ALGEBRAIC GROWTH OF THE CREMONA GROUP

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ABSTRACT. We initiate the study of the "*algebraic growth*" of groups of automorphisms and birational transformations of algebraic varieties. Our main result concerns $\mathsf{Bir}(\mathbb{P}^2)$, the Cremona group in 2 variables. This group is the union, for all degrees $d \geq 1$, of the algebraic variety $\mathsf{Bir}(\mathbb{P}^2)_d$ of birational transformations of the plane of degree d. Let N_d denote the number of irreducible components of $\mathsf{Bir}(\mathbb{P}^2)_d$. We describe the asymptotic growth of N_d as d goes to $+\infty$, showing that there are two constants A and $B > 0$ such that

$$A\sqrt{\ln(d)} \leq \ln\left(\ln\left(\sum_{e\leq d} N_e\right)\right) \leq B\sqrt{\ln(d)}$$

for all large enough degrees d. This growth type seems quite unusual and shows that computing the algebraic growth of $\mathsf{Bir}(X)$ is a challenging problem in general.

CONTENTS

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1. INTRODUCTION

1.1. **Automorphisms of the affine plane.** Consider the group $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{k}})$ of automorphisms of the affine plane $\mathbb{A}^2_{\mathbf{k}}$, over some field $\mathbf{k}$. For each degree $d \geq 1$, Friedland and Milnor proved in [5] that the automorphisms of degree d form an algebraic variety $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{k}})_d$ and the number of irreducible components of $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{k}})_d$ is equal to K_d, where K is the so called Kalmár's function; that is, K_d is the number of *ordered factorizations* $d = d_1 \cdot d_2 \cdots d_s$ *in integers* $d_i \geq 2$; that is, two factorizations are considered identical if they contain the same integers d_i written in the same order. For instance $K_6 = 3$ because $6 = 2\cdot 3 = 3\cdot 2 = 6$. Then, using the results of Deléglise, Hernane, and Nicolas described in [4], we obtain the following theorem. To state it, we denote by $\zeta(s) = \sum_{n\geq 1} n^{-s}$ the Riemann zeta function and by $\rho \simeq 1.728$ the real number defined by $\zeta(\rho) = 2$.

Theorem A. *The number K_d of irreducible components of $\mathsf{Aut}(\mathbb{A}^2_{\mathbf{k}})_d$ satisfies the following properties:*

(1) *as n goes to ∞,*

$$\sum_{d=1}^{n} K_d \simeq a n^{\rho}$$

with $\zeta(\rho) = 2$ and $a = |\rho\zeta'(\rho)|^{-1}$;

(2) *$K_n \geq 1$ for all n and $K_p = 1$ if and only if p is prime; in particular*

$$\liminf_{d} \log(K_d) = 0;$$

(3) $\limsup_d \log(K_d)/\log(d) = \rho$.

Thus, we see that K_d oscillates, with minimal values equal to 1 and maximal values growing polynomially, like $c^{st} d^{\rho}$.

1.2. **The Cremona group.** Our goal is to study a similar question for birational transformations of the plane $\mathbb{P}^2_{\mathbf{k}}$, over an algebraically closed field $\mathbf{k}$. As will be explained in Section 2.4, the birational transformations of $\mathbb{P}^2_{\mathbf{k}}$ of degree d form an algebraic variety $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})_d$, and we shall denote by N_d the number of its irreducible components. The main result of this article is the following theorem.

Theorem B. *The number N_d of irreducible components of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})_d$ satisfies*

$$0.832 \simeq \sqrt{\ln(2)} \leq \limsup_{d\to+\infty} \frac{\ln(\ln(N_d))}{\sqrt{\ln(d)}} \leq 2\sqrt{\ln(2)} \simeq 1.665.$$

For every $\varepsilon > 0$, there is an integer $D(\varepsilon)$ such that the sum $N_{\leq d} := \sum_{d'=1}^{d} N_{d'}$ satisfies

$$\sqrt{\ln(2)} - \varepsilon \leq \frac{\ln(\ln(N_{\leq d}))}{\sqrt{\ln d}} \leq 2\sqrt{\ln 2} + \varepsilon$$

for all $d \geq D(\varepsilon)$.

In particular, the growth is subexponential but faster than any polynomial function of d, so we observe a phenomenon of intermediate asymptotic growth.

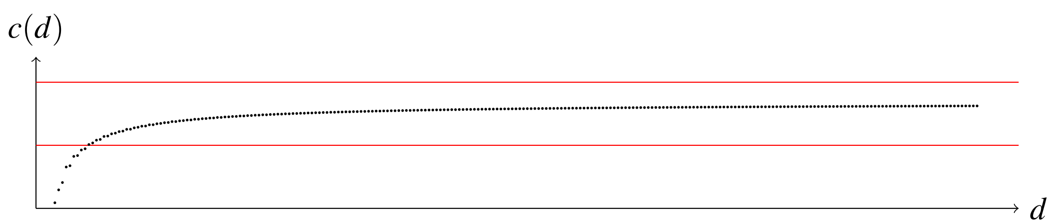
1.3. We computed the values of N_d for $d \in [1,249]$. The first ones are given by

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
N_d	1	1	1	2	5	4	5	9	10	17	19	29	34	51	63	88	102	152

and to get a rough idea of the growth rate one can consider the following list:

d	50	100	125	150	200	249
N_d	52683	9733297	59637891	287772117	3585742777	25275093795

Now, set $c(d) = \ln(\ln(N_d))/\sqrt{\ln(d)}$. In the picture below, the black curve is made of the points $(d, c(d))$ for $d \in [5,249]$. The red segments are at height $\sqrt{\ln(2)}$ and $2\sqrt{\ln(2)}$. As we can see $c(d)$ lies in between $\sqrt{\ln(2)}$ and $2\sqrt{\ln(2)}$ when $16 \leq d \leq 249$; and for d large, $c(d) \simeq 1.354$ is closer to the upper bound.



1.4. Algebraic growth type. These results are parts of a more general problem from algebraic geometry. For simplicity, consider a smooth projective variety X defined over $\mathbf{C}$ and denote by $\mathsf{Bir}(X)$ its group of birational transformations. To $f \in \mathsf{Bir}(X)$, one associates its graph in $\Gamma(f) \subset X \times X$. Let H be a polarization of X: it can be used to define a notion of degree for subvarieties of $X \times X$. Now, subvarieties of $X \times X$ of dimension $\dim(X)$ and degree $\leq d$ which are graphs of birational transformations form an algebraic subset of the Hilbert scheme of $X \times X$; let $N^X_{\leq d}$ denote the number of irreducible components of this algebraic variety. By definition, the *algebraic growth type* of $\mathsf{Bir}(X)$ is the asymptotic growth type of the sequence $d \mapsto N^X_{\leq d}$. A similar notion can be defined for automorphism groups of affine varieties. Theorems A and B show that this notion is interesting and non-trivial. See Section 8 for further questions

Example 1.1. Let X be a projective surface. Denote by $NS(X;\mathbf{Z})$ its Néron-Severi group. Fix a polarization of X by some ample class $[H] \in NS(X;\mathbf{R})$ of self intersection 1, and define the degree of an automorphism f to be the intersection product $(f^*[H]\cdot[H]) \in \mathbf{N}$. Then, $\mathsf{Aut}(X)$ acts on the Néron-Severi group $NS(X;\mathbf{Z})$ and the kernel of this action is an algebraic group (with neutral component denoted by $\mathsf{Aut}(X)^0$). More generally, the index of $\mathsf{Aut}(X)^0$ in the subgroup $\{f \in \mathsf{Aut}(X)\ ;\ f^*[H] = [H]\}$ is finite; we shall denote it by k_0. From this, one easily derives that the number of components of $\mathsf{Aut}(X)$ made of automorphisms of degree $\leq d$ is equal to the product $k_0 \times M_{\leq d}$ where

$$M_{\leq d} = |\{u \in NS(X;\mathbf{R})\ ;\ u\cdot[H] \leq d \text{ and } u \in \mathsf{Aut}(X)^*[H]\}|$$

Since $\mathsf{Aut}(X)^*$ is a subgroup of $\mathsf{GL}\,(NS(X;\mathbf{Z}))$ preserving the intersection form (which is of signature $(1,\rho(X)-1)$), one ends up studying a classical problem from hyperbolic geometry that is, the description of the critical exponent of a discrete group of isometries of a hyperbolic space. At the end, one gets examples with bounded, logarithmic, or polynomial growth for $M_{\leq d}$.

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2. Preliminaries

We introduce some vocabulary that will be used throughout this article and we describe the Hudson's test to provide a new formulation of Theorem B in terms of homaloidal types.

2.1. Homaloidal types. Let f be a birational transformation of the projective plane $\mathbb{P}^2_{\mathbf{k}}$. We can write $f[x:y:z] = [f_0(x,y,z) : f_1(x,y,z) : f_2(x,y,z)]$ where the f_i are homogeneous polynomials of the same degree d without common factor of positive degree; by definition, d is the **degree** of f. To f, one associates its homaloidal net: this is the linear system of curves obtained by pulling back the net of lines by f, their equations are $af_1 + bf_2 + cf_3 = 0$ with $[a:b:c]$ in $\mathbb{P}^2_{\mathbf{k}}$. This linear system has degree d. We shall denote by r the number of its base points (including infinitely near base points), by p_1, …, p_r the base points, and by m_i their respective multiplicities. The **homaloidal type** of f is the list

$$\mathbf{x}(f) = (d; m_1, \dots, m_r)$$

where, by convention, the m_i are organized in decreasing order $m_1 \geq m_2 \geq \cdots \geq m_r$. A second convention is that $r = r(f)$ will denote the number of non-zero multiplicities, but that zeros can always be added at the end of the homaloidal type. For instance, $(2;1,1,1)$ and $(2;1,1,1,0,0)$ both represent the homaloidal type of a quadratic birational transformation of the plane, and $r = 3$ for such a map. In what follows, we shall also denote by $\{m_1, \dots, m_r\}$ the unordered list of these numbers *repeated according to their occurrences* (for instance $\{2,3,2\}$ is different from $\{2,3\}$ and is equal to $\{2,2,3\}$). So, when listing multiplicities $\{m_i\}$ is not the standard notation usually used for sets.

Grouping multiplicities which are equal, we shall also write a homaloidal type $\mathbf{x} = (d; m_1, \dots, m_r)$ as

$$\mathbf{x} = (d; \mu_1^{\nu_1}, \dots, \mu_s^{\nu_s})$$

where $\mu_1 = m_1$ and ν_1 is the number of occurrences of m_1 in $\mathbf{x}$, then μ_2 is the largest of the $m_i < m_1$, etc. Thus, the corresponding birational map has ν_i base points of multiplicity μ_i for $i = 1, \dots, s$. The number $s = s(\mathbf{x}) \in \mathbb{N}$ will be called the **seedbed** of the homaloidal type $\mathbf{x}$.

A **block** of $\mathbf{x} = (d; m_1, \dots, m_r)$ is a sequence of equal multiplicities, and the **width** of the block is the number of elements in this sequence.

Example 2.1. The vector $\mathbf{x} = (d; d-1, 1, \dots, 1) = (d; d-1, 1^{2d-2})$ is the homaloidal type of de Jonquières maps of degree d. It contains two blocks, of respective widths 1 and $2d-2$.

2.2. The Noether equalities and inequality. The homaloidal type $(d; m_1, \dots, m_r)$ of a birational map f satisfies the **Noether equalities**

$$\sum_{i=1}^{r} m_i = 3d - 3, \quad \sum_{i=1}^{r} m_i^2 = d^2 - 1. \tag{2.1}$$

By convention, a homaloidal type will always be obtained from an element of $\mathsf{Bir}(\mathbb{P}^2)$([1]). Thus, our homaloidal types are the proper homaloidal types of [2]. We will say that $(d; m_1, \dots, m_r)$ is an **improper homaloidal type** if it satisfies the Noether equalities but is not the homaloidal type of a birational self-map of $\mathbb{P}^2$.

[1] As we shall see below, the set of all possible homaloidal types does not depend on the field of definition, provided this field be algebraically closed.

The **Noether inequality** tells us that

$$m_1 + m_2 + m_3 \geq d + 1 \tag{2.2}$$

if $(d; m_1, m_2, \ldots, m_r)$ is a (possibly improper) homaloidal type with multiplicities $m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_r$ and degree $d \geq 2$ (see [1]). In particular, $3m_1 > d$ and

$$m_1 > d/3. \tag{2.3}$$

2.3. Hudson's test and Hudson's tree. Let $\mathbf{x} = (d; m_1, \ldots, m_r)$ be a possibly improper homaloidal type. Hudson's test is an algorithm establishing whether $\mathbf{x} = (d; m_1, \ldots, m_r)$ is actually a homaloidal type. It runs as follows.

