GAPS IN DYNAMICAL DEGREES FOR ENDOMORPHISMS AND RATIONAL MAPS

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ABSTRACT. We study the ratio of dynamical degrees $\lambda_1(f)^2/\lambda_2(f)$ for regular, dominant endomorphisms of smooth complex projective surfaces, and obtain a gap property: for $\lambda_2(f) \leq D$, there is a uniform $\varepsilon(D) > 0$ such that this ratio is never contained in $]1, 1+\varepsilon(D)[$. The proof is a simple variation on the main theorems of [6].

This version is longer than the one submitted for publication. Here, I include some of the proofs of [6]*, instead of just pointing to this paper.*

1. DYNAMICAL DEGREES

Let f be a dominant rational transformation of a smooth complex projective variety X. Let m denote the dimension of X. Let H be a hyperplane section of X. The dynamical degrees $\lambda_k(f)$ are defined for each dimension $0 \leq k \leq m$ by the following limits

$$\lambda_k(f) = \lim_{n\to+\infty} \left(((f^n)^* H^k) \cdot (H^{n-k}) \right)^{1/n} \tag{1.1}$$

where $(\cdot)$ denotes the intersection product and $H^k = H \cdot H \cdots H$ (with k factors H). Thus, $\lambda_0(f) = 1$ and $\lambda_m(f)$ is the topological degree $\deg_{top}(f)$ (a positive integer since f is dominant). The sequence $k \mapsto \lambda_k(f)$ is log-concave, i.e.

$$\lambda_{k-1}(f)\lambda_{k+1}(f) \leq \lambda_k(f)^2 \tag{1.2}$$

for all $0 < k < m$. In particular, $\lambda_m(f)^{k/m} \leq \lambda_k(f) \leq \lambda_1(f)^k$. This proves the following well known result.

Theorem A. *There is a uniform lower bound*

$$\lambda_k(f) \geq \lambda_m(f)^{k/m} \geq 2^{k/m} > 1 \qquad (\forall 1 \leq k \leq m)$$

for every variety X of dimension m and every dominant rational transformation f of X with topological degree $\lambda_m(f) > 1$.

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For instance, when f is an endomorphism of the projective space $\mathbb{P}^m$ defined by polynomial formulas of degree d, one gets $\lambda_k(f) = d^k$ and the previous inequality is indeed an equality. This paper discusses whether a further uniform gap $\lambda_1(f)^m \geq \lambda_m(f)(1+\varepsilon)$ is satisfied for maps with $\lambda_1(f)^m > \lambda_m(f)$. We focus on the first interesting case, that is when X is a surface.

2. Gaps for surfaces ?

When $\dim(X) = 2$, one gets $\lambda_1(f)^2 \geq \lambda_2(f)$. If $\lambda_2(f) = 1$, i.e. if f is a birational map of X, then either $\lambda_1(f) = 1 = \lambda_2(f)$, or $\lambda_1(f) \geq \lambda_L$, where λ_L is the Lehmer number: this is an important consequence of [5] proven in [1]. This inequality may be considered as a gap for dynamical degrees, since $\lambda_L \simeq 1.17628 > 1$.

With the Inequalities (1.2) in mind, one would like to compute the infimum $R(D)$ of $\lambda_1(f)^2/\lambda_2(f)$ over all dominant rational maps of a given surface X (resp. of any surface) with a given topological degree $\lambda_2(f) = D$ and a first dynamical degree $\lambda_1(f) > \sqrt{\lambda_2(f)}$. A less precise question is the following.

Question.– Fix an integer $D \geq 2$. Does there exist a constant $\varepsilon(D) > 0$ such that

$$\lambda_1(f)^2 \geq D(1+\varepsilon(D)) \tag{2.1}$$

for all dominant rational maps of surfaces with $\lambda_2(f) = D$ and $\lambda_1(f)^2 > \lambda_2(f)$?

If the answer is positive for some family of rational maps, we say that this family satisfies the gap property for λ_1. Theorem B below provides such a gap for *regular endomorphisms* of smooth complex projective surfaces.

