DYNAMICS OF AUTOMORPHISMS OF COMPACT COMPLEX SURFACES

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ABSTRACT. Recent results concerning the dynamics of holomorphic diffeomorphisms of compact complex surfaces are described, that require a nice interplay between algebraic geometry, complex analysis, and dynamical systems.

RÉSUMÉ. Nous décrivons quelques résultats récents concernant la dynamique des difféomorphismes holomorphes des surfaces complexes compactes. Ceci nécessite des outils de géométrie algébrique, d'analyse complexe et de systèmes dynamiques.

1. INTRODUCTION

1.1. **Automorphisms.**

1.1.1. *Automorphisms.* Let M be a compact complex manifold. By definition, holomorphic diffeomorphisms $f\colon M \to M$ are called **automorphisms**; they form a group, the group $\mathsf{Aut}(M)$ of automorphisms of M. Endowed with the topology of uniform convergence, $\mathsf{Aut}(M)$ is a topological group and a theorem due to Bochner and Montgomery shows that this topological group is a complex Lie group, whose Lie algebra is the algebra of holomorphic vector fields on M (see [21]). The connected component of the identity in $\mathsf{Aut}(M)$ is denoted by $\mathsf{Aut}(M)^0$, and the group of connected components of $\mathsf{Aut}(M)$ is

$$\mathsf{Aut}(M)^\sharp = \mathsf{Aut}(M)/\mathsf{Aut}(M)^0.$$

1.1.2. *Curves.* If $M = \mathbb{P}^1(\mathbf{C})$ then $\mathsf{Aut}(M)$ is the group of linear projective transformations $\mathsf{PGL}_2(\mathbf{C})$. In particular, $\mathsf{Aut}(M)$ is connected, and the dynamics of all elements $f \in \mathsf{Aut}(M)$ is easily described. (However, the theory of Kleinian groups shows the richness of the dynamics of subgroups of $\mathsf{Aut}(\mathbb{P}^1(\mathbf{C}))$).

If $M = \mathbf{C}/\Lambda$ is an elliptic curve, the connected component of the identity $\mathsf{Aut}(M)^0$ coincides with the abelian group $\mathbf{C}/\Lambda$, acting by translations. The

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group $\mathsf{Aut}(M)$ is the semi-direct product of $\mathsf{Aut}(M)^0$ by the finite group F of similarities of $\mathbf{C}$ preserving Λ. The group F contains $z \mapsto -z$, and is generated by this involution in all cases except when Λ is similar to $\mathbf{Z}[\sqrt{-1}]$, and then $|F| = 4$, or when Λ is similar to $\mathbf{Z}[\omega]$, where ω is a cubic root of 1, and then $|F| = 6$.

If M is a connected curve of genus $g \geq 2$, Hurwitz's Theorem shows that $\mathsf{Aut}(M)$ is finite, with at most $84(g-1)$ elements.

1.1.3. *Connected components.* Starting with $\dim(M) = 2$, the group $\mathsf{Aut}(M)$ may have an infinite number of connected components.

As an example, let $E = \mathbf{C}/\Lambda$ be an elliptic curve and $M = E^n$ be the product of n copies of E; in other words, M is the torus $\mathbf{C}^n/\Lambda^n$. The group $\mathsf{GL}_n(\mathbf{Z})$ acts linearly on $\mathbf{C}^n$, preserves the lattice $\Lambda^n \subset \mathbf{C}^n$, and therefore embeds into the group $\mathsf{Aut}(M)$. All non-trivial elements B in $\mathsf{GL}_n(\mathbf{Z})$ act non-trivially on the homology of M, so that distinct matrices fall in distinct connected components of $\mathsf{Aut}(M)$. Thus, $\mathsf{Aut}(M)^\sharp$ is infinite if $n \geq 2$.

As a specific example, one can take

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The automorphism induced by B on $E \times E$ is an Anosov diffeomorphism : It expands a holomorphic linear foliation of $E \times E$ by a factor $(3+\sqrt{5})/2 > 1$ and contracts another transverse linear foliation by $(3-\sqrt{5})/2 < 1$.

This example shows that there are compact complex surfaces X for which $\mathsf{Aut}(X)$ has an infinite number of connected components and $\mathsf{Aut}(X)$ contains elements $f : X \to X$ that exhibit a rich dynamics; as we shall explain, these two properties are intimately linked together. The following paragraph provides another example that will be used all along this survey.

1.2. **An example.** Consider the affine space of dimension 3, with coordinates (x_1, x_2, x_3), and compactify it as $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Denote by π_i the projection onto $\mathbb{P}^1 \times \mathbb{P}^1$ that forgets the i-th factor; for example, in affine coordinates,

$$\pi_2(x_1, x_2, x_3) = (x_1, x_3).$$

Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth surface such that all three projections π_i induce ramified covers of degree 2, still denoted π_i, from X to $\mathbb{P}^1 \times \mathbb{P}^1$. Equivalently, X is defined in the affine space by a polynomial equation

$$P(x_1, x_2, x_3) = 0$$

that has degree 2 with respect to each variable. For instance, one can take

$$P(x_1,x_2,x_3) = (1+x_1^2)(1+x_2^2)(1+x_3^2) + Ax_1x_2x_3 - 2,$$

for all parameters $A \neq 0$, as in [101]. Since $\pi_i : X \to \mathbb{P}^1 \times \mathbb{P}^1$ is a 2 to 1 cover, there is an involutive automorphism s_i of X such that $\pi_i \circ s_i = \pi_i$. For example, if (x_1,x_2,x_3) is a point of X, then

$$s_2(x_1,x_2,x_3) = (x_1,x_2',x_3)$$

where x_2 and x_2' are the roots of the equation $P(x_1,t,x_3) = 0$ (with x_1 and x_2 fixed).

As we shall see in the following pages, there are no non-trivial relations between these involutions. In other words, the subgroup of $\mathsf{Aut}(X)$ generated by s_1, s_2, and s_3 is isomorphic to the free product $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$. Moreover, if f is a non trivial element of this group, then

- either f is conjugate to one of the s_i, and then f is an involution;
- or f is conjugate to an iterate $(s_i \circ s_j)^n$ of one of the compositions $s_i \circ s_j$, $i \neq j$, and the dynamics of f is easily described since the closure of typical orbits are elliptic curves;
- or f has a rich dynamics, with positive topological entropy, an infinite number of saddle periodic points, etc.

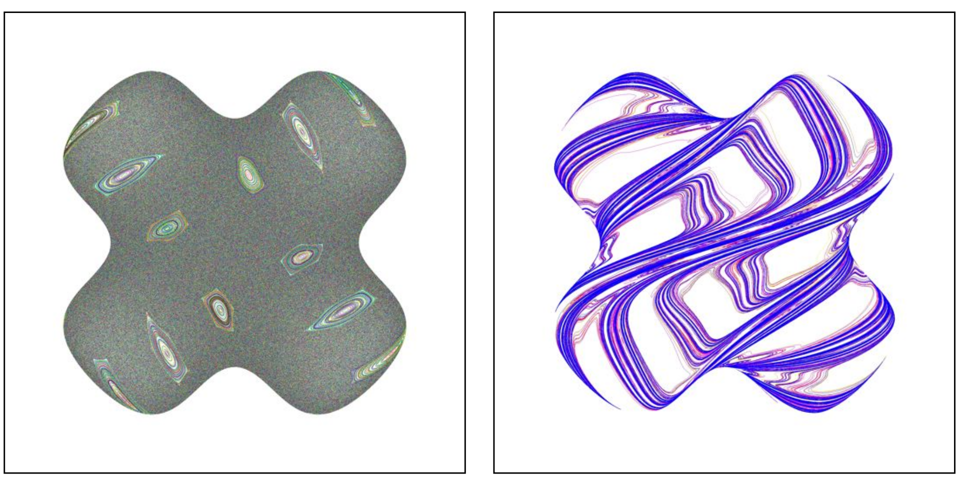

FIGURE 1. Here, X is defined by a polynomial with real coefficients. The automorphism $f = s_1 \circ s_2 \circ s_3$ preserves the real part $X(\mathbf{R})$. On the left, several orbits are plotted, while on the right, an approximation of a stable manifold of a saddle fixed point is drawn. (Picture realized by V. Pit, based on a program by C.T. McMullen [100])

1.3. **Aims and scope.** This text describes the dynamics of automorphisms of compact complex surfaces when it is rich, as in the example $f = s_1 \circ s_2 \circ s_3$ above. We restrict the study to compact Kähler surfaces. This is justified by the fact that the topological entropy of all automorphisms vanishes on compact complex surfaces which are not Kähler, as explained in Section 2.5 and the Appendix.

Not much is known, but a nice interplay between algebraic geometry, complex analysis, and dynamical systems provides a few interesting results. This leads to a precise description of the main stochastic properties of the dynamics of automorphisms, whereas topological properties seem more difficult to obtain.

Our goal is to present the main results of the subject to specialists of algebraic geometry and to specialists of dynamical systems as well; this implies that several definitions and elementary explanations need to be given that are common knowledge for a large proportion of potential readers. No proof is detailed but a few arguments are sketched in order to enlighten the interplays between algebraic geometry, complex analysis, and dynamical systems. When a result holds for automorphisms of projective surfaces over any algebraically closed field $\mathbf{k}$, we mention it.

We tried as much as possible to focus on topics which are not covered by other recent surveys on holomorphic dynamics in several complex variables; we recommend [115], [38], [78], [5], and [58], for complementary material.

1.4. **Acknowledgement.** Thanks to Eric Bedford, Olivier Debarre, Julie Déserti, Hélène Esnault, Charles Favre, Curtis T. McMullen, and Olivier Wittenberg for interesting comments and references.

Contents

2. Hodge theory and automorphisms

Let X be a connected, compact, Kähler surface. Our goal in this section is to describe the action of automorphisms $f \in \mathsf{Aut}(X)$ on the cohomology groups of X. The Hodge structure plays an important role; we refer to the four books [75], [122], [2] and [97] as general references for this topic, and to [38] for details concerning the action of $\mathsf{Aut}(X)$ on the cohomology of X.

2.1. **Hodge decomposition, intersection form, and Kähler cone.**

2.1.1. *Cohomology groups.* Hodge theory implies that the de Rham cohomology groups $H^k(X,\mathbf{C})$ split into direct sums

$$H^k(X,\mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(X,\mathbf{C}),$$

where classes in the Dolbeault cohomology groups $H^{p,q}(X,\mathbf{C})$ are represented by closed forms of type (p,q). For example, $H^{1,0}(X,\mathbf{C})$ and $H^{2,0}(X,\mathbf{C})$ correspond respectively to holomorphic 1-forms and holomorphic 2-forms. This bigraded structure is compatible with the cup product. Complex conjugation permutes $H^{p,q}(X,\mathbf{C})$ with $H^{q,p}(X,\mathbf{C})$; it defines a real structure on the complex vector space $H^{1,1}(X,\mathbf{C})$, for which the real part is

$$H^{1,1}(X,\mathbf{R}) = H^{1,1}(X,\mathbf{C}) \cap H^2(X,\mathbf{R}),$$

and on the space $H^{2,0}(X,\mathbf{C}) \oplus H^{0,2}(X,\mathbf{C})$. We denote by $h^{p,q}(X)$ the dimension of $H^{p,q}(X,\mathbf{C})$.

