ELEMENTS GENERATING A PROPER NORMAL SUBGROUP OF THE CREMONA GROUP

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ABSTRACT. Consider an algebraically closed field $\mathbf{k}$, and let $\mathsf{Cr}_2(\mathbf{k})$ be the Cremona group of all birational transformations of the projective plane over $\mathbf{k}$. We characterize infinite order elements $g \in \mathsf{Cr}_2(\mathbf{k})$ having a power g^n, $n \neq 0$, generating a proper normal subgroup of $\mathsf{Cr}_2(\mathbf{k})$.

1. INTRODUCTION

Let $\mathbf{k}$ be a field. Let $\mathsf{Cr}_2(\mathbf{k})$ be the group of birational transformations of the projective plane $\mathbb{P}^2_{\mathbf{k}}$; we shall call it the **Cremona group**. If f is an element of $\mathsf{Cr}_2(\mathbf{k})$, one can write f in homogeneous coordinates $[x : y : z]$ as

$$f[x : y : z] = [P : Q : R] \tag{1.1}$$

where P, Q, and R are three homogeneous polynomials of the same degree d in (x, y, z) with no common factor of positive degree: the integer d is called the degree of f, and is denoted $\deg(f)$.

The Cremona group acts by isometries on a hyperbolic space of infinite dimension $\mathbb{H}_\infty$ (see Subsection 2.3). There are three types of isometries of such a space, and therefore also three types of elements $f \in \mathsf{Cr}_2(\mathbf{k})$ (see Section 3):

(1) **elliptic** isometries correspond to birational transformations for which the sequence $(\deg(f^n))$ is bounded;
(2) **parabolic** isometries correspond to a polynomial growth of $(\deg(f^n))$: either $\deg(f^n)$ grows linearly and f is called a **Jonquières twist**, or $\deg(f^n)$ grows quadratically and f is called a **Halphen twist**;
(3) **loxodromic** isometries correspond to birational transformations for which $\deg(f^n)$ grows exponentially fast as n goes to $+\infty$.

The name for Jonquières and Halphen twists come from the following properties. If $\mathbf{k}$ is algebraically closed, such a twist $f \colon \mathbb{P}^2_{\mathbf{k}} \dashrightarrow \mathbb{P}^2_{\mathbf{k}}$ preserves a unique pencil of curves, of genus 0 or 1 respectively. Moreover, every pencil of rational curves is equivalent, via a birational change of coordinates, to the pencil of lines through the point $[0 : 0 : 1]$, and the group of birational transformations

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preserving this pencil is known as the Jonquières group. And every pencil of curves of genus 1 is equivalent to a Halphen pencil (see Section 4).

A **monomial** transformation is a birational transformation which, in affine coordinates $(x, y) = [x : y : 1]$, can be written $f(x, y) = (x^a y^b, x^c y^d)$ for some integers a, b, c, and d. The group of all monomial transformations is isomorphic to $\mathsf{GL}_2(\mathbf{Z})$.

In a group G, we say that an element h **generates a proper normal subgroup** if the smallest normal subgroup containing h is not equal to G.

Theorem A. *Let* $\mathbf{k}$ *be an algebraically closed field of characteristic* 0. *Let* f *be an element of* $\mathsf{Cr}_2(\mathbf{k})$ *of infinite order. The following properties are equivalent.*

  (a) For some $n \neq 0$, f^n *generates a proper normal subgroup of* $\mathsf{Cr}_2(\mathbf{k})$.
  (b) The birational transformation f *is a Halphen twist or a loxodromic element that is not conjugate to a monomial transformation.*

A version of this theorem in positive characteristic is given in Section 3.4, Theorem A'. The proof of this theorem is a combination of several recent results together with one new input. Indeed, the heart of our article is Theorem B:

Theorem B. *There exists a positive integer* n_0 *with the following property. Let* $\mathbf{k}$ *be a field. If* $g \in \mathsf{Cr}_2(\mathbf{k})$ *is a Halphen twist, then* g^{n_0} *generates a proper normal subgroup of* $\mathsf{Cr}_2(\mathbf{k})$. *Moreover, this normal subgroup is a free product of free abelian groups of rank* ≤ 8.

The proof of Theorem B combines algebraic geometry with geometric group theory, as in [12] and [8]. In Theorem B, n_0 does not depend on the Halphen twist g, but in Theorem A, the integer n must depend on f: see Theorem 3.11. We refer to Theorem B', stated in Section 6.3, for a slightly stronger result.

2. RATIONAL SURFACES AND HYPERBOLIC SPACES

2.1. **The lattice** $\mathbf{Z}^{1,n}$**.** Let n be a (maybe infinite) cardinal number; in what follows n will be either a positive integer or the cardinality of the field $\mathbf{k}$.

Denote by $\mathbf{R}^n$ a real Hilbert space of dimension n, with a fixed orthonormal Hilbert basis $(\mathbf{e}_j)_{j \in J}$, where J has cardinal n (which may be uncountable). Then, consider a one dimensional space $\mathbf{Re}_0$, and define $\mathbf{R}^{1,n}$ to be the direct

sum $\mathbf{R}\mathbf{e}_0 \oplus \mathbf{R}^n$, together with the (indefinite) scalar product defined by

$$u \cdot v = a_0 b_0 - \sum_{j \in J} a_j b_j \tag{2.1}$$

for every pair of vectors $u = a_0 \mathbf{e}_0 + \sum_j a_j \mathbf{e}_j$ and $v = b_0 \mathbf{e}_0 + \sum_j b_j \mathbf{e}_j$ with coefficients satisfying $\sum_j a_j^2 < +\infty$ and $\sum_j b_j^2 < +\infty$. We shall also set $u^2 = u \cdot u$.

The subset $\mathbb{H}_n \subset \mathbf{R}^{1,n}$ is defined as the set of vectors u in $\mathbf{R}^{1,n}$ with $u^2 = 1$ and $u \cdot \mathbf{e}_0 > 0$. With the distance dist defined by the formula

$$\cosh(\mathrm{dist}(u,v)) = u \cdot v, \tag{2.2}$$

$\mathbb{H}_n$ is isometric to the classical hyperbolic space of dimension n. Its boundary $\partial \mathbb{H}_n$ can be identified with the isotropic rays $\mathbf{R}\xi$ (where ξ is any non-zero vector in $\mathbf{R}^{1,n}$ with $\xi^2 = 0$).

The lattice $\mathbf{Z}^{1,n}$ is, by definition, the subset of vectors $u = \sum_i a_i \mathbf{e}_i \in \mathbf{R}^{1,n}$ with integer coefficients (a_i). The family $(\mathbf{e}_i)_{i \in I}$ is a basis of $\mathbf{Z}^{1,n}$ and the quadratic form associated to the scalar product satisfies

$$\mathbf{e}_0 \cdot \mathbf{e}_0 = 1, \; \mathbf{e}_j \cdot \mathbf{e}_j = -1 \text{ for } 1 \leq j \leq n, \text{ and } \mathbf{e}_i \cdot \mathbf{e}_j = 0 \text{ if } i \neq j. \tag{2.3}$$

When n is finite, this is the standard odd, unimodular quadratic form of signature $(1,n)$.

To simplify the exposition, we shall write abusively $\mathbf{R}^{1,\infty}$ instead of $\mathbf{R}^{1,n}$ for any infinite n.

2.2. **Blow-ups of the plane.** Now, let X be a rational surface which is obtained from $\mathbb{P}^2_{\mathbf{k}}$ by n successive blow-ups; thus, X comes with a birational morphism $\pi \colon X \to \mathbb{P}^2$. Denote by $\mathrm{NS}(X)$ the Néron-Severi group of X, by $\mathbf{e}_0$ the class of a line and by $\mathbf{e}_i$, $1 \leq i \leq n$, the classes of the exceptional divisors (more precisely, $\mathbf{e}_0$ is the pull-back of the class of a line by π, and each $\mathbf{e}_i$ is the pull-back in $\mathrm{NS}(X)$ of the class of an exceptional divisor). The $\mathbf{e}_i$ form a basis of $\mathrm{NS}(X)$ and the intersection products between the $\mathbf{e}_i$ are exactly as in Equation (2.3). Thus, $\mathrm{NS}(X)$ is isometric to the lattice $\mathbf{Z}^{1,n}$, by a unique isometry identifying the two bases $(\mathbf{e}_i)$. Viewed in $\mathrm{NS}(X;\mathbf{R}) = \mathrm{NS}(X) \otimes \mathbf{R}$, the hyperbolic space $\mathbb{H}_n$ will be denoted by $\mathbb{H}_X$.

2.3. **Infinitely many blow-ups.** Blowing up all possible points of $\mathbb{P}^2_{\mathbf{k}}$, including infinitely near ones, one gets a nested family of rational surfaces $\pi \colon X \to \mathbb{P}^2_{\mathbf{k}}$. The inductive limit of their Néron-Severi groups $\mathrm{NS}(X)$ is a well defined, infinite dimensional $\mathbf{Z}$-module $\mathcal{Z}$; by definition, $\mathcal{Z}$ is the **Picard-Manin space**. It comes with an intersection form and a natural basis $(\mathbf{e}_i)$, where $\mathbf{e}_0$ is the class of a line in the plane, and each $\mathbf{e}_j$, $j \neq 0$, is the class of the exceptional divisor of a point (in some rational surface $X \to \mathbb{P}^2_{\mathbf{k}}$). Their relative intersections satisfy the Equation (2.3). We refer to [6, 15, 22] for a detailed account on

this construction (and the definition of the bubble space indexing the elements $\mathbf{e}_j$ of that basis). With the notation of Section 2.1, $\mathcal{Z}$ is isometric to a lattice $\mathbf{Z}^{1,\infty} \subset \mathbf{R}^{1,\infty}$; here, $\mathbf{R}^{1,\infty}$ is the direct sum of $\mathbf{Re}_0$ and of the ℓ^2-completion of $\oplus_{i\neq 0}\mathbf{Re}_i$.

Now assume $\mathbf{k}$ to be algebraically closed. Since all points have been blown-up to construct $\mathcal{Z}$, all indeterminacy points of birational transformations are resolved, and the Cremona group acts by isometries on $\mathcal{Z}$ and on the hyperbolic space $\mathbb{H}_\infty \subset \mathbf{R}^{1,\infty}$ (see [6, 22]):

Theorem 2.1. *Let $\mathbf{k}$ be an algebraically closed field. The group $\mathsf{Cr}_2(\mathbf{k})$ acts faithfully by linear isometries on $\mathcal{Z} \simeq \mathbf{Z}^{1,\infty}$. In particular, it acts faithfully by isometries on the infinite dimensional hyperbolic space $\mathbb{H}_\infty \subset \mathbf{R}^{1,\infty}$.*

Remark 2.2. If $\mathbf{k}$ is not algebraically closed, we can choose an algebraic closure $\bar{\mathbf{k}}$ of $\mathbf{k}$, and embed $\mathsf{Cr}_2(\mathbf{k})$ into $\mathsf{Cr}_2(\bar{\mathbf{k}})$ to get a faithful action on an infinite dimensional hyperbolic space.

3. TYPES OF BIRATIONAL TRANSFORMATIONS

From Section 2.3 we know that the Cremona group $\mathsf{Cr}_2(\mathbf{k})$ acts faithfully by isometries on $\mathcal{Z}$ (with respect to the intersection product) and on $\mathbb{H}_\infty$ (with respect to dist). Since there are three types of isometries for such a space, we obtain three types of birational transformations of the plane: elliptic, parabolic, and loxodromic elements of $\mathsf{Cr}_2(\mathbf{k})$. We shall divide the proof of Theorem A (and Theorem A' below) in three cases, according to the type of $f \in \mathsf{Cr}_2(\mathbf{k})$.

