GENERATORS FOR THE CREMONA GROUP (AFTER HUDSON, PAN, DERKSEN, ...)

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ABSTRACT. We discuss a result due to Hudson and Pan concerning generators of the Cremona group in n variables. We add a few remarks based on the works of Frumkin and of Derksen, and on discussion with Blanc, Dubouloz, Lamy and Urech; these remarks concern (bi)rational transformations of projective varieties and groups of automorphisms of the affine space.

CONTENTS

1. INTRODUCTION

1.1. **Cremona groups.** Let n be a positive integer. The Cremona group in n variables over a field $\mathbf{k}$ is the group of birational transformations of the n-dimensional projective space $\mathbb{P}^n_{\mathbf{k}}$; this group, denoted $\mathsf{Cr}_n(\mathbf{k})$ in what follows, coincides with the group of $\mathbf{k}$-automorphisms of the field of rational functions $\mathbf{k}(X_1, ..., X_n)$. It contains the group of automorphisms of $\mathbb{P}^n_{\mathbf{k}}$, i.e. the group of projective linear transformations $\mathsf{PGL}_{n+1}(\mathbf{k})$.

1.2. **Degrees.** Let $[x_0 : x_1 : \ldots : x_n]$ be a system of homogeneous coordinates on $\mathbb{P}^n_{\mathbf{k}}$. If f is an element of $\mathsf{Cr}_n(\mathbf{k})$, there are homogeneous polynomial functions P_0, P_1, ... and P_n in $\mathbf{k}[x_0, \ldots, x_n]$, of the same degree d and with no common factor of positive degree, such that

$$f[x_0 : \ldots : x_n] = [P_0 : \ldots : P_n].$$

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By definition, the common degree d of the P_i is the degree of f. Automorphisms of $\mathbb{P}^n_{\mathbf{k}}$ correspond to birational transformations of degree 1.

1.3. **Increasing the dimension.** The Cremona group $\mathsf{Cr}_n(\mathbf{k})$ coincides with the group of birational transformations of the n-dimensional affine space $\mathbf{A}^n_{\mathbf{k}}$. Let f be a birational transformation of $\mathbf{A}^n_{\mathbf{k}}$. The transformation $\hat{f}$ of $\mathbf{A}^n_{\mathbf{k}} \times \mathbf{A}^1_{\mathbf{k}}$ which is defined by

$$\hat{f}(x,y) = (f(x),y)$$

for (x,y) in $\mathbf{A}^n_{\mathbf{k}} \times \mathbf{A}^1_{\mathbf{k}}$ is a birational transformation of $\mathbf{A}^{n+1}_{\mathbf{k}} = \mathbf{A}^n_{\mathbf{k}} \times \mathbf{A}^1_{\mathbf{k}}$. This defines an injective morphism of groups

$$\begin{array}{ccc} \mathsf{Cr}_n(\mathbf{k}) & \to & \mathsf{Cr}_{n+1}(\mathbf{k}) \\ f & \mapsto & \hat{f} \end{array}$$

and therefore an embedding of $\mathsf{Cr}_n(\mathbf{k})$ into $\mathsf{Cr}_{n+1}(\mathbf{k})$, so that Cremona groups are larger and larger groups when the dimension n increases.

1.4. **Infinite dimension.** In dimension $n = 1$, $\mathsf{Cr}_1(\mathbf{k})$ coincides with $\mathsf{PGL}_2(\mathbf{k})$, but $\mathsf{Cr}_2(\mathbf{k})$, and therefore all $\mathsf{Cr}_n(\mathbf{k})$ with $n \geq 2$, is a very large group : For every $k \geq 0$, it contains all transformations f of the form

$$f[x_0 : x_1 : x_2] = [x_0x_2^k + P(x_1,x_2) : x_1x_2^k : x_2^{k+1}]$$

where P is a homogeneous polynomial of degree $k+1$ (the inverse of f is obtained by replacing P by $-P$). Thus, $\mathsf{Cr}_n(\mathbf{k})$ is "infinite dimensional" for $n \geq 2$.