(1) replace $\mathbf{x}$ with $\mathbf{x}' = (d - \Delta; m_1 - \Delta, m_2 - \Delta, m_3 - \Delta, m_4, \ldots, m_r)$, where

$$\Delta = m_1 + m_2 + m_3 - d.$$

(Note that by the Noether inequality, Δ is positive.)

(2) rearrange the multiplicities of $\mathbf{x}'$ in non increasing order;

(3) if $\mathbf{x}'$ is equal to $(1; 0, \ldots)$ then stop and conclude that $\mathbf{x}$ was a homaloidal type; if one of the multiplicities of $\mathbf{x}'$ is negative, then stop, and concludes that $\mathbf{x}$ was an improper homaloidal type; otherwise, go to step 1.

Geometrically, the first step corresponds to what happens to the homaloidal type when one composes f with a standard quadratic Cremona involution centered at the three base points of highest multiplicity.

We now use this algorithm to organize all homaloidal types as the vertices of a tree with root $(1; 0, \ldots)$.

2.3.1. *Parents.* Given a homaloidal type

$$\mathbf{x} = (d; m_1, \ldots, m_r)$$

of degree $d \geq 2$, its **parent** - denoted by $\mathbf{x}'$ or by $p(\mathbf{x})$ - is the homaloidal type

$$\mathbf{x}' = (d'; m'_1, \ldots, m'_{r'})$$

obtained as follows. First, one computes

$$\Delta(\mathbf{x}) = m_1 + m_2 + m_3 - d$$

to get an integer $\Delta(\mathbf{x}) \geq 1$. Then, the degree d' of $\mathbf{x}'$ is given by

$$d' = d - \Delta(\mathbf{x}).$$

Then, the multiplicities m'_ℓ of $\mathbf{x}'$ are obtained in three steps from the multiplicities of $\mathbf{x}$: first, one replaces m_1, m_2, m_3 respectively $m_1 - \Delta(\mathbf{x}), m_2 - \Delta(\mathbf{x}), m_3 -$

$\Delta(\mathbf{x})$ and then one reorders the set of multiplicities to list them in decreasing order. Thus, there are indices i, j, k such that

$$\begin{cases} m'_i = m_1 - \Delta(\mathbf{x}) = d - m_2 - m_3 \\ m'_j = m_2 - \Delta(\mathbf{x}) = d - m_3 - m_1 \\ m'_k = m_3 - \Delta(\mathbf{x}) = d - m_1 - m_2 \end{cases}$$

and as a set (with elements repeated according to their multiplicities) we have

$$\{m'_1, m'_2 \ldots\} = \{m_1 - \Delta(\mathbf{x}), m_2 - \Delta(\mathbf{x}), m_3 - \Delta(\mathbf{x}), m_4, \ldots, m_r\}$$

Our convention is that the indices (i, j, k) will always be chosen as small as possible; in particular, they are uniquely defined. Note that we always have $r' \leq r$ and $d' \leq d - 1$.

We shall also say that $\mathbf{x}$ is the **child** of $\mathbf{x}'$ obtained from the **seed** (m'_i, m'_j, m'_k).

2.3.2. *The tree.* Hudson's algorithm tells us that, if we draw the graph with vertices labeled by homaloidal types and with an edge between a type and its parent, then we get a tree, with root at $(1;0)$. Going down the tree corresponds to taking sequences of children; when doing so, the degree increases strictly. Going up the tree corresponds to the Hudson's algorithm, or equivalently to computing the sequence of successive parents $\mathbf{x}$, $p(\mathbf{x})$, $p(p(\mathbf{x}))$,... This tree will be called the **Hudson tree**. A finite sequence $\mathbf{x}^0, \ldots, \mathbf{x}^N$ of homaloidal types such that $\mathbf{x}^n = p(\mathbf{x}^{n+1})$ for $n < N$ is called a **lineage** (or an ascending lineage, to say that we go up the tree).

2.3.3. *Children.* Conversely, given a proper homaloidal type $\mathbf{x} = (d; m_1, \ldots, m_r)$, and a triple of multiplicities (m_i, m_j, m_k) in non increasing order([2]), *with minimal possible indexes*, one can consider the homaloidal type

$$T_{i,j,k}\mathbf{x} := (d + \nabla(\mathbf{x}); \{m_i + \nabla(\mathbf{x}), m_j + \nabla(\mathbf{x}), m_k + \nabla(\mathbf{x}), (m_t)_{t \neq, i,j,k}\}),$$

where we ask the following number to be positive:

$$\nabla(\mathbf{x}) = d - (m_i + m_j + m_k) \geq 1. \tag{2.4}$$

We shall also denote this number by $\nabla_{i,j,k}(\mathbf{x})$ to make the triple precise.

A triple (m_i, m_j, m_k) is said to be **admissible** if it is a seed, i.e. if $T_{i,j,k}\mathbf{x}$ is a child of $\mathbf{x}$. It is equivalent to ask that $(m_i + \nabla(\mathbf{x}), m_j + \nabla(\mathbf{x}), m_k + \nabla(\mathbf{x}))$ are the

[2]The values may be added zeroes at the end, so that some of the multiplicities in the triple can be equal to 0.

three largest multiplicities of the set $\{m_i+\nabla(\mathbf{x}), m_j+\nabla(\mathbf{x}), m_k+\nabla(\mathbf{x}), (m_t)_{t\neq i,j,k}\}$. Since $m_i \geq m_j \geq m_k$, this is equivalent to ask that

$$\forall t \neq i,j,k, \quad m_k+\nabla(\mathbf{x}) \geq m_t,$$

or equivalently

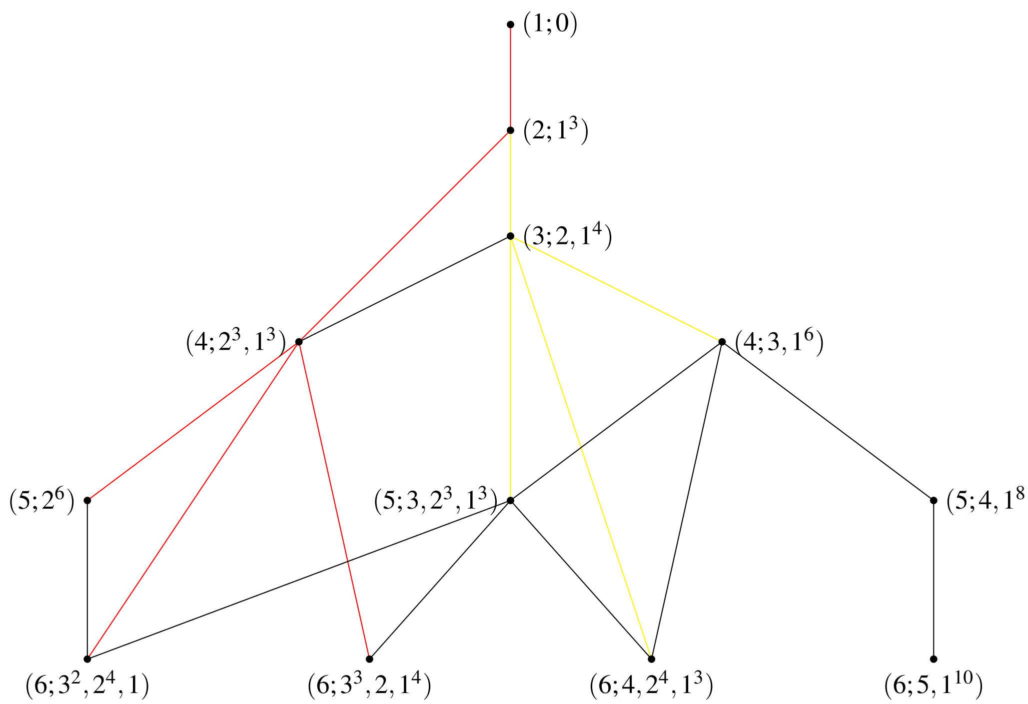
$$\forall t \neq i,j,k, \quad d-m_i-m_j \geq m_t. \tag{2.5}$$

To sum up,

- any homaloidal type $\mathbf{x}$ of degree $d \geq 2$ determines a seed (m'_i, m'_j, m'_k) from the set of multiplicities $m'_1 \geq m'_2 \geq \ldots \geq m'_{r'}$ of its parent $\mathbf{x}' = p(\mathbf{x})$;
- conversely, any triple (m_i, m_j, m_k) in the set of multiplicities of $\mathbf{x}$ that satisfies (2.5) is a seed of $\mathbf{x}$ that determines a child $T_{i,j,k}\mathbf{x}$ of $\mathbf{x}$.

In what follows, we will often consider **descending lineages** (also called simply **descendants**) $\mathbf{x}_0, \ldots, \mathbf{x}_N$, which are completely determined by $\mathbf{x}_0$ and a sequence of seeds $(m^{(n)}_{i(n)}, m^{(n)}_{j(n)}, m^{(n)}_{k(n)})$ for $0 \leq n \leq N-1$, where

$$\mathbf{x}_n = (d^{(n)}; m^{(n)}_1, m^{(n)}_2, \ldots).$$



Example 2.3. The picture above represents all possible ways to reach the homaloidal types in degree 6, via quadratic Cremona maps, from lower degree homaloidal types. We do not include children in degree ≥ 7. The red paths represent the complete lineages of $(5;2^6),(6;3^2,2^4,1),(6;4,2^4,1^3)$. They have $(4;2^3,1^2)$ as parent so that $(4;2^3,1^2)$ has $(5;2^6),(6;3^2,2^4,1),(6;4,2^4,1^3)$ has children. The parent of $(4;2^3,1^2)$ is $(2;1^3)$ which has a second child, namely $(3;2,1^4)$. The homaloidal type $(3;2,1^4)$ has in turn three children, namely $(5;3,2^3,1^3)$, $(6,3^3,2,1^4)$, $(4;3,1^6)$. Their complete lineages is represented by the yellow paths together with the red segment joining $(1;0)$ and $(2;1^3)$.

2.4. Irreducible components and the main intermediate statement. For $d \geq 1$, let $\mathsf{Bir}_d = \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})_d$ be the family of all birational transformations f of $\mathbb{P}^2_{\mathbf{k}}$ of degree equal to d. Using homogeneous formulas $[f_0 : f_1 : f_2]$ with no common factor, one can endow Bir_d with the structure of an algebraic variety (embedded as a non-closed variety in $\mathbb{P}^{M-1}$ where $M = 3\binom{d+2}{2}$, see [2, 3]).

As said in the Introduction, we denote by N_d the number of irreducible components of Bir_d. The main result we shall need, beside Hudson's test, is that *irreducible components of* Bir_d *are in* 1*-to-*1 *correspondence with homaloidal types of degree d* (see [1, 2]). Thus, N_d is just the number of such homaloidal types, and the purpose of this article is to prove the following (as well as a similar result for $\sum_{d'\leq d} N_{d'}$):

Theorem 2.4. *Let N_d be the number of (proper) homaloidal types of degree d, and $N_{\leq d} := \sum_{d'=1}^{D} N_{d'}$ the number of homaloidal type of degree $\leq d$. Then*

$$\sqrt{\ln(2)} \leq \liminf_{d\to\infty} \frac{\ln\ln N_{\leq d}}{\sqrt{\ln d}} \leq \limsup_{d\to\infty} \frac{\ln\ln N_{\leq d}}{\sqrt{\ln d}} \leq 2\sqrt{\ln 2},$$

and

$$\sqrt{\ln(2)} \leq \limsup_{d\to\infty} \frac{\ln\ln N_d}{\sqrt{\ln d}} \leq 2\sqrt{\ln 2}.$$

The strategy is to estimate precisely the maximal growth of the seedbed $s(\mathbf{x})$ of a homaloidal type $\mathbf{x}$ in terms of its degree $d(\mathbf{x})$. We shall prove the following statement.

Theorem 2.5. *For any $\alpha > 2\sqrt{\ln 2}$, there exists $C_\alpha > 0$ such that for all* $\mathbf{x}$*,*

$$s(\mathbf{x}) \leq C_\alpha e^{\alpha\sqrt{\ln(d(\mathbf{x}))}}.$$

For any $\beta < \sqrt{\ln(2)}$, there exists a homaloidal type $\mathbf{x}$ *of arbitrarily large degree such that*

$$s(\mathbf{x}) \geq e^{\beta\sqrt{\ln(d(\mathbf{x}))}}.$$

As a consequence,

$$\sqrt{\ln(2)} \leq \limsup \frac{\ln(s(d))}{\sqrt{\ln(d)}} \leq 2\sqrt{\ln(2)}$$

where $s(d)$ denotes the **maximal seedbed** among homaloidal types of degree d. The reason why $s(\mathbf{x})$ is called the seedbed of $\mathbf{x}$ is because a large seedbed corresponds to a large variety of seeds for $\mathbf{x}$, hence a large number of edges emanating from $\mathbf{x}$ in the Hudson tree.