3.1. **Elliptic elements.** A birational transformation $f \in \mathsf{Cr}_2(\mathbf{k})$ is **elliptic** if and only if its orbits on $\mathbb{H}_\infty$ are bounded, if and only if the sequence $(\deg(f^n))$ is bounded. In [2], Jérémy Blanc and Julie Déserti prove that any elliptic element f of infinite order is conjugate to an element f' of $\mathsf{PGL}_3(\mathbf{k})$ (their proof is written for $\mathbf{k} = \mathbf{C}$, but it works over any algebraically closed field). We can now follow an argument of Dominique Cerveau and Julie Déserti (see [9]). The smallest normal subgroup containing f contains also f', and because $\mathsf{PGL}_3(\mathbf{k})$ is simple, it contains $\mathsf{PGL}_3(\mathbf{k})$. In particular, it contains the involution $\eta([x:y:z]) = [-x:-y:z]$. But η is conjugate to $\sigma([x:y:z]) = [yz:zx:xy]$ in $\mathsf{Cr}_2(\mathbf{k})$ and by Noether-Castelnuovo theorem σ and $\mathsf{PGL}_3(\mathbf{k})$ generate $\mathsf{Cr}_2(\mathbf{k})$; thus the normal subgroup generated by f is equal to $\mathsf{Cr}_2(\mathbf{k})$. This proves the following lemma, and therefore also Theorem A over arbitrary algebraically closed field for elliptic elements.

Lemma 3.1. *Let $\mathbf{k}$ be an algebraically closed field. Let f be a non-trivial element of $\mathsf{PGL}_3(\mathbf{k})$ or an elliptic element of $\mathsf{Cr}_2(\mathbf{k})$ of infinite order. The smallest normal subgroup of $\mathsf{Cr}_2(\mathbf{k})$ containing f is equal to $\mathsf{Cr}_2(\mathbf{k})$.*

Remark 3.2. It would be interesting to study the finite order elements of $\mathsf{Cr}_2(\mathbf{k})$ which are not conjugate to an element of $\mathsf{PGL}_3(\mathbf{k})$, and to decide which ones generate a proper normal subgroup of $\mathsf{Cr}_2(\mathbf{k})$.

Remark 3.3. Lemma 3.1 requires $\mathbf{k}$ to be algebraically closed. When $\mathbf{k}$ is the field $\mathbf{R}$ of real numbers, the smallest normal subgroup containing $\mathsf{PGL}(3, \mathbf{R})$ is a proper normal subgroup of $\mathsf{Cr}_2(\mathbf{R})$ ([25, Theorem 1.1]). It is also the case for many other non-algebraically closed fields ([20, Theorem C]).

3.2. **Parabolic elements and Jonquières twists.** By definition, a **parabolic** element of $\mathsf{Cr}_2(\mathbf{k})$ acts on $\mathbb{H}_\infty$ without fixed point, but with a unique fixed point on the boundary $\partial \mathbb{H}_\infty$. According to [13, 17], there are in fact two types of parabolic elements:

(1) the sequence $(\deg(f^n))$ grows linearly, and in that case, one says that f is a **Jonquières twist**.
(2) the sequence $(\deg(f^n))$ grows quadratically and in that case, f is called a **Halphen twist**.

This result holds over any field.

Let us assume that f is a Jonquières twist first (we shall study Halphen twists in Section 4), and that $\mathbf{k}$ is algebraically closed. Then, it is proved in [13] that f preserves a unique pencil of rational curves in $\mathbb{P}_\mathbf{k}^2$ (this result of Jeffrey Diller and Charles Favre is stated for the field of complex numbers, but the proofs apply to any algebraically closed field). Up to conjugacy by an element of $\mathsf{Cr}_2(\mathbf{k})$, there is a unique pencil of rational curves in the plane, namely the pencil of lines through a point. Thus, every Jonquières twist is conjugate to a birational transformation of the affine plane $\mathbb{A}_\mathbf{k}^2$ that permutes the vertical lines; in affine coordinates, such a transformation can be written

$$f(x, y) = (A(x), B_x(y)) \tag{3.1}$$

where A is in $\mathsf{PGL}_2(\mathbf{k})$ and B_x is in $\mathsf{PGL}_2(\mathbf{k}(x))$. Dominique Cerveau and Julie Déserti proved in [9, Prop. 5.21] that the smallest normal subgroup of $\mathsf{Cr}_2(\mathbf{k})$ containing such an element $f \neq \mathrm{Id}$ coincides always with $\mathsf{Cr}_2(\mathbf{k})$. Thus, we obtain the following lemma ([1]).

Lemma 3.4. *Let* $\mathbf{k}$ *be an algebraically closed field. If* $f \in \mathsf{Cr}_2(\mathbf{k})$ *is a Jonquières twist, the smallest normal subgroup of* $\mathsf{Cr}_2(\mathbf{k})$ *containing* f *coincides with* $\mathsf{Cr}_2(\mathbf{k})$.

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[1]The proof works as follows. Write f as in Equation (3.1), and choose $C_x \in \mathsf{PGL}_2(\mathbf{k}(x))$ such that the transformation $h(x, y) = (x, C_x(y))$ does not commute to f. Then, the commutator $[f, h]$ is an element of the simple group $\mathsf{PGL}_2(\mathbf{k}(x))$. This proves that the smallest normal subgroup containing f contains $\mathsf{PGL}_2(\mathbf{k})$, and then one can apply Lemma 3.1.

This proves Theorem A over any algebraically closed field when f is a Jonquières twist, because all iterates f^n, $n \neq 0$, are again Jonquières twists.

Remark 3.5. Lemma 3.4 requires $\mathbf{k}$ be algebraically closed. When $\mathbf{k} = \mathbf{R}$ is the field of real numbers, there are Jonquières twists generating proper normal subgroups (see [25], and [20] for many other perfect fields).

3.3. **Loxodromic transformations.**
An element f of $\mathsf{Cr}_2(\mathbf{k})$ is **loxodromic** if it fixes two boundary points of $\mathbb{H}_\infty$ and acts as a non-trivial translation on the geodesic joining these two points. If $L(f)$ is the hyperbolic length of this translation, then the sequence $(\deg(f^n))$ grows like $\exp(|n|L(f))$ as n goes to $\pm\infty$. By definition, the number

$$\lambda(f) = \exp(L(f)) = \lim_{n \to +\infty} \deg(f^n)^{1/n} \tag{3.2}$$

is the **dynamical degree** of f (see [5]).

3.3.1. *p-Automorphisms (see [6, 24]).* Before studying all loxodromic elements, we focus on a class of birational transformations that is defined only when the characteristic of $\mathbf{k}$ is positive. Note that Theorem 3.8 has been obtained under slightly more restrictive hypotheses by Nicholas Shepherd-Barron in a recent version of [24].

Assume that $\mathbf{k}$ is a field of characteristic $\mathrm{char}(\mathbf{k}) = p > 0$. Let $\mathrm{Fr}(t) = t^p$ be the Frobenius endomorphism of $\mathbf{k}$. An element $\varphi(t)$ of $\mathbf{k}[t]$ is a **linearized polynomial** or is a *p***-polynomial** if all its monomials have degree p^j for some $j \geq 0$; in other words, one can write

$$\varphi(t) = \sum_j a_j t^{p^j} = \sum_j a_j \mathrm{Fr}^j(t) \tag{3.3}$$

where Fr^j is the j-th iterate of the Frobenius endomorphism, i.e. $t^{p^j} = \mathrm{Fr}^j(t) = \mathrm{Fr} \circ \mathrm{Fr} \circ \cdots \circ \mathrm{Fr}(t)$ (j compositions). These p-polynomials are exactly the polynomial transformations of $\mathbf{k}$ which are additive $\varphi(x+y) = \varphi(x) + \varphi(y)$. The composition of two p-polynomials φ and $\psi \colon \mathbf{k} \to \mathbf{k}$ is another p-polynomial; with the laws given by addition and composition, the set of linearized polynomials $\mathbf{k}[\mathrm{Fr}]$ is, naturally, a non-commutative $\mathbf{k}$-algebra.

Let $\mathbb{G}_a$ denote the additive group of dimension 1. Every 2×2 matrix with coefficients in $\mathbf{k}[\mathrm{Fr}]$ determines an algebraic endomorphism of the algebraic group $\mathbb{G}_a(\mathbf{k}) \times \mathbb{G}_a(\mathbf{k})$: if a, b, c and d are the coefficients of the matrix, the endomorphism is given by

$$f(x,y) = (a(x) + b(y), c(x) + d(y)). \tag{3.4}$$

It is invertible if and only if the matrix is invertible over the non-commutative ring $\mathbf{k}[\mathrm{Fr}]$ and, in fact, one easily proves the following lemma.

Lemma 3.6. *Let* **k** *be a field of characteristic* $p > 0$. *The group of algebraic automorphisms of the algebraic group* $\mathbb{G}_a(\mathbf{k}) \times \mathbb{G}_a(\mathbf{k})$ *coincides with the group* $\mathsf{GL}_2(\mathbf{k}[\mathrm{Fr}])$.

Sketch of proof. Consider an algebraic automorphism F of the algebraic group $\mathbb{G}_a(\mathbf{k}) \times \mathbb{G}_a(\mathbf{k})$. Writing $F(x,y) = (U_1(x,y), U_2(x,y))$, we see that U_i satisfies the relation $U_i(x_1 + x_2, y_1 + y_2) = U_i(x_1, y_1) + U_i(x_2, y_2)$. Looking at the highest degree terms of U_i, one proves by induction that $U_i(x,y) = a(x) + b(y)$ for some pair of linearized polynomials. Since F is invertible, one concludes that F is an element of $\mathsf{GL}_2(\mathbf{k}[\mathrm{Fr}])$. $\qquad \square$

Consider the group $\mathsf{T} \subset \mathsf{Cr}_2(\mathbf{k})$ consisting of all translations

$$t_{u,v}(x,y) = (x+u, y+v); \tag{3.5}$$

this group is isomorphic to $\mathbb{G}_a(\mathbf{k}) \times \mathbb{G}_a(\mathbf{k})$. We define the **group** $\mathsf{pAut}(\mathbb{A}_{\mathbf{k}}^2)$ **of** *p***-automorphisms** of the affine plane as the normalizer of T in $\mathsf{Cr}_2(\mathbf{k})$.