1.5. **Generators.** On the other hand, Noether-Castelnuovo theorem implies that, in terms of generators, $\mathsf{Cr}_2(\mathbf{k})$ is rather small. To state it, we need to introduce the standard quadratic involution $\sigma : \mathbb{P}^2_{\mathbf{k}} \dashrightarrow \mathbb{P}^2_{\mathbf{k}}$: In homogeneous coordinates,

$$\sigma[x_0 : x_1 : x_2] = [x_1x_2 : x_2x_0 : x_0x_1]$$

(i.e. $\sigma(X_1,X_2) = (1/X_1, 1/X_2)$ in affine coordinates).

Theorem 1.1 (M. Noether, G. Castelnuovo, see [5])**.** *If the field* $\mathbf{k}$ *is algebraically closed, the Cremona group* $\mathsf{Cr}_2(\mathbf{k})$ *is generated by the group of automorphisms* $\mathsf{PGL}_3(\mathbf{k})$ *and the standard quadratic involution* σ.

One can also describe a complete set of relations between these generators; this result is due to M. H. Gizatullin [4] (see also [2]). Our goal is to justify that the picture is much less simple if $n \geq 3$.

Theorem 1.2 (H. P. Hudson, I. Pan, [7])**.** *Let n be a natural integer with $n \geq 3$. To generate the Cremona group $\mathsf{Cr}_n(\mathbf{k})$, one needs as many algebraic families of generators, as families of smooth hypersurfaces of $\mathbb{P}^{n-1}_{\mathbf{k}}$ of degree $\geq n+2$. One cannot generate the Cremona group by generators of bounded degree.*

Obviously, this statement is loosely stated; I hope that the proof, which is quite short, will make it clearer. [1]

2. Proof

2.1. **Exceptional hypersurfaces.** Let f be a birational transformation of $\mathbb{P}^n_{\mathbf{k}}$, and let X be an irreducible hypersurface of $\mathbb{P}^n_{\mathbf{k}}$. We say that X is f-exceptional if there is an open subset of X which is mapped into a subset of codimension ≥ 2 by f (equivalently, if $\mathbf{k}$ is algebraically closed, f is not injective on any open subset of X).

Let g_1, ..., g_m be birational transformations of the projective space $\mathbb{P}^n_{\mathbf{k}}$, and let g be the composition $g = g_m \circ g_{m-1} \circ \ldots \circ g_1$. Let X be an irreducible hypersurface of $\mathbb{P}^n_{\mathbf{k}}$. If X is g-exceptional, then there is an index i, with $1 \leq i \leq m$, and a g_i-exceptional hypersurface X_i such that X is birationnaly equivalent to X. More precisely, for some index i, $g_{i-1} \circ \ldots \circ g_1$ realizes a birational isomorphism from X to X_i, and then g_i contracts X_i.

2.2. **De Jonquières transformations with prescribed exceptional hypersurfaces.** Let $[x_0 : \ldots : x_{n-1}]$ be homogeneous coordinates for $\mathbb{P}^{n-1}_{\mathbf{k}}$ and $[y_0 : y_1]$ be homogeneous coordinates for $\mathbb{P}^1_{\mathbf{k}}$.

Let Y be an irreducible hypersurface of degree d in $\mathbb{P}^{n-1}_{\mathbf{k}}$, which is not the plane $x_0 = 0$, and let h be a reduced homogeneous equation for Y. Define a birational transformation f_Y of $\mathbb{P}^{n-1}_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ by

$$f_Y(x, [y_0 : y_1]) = (x, [y_0 x_0^d : h(x_0, \ldots, x_{n-1}) y_1]).$$

The transformation f_Y preserves the natural projection of $\mathbb{P}^{n-1}_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ onto the first factor $\mathbb{P}^{n-1}_{\mathbf{k}}$. It acts by linear projective transformations on the generic fibers $\mathbb{P}^1_{\mathbf{k}}$; more precisely, f_Y is the projective linear transformation which is

[1]This text is not intended for publication. I thank Jean-Louis Collitot-Thélène who pointed out a mistake in the first version of these notes. Section 3.3 is the result of a discussion with Jérémy Blanc and Stéphane Lamy. Section 3.4 comes from a discussion with J. Blanc and Christian Urech. I also thank Adrien Dubouloz for nice discussions on polynomial automorphisms.

determined by the 2 by 2 matrix

$$\begin{pmatrix} x_0^d & 0 \\ 0 & h(x_0,\ldots,x_{n-1}) \end{pmatrix}$$

over the point $[x_0 : \ldots : x_{n-1}]$. This matrix is invertible if and only if $x_0 \neq 0$ and $h(x) \neq 0$.