3. Monomial maps

Consider a monomial map $f\colon (x,y) \mapsto (\alpha x^a y^b, \beta x^c y^d)$, viewed as a rational transformation of the projective plane. Set $\tau = a+d$ and $\delta = ad - bc$, the trace and determinant of the 2×2 matrix

$$A_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.1}$$

associated to f. Then $|\delta| = \lambda_2(f)$, and the spectral radius of A_f is $\lambda_1(f)$; changing A_f into $-A_f$ does not change the dynamical degrees, so we assume $\tau \geq 0$. The characteristic polynomial of A_f is $\chi(t) = t^2 - \tau t + \delta$. If its eigenvalues are complex conjugate, then $\lambda_1^2(f) = |\delta| = \lambda_2(f)$. So, we now assume that χ has two real roots. The largest one is $\lambda_1(f) = \frac{1}{2}\left(\tau + \sqrt{\tau^2 - 4\delta}\right)$; it satisfies

$$\lambda_1(f)^2 = \frac{1}{2}\left(\tau^2 - 2\delta + \tau\sqrt{\tau^2 - 4\delta}\right). \tag{3.2}$$

Thus, with $a = d$ and $b = c = 1$ we obtain $\tau = 2a$, $\delta = a^2 - 1$, and

$$\frac{\lambda_1(f)^2}{\lambda_2(f)} = \frac{a+1}{a-1}. \tag{3.3}$$

As $a \to +\infty$, the limit is 1. Thus, *if $D = |\delta|$ is not fixed there is no gap for λ_1*.

Now, if $D = |\delta|$ is fixed there is a gap:

Proposition 3.1. *Let D be an integer ≥ 1. If $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a dominant monomial map with $\lambda_2(f) = D$ and $\lambda_1(f)^2 > \lambda_2(f)$, the ratio $\lambda_1(f)^2/\lambda_2(f)$ is bounded from below by* $1 + (2D)^{-1}$.

Proof. As explained above, we may assume $\tau \geq 0$. Since the eigenvalues are distinct, $\tau \neq 0$ and $\tau^2 - 4\delta \geq 1$; hence, $\tau \geq 1$. Using that $\lambda_2(f) = |\delta|$ and Equation (3.2), the lower bound $\lambda_1(f)^2/\lambda_2(f) \geq 1 + (2D)^{-1}$ is equivalent to

$$\tau^2 - 2\delta + \tau\sqrt{\tau^2 - 4\delta} \geq 2|\delta| + 1. \tag{3.4}$$

If $\delta < 0$, this follows from $\tau \geq 1$. If $\delta > 0$, we denote by α and β the two eigenvalues of A_f, and we remark that (3.4) is equivalent to

$$(\alpha - \beta)^2 + \tau\sqrt{\tau^2 - 4\delta} \geq 1. \tag{3.5}$$

This is always satisfied, because τ and δ are integers, and $\tau^2 - 4\delta \geq 1$. □

Remark 3.2. Similar examples can be obtained on abelian surfaces. For instance, for any elliptic curve E, one gets linear endomorphisms of $X = E \times E$ with $\lambda_1(f)^2/\lambda_2(f) > 1$ but arbitrary close to 1.

4. Regular endomorpisms

Let us look at regular endomorphisms of projective surfaces. As explained in Section 3, we need to fix $\lambda_2(f)$ to some value D in order to get a gap.

Theorem B. *Let D be a positive integer. There is a positive real number $\varepsilon(D)$ such that*

$$\frac{\lambda_1(f)^2}{\lambda_2(f)} \geq 1 + \varepsilon(D)$$

for every smooth complex projective surface X and every dominant endomorphism f of X with $\lambda_2(f) = D$ and $\lambda_1(f)^2 > \lambda_2(f)$.

The proof occupies the rest of this section. So, f will denote a dominant endomorphism of a smooth projective surface X. Since Theorem B is known for $D = 1$, we shall always assume $2 \leq \lambda_2(f) \leq D$ for some fixed integer D.

The main arguments, described in § 4.2, are taken from the very nice paper [6] *of Noboru Nakayama (see also the companion paper* [4]*). For all necessary results on dynamical degree, we refer to* [2].

4.1. **When a bound on $\rho(X)$ is satisfied.** Since f is dominant, f_*f^* is the multiplication by $\lambda_2(f)$, so f_* and f^* are isomorphisms of the Néron-Severi group $\mathrm{NS}(X;\mathbf{Q})$. The dynamical degree $\lambda_1(f)$ is the spectral radius of f^* on $\mathrm{NS}(X;\mathbf{R})$ and is the largest eigenvalue of f^* on $\mathrm{NS}(X;\mathbf{R})$; as such, it is an algebraic integer.