Lemma 3.7. *The group* $\mathsf{pAut}(\mathbb{A}_{\mathbf{k}}^2)$ *is a subgroup of* $\mathsf{Aut}(\mathbb{A}_{\mathbf{k}}^2)$, *and*

$$\mathsf{pAut}(\mathbb{A}_{\mathbf{k}}^2) = \mathsf{GL}_2(\mathbf{k}[\mathrm{Fr}]) \ltimes (\mathbb{G}_a(\mathbf{k}) \times \mathbb{G}_a(\mathbf{k})).$$

Proof. Let $f \in \mathsf{Cr}_2(\mathbf{k})$ be an element that normalizes T. For all $(a,b) \in \mathbb{G}_a(\mathbf{k}) \times \mathbb{G}_a(\mathbf{k})$, there exists a unique $(\tilde{a}, \tilde{b}) \in \mathbb{G}_a(\mathbf{k}) \times \mathbb{G}_a(\mathbf{k})$ such that

$$f \circ t_{a,b} \circ f^{-1} = t_{\tilde{a}, \tilde{b}}. \tag{3.6}$$

The map $F : (a,b) \mapsto (\tilde{a}, \tilde{b})$ is birational, and is an automorphism of the algebraic group $\mathsf{T} = \mathbb{G}_a(\mathbf{k}) \times \mathbb{G}_a(\mathbf{k})$. Then F is in fact a regular algebraic automorphism, and Lemma 3.6 shows that $F \in \mathsf{GL}_2(\mathbf{k}[\mathrm{Fr}])$. The equation $f \circ t_{a,b} = t_{F(a,b)} \circ f$ for all $(a,b) \in \mathbb{G}_a(\mathbf{k}) \times \mathbb{G}_a(\mathbf{k})$ shows that f is a biregular map of $\mathbb{G}_a(\mathbf{k}) \times \mathbb{G}_a(\mathbf{k})$ to itself, and since $f(a,b) = F(a,b) + f(0,0)$ for all $(a,b) \in \mathbb{G}_a(\mathbf{k}) \times \mathbb{G}_a(\mathbf{k})$, we deduce that $f \in \mathsf{GL}_2(\mathbf{k}[\mathrm{Fr}]) \ltimes (\mathbb{G}_a(\mathbf{k}) \times \mathbb{G}_a(\mathbf{k}))$. $\qquad \square$

When we say that an element q of $\mathbf{k}[\mathrm{Fr}]$ has degree d (with respect to Fr), we mean that $q = \sum_{j=0}^{d} q_j \mathrm{Fr}^j$ with $q_d \neq 0$; writing $\mathrm{Fr}^j(t) = t^{p^j}$, q becomes an element of $\mathbf{k}[t]$ of degree p^d.

Let $\mathsf{Aff}(\mathbb{A}_{\mathbf{k}}^2) = \mathsf{GL}_2(\mathbf{k}) \ltimes (\mathbb{G}_a(\mathbf{k}) \times \mathbb{G}_a(\mathbf{k}))$ be the group of affine transformations of the affine plane $\mathbb{A}_{\mathbf{k}}^2$, a subgroup of $\mathsf{pAut}(\mathbb{A}_{\mathbf{k}}^2)$. Let $\mathsf{El}(\mathbb{A}_{\mathbf{k}}^2)$ be the group of all elementary automorphisms, i.e. automorphisms

$$h(x,y) = (ux + q(y), vy + w) \tag{3.7}$$

with $q \in \mathbf{k}[y]$, u, v, and w in $\mathbf{k}$, and $uv \neq 0$. The subgroup $\mathsf{pEl}(\mathbb{A}_{\mathbf{k}}^2) = \mathsf{El}(\mathbb{A}_{\mathbf{k}}^2) \cap \mathsf{pAut}(\mathbb{A}_{\mathbf{k}}^2)$ of elementary *p*-automorphisms consists of automorphisms h as in Equation (3.7) with $q \in \mathbf{k}[\mathrm{Fr}]$.

Theorem 3.8. *Let* **k** *be a field of characteristic* $p > 0$.

(1) *The group* $\mathsf{pAut}(\mathbb{A}^2_{\mathbf{k}})$ *is generated by* $\mathsf{Aff}(\mathbb{A}^2_{\mathbf{k}})$ *and* $\mathsf{pEl}(\mathbb{A}^2_{\mathbf{k}})$. *It is the amalgamated product of these two groups along their intersection.*

(2) *The dynamical degree of any p-automorphism is a non-negative power of p.*

Proof. To ligthen notations, we set $\mathsf{A} = \mathsf{Aff}(\mathbb{A}^2_{\mathbf{k}})$, $\mathsf{E} = \mathsf{El}(\mathbb{A}^2_{\mathbf{k}})$ and $\mathsf{pE} = \mathsf{pEl}(\mathbb{A}^2_{\mathbf{k}})$. We first prove that A and pE generate $\mathsf{pAut}(\mathbb{A}^2_{\mathbf{k}})$. By Lemma 3.7, it suffices to prove that $\mathsf{GL}_2(\mathbf{k}[\mathrm{Fr}]) \subset \langle \mathsf{A}, \mathsf{pE} \rangle$.

Consider $f \in \mathsf{GL}_2(\mathbf{k}[\mathrm{Fr}])$ with coefficients a, b, c, and d in $\mathbf{k}[\mathrm{Fr}]$, as in Equation (3.4). If $c = 0$, then $f(x, y) = (a(x) + b(y), d(y))$, and since f is an automorphism of $\mathbb{A}^2_{\mathbf{k}}$ the degree of $a(t)$ and $d(t)$ with respect to t must be equal to 1. Thus, f is an elementary p-automorphism, i.e. $f \in \mathsf{pE}$. If $a = 0$, consider the linear automorphism $J \in \mathsf{A}$ defined by $J(x, y) = (y, x)$. Then as above, $J \circ f \in \mathsf{pE}$ and we are done.

Assume now that $ac \neq 0$. Write $a = \sum_{i=0}^{m} a_i \mathrm{Fr}^j$ and $c = \sum_{j=0}^{n} c_j \mathrm{Fr}^j$ with $m = \deg(a)$ and $n = \deg(c)$ (the degrees are with respect to Fr). We argue by induction on the complexity $\deg(a) + \deg(c)$. Up to changing f into $J \circ f$, we may assume that $m \leq n$. Set $u = c_n(\mathrm{Fr}^{n-m}(a_m))^{-1}$, and compose f with the p-automorphism

$$g(x, y) = \left(x, y - u\mathrm{Fr}^{n-m}(x)\right) \in J \circ \mathsf{pE} \circ J^{-1} \tag{3.8}$$

to get a new element $g \circ f$ of $\mathsf{GL}_2(\mathbf{k}[\mathrm{Fr}])$ with lower complexity. We conclude by induction that $g \circ f$ and therefore f lie in $\langle \mathsf{A}, \mathsf{pE} \rangle$.

To conclude the proof of the first assertion, set $\mathsf{S} = \mathsf{A} \cap \mathsf{pE} = \mathsf{A} \cap \mathsf{E}$; this is the group of affine automorphisms of $\mathbb{A}^2_{\mathbf{k}}$ whose linear part is upper triangular. By the theorem of Jung and van der Kulk, one has a decomposition into an amalgamated product $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{k}}) = \mathsf{A} *_{\mathsf{S}} \mathsf{E}$. From Theorem 7 of [23], chapter 4, $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{k}})$ acts on a tree $\mathcal{T}$, with a fundamental domain given by an edge $e = v_1 v_2$, where the stabilizer of e, v_1 and v_2 are respectively S, A and E. Restricting this action to the subgroup $\mathsf{pAut}(\mathbb{A}^2_{\mathbf{k}})$, the stabilizers of e, v_1 and v_2 in $\mathsf{pAut}(\mathbb{A}^2_{\mathbf{k}})$ are respectively S, A and pE. The orbit of e under $\mathsf{pAut}(\mathbb{A}^2_{\mathbf{k}})$ is a subtree of $\mathcal{T}$, to which we can apply Theorem 6 of [23], chapter 4: this gives the amalgamated product structure we were looking for.

For the second assertion, take a p-automorphism f, and write it as a composition

$$f = g_\ell \circ g_{\ell-1} \circ \cdots \circ g_1 \tag{3.9}$$

where each factor g_i is an element of A or pE. We say that such a decomposition is reduced if none of the g_i lies in S, and no two consecutive g_i lie in the same factor. Every p-automorphism has such a reduced decomposition, unless it is an element of S. We say that it is cyclically reduced if, moreover, g_n and g_1 are not both in A or pE.

If f lies in pE or A, then the degree of f^n is bounded by $\deg(f)$, so the dynamical degree of f is $\lambda(f) = 1$. If f is written as a reduced composition of length ≥ 2, Theorem 2.1 in [16] asserts that $\deg(f) = \prod_i \deg(g_i)$. Up to conjugating f in $\mathsf{pAut}(\mathbb{A}^2_{\mathbf{k}})$, we can assume that this decomposition is cyclically reduced so that the composition

$$f^n = (g_\ell \circ g_{\ell-1} \circ \cdots \circ g_1) \circ \cdots \circ (g_\ell \circ g_{\ell-1} \circ \cdots \circ g_1) \tag{3.10}$$

is also reduced for all $n \geq 1$, and $\deg(f^n) = \prod_i \deg(g_i)^n = \deg(f)^n$ by Theorem 2.1 in [16]. This implies $\lambda(f) = \deg(f)$, and this number is a power of p because f is a p-automorphism. $\qquad\square$

3.3.2. *A result of Shepherd-Barron.*

In [8], a criterion is given to show that (a large iterate of) a given loxodromic element f of the Cremona group $\mathsf{Cr}_2(\mathbf{k})$ generates a proper normal subgroup of $\mathsf{Cr}_2(\mathbf{k})$. In [24], Shepherd-Barron proves that this criterion is satisfied by every loxodromic element f, except in two cases: when f is conjugate to a monomial transformation, or when $\mathrm{char}(\mathbf{k}) = p > 0$ and f is conjugate to a p-automorphism of the plane. These results prove one direction of the following theorem.

Theorem C. *Let $\mathbf{k}$ be an algebraically closed field, and let f be a loxodromic element of $\mathsf{Cr}_2(\mathbf{k})$. The following properties are equivalent:*

   (a) there is an integer $\ell \geq 1$ such that f^ℓ generates a proper normal subgroup of $\mathsf{Cr}_2(\mathbf{k})$;

   (b) f is not conjugate to a monomial transformation or, when $p := \mathrm{char}(\mathbf{k}) > 0$, to a p-automorphism of the plane.

To conclude the proof of this theorem, one needs to prove that monomial transformations and p-automorphisms do not generate proper normal subgroups. Let us do it when $f \neq \mathrm{Id}$ is a p-automorphism. Write $f(x,y) = F(x,y) \circ t_{a,b}$ where $t_{a,b}$ is a translation. Since f is loxodromic, F is not the identity. Then, for every translation $t_{u,v}$ (as in Equation 3.5), one gets

$$f \circ t_{u,v} \circ f^{-1} \circ t_{u,v}^{-1} = t_{u',v'} \tag{3.11}$$

where $(u'+u, v'+v) = F(u,v)$. One can choose $t_{u,v}$ in such a way that $t_{u',v'} \neq \mathrm{Id}$, because F is not the identity. This implies that the smallest normal subgroup containing f contains a non-trivial element $t_{u',v'}$ of $\mathsf{PGL}_3(\mathbf{k})$, and the conclusion follows from Lemma 3.1. The proof is the same for monomial transformations, replacing the translations $t_{u,v}$ by the diagonal transformations $d_{u,v} \colon (x,y) \mapsto (ux, vy)$ with $uv \neq 0$ (i.e. the additive group $\mathbb{G}_a \times \mathbb{G}_a$ by the multiplicative group $\mathbb{G}_m \times \mathbb{G}_m$).