The birational transformation f contracts the generic points of the hypersurface $Y \times \mathbb{P}^1_{\mathbf{k}}$ to the codimension 2 subset $Y \times \{[1:0]\}$.

Lemma 2.1. *For every irreducible hypersurface Y of $\mathbb{P}^{n-1}_{\mathbf{k}}$ (of degree d), there exists a birational transformation g_Y of $\mathbb{P}^n_{\mathbf{k}}$ (of degree $d+1$) and a hypersurface $X \subset \mathbb{P}^n_{\mathbf{k}}$ such that*

- *X is birationally equivalent to $Y \times \mathbb{P}^1_{\mathbf{k}}$;*
- *X is g_Y exceptional.*

Proof. The projective variety $\mathbb{P}^{n-1}_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ is birationally equivalent to $\mathbb{P}^n_{\mathbf{k}}$. An explicit birational map $\eta : \mathbb{P}^{n-1}_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}} \to \mathbb{P}^n_{\mathbf{k}}$ is given by

$$\eta([x_0 : \ldots : x_{n-1}], [y_0 : y_1]) = [y_0x_0 : y_1x_0 : y_1x_1 : \ldots : y_1x_{n-1}].$$

In the complement of $x_0 = 0$, η maps verticals $\{*\} \times \mathbb{P}^1_{\mathbf{k}}$ to lines through the point $[1 : 0 : \ldots : 0] \in \mathbb{P}^n_{\mathbf{k}}$, and contracts the hypersurface $y_1 = 0$ to that point.

Conjugate f_Y by the birational map $\eta : \mathbb{P}^{n-1}_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}} \dashrightarrow \mathbb{P}^n_{\mathbf{k}}$ and denote by X the image of $Y \times \mathbb{P}^1_{\mathbf{k}}$ by η. The result follows, because η maps the subset

$$(Y \times \{[1:0]\}) \setminus \{x_0 = 0\} \subset \mathbb{P}^{n-1}_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$$

to the point $[1 : 0 : \ldots : 0]$ of $\mathbb{P}^n_{\mathbf{k}}$. □

2.3. **Stable equivalence.** Say that Y is (m-)stably-equivalent to Y' if there is a birational map from $Y \times \mathbb{P}^m_{\mathbf{k}}$ to $Y' \times \mathbb{P}^m_{\mathbf{k}}$. There are examples of complex projective varieties Y of dimension $l \geq 3$ such that Y is not rational but Y is stably-equivalent to the projective space $\mathbb{P}^l_{\mathbf{C}}$ (see [1], [8]).

Suppose that Y and Y' are smooth hypersurfaces of $\mathbb{P}^{n-1}_{\mathbf{k}}$ of degree $\geq n+1$; denote by d the degree of Y and by d' the degree of Y'. Assume that Y is m-stably-equivalent to Y'. Let $V = Y \times \mathbb{P}^m_{\mathbf{k}}$ and $V' = Y \times \mathbb{P}^m_k$; let π and π' denote the natural projections onto Y and Y' respectively. Let $\phi \colon V \dashrightarrow V'$ be a birational map.

The variety Y is a smooth variety of general type: the canonical bundle $K_Y = \bigwedge^{n-2} T^*Y$ is very ample (by adjunction, $K_Y = O_Y(d-(n+1))$); moreover, the dimension of $H^0(Y, K_Y)$ determines the degree d. Since all regular global sections of $\bigwedge^l T^*\mathbb{P}^m_{\mathbf{k}}$ vanish identically, for all $1 \le l \le m$, the projection π determines an isomorphism

$$\pi^*\colon H^0(Y, K_Y) \to H^0(V, \bigwedge^{n-2} T^*V).$$

Similar results hold for Y'.