Remark 2.6. Since any homaloidal type of degree d satisfies the Noether Equalities (2.1), N_d is bounded from above by the number of partitions of $3d-3$ and the Hardy-Ramanujam estimate implies that $N_d \leq (a/d)\exp(b\sqrt{d})$, for some $a > 0$ and $b = \pi\sqrt{2}$. This upper bound relies only on the first Noether equality; adding the second inequality is reminiscent of the Hilbert-Kamke problem (see [8]), but with a number r of terms m_i that is not uniformly bounded. From this, one can show that the number of solutions to (2.1) grows at least as $\exp(b'\sqrt{d})$ for some $b' > 0$. In particular, Theorem 2.4 implies that the probability that a solution of (2.1) be represented by a homaloidal type goes to 0 as d goes to $+\infty$.The next table lists the values of N_d and of the number S_d of solutions to (2.1) (with non-increasing m_i) for $d \in [2,15]$:

d	2	3	4	5	6	7	8	9	10	11	12	13	14	15
N_d	1	1	2	3	4	5	9	10	17	19	29	34	51	63
S_d	1	1	2	4	5	9	16	25	42	64	107	165	256	402

3. Basic inequalities

As recalled in Section 2.2, the multiplicities of a homaloidal type satisfy certain basic inequalities. The next lemmas describe some of them.

Lemma 3.1. *Let (m_i, m_j, m_k) be a seed of a homaloidal type of degree ≥ 2. Then at most one value was chosen from the three largest ones, so that*

$$j > 3.$$

Proof. We make a case-by-case analysis to show that any 2 values taken in the set $\{m_1, m_2, m_3\}$ lead to a contradiction.

• Assume that the triple is of the form (m_1, m_2, m_k). By Equation (2.2),

$$m_1 + m_2 + m_3 > d, \tag{3.1}$$

hence $k \neq 3$; so by Equation (2.5), we obtain $d - m_1 - m_2 \geq m_3$, in contradiction with Inequality (3.1).

•Assume that the triple is of the form (m_1, m_3, m_k). By Equation (2.5),

$$d - m_1 - m_3 \geq m_2$$

and again, this contradicts Inequality (3.1).

• Assume that the triple is of the form (m_2, m_3, m_k). By Equation (2.5),

$$d - m_2 - m_3 \geq m_1$$

and for the same reason this is a contradiction. □

Lemma 3.2. *Let* $\mathbf{x} = (d; m_1, m_2, m_3, \ldots)$ *be a (proper) homaloidal type of degree* $d \geq 2$. *Then*

$$m_1 + m_2 \leq d,$$

$$m_1 + 2m_3 > d$$

and

$$m_2 \geq \frac{d^2 - 1 - m_1^2}{3d - 3 - m_1}.$$

The first two inequalities are well known. The second one is in Lemma 8.2.6 of [1], it implies Noether's inequality, and it holds for improper homaloidal types such that $m_1 + m_2 \leq d$.

Proof. Let $\mathbf{x}' = (d'; m'_1, \ldots)$ be the parent of $\mathbf{x}$ and let (m'_i, m'_j, m'_k) be the corresponding seed. Then, $m'_k = d - m_1 - m_2 \geq 0$, hence $m_1 + m_2 \leq d$. For the second inequality, note that $j > 3$ by Lemma 3.1. Thus, m_4 is equal to m'_2 if $i = 1$ or m'_1 otherwise. In both cases,

$$m_3 \geq m_4 \geq m'_2 > m'_j,$$

so

$$m_3 > m_2 + (d - m_1 - m_2 - m_3).$$

Hence the second inequality.

The last inequality is deduced from Noether's equations. By Noether's first equation,

$$d^2 - 1 = m_1^2 + m_2^2 + \sum_{i \geq 3} m_i^2,$$

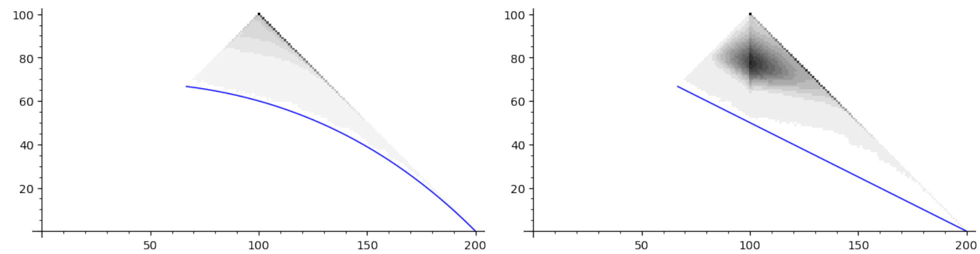


FIGURE 1. On the left, schematic representation of the distribution of (m_1, m_2) for the 3585742777 homaloidal types of degree 200, where darker indicates more homaloidal types. On the right, the distribution of (m_1, m_3) for the same degree. Both are included in triangles determined by the inequalities $m_1 + m_2 \leq d$, $m_2 \leq m_1$ and similarly for m_3. The blue curves indicates the lower bounds on m_2 and m_3 as functions of m_1 given by Lemma 3.2.

so since $m_i \leq m_2$ for $i \geq 3$,

$$d^2 - 1 \leq m_1^2 + m_2^2 + m_2(\sum_{i \geq 3} m_i),$$

using Noether's second equation,

$$d^2 - 1 \leq m_1^2 + m_2^2 + m_2(3d - 3 - m_1 - m_2) = m_1^2 + m_2(3d - 3 - m_1),$$

so

$$\frac{d^2 - 1 - m_1^2}{3d - 3 - m_1} \leq m_2,$$

As required. □

4. SEEDS, $*$-SEEDS, AND TAILS

We collect several important lemmas that concern the sequences of seeds in a descending lineage.

4.1. $*$-triples and $*$-seeds. We say that the triple (m_i, m_j, m_k) of $\mathbf{x}$ is a **$*$-triple** if $m_i = m_1$ (i.e. $i = 1$). Such a triple will often be written $(*, m_j, m_k)$, as the first value is implied. A seed defined by a $*$-triple will be called a **$*$-seed**, and the birth of a child conducted by a $*$-seed will be called a **$*$-birth**. Of particular interest to us will be lineages that are defined by sequences of successive $*$-seeds, as they admit simple combinatorial descriptions.

Being a $*$-triple can be characterized by how $d - m_1$ increases:

Lemma 4.1. *Let* $\mathbf{x} = (d; m_1, \ldots, m_r)$ *be some homaloidal type,* $[m_i, m_j, m_k]$ *be a seed and* $\mathbf{x}' = (d'; m'_1, \ldots, m'_r) = T_{i,j,k}\mathbf{x}$ *be the corresponding child. Then*

$$d' - m'_1 \geq d - m_1,$$

with equality if and only if (m_i, m_j, m_k) *is a* $*$*-seed, that is* $m_i = m_1$*.*

Proof. This follows, with $\nabla = \nabla_{i,j,k}(\mathbf{x})$, from

$$d' - m'_1 = (d + \nabla) - (m_i + \nabla) = (d - m_1) + (m_1 - m_i)$$

and the convention that $i = \inf\{\ell \,;\, m_\ell = m_i\}$. □

We will say that the first multiplicity m_1 of $\mathbf{x} = (d; m_1, m_2, \ldots)$ is **lonely** if $m_1 > m_2$. We note the following easy property:

Lemma 4.2. *If* $\mathbf{x}$ *is obtained from a* $*$*-seed of* $p(\mathbf{x})$*, then its first multiplicity is lonely.*

Proof. Write $p(\mathbf{x}) = (d'; m'_1, m'_2, \ldots)$, and let (m'_1, m'_j, m'_k) be the seed giving birth to $\mathbf{x}$. Then by Lemma 3.1, $m'_j < m'_2$ so $m'_j < m'_1$. The two largest multiplicities of $\mathbf{x}$ being $d' - m'_j - m'_k$ and $d' - m'_1 - m'_k$, they are not equal. □

A $*$-triple $(*, m_j, m_k)$ is a seed if and only if the second value m_j is small enough:

Lemma 4.3. *A* $*$*-triple* $(*, m_j, m_k)$ *is a* $*$*-seed if and only if*

$$d - m_1 - m_2 \geq m_j.$$

Proof. Assume the triple is a seed, then by Lemma 3.1, $m_2 = max_{t \neq 1,j,k} m_t$ so Inequality (2.5) proves the claim. Conversely, if

$$d - m_1 - m_2 \geq m_j,$$

then

$$d - m_1 - m_j \geq m_2 \geq max_{t \neq 1,j,k} m_t$$

because we always have $m_2 \geq max_{t \neq 1,j,k} m_t$. So Inequality (2.5) is satisfied, i.e. the triple $(*, m_j, m_k)$ is a seed. □

4.2. Tail and successive $*$-seeds. Let $\mathbf{x} = (d; m_1, m_2, \cdots, m_r)$ be a homaloidal type. Define the **tail** of $\mathbf{x}$ to be the sequence $(m_\ell, m_{\ell+1}, \ldots, m_r)$ of all multiplicities satisfying $m_\ell \leq d - m_1 - m_2$.

The tail of $\mathbf{x}$ is related to the parent $\mathbf{x}'$ of $\mathbf{x}$ in the following way. Write $\mathbf{x}' = (d'; m'_1, m'_2, \ldots)$ and let (m'_i, m'_j, m'_k) be the corresponding seed. Then $m'_k = d - m_1 - m_2$ and, in the transition from $\mathbf{x}'$ to $\mathbf{x}$, the multiplicities m'_t with $t \geq$

$k+1$ are kept unchanged in $\mathbf{x}$: this sequence of multiplicities $(m'_{k+1}, m'_{k+2}, \dots)$ coincides with the tail of $\mathbf{x}$.

Lemma 4.4. *Let (m_i, m_j, m_k) and $(*, m'_{j'}, m'_{k'})$ be two successive triples, for $\mathbf{x}_0$ and for its child $\mathbf{x}_1$ respectively, the first triple being a seed and the second being a $*$-triple. Then $(*, m'_{j'}, m'_{k'})$ is a seed*

- *if and only if*

$$m_k \geq m'_{j'} \geq m'_{k'},$$

- *if and only if $m'_{j'}, m'_{k'}$ are chosen in the tail of $\mathbf{x}_1$.*

In this case the tail of the child of $\mathbf{x}_1$ is obtained from the tail of $\mathbf{x}_1$ by removing the first elements of the sequence up to $m'_{k'}$.

Proof. Since (m_i, m_j, m_k) is a seed, the homaloidal type $\mathbf{x}_1$ is equal to

$$T_{i,j,k}(\mathbf{x}_0) = (d+\nabla; m_i+\nabla, m_j+\nabla, m_k+\nabla, m_1, \dots, \widehat{m_i}, \dots, \widehat{m_j}, \dots, \widehat{m_k}, \dots, m_r)$$

where elements with hats are omitted in the list. By Lemma 4.3, $(m'_1, m'_{j'}, m'_{k'})$ is a seed if and only if

$$(d+\nabla) - (m_i+\nabla) - (m_j+\nabla) \geq m'_{j'},$$

if and only if

$$d - m_i - m_j - (d - m_i - m_j - m_k) \geq m'_{j'},$$

if and only if

$$m_k \geq m'_{j'}. \tag{4.1}$$

Since $\nabla > 0$, the value m_k was removed once from the list of $T_{i,j,k}(\mathbf{x}_0)$. Thus, the previous inequality means that $m'_{j'}$ and $m'_{k'}$ have to be chosen in the tail

$$m_{k+1}, \dots, m_r, 0, 0, \dots$$

of $\mathbf{x}_1$, because by convention the index k was chosen as small as possible when the value m_k appears several times in the set of multiplicities. Conversely, if $m'_{j'}, m'_{k'}$ are chosen from the tail of $\mathbf{x}_1$, Equation (4.1) is satisfied so $(*, m'_{j'}, m'_{k'})$ is a seed. In this case, let $\mathbf{x}_2$ be the child of $\mathbf{x}_1$ corresponding to this seed; by definition, the tail of $\mathbf{x}_2$ is obtained by removing the first elements of the tail of $\mathbf{x}_1$ up to the first occurrence of $m'_{k'}$, or the second one if $m'_{j'} = m'_{k'}$. □

As a consequence, a descending lineage $\mathbf{x}_0, \dots, \mathbf{x}_\ell$ obtained by a sequence of $*$-seeds can be encoded by considering the tail $m_{k+1}, \dots, 0, \dots$ of $\mathbf{x}_0$, by selecting 2ℓ values in this sequence, say $\mu_1 \geq \mu'_1 \geq \mu_2 \geq \mu'_2 \geq \dots \geq \mu'_\ell \geq \mu'_\ell$, and then by choosing for $\mathbf{x_n}$ ($1 \leq n \leq \ell$) the triple $(*, \mu_i, \mu'_i)$. Thus, there are not so many possibilities for sequences $*$-births, as the tail shorten at each step by at least two elements.

