties hold, and if μ is an ergodic ν-stationary measure then by Property (2) μ is either Zariski diffuse or finitely supported. Then, Theorem 8.8 and Property (1) imply that μ is not non-expanding, and we conclude with Corollary 8.2. □

Proof of Theorem 1.6. This follows directly from Theorem 1.4 and Theorem 8.9. □

Remark 8.10. The proof of (b)⇒(a) in Theorem 8.8 relies on the following fact: *for a hyperbolic stationary measure, if the stable directions of μ depend on the itinerary, then its unstable conditionals satisfy some non-trivial translation invariance*. This is the "easy part" of the adaptation of [13] to complex surfaces; the "difficult part" would be to obtain stiffness and some SRB property from this invariance (either on X or on some totally real surface associated to the stationary measure). We did not provide a proof for this fact because the arguments of [13] can be applied directly. This program was recently carried out by Roda and will appear shortly. As a consequence, this fact is also used in the implication "uniformly expanding implies (3)" in Theorem 8.9. On the other hand, it is not used in Theorem 1.6 because in this case the condition (3) of Theorem 8.9 is automatically satisfied, thanks to Theorem 1.4; it is not used either for the part of Theorem 8.9 asserting that the assumptions (1), (2) and (3) imply uniform expansion.

Remark 8.11. Using Theorem 7.4 instead of Theorem 1.4 gives a version of Theorem 1.6 where the existence of a parabolic element in Γ is replaced by the existence of a Kummer element. The details of the adaptation are left to the interested reader.

8.2. **Uniform expansion along finite orbits.** Using classical results on random products of matrices, it is easy to characterize when a fixed point under Γ_ν is uniformly expanding. We say that a subgroup of $\mathsf{GL}_2(\mathbf{C})$ is **strictly triangular** if it is reducible with exactly one invariant direction.

Proposition 8.12. *Let X be a torus, a K3 surface, or an Enriques surface. Let ν be a probability measure on* $\mathsf{Aut}(X)$ *satisfying* (M)*, and let x_0 be a fixed point of Γ_ν. Then ν is uniformly expanding on $T_{x_0}X$ if and only if one of the following holds*

(a) the induced action of Γ_ν on $T_{x_0}X$ is non-elementary;

(b) this action is strictly triangular and its invariant direction is expanding.

If ν is symmetric, it is uniformly expanding on $T_{x_0}X$ if and only if (a) holds.

In case (b) there exists $u \in T_{x_0}X$ such that $f_\star u = \lambda_f u$ for every $f \in \Gamma_\nu$, and the expansion means that $\int \log|\lambda_f|\, d\nu(f) > 0$.

Proof (see also [48]*).* By Lemma 2.3, to prove uniform expansion it is enough to show that for every $v \in T_{x_0}X$, $\liminf_{n\to\infty} \frac{1}{n}\log\|(f^n_\omega)_\star v\| > 0$. The proof is based on the work of Furstenberg and Kifer [36] (see also [10, §3.7]). These references deal with general random products of matrices in $\mathsf{GL}_d(\mathbf{R})$; in our volume preserving situation the Lyapunov exponents $\lambda_2 \leqslant \lambda_1$ of the random product in $\mathsf{GL}_2(\mathbf{C})$ satisfy $\lambda_1 + \lambda_2 = 0$, so they can be read off directly from the action on $\mathbb{P}T_{x_0}X$. According to Theorems 3.5 and 3.9 of [36], there are two possibilities:

(i) for every $v \in T_{x_0}X$ and $\nu^{\mathbf{N}}$-almost every ω, $\frac{1}{n}\log\|(f^n_\omega)_\star v\| \to \lambda_1$;
(ii) there exists a non-random, Γ_ν-invariant filtration $\{0\} = L_2 < L_1 < L_0 = T_{x_0}X$ and $\beta_1 < \beta_0$ such that for $i = 0, 1$ for any $v \in L_i \backslash L_{i+1}$, for $\nu^{\mathbf{N}}$-almost every ω, $\frac{1}{n}\log\|(f^n_\omega)_\star v\| \to \beta_i$. Furthermore $\beta_0 = \lambda_1$.

We now compare this dichotomy with the classification of subgroups of $\mathsf{PGL}_2(\mathbf{C})$ (with a slight abuse of notation, we also denote by Γ_ν the induced subgroup of $\mathsf{PGL}_2(\mathbf{C})$).

– If Γ_ν is strongly irreducible, we are in case (i) and there are two possibilities. If Γ_ν is proximal (hence non-elementary) then $\lambda_1 > 0$ and ν is uniformly expanding. If Γ_ν is not proximal, it is contained in a compact subgroup and ν is not uniformly expanding.

– If Γ_ν is irreducible but not strongly irreducible, we are in case (i) and there are two lines which are permuted by Γ. In some affine coordinate z on $\mathbb{P}T_{x_0}X$, Γ_ν is then conjugated to a subgroup of

$$\left\{z \mapsto \lambda z^{\varepsilon}\ ;\ \lambda \in \mathbf{C}^\times,\ \varepsilon = \pm 1\right\} \tag{8.5}$$

where $\varepsilon = -1$ with positive probability. In this case $\lambda_1 = 0$ (see e.g. [29, Prop. 5.3]), so ν is not uniformly expanding.

– If Γ_ν is reducible it preserves one or two directions in $T_{x_0}X$. If Γ preserves a direction with exponent $\leqslant 0$, then ν is not uniformly expanding. So, we can assume that Γ preserves a unique direction, and that the corresponding exponent β is positive. By (i) and (ii) we see that $\lim_{n\to\infty}\frac{1}{n}\log\|(f^n_\omega)_\star v\| \geqslant \beta$ for any $v \in T_{x_0}X$ and almost every ω; so ν is uniformly expanding.

This covers all possible cases and the proof is complete. □

Let F be a finite set, viewed as a 0-dimensional manifold, and V be a real or complex vector bundle of dimension d over F; identify V with $F \times \mathbf{K}^d$, for $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let $\mathsf{GL}(V)$ be the group of bijections of V acting linearly on fibers: it is a semidirect product $\mathsf{GL}(V) \simeq \mathfrak{S}(F) \ltimes \mathsf{GL}_d(\mathbf{K})^F$ where $\mathfrak{S}(F)$ acts on $\mathsf{GL}_d(\mathbf{K})^F$ by permuting the factors. We say that a subgroup of $\mathsf{GL}(V)$ is **strongly irreducible** if it acts transitively on F and the stabilizer of any $x \in F$ acts strongly irreducibly on the fiber $\{x\} \times \mathbf{K}^d$ of V; equivalently, if there is no invariant and finite collection of subspaces of dimension $\neq 0, d$ in some fibers of V. Similar notations and notions are defined for $\mathsf{PGL}(V)$.

Assume now that F is a finite Γ-orbit on X, and consider the induced action of Γ on $TX|_F := \bigcup_{x\in F} T_xX$. We say that this action is **non-elementary** if its image in

$\mathsf{PGL}\,(TX|_F)$ is strongly irreducible and unbounded. When Γ preserves a volume form on X, its image in $\mathsf{GL}\,(TX|_F)$ is unbounded if and only if it is unbounded in $\mathsf{PGL}\,(TX)$. We say that it is **strictly triangular** if the only proper Γ-invariant subbundle in $TX|_F$ is given by a 1-dimensional subbundle $L \subset TX|_F$.

Pick a point x in F and set $\Gamma_x = \mathrm{Stab}_\Gamma(\{x\})$. Since F is an orbit, $[\Gamma : \Gamma_x] = |F|$ and the image of Γ in $\mathsf{PGL}\,(TX|_F)$ is unbounded if and only if the image of Γ_x in $\mathsf{PGL}\,(T_xX)$ is unbounded. Thus, one easily gets the following lemma.

Lemma 8.13. *If F is a finite Γ-orbit, the action of Γ on $TX|_F$ is non-elementary (resp. strictly triangular) if and only if for some, hence any, $x \in F$ the action of $\mathrm{Stab}_\Gamma(\{x\})$ on T_xX is non-elementary (resp. strictly triangular).*

Theorem 8.14. *Let X be a torus, a K3 surface, or an Enriques surface. Let ν be a probability measure on $\mathsf{Aut}(X)$ satisfying (M$_+$), and F be a finite Γ_ν-orbit. Then ν is uniformly expanding on F if and only if the induced action of Γ_ν on TF is*

(a) *either non-elementary (in the sense of the above definition);*
(b) *or strictly triangular and the field of invariant directions $L \subset TX|_F$ is uniformly expanding.*

If ν is symmetric, it is uniformly expanding on F if and only if (a) holds.

Proof. Let Γ_F be the finite index subgroup fixing every point of F. Assume that ν is uniformly expanding. Then by Proposition 3.2, for some n_0, the induced measure $(\nu^{(n_0)})_{\Gamma_F}$ is uniformly expanding. Therefore, by Proposition 8.12, Γ_F satisfies Property (a) or (b) at every point of F, and we conclude by Lemma 8.13. Conversely, assume that (a) or (b) holds. Note that by Theorem 3.1, ν_{Γ_F} satisfies (M$_+$). By Lemma 8.13 and Proposition 8.12, ν_{Γ_F} is uniformly expanding on F, hence by Proposition 3.3, ν is uniformly expanding on F, as desired. □

This theorem shows that when ν is symmetric all conditions in Theorem 8.9 depend only on Γ_ν, and not on ν. Thus we obtain:

Corollary 8.15. *Let X be a torus, a K3 surface, or an Enriques surface. Let Γ be a non-elementary subgroup of $\mathsf{Aut}(X)$. Let ν and ν' be symmetric probability measures on $\mathsf{Aut}(X)$ satisfying (M$_+$) such that $\Gamma_\nu = \Gamma_{\nu'} = \Gamma$. Then ν is uniformly expanding if and only if ν' is uniformly expanding.*

9. Examples of uniformly expanding actions

9.1. **A finitary version of Theorem 1.6 and application to Wehler surfaces.** In [24, § §7-8], Chung uses computer assistance to prove the uniform expansion of some concrete algebraic actions on real surfaces. In our situation Theorem 1.6 can be used to check uniform expansion, but this requires a description of all invariant Zariski-closed subsets. As already explained, invariant curves can be determined by cohomological computations; for instance, if X is a generic Wehler surface, there is no $\mathsf{Aut}(X)$-invariant curve. Thus the main problem is to study finite orbits.