Now, by pull-back, ϕ provides a linear isomorphism

$$\phi^*\colon H^0(V', \bigwedge^{n-2} T^*V') \to H^0(V, \bigwedge^{n-2} T^*V).$$

This implies that $d = d'$. Moreover, if

$$\theta\colon Y \to \mathbb{P}(H^0(Y, K_Y)^\vee), \quad \theta(y) = eval_y,$$

denotes the Kodaira-Iitaka embedding of Y, and θ' denotes the Kodaira-Iitaka embedding of Y', then ϕ^* induces an isomorphism from $\theta(Y)$ to $\theta'(Y')$; hence Y is isomorphic to Y'.

In other words, for hypersurfaces of $\mathbb{P}^{n-1}_{\mathbf{k}}$ of degree $\ge n+1$, the stable equivalence of Y and Y' implies that Y is isomorphic to Y'.

2.4. **Conclusion.** Put Sections 2.1, 2.2, and 2.3 together. We get the following statement: *To generate the Cremona group in n variables $\mathsf{Cr}_n(\mathbf{k})$, one needs as many generators as classes of hypersurfaces $Y \subset \mathbb{P}^{n-1}_{\mathbf{k}}$ of degree $\ge n+1$ modulo isomorphism.* This result, of course, is not properly stated, because the cardinality of those classes of hypersurfaces is the same as the cardinality of $\mathbf{k}$ (resp. of $\mathsf{PGL}_{n+1}(\mathbf{k})$).

Let $\mathsf{Cr}_n(\mathbf{k}; d)$ be the set of birational transformations of $\mathbb{P}^n_{\mathbf{k}}$ of degree exactly d. The set of rational maps from $\mathbb{P}^n_{\mathbf{k}}$ to itself of degree d is a quasi-projective variety; $\mathsf{Cr}_n(\mathbf{k}; d)$ is an algebraic subset in this variety (more precisely, it is a Zariski open subset in some algebraic subvariety). By definition, $\mathcal{G} \subset \mathsf{Cr}_n(\mathbf{k})$ is an algebraic family of birational transformations of $\mathbb{P}^n_{\mathbf{k}}$ of degree d if $\mathcal{G}$ is contained in $\mathsf{Cr}_n(\mathbf{k}; d)$ and $\mathcal{G}$ is an algebraic subset of $\mathsf{Cr}_n(\mathbf{k}; d)$.

What can be proved from the previous sections is

Theorem 2.2. *Let n be a positive integer with $n \ge 3$. Let $\mathcal{G}_i$, $i \in \mathbf{N}$, be a countable collection of algebraic families of birational transformations of $\mathbb{P}^n_{\mathbf{k}}$ such that $\cup_i \mathcal{G}_i$ generates $\mathsf{Cr}_n(\mathbf{k})$ as a group. Let $H_d(\mathbb{P}^{n-1}_{\mathbf{k}})$ be the moduli space of smooth hypersurfaces of degree d in $\mathbb{P}^{n-1}_{\mathbf{k}}$, $d \ge n+2$. Then,*

- *for every $d \ge n+2$, there is a Zariski open subset Z_d of $H_d(\mathbb{P}^{n-1}_{\mathbf{k}})$ and an integer $i \in \mathbf{N}$ such that Z_d embeds into $\mathcal{G}_i$;*

- *there are two strictly increasing sequences of integers d_j and m_j such that $\cup_i \mathcal{G}_i$ intersects $\mathsf{Cr}_n(\mathbf{k}; d_j)$ on a subset of dimension $\geq m_j$.*

In particular, one cannot generate the Cremona group in $n \geq 3$ variables by generators of bounded degree.

Sketch of the proof. If $d \geq n+2$, general hypersurfaces of degree d in $\mathbb{P}^{n-1}_{\mathbf{k}}$ are smooth varieties with very ample canonical bundle. Let Y_1 and Y_2 be two smooth hypersurfaces of degree d. If $F : Y_1 \dashrightarrow Y_2$ is birational, there exists an automorphism A of $\mathbb{P}^{n-1}_{\mathbf{k}}$ which maps Y_1 onto Y_2 and coincides with F on Y_1. Moreover, the group $\mathsf{Aut}(Y)$ is trivial for general hypersurfaces of degree d. In particular, the dimension of $H_d(\mathbb{P}^{n-1}_{\mathbf{k}})$ goes to $+\infty$ with d.