Let $\rho(X)$ denote the Picard number of X, i.e. $\rho(X) = \dim_{\mathbf{Q}} \mathrm{NS}(X;\mathbf{Q})$.

Lemma 4.1. *Let D and R be positive integers > 1. There is a positive real number $\varepsilon(R,D)$ such that $\lambda_1(f)^2/\lambda_2(f) > 1+\varepsilon(R,D)$ for every smooth projective surface X and every regular endomorphism f of X such that $\rho(X) \leq R$, $\lambda_2(f) \leq D$, and $\lambda_1(f)^2 > \lambda_2(f)$.*

Proof. We can assume $\lambda_1(f)^2 > \lambda_2(f)$ and $\lambda_1(f)^2/\lambda_2(f) \leq 2$. Thus, $\lambda_1(f)$ is bounded from above by $\sqrt{2D}$; since $\lambda_1(f)$ is the spectral radius of f^*, all eigenvalues of f^* on $\mathrm{NS}(X;\mathbf{C})$ have modulus $\leq \sqrt{2D}$. So, the characteristic polynomial χ_{f^*} of $f^*\colon \mathrm{NS}(X;\mathbf{Z}) \to \mathrm{NS}(X;\mathbf{Z})$ is a polynomial with integer coefficients, of degree R, the coefficients of which are bounded from above by $C(R)(\sqrt{2D})^R$ for some constant $C(R)$. This gives only finitely many possibilities for χ_{f^*}, and the result follows. □

4.2. **Orbits of negative curves, following Nakayama.** Consider the set $\mathrm{Neg}(X)$ of irreducible curves $C \subset X$ with $C^2 < 0$ (negative curves).

Pick $C \in \mathrm{Neg}(X)$ and set $C_1 = f(C)$; let $a > 0$ be the integer such that $f_*(C) = aC_1$ (a is the degree of f along C). If C' is another irreducible curve such that $f_*(C') = a'C_1$ for some $a' > 0$ then $aC' = a'C$ in $\mathrm{NS}(X;\mathbf{Q})$ because f_* is injective. This implies that $C' = C$ because $C^2 < 0$ and C and C' are irreducible and reduced. Thus, $f^*C_1 = bC$ with $ab = \lambda_2(f)$; together with

$$f^*(C_1)\cdot C = bC\cdot C = C_1\cdot f_*(C) = C_1\cdot(aC_1) \tag{4.1}$$

this implies

$$ab = \lambda_2(f) \quad \text{and} \quad C_1^2 = (b/a)C^2 < 0. \tag{4.2}$$

In particular, f_* *permutes the irreducible curves $C \subset X$ with negative self-intersection.* This set of curves is, a priori, infinite, but we have

Lemma 4.2 (Nakayama, see Lem. 10 and Pro. 11 in [6])**.** *Let $R(f)$ be the ramification divisor of f. Let $\mathrm{Neg}(X;R(f))$ be the set of irreducible components of $R(f)$ with negative self-intersection. Let C be an element of $\mathrm{Neg}(X)$.*

(1) *There is an integer $0 \leq m \leq \log(|C^2|)$ such that $f^m(C) \in \mathrm{Neg}(X;R(f))$.*

(2) *If* $C^2 = -1$ *then* $C \in \mathrm{Neg}(X; R(f))$, $f(C)^2 \leq -\lambda_2(f)$, *and* $f^m(C) \in R(f)$ *for a positive integer* $m \leq \log(\lambda_2(f))$.
(3) *The set* $\mathrm{Neg}(X)$ *is finite.*
(4) *There is an integer* $N > 0$ *such that* $f^N(C) = C$ *for every* C *in* $\mathrm{Neg}(X)$ *and* $\mathrm{Neg}(X) = \mathrm{Neg}(X; R(f^N))$.