If the group Γ is non-elementary, contains parabolic elements, and has no invariant curve, the main result of [22] says that Γ admits only finitely many finite orbits, except when (X, Γ) is a Kummer example. However, the proof given in [22] does not provide any bound on the number or the lengths of such orbits; so, there is *a priori* no hope of numerically checking uniform expansion along all of them, nor proving that there are no finite orbits. The next result explains how to overcome this issue. To state it, we denote by $\mathrm{NS}(X; \mathbf{R})$ the Néron-Severi group of X that is, the subgroup of $H^{1,1}(X; \mathbf{R})$ obtained by all Chern classes of holomorphic line bundles on X; it coincides with the intersection of $H^{1,1}(X; \mathbf{R})$ with the torsion free part of $H^2(X; \mathbf{Z})$ (see [37], p. 163).

Theorem 9.1. *Let X be a smooth projective surface and Γ be a non-elementary subgroup of $\mathsf{Aut}(X)$ containing parabolic elements, which does not preserve any algebraic curve. Assume that we are given:*

(i) algebraic equations for X, and the formulas defining a generating subset S of Γ;
(ii) a basis of $\mathrm{NS}(X; \mathbf{R})$ and the matrices of $s^\star \colon \mathrm{NS}(X; \mathbf{R}) \to \mathrm{NS}(X; \mathbf{R})$, for s in S;
(iii) a parabolic element $g \in \Gamma$, given as a word in the generators $s \in S$, and its invariant fibration $\pi \colon X \to B$.

Then, there is an analytically computable integer $N(X, \Gamma)$ such that every finite Γ-orbit of length greater than $N(X, \Gamma)$ is uniformly expanding (in the sense of Section 1.3).

By **analytically computable**, we mean computable by a computer able to solve real analytic equations; by **algebraically computable**, we mean computable by a computer able to solve algebraic equations. The proof will provide an analytically computable subset containing all possible non-expanding finite orbits.

Example 9.2. Let g be a parabolic element of Γ, and let $h \in \Gamma$ be a conjugate of g with a distinct invariant fibration. Denote by $\mathrm{Tor}_N(g)$ the finite set of fibers of the g-invariant fibration in which g is a periodic translation of period $\leqslant N$. Then, the set of finite orbits of Γ of length $\leqslant N$ is algebraically computable since it is contained in

$$\mathrm{Tor}_N(g) \cap \mathrm{Tor}_N(h) = \{x \in X \ ; \ g^N(x) = h^N(x) = x\}. \tag{9.1}$$

A typical application of Theorem 9.1 is to the Wehler family. Recall from § 1.1 that $\mathcal{W}_0$ is the family of Wehler surfaces which are smooth and do not contain any fiber of the three natural projections $(\mathbb{P}^1)^3 \to (\mathbb{P}^1)^2$. Under these assumptions the group Γ generated by the three basic involutions σ_1, σ_2 and σ_3 is non-elementary and has no invariant curve (see [22, Prop. 2.2]). It turns out that in this case $N(X, \Gamma)$ is constant on a Zariski dense open subset (see Proposition 9.7 below). This leads to the following theorem, which will be proved in Section 9.1.3:

Theorem 9.3. *There is a dense Zariski open subset of $\mathcal{W}_0$ (resp. of the family $\mathcal{W}_0(\mathbf{R})$ of real Wehler surfaces), in which the action of $\Gamma = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ is uniformly expanding on X.*

9.1.1. *Preliminaries on Halphen twists.* Let us resume the discussion from § 6.1 and add a few preliminaries on Betti foliations and the non-twisting locus. Let h be a Halphen twist with associated fibration $\pi\colon X \to B$. Consider a simply connected open subset U of B° together with a section $\sigma\colon U \to X$ of π and a continuous frame for the homology of the fibers above U. For $w \in U$, one can identify the fiber X_w to $\mathbf{C}/\mathrm{Lat}(w)$ ($\sigma(w)$ corresponding to the zero of $\mathbf{C}/\mathrm{Lat}(w)$), as in § 6.1. Then, above U, there is a unique real-analytic diffeomorphism $\Psi : \pi^{-1}(U) \to U \times \mathbf{R}^2/\mathbf{Z}^2$ such that

(a) $\pi \circ \Psi = \Psi \circ \pi_U$, where π_U is the projection onto U;
(b) Ψ maps σ to the zero section $w \mapsto (w, (0, 0))$ of π_U, and maps the basis of $H_1(X_w; \mathbf{Z})$ to the standard basis of $H_1(\mathbf{R}^2/\mathbf{Z}^2; \mathbf{Z}) = \mathbf{Z}^2$;
(c) on each fiber, Ψ is a real analytic isomorphism of real Lie group.

Above U, the **Betti foliation** is the foliation by submanifolds of the form $\Psi^{-1}(U \times \{(x, y)\})$; these leaves are local holomorphic sections of π, with σ corresponding to $\Psi^{-1}(U \times \{(0, 0)\})$. Conjugating by Ψ, we get

$$\Psi \circ h \circ \Psi^{-1}\colon (w, (x, y)) \mapsto (w, (x, y) + T(w)), \tag{9.2}$$

where $T\colon U \to \mathbf{R}^2/\mathbf{Z}^2$ is real analytic. By [20, Lem. 3.9], the map T is an (orientation preserving) branched covering, so it behaves topologically like $w \mapsto w^k$. In U, the **non-twisting locus** NT_h is the set $\{w \in U \ ; \ D_wT = 0\}$; equivalently, $\mathrm{NT}_h \cap U = \pi(\{t_1, \ldots, t_q\})$, where $\{t_1, \ldots, t_q\}$ is the set of tangencies between the Betti foliation and the section $h \circ \sigma$. These definitions do not depend on the above choices and NT_h can indeed be defined globally on B°. A key fact is that NT_h is a finite subset of B° (see [20, Prop. 3.14] or [28, Cor. 7.7.10]). We denote by $|\mathrm{NT}_h|$ its cardinality, and by $\mathrm{mult}(\mathrm{NT}_h)$ its cardinality counted with multiplicity, that is, taking into account the degree of the local branched covering T.

Note that, once h and π are given, *the set* $\mathrm{NT}_h \subset B^\circ$ *is analytically computable*: one has to compute the periods of X_w to get $\mathrm{Lat}(w)$, then Ψ is $\mathbf{R}$-linear from $\mathbf{C}/\mathrm{Lat}(w)$ to $\mathbf{R}^2/\mathbf{Z}^2$, and T is then obtain from h by conjugacy.

9.1.2. *Proof of Theorem 9.1.* As in [20], for $(g, h) \in \mathrm{Hal}(\Gamma)^2$ we set

$$\mathrm{STang}(\pi_g, \pi_h) = \mathrm{Sing}(\pi_g) \cup \mathrm{Sing}(\pi_h) \cup \mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h), \tag{9.3}$$

where $\mathrm{Sing}(\pi_g)$ is the union of all singular and multiple fibers, and $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h)$ is the part of the tangency locus of π_g and π_h which is not contained in $\mathrm{Sing}(\pi_g) \cup \mathrm{Sing}(\pi_h)$. Put $\mathrm{NT}_g^X = \pi_g^{-1}(\mathrm{NT}_g)$ (so that NT_g^X is a curve in X) and likewise $\mathrm{NT}_h^X = \pi_h^{-1}(\mathrm{NT}_h)$.

Lemma 9.4. *Let g, h be a pair of Halphen twists in Γ with distinct invariant fibrations, and let $x \in X$ be a point with a finite Γ-orbit. If this orbit is not uniformly expanding, then it is contained in* $\mathrm{STang}(\pi_g, \pi_h) \cup \mathrm{NT}_g^X \cup \mathrm{NT}_h^X$.

Proof. We argue by contraposition: replacing x by another point in its orbit if necessary, we assume that $x \notin \mathrm{STang}(\pi_g, \pi_h) \cup \mathrm{NT}_g^X \cup \mathrm{NT}_h^X$, and we want to show that its orbit is uniformly expanding. Since $\Gamma(x)$ is finite, there are positive integers k and ℓ such that g^k and h^ℓ are in $\mathrm{Stab}_\Gamma(x)$. By definition of the non-twisting locus, g^k and h^ℓ

induce parabolic homographies on $\mathbb{P}(T_xX)$; and since $x \notin \mathrm{Tang}^{tt}(\pi_g, \pi_h)$, the fixed points of these homographies are distinct; thus, the action of $\langle g^k, h^\ell \rangle$ on $\mathbb{P}(T_xX)$ is non-elementary. By Proposition 8.12 and Lemma 8.13, the orbit of x is uniformly expanding. □

The intersection number of $\mathrm{NT}_g^X \cup \mathrm{STang}(\pi_g, \pi_h)$ (resp. $\mathrm{NT}_h^X \cup \mathrm{STang}(\pi_g, \pi_h)$) with a smooth fiber X_w^h (resp. X_w^g) does not depend on the fiber. Let $n_0(g,h)$ be the maximum of these intersection numbers:

$$n_0 = \max\{\, [\mathrm{NT}_g^X \cup \mathrm{STang}(\pi_g, \pi_h)] \cdot [X_w^h] \,;\, [\mathrm{NT}_h^X \cup \mathrm{STang}(\pi_g, \pi_h] \cdot [X_w^g])\}. \quad (9.4)$$

The set $\mathrm{STang}(\pi_g, \pi_h)$ can be computed algebraically, thus

$$n_0(g,h) \leqslant A(g,h) \max(|\mathrm{NT}_g| \,;\, |\mathrm{NT}_h|) + B(g,h) \quad (9.5)$$

where $A(g,h)$ and $B(g,h)$ can be computed algebraically (by computing the tangency loci and intersection numbers). Then, we set

$$n(g,h) = n_0(g,h)! \quad (9.6)$$

Lemma 9.5. *Let g, h be a pair of Halphen twists in Γ with distinct invariant fibrations. Let $x \in X$ be such that $\Gamma(x)$ is finite and not uniformly expanding. Then $\Gamma(x)$ is contained in*

$$\mathrm{STang}(\pi_g, \pi_h) \cup (\mathrm{NT}_g^X \cap \mathrm{NT}_h^X) \cup (\mathrm{NT}_g^X \cap \mathrm{NT}_{h^n g h^{-n}}^X) \cup (\mathrm{NT}_h^X \cap \mathrm{NT}_{g^n h g^{-n}}^X),$$

where $n = n(g,h)$ is defined by (9.6).