2.1.2. *Intersection form.* Since X is canonically oriented by its complex structure, it admits a natural fundamental class $[X] \in H_4(X,\mathbf{Z})$; this provides an identification of $H^4(X,\mathbf{Z})$ with $\mathbf{Z}$. Hence, the **intersection form** defines an integral bilinear form on $H^2(X,\mathbf{Z})$. We denote by $\langle \cdot | \cdot \rangle$ the bilinear form which is induced on $H^{1,1}(X,\mathbf{R})$ by the intersection form:

$$\forall u,v \in H^{1,1}(X,\mathbf{R}), \quad \langle u|v\rangle = \int_X u\wedge v. \,(^1)$$

[1] here u and v are implicitly represented by $(1,1)$-forms and the evaluation of the cup product of u and v on the fundamental class $[X]$ is identified to the integral of $u\wedge v$ on X

Theorem 2.1 (Hodge index Theorem). *Let X be a connected compact Kähler surface. On the space $H^{1,1}(X,\mathbf{R})$, the intersection form $\langle\cdot|\cdot\rangle$ is non-degenerate and of signature $(1,h^{1,1}(X)-1)$.*

In particular, $\langle\cdot|\cdot\rangle$ endows $H^{1,1}(X,\mathbf{R})$ with the structure of a Minkowski space that will play an important role in the following Sections.

Remark 2.2. If Ω is a non-zero holomorphic 2-form then

$$\int_X \Omega\wedge\overline{\Omega} > 0, \tag{2.1}$$

where $\overline{\Omega}$ is the complex conjugate. As a consequence, the intersection form is positive definite on the real part of $H^{2,0}(X,\mathbf{C})\oplus H^{0,2}(X,\mathbf{C})$, and the signature of the intersection form on $H^2(X,\mathbf{R})$ is $(2h^{2,0}(X)+1,h^{1,1}(X)-1)$.

Example 2.3. Let X be a smooth surface of degree $(2,2,2)$ in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, as in Section 1.2. In the affine space $\mathbf{C}\times\mathbf{C}\times\mathbf{C}$, with coordinates (x_1,x_2,x_3), X is defined by a polynomial equation $P(x_1,x_2,x_3)=0$. Since X is smooth, every point of X is contained in an open set where one of the partial derivatives of P does not vanish; hence, we can define a holomorphic 2-form Ω on the affine part of X by

$$\Omega = \frac{dx_1\wedge dx_2}{\partial P/\partial x_3} = \frac{dx_2\wedge dx_3}{\partial P/\partial x_1} = \frac{dx_3\wedge dx_1}{\partial P/\partial x_2}.$$

As the reader can check, this form extends to X as a non-vanishing holomorphic 2-form Ω_X, because P has degree 2 with respect to each variable (an instance of the "adjunction formula", see [75]). Now, if Ω' is another holomorphic 2-form, then $\Omega'=\psi\Omega_X$ for some holomorphic, and therefore constant, function $\psi\colon X\to\mathbf{C}$. Thus, $H^{2,0}(X,\mathbf{C})$ is generated by $[\Omega_X]$ and $h^{2,0}(X)=1$; by conjugation, $h^{0,2}(X)=1$.

The generic fibers of the projection

$$\sigma_1\colon (x_1,x_2,x_3)\in X\to x_1\in\mathbb{P}^1$$

are elliptic, with a finite number of singular fibers – for a generic choice of P, one sees that π has exactly 24 singular fibers which are isomorphic to a rational curve with a double point. This implies that X is simply connected with Euler characteristic 24. Thus, $H^1(X,\mathbf{Z})$, $H^{1,0}(X,\mathbf{C})$, and $H^{0,1}(X,\mathbf{C})$ vanish, and $H^{1,1}(X,\mathbf{C})$ has dimension 20. Consequently, the signature of the intersection form on $H^2(X,\mathbf{R})$ is $(3,19)$.

2.1.3. *Kähler and nef cones.* Classes $[\kappa] \in H^{1,1}(X;\mathbf{R})$ of Kähler forms are called **Kähler classes**. The **Kähler cone** of X is the subset $\mathcal{K}(X) \subset H^{1,1}(X,\mathbf{R})$ of all Kähler classes. This cone is convex and is contained in one of the two connected components of the cone $\{u \in H^{1,1}(X,\mathbf{R}) \,|\, \langle u|u\rangle > 0\}$. Its closure $\overline{\mathcal{K}}(X) \subset H^{1,1}(X,\mathbf{R})$ is the **nef cone** (where "nef" simultaneously stands for "numerically eventually free" and "numerically effective").

2.1.4. *The Néron-Severi group.* The **Néron-Severi group** of X is the discrete subgroup of $H^{1,1}(X,\mathbf{R})$ defined by

$$NS(X) = H^{1,1}(X,\mathbf{R}) \cap H^2(X,\mathbf{Z}).$$

Lefschetz Theorem on $(1,1)$-classes asserts that this space coincides with the group of Chern classes of holomorphic line bundles on X. The dimension $\rho(X)$ of $NS(X)$ is the **Picard number** of X; by definition $\rho(X) \leq h^{1,1}(X)$. Similarly, we denote by $NS(X,\mathbf{A})$ the space $NS(X) \otimes_{\mathbf{Z}} \mathbf{A}$ for $\mathbf{A} = \mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$.

When Y is a projective surface defined over an algebraically closed field $\mathbf{k}$, the Néron-Severi group $NS(Y)$ is defined as the group of classes of curves modulo numerical equivalence; this definition coincides with the definition just given when $\mathbf{k} = \mathbf{C}$ and $Y = X$ is a complex projective surface.

Remark 2.4. Let X be a projective surface, embedded as a degree d surface in some projective space $\mathbb{P}^n(\mathbf{C})$. Recall that the line bundle $O(1)$ on $\mathbb{P}^n(\mathbf{C})$ is the inverse of the tautological line bundle: Holomorphic sections of $O(1)$ are given by linear functions in homogeneous coordinates and their zero-sets are hyperplanes of $\mathbb{P}^n(\mathbf{C})$; the Chern class of $O(1)$ is a Kähler class (represented by the Fubini-Study form). Restricting $O(1)$ to X, we obtain a line bundle, the Chern class of which is an integral Kähler class with self-intersection d. Equivalently, intersecting X with two generic hyperplanes, one gets exactly d points. This shows that $\langle\cdot|\cdot\rangle$ restricts to a non-degenerate quadratic form of signature $(1, \rho(X)-1)$ on $NS(X,\mathbf{R})$ when X is projective.

In the other direction, the description of Kähler cones for surfaces (see [24, 95]) and Kodaira's embedding Theorem imply that X is a projective surface as soon as $NS(X)$ contains classes with positive self-intersection.

Example 2.5. Let X be a smooth surface of degree $(2,2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Each projection of the three projections $\sigma_i \colon X \to \mathbb{P}^1$ determines a fibration of X by curves of genus 1 (with singular fibers). Denote by $[C_i]$ the class of the generic fiber of σ_i; these classes generate a free abelian subgroup $\mathbf{Z}^3$ of $NS(X)$.

Noether-Lefschetz Theorem (see [123], or [122], Theorem 15.33) implies that $NS(X)$ coincides with this subgroup for generic surfaces of degree $(2,2,2)$.

When X is a torus $\mathbf{C}^2/\Lambda$, the second cohomology group has dimension 6. Since $H^{2,0}(X,\mathbf{C})$ has dimension 1 (it is generated by the class of $dx \wedge dy$), the Picard number $\rho(X)$ is at most 4. For example, generic tori have Picard number 0 and $\rho(\mathbf{C}/\mathbf{Z}[\sqrt{-1}] \times \mathbf{C}/\mathbf{Z}[\sqrt{-1}]) = 4$. (see [17])

2.2. **Automorphisms.**

2.2.1. *Action on cohomology groups.* The group $\mathsf{Aut}(X)$ acts by pull-back on $H^*(X,\mathbf{Z})$, where $H^*(X,\mathbf{Z})$ stands for the graded direct sum of the cohomology groups $H^k(X,\mathbf{Z})$. This action provides a morphism

$$f \in \mathsf{Aut}(X) \mapsto (f^*)^{-1} \in \mathsf{GL}\,(H^*(X,\mathbf{Z})), \tag{2.2}$$

the image of which preserves

(1) the graded structure, *i.e.* each subspace $H^k(X,\mathbf{Z})$, acting trivially on $H^0(X,\mathbf{Z})$ and $H^4(X,\mathbf{Z})$;
(2) the Poincaré duality;
(3) the Hodge decomposition, commuting with complex conjugation;
(4) the Kähler cone $\mathcal{K}(X)$.

Moreover,

(5) the cup product is equivariant with respect to the action of $\mathsf{Aut}(X)$; in particular, $\mathsf{Aut}(X)$ preserves the intersection form $\langle \cdot | \cdot \rangle$ on $H^{1,1}(X,\mathbf{R})$.

The connected component of the identity $\mathsf{Aut}(X)^0 \subset \mathsf{Aut}(X)$ acts trivially on the cohomology of X; the following Theorem shows that this group has finite index in the kernel of the morphism (2.2).

Theorem 2.6 (Lieberman [98], see also Fujiki [71])**.** *Let M be a compact Kähler manifold. If $[\kappa]$ is a Kähler class on M, the connected component of the identity $\mathsf{Aut}(X)^0$ has finite index in the group of automorphisms of M fixing $[\kappa]$.*

In other words, the group of connected components $\mathsf{Aut}(M)^\sharp$ almost embeds into $\mathsf{GL}\,(H^*(M,\mathbf{R}))$. When M is a curve, the group $\mathsf{Aut}(X)^0$ coincides with the kernel of the representation $\mathsf{Aut}(M) \to \mathsf{GL}\,(H^*(M,\mathbf{R}))$, but already for surfaces, there are (few) examples of automorphisms which are not isotopic to the identity but act trivially on $H^*(M,\mathbf{Q})$ (see [113], [26]).

2.2.2. *Eigenvalues and dynamical degree.* On the space $H^{2,0}(X,\mathbf{C})$ (resp. on $H^{0,2}(X,\mathbf{C})$), the group $\mathsf{Aut}(X)$ preserves the positive hermitian product

$$([\Omega],[\Omega']) \mapsto \int_X [\Omega] \wedge [\overline{\Omega'}]$$

from Equation (2.1). This shows that the image of $\mathsf{Aut}(X)$ in $\mathsf{GL}\,(H^{2,0}(X,\mathbf{C}))$ (resp. in $\mathsf{GL}\,(H^{0,2}(X,\mathbf{C}))$) is contained in a unitary group.

Since $\mathsf{Aut}(X)$ preserves the Hodge decomposition and the integral structure of the cohomology, it preserves the Néron-Severi group. When X is projective, $NS(X)$ intersects the Kähler cone (see Remark 2.4) and by Hodge index Theorem, the intersection form is negative definite on its orthogonal complement $NS(X)^{\perp} \subset H^{1,1}(X,\mathbf{R})$.