Remark 3.9. When the field is not algebraically closed, the theorem of Shepherd-Barron may be stated as follows: let f be a loxodromic element of $\mathsf{Cr}_2(\mathbf{k})$; if

the normal subgroup generated by f^n coïncides with $\mathsf{Cr}_2(\mathbf{k})$ for all $n \neq 0$, then f normalizes a commutative algebraic subgroup $G \subset \mathsf{Cr}_2(\mathbf{k})$ such that $G(\mathbf{k})$ is infinite and G becomes isomorphic to $\mathbb{G}_a^2$ or $\mathbb{G}_m^2$ on any algebraic closure of $\mathbf{k}$. In $\mathsf{Cr}_2(\mathbf{R})$, the derived subgroup is a subgroup of infinite index that contains all monomial transformations (this follows from [25, Theorem 1.1] because the group of monomial transformations is generated by elements of $\mathsf{Bir}(\mathbb{P}_{\mathbf{k}}^2)$ of degree ≤ 2). If $\mathbf{k}$ is any perfect field with an algebraic Galois extension of degree 8 (for instance any number field), Stéphane Lamy and Susanna Zimmermann show that the smallest normal subgroup of $\mathsf{Cr}_2(\mathbf{k})$ containing any given monomial transformation is a proper subgroup of $\mathsf{Cr}_2(\mathbf{k})$ ([20]).

3.3.3. *An example.* Let m, and q be positive integers. Set $n = m^2 - 1$. Consider the element h of $\mathsf{Cr}_2(\mathbf{k})$ defined in affine coordinates by

$$h(x,y) = (y, x + y^e) \tag{3.12}$$

with $e = m$ modulo n. Let α be a root of unity such that $\alpha^n = 1$ (i.e. $\alpha^{m^2} = \alpha$). Set $\beta = \alpha^m$ and consider the diagonal linear automorphism $g(x,y) = (\alpha x, \beta y)$. Then

$$h \circ g \circ h^{-1} = g^m. \tag{3.13}$$

Now, compose h with a monomial transformation $f(x,y) = (x^a y^b, x^c y^d)$. Then, $(f \circ h) \circ g \circ (f \circ h)^{-1} = g^q$ if and only if the integers a, b, c, and d satisfy

$$am + b = q \quad \text{and} \quad cm + d = qm \qquad \text{mod } (n). \tag{3.14}$$

We can choose $c = q$ and $d = 0$ modulo n, and then b such that $bq = \pm 1$ modulo n (this is possible if q is invertible modulo n). Since m is invertible modulo n, we then set $a = (q - b)/m$ modulo n. Thus, the system of congruences (3.14) has solutions for which $ad - bc = \pm 1$ modulo n; then, one can lift these solutions to elements of $\mathsf{GL}_2(\mathbf{Z})$. One gets more solutions by composition with elements in the kernel of the projection $\mathsf{GL}_2(\mathbf{Z}) \to \mathsf{GL}_2(\mathbf{Z}/n\mathbf{Z})$.

Proposition 3.10. *Let $\mathbf{k}$ be a field. Let m and q be positive integers, set $n = m^2 - 1$, and assume that q is invertible modulo n. Then, there is a loxodromic element $s \in \mathsf{Cr}_2(\mathbf{k})$ and an elliptic element $g \in \mathsf{PGL}_3(\mathbf{k}) \subset \mathsf{Cr}_2(\mathbf{k})$ such that*

(1) g is of order n;
(2) $s \circ g \circ s^{-1} = g^q$;
(3) s is not conjugate to a monomial map or a p-automorphism (if $\mathrm{char}(\mathbf{k}) = p > 0$).

Sketch of proof. Take f, g, and h as above, and set $s = f \circ h$. All we need to show is that one can choose f in such a way that s is not conjugate to a monomial map or a p-automorphism. Composing f with monomial maps given by matrices $M \in \mathsf{GL}_2(\mathbf{Z})$ which are equal to Id modulo n, and choosing e large, we

may assume that the entries of f satisfy $e > c > d > a > b > 1$. Then, a simple recursion shows that the degree of $(f \circ h)^k$ is equal to $(c + de)^k$; in particular, the dynamical degree of $f \circ h$ is equal to the integer $c + de$. Thus, $f \circ h$ is not conjugate to a monomial map because the dynamical degree of a loxodromic monomial map is not an integer: it is a quadratic unit. And, changing e if necessary, one sees that $f \circ h$ is not conjugate to a p-automorphism because the dynamical degree of such an automorphism of the plane is a power of p (see Theorem 3.8). $\qquad\square$

Pick such a pair of elements (s, g) in the Cremona group. The smallest normal subgroup generated by s^k contains

$$[s^k, g] = s^k \circ g \circ s^{-k} \circ g^{-1} = g^{q^k - 1}. \tag{3.15}$$

Thus, if $q^k \neq 1$ modulo n for all $1 \leq k \leq \ell$, the smallest normal subgroup generated by s^k, for any $k \leq \ell$, contains a non-trivial elliptic element and coïncides with $\mathrm{Cr}_2(\mathbf{k})$ if $\mathbf{k}$ is algebraically closed (Lemma 3.1). We obtain:

Theorem 3.11. *Let $\mathbf{k}$ be an algebraically closed field. For every integer ℓ, there is a loxodromic element s in $\mathrm{Cr}_2(\mathbf{k})$ such that s^n generates a proper normal subgroup of $\mathrm{Cr}_2(\mathbf{k})$ for some $n > \ell$, but not for $n \leq \ell$.*

This result shows that, unlike in Theorem B, one cannot take n independent of f in Theorem A and Theorem A' below.

3.4. **Theorem A'.** We can now state the extension of Theorem A' to algebraically closed fields of arbitrary characteristic.

Theorem A'. *Let $\mathbf{k}$ be an algebraically closed field. Let f be an element of $\mathrm{Cr}_2(\mathbf{k})$ of infinite order. The following properties are equivalent.*

    (a) There exists an integer $n > 0$ such that f^n generates a non-trivial proper normal subgroup of $\mathrm{Cr}_2(\mathbf{k})$.

    (b) The birational transformation f is a Halphen twist or it is a loxodromic element that is not conjugate to a monomial transformation or to a p-automorphism when $p := \mathrm{char}(\mathbf{k}) > 0$.

We already proved this result for elements $f \in \mathrm{Cr}_2(\mathbf{k})$ which are not Halphen twists. The rest of this paper is devoted to this last case.

4. HALPHEN TWISTS, HALPHEN PENCILS, AND TRANSLATION LENGTHS

By definition, a Halphen twist f in $\mathrm{Cr}_2(\mathbf{k})$ is a birational transformation $f : \mathbb{P}^2_{\mathbf{k}} \to \mathbb{P}^2_{\mathbf{k}}$ such that $\deg(f^n)$ grows quadratically. In this section, we describe

a result of Marat Gizatullin showing that such a birational transformation preserves a unique pencil of curves of genus 1. But before that, we start with a description of all such pencils and their geometry.

4.1. Horoballs.

As in Section 2.1, consider the space $\mathbf{R}^{1,n}$, together with its lattice $\mathbf{Z}^{1,n}$ and hyperbolic subset $\mathbb{H}_n$. Assume that the vector $\xi = a_0\mathbf{e}_0 - \sum_j a_j\mathbf{e}_j$ of $\mathbf{R}^{1,n}$ satisfies $a_0 > 0$, and $\xi \cdot \xi = a_0^2 - \sum_j a_j^2 = 0$. Then, ξ determines a boundary point of the hyperbolic space $\mathbb{H}_n$. Let ε be a positive real number. The **horoball** $H_\xi(\varepsilon)$ in $\mathbb{H}_n$ is the subset

$$H_\xi(\varepsilon) = \{v \in \mathbb{H}_n \; ; \; 0 < v \cdot \xi \leq \varepsilon\}.$$

It is a limit of balls with centers converging to the boundary point ξ. The horosphere is the boundary

$$\partial H_\xi(\varepsilon) = \{v \in \mathbb{H}_n \; ; \; v \cdot \xi = \varepsilon\}. \tag{4.1}$$

Remark 4.1. If h is any isometry of $\mathbb{H}_n$, then

$$h(H_\xi(\varepsilon)) = H_{h(\xi)}(\varepsilon). \tag{4.2}$$

If h fixes the boundary point $\mathbf{R}\xi \in \partial\mathbb{H}_n$ then h maps $H_\xi(\varepsilon)$ to $H_\xi(e^{\pm L(h)}\varepsilon)$, where $L(h)$ is the translation length of h and the sign is positive (resp. negative) when ξ is attracting (resp. repelling).

4.2. The lattice $\mathbf{Z}^{1,9}$ and the group W_9.

Consider the lattice $\mathbf{Z}^{1,n}$ introduced in Subsection 2.1, but now for the specific dimension $n = 9$. The *anti-canonical vector* $\xi_9 = 3\mathbf{e}_0 - \sum_{j=1}^9 \mathbf{e}_i$ is isotropic, and the ray $\mathbf{R}\xi_9$ determines a boundary point of the hyperbolic space $\mathbb{H}_9$. The horoballs centered at this boundary point are the subsets $H_{\xi_9}(\varepsilon) = \{u \in \mathbb{H}_9 \; ; \; 0 < u \cdot \xi_9 \leq \varepsilon\}$. The metric induced on $\partial H_{\xi_9}(\varepsilon)$ by the hyperbolic metric is euclidean, and we can identify $\partial H_{\xi_9}(\varepsilon)$ to $\mathbf{R}^8$ with a euclidean metric dist_{euc} (see [4, Part II, Lemma 11.32]).

In the orthogonal group $O(\mathbf{Z}^{1,9})$, there is an index 2 subgroup $O(\mathbf{Z}^{1,9})^+$ preserving $\mathbb{H}_9$, and thus acting by isometries on $\mathbb{H}_9$. The subgroup $O(\mathbf{Z}^{1,9};\xi_9)^+$ fixing the isotropic line $\mathbf{R}\xi_9$ fixes also the class ξ_9, because it is the unique primitive integral vector on this line with $a_0 > 0$; this group preserves every horosphere $\partial H_{\xi_9}(\varepsilon)$, acting on it as a group of affine isometries for the euclidean metric dist_{euc}. Such an isometry is made of a linear part and a translation part; since the group is contained in the discrete group $O(\mathbf{Z}^{1,9})$, Bieberbach theorem shows that there is a finite index subgroup P^* in $O(\mathbf{Z}^{1,9};\xi_9)^+$ acting by pure translations on the horospheres $\partial H_{\xi_9}(\varepsilon)$. Denoting by P_n^* the subgroup of P^* consisting of all n-th powers of elements of P^*, one gets the following result.

Lemma 4.2. *There is a finite index, normal subgroup $P^* \subset \mathrm{O}(\mathbf{Z}^{1,9}; \xi_9)^+$ with the following properties. For every $\varepsilon > 0$, there is a positive real number $\Delta(\varepsilon)$, such that*

    (1) the group P^* is isomorphic to the abelian group $\mathbf{Z}^8$;

    (2) it acts by translations on the euclidean horosphere $\partial H_{\xi_9}(\varepsilon)$;

    (3) the translation length of any element of $P_n^* \setminus \{\mathrm{Id}\}$ on $\partial H_{\xi_9}(\varepsilon)$ for the euclidean distance is at least $n\Delta(\varepsilon)$.