Proof. The statement of the lemma concerns the orbit $\Gamma(x)$, but we only have to prove it for x itself. If $x \in \mathrm{STang}(\pi_g, \pi_h) \cup (\mathrm{NT}_g^X \cap \mathrm{NT}_h^X)$ we are done. Otherwise by Lemma 9.4, x belongs to $\mathrm{NT}_g^X \setminus \mathrm{NT}_h^X$ or $\mathrm{NT}_h^X \setminus \mathrm{NT}_g^X$. Assume that $x \in \mathrm{NT}_g^X \setminus \mathrm{NT}_h^X$. The h-orbit of x is finite and by Lemma 9.4 again, for every q, $h^q(x)$ is contained in $X_x^h \cap (\mathrm{NT}_g^X \cup \mathrm{STang}(\pi_g, \pi_h))$ (here we abuse notation and write X_x^h for $X_{\pi_h(x)}^h$). Thus, $h^n(x) = x$, where $n = n(g,h)$. Set $f = h^n g h^{-n}$. The fiber X_x^f associated to f through x is $h^n(X_x^g)$, and since $x \notin \mathrm{NT}_h^X$, X_x^f is transverse to X_x^g at x, as well as to X_x^h. Moreover, x belongs to NT_f^X, because x belongs to NT_g^X and $h^n(x) = x$. Hence $x \in \mathrm{NT}_g^X \cap \mathrm{NT}_{h^n g h^{-n}}^X$. Doing the same in the case where $x \in \mathrm{NT}_h^X \setminus \mathrm{NT}_g^X$ completes the proof. □

The set NT_g is analytically computable (by § 9.1.1), and $\mathrm{Crit}(\pi_g)$ is algebraically computable. Similarly, if h is in $\mathrm{Hal}(\Gamma)$, $\mathrm{Tang}^{tt}(\pi_g, \pi_h) \cap \mathrm{NT}_f^X$ is analytically computable. The previous lemma shows that all non uniformly expanding finite orbits are contained in

$$\mathrm{Bad}(g,h) := \mathrm{STang}(\pi_g, \pi_h) \cup (\mathrm{NT}_g^X \cap \mathrm{NT}_h^X) \cup (\mathrm{NT}_g^X \cap \mathrm{NT}_{h^n g h^{-n}}^X) \cup (\mathrm{NT}_h^X \cap \mathrm{NT}_{g^n h g^{-n}}^X)$$

for every pair $(g,h) \in \mathrm{Hal}(\Gamma)^2$ with distinct invariant fibrations, where $n = n(g,h)$ as in Equation (9.6). Intersecting these sets for various choices of (g,h), we expect to get a finite analytically computable set. Observe that $\mathrm{Bad}(g,h)$ is the union of $\mathrm{STang}(\pi_g, \pi_h)$

and a finite set, because $\mathrm{NT}_g^X \cap \mathrm{NT}_h^X$ is finite when π_g and π_h are distinct. So, what remains to do is to *exhibit an explicit finite set of pairs* (g, h) *such that the intersection of the* $\mathrm{STang}(\pi_g, \pi_h)$ *is finite*. We first treat the case of Wehler surfaces, which is sufficient to proceed with Theorem 9.3.

Conclusion of the proof of Theorem 9.1 in the Wehler case. Fix a Wehler surface $X \in \mathcal{W}_0$ and consider the three pairs (g_1, g_2), (g_2, g_3), (g_3, g_1), where $g_1 = \sigma_2 \circ \sigma_3$, $g_2 = \sigma_3 \circ \sigma_1$ and $g_3 = \sigma_1 \circ \sigma_2$. Note that the g_i-invariant fibration is the i-th projection π_i.

Assume that the intersection of the divisors $\mathrm{STang}(\pi_i, \pi_j)$ contains an irreducible curve $D \subset X$. If D is contained in $\mathrm{Sing}(\pi_i) \cap \mathrm{Sing}(\pi_j)$ with $i \neq j$, then (π_i, π_j) maps D onto a point and this contradicts the fact that $X \in \mathcal{W}_0$. If D is contained in, say, $\mathrm{Tang}^{tt}(\pi_1, \pi_2)$ and $\mathrm{Tang}^{tt}(\pi_2, \pi_3)$, the three fibrations are pairwise tangent along D, and we obtain a contradiction because there is no tangent vector $v \neq 0$ to $(\mathbb{P}^1)^3$ which is mapped to 0 by each $D\pi_i$. The last possibility is that D is contained in, say, $\mathrm{Tang}^{tt}(\pi_1, \pi_2)$ and $\mathrm{Sing}(\pi_3)$. In this case, there is a point p on D at which $D_p\pi_3 \colon T_pX \to T_{\pi_3(p)}\mathbb{P}^1$ is equal to 0, and at such a point, the same contradiction applies. This shows that

$$F(g_1, g_2, g_3) := \mathrm{Bad}(g_1, g_2) \cap \mathrm{Bad}(g_2, g_3) \cap \mathrm{Bad}(g_3, g_1) \tag{9.7}$$

is finite, with an analytically computable cardinality, and the proof is complete. □

Remark 9.6. The above proof provides a computation of the integer $N(X, \Gamma)$ involving:

(1) algebraic quantities that are constant on $\mathcal{W}_0$, like $\mathrm{STang}(\pi_i, \pi_j) \cdot \mathrm{STang}(\pi_j, \pi_k)$,
(2) $|\mathrm{NT}_{g_i}|$ for $i = 1, 2, 3$.

Therefore, if $|\mathrm{NT}_{g_i}| \leqslant B$*, then* $N(X, \Gamma) \leqslant N(B)$ *for some* $N(B)$ *depending only on* B.

Indeed, the number n in Lemma 9.5 depends only on n_0 (see Equations (9.6)) and by Equation (9.4)) n_0 is bounded by a function of B. Then, because the norm of $(g_i^{n_0})^*\colon \mathrm{NS}(X) \to \mathrm{NS}(X)$ is bounded by Cn_0^2 for some uniform constant C, we obtain $\mathrm{NT}_{g_i}^X \cap \mathrm{NT}^X_{g_j^n g_i g_j^{-n}} \leqslant C' n_0^2 B^2$ for some constant C' and the result follows. □

Conclusion of the proof of Theorem 9.1 in the general case. By assumption, Γ is non-elementary and has no invariant curve. Let Γ^* be its image in $\mathsf{GL}(\mathrm{NS}(X; \mathbf{Z}))$.

If g is the parabolic element given in assumption (iii) of the theorem, up to sign, there is a unique integral primitive class $c(g) \in \mathrm{NS}(X)$ such that $g^\star c(g) = c(g)$ and $c(g) \cdot c(g) = 0$. By the assumptions (ii) and (iii), this class can be computed explicitly. An element f of $\mathsf{Aut}(X)$ preserves the g-invariant fibration π (permuting its fibers) if and only if it fixes $c(g)$. Since Γ is non-elementary, it is not contained in the stabilizer of $c(g)$. Thus, according to Proposition 3.2 of [33], there is a computable integer N, and a composition f of length N in the generators $s \in S$ that does not preserve $c(g)$. Then, $h := f \circ g \circ f^{-1}$ is a parabolic element of Γ with invariant fibration $\pi \circ f \neq \pi$.

Since g and its invariant fibration π, as well as f, are explicit, we can compute the degree of the subvariety $\mathrm{STang}(\pi_g, \pi_h)$ (for the embedding $X \subset \mathbb{P}^m(\mathbf{C})$ given by assumption (i)). Denote by $(C_i)_{i \in I}$ the irreducible components of $\mathrm{STang}(\pi_g, \pi_h)$; we have $|I| \leqslant \deg(\mathrm{STang}(\pi_g, \pi_h))$.

Suppose that for each C_i, one can exhibit some $f_i \in \Gamma$ for which $f_i(C_i) \not\subset \mathrm{STang}(\pi_g, \pi_h)$. Then the set of pairs

$$\{(g,h)\} \cup \left\{(f_i g f_i^{-1}, f_i h f_i^{-1}) \ ; \ \ i \in I\right\} \tag{9.8}$$

satisfies $|\bigcap_i \mathrm{STang}(\pi_{g_i}, \pi_{h_i})| < +\infty$, and we are done because the cardinality of this finite set is algebraically computable. So, we now fix such an irreducible component C_i, and we construct such an f_i.

First, assume that C_i is an irreducible component of $\mathrm{Tang}^{\mathrm{tt}}(\pi_g, \pi_h)$. Then C_i is generically transverse to π, hence $\deg(g^n(C_i))$ tends to infinity. We can thus set $f_i = g^{n_i}$ for some large enough n_i (explicitely computable from the action on $\mathrm{NS}(X)$).