The image W_d of the map $Y \mapsto g_Y$ described in Section 2.2, is generated by the family $\mathcal{G}_i$, $i \in \mathbf{N}$, and W_d is an algebraic subset of $\mathsf{Cr}_n(\mathbf{k}; d+1)$. Thus, there is an integer m such that the image of $\mathcal{G}_1 \times \ldots \times \mathcal{G}_m$ by

$$(g_1, \ldots, g_m) \mapsto g_1 \circ \ldots \circ g_m$$

contains W_d. From section 2.1, one deduces that there is an open subset Z of $H_d(\mathbb{P}^{n-1}_{\mathbf{k}})$ and an integer i between 1 and m, such that Z embeds into $\mathcal{G}_i$. □

3. Remarks

3.1. The previous proof may apply to other types of projective varieties, beside $\mathbb{P}^n_{\mathbf{k}}$. Unfortunately, I am not aware of any non-trivial example. For instance, I don't know whether the group of birational transformations of a smooth cubic hypersurface of $\mathbb{P}^4_{\mathbf{C}}$ ($\mathbf{C}$ the field of complex numbers) is generated by transformations of bounded degree.

3.2. For simplicity, consider the case $n = 3$. Given f in the Cremona group $\mathsf{Cr}_3(\mathbf{k})$, consider the set of irreducible components $\{X_i\}_{1 \leq i \leq m}$ of the union of the exceptional loci of f and of its inverse f^{-1}. Each X_i is birationally equivalent to a product $\mathbb{P}^1_{\mathbf{k}} \times C_i$, where C_i is a smooth irreducible curve. Define $g(X_i)$ as the genus of C_i, and the **genus** of f as the maximum of the $g(X_i)$, $1 \leq i \leq m$. Then, the subset of $\mathsf{Cr}_3(\mathbf{k})$ of all birational transformations f of genus at most g_0 is a subgroup of $\mathsf{Cr}_3(\mathbf{k})$: In this way, one obtains a filtration of the Cremona group by an increasing sequence of strict subgroups. (see [3] for related ideas and complements[2])

[2]See also [6], which simplifies [3] and contains nice complements concerning the genus of a birational transformation.

3.3. Now, consider the case $n = 2$, but with a field which is not algebraically closed; for simplicity, take $\mathbf{k} = \mathbf{Q}$, the field of rational numbers. Given f in $\mathsf{Cr}_2(\mathbf{Q})$, the indeterminacy locus $\mathrm{Ind}(f)$ of f is a finite subset of $\mathbb{P}^2(\overline{\mathbf{Q}})$, where $\overline{\mathbf{Q}}$ is a fixed algebraic closure of $\mathbf{Q}$. Fix a number field $\mathbf{k}$, and consider the subset of all f in $\mathsf{Cr}_2(\mathbf{Q})$ such that each indeterminacy point of f or f^{-1} (I include infinitesimally closed points) is defined over $\mathbf{k}$; for instance, if $p \in \mathbb{P}^2(\mathbf{C})$ is an indeterminacy point of f^{-1}, then $p = [a_0 : a_1 : a_2]$ with a_i in $\mathbf{k}$. This subset is a subgroup of $\mathsf{Cr}_2(\mathbf{Q})$; one thus gets an inductive net of subgroups of $\mathsf{Cr}_2(\mathbf{Q})$.

More generally, let us fix a field $\mathbf{k}$ together with an algebraic closure $\overline{\mathbf{k}}$ of $\mathbf{k}$. To an element f of $\mathsf{Cr}_2(\mathbf{k})$, one can introduce the field $\mathbf{k}_f$: the smallest field $\mathbf{k}_f \subset \overline{\mathbf{k}}$ on which (i) f and f^{-1} are defined and (ii) all base points of f and f^{-1} are defined. Note that, with this definition, $\mathbf{k}_f$ may be smaller that $\mathbf{k}$. Then, the field $\mathbf{k}_{f\circ g}$ is contained in the extension generated by $\mathbf{k}_f$ and $\mathbf{k}_g$. Thus, $\mathbf{k}_f$ provides a measure for the arithmetic complexity of f, and this measure behaves sub-multiplicatively (as the degree $\deg(f)$ does).