Proof. It suffices to prove (1). With the above notation, the condition $C \subset R(f)$ is equivalent to $b \geq 2$. On the other hand, $b = 1$ means $a = \lambda_2(f)$; and then $C_1^2 = \lambda_2(f)^{-1}C^2 > C^2$, hence $f^m(C) \subset R(f)$ for an $m \leq \log(-C^2)/\log(a)$. □

This lemma shows that one can contract a sequence of (-1)-curves in an f^N-equivariant way to reach a minimal model of X:

Theorem 4.3 (Nakayama)**.** *If f is an endomorphism of a smooth projective surface X, there is an integer $N > 0$, a birational morphism $\pi\colon X \to X_0$ onto a minimal model X_0 of X, and an endomorphism f_0 of X_0 such that $\pi \circ f^N = f_0 \circ \pi$.*

Remark 4.4. Since the dynamical degrees are invariant under birational conjugacy, we have

$$\frac{\lambda_1(f)^2}{\lambda_2(f)} = \left(\frac{\lambda_1(f_0)^2}{\lambda_2(f_0)}\right)^{1/N}; \tag{4.3}$$

without any control on N, one can not deduce a gap for f from a gap for f_0. But if N and $\rho(X_0)$ are bounded, then we automatically get a gap from Theorem 4.3 and Lemma 4.1. So, *we shall either control N and $\rho(X_0)$, and for this we follow closely* [6]*, or reduce the computation to the case of monomial maps (with the same value of D).*

Let us replace f by $g := f^N$ to assume that $\mathrm{Neg}(X) = \mathrm{Neg}(X; R(g))$ and that g fixes each irreducible curve $C \subset R(g)$. From Equations (4.1) and (4.2) we obtain

(1) $\lambda_2(g)$ is a square: there is an integer $a_g > 0$ such that $a_g^2 = \deg_{top}(g)$;
(2) $g^*(C) = g_*(C) = a_g C$;
(3) the multiplicity of C in $R(g)$ is $a_g - 1$.

Thus, if we set

$$N_X = \sum_{C \in \mathrm{Neg}(X)} C, \tag{4.4}$$

we can write $R(g) = (a_g - 1)N_X + R^+(g)$ for some effective divisor $R^+(g)$, the components of which have non-negative self-intersection (these components are numerically effective). For C in $\mathrm{Neg}(X)$, we obtain the following linear equivalence

$$K_X + C \simeq g^*(K_X + C) + R^+(g) + (a_g - 1)(N_X - C). \tag{4.5}$$

Thus, by adjunction formula, the ramification divisor $R(g_{|C})$ of $g_{|C}\colon C\to C$ satisfies $R(g_{|C})\simeq R^+(g)_{|C}+(a_g-1)(N_X-C)_{|C}$. And the Equality (4.5) gives

$$(a_g-1)(K_X\cdot C+C^2)+R^+(g)\cdot C+(a_g-1)(N_X-C)\cdot C=0. \tag{4.6}$$

Lemma 4.5 (Nakayama, Lem. 13 of [6])**.**

(1) *Let C be an element of* $\mathrm{Neg}(X)$*. The arithmetic genus of C is ≤ 1, and if it is equal to 1 then C is a connected component of the support of $R(g)$ (hence also of N_X), where $g=f^N$.*

(2) *A connected component of the support of N_X is an irreducible curve, or a chain of rational curves, or a cycle of rational curves.*

Proof. The arithmetic genus $p_a(C)$ is defined by $2p_a(C)-2=K_X\cdot C+C^2$. Thus, Equation (4.6) gives

$$2(a_g-1)(p_a(C)-1)=-(a_g-1)(N_X-C)\cdot C-R^+(g)\cdot C. \tag{4.7}$$

Since $R^+(g)$ is numerically effective and C has multiplicity 1 in N_X, we get $p_a(C)-1\leq 0$, with equality if and only if $(N_X-C)\cdot C=0=R^+(g)\cdot C$. The first assertion follows.

Now, if C and C' are two elements of $\mathrm{Neg}(X)$ with $C\cdot C'>0$, then the arithmetic genus of both C and C' is 0; this implies that C and C' are smooth rational curves. Equation (4.7) implies that $C\cdot C'\leq C\cdot(N_X-C)\leq 2$ and $R^+(g)\cdot C=0$ in case of equality. Thus, if $C\cdot C'=2$, $C\cup C'$ is a connected component of the support of both N_X and $R(g)$. If, moreover, C and C' are tangent at some point p, then $R(g_{|C})=(a_g-1)C'_{|C}$, $g_{|C}^{-1}(p)=p$ with multiplicity $a_g=\deg_{top}(g_{|C})$, and $g_{|C}$ is unramified on $C\setminus\{p\}$: this is a contradiction because a polynomial transformation of the affine line of degree $a_g>1$ has at least one ramification point. Thus, if $C\cdot C'=2$, $C\cup C'$ is a cycle of two smooth rational curves and it coïncides with a connected component of the support of both N_X and $R(g)$. If C intersects another element C'' of $\mathrm{Neg}(X)$, then the two points of intersection are distinct, by the same argument, and $C\cdot R^+(g)=0$. Thus, a connected component of the support of N_X is a chain or a cycle of smooth rational curves. If it is a cycle, it is also a connected component of the support of $R()$. □