The second case is when C_i is an irreducible component of $\mathrm{Sing}(\pi_g) \cup \mathrm{Sing}(\pi_h)$; in particular, its self-intersection C_i^2 is $\leqslant 0$. By [22, Thm D], there exists a loxodromic element $f_0 \in \Gamma$ without invariant curve; in particular $f_0^{|I|!}(C_i) \neq C_i$. Since f_0 is loxodromic, this inequation is equivalent to $(f_0^{|I|!})_*[C_i] \neq [C_i]$. Indeed, either $C_i^2 = 0$ and we readily get a contradiction since a loxodromic element does not fix any non-zero isotropic class, or $C_i^2 < 0$ and this follows from $f_0^{|I|!}(C_i) \neq C_i$ since $[C_i]$ determines C_i when the self-intersection is negative. Thus, if we set

$$W_i := \left\{ f \in \mathsf{GL}(\mathrm{NS}(X, \mathbf{R})) : f_*^{|I|!}[C_i] = [C_i] \right\}, \tag{9.9}$$

we see that Γ^* is not contained in W_i. Proposition 3.2 of [33] then provides a computable element $f \in \Gamma$ such that $f \notin W_i$. Now, if $f^q(C_i)$ were contained in $\mathrm{STang}(\pi_g, \pi_h)$ for $0 \leqslant q \leqslant |I|$, we would find two integers $q_1 < q_2 \leqslant |I|$ such that $f^{q_2 - q_1}(C_i) = C_i$; in particular, $f^{|I|!}(C_i)$ would be equal to C_i, a contradiction. Thus, there is an iterate $f_i := f^{q_i}$, with $q_i \leqslant |I|$, such that $f_i(C_i) \not\subset \mathrm{STang}(\pi_g, \pi_h)$, and the proof is complete. □

9.1.3. *Proof of Theorem 9.3.* Recall that, for Wehler surfaces, $\Gamma = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$.

Proposition 9.7. *There exists an analytically computable integer N such that for any Wehler surface $X \in \mathcal{W}_0$, any finite Γ-orbit of length $> N$ is non-elementary (hence uniformly expanding by Theorem 8.14).*

This uniform bound is the main step towards Theorem 9.3. In view of Remark 9.6, this proposition follows from Theorem 9.1 and the following uniformity result.

Proposition 9.8. *For any $g \in \{g_1, g_2, g_3\}$, the cardinality of NT_g is uniformly bounded in $\mathcal{W}_0$.*

Let $\mathcal{X} \subset \mathcal{W}_0 \times (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ be the universal family of Wehler surfaces, as in [22, §2]. As X varies in $\mathcal{W}_0$, the automorphisms g_i and their invariant fibrations π_i depend on X, but for notational simplicity we drop the dependence in X.

From now on, we fix $g \in \{g_1, g_2, g_3\}$; its invariant fibration $\pi\colon X \to \mathbb{P}^1$ is the restriction of one of the projections π_i to X; its base does not depend on X.

Lemma 9.9. *Let $X_0 \in \mathcal{W}_0$, $w_0 \in \mathrm{NT}_g$, and k be the multiplicity of w_0 in NT_g. Let $U \subset \mathbb{P}^1$ be a topological disk such that $\overline{U} \cap \mathrm{NT}_g = \{w_0\}$ and $\overline{U} \cap \mathrm{Crit}(\pi) = \varnothing$.*

Then, there exists a neighborhood V of X_0 in $\mathcal{W}_0$ such that for any X in V, the total multiplicity of NT_g *in U is equal to k.*

Proof. Fix an open connected neighborhood V of X_0 such that for X in V,

– U does not intersect any of the sets $\mathrm{Crit}(\pi)$;
– there is a section $w \mapsto \varsigma_X(w)$ of $\mathcal{X} \to \mathcal{W}_0 \times \mathbb{P}^1$ above $V \times U$, together with a continuous choice of basis for the homology of the fibers of π above U.

Then the sections, the Betti foliations (above U), and their lifts to $U \times \mathbf{C}$ all depend continuously on X in V. In particular, we can find a disk $U' \subset \mathbb{P}^1$, with $w_0 \in U' \Subset U$, whose boundary is a smooth Jordan curve γ, and such that for any $X \in V$, the Betti foliation is transverse to $g \circ \varsigma_X$ above γ. In particular, NT_g is disjoint from γ.

Now, recall that the map T defined in Equation (9.2) behaves topologically like $w \mapsto w^{k+1}$; in such a local coordinate, $k+1$ is the winding number of the curve $T \circ \gamma$ around $T(0)$. Since NT_g stays disjoint from γ for X in V, this winding number is constant in V; thus, the number of points of NT_g enclosed by γ (counted with multiplicity) stays constant on V. The lemma follows. □

Lemma 9.10. *There exists a proper semi-algebraic subvariety $\mathcal{Z}_g \subset \mathcal{W}_0$ of positive codimension such that* $\mathrm{mult}(\mathrm{NT}_g)$ *is locally constant in $\mathcal{W}_0 \backslash \mathcal{Z}_g$.*

Proof.
Step 1: *Keeping away from the singular fibers.* Fix $X_0 \in \mathcal{W}_0$ and $w_0 \in \mathbb{P}^1$ a critical value of π. It is shown in [20, Lem. 3.11] that NT_g does not accumulate w_0. Here, we show that outside a semi-algebraic subvariety $\mathcal{Z}_g \subset \mathcal{W}_0$ this non-accumulation holds uniformly with respect to X: we shall construct a neighborhood $V \times U$ of (X_0, w_0) such that U is disjoint from NT_g for every X in V. For this, we review the proof of [20, Lem. 3.11] and make it locally uniform in X under appropriate hypotheses on X_0.

Define $\mathcal{W}_1$ to be the dense, Zariski open subset ([8]) of $\mathcal{W}_1$ such that for any $X \in \mathcal{W}_1$ and any $i \in \{1, 2, 3\}$, all singular fibers of π_i are of type I_1. In this case there are 24 such fibers (the Euler characteristic of a K3 surface is 24, the contribution to the Euler characteristic of a smooth fiber is 0, and the contribution of an I_1 fiber is 1). Suppose that $X_0 \in \mathcal{W}_1$.

Fix a small disk $U \subset \mathbb{P}^1$ centered at w_0 and containing no other singular value of $\pi \colon X_0 \to \mathbb{P}^1$. Fix a neighborhood V of X_0 in $\mathcal{W}_1$, and local coordinates on U (depending on X), so that (i) this property persists for $X \in V$ and (ii) the unique singular value

[8]To show that $\mathcal{W}_1$ is dense, we only have to show that it is non-empty. This is a consequence of the following fact. *Let X be in $\mathcal{W}_0$, let $\pi_1 \colon X \to \mathbb{P}^1$ be the first projection, and let m be a critical point of π_1. Let F be the fiber of π_1 containing m. Then, each of the conditions*

(1) the singularity of F at m is degenerate (in the sense of Morse, i.e. it is not a A_1-singularity);
(2) F contains a second singular point m'

defines a proper subset of $\mathcal{W}_0$. In other words, these properties (1) and (2) disappear after a generic small perturbation of X in $\mathcal{W}_0$, which can be checked directly.

of π in U is $w_0 = 0$. Let $X_U^{\#}$ be the complement in $X_U^g := \pi^{-1}(U)$ of the unique singular point of $X_{w_0}^g$. We fix a reference section $\varsigma_X : U \to X_U^{\#}$ depending holomorphically on $X \in V$ and $w \in U$.

For $X \in V$ and $w \in U \backslash \{w_0\}$ we can write $X_w^g \simeq \mathbf{C}/\mathbf{Z} \oplus \mathbf{Z}\tau_X(w)$, as in § 6.1. Since the singular fiber $X_{w_0}^g$ is of type I_1 and $w_0 = 0$, the monodromy along a simple loop around 0 maps the basis $(1, \tau_X(w))$ to $(1, \tau_X(w) + 1)$. Moreover, $X_U^{\#}$ is biholomorphic to the quotient of $U \times \mathbf{C}$ by the family of lattices $\mathbf{Z} \oplus \mathbf{Z}\tau_X(w)$, where $\tau_X(w) = \frac{1}{2\mathrm{i}\pi}\log(k_X(w))$ for a function $k_X : U \to \mathbf{C}$ which has a single zero at the origin and depends holomorphically on $X \in V$ and $w \in U$. Since $g \circ \varsigma_X$ is another section of π above U, there is a holomorphic function $t_X(w)$ of X and w such that the lift of g to $U \times \mathbf{C}$ is given by $(w, z) \mapsto (w, z + t_X(w))$. The calculations of [20] (see §3.3.2 and Lemma 3.11 there) show that the equation for NT_g in U is

$$-\mathrm{i}\log(|k_X(w)|)k_X(w)\, t'_X(w) = k'_X(w)\mathrm{Im}(t_X(w)). \tag{9.10}$$

We claim that if $\mathrm{Im}(t_X(0)) \neq 0$, then by reducing V and U if necessary, $\mathrm{NT}_g \cap X_U^g = \varnothing$. Indeed if U is small enough, there exist positive constants ε, c such that for any $X \in V$,

$$|k_X(w)| \leqslant \varepsilon,\ \ |k'_X(w)| \geqslant c,\ \ |\mathrm{Im}(t_X(w))| \geqslant c,\ \text{ and }\ |t'_X(w)| \leqslant c^{-1}. \tag{9.11}$$

Reducing U further, ε can be chosen arbitrary small while c remains bounded away from 0. If $\varepsilon \log \varepsilon < c^3$, this is not compatible with the equality (9.10), so $\mathrm{NT}_g \cap U = \varnothing$.

Lemma 9.11. *The locus*

$$\{X \in V :\ \mathrm{Im}(t_X(w_0)) = 0\} \tag{9.12}$$

is a semi-algebraic subset of positive codimension.