These facts imply the following Lemma.

Lemma 2.7. *Let X be a compact Kähler surface, and f be an automorphism of X. Let $u \in H^2(X,\mathbf{C})$ be a non-zero eigenvector of f^*, with eigenvalue λ. If $|\lambda| > 1$, then u is contained in $H^{1,1}(X,\mathbf{C})$, and is contained in $NS(X,\mathbf{C})$ when X is projective.*

Let now u be an eigenvector of f^* in $H^1(X,\mathbf{C})$ with eigenvalue β. Its $(1,0)$ and $(0,1)$ parts $u_{1,0}$ and $u_{0,1}$ are also eigenvectors of f^*, with the same eigenvalue β. Since $u_{1,0}$ is represented by a holomorphic 1-form, we have

$$u_{1,0} \wedge \overline{u_{1,0}} \neq 0$$

as soon as $u_{1,0} \neq 0$. Thus $\beta\overline{\beta}$ is an eigenvector of f^* in $H^{1,1}(X,\mathbf{R})$.

Lemma 2.8. *The square of the spectral radius of f^* on $H^1(X,\mathbf{C})$ is bounded from above by the largest eigenvalue of f^* on $H^{1,1}(X,\mathbf{R})$. The spectral radius of f^* on $H^*(X,\mathbf{C})$ is equal to the spectral radius of f^* on $H^{1,1}(X,\mathbf{R})$.*

We shall denote by $\lambda(f)$, or simply λ, the spectral radius of f^*; this number is the **dynamical degree** of f. As we shall see in Sections 2.4.3 and 4.4.2, $\lambda(f)$ is an eigenvalue of f^*, $\lambda(f)$ is and algebraic integer, and its logarithm is equal to the topological entropy of f as a transformation of the complex surface X.

Example 2.9 (see Section 7.3 for explicit examples)**.** Let f_0 be a birational transformation of the plane $\mathbb{P}^2(\mathbf{C})$. By definition, the degree $\deg(f_0)$ of f_0 is the degree of the pre-image of a generic line by f_0. Equivalently, there are homogeneous polynomials P, Q, and R of the same degree d and without common factors of degree >1 such that $f_0[x:y:z] = [P:Q:R]$ in homogeneous coordinates; this number d is equal to $\deg(f_0)$.

Assume, now, that there is a birational map $\varphi\colon X \dashrightarrow \mathbb{P}^2(\mathbf{C})$ such that $f := \varphi^{-1} \circ f_0 \circ \varphi$ is an automorphism of X. Then $\lambda(f) = \lim_n \deg(f_0^n)^{1/n}$. This formula justifies the term "dynamical degree".

2.3. **Isometries of Minkowski spaces.** This paragraph is a parenthesis on the geometry of Minkowski spaces and their isometries.

2.3.1. *Standard Minkowski spaces.* The standard Minkowski space $\mathbf{R}^{1,m}$ is the real vector space $\mathbf{R}^{1+m}$ together with the quadratic form

$$x_0^2 - x_1^2 - x_2^2 - \ldots - x_m^2.$$

Let $\langle \cdot | \cdot \rangle_m$ be the bilinear form which is associated to this quadratic form. Let w be the vector $(1, 0, \ldots, 0)$; it is contained in the hyperboloid of vectors u with $\langle u|u\rangle_m = 1$. Define $\mathbf{H}_m$ to be the connected component of this hyperboloid that contains w, and let dist_m be the distance on $\mathbf{H}_m$ defined by (see [15, 85, 118])

$$\cosh(\mathsf{dist}_m(u, u')) = \langle u|u'\rangle_m.$$

The metric space $(\mathbf{H}_m, \mathsf{dist}_m)$ is a riemannian, simply-connected, and complete space of dimension m with constant sectional curvature -1; these properties uniquely characterize it up to isometry. ([2])

The projection of $\mathbf{H}_m$ into the projective space $\mathbb{P}(\mathbf{R}^{1,m})$ is one-to-one onto its image. In homogeneous coordinates, its image is the ball $x_0^2 > x_1^2 + \ldots + x_m^2$, and the boundary is the sphere obtained by projection of the isotropic cone $x_0^2 = x_1^2 + \ldots + x_m^2$. In what follows, $\mathbf{H}_m$ is identified with its image in $\mathbb{P}(\mathbf{R}^{1,m})$ and its boundary is denoted by $\partial \mathbf{H}_m$; hence, boundary points correspond to isotropic lines in $\mathbf{R}^{1,m}$.

2.3.2. *Isometries.* Let $\mathsf{O}_{1,m}(\mathbf{R})$ be the group of linear transformations of $\mathbf{R}^{1,m}$ preserving the bilinear form $\langle \cdot | \cdot \rangle_m$. The group of isometries $\mathsf{Isom}(\mathbf{H}_m)$ coincides with the subgroup of $\mathsf{O}_{1,m}(\mathbf{R})$ that preserves the chosen sheet $\mathbf{H}_m$ of the hyperboloid $\{u \in \mathbf{R}^{1,m} \,|\, \langle u|u\rangle_m = 1\}$. This group acts transitively on $\mathbf{H}_m$, and on its unit tangent bundle.

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[2]The riemannian structure is defined as follows. If u is an element of $\mathbf{H}_m$, the tangent space $T_u\mathbf{H}_m$ is the affine space through u that is parallel to $u^\perp$, where $u^\perp$ is the orthogonal complement of $\mathbf{R}u$ with respect to $\langle \cdot | \cdot \rangle_m$; since $\langle u|u\rangle_m = 1$, the form $\langle \cdot | \cdot \rangle_m$ is negative definite on $u^\perp$, and its opposite defines a positive scalar product on $T_u\mathbf{H}_m$; this family of scalar products determines a riemannian metric, and the associated distance coincides with dist_m (see [15]).

If $h \in \mathsf{O}_{1,m}(\mathbf{R})$ is an isometry of $\mathbf{H}_m$ and $v \in \mathbf{R}^{1,m}$ is an eigenvector of h with eigenvalue λ, then either $\lambda^2 = 1$ or v is isotropic. Moreover, since $\mathbf{H}_m$ is homeomorphic to a ball, h has at least one eigenvector v in $\mathbf{H}_m \cap \partial\mathbf{H}_m$. Thus, there are three types of isometries: **Elliptic** isometries, with a fixed point u in $\mathbf{H}_m$; **parabolic** isometries, with no fixed point in $\mathbf{H}_m$ but a fixed vector v in the isotropic cone; **loxodromic** (or **hyperbolic**) isometries, with an isotropic eigenvector v corresponding to an eigenvalue $\lambda > 1$. They satisfy the following additional properties (see [15]).

(1) An isometry h is elliptic if and only if it fixes a point u in $\mathbf{H}_m$. Since $\langle \cdot | \cdot \rangle_m$ is negative definite on the orthogonal complement $u^\perp$, the linear transformation h fixes pointwise the line $\mathbf{R}u$ and acts by rotation with respect to $\langle \cdot | \cdot \rangle_m$ on the orthogonal complement $u^\perp$.

(2) An isometry h is parabolic if it is not elliptic and fixes a vector v in the isotropic cone. The line $\mathbf{R}v$ is uniquely determined by the parabolic isometry h. For all points u in $\mathbf{H}_m$, the sequence $h^n(u)$ converges towards the boundary point $\mathbf{R}v$ in the projective space $\mathbb{P}(\mathbf{R}^{1,m})$ as n goes to $+\infty$ and $-\infty$.

(3) An isometry h is hyperbolic if and only if h has an eigenvector v_h^+ with eigenvalue $\lambda > 1$. Such an eigenvector is unique up to scalar multiplication, and there is another, unique, isotropic eigenline $\mathbf{R}v_h^-$ corresponding to an eigenvalue < 1; this eigenvalue is equal to $1/\lambda$. If u is an element of $\mathbf{H}_m$,

$$\frac{1}{\lambda^n} h^n(u) \longrightarrow \frac{\langle u | v_h^- \rangle_m}{\langle v_h^+ | v_h^- \rangle_m} v_h^+$$

and

$$\frac{1}{\lambda^n} h^{-n}(u) \longrightarrow \frac{\langle u | v_h^+ \rangle_m}{\langle v_h^+ | v_h^- \rangle_m} v_h^-$$

as n goes to $+\infty$. On the orthogonal complement of $\mathbf{R}v_h^+ \oplus \mathbf{R}v_h^-$, h acts as a rotation with respect to $\langle \cdot | \cdot \rangle_m$.

The type of h is also characterized by the growth of the iterates h^n: For any norm $\| \cdot \|$ on the space $\mathsf{End}(\mathbf{R}^{1,m})$, the sequence $\| h^n \|$ is bounded if h is elliptic, grows like $\mathsf{C}^{\mathsf{ste}} n^2$ if h is parabolic, and grows like λ^n if h is hyperbolic (with $\lambda > 1$ as in (3) above).

2.4. **Types of automorphisms and geometry.** Let X be a connected, compact, Kähler surface.

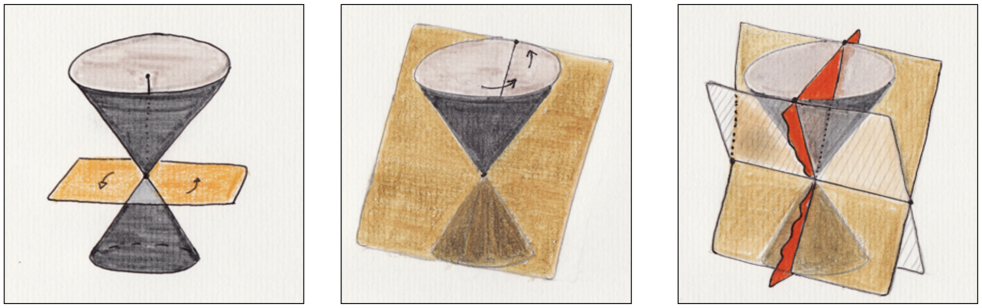

FIGURE 2. Three types of automorphisms (from left to right): Elliptic, parabolic, and loxodromic. Elliptic isometries preserve a point in $\mathbf{H}_m$ and act as a rotation on the orthogonal complement. Parabolic isometries fix an isotropic vector v; all positive and negative orbits in $\mathbf{H}_m$ converge towards the line $\mathbf{R}v$; the orthogonal complement of $\mathbf{R}v$ contains it, and is tangent to the isotropic cone. Loxodromic isometries dilate an isotropic line, contract another one, and act as a rotation on the intersection of the planes tangents to the isotropic cone along those lines (see also Figure 3 below).

2.4.1. *The hyperbolic space* $\mathbf{H}_X$. The intersection form on $H^{1,1}(X,\mathbf{R})$ is non-degenerate of signature $(1,h^{1,1}(X)-1)$; as such, it is isometric to the standard Minkowski form in dimension $h^{1,1}(X)$. One, and only one sheet of the hyperboloid $\{u \in H^{1,1}(X,\mathbf{R}) \,|\, \langle u|u\rangle = 1\}$ intersects the Kähler cone $\mathcal{K}(X)$: We denote by $\mathbf{H}_X$ this hyperboloid sheet ; as in Section 2.3, the intersection form endows $\mathbf{H}_X$ with the structure of a hyperbolic space $\mathbf{H}_m$ of dimension $m = h^{1,1}(X)-1$.