4.3. **Rational surfaces.** Assume that X is rational. We follow the proof of Theorem 17 in [6]. If $\rho(X)\leq 3$, § 4.1 shows that the endomorphisms of X satisfy a gap for λ_1. Thus, we assume that $\rho(X)\geq 4$. Since X is the blow-up of a minimal rational surface (the plane, the quadric, or a Hirzebruch surface), there is a fibration $\pi\colon X\to B$ such that

(i) B is the projective line $\mathbb{P}^1$ and the generic fiber of π is a projective line;

(ii) there is at least one singular fiber F;
(iii) every singular fiber is a tree of smooth rational curves with negative self-intersection;
(iv) there is at least one section S of π with self-intersection $S^2 < 0$.

Since X admits an endomorphism with $\lambda_2(f) > 1$, we also know that

(iii') every singular fiber is a chain of smooth rational curves with negative self-intersections.

Indeed, such a fiber is entirely contained in N_X. Since S is also contained in N_X, we see that N_X is connected and contains at least three irreducible components. Thus, Lemma 4.5 implies that (iii') holds and that

(v) π has at most 2 singular fibers and N_X is connected and is either a chain or a cycle of rational curves.

Case of a chain.– Assume that N_X is a chain of rational curves. Either f or f^2 fixes each irreducible component of N_X (because $f(C) \cap f(C') = f(C \cap C')$). Thus, when contracting (-1)-curves, we can do it f^2-equivariantly up to a minimal model of X. Since a minimal rational surface satisfies $\rho(X_0) = 2$, the gap follows from Remark 4.4.

Case of a cycle.– Now, assume that N_X is a cycle of rational curves. There are two possibilities :

(1) N_X is the union of two singular fibers F and F' and two sections S and S';
(2) N_X is the union of the unique singular fiber F of π and two sections S and S', with $S \cap S' = \{p\}$ for some $p \notin F$.

In the first case, we can contract (-1)-curves contained in the two singular fibers to reach a minimal model $\eta\colon X \to X_0$ on which $g := f^N$ induces an endomorphism g_0 and $\pi\colon X \to B$ induces a rational fibration $\pi_0\colon X_0 \to B$ such that

- $F_0 := \eta(F)$, $F_0' := \eta(F')$ are two (smooth) fibers of π_0, $S_0 := \eta(S)$ and $S_0' := \eta(S')$ are two sections of π_0;
- f induces a rational transformation f_0 of X_0 such that $f_{|X_0 \setminus R(g_0)}$ is regular and $f_0^N = g_0$;
- $F_0 \cup F_0' \cup S_0 \cup S_0'$ is f_0-invariant and coïncides with $R(g_0)$.

Then, the complement of $R(g_0)$ in X_0 is a torus $T \simeq \mathbb{G}_m \times \mathbb{G}_m$ on which f_0 and g_0 act as regular endomorphisms. The restriction of f_0 to $T \simeq \mathbb{G}_m \times \mathbb{G}_m \simeq \mathbf{C}^\times \times \mathbf{C}^\times$ is monomial: one can find integers a, b, c, d and elements α, β in $\mathbf{C}^*$ such that $f_0(x,y) = (\alpha x^a y^b, \beta x^c y^d)$. From Section 3, we know that such transformations satisfy the gap property for λ_1.

Let us show that the second case does not occur. We shall need the following lemma (see Lem. 16 of [6]).

Lemma 4.6. *Let $U = \sum_{i=1}^{k} a_i C_i$ be an effective divisor on a smooth projective surface such that $a_i > 0$ for $1 \leq i \leq k$ and the C_i form a chain of smooth rational curves starting with C_1 and ending with C_k. If*

$$K_X \cdot U + 2 = 0 \quad \text{and} \quad U \cdot C_i = 0 \ \text{for all} \ \ i$$

then $a_1 = a_k = 1$, $C_i^2 = -1$ for some $i < k$ and $C_j^2 = -1$ for some $j > 1$.