Proof of Lemma 9.11. Consider the Wehler surfaces $X \subset V \subset \mathcal{W}_1$, and their equations

$$A_{222}x^2y^2z^2 + A_{221}x^2y^2z + \cdots + A_{100}x + A_{010}y + A_{001}z + A_{000} = 0. \tag{9.13}$$

Permuting coordinates if necessary, we suppose that $\pi\colon X \to \mathbb{P}^1$ is the projection onto the first coordinate. As X varies near X_0, the critical value of π near w_0 and the corresponding critical point in X can be computed algebraically in terms of the A_{ijk}. Using the action of $\mathsf{PGL}(2, \mathbf{C})^3$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, we may assume that $w_0 = 0$ (as above) and the unique singular point of the fiber $X_{w_0}^g := X \cap \{x = 0\}$ is $(0, 0)$. So, the equation of $X_{w_0}^g$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is

$$ay^2z^2 + by^2z + cyz^2 + dyz + ey^2 + fz^2 = 0, \tag{9.14}$$

for some coefficients $a, \ldots, f$ given by algebraic expressions in the A_{ijk}. Since $X \in \mathcal{W}_1$, $X_{w_0}^g$ has two transverse branches at $(0, 0)$: their tangent directions are given by the solutions of $dyz + ey^2 + fz^2 = 0$ in $\mathbb{P}^1$.

One can also write $X_0^g \backslash \{0, 0\}$ as the quotient of $\{0\} \times \mathbf{C}$ by the lattice $\mathrm{Lat}(0) = \mathbf{Z}$; in this coordinate, g acts as multiplication by $\exp(2\mathrm{i}\pi t_X(0))$. Thus, $\mathrm{Im}(t_X(0)) = 0$ means that g induces a rotation, instead of a loxodromic homography, on the rational curve X_0^g. Writing down $g = \sigma_y \circ \sigma_z$ in coordinates, we obtain

$$g(y,z) = \begin{pmatrix} -1 - \frac{d^2}{ef} & \frac{d}{e} \\ -\frac{d}{f} & -1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + O\left(\|(y,z)\|^2\right) \tag{9.15}$$

for $(y,z) \in X_0^g$. Thus, $D_{(0,0)}g \in \mathsf{GL}(T_{0,0}X)$ has determinant 1 and trace $-2 - \frac{d^2}{ef}$. As a consequence, g acts as a rotation on X_0^g if and only if $2 + \frac{d^2}{ef} \in [-2, 2]$: this is a semi-algebraic condition. □

To conclude, we let $\mathcal{W}_g \subset \mathcal{W}_0$ be the intersection of $\mathcal{W}_1$ with the complement of the subsets defined by

$$\{X \in \mathcal{W}_1;\ \mathsf{Im}(t_X(w_i) = 0\} \tag{9.16}$$

for each of the 24 singular values w_i of π. We finally define $\mathcal{Z}_g$ to be the complement of $\mathcal{W}_g$; by Lemma 9.11, it is a proper semi-algebraic set of positive codimension.

Step 2: *Conclusion.* Pick $X_0 \in \mathcal{W}_0 \backslash \mathcal{Z}_g$ and cover $\mathbb{P}^1$ by a finite family F of topological disks, such that for every $U \in F$, $\overline{U}$ contains at most one point of $\mathrm{Crit}(\pi) \cup \mathrm{NT}_g$. If $U \in F$ contains a critical value of π (and no point of NT_g), then, as already explained, this property persists in a neighborhood of X_0. By Step 1, for X sufficiently close to X_0, $\overline{U}$ is disjoint from NT_g as well. For the remaining disks, the local constancy of $\mathrm{mult}(\mathrm{NT}_g)$ follows from Lemma 9.9. The proof is complete. □

Proof of Proposition 9.8. We use a semi-continuity argument. Since the exceptional set $\mathcal{Z}_g$ defined in Lemma 9.10 is semi-algebraic, the open set $\mathcal{W}_0 \backslash \mathcal{Z}_g$ is also semi-algebraic, so it admits finitely many connected components (see [7, Cor. 2.7] for instance). Thus, by Lemma 9.10, $\mathrm{mult}(\mathrm{NT}_g)$ and therefore $|\mathrm{NT}_g|$ are uniformly bounded on $\mathcal{W}_0 \backslash \mathcal{Z}_g$, say $|\mathrm{NT}_g| \leqslant B$. Now, pick $X_0 \in \mathcal{Z}_g$ (thus $X_0 \in \mathcal{W}_0$) and assume that for X_0 one has $|\mathrm{NT}_g| > B$. We can then consider a finite number of small topological disks U_i with disjoint closures in $\mathbb{P}^1$, such that $|\mathrm{NT}_g| \cap \bigcup U_i > B$. By Lemma 9.9, these non-twisting points persist for X close enough to X_0. Since $\mathcal{W}_0 \backslash \mathcal{Z}_g$ is dense in $\mathcal{W}_0$, this contradicts the definition of B and the proof is complete. □

Proof of Theorem 9.3. The main point of [22, Thm A] is that the set of $X \in \mathcal{W}_0$ possessing a finite orbit of length $\leqslant B$ is a proper Zariski closed subset $\mathcal{Z}_B$ of $\mathcal{W}_0$. For N as in Proposition 9.7, for any $X \in \mathcal{W}_0 \backslash \mathcal{Z}_N$, all finite orbits of Γ are uniformly expanding. We conclude by applying Theorem 1.6 (with $\nu = \frac{1}{3}(\delta_{\sigma_1} + \delta_{\sigma_2} + \delta_{\sigma_3})$). The proof of the corresponding statement in $\mathcal{W}_0(\mathbf{R})$ is identical. □

Remark 9.12. We expect that an analogue of Theorem 9.3 holds for other families with large automorphism groups containing parabolic elements, like Enriques surfaces, or the family associated to pentagon folding (see [18]).

Remark 9.13. The proof of Proposition 9.8 suggests that there should exist a notion of multiplicity, *including singular fibers*, for which $\mathrm{mult}(\mathrm{NT}_g)$ would be constant on $\mathcal{W}_0$

and would be an algebraically computable invariant of the parabolic automorphism g. A variant of this question is mentioned in [28, Rmk 7.7.4].

9.2. **Thin subgroups.** In this section we consider the total space $\mathcal{W}$ of all Wehler surfaces and the universal family $\mathcal{X} \subset \mathcal{W} \times (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$. We change a little bit the notation: Γ will be a subgroup of $\mathbf{Z}/2\mathbf{Z}*\mathbf{Z}/2\mathbf{Z}*\mathbf{Z}/2\mathbf{Z}$, and Γ_X will be the corresponding subgroup of $\mathsf{Aut}(X)$.

Let E be an elliptic curve. Consider the following classical Kummer construction (see [22, §4]): let η be the involution $\eta(x, y) \mapsto (-x, -y)$ on $A := E\times E$; the associated Kummer surface is the desingularization $\widehat{E \times E/\eta}$; the natural $\mathsf{GL}(2, \mathbf{Z})$ action on $E\times E$ descends to $E \times E/\eta$ and induces a non-elementary automorphism group of $\widehat{E \times E/\eta}$. The surface $E \times E/\eta$ can be realized as a singular Wehler example (see [16, §8.2]); in addition the action of $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ is induced by a finite index subgroup of $\mathsf{GL}(2, \mathbf{Z})$. Let us briefly recall the construction: write E in Weierstrass form $y^2 = 4x^3 - g_2x - g_3$, with the neutral element of the group law on E located at infinity. To $(m_1, m_2) \in E \times E$, $m_i = (x_i, y_i)$, we associate $m_3 = -(m_1 + m_2)$ and $\phi(m_1, m_2) = (x_1, x_2, x_3)$, where $m_3 = (x_3, y_3)$. Then, ϕ is η-invariant and determines a biregular map $\phi : E \times E/\eta \to X_E$ onto a singular Wehler surface X_E with 16 nodal singularities.

Assume that $\Gamma \subset \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ is not virtually cyclic. Then for $X \in \mathcal{W}_0$, Γ_X is non-elementary (see [21, §3]).

Theorem 9.14. *Let Γ be a subgroup of $\mathbf{Z}/2\mathbf{Z}*\mathbf{Z}/2\mathbf{Z}*\mathbf{Z}/2\mathbf{Z}$ which is not virtually cyclic. For $X \in \mathcal{W}_0$ sufficiently close to X_E, the subgroup Γ_X is uniformly expanding on X.*

Thus for every "abstract" non-elementary subgroup Γ of $\mathbf{Z}/2\mathbf{Z}*\mathbf{Z}/2\mathbf{Z}*\mathbf{Z}/2\mathbf{Z}$, the open subset $\mathcal{W}_{\rm exp}(\Gamma)$ of those $X \in \mathcal{W}_0$ for which the action of Γ_X is uniformly expanding is non-empty. The group Γ can be arbitrarily thin, in particular it is not assumed to contain parabolic elements. In view of Theorem 8.9, it is natural to expect that $\mathcal{W}_{\rm exp}(\Gamma)$ is actually dense in the Euclidean topology.

Proof. The difficulty is that we cannot directly argue that uniform expansion is an open property, because X_E is singular.