Remark 4.3 (See [7])**.** One can, in fact, be much more precise. The orthogonal complement of the vector ξ_9 is a sublattice $\mathbf{E}_9 \subset \mathbf{Z}^{1,9}$. A basis of $\mathbf{E}_9$ is formed by the vectors $\alpha_0 = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$ and $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$, for $i = 1, \ldots,$ 8. The intersection matrix $(\alpha_i \cdot \alpha_j)$ is equal to $\Gamma_9 - 2\mathrm{Id}_9$, where Γ_9 is the incidence matrix of the Dynkin diagram of type E_9. In particular, each class α_i has self-intersection -2 and determines an involutive isometry of $\mathbf{Z}^{1,9}$, namely $s_i\colon x \mapsto x + (x \cdot \alpha_i)\alpha_i$. By definition, these involutions generate the Weyl (or Coxeter) group W_9, and one shows that W_9 has finite index in $\mathrm{O}(\mathbf{Z}^{1,9}; \xi_9)^+$. When restricted to $\mathbf{E}_9$, the intersection form is degenerate; its radical is generated by the vector ξ_9 and the lattice $\mathbf{E}_8 \simeq \mathbf{E}_9/\mathbf{Z}\xi_9$ is isomorphic to the root lattice of finite type E_8. Then, W_9 is isomorphic to the affine Weyl group of type E_8, and fits in the extension $0 \to \mathbf{E}_8 \to W_9 \to W_8 \to 1$ where the injection $\iota\colon \mathbf{E}_8 \to W_9$ of the additive group $\mathbf{E}_8 \simeq \mathbf{Z}^8$ into W_9 is defined by the following formula: for $w \in \mathbf{E}_8$,

$$\iota(w)\colon v \in \mathbf{Z}^{1,9} \mapsto v - (v \cdot \xi_9)w + \left((w \cdot v) - \frac{1}{2}(v \cdot \xi_9)(w \cdot w) \right)\xi_9. \qquad (4.3)$$

The quotient W_8 is a finite reflection group. In Lemma 4.2, one can take $P^* = \iota(\mathbf{E}_8)$.

4.3. **Halphen pencils and Halphen surfaces (see** [14, 7])**.** A **Halphen pencil** $\mathfrak{p}$ of index m is an irreducible pencil of plane curves of degree $3m$ with 9 base-points of multiplicity m; here, infinitely near points may be included in the list of base-points. In particular, a Halphen pencil of index 1 is just a pencil of cubic curves with no fixed component. A smooth rational surface X is a **Halphen surface** if there exists an integer $m > 0$ such that the linear system $|-mK_X|$ is of dimension 1, has no fixed component, and has no base-point (here, K_X denotes the canonical divisor). The index of a Halphen surface is the smallest possible value for such a positive integer m.

    As shown in [7], every Halphen surface X is obtained by blowing-up the nine base-points of a Halphen pencil $\mathfrak{p}$. In particular, the Picard number of X is equal to 10: there is a basis of the Néron-Severi group given by the class $\mathbf{e}_0$ of a line in $\mathbb{P}^2$, and the classes $\mathbf{e}_j$ of the 9 exceptional divisors (coming from the 9 base-points). The pencil $\mathfrak{p}$ determines a fibration $\pi\colon X \to \mathbb{P}^1$; this fibration

is the same as the one given by the linear system $|-mK_X|$. Two cases may appear: if the general fiber of π is a smooth curve of genus 1 we say that π is elliptic; otherwise, the general fiber is a rational curve with a cusp and one says that π is quasi-elliptic. Quasi-elliptic examples occur only in characteristic 2 and 3 (see the first sections of [3] or the last chapter of [10]); the cusps of the fibers form a smooth curve which will be denoted by $\Sigma_X \subset X$.

When $m > 1$, the pencil contains a unique cubic curve with multiplicity m. The class of this curve in the Picard-Manin space is the class of $-K_X$ and it is equal to $\xi_X = 3\mathbf{e}_0 - \mathbf{e}_1 - \ldots - \mathbf{e}_9$; if we want to refer to the pencil, rather than to the surface, we shall denote this class by $\xi_{\mathfrak{p}}$ (it may be considered as a point of $\mathrm{NS}(X)$ or of $\mathcal{Z}$). Thus, ξ_X corresponds to the anti-canonical vector ξ_9 under the natural isometry $\mathbf{Z}^{1,9} \to \mathrm{NS}(X)$. The model $\mathbf{Z}^{1,9}$ does not depend on the pencil $\mathfrak{p}$ (or on the Halphen surface X).

The group of automorphisms $\mathrm{Aut}(X)$ acts on $\mathrm{NS}(X)$ and its image is a group of isometries fixing the anti-canonical class ξ_X. Since $\mathrm{Aut}(X)$ preserves the canonical bundle, it permutes the sections of $|-mK_X|$, and it preserves the fibration π (permuting its fibers).

Consider the group $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}; \mathfrak{p})$ of birational transformations of the plane preserving the pencil $\mathfrak{p}$. After conjugacy by the blow-up $X \to \mathbb{P}^2_{\mathbf{k}}$, $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}; \mathfrak{p})$ becomes a subgroup of $\mathrm{Bir}(X)$ that permutes the fibers of π. The fibration π is relatively minimal, which amounts to say that there is no exceptional divisor of the first kind in the fibers of π (see [18]); this implies that $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}; \mathfrak{p})$ is contained in $\mathrm{Aut}(X)$ (see [19]). Since $\mathrm{Aut}(X)$ permutes the fibers of π, we conclude that $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}; \mathfrak{p}) = \mathrm{Aut}(X)$ (after conjugacy by the blow-up $X \to \mathbb{P}^2$):

Lemma 4.4. *Let $\mathfrak{p}$ be a Halphen pencil, and let $\pi\colon X \to \mathbb{P}^1_{\mathbf{k}}$ be the Halphen surface defined by $\mathfrak{p}$. Then $\mathrm{Aut}(X)$ permutes the fibers of π, and is conjugate to $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}; \mathfrak{p})$ by the blow-up of the nine base-points of $\mathfrak{p}$.*

Let $P_0(X) \subset \mathrm{Aut}(X)$ (resp. $P_0(\mathfrak{p}) \subset \mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}, \mathfrak{p})$) be the subgroup of automorphisms f such that the action $f^*\colon \mathrm{NS}(X) \to \mathrm{NS}(X)$ corresponds to an element of the abelian group $P^* \subset \mathrm{O}(\mathbf{Z}^{1,9})^+$ given by Lemma 4.2.

In the following lemma, ξ_X is viewed as a boundary point of $\mathbb{H}_\infty$, via the natural embedding $\mathrm{NS}(X) \to \mathcal{Z}$, the horosphere $\partial H_{\xi_X}(\varepsilon)$ is the horosphere in $\mathbb{H}_\infty$ centered at ξ_X, and the real number $\Delta(\varepsilon)$ is the same as in Lemma 4.2.

Lemma 4.5. *The group $P_0(\mathfrak{p})$ is a normal subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}; \mathfrak{p})$. For every $\varepsilon > 0$, every non-trivial element f of this group acts*

- *on $\partial H_{\xi_X}(\varepsilon) \cap \mathbb{H}_X$ as a euclidean translation, of length $\geq \Delta(\varepsilon)$;*
- *on $\partial H_{\xi_X}(\varepsilon) \subset \mathbb{H}_\infty$ as a fixed point free isometry that satisfies*

$$\mathrm{dist}(u, f(u)) \geq \Delta(\varepsilon)$$

for every $u \in \partial H_{\xi_X}(\varepsilon)$.

Moreover, for each $l > 0$, there exists an integer $n \geq 1$ that does not depend on the pencil $\mathfrak{p}$, such that

$$\text{dist}_{\mathbb{H}_\infty}(x, g^n(x)) \geq l$$

for each $g \in P_0(\mathfrak{p}) \setminus \{1\}$ and each point x in the horosphere $\partial H_{\xi_X}(\varepsilon)$, .

Proof. The first property follows directly from the definition of $P_0(\mathfrak{p})$. The second one follows from the fact that the projection from the horosphere in $\mathbb{H}_\infty$ to its intersection with $\mathbb{H}_X$ is 1-lipschitz. $\qquad\square$

4.4. Gizatullin's theorem.

The following result encapsulates a theorem of Marat Gizatullin (see [17]) and results of Georges H. Halphen and Igor Dolgachev (see [14]).

Theorem 4.6. *There is an integer $B_H \leq 8!$ with the following property. Let $\mathbf{k}$ be an algebraically closed field. Let $f \in \text{Cr}_2(\mathbf{k})$ be a Halphen twist. Then, up to conjugacy, f preserves a unique Halphen pencil $\mathfrak{p}$ (of some index $m \geq 1$), and:*

- *this Halphen pencil provides an elliptic fibration of the corresponding Halphen surface X,*
- *the kernel of the homomorphism $\text{Aut}(X) \to \text{GL}(\text{NS}(X))$ is finite, of order $\leq B_H$.*

In other words, one can conjugate f by a birational transformation $g \colon \mathbb{P}^2_{\mathbf{k}} \dashrightarrow \mathbb{P}^2_{\mathbf{k}}$ and find a Halphen pencil $\mathfrak{p}$, of some index m, such that $f' := g \circ f \circ g^{-1}$ is contained in $\text{Bir}(\mathbb{P}^2_{\mathbf{k}}; \mathfrak{p})$. Blowing-up the base-points of $\mathfrak{p}$, f' becomes an automorphism of a rational surface X, preserving a relatively minimal fibration $\pi \colon X \to \mathbb{P}^1_{\mathbf{k}}$ of genus 1. The class of a general fiber is equal to $m\xi_X$, where $\xi_X = \xi_{\mathfrak{p}} = 3\mathbf{e}_0 - \mathbf{e}_1 - \ldots - \mathbf{e}_9$. Viewed in $\mathbb{H}_\infty$, this class $\xi_{\mathfrak{p}}$ gives the unique boundary point fixed by f'. Conversely, if h is an element of $\text{Cr}_2(\mathbf{k})$ that fixes $\xi_{\mathfrak{p}}$, then h determines an automorphism of X preserving the fibration π; then, either h is elliptic, or it is a Halphen twist. If h is a Halphen twist, then some iterate of h is in the group $P_0(X)$.

Proof. The first assertion, i.e. the existence of an invariant Halphen pencil $\mathfrak{p}$ (up to conjugacy) is due to Gizatullin (see [17]): his proof works also in characteristic 2 or 3, as noted in [7]. Blowing-up the base-points of $\mathfrak{p}$, one gets a Halphen surface $X \to \mathbb{P}^2$, and f is conjugate to an automorphism f_X of X; this automorphism preserves the fibration $\pi \colon X \to \mathbb{P}^1$ given by $\mathfrak{p}$.

If π were quasi-elliptic, f_X would preserve the cusp curve Σ_X. Thus, f_X would preserve the class $[\Sigma_X]$ of the curve of cusps, as well as the class ξ_X of a general fiber. Since the intersection product $[\Sigma_X] \cdot \xi_X$ is positive, f_X should

preserve a class $[\Sigma_X] + m\xi_X$ with positive self-intersection, and f_X should be elliptic, in contradiction with f being a Halphen twist. Thus, π is an elliptic fibration: its general fibers are curves of genus 1.