Proof. From $U \cdot C_i = 0$ we get $a_1 C_1^2 + a_2 = 0$, $a_{k-1} + a_k C_k^2 = 0$ and $a_{i-1} + a_i C_i^2 + a_{i+1} = 0$ for $1 < i < k$. This implies that $C_i^2 < 0$ for all i and that a_1 divides a_j for every $j \geq 1$. From $K_X \cdot U + 2 = 0$ we deduce that a_1 is equal to 1 or 2. If $a_1 = 2$, then $U_0 = (1/2)U$ is an effective divisor such that $K_X \cdot U_0 = -1$ and $U_0^2 = 0$, so that the arithmetic genus of U_0 should be $1/2$, and we get a contradiction. So $a_1 = 1$ and by symmetry $a_k = 1$ as well.

If $C_i^2 \leq -2$ for each $i > 1$, then $K_X \cdot C_i \geq 0$ for each $i > 1$ (by the genus formula) and $2 + K_X \cdot U \geq 2 + K_X \cdot C_1 = -C_1^2$, which gives $C_1 \geq 2$, a contradiction. □

Let F be the singular fiber of π. Then $F = \sum_i a_i C_i$ for a chain of rational curves C_i, and moving F to a nearby smooth fiber we see that $F \cdot C_i = 0$ for each i, and $K_X \cdot F = -2$. Thus, Lemma 4.6 can be applied to $U = F$.

First, one applies this lemma to contract a (-1)-curve contained in F that does not intersect S'; then we repeat this step until we reach a model X_1 of X in which the image S_1 of S satisfies $S_1^2 = 0$. This is always possible, at least after permutation of S and S', since otherwise we would reach a minimal model X_0 with two sections of negative self-intersection, but no such surface exists.

Then, one applies Lemma 4.6 to contract (-1) curves of the singular fiber that do not intersect S_1 in order to reach a relatively minimal model X_0 of X in which S becomes a section S_0 with $S_0^2 = 0$ and S' provides a second section S'_0.

The existence of a section S_0 with self-intersection 0 implies that X_0 is $\mathbb{P}^1 \times \mathbb{P}^1$. Since S'_0 intersects S_0, $(S'_0)^2 \geq 2$. By construction, $q_0 := F_0 \cap S'_0$ is not contained in S_0. Consider the section containing q_0 which is horizontal, i.e. linearly equivalent to S_0. This section is not S_0 and, its self-intersection being 0, it is not equal to S'_0 either. Its proper transform in X is a negative curve; this proper transform should be in $\mathrm{Neg}(X)$, and we get a contradiction.

4.4. **Ruled surfaces.** If X is ruled but not rational, the Albanese map $\alpha \colon X \to B$ is a surjective morphism onto a curve B of genus ≥ 1 ([1]). There is an endomorphism

[1]Moreover, by a theorem of M. Segami, α endows X with the structure of a $\mathbb{P}^1$-bundle, i.e. X is ruled and the ruling is relatively minimal (see Pro. 14 of [6]).

f_B of B such that $\alpha \circ f = f_B \circ \alpha$; in particular, each fiber $X_b := \alpha^{-1}(b)$ is mapped to the fiber $X_{f_B(b)}$ by f. Then $\lambda_1(f_B)$ is an integer, the topological degree δ of $f_{|X_b}\colon X_b \to X_{f_B(b)}$ for a generic point $b \in B$ is also an integer, and we have

$$\lambda_1(f) = \max\{\lambda_1(f_B), \delta\} \text{ and } \lambda_2(f) = \lambda_1(f_B)\delta; \tag{4.8}$$

see [3] for the general setting of rational maps permuting the fibers of a fibration. Thus, we obtain the gap property for $\lambda_1(f)$ with $\varepsilon(D) = \frac{1}{D-1}$, i.e. $\lambda_1(f)^2 \geq \lambda_2(f)(1+\frac{1}{D-1})$ if $\lambda_2(f) \leq D$ and $\lambda_1(f)^2 > \lambda_2(f)$.