Lemma 9.15. *Fix $f \in \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$, and denote also by f the induced fibered map on the universal family of $(2, 2, 2)$-surfaces in $(\mathbb{P}^1)^3$. Then f is regular on a neighborhood of X_E.*

Proof of Lemma 9.15. Pick a $(2, 2, 2)$ surface X. If X does not contain any fiber of the projection $\pi_{12} = (\pi_1, \pi_2)\colon (\mathbb{P}^1)^3 \to (\mathbb{P}^1)^2$, then the same property holds in a neighborhood $\mathcal{V}$ of X in the universal family of $(2, 2, 2)$-surfaces; furthermore, σ_3 determines an automorphism([9]) of $\mathcal{V}$. Thus, we only have to prove that X_E does not contain

[9]Indeed, denote by $\mathcal{V}_0$ the projection of $\mathcal{V}$ in the space of Wehler surfaces $\mathcal{W}_0$ (see § 1.1), and let $o \in \mathcal{W}_0$ be the image of X. For $v \in \mathcal{V}_0$, let $X_v \subset \mathcal{V}$ be the corresponding Wehler surface. Pick a point $(x, y, z) \in X$, say in an affine chart of $(\mathbb{P}^1)^3$ where none of the coordinates is ∞. In other words, $z = [z_0 : z_1] \in \mathbb{P}^1$ with $z_1 \neq 0$ and $z = z_0/z_1$, and similarly for x and y. For simplicity, we use

any fiber of the projections π_{ij}. Let us show that X_E does not contain any vertical line $\{x = x_0, y = y_0\}$. Such a line would provide a family of points (m_1, m_2) on $E \times E$ with fixed first coordinates $x_1 = x_0$, $x_2 = y_0$, for which the first coordinate of $m_3 := -(m_1 + m_2)$ takes arbitrary values. This is impossible. The same argument applies to the lines $\{y = y_0, z = z_0\}$ and $\{z = z_0, x = x_0\}$ because the relation $m_1 + m_2 + m_3 = 0$ is symmetric (equivalently, the equation of X_E given in [16, §8.2] is symmetric in (x, y, z)). □

There is a finite index subgroup of Γ that fixes each singularity of X_E. By Proposition 3.3 and the fact that uniform expansion does not depend on the measure, we can replace Γ with this finite index subgroup, endow Γ with a finitely supported, symmetric measure ν with $\Gamma = \Gamma_\nu$, and then we have to prove that (Γ_X, ν_X) is uniformly expanding for $X \in \mathcal{W}_0$ near X_E; here, ν_X is the measure induced by ν on Γ_X.

Endow $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with the Fubini study metric, and the Wehler surfaces X with the induced metric. Recall that T^1X denotes the unit tangent bundle.

Assume, by way of contradiction, that there is a sequence $X_n \to X_E$ along which ν_{X_n} is not uniformly expanding. For each n, let ϖ denote the natural projection $T^1X_n \to X_n$ (resp. $T^1X_E \to X_E$). Denote by T^1X_E the subset of $T^1(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ which coincides with $T^1\mathrm{Reg}(X_E)$ above the regular part of X_E and coincides with $T^1_x(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ above each singularity $x \in \mathrm{Sing}(X_E)$. With this definition it is obvious that if x_k is any sequence such that $x_k \in \bigcup X_n \cup X_E$ and $x_n \to x \in X_E$ then we have the semicontinuity $\limsup T^1_{x_n} X_n \subset T^1X_E$. Theorem 5.1 provides a sequence of stationary measures $\hat\mu_{X_n}$ on T^1X_n (with projections $\mu_{X_n} := \varpi_*\hat\mu_{X_n}$) such that

$$\int \log \|f_\star u\| d\nu_{X_n}(f)\, d\hat\mu_{X_n}(u) \leqslant 0. \tag{9.17}$$

From Lemma 9.15, we can extract a subsequence, still denoted by (X_n), such that $(\hat\mu_{t_n})$ converges to a stationary measure $\hat\mu_{X_E}$ on T^1X_E satisfying

$$\int \log \|f_\star u\| d\nu_{X_E}(f)\, d\hat\mu_{X_E}(u) \leqslant 0. \tag{9.18}$$

By iterating and using the stationarity of $\hat\mu_{X_E}$, the same inequality holds with $\nu_{X_E}^{(m)}$ instead of ν_{X_E} for every $m > 0$. To get the desired contradiction, we shall show that no such measure exists.

---

affine coordinates $\mathbf{x}$, $\mathbf{y}$ for the first two coordinates and homogeneous coordinates $[\mathbf{z}_0 : \mathbf{z}_1]$ for the third one. The equation for $X_v \subset \mathcal{V}$ can be written $A_v(\mathbf{x},\mathbf{y})\mathbf{z}_0^2 + B_v(\mathbf{x},\mathbf{y})\mathbf{z}_0\mathbf{z}_1 + C_v(\mathbf{x},\mathbf{y})\mathbf{z}_1^2 = 0$ and then $\sigma_3(\mathbf{x},\mathbf{y},[\mathbf{z}_0 : \mathbf{z}_1]) = (\mathbf{x},\mathbf{y},[-B_v(\mathbf{x},\mathbf{y})\mathbf{z}_1 - A_v(\mathbf{x},\mathbf{y})\mathbf{z}_0 : A_v(\mathbf{x},\mathbf{y})\mathbf{z}_1])$. We have to show that this map is regular near (x, y, z). Since we work near a point at which $z_1 \neq 0$, the only problem could be that, at some point $(x', y', [z_0' : z_1'])$ of some X_v, we have both $A_v(x', y') = 0$ and $B_v(x', y') = 0$, but then $C(x', y')$ should be 0 too, and X_v should contain a vertical line.

Step 1: *near the singularities*.– Here we show that there exists $n_0 \in \mathbf{N}$, $c_0 > 0$, and an open neighborhood U of $\mathrm{Sing}(X_E)$ such that if $u \in T^1X_E$ and $\varpi(u) \in U$, then

$$\int \log \|f_\star u\| \, d\nu_{X_E}^{(n_0)}(f) \geqslant c_0. \tag{9.19}$$

By Lemma 9.15 and the above mentioned semicontinuity of unit tangent bundles it is enough to prove this when $x = \varpi(u) \in \mathrm{Sing}(X_E)$. Recall that Γ_{X_E} fixes $\mathrm{Sing}(X_E)$ pointwise. Around each of its singularities, X_E is locally isomorphic to the quotient $\mathbf{C}^2/\eta$, $\eta(u,v) = (-u,-v)$, standardly embedded in $\mathbf{C}^3$ by

$$\phi : (u,v) \mapsto (u^2, uv, v^2) = (\xi, \eta, \zeta), \tag{9.20}$$

whose image is the quadratic cone $\{\xi\zeta - \eta^2 = 0\} \subset \mathbf{C}^3$. The level-2 congruence subgroup G of $\mathsf{GL}_2(\mathbf{Z})$ fixes each torsion point of $A := E \times E$ of order $\leqslant 2$, and Γ_{X_E} is induced by a non-elementary subgroup G_0 of G. The standard linear action of G on $\mathbf{C}^2$ (or more precisely on a neighborhood of any 2-torsion point of A) commutes to η and induces a linear action on $\mathbf{C}^3$ via the homomorphism

$$\phi_\star : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bc \\ c^2 & 2cd & d^2 \end{pmatrix}. \tag{9.21}$$

Thus, the action of Γ_{X_E} on the tangent cone of X_E at the origin (which is naturally identified with $\{\xi\zeta - \eta^2 = 0\}$) is, up to a linear conjugacy, given by $\phi_\star(G_0)$. Since this cone is Zariski-dense in $T_xX_E = T_x((\mathbb{P}^1)^3)$ and the action of Γ on the universal family of $(2,2,2)$ surfaces is smooth at (X_E, x), we deduce that the action of Γ on $T^1_xX_E$ is also induced by $\phi_\star(G_0)$.

This is a subgroup of $O(q;\mathbf{R}) \simeq O_{2,1}(\mathbf{R})$, where q is the quadratic form $q(x,y,z) = xz - y^2$. By assumption, it is a non-elementary group of isometries of q, hence it acts strongly irreducibly and proximally on $\mathbf{R}^3 \subset \mathbf{C}^3$ (loxodromic elements of $\mathsf{GL}_2(\mathbf{Z})$ are mapped to loxodromic elements in $O(q;\mathbf{R})$). It preserves the real decomposition $\mathbf{C}^3 = \mathbf{R}^3 \oplus_{\mathbf{R}} \mathsf{i}\mathbf{R}^3$ and the action on $\mathbf{R}^3$ and $\mathsf{i}\mathbf{R}^3$ are linearly conjugate (by multiplication by i). Therefore, as in § 8.2, the Inequality (9.19) follows from [36] (see also [10], Chap. III, Cor. 3.4(iii)).

Step 2: *away from the singularities*.– We shall show that there exists a neighborhood $U' \subset U$ of $\mathrm{Sing}(X_E)$ and $c > 0$ such that for any fixed $u \in T^1X_E$ such that $\varpi(u) \notin U'$,

$$\mathbb{P}\left(\frac{1}{m}\log\|(f^m_\omega)_\star u\| \geqslant c\right) \underset{m\to\infty}{\longrightarrow} 1. \tag{9.22}$$

By Lemma 2.3 (see also Remark 2.4), this implies that $\mathbb{E}\left(\log\|(f^m_\omega)_\star u\|\right) \geqslant mc/2$ for large m. Then, the first step and a compactness argument identical to that of Lemma 2.2 show that uniform expansion holds on T^1X_E, which is the desired contradiction.

Let U' be an open neighborhood of $\mathrm{Sing}(X_E)$ which will be specified later. There is a constant $\delta = \delta(U')$ such that

$$\text{if } \varpi(u) \notin U' \text{ and } \varpi(f_\star u) \notin U', \text{ then } \log\|f_\star u\| \geqslant \log\|f_\star u\|_{\mathsf{flat}} - \delta, \tag{9.23}$$

where $\|\cdot\|_{\text{flat}}$ is the Riemannian metric on $\mathrm{Reg}(X_E)$ induced by the flat metric of $E \times E$.