Consider the kernel K of the homomorphism $\mathsf{Aut}(X) \to \mathsf{GL}(\mathrm{NS}(X))$. Since $\mathrm{Pic}^0(X)$ is trivial, K is a linear algebraic group; we denote by K^0 the connected component of the identity in K. First, we show that K is finite, or equivalently that K^0 is trivial. Recall from Lemma 4.4 that $\mathsf{Aut}(X)$ permutes the fibers of $\pi\colon X \to \mathbb{P}^1_{\mathbf{k}}$: this provides an algebraic homomorphism $\rho\colon K \to \mathsf{Aut}(\mathbb{P}^1)$ such that $\pi \circ g = \rho(g) \circ \pi$ for all $g \in K$. The kernel of ρ is a linear algebraic group, preserving each of the fibers of π. Since every algebraic homomorphism of a connected linear algebraic group to a curve of genus 1 is trivial, the kernel of ρ is finite. The Euler characteristic of X (using ℓ-adic cohomology if $\mathrm{char}(\mathbf{k}) > 0$) is equal to 12 because X is obtained from $\mathbb{P}^2_{\mathbf{k}}$ by a sequence of 9 successive blow-ups; hence, π has at least one singular fiber (see [10], Proposition 5.1.6, p. 290). Since, the group $\rho(K^0) \subset \mathsf{PGL}_2(\mathbf{k})$ fixes all the critical values of π, it fixes at least one point; in particular, $\dim(K^0) = 0$, 1, or 2. If $\dim(K^0) = 1$, the (Zariski closures) of the orbits of K^0 form a pencil of rational curves, and this pencil is $\mathsf{Aut}(X)$-invariant, because K^0 is a normal subgroup of $\mathsf{Aut}(X)$. Then, f_X would preserve two pencils, the Halphen pencil and this pencil of rational curves, and f would not be a Halphen twist: the two pencils would provide two fixed points of f on the boundary of $\mathbb{H}_\infty$, and f would not be a parabolic isometry of $\mathbb{H}_\infty$. This contradiction shows that $\dim(K^0) \neq 1$. If the dimension of K^0 is 2, then $\rho(K^0)$ is the group of affine transformations $z \mapsto az + b$ of the line; the subgroup of elements g in K^0 such that $\rho(g)$ is a translation $z \mapsto z + b$ is invariant under the action of $\mathsf{Aut}(X)$ on K^0 by conjugacy, its dimension is equal to 1, and we also get a contradiction. Thus, K is finite.

Now, we show that $|K| \leq 8!$. Consider the birational morphism $\tau\colon X \to \mathbb{P}^2_{\mathbf{k}}$ given by (the inverse) of the blow-up of the base-points of the pencil $\mathfrak{p}$. Since K acts trivially on $\mathrm{NS}(X)$, it preserves the classes of the 9 irreducible curves contracted by τ, and since these curves have negative self-intersections it preserves each of these curves. Thus, there is a finite subgroup K_0 of $\mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}})$ and an isomorphism $\varphi\colon K \to K_0$ such that $\varphi(g) \circ \tau = \tau \circ g$ for every $g \in K$. The group K_0 preserves $\mathfrak{p}$, permuting its members but fixing each of its base-points (including infinitely near ones). Conversely, the subgroup G of $\mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}})$ that preserves each of these 9 points is an algebraic subgroup that lifts to a subgroup of $\mathsf{Aut}(X)$, and this subgroup acts trivially on $\mathrm{NS}(X)$ because it preserves the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_9$ of the corresponding geometric basis as well as the vector $\mathbf{e}_0$ (since G preserves the class of a line in $\mathbb{P}^2_{\mathbf{k}}$). Thus, G coincides with K_0 and is finite. So, we just have to prove that $|G| \leq 8!$. For that, we identify $\mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}}) = \mathsf{PGL}_3(\mathbf{k})$ to the open subset of $\mathbb{P}^8_{\mathbf{k}}$ corresponding to the coefficients

$[h_{i,j}]$ of matrices such that $\det(h_{i,j}) \neq 0$. The finite group G is determined by 9 algebraic equations $h(q_k) = q_k$, where the q_k are the base-points of $\mathfrak{p}$. Let us show that the degrees of these equations is bounded by 8. Indeed, if $h = (h_{i,j})$ is an element of $\mathsf{PGL}_3(\mathbf{k})$ and q_0 is a point of $\mathbb{P}^2_{\mathbf{k}}$, then the equation $h(q_0) = q_0$ is linear in the coefficients $h_{i,j}$. Then, if we blow-up q_0 and choose a point q_1 on the exceptional divisor, the equation $h(q_1) = q_1$ is linear in the coefficients of Dh_{q_0}, and therefore also in the coefficients of h. Going on one step further, blowinq-up q_1 and choosing a point q_2 on the exceptional divisor, the equation $h(q_2) = q_2$ is a linear constraint on the second jet of h at q_0; thus, it is a quadratic equation in the coefficients $[h_{i,j}]$ (^{2}). If we do a tower of k successive blow-ups, with q_{i+1} infinitely near q_i, then the equation $h(q_k) = q_k$ has degree $\max(k-1, 1)$ in the coefficients of h. Since, in our setting, we have 9 base-points, we get 9 equations of respective degrees $d_1, \ldots, d_9$ in the coefficients $[h_{i,j}]$. By the Bezout Theorem, the number of solutions of this system of equations, if finite, is at most $d_1 \times \cdots \times d_9$. The worst case is for

$$(d_1, d_2, d_3, \ldots, d_9) = (1, 1, 2, \ldots, 8) \tag{4.4}$$

and gives the upper bound $8! = 40320$. $\qquad\square$

Example 4.7. Keep the notation of the proof. Given any Halphen pencil, the group K can be described explicitly. We just give two examples. First, assume that the 9 base-points of the pencil $\mathfrak{p}$ contain a projective basis (q_0, q_1, q_2, q_3) of $\mathbb{P}^2_{\mathbf{k}}$. Then K is trivial. This corresponds to the generic situation.

Now, assume that the index m of $\mathfrak{p}$ is larger than 1 and the unique multiple fiber C of the fibration $\pi \colon X \to \mathbb{P}^1_{\mathbf{k}}$ is smooth. Then, the projection $C_0 = \tau(C) \subset \mathbb{P}^2_{\mathbf{k}}$ is a smooth cubic curve, and is invariant under K_0. Since K_0 fixes at least one point of C_0, it acts as a group automorphisms on C_0. Moreover, an element of $\mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}})$ which preserves C_0 pointwise is the identity. So, $|K_0| = |K|$ is bounded by 6 if $\mathrm{char}(\mathbf{k}) \neq 2, 3$, by 12 if $\mathrm{char}(\mathbf{k}) = 3$, and by 24 if $\mathrm{char}(\mathbf{k}) = 2$.

Recall that $P_0(\mathfrak{p}) \subset \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}}; \mathfrak{p})$ is the subgroup of automorphisms $f \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}}; \mathfrak{p})$ whose image in $\mathsf{GL}(\mathsf{NS}(X))$ lies in the abelian group $P^* \subset \mathsf{O}(\mathbf{Z}^{1,9})^+$ given by Lemma 4.2.

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^{2} To see this, assume that h fixes the point $[0:0:1]$, then in the affine coordinates (x,y) we have

$$h\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{ax+by}{1+ux+vy} \\ \frac{cx+dy}{1+ux+vy} \end{pmatrix} = \begin{pmatrix} (ax+by)(1-(ux+vy)+(ux+vy)^2+\cdots) \\ (cx+dy)(1-(ux+vy)+(ux+vy)^2+\cdots) \end{pmatrix}.$$

Thus, the coefficients of the jet of degree d of h at the origin $(0,0)$ are polynomial functions of degree $\leq d$ in the coefficients of h.

Corollary 4.8. *Let f be a Halphen twist, and let $\mathfrak{p}$ be the unique f-invariant Halphen pencil. The subgroup $P_0(\mathfrak{p}) \subset \mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}; \mathfrak{p})$ is an extension*

$$1 \to K \to P_0(\mathfrak{p}) \to \mathbf{Z}^s \to 0$$

of an abelian group of rank $s \leq 8$ by a finite group K with $|K| \leq B_H \leq 8!$.

There is a normal subgroup $P(\mathfrak{p}) \triangleleft P_0(\mathfrak{p})$ of index $\leq s^{2|K|}((|K|+1)!)$ which is a free abelian group of rank s.

This subgroup $P(\mathfrak{p})$ of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}; \mathfrak{p})$ will play an important role in the next sections.

Proof. The previous theorem shows the first assertion. To prove the second assertion, consider any exact sequence $1 \to K \to P_0 \to \mathbf{Z}^s \to 0$ where K is a finite group, and $|K| \leq D$ for some positive integer D. The group P_0 normalizes K, acting by conjugation on it. There is a subgroup P_1 of P_0 of index $\leq D!$ that centralizes K: every element of P_1 commutes to every element of K. Set $n = 2|K|$; hence $|K|$ divides n and $n(n+1)/2$. Then, set $P = \{f^n \mid f \in P_1\}$. The first remark is that P is a subgroup of P_1. Indeed, given any pair of elements f, and g in P_1, we have $fg = gfc$ for some c in K; then, for every $m \geq 1$, $f^m g = gf^m c^m$ and $fg^m = g^m fc^m$ because c is central, and this gives $(fg)^m = g^m f^m c^{1+2+3+\dots+m}$; with $m = n$ we obtain that $f^n g^n = (fg)^n$ and P is stable under multiplication. The second remark is that $P \cap K = \{1\}$. The third remark is that the projection of P in $\mathbf{Z}^s$ is an injective homomorphism, the image of which is exactly $n\mathbf{Z}^s$. Thus, P is a free abelian subgroup of P_1 of rank s and index $\leq s^n D$. Since $n = 2|K|$ we see that the index of P in P_0 is $\leq s^{2D} D(D!) \leq s^{2D}((D+1)!)$. Moreover, P is a normal subgroup of P_0 (because P_1 is normal and P is characteristic). $\qquad\square$

5. DISJONCTION OF HOROBALLS

In this section, we prove Corollary 5.3 which says that horoballs associated to Halphen twists are disjoint. This is the first technical input to apply results of small cancellation and geometric group theory.

Consider a family $\mathcal{C}$ of convex subsets of $\mathbb{H}_\infty$. Let R be a positive real number. By definition, $\mathcal{C}$ is **R-separated** if $\mathrm{dist}_{\mathbb{H}_\infty}(C, C') > R$ for all pairs of distinct elements $C, C' \in \mathcal{C}$. A vector $v \in \mathbf{Z}^{1,\infty}$ is **primitive** if it is not a non-trivial multiple of some vector $v' \in \mathbf{Z}^{1,\infty}$.

Theorem 5.1. *Let $\mathcal{P}$ be the set of integral, isotropic and primitive vectors v in $\mathbf{Z}^{1,\infty}$ such that $v \cdot \mathbf{e}_0 > 0$. If $0 < \varepsilon < 1/\sqrt{2}$, the family of horoballs $\{H_v(\varepsilon)\}_{v \in \mathcal{P}}$ is $\mathrm{arcosh}(\frac{1}{4\varepsilon^2})$-separated.*

Remark 5.2. The set of horoballs $\{H_v(\varepsilon)\}_{v \in \mathcal{P}}$ is invariant by the group of isometries of $\mathbb{H}_\infty$ preserving integral points.