Example 4.5. If we begin with

$$\mathbf{x}_1 = (80;43,31,27,26,26,21,21,18,17,2,2,2,1)$$

and try to list its possible descendants by $*$-seeds, we first look at its parent,

$$\mathbf{x}_0 = p(\mathbf{x}_1) = (59;26,26,22,21,21,18,17,10,6,2,2,2,1),$$

and the seed giving birth to $\mathbf{x}_1$ is $(22,10,6)$; thus, the remaining tail is $(2,2,2,1,0,\ldots)$. The choices for $*$-seeds are

(a) $(*,2,2)$ with remaining tail $(2,1,0,\ldots)$,
(b) or $(*,2,1)$, $(*,2,0)$, $(*,1,0)$, $(*,0,0)$, each with the zero remaining tail $(0,\ldots)$.

Unless we made the first choice $(*,2,2)$, the tail vanishes so the next choices of triple have to be $(*,0,0)$. If we did the first choice $(*,2,2)$, so that

$$\mathbf{x}_2 = (113;76,35,35,31,27,26,26,21,21,18,17,2,1),$$

then the remaining tail is $(2,1,0\ldots)$ so the next possible choices are $(*,2,1)$, $(*,2,0)$, $(*,1,0)$ and $(*,0,0)$, and each of them reduces the tail to zero, so the next choices of triple have to be $(*,0,0)$. So basically, there is only seven sequences of $*$-seeds originating from $\mathbf{x}_1$, which are given by :

$$(*,2,2),(*,2,1),(*,0,0)\ldots \quad \text{and} \quad (*,2,2),(*,2,0),(*,0,0)\ldots$$

$$(*,2,2),(*,0,0)\ldots \quad \text{and} \quad (*,2,1),(*,0,0)\ldots$$

$$(*,2,0),(*,0,0)\ldots \quad \text{and} \quad (*,1,0),(*,0,0)\ldots$$

$$(*,0,0)\ldots$$

4.3. Successive $*$-triples and degrees. In this section, we consider $\mathbf{x}_1,\ldots,\mathbf{x}_\ell$, a descending lineage such $d(\mathbf{x}_1)\geq 2$ and the corresponding (admissible) triples $(*,m^{(n)}_{j(n)},m^{(n)}_{k(n)})$ are all $*$-seeds. First, we can precisely estimate the increase in degree of this sequence, which is roughly linear in ℓ:

Lemma 4.6. *Let $\mathbf{x}_1,\ldots,\mathbf{x}_\ell$ be a lineage such that $d(\mathbf{x}_1)\geq 2$ and all corresponding seeds are $*$-seeds. Then*

$$d(\mathbf{x}_\ell) \leq \left(\frac{2\ell+1}{3}\right) d(\mathbf{x}_1).$$

If moreover the ratio between the first multiplicity of $\mathbf{x}_1$ *and its degree is at most* $5/6$ *then*

$$d(\mathbf{x}_\ell) \geq \max\left(\frac{\ell-13}{6}, 1\right) d(\mathbf{x}_1).$$

When $m_1 \leq 5d(\mathbf{x}_1)/6$, we shall say that $\mathbf{x}_1$ does not have large first multiplicity, see Section 6.1.

Proof. The degree increase between $\mathbf{x_n}$ and $\mathbf{x}_{n+1}$ is given by

$$\Delta(\mathbf{x}_{n+1}) = d(\mathbf{x}_n) - m_1^{(n)} - m_{j(n)}^{(n)} - m_{k(n)}^{(n)},$$

and by Lemma 4.1, the difference between the degree and the first value is constant along a sequence of $*$-births. So

$$\Delta(\mathbf{x}_{n+1}) = \left(d(\mathbf{x}_1) - m_1^{(1)}\right) - m_{j(n)}^{(n)} - m_{k(n)}^{(n)}.$$

Summing over ℓ gives

$$d(\mathbf{x}_\ell) - d(\mathbf{x}_1) = \sum_{n=1}^{\ell-1} \Delta(\mathbf{x}_{n+1}) \tag{4.2}$$

$$= (\ell-1)(d(\mathbf{x}_1) - m_1^{(1)}) - \sum_{n=1}^{\ell-1} (m_{j(n)}^{(n)} + m_{k(n)}^{(n)}). \tag{4.3}$$

Using that $m_1^{(1)} > d(\mathbf{x}_1)/3$, we get the desired upper bound

$$d(\mathbf{x}_\ell) - d(\mathbf{x}_1) \leq \frac{2}{3}(\ell-1) d(\mathbf{x}_1).$$

This proves the first inequality; and this is optimal since we can always choose the $m_{k(n)}^{(k)}$ and $m_{k(n)}^{(n)}$ to be 0.

By the Lemma 4.4, the decreasing sequence

$$m_{j(1)}^{(1)}, m_{k(1)}^{(1)}, m_{j(2)}^{(2)}, m_{k(2)}^{(2)}, \ldots, m_{j(\ell-1)}^{(\ell-1)}, m_{k(\ell-1)}^{(\ell-1)}$$

is extracted from the tail of $\mathbf{x}_1$. Therefore,

$$\sum_{n=1}^{\ell-1} (m_{j^{(n)}}^{(n)} + m_{k^{(n)}}^{(n)}) \leq \sum_p m_p^{(1)} < 3d(\mathbf{x}_1)$$

by Noether's first equation (see Equation (2.1)). By Equation (4.3), we obtain

$$d(\mathbf{x}_\ell) - d(\mathbf{x}_1) > (\ell-1)(d(\mathbf{x}_1) - m_1^{(1)}) - 3d(\mathbf{x}_1).$$

and since $m_1^{(1)} \leq 5d(\mathbf{x}_1)/6$, this gives

$$d(\mathbf{x}_\ell) - d(\mathbf{x}_1) \geq \left(\frac{1}{6}(\ell-1) - 3\right) d(\mathbf{x}_1).$$

Since $d(\mathbf{x}_\ell) \geq d(\mathbf{x}_1)$, this proves the second inequality. □

4.4. Large seedbed and large progeny. The following proposition shows that a single homaloidal type with large seedbed $s(\mathbf{x})$ produces many distinct descendants of small degrees. This proposition is at the heart of the proof of the lower bound on N_d. We prove it now to illustrate the mechanism of how sequences of $*$-births behave.

Proposition 4.7. *Let* $\mathbf{x}$ *be a homaloidal type of seedbed* $s(\mathbf{x})$ *and degree* d. *Let* $s \leq s(\mathbf{x})$ *be an integer. Then* $\mathbf{x}$ *has a least* 2^s *distinct descendants of degree* $\leq \left(\frac{5s+10}{3}\right) d$.

Proof. Let $m_{i_1} > \ldots > m_{i_{s(\mathbf{x})}}$ be indices corresponding to non-zero distinct multiplicities, chosen minimally.

Construction of descendants.– For each of the 2^s subsets E of $\{i_1, \ldots, i_s\}$, we construct a descendant $\mathbf{y}(E)$, by considering the following sequence of seeds.

First 3 (uniform) moves.– The first three choices of triples are made so that further triples will be admissible, while having the tail $m_1, \ldots, m_r$. We begin with

$$\mathbf{x}_1 = \mathbf{x} = (d; m_1, \ldots, m_r),$$

and the children we shall construct will be labeled $\mathbf{x}_i$, $i = 2, 3, \ldots$. We first use the triple: $(0,0,0)$. It is a seed by Inequality (2.5), and $\Delta(\mathbf{x}_2) = d$. We obtain

$$\mathbf{x}_2 = (2d; d, d, d, m_1, \ldots, m_r),$$

Now, we use the seed $(0,0,0)$, with $\Delta(\mathbf{x}_3) = 2d$, to get

$$\mathbf{x}_3 = (4d; 2d, 2d, 2d, d, d, d, m_1, \ldots, m_r).$$

Next, we choose the triple (d,d,d), it is a seed by (2.5) because $4d - d - d \geq 2d$, and $\Delta(\mathbf{x}_4) = d$; so

$$\mathbf{x}_4 = (5d; (2d)^6, m_1, \ldots, m_r),$$

where the exponent 6 indicates repetition. Now the tail of $\mathbf{x}_4$ is precisely $(m_1, \ldots, m_r, 0, \ldots)$, and in the next step we start choosing $*$-seeds from this sequence.

Sequence of moves associated to E.– We choose a subset $E \subset \{i_1, \ldots, i_s\}$, which will act as a parameter, to construct a homaloidal type $\mathbf{y}(E)$. Consider the values $(m_j)_{j \in E}$; in the case where E has odd cardinality, we add a 0 at the end of these values. Then we order them with least possible indices to obtain a subsequence $m_{j_1} > \ldots > m_{j_{2n}}$ of even length, and we consider the sequence of triples

$$(*, m_{j_1}, m_{j_2}), (*, m_{j_3}, m_{j_4}), \ldots, (*, m_{j_{2n-1}}, m_{j_{2n}}).$$

By construction, m_{j_1}, m_{j_2} are in the tail of $\mathbf{x}_4$, hence $(*, m_{j_1}, m_{j_2})$ is admissible by Lemma 4.3. Now (m_{j_3}, m_{j_4}) are also chosen from this tail so $(*, m_{j_3}, m_{j_4})$ is, again, an admissible triple for $\mathbf{x}_5$. By induction, the whole sequence of triples

$$(*, m_{j_1}, m_{j_2}), (*, m_{j_3}, m_{j_4}), \ldots, (*, m_{j_{2n-1}}, m_{j_{2n}})$$

starting from $\mathbf{x}_4$ is a sequence of $*$-seeds and defines a final homaloidal type $\mathbf{y}(E) := \mathbf{x}_{4+n}$.

Degree estimate, injectivity, and conclusion.– First, let us estimate the degree of $\mathbf{y}(E)$. By Lemma 4.6,

$$d(\mathbf{y}(E)) \leq \left(\frac{2n+1}{3}\right) d(\mathbf{x_4}).$$

Since $n \leq (s+1)/2$ by construction, we get

$$d(\mathbf{y}(E)) \leq \left(\frac{s+2}{3}\right) 5d.$$

Second, let us show that the association $E \mapsto \mathbf{y}(E)$ is injective: given $\mathbf{y}(E)$, we can recover the whole sequence

$$(*, m_{j_1}, m_{j_2}), (*, m_{j_3}, m_{j_4}), \ldots, (*, m_{j_{2n-1}}, m_{j_{2n}}),$$

in reverse order by applying Hudson's algorithm to it, until we arrive to $\mathbf{x}_4$. Since the numbers $(m_{j_k})_{0 \leq k \leq 2n}$ are distinct, the corresponding indices j_p, j_{p+1} are uniquely determined, and E is the set of such indices. Since this map is injective, the conclusion follows from the degree estimate. □

5. Constructing homaloidal types of large seedbed

In this section, we construct homaloidal types of large seedbed, and use them to deduce the lower bound in Theorem 2.5 and Theorem B.

5.1. Splitting blocks. Recall that a block in $(d; m_1, \ldots, m_r)$ is a maximal set of identical nonzero values $m_i = m_{i+1} = \ldots = m_j \neq 0$, and its width is the number of its elements.

We will produce homaloidal types of large seedbed by beginning with a de Jonquières homaloidal type $(2^{N-1}+1; 2^{N-1}, 1^{(2^N)})$, which has one block of width 2^N. Such a de Jonquiéres homaloidal type is obtained from the root $(1;0)$ by a series of 2^{N-1} $*$-births.

The following lemma shows that if we begin with an $\mathbf{x}$ that has long blocks, then we can split each block in two while controlling the increase in degree.