The pull-back of ν to $\mathsf{GL}(2, \mathbf{Z})$ generates G_0 and its support is finite. Since $G_0 \subset \mathsf{GL}(2, \mathbf{Z})$ is non-elementary, we have uniform expansion with respect to the flat metric. By Lemma 2.3, there exists a constant $c_1 > 0$ and sets of trajectories $\Omega^1_m \subset \Omega$ such that $\mathbb{P}(\Omega^1_m) \to 1$ as $m \to \infty$ and

$$\text{if } \omega \in \Omega^1_m, \quad \frac{1}{m} \log \|(f^m_\omega)_\star u\|_{\text{flat}} \geqslant c_1. \tag{9.24}$$

Fix $\varepsilon > 0$ and $0 < c < c_1$. We claim that there is a Margulis function on X_E with poles at $\mathrm{Sing}(X_E)$. Indeed, to construct it we view X_E as $E \times E/\eta$. Let $\pi : E \times E \to E \times E/\eta \simeq X_E$ be the natural map. On $E \times E$, $\pi^{-1}(\mathrm{Sing}(X_E)) = \mathrm{Fix}(\eta)$ is invariant under the $\mathsf{GL}_2(\mathbf{Z})$-action, which is uniformly expanding, so $u : x \mapsto -\log d_{\text{flat}}(x, \mathrm{Fix}(\eta))$ defines a Margulis function. Since η is an isometry for the flat metric, u is η-invariant so it descends to a Margulis function on X_E, as asserted. We deduce that there is an open set $U' = U'(\varepsilon) \subset U$ with the following property: for large enough m, the set Ω^2_m of trajectories $\omega \in \Omega$ such that $(f^m_\omega)(\varpi(u)) \notin U'$ satisfies $\mathbb{P}(\Omega^2_m) \geqslant 1 - \varepsilon/2$. Now, U' being fixed, for m large enough we have that $\mathbb{P}(\Omega^1_m \cap \Omega^2_m) \geqslant 1 - \varepsilon$ and by (9.23) and (9.24), if $\omega \in \Omega^1_m \cap \Omega^2_m$,

$$\frac{1}{m} \log \|(f^m_\omega)_\star u\| \geqslant c_1 - \frac{\delta(U')}{m} \geqslant c. \tag{9.25}$$

Thus, the convergence (9.22) holds and the proof is complete. □

10. Applications

10.1. **Orbit closures.** The following is a version of the orbit closure Theorem E of [20] in which periodic orbits are allowed. Combined with Theorem 9.3, it gives Theorem 1.1.

Theorem 10.1. *Let X be a torus, a K3 surface, or an Enriques surface. Let $\Gamma \subset \mathsf{Aut}(X)$ be a non-elementary subgroup which contains parabolic elements and does not preserve any algebraic curve. Assume that for any finite orbit O, the induced action of Γ on $TX|_O$ is non-elementary. Then there exists a finite set F and a real analytic, totally real, and Γ-invariant surface $Y \subset X$ with $\mathrm{Sing}(Y) \subset F$ such that for every $x \in X$:*

(a) *either x belongs to F (and its orbit is finite);*
(b) *or x belongs to $Y \setminus F$ and $\overline{\Gamma(x)}$ is a union of components of Y;*
(c) *or $\overline{\Gamma(x)} = X$.*

Proof. First observe that under these assumptions, [22, Thm C] implies that there exists a maximal finite invariant subset F. Fix a symmetric measure ν such that $\Gamma_\nu = \Gamma$ and satisfying the moment condition (M_+). By Theorems 1.6 and 8.14, ν is uniformly expanding. We now resume the discussion from [20, §8], in particular Remark 8.6 there: STang_Γ is a finite invariant set and if $x \in X$ is such that $\Gamma(x)$ is infinite but not dense, then there are two possible situations:

(1) either $\overline{\Gamma(x)} \setminus \mathrm{STang}_\Gamma$ is discrete outside STang_Γ;

(2) or $\overline{\Gamma(x)} \backslash \mathrm{STang}_\Gamma =: Y(x)$ is a totally real analytic surface, whose singular locus $\mathrm{Sing}(Y)$ is discrete outside STang_Γ.

In Case (1), $\Gamma(x)$ is finite. Indeed $\overline{\Gamma(x)}$ is at most countable, so if μ is any cluster value of $\frac{1}{n}\sum_{k=0}^{n-1} \nu^k \star \delta_x$, then μ is a purely atomic stationary measure. In this case it follows from Theorem 4.3 that the orbit of x must be finite, hence contained in F.

If Case (2) holds, we first claim that $\mathrm{Sing}(Y(x))$ is finite. Indeed, $\overline{\mathrm{Sing}(Y(x))}$ is a Γ-invariant countable set, which clusters only at STang_Γ. By the previous argument, every orbit $\Gamma(y)$ in $\overline{\mathrm{Sing}(Y(x))}$ is finite, so by the finiteness of the set of finite orbits [22, Thm C] we conclude that $\mathrm{Sing}(Y(x))$ itself is finite.

Now, let μ' be a cluster value of $\frac{1}{n}\sum_{k=0}^{n-1} \nu^k \star \delta_x$. By Theorem 4.3, μ' is an atomless stationary measure supported on $Y(x)$ such that $\mu'(\mathrm{Reg}(Y(x))) = 1$. Since Γ has no invariant curve, μ' is Zariski diffuse. Let μ be any ergodic component of μ'. Theorems 8.8 and 1.4 imply that μ is hyperbolic and its stable directions depend genuinely on the itinerary. Then the argument of [13, Thm 3.1] adapts immediately to show that μ is SRB([10]). The canonical invariant 2-form of X induces a Γ-invariant measure $\mathsf{vol}_{Y(x)}$ on $Y(x)$ (see Lemma 6.3). Since $\mathrm{Reg}(Y(x))$ admits a Margulis function, we conclude from Proposition 4.4 that the volume $\mathsf{vol}_{Y(x)}$ is finite. Therefore we can copy verbatim the argument of [13, Thm 3.4] to conclude that μ is Γ-invariant. Since [20, Thm C] says that there are only finitely many Γ-invariant measures, there are only finitely many possible surfaces $Y(x)$. Taking Y to be their union, the proof is complete. □

10.2. **Ergodicity.** In [26], the original motivation to introduce uniform expansion was a criterion for ergodicity. The same holds in our setting, with a few caveats which will be explained below.

Theorem 10.2 (Dolgopyat-Krikorian [26, Cor. 2], see also [45, 24])**.** *Let X be a torus, a K3 surface, or an Enriques surface. Let $\Gamma \subset \mathsf{Aut}(X)$ be a non-elementary subgroup with a uniformly expanding action on X. Then vol_X is Γ-ergodic.*

Likewise, if $Y \subset X$ is a Γ-invariant totally real analytic subset such that Γ acts transitively on the set of irreducible components of Y, then vol_Y is Γ-ergodic.

Note that the notion of irreducible component in real analytic geometry is not well-behaved in general (see [20, §5.1] for a short discussion). Here we content ourselves with saying that Y is irreducible when $\mathrm{Reg}(Y)$ is connected. Observe also that the ergodicity of vol_X follows directly from Theorem 6.4 when Γ contains a parabolic element.

[10]We are *not* claiming that we can extend [13] to non-compact surfaces here. All the necessary estimates on the Lyapunov norms and Pesin charts hold by viewing μ as a hyperbolic stationary measure on the *compact* complex manifold X. The only issue appears when considering the size and intersection properties of real stable and unstable manifolds in $Y(x)$, starting from §9.7 of [13]. At this stage Brown and Rodriguez-Hertz already discard a set of small measure of points with bad properties (see the definition of $\Lambda(\gamma_1)$ on p. 1087); so it is enough to remove from this $\Lambda(\gamma_1)$ the set of small measure of points too close to $\mathrm{Sing}(Y(x))$, and proceed with their argument.

Proof (sketch). The proof in [26] is a bit sketchy, but it was already expanded in [45, 24] (see also [52]). Here we just make a few comments on (1) the extension to the holomorphic case for the action on X, and (2) how to deal with the possible singularities for the action on totally real surfaces Y.

Regarding the action on X, let us recall that the proof of [26] is a variation on the Hopf argument in which the asymptotic behavior of the Birkhoff sums $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f_\omega^n(x)}$ is propagated along chains of local stable manifolds (associated to different ω's), to ultimately conclude that there is a uniform r so that almost every point x is located at distance at least r from the boundary of its ergodic component. The key technical ingredients are the facts that under the uniform expansion assumption:

– stable directions at a given point do not concentrate, more precisely there exists $\alpha > 0$ such that for any $x \in X$ and any $[v] \in \mathbb{P}(T_xX)$, the probability that $d_{\mathbb{P}^1}([v], [E^s(x,\omega)]) < \alpha$ is smaller than 1/100: this follows from a compactness argument (see [45, Prop. 4.4.4]);
– the Pesin local stable manifolds have uniformly bounded geometry (e.g. uniformly bounded size in the sense of [21, §7.4]): this follows from the usual proof of the local stable manifold theorem;
– the absolute continuity of the local stable foliation in Pesin charts: we can copy the usual proof or notice that in the holomorphic case this follows from the fact that the holonomy of a holomorphic motion is quasiconformal.

Given these facts, we can copy the proof of [26] by plugging in §10.4 the following elementary geometric property, whose proof is left to the reader: let $w = (w_1, w_2) \in \mathbf{C}^2$ with $\|w\| < 1$ (possibly close to 1) and E_w be the direction perpendicular to the line $(0w)$; then if L is a complex line containing w, such that the angle in $\mathbb{P}^1$ between the direction of L and E_w is greater than α, then $L \cap B(0,1)$ contains a disk of radius $r \geqslant r(\alpha)$.

For the second statement of the theorem we can directly resort to [45, 24], except that we have to take into account the possibility of singular points on Y, which affect the size and geometry of local stable manifolds on Y. For this, we may argue exactly as in Theorem 10.1: first, the existence of a Margulis function guarantees that vol_Y is finite. Next, since uniform expansion holds on X, the size and angle change of local complex stable manifolds is uniformly controlled. Thus, when restricting to Y, we also have a uniform control of this geometry outside any δ-neighborhood of $\mathrm{Sing}(Y)$. Since the Hopf argument is local, we get that there is a single ergodic component outside a δ-neighborhood of $\mathrm{Sing}(Y)$, for every $\delta > 0$, and we conclude by letting δ tend to zero. □

Remark 10.3. The argument of [26] works for a random dynamical system on a (real) compact $2d$-dimensional manifold enjoying a uniform expansion property along d dimensional tangent subspaces. This assumption does *not* hold in our setting since along a totally real subspace one may witness both expansion and contraction. In particular the complex uniform expansion condition is not stable under C^1 perturbations by (real) volume preserving diffeomorphisms of X. Still, the philosophy of the above proof is that the argument is robust enough so that uniform expansion along complex 1-dimensional tangent subspaces in a 2-dimensional complex surface guarantees ergodicity.