3.4. As J. Blanc noticed, the previous remark implies the following[3].

Proposition 3.1. *Let* $\mathbf{k}$ *be a field. The Cremona group* $\mathsf{Cr}_2(\mathbf{k})$ *is not finitely generated.*

Proof. Let p be the characteristic of $\mathbf{k}$, let $\mathbf{k}_0$ be the prime subfield of $\mathbf{k}$ (so that $\mathbf{k}_0 \simeq \mathbf{F}_p$ if $p > 0$ and $\mathbf{k}_0 = \mathbf{Q}$ if $p = 0$). Fix an algebraic closure $\overline{\mathbf{k}}$ of $\mathbf{k}$.
-a- Let $\mathcal{F}$ be a finite subset of $\mathsf{Cr}_2(\mathbf{k})$. Let $\mathbf{k}_{\mathcal{F}} \subset \overline{\mathbf{k}}$ be the finite extension of $\mathbf{k}$ which is generated by the fields $\mathbf{k}_f$, $f \in \mathcal{F}$. Let G be the subgroup of $\mathsf{Cr}_2(\mathbf{k})$ generated by the $\mathcal{F}$. Then $\mathbf{k}_g \subset \mathbf{k}_{\mathcal{F}}$ for all elements g of G. Assume now that G coincides with $\mathsf{Cr}_2(\mathbf{k})$. Then $\mathbf{k} = \mathbf{k}_{\mathcal{F}}$ is finitely generated; indeed, for each a in $\mathbf{k}$, the transformation $[x : y : z] \mapsto [x + ay : y : z]$ is defined over $\mathbf{k}_0(a)$ but not on a smaller subfield of $\overline{\mathbf{k}}$.
-b- Let R be an element of $\mathbf{k}[x]$, and consider the Jonquières transformation g_R which is defined in affine coordinates by

$$g_R(X,Y) = (X, R(X)Y).$$

Then each roots α_i of R gives rise to an indeterminacy point $(0, \alpha_i)$ of g_R^{-1}. Thus, if g_R belongs to the group G then all roots of R are contained in $\mathbf{K}$ and if G coincides with $\mathsf{Cr}_2(\mathbf{k})$ then $\mathbf{K}$ coincides with the algebraically closure $\overline{\mathbf{k}}$ of $\mathbf{k}$.

[3] I added this sub-section and the previous 7 lines in 2013 after a discussion with Jérémy Blanc and Christian Urech.

-c- Thus, if $\mathsf{Cr}_2(\mathbf{k})$ is finitely generated then $\mathbf{k}$ is finitely generated and a finite extension of $\mathbf{k}$ is algebraically closed. There is no such field. □

3.5. Consider the semi-group $\mathsf{Rat}(\mathbb{P}^2_{\mathbf{C}})$ of all rational transformations of the projective plane $\mathbb{P}^2_{\mathbf{C}}$. Since the topological degree is multiplicative, one needs rational transformations of degree p for all prime numbers p to generate $\mathsf{Rat}(\mathbb{P}^2_{\mathbf{C}})$. This remark can be strengthen with the strategy of the previous paragraphs: to generate $\mathsf{Rat}(\mathbb{P}^2_{\mathbf{C}})$, one needs as many parameters as parameters for curves (of arbitrary genus).

3.6. Let X be a smooth projective threefold. Consider, as in paragraph **3.2**, the genus $g(f)$ of each element f in $\mathsf{Bir}(X)$: this provides a subset $\mathbf{g}-\mathbf{bir}(X) = \{g(f) \mid f \in \mathsf{Bir}(X)\}$ of the set of integers which is canonically associated to X. This set is invariant under birational conjugacy, and the question arises to compute $\mathbf{g}-\mathbf{bir}(X)$ for, say, the generic smooth cubic volume in $\mathbb{P}^4_{\mathbf{C}}$.

