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Linearity of the exceptional set for maps of $\mathsf{P}_k(\mathbf{C})$

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Abstract. *Let f be a holomorphic self-map of $\mathsf{P}_k(\mathbf{C})$, with degree larger than 2. We show that its exceptional set is a finite union of linear subspaces.*

1 introduction

This note is about the exceptional set of holomorphic self-maps of the complex projective space $\mathsf{P}_k(\mathbb{C})$. In [BD], it is proved that the following definition makes sense :

Definition 1.1. — *Let f be a holomorphic self-map of $\mathsf{P}_k(\mathbb{C})$, of degree $d \geq 2$. The exceptional set* E *of f is the largest completely invariant proper algebraic subset of $\mathsf{P}_k(\mathbb{C})$.*

The exceptional set is of importance in the dynamical study of f as points outside E have iterated preimages asymptotically distributed on the unique maximal entropy measure of f. It is therefore interesting to know more about this set, and especially what kind of irreducible components it can have. Firstly, it follows from the definition that E is the same for f or any of its iterates, that E is a subset of the critical set of f, and that moreover, taking a sufficiently high iterate of f, each irreducible component of E is itself completely invariant. The subject of completely invariant subvarieties of $\mathsf{P}_k(\mathbb{C})$ has been addressed several times before the publication of [BD]. Let us recall those various works briefly before stating our theorem[1].

The first property of E is that it is empty for a generic f. In [FS1], John-Erik Fonæss and Nessim Sibony show that a completely invariant hypersurface can have

[1] For a general introduction to the subject of iteration in several complex variables, we refer the reader to the survey of Nessim Sibony [S] and to the references therein.

at most $k+1$ irreducible components. Then they give some results for the case of $\mathsf{P}_2(\mathbb{C})$. Dominique Cerveau and Alcides Lins Neto in [CLN] generalize further those results, showing for instance that a smooth quadric in $\mathsf{P}_k(\mathbb{C})$ can never be exceptional, and that an exceptional hypersurface of degree larger than 3 can never be smooth. Finally, in [SSU], Bernard Shiffman, Mitsuhiro Shishikura and Tetsuo Ueda show that an exceptional curve in $\mathsf{P}_2(\mathbb{C})$ is the union of at most three lines. For this they show that in general for a completely invariant hypersurface H in $\mathsf{P}_k(\mathbb{C})$, if a line is sufficiently tangent to H, then it is actually contained in H. In some sense, the proof of the theorem below gives the generalization of this last result.

We will show in this note that one can in fact prove a better characterization of the exceptional set, building on the proofs in [BD].

Theorem 1.1. — *Let f be a holomorphic self-map of $\mathsf{P}_k(\mathbb{C})$. Any totally invariant irreducible algebraic subvariety of $\mathsf{P}_k(\mathbb{C})$ is a linear space. In particular, the exceptional set of f is a finite union of linear subspaces.*

We end this note by some considerations about hyperplanes, proving in particular that when f admits $k+1$ completely invariant hyperplanes, they have to be in general position. The case of lower dimensional components of E is quite mysterious and should be investigated.

2 Proof of the theorem

In the following, all the objects that we will consider, except those strictly related to f, like E, will be sufficiently generic : we allow ourselves to move them a little bit if necessary. We keep the notations of [BD]. A generic point $x \in \mathsf{P}_k(\mathbb{C})$ has d^k preimages, therefore the *topological degree* of f is d^k. The *local degree* of f at the point $x \in \mathsf{P}_k(\mathbb{C})$ is defined as the number of distinct preimages of a generic point y close to $f(x)$ in a neighborhood of x. In other terms,

$$\deg_x(f) = \max\left\{\mathrm{Card}\left(f^{-1}(y) \cap \mathrm{B}(x,\varepsilon)\right),\ y \in \mathrm{B}(f(x),\varepsilon)\right\},$$

for all sufficiently small ε. This gives a natural stratification $\mathrm{A}_s = \{x,\ \deg_x(f) \geq s\}$ in algebraic sets, with $\mathrm{A}_1 = \mathsf{P}_k(\mathbb{C})$, $\mathrm{A}_2 = \mathrm{C}$ the critical set of f, and $\mathrm{A}_s = \emptyset$ for $s > d^k$. If A is an irreducible algebraic variety, one can define

$$\deg_{\mathrm{A}}(f) = \min\{\deg_x(f),\ x \in \mathrm{A}\} = \deg_x(f) \text{ for a generic } x \in \mathrm{A}.$$

It is proved in [BD] that on a completely invariant variety, this degree is controlled by the codimension. Let A be an irreducible component of E of codimension p. We can assume, taking an iterate of f, that A is completely invariant. Then $\deg_{\mathrm{A}}(f) = d^p$, and we now adapt the proof of this fact in [BD] to prove our theorem.