Lemma 5.1. *Let $k \geq 2$ and $n \geq 1$ be integers. Let $\mathbf{x}$ be a proper homaloidal type that contains at least n distinct blocks of width $\geq 2^k$ each. Then there exists a descendant of $\mathbf{x}$ of degree $\leq 5n2^k d(\mathbf{x})$ with at least $2n$ distinct blocks of width $\geq 2^{k-1}$.*

Proof. Let $d = d(\mathbf{x})$. By assumption, $\mathbf{x}$ can be written

$$\mathbf{x} = (d;\square,\ldots,\square,(\mu_1)^{2^k},\square,\ldots,\square,(\mu_2)^{2^k},\square\ldots\square,(\mu_n)^{2^k},\square,\ldots,\square)$$

where the exponents 2^k indicate repetition, with $\mu_1 > \mu_2 > \ldots > \mu_n$, and $\square$ indicates values that will be ignored. The number of occurrences of one of these μ_i can, of course, be larger than 2^k. As in the proof of Proposition 4.7, we first apply the 3 admissible triples $(0,0,0)$, $(0,0,0)$, (d,d,d) to obtain

$$(5d;(2d)^6,\square,\ldots,\square,(\mu_1)^{2^k},\square,\ldots,\square,(\mu_2)^{2^k},\square\ldots\square,(\mu_n)^{2^k},\square,\ldots,\square).$$

This is done in order to insure that the tail is now the full initial sequence of multiplicities of $\mathbf{x}$, at the price of a factor 5 in the degree. We now apply $(*,\mu_1,\mu_1)$, which is admissible since (μ_1,μ_1) is in the tail. We get $\nabla_1 = 3d - 2\mu_1$, and a new homaloidal type

$$(5d+\nabla_1;2d+\nabla_1,(3d-\mu_1)^2,(2d)^5,\square\ldots\square,(\mu_1)^{2^k-2},\square\ldots\square,(\mu_2)^{2^k},\\ \square\ldots\square,(\mu_n)^{2^k},\square\ldots\square)$$

and we may continue with $(*,\mu_1,\mu_1)$ until it has been applied 2^{k-2} times. Each application has the same $\nabla = \nabla_1$, and we get

$$(5d+2^{k-2}\nabla_1;2d+2^{(k-2)}\nabla_1,(3d-\mu_1)^{2^{k-1}},(2d)^5,\square,\ldots,\square,(\mu_1)^{2^{k-1}},\\ \square,\ldots,\square,(\mu_2)^{2^k},\square\ldots)$$

Now we do the same with $(*,\mu_2,\mu_2)$, again applied 2^{k-2} times. This time $\nabla_2 = 3d - 2\mu_2$, so we get the new homaloidal type

$$(5d+2^{k-2}\nabla_1+2^{k-2}\nabla_2;\ 2d+2^{k-2}\nabla_1+2^{k-2}\nabla_2,(3d-\mu_2)^{2^{k-1}},(3d-\mu_1)^{2^{k-1}},\\ (2d)^5,\square\ldots\square,(\mu_1)^{2^{k-1}},\square\ldots\square,(\mu_2)^{2^{k-1}},\square\ldots\square,(\mu_n)^{2^k},\square\ldots\square)$$

Continuing this way for the values $\mu_3,\ldots,\mu_n$, we obtain something of the form

$$(d';m_1',(3d-\mu_n)^{2^{k-1}},\ldots,(3d-\mu_1)^{2^{k-1}},\square\ldots\square,(\mu_1)^{2^{k-1}},\\ \square\ldots\square,(\mu_n)^{2^{k-1}},\square\ldots\square),$$

which has $2n$ distinct blocks of width $\geq 2^{k-1}$. The blocks are distinct since $3d-\mu_n > \ldots > 3d-\mu_1 > d > \mu_1 > \ldots > \mu_n$. By Lemma 4.6, the degree d' obtained after this sequence of $\ell - 1 = n2^{k-2}$ $*$-seeds is at most

$$d' \leq \left(\frac{n2^{k-1}+3}{3}\right)(5d) \leq 5n2^k d.$$

□

5.2. Lower bound on length increase. Here we prove the lower bound in Theorem 2.5, which we recall in the next proposition.

Proposition 5.2. *There exists a sequence of proper homaloidal types $(\mathbf{y}_N)_{N\geq 1}$ of degrees $d(\mathbf{y}_N) \leq 2^{N^2}10^N$, such that $s(\mathbf{y}_N) \geq 2^{N-1}$. In particular, for any $c < \sqrt{\ln(2)}$ and N large enough,*

$$s(\mathbf{y}_N) \geq \exp(c\sqrt{\ln(d(\mathbf{y}_N))})$$

Proof. Let $N \geq 1$ be an integer parameter. We begin by constructing recursively an auxiliary finite sequence of homaloidal types $(\mathbf{x}_{0,N}, \ldots, \mathbf{x}_{N-1,N})$ whose last element will be $\mathbf{y}_N$. If we start with the de Jonquières homaloidal type

$$\mathbf{x}_{0,N} := (2^{N-1}+1;\ 2^{N-1}, (1)^{2^N})$$

which has 1 block of width 2^N, we can apply Lemma 5.1 successively $N-1$ times with $n = 2^i$ and $k = N-i$, where $i = 0, \ldots, N-2$. Doing so, we obtain a sequence of homaloidal types $(\mathbf{x}_{i,N}) = (\mathbf{x}_{0,N}, \ldots, \mathbf{x}_{N-1,N})$ of degrees $d_{0,N} = 2^{N-1}+1, d_{1,N}, \ldots, d_{N-1,N}$ such that each $\mathbf{x}_{i,N}$ has 2^i distinct blocks of width $\geq 2^{N-i}$. The successive degrees satisfy the inequality

$$d_{i+1,N} \leq 5 \cdot 2^{N-i} \cdot 2^i \cdot d_{i,N} = 5 \cdot 2^N \cdot d_{i,N}.$$

At the end,

$$d_{N-1,N} \leq 5^N 2^{N^2}(2^{N-1}+1) \tag{5.1}$$

$$\leq 2^{N^2}10^N. \tag{5.2}$$

We now define the homaloidal type $\mathbf{y}_N := \mathbf{x}_{N-1,N}$, so $d(\mathbf{y}_N) \leq 2^{N^2}10^N$. Its seedbed $s(\mathbf{y}_N)$ is larger than the number 2^{N-1} of blocks of width 2 that have been created. The last statement follows from the fact that when N is large, we have

$$N \geq \sqrt{\frac{\ln(d(\mathbf{y}_N))}{\ln(2)}}(1+o(1)), \tag{5.3}$$

and the seedbed satisfies

$$s(\mathbf{y}_N) \geq 2^{N-1} = \exp(N\ln(2) + O(1)). \tag{5.4}$$

□

5.3. Lower bound on the number of proper homaloidal types. Combining Proposition 4.7 and 5.2, we now prove the lower bound of Theorem 2.4.

Theorem 5.3. *Let N_d be the number of proper homaloidal types of degree d, and $N_{\leq d}$ the number of proper homaloidal types of degree $\leq d$. Then*

$$\sqrt{\ln(2)} \leq \liminf_{d\to\infty} \frac{\ln\ln N_{\leq d}}{\sqrt{\ln d}},$$

and for every $\beta < \sqrt{\ln(2)}$, there exists arbitrary large degrees d for which

$$\ln\ln N_d \geq \beta\sqrt{\ln d}.$$

Proof. By Proposition 5.2, there is a sequence of homaloidal types $\mathbf{y}_k$ of degree $d_k \leq 2^{k^2}10^k$, with seedbed $s_k \geq 2^{k-1}$. We choose the auxiliary parameter $s = 2^{k-1}$, and apply Proposition 4.7. So $\mathbf{y}_k$ has at least 2^s children of degree $\leq D_k := \binom{5s+10}{3} d_k$, so as k tends to infinity,

$$\ln(D_k) \leq \ln(s) + O(1) + \ln(d_k) \leq k^2\ln 2 + k\ln(20) + O(1).$$

so

$$\sqrt{\ln(D_k)} \leq k\sqrt{\ln 2 + o(1)},$$

Let $\beta < \sqrt{\ln(2)}$, then for k large enough, say $k \geq k_\beta$ for some $k_\beta > 0$, we have

$$\sqrt{\ln(D_k)} \leq k\frac{\ln(2)}{\beta}. \tag{5.5}$$

Let d be a large integer. and choose the unique integer k (that depends now on d) such that

$$\frac{\beta}{\ln(2)}\sqrt{\ln(d)} - 1 < k \leq \frac{\beta}{\ln(2)}\sqrt{\ln(d)}. \tag{5.6}$$

Then if d is large enough, k is larger than k_β, so by (5.5) and (5.6), we have $D_k \leq d$. In particular, remembering that $\mathbf{y}_k$ has at least $2^{2^{k-1}}$ children of degree $\leq D_k$, we get

$$2^{2^{k-1}} \leq N_{\leq D_k} \leq N_{\leq d}.$$

Using (5.6), we obtain

$$\ln\ln N_{\leq d} \geq (k-1)\ln(2) + \ln\ln(2) \tag{5.7}$$

$$\geq \beta\sqrt{\ln(d)} + O(1). \tag{5.8}$$

Therefore, as d tends to $+\infty$, we have

$$\liminf_{d\to+\infty} \frac{\ln\ln N_{\leq d}}{\sqrt{\ln(d)}} \geq \beta,$$

which is valid for all $\beta < \sqrt{\ln(2)}$, thus implying the lower bound:

$$\liminf_{d\to+\infty} \frac{\ln\ln N_{\leq d}}{\sqrt{\ln(d)}} \geq \sqrt{\ln(2)}. \tag{5.9}$$

Now assume, by contradiction, that there exists $\beta < \sqrt{\ln(2)}$ such we have $\ln\ln N_d < \beta\sqrt{\ln d}$ for all d sufficiently large, say $d \geq d_1$. Pick c such that $\beta < c < \sqrt{\ln(2)}$. Then for D sufficiently large so that $\max_{d<d_1} N_d < \exp(\exp(\beta\sqrt{\ln D}))$, we would have

$$N_{\leq D} \leq D\exp\left(\exp(\beta\sqrt{\ln D})\right),$$
$$N_{\leq D} = o\left(\exp\left(\exp(c\sqrt{\ln D}\right)\right),$$

because $\beta < c$, and this would contradict the lower bound (5.9). □

5.4. Upper bounds on the seedbed increase. In the previous paragraphs, we managed to produce homaloidal types of large seedbed; to do this, we could double the seedbed, at each step, along a sequence of $*$-births. The main point of this section is to prove that through such a sequence of $*$-births, the seedbed can at most be multiplied by 4. In the proof of Lemma 5.1, we could notice that with a $*$-seed $(*,\mu,\mu)$ we might have created a new multiplicity value; however, applying the same type of seed $(*,\mu,\mu)$ a second time did not: it enlarged the width of an existing block, but did not create a new multiplicity. More formally, we have the following lemma.

Lemma 5.4. *Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be a lineage obtained from $\mathbf{x}_1$ by application of two successive $*$-seeds $(*,\mu,\mu)$, for some multiplicity μ. Then*

$$s(\mathbf{x}_3) \leq s(\mathbf{x}_2).$$

Proof. We denote

$$\mathbf{x}_1 = (d; m_1, m_2, m_3, \ldots, (\mu)^4, \ldots)$$

where the block of value μ is of width at least 4. Since the assumption is that the triples $(*,\mu,\mu), (*,\mu,\mu)$ are seeds, we know that $(\mu)^4$ must be part of the tail of $\mathbf{x}_1$ (see Lemma 4.4). So, applying $(*,\mu,\mu)$, we get the first child

$$\mathbf{x}_2 = (2d - m_1 - 2\mu;\ d - 2\mu, (d - m_1 - \mu)^2, m_2, m_3, \ldots, (\mu)^2, \ldots)$$

Recall that by Lemma 4.2, the first multiplicity is then lonely. We apply $(*,\mu,\mu)$ again, assuming this is an admissible triple. The increase in degree is the same as in the previous step, so we get

$$\mathbf{x}_3 = (3d - 2m_1 - 2\mu;\ d - 2\mu, (d - m_1 - \mu)^4, m_2, m_3, \ldots),$$

where the first multiplicity is still lonely, and no new multiplicity value is created (but, maybe, the multiplicity μ disappeared). So in any case, $s(\mathbf{x}_3) \leq s(\mathbf{x}_2)$. □

Proposition 5.5. *Let $\mathbf{x}_1, \ldots, \mathbf{x}_\ell$ be a descending lineage with $d(\mathbf{x}_1) \geq 2$, such that all corresponding seeds are $*$-seeds. Then*

$$s(\mathbf{x}_\ell) \leq \min\left(4s(\mathbf{x}_1) - 1,\, s(\mathbf{x}_1) + 3(\ell - 1)\right).$$

Heuristical Remark. The factor 4 in $s(\mathbf{x}_\ell) \leq 4s(\mathbf{x}_1) - 1$ is directly related to the constant $2\sqrt{\ln(2)} = \sqrt{2\ln(4)}$ in the upper bound given in Theorem B. Heuristically, imagine we could produce homaloidal types in the spirit of Proposition 5.2 by constructing a sequence $\mathbf{y}_0, \ldots, \mathbf{y}_n, \ldots$ of respective degrees $d_0, \ldots, d_n, \ldots$ and seedbeds $s_0, \ldots, s_n, \ldots$, where $\mathbf{y}_{n+1}$ is obtained from $\mathbf{y}_n$ by a small, bounded number of birth followed by a long sequence of $*$-birth, of length ℓ_n. Let's assume that the worst case scenario of seedbed increase is when the two upper bounds given by Proposition 5.5 are equal, that is when $\ell_n \simeq s_n$, and that it is realized by our sequence $(\mathbf{y}_n)_n$. In this case, we would have $s_{n+1} \simeq 4s_n$, so $s_n \simeq 4^n$, and by Lemma 4.6, $d_{n+1} \simeq \ell_n d_n$ so $d_n \simeq 4^{\frac{n(n+1)}{2}}$. In this heuristic computation, we would get

$$\lim_{n\to+\infty} \frac{\ln(s_n)}{\sqrt{\ln(d_n)}} = \sqrt{2\ln(4)}.$$

Our task later will be to explain that this heuristic is, somehow, the worst case scenario authorized by Proposition 5.5, namely to show that for homaloidal types $\mathbf{x}$ of large degrees,

$$\limsup_{d(\mathbf{x})\to+\infty} \frac{\ln(s(\mathbf{x}))}{\sqrt{\ln(d(\mathbf{x}))}} \leq \sqrt{2\ln(4)}.$$

Improving the factor 4 would improve this upper bound. More precisely, what may happen is that when one concatenates m descending lineages that start and end in the region of average first multiplicity but otherwise are always given by $*$-births in the region of large first multiplicity, then the obvious factor 4^m could possibly be replaced by a smaller quantity.