2.4.2. *Isometries induced by automorphisms.* Since automorphisms of X act by isometries with respect to the intersection form and preserve the Kähler cone, they preserve the hyperbolic space $\mathbf{H}_X$. This provides a morphism

$$\mathsf{Aut}(X) \to \mathsf{Isom}\,(\mathbf{H}_X).$$

By definition, an automorphism f is either **elliptic**, **parabolic**, or **loxodromic**, according to the type of $f^* \in \mathsf{Isom}\,(\mathbf{H}_X)$.([3])

2.4.3. *Loxodromic automorphisms.* Let f be a loxodromic automorphism, and let $\lambda(f)$ be its dynamical degree. We know from Sections 2.2.2 and 2.3.2 that $\lambda(f)$ is the unique eigenvalue of f^* on $H^2(X,\mathbf{C})$ with modulus > 1. It is real,

[3]In the literature, the terminology used for "loxodromic" is either "hyperbolic" or "hyperbolic on the cohomology", or "with positive entropy" depending on the authors.

positive, and its eigenspace is a line: This line is defined over $\mathbf{R}$, is contained in $H^{1,1}(X,\mathbf{C})$, and is isotropic with respect to the intersection form. Moreover, $\lambda(f)$ is an algebraic number because $\lambda(f)$ is an eigenvalue of f^* and f^* preserves the lattice $H^2(X,\mathbf{Z})$. Since the other eigenvalues of f^* on $H^2(X,\mathbf{C})$, beside $\lambda(f)$ and its inverse $1/\lambda(f)$, have modulus 1, this implies that $\lambda(f)$ *is either a quadratic integer or a Salem number.*([4])

Remark 2.10. The set of Salem numbers is not well understood. In particular, its infimum is unknown. However, dynamical degrees of automorphisms provide only a small subset of the set of Salem numbers, and McMullen proved that the minimum of all dynamical degrees $\lambda(f)$, for f describing the set of all loxodromic automorphisms of compact Kähler surfaces, is equal to Lehmer's number $\lambda_{10} \simeq 1.17628$, the largest root of

$$x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1.$$

This is the smallest known Salem number; for comparison, the smallest quadratic integer is the golden mean, $\lambda_G \simeq 1.61803$. Lehmer's number is realized as the dynamical degree of automorphisms on some rational surfaces (see [7, 102]) and K3 surfaces (both on projective, and non-projective K3 surfaces, see [103] and [101] respectively).

The Kähler cone $\mathcal{K}(X)$ is contained in the convex cone $\mathbf{R}_+\mathbf{H}_X$. Let $[\kappa]$ be an element of $\mathcal{K}(X)$. Then, from §2.3.2, $(1/\lambda(f)^n)(f^*)^n[\kappa]$ converges towards a non-zero eigenvector of f^* for the eigenvalue $\lambda(f)$. Thus, there exists a non-zero nef vector $v_f^+ \in H^{1,1}(X,\mathbf{R})$ such that

$$f^*v_f^+ = \lambda(f)v_f^+.$$

We fix such an eigenvector v_f^+ in what follows; this choice is unique up to a positive scalar factor, because the eigenspace for $\lambda(f)$ is a line. The same argument, applied to f^{-1}, provides a nef vector v_f^- such that $f^*v_f^- = (1/\lambda(f))v_f^-$. Changing v_f^- in a scalar multiple, we assume that

$$\langle v_f^+ | v_f^- \rangle = 1.$$

The orthogonal complements of v_f^+ and of v_f^- intersect along a codimension 2 subspace

$$N_f := (v_f^+)^\perp \cap (v_f^-)^\perp \subset H^{1,1}(X,\mathbf{R});$$

[4]By definition, an algebraic number λ is a **Salem number** if λ is real, $\lambda > 1$, its degree is ≥ 4, and the conjugates of λ are $1/\lambda$ and complex numbers of modulus 1. In particular, quadratic integers are not considered as Salem numbers here.

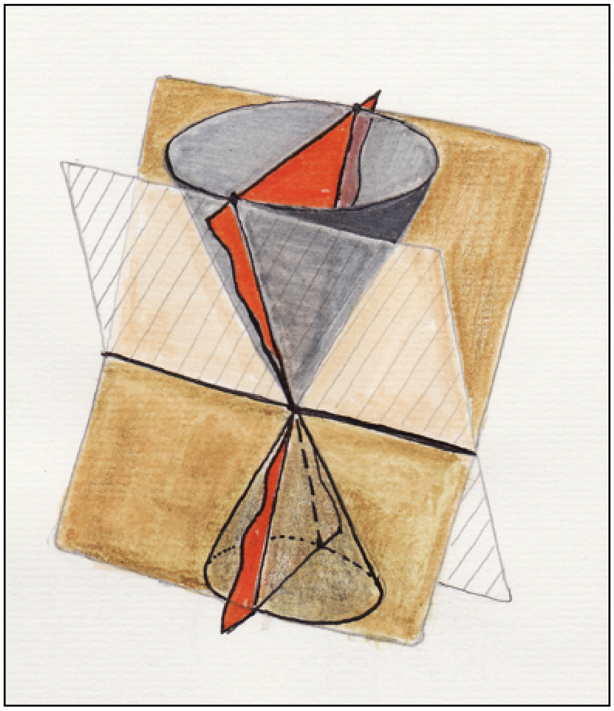

FIGURE 3. Action of a loxodromic automorphism on $H^{1,1}(X,\mathbf{R})$. The two invariant isotropic lines $\mathbf{R}v_f^+$ and $\mathbf{R}v_f^-$ generate on orange plane; the vector spaces $(\mathbf{R}v_f^+)^\perp$ and $(\mathbf{R}v_f^-)^\perp$ are tangent to the isotropic cone, and their intersection N_f is the orthogonal complement to the orange plane.

the intersection form $\langle\cdot|\cdot\rangle$ is negative definite on N_f.

2.4.4. *Elliptic and parabolic automorphisms.* The following result provides a link between this classification in types and the geometry of the transformation $f : X \to X$.

Theorem 2.11 (Gizatullin, Cantat [74, 34])**.** *Let X be a connected, compact, Kähler surface. Let f be an automorphism of X.*

(i) *If f is elliptic, a positive iterate f^k of f is contained in the connected component of the identity $\mathsf{Aut}(X)^0$; in particular $f^* \in \mathsf{GL}(H^*(X,\mathbf{Z}))$ has finite order.*
(ii) *If f is parabolic, there is an elliptic fibration $\pi_f : X \to B$, and an automorphism $\overline{f}$ of the curve B such that $\pi_f \circ f = \overline{f}$. If C is a fiber of the fibration, its class $[C]$ is contained in the unique isotropic line which*

is fixed by f^*; in particular, this line intersects $NS(X)\setminus\{0\}$. If $\overline{f}$ does not have finite order, then X is isomorphic to a torus $\mathbf{C}^2/\Lambda$.

Moreover, f is elliptic if and only if $\parallel (f^n)^* \parallel$ is a bounded sequence, f is parabolic if and only if $\parallel (f^n)^* \parallel$ grows quadratically, and f is loxodromic if and only if $\parallel (f^n)^* \parallel$ grows exponentially fast, like $\lambda(f)^n$.

2.4.5. *Projective surfaces over other fields.* Assume that X is a complex projective surface. Since $NS(X,\mathbf{R})$ intersects the Kähler cone, it intersects also $\mathbf{H}_X$ on an $\mathsf{Aut}(X)$-invariant totally isometric subspace. Thus, *the type of every automorphism f is the same as the type of f^* as an isometry of the hyperbolic subspace $\mathbf{H}_X\cap NS(X,\mathbf{R})$*. In particular, if f is loxodromic, the two isotropic eigenlines are contained in $NS(X,\mathbf{R})$. On the other hand, there is no vector u in $NS(X)$ such that $f^*u=\lambda u$ with $\lambda>1$, because f^* determines an automorphism of the lattice $NS(X)$. Hence, when f is loxodromic, the eigenline corresponding to the eigenvalue $\lambda(f)$ is irrational with respect to the lattice $NS(X)$.

Let now Y be a smooth projective surface defined over an algebraically closed field $\mathbf{k}$. Let f be an automorphism of Y. Then f acts on the Néron-Severi group $NS(Y)$ by isometries with respect to the intersection form, where $NS(Y)$ is defined as the group of numerical classes of divisors (see [79] for Néron-Severi groups). Hodge index Theorem applies, and shows that $NS(Y,\mathbf{R})$ is a Minkowski space with respect to its intersection form. Consequently, automorphisms of Y can also be classified in three categories in accordance with the type of the isometry f^* of $NS(Y,\mathbf{R})$; as said above, this definition is compatible with the previous one – which depends on the action of f^* on $H^{1,1}(X,\mathbf{R})$ – when X is a smooth complex projective surface.

Theorem 2.11 also applies to this setting, as shown by Gizatullin in [74].

2.4.6. *Two examples.*

A family of complex tori.– Consider an elliptic curve $E=\mathbf{C}/\Lambda$, and the abelian surface $X=E\times E$, as in Section 1.1.3. The group $\mathsf{SL}_2(\mathbf{Z})$ acts linearly on $\mathbf{C}^2$ and this action preserves the lattice $\Lambda\times\Lambda$, so that $\mathsf{SL}_2(\mathbf{Z})$ embeds as a subgroup of $\mathsf{Aut}(X)$. Let B be an element of $\mathsf{SL}_2(\mathbf{Z})$ and $\mathsf{tr}(B)$ be the trace of B. Then, the automorphism f_B of X induced by B is

- elliptic if and only if $B=\pm\mathrm{Id}$ or $\mathsf{tr}(B)=-1$, 0, or 1;
- parabolic if and only if $\mathsf{tr}(B)=-2$ or 2 and $B\neq\pm\mathrm{Id}$;

- loxodromic if and only if $|\mathsf{tr}(B)| > 2$; in this case, the dynamical degree of f_B is the square of the largest eigenvalue of B.

Since the type of an automorphism depends only on its action on the cohomology of X, all automorphisms of the form $t \circ f_B$ where $t \in \mathsf{Aut}(X)^0$ is a translation have the same type as f_B.

Remark 2.12. The appendix of [73] list all 2-dimensional tori with a loxodromic automorphism.

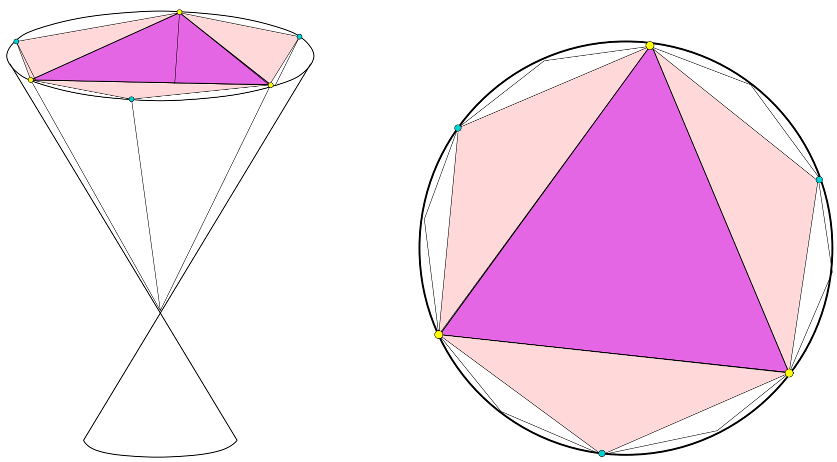

FIGURE 4. Action of involutions.– On the left, a picture of N_X with the triangular cone $\mathbf{R}^+[C_1] + \mathbf{R}^+[C_2] + \mathbf{R}^+[C_3]$ and its images under the three involutions. On the right, a projective view of the same picture: The triangular cone becomes a pink ideal triangle Δ.