10.3. **Equidistribution.** In the following results, given an action of (Γ, ν) on M we say that random trajectories from x equidistributes towards μ if $\frac{1}{n}\sum_{k=1}^{n}\delta_{f_\omega^k} \to \mu$ for $\nu^{\mathbf{N}}$-almost every ω, where the convergence is in the weak* topology. By averaging with respect to $\nu^{\mathbf{N}}$ and applying the Dominated Convergence Theorem, this implies that $\frac{1}{n}\sum_{k=1}^{n}\nu^k * \delta_x \to \mu$ as well.

The following theorem already appears under stronger moment assumptions in [24, Thm D].

Theorem 10.4. *Let $X_{\mathbf{R}}$ be a smooth real projective surface and ν a probability measure on $\mathrm{Aut}(X_{\mathbf{R}})$ satisfying* (M_+). *Assume that Γ_ν preserves a smooth volume form* vol *on $X(\mathbf{R})$ and that ν is uniformly expanding on $X(\mathbf{R})$. Then for any $x \in X$ one of the following alternatives holds:*

(a) either $\Gamma_\nu \cdot x$ is finite;
(b) or the random trajectories from x equidistribute towards $\mathsf{vol}_{X'(\mathbf{R})}$, the normalized induced volume on a union of components of $X(\mathbf{R})$.

Recall from Theorem 1.6 that the uniform expansion assumption holds when Γ_ν contains parabolic elements, has no invariant curve, and that the induced action of Γ_ν on finite orbits is uniformly expanding. In this case by [22, Thm C], the number of finite orbits is finite. By Theorem 9.3 this applies to generic real Wehler surfaces and yields Theorem 1.3.

Proof. Breiman's ergodic theorem says that that for $\nu^{\mathbf{N}}$-almost every ω, any cluster limit μ of the sequence of empirical averages $\frac{1}{n}\sum_{k=1}^{n}\delta_{f_\omega^k(x)}$ is stationary. Since ν is uniformly expanding, the existence of a Margulis function (Theorem 4.3) shows that if $\Gamma_\nu \cdot x$ is infinite, μ gives no mass to finite orbits. Since ν is uniformly expanding, any ergodic stationary measure μ' is hyperbolic and its stable directions are non-random, so by [13, Thm 3.4], μ' is absolutely continuous with respect to $\mathsf{vol}_{X(\mathbf{R})}$. The ergodicity Theorem 10.2 shows that for any component $X_0(\mathbf{R})$ of $X(\mathbf{R})$, $\mathsf{vol}_{X_0(\mathbf{R})}$ is ergodic, so we conclude that, up to scaling, μ is a finite combination of measures of this type. □

The next result is conditional to the ν-stiffness property of complex non-elementary uniformly expanding actions. We expect that it will be established in the near future.

Theorem 10.5. *Let X be a K3 or Enriques surface and ν be a probability measure on $\mathrm{Aut}(X)$ satisfying* (M_+). *Assume that*

(1) *Γ_ν is non-elementary, contains parabolic elements, has no invariant curve, and every finite Γ-orbit is uniformly expanding;*
(2) *ν-stiffness holds, that is, every ν-stationary measure is invariant;*
(3) *every compact, real analytic, totally invariant surface $Y \subset X$ is smooth.*

Then there exists a finite set F and a (possibly singular) totally real analytic surface Y such that for every $x \in X$:

(a) *either x belongs to F;*
(b) *or x belongs to $Y \setminus F$ and its orbit equidistributes towards $\mathsf{vol}_{Y'}$, where Y' is a union of components of Y;*
(c) *or $x \notin F \cup Y$ and its orbit equidistributes towards vol_X.*

The third hypothesis is a weakness of this statement, since we do not know how to study the singularities of invariant real analytic surfaces (except, of course, when we know how to exclude the existence of finite orbits).

Proof. The sets F and Y were already constructed in Theorem 10.1, whose proof also implies property (b). The classification of invariant measures (Theorem 6.4) and the stiffness property show that the only ν-stationary measure giving no mass to $Y \cup F$ is vol_X. Therefore the equidistribution property (c) follows from Breiman's ergodic theorem and the existence of a Margulis function associated to finite orbits and totally real surfaces (Theorems 4.1 and 4.5). □

Appendix A. Rigidity of zero entropy measures

We complete the proof of Theorem 8.8 with the following result of independent interest.

Theorem A.1. *Let X be a torus or a K3 or Enriques surface, and ν be a probability measure on $\mathsf{Aut}(X)$ such that Γ_ν is non-elementary. Assume that μ is a Zariski diffuse ν-stationary measure such that $h_\mu(X,\nu) = 0$. Then μ is Γ_ν-invariant and for every $f \in \Gamma_\nu$, $h_\mu(f) = 0$.*

Proof. As in Theorem 8.8, we may assume that μ is ergodic as a stationary measure, and its Γ_ν-invariance was already established there. Pick $f \in \Gamma_\nu$. Assume by way of contradiction that $h_\mu(f) > 0$, in particular f must be loxodromic. If μ is ergodic for f, then the result follows rather immediately from the measure rigidity theorem 11.1 in [21]. Indeed in that theorem we consider an ergodic measure μ of positive entropy for f and study the group of automorphisms of X preserving μ, under the additional assumption that μ is supported on a real surface. We reduce the argument to the case of $\Gamma = \langle f, g\rangle$ for some g, and divide the proof into 3 cases: (1) either there is a Γ-invariant measurable line field, or (2) there is a Γ-invariant pair of measurable line fields, or (3) none of the above. In cases (1) and (2) we conclude that Γ is elementary by adapting the argument of [21, Thm. 9.1]: this does not rely on the additional real structure. In case (3), since μ is hyperbolic for f, Theorems 7.2 and 7.3 imply that μ is hyperbolic as a stationary measure and as in the proof of Theorem 8.8 we deduce that $h_\mu(X,\nu) > 0$, which is contradictory. Thus, case (3) does not happen, and we deduce that Γ is elementary for every $g \in \Gamma_\nu$, which is a contradiction. Therefore $h_\mu(f) = 0$.

What remains to do is to adapt this argument to the case where μ is not ergodic under f. So consider $f \in \Gamma_\nu$ and assume that $h_\mu(f) > 0$ so that f is loxodromic. As before there are 3 cases: either (1) there is a Γ_ν-invariant measurable line field, or (2) there is a Γ_ν-invariant pair of measurable line fields, or (3) none of (1) and (2). We first observe that as before Case 3 does not happen: indeed if there is no invariant line field or pair of invariant line fields, by Theorems 7.2 and 7.3, either μ is hyperbolic as a ν-stationary measure, or the projectivized tangent action of Γ_ν reduces to a compact subgroup. But since $h_\mu(f) > 0$, f admits non-zero Lyapunov exponents on a set of positive measure so the latter is impossible. Hence μ is hyperbolic as a ν-stationary measure, and since there is no invariant line field, stable directions depend on the itinerary and as before we conclude that $h_\mu(X, \nu) > 0$, a contradiction. So one of Cases (1) or (2) holds.

So assume there exists a measurable Γ_ν-invariant line field $x \mapsto [E(x)] \in \mathbb{P}(T_xX)$ and pick $g \in \Gamma_\nu$. Assume further that g is loxodromic. We will derive a contradiction by showing that $\langle f, g\rangle$ must be elementary: this is a contradiction because any non-elementary subgroup of $\mathsf{Aut}(X)$ contains a purely loxodromic non-elementary subgroup. Let $\mathcal{P}$ be the measurable partition into ergodic components (under f) and denote by μ_P the conditional measure on $P \in \mathcal{P}$, so that that $\mu = \int \mu_{\mathcal{P}(x)} d\mu(x)$ is the ergodic decomposition of μ. Since the entropy function is affine, there exists a f-invariant set B of positive measure such that for any $x \in B$, $h_{\mu_{\mathcal{P}(x)}}(f) > 0$. In particular f is non-uniformly hyperbolic along B, so along B, E must coincide almost everywhere with one of E^s_f or E^u_f. Reducing B to a smaller invariant subset we may assume that $E = E^s_f$ almost everywhere along B. For every $n \in \mathbf{Z}$, the automorphism $g^{-n}fg^n$ is loxodromic, preserves μ, is non-uniformly hyperbolic along $g^{-n}(B)$, and E coincides with $E^s_{g^{-n}fg^n}$ almost everywhere. By measure preservation there exists $m \neq n$ such that $\mu(g^{-n}(B) \cap g^{-m}(B)) > 0$, so $\mu(B \cap g^{m-n}(B)) > 0$. Letting $h = g^{m-n}fg^{-(m-n)}$ and $A = B \cap g^{m-n}(B)$ we are exactly in the situation of Lemma 11.2 of [21], and we conclude that $W^s(f, x) = W^s(h, x)$ for μ-almost every $x \in A$, from which it follows that $T^+_f = T^+_h$ and finally $(g^{m-n})^* T^+_f = cT^+_f$. Since g is loxodromic, this implies that $T^+_f = T^+_g$ or $T^+_f = T^-_g$, and finally that $\langle f, g\rangle$ is elementary, which is the sought-after contradiction.

Finally, if there is a measurable pair $\{E_1, E_2\}$ of line fields which is ν-a.s. invariant, we get a f-invariant set B of positive measure along which $\{E_1(x), E_2(x)\} = \left\{E^s_f(x) = E^u_f(x)\right\}$, and a set $A = B \cap g^{m-n}(B)$ of positive measure along which $\left\{E^s_f(x) = E^u_f(x)\right\} = E^s_h(x) = E^u_h(x)$, where $h = g^{m-n}fg^{-(m-n)}$, and we conclude as before. □

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