Proof of Proposition 5.5. Denote by $\left(*, m^{(n)}_{j(n)}, m^{(n)}_{k(n)}\right)$, $1 \le n \le \ell - 1$, the corresponding $*$-seeds. By Lemma 4.4, the non-increasing sequence

$$m^{(1)}_{j(1)}, m^{(1)}_{k(1)}, m^{(2)}_{j(2)}, m^{(2)}_{k(2)}, \dots, m^{(\ell-1)}_{j(\ell-1)}, m^{(\ell-1)}_{k(\ell-1)} \tag{5.10}$$

is an extracted subsequence of the tail of $\mathbf{x}_1$.

Let $\alpha_1 > \ldots > \alpha_t$ be the values that appear in the sequence (5.10). This means that the successive $*$-seeds must be of the form $(*, \alpha_i, \alpha_i)$ or $(*, \alpha_i, \alpha_{i+1})$, and the indices must be nondecreasing. Note that each triple $(*, \alpha_i, \alpha_{i+1})$ can happen only once; and it may happen that $(*, \alpha_i, \alpha_i)$ appears several times, but always consecutively. We have the obvious bound $t \le s(\mathbf{x}_1) + 1$ (taking care of the possibility that $\alpha_t = 0$), but in this can be improved to $t \le s(\mathbf{x}_1)$ because the third multiplicity m_3 of $\mathbf{x}_1$ is never in the tail of $\mathbf{x}_1$.

At each of the $\ell - 1$ steps, we change the values of three multiplicities, so clearly

$$s(\mathbf{x}_\ell) \le s(\mathbf{x}_1) + 3(\ell - 1).$$

When applying a triple of the form $(*, \alpha_i, \alpha_{i+1})$, one adds at most three new multiplicities. However, by Lemma 4.2, the first multiplicity is lonely whenever $n \ge 2$, so apart possibly from the first one, seeds of the form $(*, \alpha_i, \alpha_{i+1})$ add at most two new multiplicities each. When we apply successively $(*, \alpha_i, \alpha_i)$ any number of times, by Lemma 5.4, we may have added only one multiplicity the first time and none the others, unless it was $(*, \alpha_1, \alpha_1)$ and the first multiplicity wasn't lonely in $\mathbf{x}_1$. In the end,

$$s(\mathbf{x}_\ell) - s(\mathbf{x}_1) \le 2(t-1) + t + 1,$$

the final $+1$ counting the possibility that $\mathbf{x}_1$ did not have a lonely first multiplicity. Since $t \le s(\mathbf{x}_1)$, we obtain $s(\mathbf{x}_\ell) - s(\mathbf{x}_1) \le 3s(\mathbf{x}_1) - 1$, as desired. □

From the above upper bound, we now derive an inequality on the seedbed increase that will be more suitable for iterative applications.

Corollary 5.6. *For any $\alpha > \sqrt{2\ln(4)}$, there exist $S_0(\alpha) > 0$ and $L(\alpha) \ge 5$, such that*

$$(\ln s(\mathbf{x}_{\ell+1}))^2 - (\ln s(\mathbf{x}_1))^2 \le \alpha^2 \left(\ln d(\mathbf{x}_{\ell+1}) - \ln d(\mathbf{x}_1)\right) \tag{5.11}$$

for any descending lineage $\mathbf{x}_1, \ldots, \mathbf{x}_\ell$, $\mathbf{x}_{\ell+1}$ such that

(i) *$s(\mathbf{x}_1) \ge S_0(\alpha)$, $\ell \ge L(\alpha)$, and $d(\mathbf{x}_1) \ge 2$;*
(ii) *$\mathbf{x}_1$ does not have large first multiplicity, i.e. $m_1(\mathbf{x}_1) \le 5d(\mathbf{x}_1)/6$;*
(iii) *all the corresponding seeds are $*$-seeds, except possibly the last one.*

Remark 5.7. The inequality $L(\alpha) \geq 5$ is arbitrary, we choose it because it will be used in the proof of Proposition 7.1.

Proof. Fix $\alpha > 2\sqrt{\ln 2}$ and define $\varepsilon > 0$ by

$$2\ln(4) + 2\varepsilon = \alpha^2.$$

Set $\delta' = \ln d(\mathbf{x}_{\ell+1}) - \ln d(\mathbf{x}_1)$. By Lemma 4.6,

$$\ln d(\mathbf{x}_{\ell+1}) \geq \ln d(\mathbf{x}_\ell) \geq \ln\left(\frac{\ell-13}{6}\right) + \ln d(\mathbf{x}_1),$$

so

$$\ln\left(\frac{\ell-13}{6}\right) \leq \delta'. \tag{5.12}$$

The increase of the seedbed is captured by the quantity

$$\begin{aligned}\delta &= (\ln s(\mathbf{x_{\ell+1}}))^2 - (\ln s(\mathbf{x_1}))^2 \\ &= \ln\left(\frac{s(\mathbf{x_{\ell+1}})}{s(\mathbf{x_1})}\right) \times \ln\left(s(\mathbf{x_{\ell+1}})s(\mathbf{x_1})\right)\end{aligned}$$

and the goal is to prove that $\delta \leq \alpha^2\delta'$. We write

$$\ln\left(s(\mathbf{x_{\ell+1}})s(\mathbf{x_1})\right) = 2\ln\left(\frac{\ell-13}{6}\right) + 2\ln\left(\frac{6\ell}{\ell-13}\right) + 2\ln\left(\frac{s(\mathbf{x_1})}{\ell}\right) + \ln\left(\frac{s(\mathbf{x_{\ell+1}})}{s(\mathbf{x_1})}\right).$$

and obtain

$$\begin{aligned}\delta = {} & 2\ln\left(\frac{s(\mathbf{x_{\ell+1}})}{s(\mathbf{x_1})}\right)\cdot\ln\left(\frac{\ell-13}{6}\right) + 2\ln\left(\frac{s(\mathbf{x_{\ell+1}})}{s(\mathbf{x_1})}\right)\cdot\ln\left(\frac{6\ell}{\ell-13}\right) \\ & + 2\ln\left(\frac{s(\mathbf{x_{\ell+1}})}{s(\mathbf{x_1})}\right)\cdot\ln\left(\frac{s(\mathbf{x_1})}{\ell}\right) + \left(\ln\frac{s(\mathbf{x_{\ell+1}})}{s(\mathbf{x_1})}\right)^2.\end{aligned}$$

By Proposition 5.5,

$$s(\mathbf{x}_\ell) \leq \min\left(4s(\mathbf{x}_1) - 1),\, s(\mathbf{x}_1) + 3(\ell-1)\right),$$

so

$$\begin{cases} s(\mathbf{x}_{\ell+1}) \leq 4s(\mathbf{x}_1) + 2 \\ s(\mathbf{x}_{\ell+1}) \leq s(\mathbf{x}_1) + 3\ell. \end{cases}$$

Since $s(\mathbf{x_1}) \geq 1$, we have the upper bound $s(\mathbf{x}_{\ell+1})/s(\mathbf{x}_1) \leq 6$. If $\ell \geq 26$, then $\frac{6\ell}{\ell-13} \leq 12$. So, with Inequality 5.12, we obtain

$$\delta \leq 2\ln\left(4 + \frac{2}{s(\mathbf{x_1})}\right)\cdot\delta' + 2\ln 6\cdot\ln 12 \ + 2\ln\left(1 + \frac{3\ell}{s(\mathbf{x_1})}\right)\cdot\ln\frac{s(\mathbf{x_1})}{\ell} + (\ln 6)^2.$$

The function $f(x) = 2\ln(1+3x)\cdot\ln(1/x)$ is bounded from above on $\mathbb{R}^+$ by a constant $M < +\infty$ (numerically, $M < 1.6$). Hence, if $M' = M + 2\ln 6 \cdot \ln 12 + (\ln 6)^2$, we obtain

$$\delta \leq 2\ln\left(4 + \frac{2}{s(\mathbf{x_1})}\right)\cdot\delta' + M'.$$

Let $L(\alpha) > 26$ be such that for all $\ell \geq L(\alpha)$,

$$M' < \varepsilon \ln\left(\frac{\ell - 13}{6}\right)$$

Then, by the Estimate (5.12)

$$\delta \leq 2\ln\left(4 + \frac{2}{s(\mathbf{x_1})}\right)\cdot\delta' + \varepsilon\times\delta'. \tag{5.13}$$

We choose now $S_0(\alpha) \geq 1$ such that

$$2\ln\left(4 + \frac{2}{S_0(\alpha)}\right) \leq \varepsilon + 2\ln 4.$$

Then

$$\delta \leq (2\ln 4 + 2\varepsilon)\delta',$$

as required. □

6. SMALL, AVERAGE, AND LARGE FIRST MULTIPLICITY

In this section, we introduce three regimes: small, average, and large first multiplicities. We study how the degree and seedbed vary when a descending lineage stays in one of these regimes.

6.1. Definition. Consider a homaloidal type $\mathbf{x} = (d; m_1, \ldots)$ of degree $d \geq 2$. Note that the ratio m_1/d is always between $1/3$ and 1 (see Equation (2.3)). We say that $\mathbf{x}$ has **large first multiplicity** if

$$m_1/d > 5/6.$$

We say that $\mathbf{x}$ has **small first multiplicity** if

$$m_1/d < 7/20.$$

Otherwise, the ratio is in $[7/20, 5/6]$, and we say that $\mathbf{x}$ has **average first multiplicity**. Observe that $(2;1,1,1)$ has average first multiplicity and is a common ancestor for all nontrivial homaloidal types. The goal of this section is to describe different behaviors related to the parent of $\mathbf{x}$ according to the size of the first multiplicity.

Remark 6.1. The cutoff values $5/6$ and $7/20$ are rather arbitrary choices. See for instance the discussion around the proof of Lemma 6.3.

6.2. Large first multiplicity.

Lemma 6.2. *Let* $\mathbf{x} = (d;\, m_1,\dots)$ *be a homaloidal type of large first multiplicity. Denote by* $\mathbf{x}' = (d';\, m'_1,\dots)$ *its parent and by* (m'_i, m'_j, m'_k) *the corresponding seed. Then*

- *the seed* (m'_i, m'_j, m'_k) *is a* $*$*-seed, and*
- *the first multiplicity of* $\mathbf{x}'$ *is not small (except if* $d = 2$ *and* $\mathbf{x}' = (1;0)$*).*

Proof. We may assume $d \geq 3$, so that $d' \geq 2$ too. Then, by Lemma 3.2, $m_1 + m_2 \leq d$ and $m'_1 + m'_2 \leq d'$. Since $m_1 > 5d/6$, we get $m_2 < d/6$ and since $m_3 \leq m_2$, we get $m_3 < d/6$. So the degree increase between $\mathbf{x}'$ and $\mathbf{x}$ is at most

$$\Delta(\mathbf{x}) = m_1 + m_2 + m_3 - d \leq d + d/6 + d/6 - d = d/3.$$

This implies that

$$m'_i = m_1 - \Delta(\mathbf{x}) \geq 5d/6 - d/3 = d/2.$$

Assume $i \neq 1$, i.e. (m'_i, m'_j, m'_k) is not a $*$-seed. In this case $m'_1 \geq m'_2 \geq m'_i$, so $m'_1 + m'_2 \geq 2m'_i \geq d > d'$. This contradicts the inequality $m'_1 + m'_2 \leq d'$, and we conclude that (m'_i, m'_j, m'_k) is indeed a $*$-seed.