Surfaces of degree (2,2,2) in $\mathbb{P}^1(\mathbf{C}) \times \mathbb{P}^1(\mathbf{C}) \times \mathbb{P}^1(\mathbf{C})$.– Let X be a smooth surface of degree $(2,2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Let $N_X \subset NS(X)$ be the subgroup of the Néron-Severi group which is generated by the three classes $[C_i]$, $i = 1,2,3$, where $[C_i]$ is the class of the fibers of the projection $\sigma_i \colon X \to \mathbb{P}^1$ defined by $\sigma_i(x_1,x_2,x_3) = x_i$. One easily checks that the three involutions s_i^* preserve the space N_X; on N_X, the matrix of s_1^* in the basis $([C_1],[C_2],[C_3])$ is equal to

$$\begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix},$$

and the matrices of s_2^* and s_3^* are obtained from it by permutation of the coordinates. Thus, on N_X, s_i is the orthogonal reflexion with respect to the plane

$\mathsf{Span}([C_j],[C_k])$ (for $\{i,j,k\}=\{1,2,3\}$). The space $\mathbf{H}_X\cap(N_X\otimes\mathbf{R})$ is isometric to the Poincaré disk. Denote by Δ the ideal triangle of the disk with vertices $[C_1]$, $[C_2]$, $[C_3]$. Then Δ is a fundamental domain for the action of the group generated by the s_i^*, as shown on Figure 4. The group generated by the involution s_i^* acts by symmetries of the tessellation of the disks by ideal triangle. This proves that there are no non-obvious relations between the involutions, as stated in the Introduction.

The transformation $s_i\circ s_j$, for $i\neq j$, is parabolic, and all parabolic elements in the group $\langle s_1,s_2,s_3\rangle$ are conjugate to some iterate of one of these parabolic automorphisms. A prototypical example of a loxodromic automorphism is the composition $g=s_3\circ s_2\circ s_1$. Its action on N_X is given by the matrix

$$\begin{pmatrix} -1 & -2 & -6 \\ 2 & 3 & 10 \\ 2 & 6 & 15 \end{pmatrix},$$

and the eigenvalues of this matrix are

$$\lambda=9+4\sqrt{5},\quad \frac{1}{\lambda}=9-4\sqrt{5},\quad \text{and } -1.$$

Thus, the dynamical degree of g is $\lambda(g)=9+4\sqrt{5}$.

2.5. **Classification of surfaces.** Compact complex surfaces have been classified (see [2]), and this classification, known as Enriques-Kodaira classification, has been extended to projective surfaces over algebraically closed fields by Mumford and Bombieri. This classification can be used to list all types of surfaces that may admit a loxodromic automorphism. Since this classification is not used in the sequel, we postpone the statement to an Appendix to this survey.

All we need to know is that, after contraction of smooth periodic curves with self-intersection -1, there are four main types of surfaces with loxodromic automorphisms: Rational surfaces obtained from $\mathbb{P}^2$ by a finite sequence of at least ten blow-ups, tori, K3 surfaces and Enriques surfaces. Complex Enriques surfaces are quotients of K3 surfaces by a fixed point free involution, so that the main examples, beside the well known case of tori, are given by rational surfaces and K3 surfaces.

Surfaces of degree $(2,2,2)$ are examples of K3 surfaces; Section 7.3 provides examples on rational surfaces.

3. Groups of Automorphisms

In order to illustrate the strength of our knowledge of $\mathsf{Isom}\,(\mathbf{H}_X)$, let us study the structure of subgroups of $\mathsf{Aut}(X)^\sharp$. In this Section, we denote by $\mathsf{Aut}(X)^*$ the image of $\mathsf{Aut}(X)$ in $\mathsf{GL}\,(H^*(M,\mathbf{Z}))$; up to finite index, $\mathsf{Aut}(X)^*$ coincides with $\mathsf{Aut}(X)^\sharp$.

This section is a parenthesis which is not used in the rest of this article.

3.1. **Torsion.** Let us start with a remark concerning torsion in $\mathsf{Aut}(X)^\sharp$. Let A be a subgroup of $\mathsf{Aut}(X)$, and A^* be its image in $\mathsf{GL}\,(H^*(X,\mathbf{Z}))$. The subgroup G_3 of all elements g in $\mathsf{GL}\,(H^*(X,\mathbf{Z}))$ such that

$$g = \mathrm{Id} \bmod (3)$$

is a finite index, torsion free, subgroup of $\mathsf{GL}\,(H^*(X,\mathbf{Z}))$.([5]) Denote by A_0^* its intersection with A^* and by A_0 its pre-image in A. Then A_0 is a finite index subgroup of A and A_0^* is torsion free.

Lemma 3.1. *Let X be a connected, compact Kähler surface. Up to finite index in the group $\mathsf{Aut}(X)$, every elliptic element of $\mathsf{Aut}(X)$ acts trivially on the cohomology of X.*

The same statement holds for arbitrary compact Kähler manifolds M if “elliptic” is replaced by “with finite order on $H^{1,1}(M,\mathbf{R})$".

3.2. **Free subgroups and dynamical degrees.** We can now prove the following result that provides a strong form of Tits alternative for subgroups of $\mathsf{Aut}(X)^*$.

Theorem 3.2 (Strong Tits Alternative, see [27, 30, 111, 126])**.** *Let X be a connected compact Kähler surface. If A is a subgroup of $\mathsf{Aut}(X)^*$, there is a finite index subgroup A_0 of A which satisfies one of the following properties*

- *A_0 contains a non-abelian free group, all of whose elements $g^* \neq Id$ are loxodromic isometries of $H^{1,1}(X,\mathbf{R})$;*
- *A_0 is cyclic and acts by loxodromic isometries on $H^{1,1}(X,\mathbf{R})$;*

[5] To prove that it is torsion free, suppose that there is a finite order element g in $G_3 \setminus \{\mathrm{Id}\}$. Changing g into an iterate g^k we assume that the order of g is prime: $g^p = \mathrm{Id}$ for some prime integer p. Write $g = \mathrm{Id} + 3^l A$ where A is a matrix with integer coefficients and one of them is not divisible by 3. Then $g^p = \mathrm{Id} + 3^l pA + 3^{2l} p(p-1)/2A^2 + \ldots = \mathrm{Id}$. Thus 3 divides p, hence $3 = p$, and one obtains that $A = 0$ modulo 3^{l+1}, a contradiction. (this argument is due to Minkowski)

- *A_0 is a free abelian group of rank at most $h^{1,1}(X,\mathbf{R})-2$ whose elements $g^*\neq Id$ are parabolic isometries of $H^{1,1}(X,\mathbf{R})$ (fixing a common isotropic line).*

Remark 3.3. From the classification of compact Kähler surfaces, and of holomorphic vector fields on surfaces, one easily proves the following: If $\mathsf{Aut}(X)^\sharp$ contains a loxodromic element, either X is a torus, or $\mathsf{Aut}(X)^0$ is trivial (see [39, 23]). Consequently, Theorem 3.2 can be used to describe subgroups of $\mathsf{Aut}(X)$ (instead of $\mathsf{Aut}(X)^\sharp$ or $\mathsf{Aut}(X)^*$).

Proof. By §3.1, we can assume that A is torsion free, so that it does not contain any elliptic element. Thus, either A contains a loxodromic element, or all elements of $A\setminus\{\mathrm{Id}\}$ are parabolic.

Assume A contains a loxodromic element h^*. If A does not fix any isotropic line of $H^{1,1}(X,\mathbf{R})$, then the ping-pong Lemma (see [43, 30]) implies that A contains a free non-abelian subgroup all of whose elements $f\neq \mathrm{Id}$ are loxodromic. Otherwise, A fixes an isotropic line $\mathbf{R}v$. Denote by

$$\alpha : A\to\mathbf{R}_+$$

the morphism defined by $g^*v=\alpha(g^*)v$ for all g^* in A. Since h^* fixes $\mathbf{R}v$, and h^* is loxodromic, $\alpha(h^*)=\lambda(h)^{\pm1}$ and $\mathbf{R}v$ is an irrational line with respect to $H^2(X,\mathbf{Z})$. Since this line is A-invariant and irrational, we obtain: A contains no parabolic element, $\alpha(g^*)=\lambda(g)^{\pm1}$ for all g in A, and α is injective. Moreover, all values of α in an interval $[a,b]\subset\mathbf{R}_+^*$ are algebraic integers of degree at most $\dim(H^2(X,\mathbf{Z}))$ whose conjugates are bounded by $\max(b,1/a)$. Consequently, α takes only finitely many values in compact intervals, its image is discrete, hence it is cyclic. Thus, either A contains a non-abelian free group, or A is cyclic.

Assume A does not contain any loxodromic element. Then all elements of $A\setminus\{\mathrm{Id}\}$ are parabolic. As in [30], this implies that A preserves a unique isotropic line $\mathbf{R}u\subset H^{1,1}(X,\mathbf{R})$. If g^* is an element of A, its eigenvalues in $H^2(X,\mathbf{Z})\otimes\mathbf{C}$ are algebraic integers, and all of them have modulus 1. By Kronecker Lemma, all of them are roots of 1. This implies that a finite index subgroup of A acts trivially on $u^\perp/(\mathbf{R}u)$. From this, it follows easily that, up to finite index, A is abelian of rank at most $h^{1,1}(X)-2=\dim(u^\perp)-1$. □

3.3. **Mapping class groups.** Let S be a connected, closed, and oriented surface of genus $g\geq 2$. The modular group, or mapping class group, of S is the group $\mathsf{Mod}\,(S)$ of isotopy classes of homeomorphisms of S; thus, $\mathsf{Mod}\,(S)$ is the group of connected components of the group of homeomorphisms of S, and is a natural analogue of the group $\mathsf{Aut}(X)^\sharp$. Let us list a few useful analogies between modular groups $\mathsf{Mod}(S)$ and groups of automorphisms $\mathsf{Aut}(X)$.