Proof. Consider two different horoballs $H_v(\varepsilon)$ and $H_{v'}(\varepsilon)$ of $\mathcal{P}$. Consider $x \in [v,v'] \cap \partial H_v(\varepsilon)$ and $y \in [v,v'] \cap \partial H_{v'}(\varepsilon)$, where $[v,v']$ is the geodesic from v to v'. Set $m = v \cdot v'$ (this is a positive integer), and write $x = \alpha v + \alpha' v'$ and $y = \beta v + \beta' v'$. The points x and y belong to $\mathbb{H}_\infty$ so $x \cdot x = y \cdot y = 1$, and by definition of the horospheres $x \cdot v = y \cdot v' = \varepsilon$. This gives

$$2\alpha\alpha' m = 2\beta\beta' m = 1, \tag{5.1}$$

$$\alpha' m = \beta m = \varepsilon. \tag{5.2}$$

Using these equalities we get:

$$x \cdot y = (\alpha\beta' + \alpha'\beta)m = \frac{m}{4\varepsilon^2} + \frac{\varepsilon^2}{m}. \tag{5.3}$$

Since $m \geq 1$, we obtain:

$$x \cdot y > \frac{1}{4\varepsilon^2}. \tag{5.4}$$

For $4\varepsilon^2 > 1$ the two horoballs overlap, for $4\varepsilon^2 = 1$ they are tangent at $x = y$, and for $4\varepsilon^2 < 1$ they are disjoint: The distance between $H_v(\varepsilon)$ and $H_{v'}(\varepsilon)$ is the distance between x and y and is equal to

$$\text{dist}_{\mathbb{H}_\infty}(H_v(\varepsilon), H_{v'}(\varepsilon)) = \text{dist}_{\mathbb{H}_\infty}(x,y) \geq \text{arcosh}(\frac{1}{4\varepsilon^2}). \tag{5.5}$$

This concludes the proof. $\qquad\square$

Let $\mathfrak{p}$ be a Halphen pencil. Recall from section Section 4.3 that $\xi_\mathfrak{p}$ denotes the anticanonical class of the surface obtained by blowing up the nine base-points of $\mathfrak{p}$. Since $\xi_\mathfrak{p} \in \mathcal{Z}$ is primitive and isotropic, and the Cremona group permutes the set of primitive, isotropic vectors, Theorem 5.1 provides the following result.

Corollary 5.3. *Consider the set of all horoballs of the form*

$$f(H_{\xi_\mathfrak{p}}(\varepsilon)) = H_{f(\xi_\mathfrak{p})}(\varepsilon),$$

where $\mathfrak{p}$ is any Halphen pencil and f is any element of $\text{Cr}_2(\mathbf{k})$. Then for $0 < \varepsilon < 1/\sqrt{2}$, this set of horoballs is $\text{arcosh}(\frac{1}{4\varepsilon^2})$-separated.

6. GEOMETRIC GROUP THEORY AND CONCLUSION

To prove Theorem B, we rely on the work of François Dahmani, Vincent Guirardel and Denis Osin (see [12]).

6.1. **Hyperbolic spaces and rotating families.** First we recall one of the main results of [12].

Consider a geodesic metric space X and a non-negative constant σ. A subset Q of X is σ-**strongly quasiconvex** if for any two points x, y of Q there exist x', y' in Q and geodesics $[x', y']$, $[x', x]$ and $[y, y']$ included in Q such that $\max\{\text{dist}(x, x'), \text{dist}(y, y')\} \leq \sigma$.

Let δ be a non-negative constant. A geodesic metric space X is δ-**hyperbolic** if every triangle in X is δ-thin, meaning that each side of the triangle is included in the δ-neighborhood of the union of the two remaining sides. We shall say that X is **Gromov-hyperbolic** if it is δ-hyperbolic for some $\delta > 0$.

Let G be a group acting by isometries on a δ-hyperbolic space X, for some $\delta > 0$. Consider a G-invariant subset $C \subset X$, which we call the set of **apices**. For each $c \in C$, consider $G_c \subset \text{Stab}(c)$ a subgroup of the stabilizer of c in G, called the **rotation subgroup**. The family $(C, \{G_c\}_{c \in C})$ is a **very rotating family** if it satisfies the following conditions:

(1) for every $g \in G$, for every $c \in C$, $G_{gc} = gG_cg^{-1}$,
(2) for every $c \in C$, for every $g \in G_c \setminus \{1\}$, and for every $x, y \in X$ satisfying

$$\text{dist}(x, c) \in [20\delta, 40\delta], \ \text{dist}(y, c) \in [20\delta, 40\delta], \ \text{and dist}(gx, y) \leq 15\delta,$$

any geodesic between x and y contains c.

Moreover, as in Section 5, the family is ρ-separated, for some $\rho > 0$, if the distance between distinct apices is always strictly bigger than ρ.

Theorem 6.1 ([12, Theorem 5.3]). *Let G be a group acting by isometries on a δ-hyperbolic geodesic space X. Let $(C, \{G_c\}_{c \in C})$ be a ρ-separated very rotating family, for some $\rho > 200\delta$. Then the normal subgroup N_G of G generated by the groups $\{G_c\}_{c \in C}$ is a free product of a (usually infinite) subfamily of $\{G_c\}_{c \in C}$. Moreover, for every $c \in C$, $N_G \cap \text{Stab}(c) = G_c$.*

Remark 6.2. If G_c is a proper subgroup of $\text{Stab}(c)$, or if G_c is abelian and G has no non-trivial homomorphism onto an abelian group, then the normal subgroup generated by the groups $\{G_c\}_{c \in C}$ is a proper subgroup of G. In particular, if $f \in G_c \setminus \{\text{Id}\}$, the smallest normal subgroup of G containing f is proper and non-trivial. See [12] for other consequences.

6.2. **Cone-off construction.** In our case, we shall apply Theorem 6.1 to $G = \text{Cr}_2(\mathbf{k})$, and the role of G_c will be played by the subgroup $P(\mathfrak{p})$ of $\text{Bir}(\mathbb{P}^2_\mathbf{k}, \mathfrak{p})$. This group fixes a point $\xi_\mathfrak{p} \in \partial\mathbb{H}_\infty$, but has no fixed point inside $\mathbb{H}_\infty$. We shall modify $\mathbb{H}_\infty$ to bring this fixed point inside the space: this is done in Section 6.2.2. Then, we need to modify this cone-off, because if we go deeply inside the horoballs the very rotating property (2) above will fail. For that, we dig a hole

in the cones, and this is explained in Section 6.2.3. For more details on this cone-off constructions, see [12, Sections 5.3 and 7.1] or [11, Section 3].

6.2.1. *Hyperbolic cones (see [4])*. Consider a metric space Y and a real number $r_0 > 0$. The **hyperbolic cone** over Y of height r_0, denoted by $\mathrm{Cone}(Y, r_0)$, is the quotient of $Y \times [0, r_0]$ by the equivalence relation:

$$(y_1, r_1) \sim (y_2, r_2) \text{ if and only if } r_1 = r_2 = 0 \text{ or } (y_1, r_1) = (y_2, r_2). \quad (6.1)$$

By definition, the point $(y, 0)$ is the **apex** of $\mathrm{Cone}(Y, r_0)$. The metric on this cone is defined by the formula

$$\mathrm{dist}_{\mathrm{Cone}(Y, r_0)}\big((y, r), (y', r')\big) = \mathrm{arcosh}\big(\cosh r \cosh r' - \sinh r \sinh r' \cos \theta(y, y')\big),$$

where $\theta(y, y') = \min\left(\pi, \frac{\mathrm{dist}_Y(y, y')}{\sinh r_0}\right)$. By [4, Chap. I.5 Prop 5.10], $\mathrm{Cone}(Y, r_0)$ is a geodesic metric space if and only if Y is geodesic.

6.2.2. *Cone-off*. Consider a metric space X, a family $Y = \{Y_i\}_i$ of subsets of X, and r_0 a positive constant. The **cone-off** of Y over X, denoted by $C(X, Y, r_0)$, is the space obtained from the disjoint union of X and of the cones $\mathrm{Cone}(Y_i, r_0)$, for all i, by gluing each subspace Y_i in X to $Y_i \times \{r_0\}$ in $\mathrm{Cone}(Y_i, r_0)$ via the identification

$$y \in Y_i \mapsto (y, r_0) \in \mathrm{Cone}(Y_i, r_0). \quad (6.2)$$

By [11, Proposition 2.1.5], for (y, r_0) and $(y', r_0) \in \mathrm{Cone}(Y_i, r_0)$ we have

$$\mathrm{dist}_{\mathrm{Cone}(Y_i, r_0)}\big((y, r_0), (y', r_0)\big) \leq \mathrm{dist}_X(y, y'). \quad (6.3)$$

Let us construct a metric on $C(X, Y, r_0)$. Given x and x' in $C(X, Y, r_0)$, set

$$\|x - x'\| = \begin{cases} \mathrm{dist}_X(x, x') & \text{if } x, x' \in X \text{ and there is no } i \text{ such that } x, x' \in Y_i \\ \mathrm{dist}_{\mathrm{Cone}(Y_i, r_0)}(x, x') & \text{if there exists } Y_i \text{ such that } x, x' \in \mathrm{Cone}(Y_i, r_0) \\ +\infty & \text{otherwise.} \end{cases}$$

Then, consider a chain C between x and x', i.e. a finite sequence of points $x = z_1, z_2, \ldots, z_n = x'$ in $C(X, Y, r_0)$. Define its length to be $\ell(C) = \sum_{i=1}^{n-1} \|z_{i+1} - z_i\|$. By [11, Proposition 3.1.7], a distance is defined on the cone-off as follows:

$$\mathrm{dist}_{C(X, Y, r_0)} = \inf\{\ell(C) \mid C \text{ is a chain between } x \text{ and } x'\}. \quad (6.4)$$

The following theorem concerns metric graphs with isometric edges (edges of the same length); this ensures that the cone-off is geodesic ([4, Theorem I.7.19]). As in [12], page 77, set

$$\delta_U = 900, \text{ and } r_U = 5 \times 10^{12}. \quad (6.5)$$

Theorem 6.3. *Given any $r_0 \geq r_U$, there exists $\delta_c > 0$ such that : if X is a δ_c-hyperbolic metric graph with pairwise isometric edges, and if Q is a $40\delta_c$-separated family of $10\delta_c$-strongly quasiconvex subsets of X, then the cone-off $C(X, Q, r_0)$ is globally δ_U-hyperbolic.*

Proof. This statement follows from Corollary 5.39 of [12]. Indeed, with the notation from [12], the fellow traveling constant of the family Q satisfies $\Delta(Q) = 0$ because Q is $40\delta_c$-separated. Hence, all hypotheses of Theorem 5.38 of [12] are satisfied. □

6.2.3. *Parabolic cone-off (see* [12, §7.1]*).* For any $Q \in Q$, we consider the hyperbolic cone $\mathrm{Cone}(Q, r_0)$. For any $p, q \in \partial Q$ with $\mathrm{dist}_X(p,q) \leq \pi \sinh(r_0)$, and for any geodesic $[p, q]$ in $\mathrm{Cone}(Q, r_0)$, we consider the filled triangle $T_{[p,q]} \subset \mathrm{Cone}(Q, r_0)$ bounded by the geodesics $[p,q]$, $[p, c_Q]$ and $[q, c_Q]$; this triangle is a cone over $[p, q]$. For any $Q \in Q$, we define

$$B_Q = \bigcup T_{[p,q]} \tag{6.6}$$

where the union runs over all geodesics $[p, q]$ in $\mathrm{Cone}(Q, r_0)$ of length $\leq \pi \sinh(r_0)$ and with endpoints in ∂Q.

Definition 6.4. Consider X a Gromov-hyperbolic space and Q a R-separated family of convex subsets. The **parabolic cone-off** $C'(X, Q, r_0)$ is the subset of $C(X, Q, r_0)$ given by:

$$\left(C(X, Q, r_0) \setminus \bigcup_{Q \in Q} \mathrm{Cone}(Q, r_0) \right) \cup \left(\bigcup_{Q \in Q} B_Q \right).$$

It is endowed with the induced path metric.