Let x be a generic point of A, in the sense that A is smooth at x, x is not a critical point of $f_{|\mathrm{A}}$, $\deg_x(f) = d^p$ and $y = f(x)$ is not a critical value of $f_{|\mathrm{A}}$. We choose an $\varepsilon > 0$ small enough so that an ε neighborhood of $f^{-1}(y)$ consists of d^{k-p} disjoint balls. Let now Q be a dimension p plane cutting transversaly A near x.

Then in a neighborhood $U \subset B(x, \varepsilon)$ of x, one can define a holomorphic retraction $\pi : U \longrightarrow U \cap A$ with fibers close to $Q \cap U$. Let P be a generic codimension p plane close to the tangent plane of A at $f(y)$. Then π, restricted to $f^{-1}(P) \cap U$, defines a degree d^p ramified covering of $f^{-1}(P) \cap U$ to $A \cap U$. Thus $f^{-1}(P)$ cuts Q in d^p points contained in U. Suppose now that Q cuts A transversally near another preimage x' of y (see the figure below). Then, by the same argument, Q will cut $f^{-1}(P)$ near this other point, and this would imply that

$$\mathrm{Card}(f^{-1}(P) \cap Q) \geq d^p + 1,$$

which is impossible due to Bezout's theorem. Thus a generic dimension p plane Q can cut A only near one of the preimages of y. By linear algebra this yields that the preimages of a generic point y of A are included in a codimension p plane. One can furthermore take any iterate of f (without changing A), and thus get as many points as we want in $f^{-1}(y)$, that are all contained in the intersection of a $(k-p)$-dimensional plane and A.

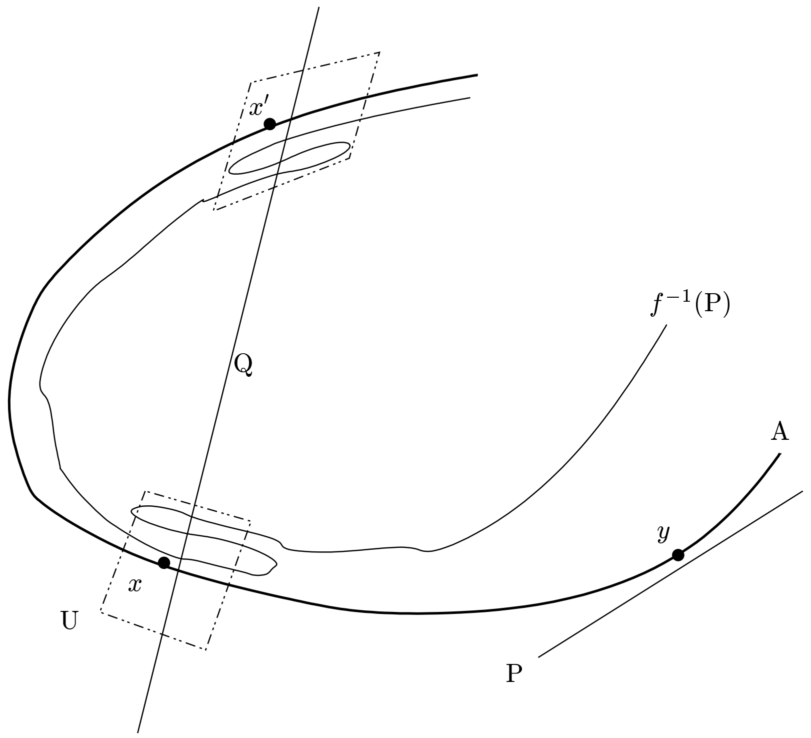

Figure 1: Idea of the proof

Let l be a positive integer such that $\deg(A) < d^l$ and y be a generic point of A. Let N_y be the smallest linear subspace containing the $d^{l(k-p)}$ points of $f^{-l}(y)$. We just proved that the dimension of N_y is at most $k-p$. If the dimension of $A \cap N_y$ is $k-p$ then A coincides with N_y and A is therefore a linear subspace. If not, $A \cap N_y$ is a subvariety of degre $\deg(A)$ and codimension $r > p$. Let us take a generic codimension r plane R through y. One then has