TABLE 1. Automorphisms versus mapping classes.– Here, f is an automorphism of a connected, compact Kähler surface X, and h is a pseudo Anosov homeomorphism of a closed oriented surface S. (see the following Sections for topological entropy and the laminar currents $T_f^\pm$)

Compact Kähler surface X	Higher genus, closed surface S
f acts on the hyperbolic space $\mathbf{H}_X$	h acts on the Teichmüller space $T(S)$ (resp. on the complex of curves)
f is loxodromic	h is pseudo-Anosov
dynamical degree $\lambda(f)$	dilatation factor $\lambda(h)$
cohomology classes v_f^+ and v_f^-	fixed points of h on $\partial T(S)$
$\mathsf{h}_{top}(f) = \log \lambda(f)$	$\mathsf{h}_{top}(h) = \log \lambda(h)$
closed laminar currents T_f^+ and T_f^-	measured stable, unstable foliations of h

On one hand, $\mathsf{Aut}(X)^\sharp$ acts almost faithfully on the cohomology of X; on the other hand, $\mathsf{Mod}(S)$ coincides with the group of outer automorphisms of the fundamental group $\pi_1(S)$. Thus, both $\mathsf{Aut}(X)^\sharp$ and $\mathsf{Mod}(S)$ are determined by their respective action on the algebraic topology of the surface. For instance

- $\mathsf{Aut}(X)^\sharp$ acts by isometries on the hyperbolic space $\mathbf{H}_X$ and we derived from this action a strong form of Tits Alternative for subgroups of $\mathsf{Aut}(X)^\sharp$ (see Theorem 3.2);
- similarly, $\mathsf{Mod}(S)$ acts on the complex of curves of S, a Gromov hyperbolic space (see [87, 99]), and $\mathsf{Mod}(S)$ satisfies also a strong form of Tits alternative (see [86, 16]). For example, solvable subgroups are

almost abelian, and torsion free abelian subgroups have rank at most $3g$ (see [18]).

Thus, subgroups of $\mathsf{Mod}\,(S)$ satisfy properties which are similar to those listed in Theorem 3.2.

If f^* is an element of $\mathsf{Aut}(X)^*$, we know that f^* is either elliptic, parabolic, or loxodromic; this classification parallels Nielsen-Thurston classification of mapping classes $g \in \mathsf{Mod}\,(S)$: Elliptic automorphisms correspond to finite order elements of $\mathsf{Mod}\,(S)$, parabolic to composition of Dehn twists along pairwise disjoint simple closed curves, and loxodromic to pseudo-Anosov classes (there are no "reducible" transformations beside "Dehn twists" in the realm of automorphisms of compact Kähler surfaces). As we shall see, when f is a loxodromic automorphism, the classes $[v_f^{\pm}]$ are represented by laminar currents on the surface X, that will play a role similar to the stable and unstable foliations for pseudo-Anosov homeomorphisms.

As explained in [30], this analogy is even more fruitful for birational transformations of X.

3.4. **Construction of automorphisms.** Most surfaces do not admit loxodromic automorphisms (or even automorphisms $f \neq \mathrm{Id}$). For example, if one blows up $n \geq 10$ generic points in $\mathbb{P}^2(\mathbf{C})$, the group of automorphisms of the surface that one gets is trivial (see [81, 94]). As explained in Section 2.5 and in the Appendix, surfaces with loxodromic automorphisms fall in four classes: Rational surfaces, tori, K3 surfaces and Enriques surfaces.

For simplicity, let us work over the field of complex numbers $\mathbf{C}$. Since tori are well understood and Enriques surfaces are quotient of K3 surfaces by a fixed point free involution, we focus on K3 surfaces and rational surfaces.

To construct K3 surfaces with loxodromic automorphisms, one can apply the so-called Torelli Theorem, which asserts that the automorphisms of a K3 surface X are in bijections with the invertible linear transformations of $H^*(X,\mathbf{Z})$ that preserve the Hodge structure, the intersection form, and the set of homology classes of smooth rational curves on X. This result is difficult to use in practice, because one needs a precise understanding of (i) the interplay between the Hodge structure and the integral structure of the cohomology and (ii) the set of rational curves (or, what is the same, classes in $NS(X)$ with self-intersection -2). Good references to see how automorphisms of K3 surfaces can be cooked up with this method are [101, 104, 103], [112].

The case of rational surfaces is more delicate, but leads to more examples. Let us describe one of them. Let C be a smooth cubic curve in $\mathbb{P}^2$. If p is a point on C, a birational involution $\sigma_p \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ can be defined as follows. For each point $m \in \mathbb{P}^2$, draw the line (mp) joining m to p; if not tangent to C, this line intersects C in three points, p, q and r; there is a unique projective linear involution on this line which fixes q and r (it is conjugate to $z \mapsto -z$, fixing 0 and ∞); the image of m by this involution is the point $\sigma_p(m)$. Thus, σ_p is a birational transformation of $\mathbb{P}^2$ which fixes C point-wise and preserves the pencil of lines through p. One can resolve the indeterminacies of σ_p by blowing up a finite number of points of C: The point p, and the four points $z \in C$ such that (pz) is tangent to C at z. Once these points have been blown up, σ_p is an automorphism that fixes (the strict transform of) C point-wise. Thus, starting with l distinct points p_i, one can do successively all necessary blow-ups to lift the transformations σ_{p_i} to automorphisms of a rational surface. Blanc proves in [19] that these automorphisms generate a free product $\mathbf{Z}_2/\mathbf{Z} * \ldots \mathbf{Z}_2/\mathbf{Z}$ of l copies of $\mathbf{Z}_2/\mathbf{Z}$. If $l \geq 3$, it follows from Theorem 3.2 that this group contains loxodromic elements.

There is no clear understanding yet on the condition that a rational surface X must satisfy to have a loxodromic automorphism, or a large group of automorphisms. We refer to [80, 7, 9, 102] for constructions of examples, to [8, 48] for their deformations, and to [31] for restrictions on the size of such groups of automorphisms.

4. Periodic curves, periodic points, and topological entropy

We now focus on the dynamics of loxodromic automorphisms on connected compact Kähler surfaces.

The main goal of this Section is to explain how ideas of algebraic geometry, including geometry over finite fields, of topology, and of dynamical systems can be used to study periodic curves and periodic points of loxodromic automorphisms.

4.1. **Periodic curves.** Let $E \subset X$ be a curve which is invariant under the loxodromic automorphism f. We denote by $[E]$ its class in $H^{1,1}(X, \mathbf{R})$. ([6]) Since $f^*[E] = [E]$, $[E]$ is contained in the orthogonal complement N_f of the plane $\mathbf{R}v_f^+ \oplus \mathbf{R}v_f^-$. Thus, the intersection form is negative definite on the subspace

[6]This class is the dual of the homology class of E. Equivalently, $[E]$ is the Chern class of the line bundle $O_X(E)$.

of $H^{1,1}(X,\mathbf{R})$ generated by the classes of all f-invariant or f-periodic curves. Grauert-Mumford contraction Theorem (see [2]) can therefore be applied to this set of curves, and provides the following result.

Proposition 4.1 (Cantat, Kawaguchi, see [34, 37, 93])**.** *Let f be a loxodromic automorphism of a connected, compact, Kähler surface X. There exist a (singular) surface X_0, a birational morphism $\pi : X \to X_0$, and an automorphism f_0 of X_0 such that*

(1) $\pi \circ f = f_0 \circ \pi$;
(2) *a curve $E \subset X$ is contracted by π if and only if E is f-periodic, if and only if $[E]$ is contained in N_f.*

This implies that the number of f-periodic curves is finite when f is loxodromic. Moreover, we can assume that f does not have any periodic curve if we admit singular models X_0 for the surface X. When f is an automorphism of a projective surface Y defined over an algebraically closed field $\mathbf{k}$, as in Section 2.4.5, the same result holds.

Theorem 4.2 (Castelnuovo, see [20, 42, 53])**.** *Let f be a loxodromic automorphism of a connected, compact, Kähler surface X. If E is a connected periodic curve of f, then E has genus* 0 *or* 1.

In his initial statement, Castelnuovo assumed the curve E to be irreducible, but this hypothesis has been removed by Diller, Jackson and Sommese. The genus of E can be defined in terms of the genus formula $g(E) = E \cdot (E + K_X)/2 + 1$, where K_X is the canonical class of X; it coincides with $1 - \chi(O_C)$, where χ denotes the Euler characteristic (see [53]).

The inequality $g(E) \leq 1$ imposes drastic constraints on E. First, E has genus 0 if and only if E is a tree of smooth rational curves. Assume now that E has genus 1. Then, there exists a birational morphism $\eta \colon X \to X_1$ such that (i) X_1 is smooth, i.e. η is a composition of contractions of exceptional curves of the first kind, (ii) f induces an automorphism $\eta \circ f \circ \eta^{-1}$ of X_1, (iii) there is a meromorphic 2-form on X_1 which does not vanish and whose divisor of poles coincides with $\eta(E)$. The curve $\eta(E)$ has genus 1 and is one of the following curves:

- a smooth curve of genus 1, or a rational curve with a node, or a rational curve with a cusp;
- a union of two smooth rational curves meeting tangentially, or intersecting transversely at two distinct points;

- a union of three smooth rational curves intersecting transversely at a single point;
- a cycle of $k \geq 3$ smooth rational curves.

In particular, the singularities of the surface X_0, obtained by blowing down the periodic curves of f, are very special.

Remark 4.3. Loxodromic automorphisms of complex tori have no periodic curve. On a K3 surface (resp. on an Enriques surface), the genus formula shows that all irreducible periodic curves are smooth rational curves. There are examples of rational surfaces X with an automorphism f such that f is loxodromic and f fixes an elliptic curve point-wise (see §3.4 above, or Example 3.1 and Remark 3.2 in [37]).

4.2. **Fixed points formulae.** Lefschetz Formula provides a link between fixed points of f and its action on the cohomology of X.

4.2.1. *Lefschetz Formula (see* [75]*).* Let M be a smooth oriented manifold and g be a smooth diffeomorphism of M. Let p be an isolated fixed point of g, and U be a chart around p. One defines the index $\mathsf{Ind}(g;p)$ of g at p as the local degree of the map $\mathrm{Id}_M - g$. The graph $\Gamma_g \subset M \times M$ of g intersects the diagonal Δ at (p,p). This intersection is transversal if and only if 1 is not an eigenvalue of the tangent map Dg_p, if and only if $\mathsf{det}(Dg_p - \mathrm{Id}) \neq 0$. In this case, the index of g at p satisfies

$$\mathsf{Ind}(g;p) = \mathrm{sign}(\mathsf{det}(Dg_p - \mathrm{Id})).$$

Another equivalent definition of $\mathsf{Ind}(g;p)$ is as follows. Orient Γ_g around (p,p) in such a way that the map $x \mapsto (x, g(x))$ preserves the orientation; then $\mathsf{Ind}(g;p)$ is the intersection number of Γ_g with the diagonal Δ at (p,p). Thus, one gets

$$\sum_{g(p)=p} \mathsf{Ind}(g;p) = \Delta \cdot \Gamma_g$$

where $\Delta \cdot \Gamma_g$ denotes the intersection number of Δ and Γ_g, a quantity which can be computed in terms of the action of g^* on the cohomology of M. One obtains $\Delta \cdot \Gamma_g = L(g)$ where $L(g)$ denotes the Lefschetz number

$$L(g) := \sum_{k=0}^{\dim M} (-1)^k \mathsf{tr}(g^*_{|H^k(M,\mathbf{R})}).$$

Thus,

$$\sum_{g(p)=p} \mathsf{Ind}(g;p) = L(g)$$

when all fixed points of M are isolated. If all fixed points of g are non-degenerate, one gets the estimate $|\mathrm{Fix(g)}| \geq |L(g)|$.