Similarly, one can define the genus $\mathbf{g}-\mathbf{rat}(X)$ with respect to $\mathsf{Rat}(X)$ for X a smooth projective surface, and one of the first question is to compute this set for X a K3 surface (for instance, for X a generic $(2,2,2)$ surface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$).

4. Polynomial automorphisms

The proof of Hudson-Pan Theorem seems to say more than what the statement provides, but one needs to be careful when using this circle of ideas. Here is a sample example, which concerns the group $\mathsf{Aut}[\mathbf{k}^n]$ of polynomial automorphisms of the affine space $\mathbf{k}^n$.

Let p be a polynomial function in the variables $x_2, \ldots, x_n$. Let f be the automorphism of $\mathbf{k}^n$ defined by

$$f(x_1, \ldots, x_n) = (x_1 + p(x_2, \ldots, x_n), x_2, \ldots, x_n).$$

Such a transformation, and all similar transformations obtained by permuting the coordinates, are called elementary automorphisms. The subgroup generated by the group of affine transformations and the set of elementary automorphisms is the group of tame automorphisms of $\mathbf{k}^n$ (for $n = 2$, Jung theorem asserts that all automorphisms are tame; for $n = 3$, this is not true anymore since the Nagata automorphism is not tame [9]).

Assume, for the sake of simplicity, that p is homogeneous of degree $d + 1 \geq 2$, and consider the birational extension F of f to $\mathbb{P}^n_{\mathbf{k}}$. In homogeneous coordinates $[x_0 : x_1 : \ldots : x_n]$ (with the plane at infinity defined by $x_0 = 0$), one gets

$$F[x_0 : x_1 : \ldots : x_n] = [x_0^{d+1} : x_1 x_0^d + p(x_1, \ldots, x_n) : x_1 x_0^d : \ldots : x_n x_0^d].$$

Its indeterminacy locus $\mathrm{Ind}(F)$ is the set of points such that

$$x_0 = 0 \quad \text{and} \quad p(x_2, \ldots, x_n) = 0.$$

Thus, the indeterminacy locus of F is the subset Σ_p of codimension 2 of $\mathbb{P}^n_{\mathbf{k}}$ that is contained in the hyperplane at infinity and has equation $p = 0$ in this hyperplane. In the plane of codimension 2 given by $x_0 = x_1 = 0$, p defines a hypersurface, and Σ_p is the cone over this set, with vertex $[0 : 1 : 0 : \ldots : 0]$. Since p is any homogeneous polynomial in $n-1$ variables, all cones over all hypersurfaces of $\mathbb{P}^{n-2}_{\mathbf{k}}$ appear as indeterminacy set of some elementary automorphism of $\mathbf{k}^n$.

Does this imply that the group of tame automorphisms cannot be generated by automorphisms of bounded degrees ? No, as the following result shows.

Theorem 4.1 (Derksen)**.** *Let $n \geq 3$ be a natural integer. The group of tame automorphisms of $\mathbf{k}^n$ is generated by the group of affine transformations of $\mathbf{k}^n$ and the elementary automorphism*

$$(x_1, \ldots, x_n) \mapsto (x_1 + x_2^2, x_2, \ldots, x_n).$$

This result is proved in [10], chapter 5.2; the proof is not difficult (much less than Castelnuovo-Noether Theorem).

Example 4.2. Consider the elementary map $f(w,x,y,z) = (w + x^3 + y^3 + z^3, x, y, z)$. Note that f is the composition of

$$g(x,y,z) = (w + x^3 + y^3, y, z) \quad \text{and} \quad h(x,y,z) = (x + z^3, y, z).$$

The indeterminacy set of g is given by the equation $x^3 + y^3 = 0$ in the hyperplane at infinity; it is a union of three planes $\mathbb{P}^2_{\mathbf{k}}$. The indeterminacy set of h is also rational. On the other hand, the indeterminacy locus of f is a cone over the planar cubic curve $x^3 + y^3 + z^3 = 0$; as such, it is not rational.

This does not contradict Section 2: The three maps f, g, and h contract the same hypersurface, namely, the hyperplane at infinity.

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