$$\mathrm{Card}(f^{-l}(R) \cap (N_y \cap A)) \geq d^{l(k-p)},$$

since the preimages of y are contained in N_y. But Bezout's theorem implies that

$$\deg(f^{-1}(\mathrm{R}) \cap (\mathrm{N}_y \cap \mathrm{A})) \leq \deg(\mathrm{A}) d^{l(k-r)}.$$

These two inequalities contradict the choice of l. This ends the proof of the theorem.

3 Further remarks

Now everybody wonders what kind of situations we can get with all these linear exceptional sets. Let us examine the case of the maximum possible number of hyperplanes. In [FS], it is showed that there can be no more than $k+1$ different hyperplanes in E. In that case, as the jacobian of f along those planes vanishes at order $d-1$, and as the critical set is of degree $(k+1)(d-1)$, we have that E is equal to the whole critical set. If those hyperplanes are in general position, then, exactly as in [FS], f is conjugate to $[z_0^d : \cdots : z_k^d]$. In the following, we will work up to linear conjugacy for the map f.

Suppose now that E contains $k+1$ hyperplanes having in common a point and nothing else. Then, sending one of them to the plane at infinity, f induces a self-map of $\mathbb{C}^k = \mathbb{C} \times \mathbb{C}^{k-1}$ preserving k hyperplanes cutting $\mathbb{C}^{k-1}$ in general position. Write the k planes as $x_2 = 0, \ldots, x_k = 0, x_2 + \cdots + x_k = 0$ and f as $(f_1, f_2, \ldots, f_k)$. The complete invariance of the first $k-1$ planes can be rewritten as $f_i = 0$ iff $x_i = 0$, giving that $f_2 = x_2^d, \ldots, f_k = x_k^d$. The complete invariance of the last then states that $x_2^d + \cdots x_k^d = 0$ iff $x_2 + \cdots + x_k = 0$, which is impossible because $d \geq 1$.

In general, if E contains $k+1$ hyperplanes having in common a l-plane, $1 \leq l \leq k-2$ and nothing else, we can apply the same argument. By sending one of the hyperplanes to infinity and writing the remainder as $\mathbb{C}^k = \mathbb{C}^{l+1} \times \mathbb{C}^{k-l-1}$, we get a map preserving k hyperplanes, any $k-l$ among them cutting $\mathbb{C}^{k-l-1}$ in general position. Then we can write them as $x_{l+2} = 0, \ldots, x_k = 0, x_{l+2} + \cdots + x_k = 0$, and the complete invariance of the first $k-l-1$ implies that $f_{l+2} = x_{l+2}^d, \ldots, f_k = x_k^d$. Entering that data for the complete invariance of the last hyperplane gives $x_{l+2}^d + \cdots x_k^d = 0$ iff $x_{l+1} + \cdots + x_k =$, which is impossible because $d \geq 2$.

There remains to consider the case of $k+1$ hyperplanes arranged in "open book" fashion, that is having a $(k-2)$-plane as common intersection. Send one of them to infinity, and we get a map from $\mathbb{C}^k$ with k parallel hyperplanes completely invariant. In fact, f induces a covering map of degree d^k from $\mathbb{C} - \{p_1, \ldots, p_k\} \times \mathbb{C}^{k-1}$ to itself, where the p_i's are distinct points in $\mathbb{C}$. But then f must preserve the foliation in horizontal planes, and thus induces a rational map with $k+1$ exceptional points : one at each p_i, plus one at infinity. But if $k \geq 2$ this is impossible. We have thus proved the

Proposition 3.1. — *If the exceptional set* E *of* f *contains* $k+1$ *hyperplanes, then they are in general position and* f *is conjugate to* $[z_0^d : \cdots : z_k^d]$ *by a linear change of coordinates. More generally the exceptional set cannot contain more than* 3 *hyperplanes having a* $k-2$*-plane in common.*

In a sense, this last proposition generalizes the fact that if a rational map has 2 exceptional points, then it is of the form $[z^d : w^d]$.

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