Now, the second assertion follows from

$$\frac{m'_1}{d'} = \frac{m_1 - \Delta(\mathbf{x})}{d - \Delta(\mathbf{x})} \geq \frac{d/2}{d} = \frac{1}{2}.$$

□

6.3. Small first multiplicity.

Lemma 6.3. *Let* $\mathbf{x} = (d; m_1,\dots)$ *be a homaloidal type of small first multiplicity. Denote by* $\mathbf{x}' = (d'; m'_1,\dots)$ *its parent and by* (m'_i, m'_j, m'_k) *the corresponding seed. Then*

- *the seed was chosen among the first* 11 *multiplicities of* $\mathbf{x}'$*, i.e.* $k \leq 11$.
- *the multiplicity of* $\mathbf{x}'$ *is not large.*

We shall prove a slightly stronger statement. Indeed, assume we replace $7/20$ by $17/50$ in the definition of small first multiplicity; then, $k \leq 11$ could be replaced by $k \leq 9$ in the lemma, and this is optimal.

Proof. Let ε be a positive real number $\leq 1/20$, and assume that $m_1 < (1+\varepsilon)d/3$. With $\varepsilon = 1/20$ the assumption becomes $m_1 < 7d/20$.

Then, $m_3 \leq m_2 \leq m_1 < (1+\varepsilon)d/3$ and

$$\Delta(\mathbf{x}) = m_1 + m_2 + m_3 - d < \varepsilon d.$$

However, $m_1 + m_2 + m_3 > d$, so $2m_1 + m_3 > d$ and then $m_3 > (1-2\varepsilon)d/3$ (so, in particular, the 3 first multiplicities must be close to $d/3$). Now

$$m'_k = m_3 - \Delta(\mathbf{x}) > (1-2\varepsilon)d/3 - \varepsilon d = (1-5\varepsilon)d/3.$$

By Noether's first equation $\sum_\ell m'_\ell = 3d' - 3 \leq 3d$, hence the sum over all multiplicities m'_ℓ from m'_1 to m'_k is strictly less that $3d$ but larger than $k \times (1-5\varepsilon)d/3$. This gives

$$k(1-5\varepsilon) < 9.$$

With $\varepsilon = 1/20$, we obtain $k \leq 11$, and for $\varepsilon \leq 1/50$ we obtain $k \leq 9$. This proves the first assertion of the lemma.

For the second assertion, we choose $\varepsilon = 1/20$ and estimate m'_1/d':

$$\frac{m'_1}{d'} = \frac{m_1 - \Delta(\mathbf{x})}{d - \Delta(\mathbf{x})} \leq \frac{21d/60}{d - d/20} = \frac{7}{19} < 5/6,$$

so $\mathbf{x}'$ does not have large first multiplicity. □

Remark 6.4. By Lemma 6.2 and Lemma 6.3, when we look at a lineage, then the first multiplicity cannot go from large to small or small to large directly without passing through an average first multiplicity.

6.4. Average first multiplicity. This can be considered somewhat the "good" case, where an exponential growth in degree is observed; the precise result we obtained is summarized by the following proposition.

Proposition 6.5. *There exists $\eta_0 > 0$ such that for any homaloidal type $\mathbf{x}$ of average first multiplicity and degree ≥ 2, we have*

$$d(\mathbf{x}) \geq (1+\eta_0)d(\mathbf{x}')$$

where $\mathbf{x}'$ is its parent.

Proof of Proposition 6.5. If we denote by x the ratio m_1/d of the homaloidal type $\mathbf{x} = (d; m_1, m_2, m_3, \ldots)$ then by Lemma 3.2

$$\frac{\Delta(\mathbf{x})}{d} = \frac{m_1 + m_2 + m_3 - d}{d} \geq x + \frac{1 - x^2 - 1/d^2}{3 - x - 3/d} + \frac{1-x}{2} - 1,$$

$$\frac{\Delta(\mathbf{x})}{d} \geq \frac{x-1}{2} + \frac{1-x^2-1/d^2}{3-x},$$

because $1/(3-x) \leq 1/2$, so

$$\frac{\Delta(\mathbf{x})}{d} \geq \frac{-3x^2+4x-1}{2(3-x)} - \frac{1}{2d^2},$$

$$\frac{\Delta(\mathbf{x})}{d} \geq \frac{(1-x)(3x-1)}{2(3-x)} - \frac{1}{2d^2}.$$

Clearly, the map $x \mapsto \frac{(1-x)(3x-1)}{2(3-x)}$ has a strictly positive lower bound on $(7/20, 5/6)$, say $\varepsilon > 0$. For d sufficiently large, $\frac{1}{2d^2} < \varepsilon/2$, so

$$\frac{\Delta(\mathbf{x})}{d} \geq \varepsilon/2.$$

Since

$$d(\mathbf{x}') = d(\mathbf{x}) - \Delta(\mathbf{x}) \leq (1-\varepsilon/2)d(\mathbf{x}).$$

This implies the result provided $d(\mathbf{x})$ is large enough. Now, we can adjust the constant η_0 to fit the inequality for the finitely many remaining cases of low degree. □

6.5. Sequences of small highest multiplicity.

Lemma 6.6. *Consider a descending lineage $\mathbf{x}_1, \dots, \mathbf{x}_N$, all of which have small first multiplicity, and let $\mathbf{x}_0$ be the parent of $\mathbf{x}_1$. Then*

$$s(\mathbf{x}_N) \leq s(\mathbf{x}_0) + 11.$$

Proof. By Lemma 6.3, for $n = 0, \dots, N-1$ the seed of $\mathbf{x}_n$ used to construct $\mathbf{x}_{n+1}$ is always made of multiplicities taken among the eleven first. Thus the multiplicities m_k with $k \geq 12$ are the same all along the lineage and the seedbed may only vary by separating values which were equal between the first 12 ones. Hence the bound. □

6.6. Suitable bound on seedbed increase. The following proposition will be useful later to deduce from an additive inequality $s(\mathbf{x}_\ell) \leq s(\mathbf{x}_1) + C$, like in Lemma 6.6, an inequality of the form (5.11).

Proposition 6.7. *Given $\alpha > \sqrt{2\ln 4}$, there exists $S_1(\alpha) > 0$ with the following property. Let $\mathbf{x}_1, \dots, \mathbf{x}_\ell$ ($\ell \geq 2$) be a descending lineage such that*

- *$s(\mathbf{x}_1) \geq S_1(\alpha)$,*
- *$\mathbf{x}_\ell$ has average first multiplicity,*
- *$s(\mathbf{x}_\ell) \leq s(\mathbf{x}_1) + 3L(\alpha)$, where $L(\alpha)$ was given in Corollary 5.6.*

Then

$$(\ln s(\mathbf{x}_\ell))^2 - (\ln s(\mathbf{x_1}))^2 \le \alpha^2 \left(\ln d(\mathbf{x}_\ell) - \ln d(\mathbf{x}_1)\right).$$

Proof. This is similar to the proof of Corollary 5.6, but much cruder. Consider

$$\begin{aligned} \delta' &= \ln d(\mathbf{x}_\ell) - \ln d(\mathbf{x}_1) && (6.1)\\ &\ge \ln d(\mathbf{x}_\ell) - \ln d(\mathbf{x}_{\ell-1}). && (6.2)\end{aligned}$$

Since $\mathbf{x}_\ell$ has average first multiplicity, Proposition 6.5 tells us that this difference is bounded from below $\ln(1+\eta_0)$, hence

$$\delta' \ge \ln(1+\eta_0).$$

Now we set $\delta = (\ln s(\mathbf{x}_\ell))^2 - (\ln s(\mathbf{x_1}))^2$ and we wish to prove that $\delta \le \alpha^2 \ln(1+\eta_0)$, since then we can conclude that $\delta \le \alpha^2\delta'$. For this, write

$$\begin{aligned} \delta &= (\ln s(\mathbf{x}_\ell))^2 - (\ln s(\mathbf{x_1}))^2\\ &= \ln\left(\frac{s(\mathbf{x}_\ell)}{s(\mathbf{x_1})}\right)\cdot \ln\left(s(\mathbf{x}_\ell)s(\mathbf{x_1})\right)\\ &= 2\ln\left(\frac{s(\mathbf{x}_\ell)}{s(\mathbf{x_1})}\right)\cdot \ln s(\mathbf{x_1}) + \left(\ln\left(\frac{s(\mathbf{x}_\ell)}{s(\mathbf{x_1})}\right)\right)^2\\ &\le 2\ln\left(1+\frac{3L(\alpha)}{s(\mathbf{x_1})}\right)\cdot \ln s(\mathbf{x_1}) + \left(\ln\left(1+\frac{3L(\alpha)}{s(\mathbf{x_1})}\right)\right)^2\\ &\le 6L(\alpha)\frac{\ln s(\mathbf{x_1})}{s(\mathbf{x_1})} + \left(\frac{3L(\alpha)}{s(\mathbf{x_1})}\right)^2,\end{aligned}$$

where we used $\ln(1+x)\le x$. Now, the function

$$f(x) = 6L(\alpha)\frac{\ln x}{x} + \left(\frac{3L(\alpha)}{x}\right)^2,$$

goes to 0 when x goes to $+\infty$, so there exists an integer $S_1(\alpha)\ge 1$ such that $f(x) < \alpha^2\ln(1+\eta_0)$ when $x \ge S_1(\alpha)$. With such a choice,

$$\delta < \alpha^2 \ln(1+\eta_0),$$

provided $s(\mathbf{x}_1) \ge S_1(\alpha)$, as required. □

7. Bound on the length

In this section, we prove the upper bounds in Theorems 2.5 and 2.4. Altogether, this will conclude the proof of Theorem B.

7.1. Upper estimate on the seedbed growth. The next proposition recalls the first assertion of Theorem 2.5.

Proposition 7.1. *For all* $\alpha > \sqrt{2\ln(4)}$*, there exists* $C_\alpha > 0$ *such that for every homaloidal type* $\mathbf{x}$*, we have* $s(\mathbf{x}) \leq C_\alpha \exp(\alpha\sqrt{\ln d(\mathbf{x})})$.

Proof. Fix $\alpha > 2\sqrt{\ln 2}$, and set

$$S_2(\alpha) = 4\max(S_0(\alpha), S_1(\alpha)) + 14,$$

where $S_0(\alpha)$ and $S_1(\alpha)$ are given by Corollary 5.6 and Proposition 6.7 respectively. We will always require $C_\alpha \geq S_2(\alpha)$.

Step 1. Preliminary reductions– Let $\mathbf{x}$ be a homaloidal type of degree ≥ 2. If $s(\mathbf{x}) < S_2(\alpha)$, we are done, so *we can assume that* $s(\mathbf{x}) \geq S_2(\alpha)$. We can also *assume that* $\mathbf{x}$ *has average first multiplicity and that the seed corresponding to* $p(\mathbf{x})$ *is not a* $*$*-seed.* Indeed, if it is not the case, we choose the seed $(0,0,0)$ of $\mathbf{x} = (d; m_1, m_2, \ldots)$ and consider the corresponding child $\mathbf{x}' = (2d; d, d, d, m_1, m_2, \ldots)$. If we manage to prove that

$$s(\mathbf{x}') \leq C'_\alpha \exp\left(\alpha\sqrt{\ln d(\mathbf{x}')}\right)$$

for some $C'_\alpha > 0$ that depends only on α, then

$$s(\mathbf{x}) = s(\mathbf{x}') - 1 \leq C'_\alpha \exp\left(\alpha\sqrt{\ln(2d(\mathbf{x}))}\right).$$

Then, $d \geq 2$ implies successively that $\ln(d) > 0.6$ and

$$\sqrt{\ln(d) + \ln(2)} \leq \sqrt{\ln(d)} + \ln(2),$$

thus we obtain

$$s(\mathbf{x}) \leq (C'_\alpha e^{\alpha \ln 2}) \exp(\alpha\sqrt{\ln d(\mathbf{x})}),$$

which proves the proposition with $C_\alpha = C'_\alpha e^{\alpha \ln 2}$.

Step 2.– According to the first step, we can now assume that $\mathbf{x}$ has average first multiplicity, the seed of $p(\mathbf{x})$ is not a $*$-seed, and $s(\mathbf{x}) \geq S_2(\alpha)$. Let's consider the whole descending lineage from $\mathbf{x}_1 = (2;1,1,1)$ up to $\mathbf{x}_N = \mathbf{x}$, and let's cut this sequence as follows.