4.2.2. *Shub-Sullivan Theorem, and automorphisms.* In order to apply Lefschetz fixed points Formula to count periodic points, one needs to control the indices of the iterates g^n. This is exactly what the following result does.

Theorem 4.4 (Shub-Sullivan, [114])**.** *Let $g : U \to \mathbf{R}^m$ be a map of class $\mathcal{C}^1$, where U is an open subset of $\mathbf{R}^m$ that contains the origin 0. Assume that 0 is an isolated fixed point of all positive iterates g^n, $n > 0$. Then $\mathsf{Ind}(g^n;0)$ is bounded as a function of n.*

Let f be a loxodromic automorphism of a compact Kähler surface X. Suppose that f does not have any curve of periodic points; then all periodic points are isolated, because the set of periodic points of period $n > 0$ is an analytic subset of X without components of positive dimension. From Lefschetz Formula and Shub-Sullivan Theorem, there is an infinite number of periodic points, because $L(f^n)$ grows like $\lambda(f)^n$ as n goes to $+\infty$. As a simple corollary, we obtain

Corollary 4.5. *If f is a loxodromic automorphism of a compact Kähler surface, the set $\mathsf{Per}(f)$ of periodic points of f is infinite.*

To prove the existence of an infinite number of isolated periodic points (*i.e.* of periodic points that are not contained in curves of periodic points), one needs (i) a Lefschetz Formula that would take into account curves of fixed points and (ii) a control of the indices along such curves; this is done in [88] for area preserving automorphisms (see [88] for examples showing that indices of f^k along curves of fixed points are not always bounded).

4.2.3. *Holomorphic fixed point formulae.* In the holomorphic setting, one can derive more precise formulae. Let f be a holomorphic endomorphism of a compact complex manifold M. For each integer $r \in \{0, \ldots, \dim_{\mathbf{C}}(M)\}$, define Lefschetz number of index r by

$$L^r(f) = \sum_{s=0}^{s=\dim_{\mathbf{C}}(M)} (-1)^s \mathsf{tr}(f^*_{|H^{r,s}(M,\mathbf{C})}).$$

For example, when M is a complex surface, Poincaré duality implies

$$L^0(f) = \overline{L^2(f)} = 1 - \mathsf{tr}(f^*_{|H^{0,1}}) + \mathsf{tr}(f^*_{|H^{0,2}})$$

$$L^1(f) = 2\mathsf{tr}(f^*_{|H^{1,0}}) - \mathsf{tr}(f^*_{|H^{1,1}}).$$

Theorem 4.6 (Atiyah-Bott fixed point Theorem, [1]). *Let f be a holomorphic endomorphism of a compact complex manifold M. If all fixed points of f are non-degenerate, then*

$$L^r(f) = \sum_{f(p)=p} \frac{\mathsf{tr}(\wedge^r Df_p)}{\mathsf{det}(Id - Df_p)}.$$

As a consequence, on a compact Kähler surface with no non-zero holomorphic form every endomorphism has at least one fixed point. This remark applies, for example, to surfaces obtained from the projective plane by a finite sequence of blow-ups. See [120, 101] for applications.

4.3. **Periodic points are Zariski dense.** As explained in the previous paragraph, every loxodromic automorphism of a compact Kähler surface has an infinite number of periodic points. Here is a stronger result for projective surfaces, which is taken from works of Fakhruddin, Hrushowski, and Xie.

Theorem 4.7 (see [69] and [124]). *Let $\mathbf{k}$ be an algebraically closed field. Let X be an irreducible projective surface and f be an automorphism of X, both defined over $\mathbf{k}$. If f is loxodromic, the set $\mathsf{Per}(f) \subset X(\mathbf{k})$ is Zariski dense in X. Moreover, for every curve $Z \subset X$ there is a periodic orbit of f in $X \setminus Z$.*

Finer results hold when f is an automorphism of a connected compact Kähler surface (see below §4.4.3).

Let us try to convey some of the ideas that lead to a proof of this Theorem. First, recall that loxodromic automorphisms have a finite number of periodic curves, as shown in Section 4.1.

4.3.1. *Finite fields.* Let us first assume that both X and f are defined over a finite field $\mathbf{F}_q$, with q elements. Pick a point x in X and choose a finite extension $\mathbf{F}_{q^l}$ of $\mathbf{F}_q$ such that $x \in X(\mathbf{F}_{q^l})$. Since f is defined over $\mathbf{F}_q$, f permutes the points of the finite set $X(\mathbf{F}_{q^l})$, so that the orbit of x is finite. This shows that all points are periodic !

There is another, more powerful, technique to construct periodic points over finite fields. Let Z be any Zariski closed proper subset of X. We shall construct a periodic orbit of f which is entirely contained in the complement of Z, a result that is stronger than the existence of a periodic point in $X \setminus Z$.

Let $\overline{\mathbf{F}}_q$ be an algebraic closure of $\mathbf{F}_q$, and let $\Phi_q \colon X \to X$ be the geometric Frobenius automorphism (on $\overline{\mathbf{F}}_q$, Φ_q raises numbers t to the power t^q). First, note that the orbit of Z under the action of the Frobenius morphism is Zariski closed: There is an integer $k \geq 0$ such that

$$\bigcup_n \Phi_q^n(Z) = Z \cup \Phi_q(Z) \cup \ldots \cup \Phi_q^k(Z)$$

because Z is defined over a finite extension of $\mathbf{F}_q$. Denote by Z' this proper, Φ_q-invariant, Zariski closed subset of X. Then, fix an affine Zariski open subset $U \subset X$ that does not intersect Z'. Denote by $\Gamma_f \subset X \times X$ the graph of f, and by $\Gamma_f(U)$ its intersection with $U \times U$. We can apply the following Theorem to $S = \Gamma_f(U)$.

Theorem 4.8 (Hrushovski)**.** *Let U be an irreducible affine variety over $\mathbf{F}_q$. Let $S \subset U \times U$ be an irreducible variety over $\overline{\mathbf{F}}_q$, and let Φ_q be the Frobenius automorphism on U. If the two projections of S on U are dominant, the set of points of S of the form $(x, \Phi_q^m(x))$, for x in U and $m \geq 1$, is Zariski dense in S.*

Thus, there exists a positive integer m and a point $x \in U$ such that $(x, \Phi_q^m(x))$ is contained in $\Gamma_f(U)$; in other words,

$$f(x) = \Phi_q^m(x).$$

Since f is defined over $\mathbf{F}_q$, it commutes to Φ_q, and

$$f^n(x) = \Phi_q^{mn}(x) \in X \setminus Z'$$

for all $n \geq 1$. But x is periodic under Φ_q, because its coordinates live in a finite extension of $\mathbf{F}_q$, hence $f^n(x) = x$ for some positive integer n. This provides a periodic orbit in the complement of Z', as desired.

4.3.2. *Arbitrary fields.* Assume now that X and f are defined over the field of rational numbers $\mathbf{Q}$. After reduction modulo a sufficiently large prime power $q = p^l$, one gets an automorphism

$$f_q : X_{\mathbf{F}_q} \to X_{\mathbf{F}_q}.$$

The Néron-Severi group of $X(\mathbf{C})$ is generated by classes of curves which are defined on a finite extension K of $\mathbf{Q}$. Thus, if p and l are large enough, the action of f_q on $NS(X_{\mathbf{F}_q})$ is loxodromic, with the same dynamical degree as $f\colon X_{\mathbf{C}} \to X_{\mathbf{C}}$. In particular, f_q has a finite number of periodic curves and $\mathsf{Per}(f_q) \subset X_{\mathbf{F}_q}(\overline{\mathbf{F}}_q)$ is Zariski dense in $X_{\mathbf{F}_q}$. Pick an isolated periodic point m of some period n, *i.e.* a periodic point $m \in X_{\mathbf{F}_q}(\overline{\mathbf{F}}_q)$ which is not contained in a curve of periodic points. Then one can lift m to a periodic point $\hat{m} \in X(\overline{\mathbf{Q}})$ (roughly speaking, the equation $f^n(m) = m$ determines a scheme of dimension

1, which is not contained in the special fiber $X_{\mathbf{F}_q}$ because m is an isolated fixed point of f^n; thus, its intersection with the generic fiber provides a periodic point). Since the set of such points m is Zariski dense, the lifts $\hat{m}$ form a Zariski dense subset of $X(\overline{\mathbf{Q}})$.

When $\mathbf{k}$ is an arbitrary, algebraically closed field, one first replaces it by a finitely generated subring over which X and f are defined. Then standard techniques show that the same strategy – reduction plus lift – can be applied.

4.4. **Topological entropy and saddle periodic points.** Let us come back to the dynamics of loxodromic automorphisms on compact Kähler surfaces, and apply tools from dynamical systems to understand periodic points.

4.4.1. *Entropy.* Let g be a continuous transformation of a compact metric space Z, with distance dist. The **topological entropy** $\mathsf{h}_{top}(g)$ is defined as follows. Let ε be a positive number and n be a positive integer. One says that a finite subset A of Z is separated at scale ε during the first n iterations, or simply that A is (ε, n)-separated, if and only if, for any pair of distinct points a and b in A, there exists a time $0 \leq k < n$ such that

$$\mathsf{dist}(g^k(a), g^k(b)) \geq \varepsilon.$$

The maximum number of elements in (ε, n)-separated subsets is denoted by $N(\varepsilon, n)$. Then, one defines successively

$$\mathsf{h}_{top}(g;\varepsilon) = \limsup_{n\to+\infty} \frac{1}{n} \log N(\varepsilon, n)$$

and, taking finer and finer scales of observation of the dynamics,

$$\mathsf{h}_{top}(g) = \lim_{\varepsilon\to 0} \mathsf{h}_{top}(g;\varepsilon).$$

So, topological entropy measures the rate at which the dynamics of g creates distinct orbits, when observed with an arbitrarily small, but positive, scale. As an example, the transformation $z \mapsto z^d$ of the unit circle $\{z \in \mathbf{C}; |z| = 1\}$, has entropy $\log(d)$.

4.4.2. *Gromov-Yomdin formula.* Computing topological entropy is a difficult problem in practice, but for holomorphic transformations $f : M \to M$ of compact Kähler manifolds, entropy coincides with the logarithm of the spectral radius of $f^* \in \mathsf{GL}(H^*(M, \mathbf{C}))$:

Theorem 4.9 (Gromov [77], Yomdin [125, 76])**.** *Let f be a diffeomorphism of a compact manifold M. Let $\lambda(f)$ be the spectral radius of the linear transformation $f^* : H^*(M,\mathbf{C}) \to H^*(M,\mathbf{C})$.*

- *If M and f are of class $\mathcal{C}^\infty$, then $\mathsf{h}_{top}(f) \geq \log\lambda(f)$.*
- *If M is a Kähler manifold and f is holomorphic, $\mathsf{h}_{top}(f) = \log\lambda(f)$.*

For automorphisms of compact Kähler surfaces, one gets

$$\mathsf{h}_{top}(f) = \log\lambda(f)$$

where $\lambda(f)$ is the dynamical degree of f^*.