4.5. **Surfaces with non-negative Kodaira dimension.** Assume that $\mathrm{kod}(X) \geq 0$. Since every dominant rational transformation of a surface X of general type is a birational transformation of finite order, we have $\mathrm{kod}(X) \in \{0,1\}$.

When $\mathrm{kod}(X) = 1$ the Kodaira-Iitaka fibration $\Phi\colon X \dashrightarrow B$ maps X onto a smooth curve B and there is an automorphism f_B of B such that $\Phi \circ f = f_B \circ \Phi$; by a superb theorem of Noboru Nakayama and De-Qi Zhang, f_B has finite order (see [7]). Then, one easily shows that $\lambda_1(f) = \lambda_2(f)$. In particular, $\lambda_1(f)^2 = \lambda_2(f)^2$ and we have a gap property as in Theorem B with $\varepsilon(D) = D - 1$.

When $\mathrm{kod}(X) = 0$, the unique minimal model X_0 of X must be a torus, a hyperelliptic surface, an Enriques or a K3 surface. Up to multiplication by an element of $\mathbf{C}^\times$, there is a unique non-zero section Ω of K_X, $f^*\Omega = \delta\Omega$ with $\delta^2 = \lambda_2(f)$, and the exceptional locus of the birational morphism $\pi\colon X \to X_0$ is the zero locus of Ω. Thus, f preserves this locus and induces a regular endomorphism of X_0. Since K3 and Enriques surfaces do not admit endomorphisms with $\lambda_2(f) > 1$, we have $\rho(X_0) \leq 6$ (it would be ≤ 22 for K3 surfaces). Thus, the gap property follows when $\mathrm{kod}(X) = 0$.

4.6. **Conclusion.** The last three subsections establish the gap property when X is rational, when X is ruled but not rational, and when $\mathrm{kod}(X) \geq 0$. From the classification of surfaces, this covers all possible cases, and Theorem B is proven.

5. Final comments

5.1. It would be nice to determine the infimum of $\lambda_1(f)^2/\lambda_2(f)$ for dominant endomorphisms of complex projective surfaces with a fixed $\lambda_2(f) = D$, say for $D = 2, 3, 4$. The proof of Theorem B shows that this is a tractable problem.

5.2. It seems reasonable to expect that Theorem B extends to projective surfaces over fields of positive characteristic, and to singular surfaces too.

5.3. As explained in § 2, the natural question is to decide whether a similar gap property holds for rational transformations of surfaces. This question was originally asked by Curtis T. McMullen, who also suggested Theorem B in a private communication. The difficult case is the one of rational transformations of the projective plane. I don't know what to expect in this more general context (see [1] for birational maps).

5.4. One can also ask similar questions for any fixed pair $(\dim(X), \deg_{top}(f)) = (m, D)$, the first ratios to consider being $\lambda_1(f)^m/\deg_{top}(f)$ and $\lambda_1^2/\lambda_2(f)$.

REFERENCES

[1] Jérémy Blanc and Serge Cantat. Dynamical degrees of birational transformations of projective surfaces. *J. Amer. Math. Soc.*, 29(2):415–471, 2016.

[2] Nguyen-Bac Dang. Degrees of iterates of rational maps on normal projective varieties. *Proc. Lond. Math. Soc. (3)*, 121(5):1268–1310, 2020.

[3] Tien-Cuong Dinh and Viêt-Anh Nguyên. Comparison of dynamical degrees for semi-conjugate meromorphic maps. *Comment. Math. Helv.*, 86(4):817–840, 2011.

[4] Yoshio Fujimoto and Noboru Nakayama. Complex projective manifolds which admit non-isomorphic surjective endomorphisms. In *Higher dimensional algebraic varieties and vector bundles*, RIMS Kôkyûroku Bessatsu, B9, pages 51–79. Res. Inst. Math. Sci. (RIMS), Kyoto, 2008.

[5] Curtis T. McMullen. Coxeter groups, Salem numbers and the Hilbert metric. *Publ. Math. Inst. Hautes Études Sci.*, (95):151–183, 2002.

[6] Noboru Nakayama. Ruled surfaces with non-trivial surjective endomorphisms. *Kyushu J. Math.*, 56(2):433–446, 2002.

[7] Noboru Nakayama and De-Qi Zhang. Building blocks of étale endomorphisms of complex projective manifolds. *Proc. Lond. Math. Soc. (3)*, 99(3):725–756, 2009.

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