We extract from $\mathbf{x}_1, \ldots, \mathbf{x}_N$ the subsequence $(\mathbf{y}_n)_{1 \leq n \leq p}$ made of the ancestors of $\mathbf{x}$ with average first multiplicities. Hence $p \geq 2$ since $\mathbf{y}_1 = (2;1,1,1)$ and $\mathbf{y}_p = \mathbf{x}$. We define $k \leq p$ to be the smallest index of this sequence such that, for all n between $k+1$ and q, we have

$$s(\mathbf{y}_n) \geq S_2(\alpha).$$

By definition,

$$s(\mathbf{y}_k) < S_2(\alpha).$$

Our goal in this second step is to show that there is no big jump in the sequence $s(\mathbf{y}_n)$, more precisely, for $n = 1, \ldots, p-1$,

$$s(\mathbf{y}_{n+1}) \leq 4s(\mathbf{y}_n) + 14. \tag{7.1}$$

Let's look at the lineage between $\mathbf{y}_n$ and $\mathbf{y}_{n+1}$, say $\mathbf{x}_q = \mathbf{y}_n, \ldots, \mathbf{x}_{q+\ell} = \mathbf{y}_{n+1}$. Since the sequence $\mathbf{x}_n$ cannot jump from large to small first multiplicity or vice-versa in one birth, by definition of the sequence $\mathbf{y}_n$ as the subsequence of average first multiplicities, there are only three cases to consider:

- $\ell = 1$, that is $\mathbf{y}_{n+1}$ is the children of $\mathbf{y}_n$. In this case,

  $$s(\mathbf{y}_{n+1}) \leq s(\mathbf{y}_n) + 3.$$

- $\mathbf{x}_{q+1}, \ldots, \mathbf{x}_{q+\ell-1}$ all have small first multiplicity. By Lemma 6.6, we have

  $$s(\mathbf{x}_{q+\ell-1}) \leq s(\mathbf{y}_n) + 11,$$

  so

  $$s(\mathbf{y}_{n+1}) \leq s(\mathbf{y}_n) + 14.$$

- $\mathbf{x}_{q+1}, \ldots, \mathbf{x}_{q+\ell-1}$ all have large first multiplicity. Then by Lemma 6.2, $\mathbf{y}_n = \mathbf{x}_q$, …, $\mathbf{x}_{q+\ell-1}$ must be a lineage entirely made of $*$-births, so by Proposition 5.5,

  $$s(\mathbf{x}_{q+\ell-1}) \leq 4s(\mathbf{y}_k) - 1;$$

  thus,

  $$s(\mathbf{y}_{n+1}) \leq 4s(\mathbf{y}_n) + 2,$$

  which again implies Inequality (7.1).

Step 3.– It follows from Inequality (7.1) and the definition of the index k that

$$4s(\mathbf{y}_k) + 14 \geq s(\mathbf{y}_{k+1}) \geq S_2(\alpha) = 4\max(S_0(\alpha), S_1(\alpha)) + 14,$$

so $s(\mathbf{y}_k) \geq \max(S_0(\alpha), S_1(\alpha))$. Then, this inequality remains valid for all indices $n > k$:

$$\forall n \geq k,\, s(\mathbf{y}_n) \geq \max(S_0(\alpha), S_1(\alpha)).$$

This lower bound will be required to apply Corollary 5.6 or Proposition 6.7.

We now prove that for $n = k, k+1, \ldots, p-1$, we have the inequality

$$(\ln s(\mathbf{y}_{n+1}))^2 - (\ln s(\mathbf{y}_n))^2 \leq \alpha^2 \left(\ln d(\mathbf{y}_{n+1}) - \ln d(\mathbf{y}_n)\right). \tag{7.2}$$

For this, we look at the lineage between $\mathbf{y}_n$ and $\mathbf{y}_{n+1}$, namely $\mathbf{x}_q = \mathbf{y}_n$, ..., $\mathbf{x}_{q+\ell} = \mathbf{y}_{n+1}$. So, ℓ is the number of births from $\mathbf{y}_n$ to $\mathbf{y}_{n+1}$ in Hudson's tree.

If $\mathbf{y}_n$ and $\mathbf{y}_{n+1}$ are parent and child, or if the intermediate lineage consists of homaloidal types all of small multiplicity, as we observed before,

$$s(\mathbf{y}_{n+1}) \leq s(\mathbf{y}_n) + 14 \leq s(\mathbf{y}_n) + 3L(\alpha),$$

because $L(\alpha) \geq 5$ by definition. Therefore, Proposition 6.7 applies and inequality (7.2) follows.

If the intermediate lineage consists of homaloidal types all of large first multiplicity, we distinguish between the two cases $\ell \leq L(\alpha)$ and $\ell > L(\alpha)$. In the first case, since $s(\mathbf{y}_{n+1}) \leq s(\mathbf{y}_n) + 3\ell \leq s(\mathbf{y}_n) + 3L(\alpha)$, we may apply Proposition 6.7 to get Inequality (7.2). In the second case, Corollary 5.6 applies, and again (7.2) is satisfied.

Step 4. Conclusion– Thus, summing the Inequalities (7.2) from $n = k$ to $p-1$, we obtain

$$\big(\ln s(\mathbf{x})\big)^2 - \big(\ln s(\mathbf{y}_k)\big)^2 \leq \alpha^2 \left(\ln d(\mathbf{x}) - \ln d(\mathbf{y}_k)\right).$$

So, since by definition of k, $s(\mathbf{y}_k) < S_2(\alpha)$,

$$(\ln s(\mathbf{x}))^2 \leq \alpha^2 \ln d(\mathbf{x}) + (\ln s(\mathbf{y}_k))^2.$$

This gives

$$\begin{aligned} \ln s(\mathbf{x}) &\leq \sqrt{\alpha^2 \ln d(\mathbf{x}) + (\ln S_2(\alpha))^2} \\ &\leq \alpha\sqrt{\ln d(\mathbf{x})} + \ln S_2(\alpha), \end{aligned}$$

because $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for all $a, b \geq 0$. Thus $s(\mathbf{x}) \leq S_2(\alpha)\exp(\alpha\sqrt{\ln d(\mathbf{x})})$, as desired. □

7.2. Upper estimate on N_d. We can now prove the upper bound on N_d from Theorem 2.4.

Let $d \geq 1$ be a large integer, and let X_d be the set of (proper) homaloidal types of degree d. Let $\alpha > 2\sqrt{\ln 2}$, and σ the smallest integer $> C_\alpha e^{\alpha\sqrt{\ln d}}$, where C_α is the constant in Proposition 7.1. We assume that d is large enough to satisfy $\sigma < d$.

By Proposition 7.1, if $\mathbf{x} \in X_d$ then $s(\mathbf{x}) \leq \sigma$, thus $\mathbf{x}$ can be written as a series of blocks

$$\mathbf{x} = (d; (\mu_1)^{\nu_1}, \ldots, (\mu_\sigma)^{\nu_\sigma}),$$

where blocks of zeros may be added at the end. We consider the map $\pi : X_d \to \mathbf{N}^\sigma$ defined by

$$\pi\colon\ (d;(\mu_1)^{\nu_1},\dots,(\mu_\sigma)^{\nu_\sigma}) \mapsto (p_1,\dots,p_\sigma) = (\mu_1\nu_1,\dots,\mu_\sigma\nu_\sigma).$$

By the first Noether equation, the image of π is a subset of the set of non-negative integers $(p_1,\dots,p_\sigma)$ such that $\sum_{i=1}^{\sigma} p_i = 3d-3$. Therefore, the image of π is in a set of cardinality

$$\binom{\sigma-1+3d-3}{\sigma-1} \le (4d)^\sigma.$$

We can bound the number of elements of each fiber of π: given $p_1,\dots,p_\sigma$ in $\mathbf{N}$, then if $p_i=0$ we can take $\mu_i=0$ too, and if $p_i\neq 0$, $p_i=\mu_i\nu_i$ is a decomposition of the integer $p_i\le 3d$ into the product of two integers ≥ 1; there are at most $3d$ such decompositions for each i, thus,

$$|\pi^{-1}(p_1,\dots,p_\sigma)| \le (3d)^\sigma.$$

Thus $N_d = |X_d| \le (3d)^\sigma(4d)^\sigma$, hence $\ln N_d \le \sigma\ln(12d^2)$, and then

$$\begin{aligned}\ln\ln N_d &\le \ln\sigma + \ln\ln(12d^2),\\ &\le \ln(C_\alpha+1)+\alpha\sqrt{\ln d}+\ln\ln(12d^2)\end{aligned}$$

Dividing by $\sqrt{\ln(d)}$, this gives

$$\frac{\ln\ln N_d}{\sqrt{\ln d}} \le \alpha + o(1)$$

when $d\to+\infty$, as was to be proved.

7.3. Upper estimate on $N_{\le d}$. The upper bound on $N_{\le d}$ from Theorem 2.4 is deduced as follows from the upper bound on N_d. Let $\alpha > 2\sqrt{\ln(2)}$, choose a smaller exponent α' such that $\alpha>\alpha'>2\sqrt{\ln(2)}$. Then for d large enough, say $d\ge d_0$, we have

$$N_d \le \exp(\exp(\alpha'\sqrt{\ln d})).$$

Assume d is large enough so that for the finitely many $d'\le d_0$, we have

$$N_{d'} \le \exp(\exp(\alpha'\sqrt{\ln d})).$$

Thus

$$N_{\le d} = \sum_{d'=1}^{d} N_d \le d\exp(\exp(\alpha'\sqrt{\ln d})).$$

so, since $\alpha'<\alpha$,

$$N_{\le d} = o(\exp(\exp(\alpha\sqrt{\ln d})).$$

8. QUESTIONS

8.1. Monotonicity. We don't know whether (N_d) is increasing (in the range $1 \leq d \leq 249$, it is). Theorem A shows that closely related examples provide oscillating sequences. Similar oscillations occur for finitely generated groups (see [9]).

8.2. Finitely generated subgroups of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$. Let G be a subgroup of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$. For $d \geq 1$, denote by $c_G(d)$ the number of components of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})_d$ containing at least one element of G; thus, $c_G(d) \leq N_d$ with equality when G is equal to $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$, or when $\mathrm{char}(\mathbf{k}) = 0$ and G contains $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{Q}})$ (this follows from Hudson's algorithm). On the other hand, we do not know whether a finitely generated subgroup of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ can visit all components of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$. As a more general problem, one can ask for a description of the possible growth rates of $(c_G(d))_{d\geq 1}$ as G varies among all (resp. all finitely generated) subgroups of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ or when G is some explicit subgroups, for instance the group of monomial transformations or the subgroup of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{F}_p})$ generated by the standard quadratic involution and $\mathsf{PGL}_3(\mathbf{F}_p)$.

8.3. The minimal and maximal dimensions. Let $\mathbf{x} = (d; m_1, \ldots, m_r)$ be a homaloidal type and denote by $r(\mathbf{x}) = r$ the number of non-zero multiplicities of $\mathbf{x}$ (with repetition). The dimension of the irreducible component I_x of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ corresponding to $\mathbf{x}$ is equal to

$$\dim(\mathbf{x}) = 8 + 2r(x) \tag{8.1}$$

because the general point of I_x is determined by its $r(\mathbf{x})$ base points up to post composition by an element of $\mathsf{PGL}_3(\mathbf{k})$ and $\dim(\mathsf{PGL}_3) = 8$. The component of maximal dimension is unique, it corresponds to the Jonquiéres type $(d; d-1, 1^{2d-2})$, and its dimension is $6+4d$. The minimal possible dimension is obtained with $r(\mathbf{x}) = 9$ and is equal to 26. Using Halphen surfaces and their automorphisms, it is easy to show that this minimal dimension occurs for a subset of $\mathbf{N}$ of positive density; in fact, numerical simulations suggest that it occurs for all large enough d. It would be great to have a description of the asymptotic shape of the discrete curve

$$D(m) = \mathrm{card}\{\mathbf{x}\,;\, \mathbf{x} \text{ is a homaloidal type of degree } d \text{ with } \dim(\mathbf{x}) = m\}. \tag{8.2}$$

8.4. Densities. Figure 3 suggests that, for a homaloidal type of degree d taken with probability $1/N_d$, the vector $(x, y, z) = \frac{1}{d}(m_1, m_2, m_3)$ determines a random variable in $\mathbf{R}^3$ which, for d large, equidistributes towards a probability measure μ. It would be great to find the exact asymptotic of N_d and then show that such a measure exists.

8.5. Other varieties. We don't know what is the growth rate for the number of irreducible components $N_d^{\mathbb{P}^m}$ of $\mathsf{Bir}(\mathbb{P}^m_{\mathbf{k}})_d$. We don't know if there is a variety X for which $N^X_{\leq d} \simeq \ln(\ln(d))$ or another one for which $N^X_{\leq d} \simeq \exp(d)$.

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