In the following lemma, $r_0 \geq r_U$ and δ_c is given by Theorem 6.3. For the definition of horoballs in a Gromov-hyperbolic graph see [12, Section 7.1 p.111].

Lemma 6.5. [12, Lemma 7.4] *Let X be a δ_c-hyperbolic graph with isometric edges and $\mathcal{H}$ a $40\delta_c$-separated system of horoballs. Then the parabolic cone-off $C'(X, Q, r_0)$ is $16\delta_U$-hyperbolic.*

The following Lemma is proved in [12]. Although the statement is given in the context of relatively hyperbolic group, its proof just needs a separated set of horoballs.

Lemma 6.6. [12, Lemma 7.5] *Let G be a group acting isometrically on a δ_c-hyperbolic graph X with isometric edges. Let $H_i \subset X$, $i \in I$, be a family of horoballs indexed by a set I. For each index i, let $G(H_i) \subset G$ be the stabilizer of H_i and $N_i \triangleleft G(H)$ be a normal subgroup of $G(H_i)$. Assume that*

(i) the H_i are in pairwise disjoint orbits under the action of G;

(ii) the set of distinct horoballs

$$\mathcal{H} = \{g(H_i) \mid i \in I, \ g \in G/G(H_i)\}$$

is $40\delta_c$-separated.

(iii) $\mathrm{dist}_X(x, gx) \geq 4\pi \sinh(r_0)$ for each $g \in N_i \setminus \{1\}$ and each $x \in \partial H_i$.

Then the family of all G-conjugates of the N_i defines a $2r_0$-separated very rotating family on $C'(X, \mathcal{H}, r_0)$.

6.3. **Conclusion.** We can now prove a strong form of Theorem B.

Theorem B'. *There exists a positive integer n_0 with the following property. Let $\mathbf{k}$ be a field. Consider the collection of all iterates g^{n_0} of all Halphen twists $g \in \mathrm{Cr}_2(\mathbf{k})$. Then the normal subgroup generated by this collection is a nontrivial proper subgroup of $\mathrm{Cr}_2(\mathbf{k})$.*

Moreover, this normal subgroup is a free product of free abelian groups of rank ≤ 8.

Proof of Theorem B'. First, assume that the field $\mathbf{k}$ is algebraically closed.

Consider the action of $\mathrm{Cr}_2(\mathbf{k})$ on $\mathbb{H}_\infty$. In order to apply the results of [12], which are stated in the case of Gromov-hyperbolic graphs, we need the following classical construction. Let $\mathcal{G}$ be the metric graph defined as follows. The vertices of $\mathcal{G}$ are the points of $\mathbb{H}_\infty$ and we put an edge between two vertices x and y if their distance in $\mathbb{H}_\infty$ is at most 1. This graph is endowed with the path metric for which each edge is isometric to the interval $[0, 1]$. The Cremona group acts on $\mathcal{G}$ by isometries. We denote by $i : \mathbb{H}_\infty \hookrightarrow \mathcal{G}$ the inclusion. It is a $(1, 1)$-quasi-isometry:

$$\mathrm{dist}_\mathcal{G}(i(x), i(y)) - 1 \leq \mathrm{dist}_{\mathbb{H}_\infty}(x, y) \leq \mathrm{dist}_\mathcal{G}(i(x), i(y)) \tag{6.7}$$

for any $x, y \in \mathbb{H}_\infty$. Then $\mathcal{G}$ is $\delta_\mathcal{G}$-hyperbolic with $\delta_\mathcal{G} = \ln(1 + \sqrt{2}) + 1$.

Fix $\frac{1}{\sqrt{2}} > \varepsilon > 0$ such that $\mathrm{arcosh}(\frac{1}{4\varepsilon^2}) \geq 40\delta_\mathcal{G}$. For every Halphen pencil $\mathfrak{p}$, consider the horoball $H_\mathfrak{p}(\varepsilon)$ in $\mathbb{H}_\infty$. By Corollary 5.3, the family of translates of all the $H_\mathfrak{p}(\varepsilon)$ under $\mathrm{Cr}_2(\mathbf{k})$ is R_ε-separated for $R_\varepsilon = \mathrm{arcosh}(\frac{1}{4\varepsilon^2}) \geq 40\delta_\mathcal{G}$. To each horoball $H_\mathfrak{p}(\varepsilon)$ of $\mathbb{H}_\infty$, we associate the subgraph $H_\mathfrak{p}$ of $\mathcal{G}$ spanned by the vertices $\{i(x) \mid x \in H_\mathfrak{p}(\varepsilon)\}$. This is a 0-strongly quasiconvex subgraph of $\mathcal{G}$ and the family of all translates $gH_\mathfrak{p}$ (for g in $\mathrm{Cr}_2(\mathbf{k})$ and $\mathfrak{p}$ a Halphen pencil) is again R_ε-separated.

Set $r_0 = r_U$. With $\lambda = \delta_c/\delta_\mathcal{G}$, the scaled graph $\lambda\mathcal{G}$ (in which each edge has length λ) is δ_c-hyperbolic. We call it $\mathcal{G}'$, and we denote by $H' \subset \mathcal{G}'$ the subset corresponding to $H \subset \mathcal{G}$. Now the set $\mathcal{H}'$ of translates of $H'_\mathfrak{p}$ under $\mathrm{Cr}_2(\mathbf{k})$, for all Halphen pencils invariant by some Halphen twists, is λR_ε-separated; the choice for ε implies that it is $40\delta_c$-separated. By Lemma 6.5, the parabolic cone-off $C'(\mathcal{G}', \mathcal{H}', r_0)$ is $16\delta_U$-hyperbolic.

Let $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}}; \mathfrak{p})$ be the stabilizer of $\mathfrak{p}$ in $\mathsf{Cr}_2(\mathbf{k})$, and let $P(\mathfrak{p}) \triangleleft P$ be the normal free abelian subgroup given by Corollary 4.8. Using Lemma 4.5, we find an integer n, that does not depend on $\mathfrak{p}$, such that

$$\mathrm{dist}_{G'}(x, g^n x) \geq 4\pi \sinh(r_0) \tag{6.8}$$

for any $g \in P(\mathfrak{p}) \setminus \{\mathrm{Id}\}$ and for any $x \in \partial H_{\mathfrak{p}}(\varepsilon)$. Then, we set $N_{\mathfrak{p}} = \{g^n \mid g \in P(\mathfrak{p})\}$. This is a subgroup because $P(\mathfrak{p})$ is abelian, and it is clearly normal in $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}}; \mathfrak{p})$. Moreover, the Corollary 4.8 shows that there is a uniform integer

$$n_0 = n \times 8^{2B_H} \times (B_H + 1)! \tag{6.9}$$

such that g^{n_0} is in some $N_{\mathfrak{p}}$ for every Halphen twist g.

The family of horoballs $H_{\mathfrak{p}}$ and normal subgroups $N_{\mathfrak{p}} \subset \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}}; \mathfrak{p})$ satisfies the hypotheses of Lemma 6.6, so the family of $\mathsf{Cr}_2(\mathbf{k})$-conjugates of the $N_{\mathfrak{p}}$ defines a $2r_0$-separated very rotating family on $C'(G', \mathcal{H}', r_0)$. By hypothesis on $r_0 = r_U = 10^6 \delta_U$, $2r_0 > 200 \times (16\delta_U)$ (see Equation (6.5)). Applying Theorem 6.1, we get the expected result.

If the field $\mathbf{k}$ is not algebraically closed, the first step is to replace it by an algebraic closure $\mathbf{k}'$, so as to apply Theorem 6.1. The normal subgroup F of $\mathsf{Cr}_2(\mathbf{k}')$ generated by the iterates g^{n_0} for all Halphen twists $g \in \mathsf{Cr}_2(\mathbf{k})$ is a free product of abelian groups. Since $\mathsf{PGL}_3(\mathbf{k})$ does not embed into such a group, the intersection $F \cap \mathsf{Cr}_2(\mathbf{k})$ is a proper normal subgroup of $\mathsf{Cr}_2(\mathbf{k})$.

To conclude, we need to show that $\mathsf{Cr}_2(\mathbf{k})$ always contain Halphen twists: this is done in the examples below. $\qquad\square$

Example 6.7. Let $\mathbf{k}$ be a field of characteristic $p \neq 2$. Let $C \subset \mathbb{P}^2_{\mathbf{k}}$ be a smooth cubic curve defined over $\mathbf{k}$, together with a point $q \in C(\mathbf{k})$. By definition, the Jonquières involution s_q is a birational involution of the plane that preserves the pencil of lines through the point q and fixes C pointwise. If L is a line containing q, it intersects C on a pair of points (except if the line is tangent to C) and the restriction $s_{q|L}$ is the unique involution fixing these two points (this involution does not exist if $\mathrm{char}(\mathbf{k}) = 2$). This involution is defined over $\mathbf{k}$. If one blows up the point q and the points of tangency of the 4 lines containing q which are tangent to C, the involution s_q lifts to a regular involution of a rational surface $X_q \to \mathbb{P}^2_{\mathbf{k}}$.

Now, if we start with two points q and q' in $C(\mathbf{k})$, we can lift s_q and $s_{q'}$ simultaneously as a pair of automorphisms s_Y and s'_Y of a rational surface Y (Y is the blow-up of the plane in 10 points that dominates X_q and $X_{q'}$). Jérémy Blanc proves in [1] that

- there are no relations between s_Y and s'_Y: the group they generate is isomorphic to $\mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z}$.

- the composition $s_Y \circ s'_Y$ is a parabolic automorphism (hence a Halphen twist).

Thus, to construct examples of Halphen twists in characteristic $\neq 2$, we only need to construct a smooth cubic curve and two points q, q' of $C(\mathbf{k})$. If $\mathrm{char}(\mathbf{k}) \neq 7$, one can take the curve defined by $x^3 + y^3 + z^3 + xyz = 0$, with $q = [1 : -1 : 0]$ and $q' = [1 : 0 : -1]$. If $\mathrm{char}(\mathbf{k}) = 7$, one can take the curve $y^2 z = x^3 + z^3$, with $q = [-1 : 0 : 1]$ and $q' = [0 : 1 : 0]$.

Example 6.8. Let us describe another strategy that works in characteristic 2. Consider a field $\mathbf{k}$ of characteristic 2. In $\mathbb{P}^1 \times \mathbb{P}^2$, with affine coordinates $(t) \times (x,y)$, consider the surface $X \subset \mathbb{P}^1 \times \mathbb{P}^2$ given by the equation

$$y^2 + xy + t^3 y = x^3 + 1. \tag{6.10}$$

Let $\pi \colon X \to \mathbb{P}^1$ be the first projection. For $t \neq \infty$, the fiber $F_t = \pi^{-1}(t)$ is a cubic curve in Weierstrass form. There are two sections of π, given by $t \mapsto (-1, 0) = [-1 : 0 : 1]$ and $t \mapsto [0 : 1 : 0]$. Let $f \colon X \to X$ be the birational transformation of X that preserves each of the fibers of π and translates the first section to the second. In [21], William E. Lang shows that this Weierstrass pencil is a Halphen pencil with twelve singular irreducible curves. Thus, there is no -2 curve on X, and this implies that f is a Halphen twist (see [7]).

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