Remark 4.10. a.– When f is an automorphism of a compact Kähler manifold M, one can replace $\lambda(f)$ by the largest eigenvalue of f^* on the sum $\bigoplus_p H^{p,p}(M,\mathbf{R})$ in Gromov-Yomdin Theorem.
b.– Let f be an automorphism of a projective variety M, both defined over a finite field $\mathbf{k}$. Let l be a prime integer, distinct from the characteristic of $\mathbf{k}$. One can define étale cohomology groups $H^*_{\text{ét}}(M,\mathbf{Q}_l)$ and look at the eigenvalues of f^* on these groups. For surfaces, all eigenvalues on the orthogonal complement of the Néron-Severi group $NS(X)$ are roots of unity: See [68] for a proof and interesting questions.

Example 4.11. Let $M = \mathsf{SL}_2(\mathbf{C})/\Gamma$ where Γ is a co-compact lattice in $\mathsf{SL}_2(\mathbf{C})$. Let t be a positive real number. The automorphism f_t of M defined by left multiplication by

$$\left(\begin{array}{cc} \exp(t) & 0 \\ 0 & \exp(-t) \end{array}\right)$$

is isotopic to the identity (let t go to 0), but has positive entropy (the flow defined by f_t is the geodesic flow on the unit tangent bundle to a hyperbolic manifold of dimension 3). This does not contradict Gromov's Theorem because M is not Kähler (see [72], page 120).

4.4.3. *Saddle periodic points.* Let p be a periodic point of the automorphism f and let k be its period. One says that p is a saddle (or hyperbolic) periodic point if one eigenvalue of the tangent map $D(f^k)_p$ has modulus > 1 and the other has modulus < 1. Since f has topological entropy $\log(\lambda(f))$ and X has dimension 2, one can apply a result due to Katok.

Theorem 4.12 (Katok, [90])**.** *Let f be a loxodromic automorphism of a compact Kähler surface. The number $N(f,k)$ of saddle periodic points of f of* *period at most k grows like* $\lambda(f)^k$*: For all* $\varepsilon > 0$*,*

$$\limsup \frac{1}{k}\log(N(f,k)) \geq \log(\lambda(f)-\varepsilon).$$

The same result holds for isolated periodic points in place of saddle periodic points.

In particular, f has an infinite number of isolated periodic points. This implies that periodic points of f are Zariski dense: If the Zariski closure were contained in a curve Z, this curve would be invariant by f and, by definition, it would contain all periodic points; but an automorphism of a curve has only finitely many isolated periodic points.

Katok's proof requires several non trivial dynamical constructions, including the full strength of Pesin theory. It provides f-invariant compact subsets $\Lambda_l \subset X$, $l \geq 1$, and numbers ε_l going to 0 when l goes to $+\infty$, such that (i) the restriction of f to Λ_l is conjugate to a horse-shoe map (and is therefore well understood, see [91]), and (ii) the number of periodic points of period n in each of these sets grows like $(\lambda(f)-\varepsilon_l)^n$ with n.

5. Invariant currents

5.1. **Currents (see** [75, 46]**).**

5.1.1. *Definitions.* Let X be a compact Kähler surface, and $\wedge^{1,1}(X,\mathbf{R})$ be the space of smooth real valued $(1,1)$-forms on X with its usual Fréchet topology. By definition, a $(1,1)$-current is a continuous linear functional on $\wedge^{1,1}(X,\mathbf{R})$. For simplicity, $(1,1)$-currents are called **currents** in this text. The value of a current T on a form ω is denoted by $(T|\omega)$.

Example 5.1. a.– Let α be a continuous $(1,1)$-form, or more generally a $(1,1)$-form with distribution coefficients. Then α defines a current $\{\alpha\}$ (also denoted by α in what follows):

$$(\{\alpha\}|\omega) = \int_X \alpha \wedge \omega.$$

b.– Let $C \subset X$ be a curve. The current of integration on C is defined by

$$(\{C\}|\omega) = \int_C \omega.$$

This is well defined even if C is singular; moreover, $\{C\}$ extends to a linear functional on the space of continuous forms for the topology of uniform convergence.

Recall that a $(1,1)$-form ω is positive if $\omega(u, \sqrt{-1}u) \geq 0$ for all tangent vectors u. A current T is **positive** if it takes non-negative values on the convex cone of positive forms. When positive, T extends as a continuous linear functional on the space of continuous $(1,1)$-forms with the topology of uniform convergence. Given two currents T and T', one says that T is larger than T', written $T \geq T'$, if the difference $T - T'$ is a positive current.

A current is **closed** if it vanishes on the space of exact forms. For example, the current associated to a smooth $(1,1)$-form α is positive (resp. closed) if and only if α is a positive (resp. closed) form. The current of integration on a curve $C \subset X$ is positive and closed (because C has empty boundary).

5.1.2. *Cohomology classes.* Let T be a closed current. Then T defines a linear form on the space $H^{1,1}(X, \mathbf{R})$, and there is a unique cohomology class $[T]$ such that

$$(T|\omega) = \langle [T] | [\omega] \rangle$$

for all closed forms ω of type $(1,1)$. By definition, $[T]$ is the cohomology class of T.

5.1.3. *Mass and compact sets of currents.* Let T be a positive current on a Kähler surface X. Let κ be a Kähler form on X. The trace measure of T is the positive measure $\parallel T \parallel$ defined by

$$\int_X \xi \parallel T \parallel = (T|\xi\kappa)$$

for all smooth functions ξ; it depends on the choice of the Kähler form κ. The mass $M(T)$ of T is the total mass of the trace measure $\parallel T \parallel$. When T is closed, we obtain

$$M(T) = \langle [T] | [\kappa] \rangle,$$

so that the mass depends only on the cohomology class $[T]$.

The space of currents is endowed with the weak topology: A sequence of currents (T_i) converges towards a current T if $(T_i|\omega)$ converges towards $(T|\omega)$ for all smooth forms. The set of positive currents with mass at most B (B any positive real number) is a compact convex set for this topology. In particular, if T_i is a sequence of closed positive currents with uniformly bounded cohomology classes, one can extract a converging subsequence.

5.1.4. *Potentials.* The differential operator d decomposes as $d = \partial + \overline{\partial}$ where, in local coordinates $z_i = x_i + \sqrt{-1}y_i$, the operators ∂ and $\overline{\partial}$ are given by

$$\partial = \sum_i \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right) dz_i, \quad \overline{\partial} = \sum_i \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right) d\overline{z_i}.$$

Denote by d^c the operator $\frac{1}{2\pi}(\overline{\partial} - \partial)$; then

$$dd^c = \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}.$$

Let T be a closed and positive current. Locally, T can be written as

$$T = dd^c u$$

for some function u, called a local **potential** of T (see [75], §3.2). The positivity of T is equivalent to the **pluri-subharmonicity** of u, which means that (i) u is upper semi-continuous with values into $\{-\infty\} \cup \mathbf{R}$, (ii) u is not identically $-\infty$, and (ii) u is subharmonic along all holomorphic disks $\varphi \colon \mathbb{D} \to X$ (*i.e.* $u \circ \varphi$ is either identically $-\infty$ or subharmonic on $\mathbb{D}$). Pluri-subharmonic functions are locally integrable, and the equation $T = dd^c u$ means that

$$(T|\omega) = \int_X u \, dd^c \omega$$

for all smooth forms ω with support in the open set where the equality $T = dd^c u$ is valid. When the local potentials of T are continuous (resp. smooth, Hölder continuous, etc), one says that T has continuous (resp. smooth, Hölder continuous, etc) potentials.

5.1.5. *Multiplication (see* [4, 46]*).* In general, distributions, and currents as well, can not be multiplied, but Bedford and Taylor introduced a pertinent way to multiply two closed positive currents T_1 and T_2 when one of them, say T_2, has continuous potentials. The product, a positive measure $T_1 \wedge T_2$, is defined by the following local formula:

$$(T_1 \wedge T_2|\psi) = (T_1|u_2 dd^c(\psi))$$

for all smooth functions ψ with support on open sets where $T_2 = dd^c(u_2)$; when both T_1 and T_2 have continuous potentials, this definition is symmetric in T_1 and T_2.

Cohomology classes and products of currents are compatible, which means that the total mass of the measure $T_1 \wedge T_2$ is equal to the intersection of the classes $[T_1]$ and $[T_2]$ (for closed positive currents with continuous potentials).

5.1.6. *Automorphisms.* Let f be an automorphism of X, and T be a current. Define f_*T by

$$(f_*T|\omega) = (T|f^*\omega), \quad \forall\, \omega \in \wedge^{1,1}(X,\mathbf{R}).$$

The operator f_* maps closed (resp. positive) currents to closed (resp. positive) currents. Define f^* by $f^* = (f^{-1})_*$; it satisfies $[f^*T] = f^*[T]$ where the right hand side corresponds to the action of f on the cohomology group $H^{1,1}(X,\mathbf{R})$.

Example 5.2. If $C \subset X$ is a curve, then $f_*\{C\}$ is the current of integration on the curve $f(C)$. If α is a $(1,1)$-form, then $f^*\{\alpha\} = \{f^*\alpha\}$.

5.2. **The currents T_f^+ and T_f^- and the probability measure μ_f.**

Theorem 5.3 (see [34, 57, 59])**.** *Let f be a loxodromic automorphism of a compact Kähler surface X. There is a unique closed positive current T_f^+ such that $[T_f^+] = v_f^+$. The local potentials of T_f^+ are Hölder continuous,*

$$f^*T_f^+ = \lambda(f)T_f^+,$$

and $\mathbf{R}^+T_f^+$ is an extremal ray in the convex cone of closed positive currents.

The extremality means that a convex combination $sT + (1-s)T'$ of two closed positive currents T and T' is proportional to T_f^+ if and only if both T and T' are proportional to T_f^+.

Applied to f^{-1}, this result shows that there is a unique closed positive current T_f^- such that $[T_f^-] = v_f^-$. This current has Hölder continuous potentials, satisfies

$$f^*T_f^- = \frac{1}{\lambda(f)}T_f^-,$$

and the ray $\mathbf{R}^+T_f^-$ is also extremal.

Corollary 5.4. *Let f be a loxodromic automorphism of a compact Kähler surface X. Let $C \subset X$ be a curve, and $\{C\}$ be the current of integration on C. Then*

$$\frac{1}{\lambda(f)^n}(f^n)^*\{C\} \to \langle [C]|v_f^-\rangle\, T_f^+$$

as n goes to $+\infty$. Let κ be a Kähler form on X, and $\{\kappa\}$ the current determined by this form. Then

$$\frac{1}{\lambda(f)^n}(f^n)^*\kappa \to \langle [\kappa]|v_f^-\rangle\, T_f^+$$

as n goes to $+\infty$.

Proof of Corollary 5.4. Let C be a curve, and $[C]$ be its cohomology class. Decompose $[C]$ as

$$[C] = [C]_+ + [C]_- + [C]_N$$

where $[C]_\pm$ is contained in $\mathbf{R}v_f^\pm$ and $[C]_N$ is in the orthogonal complement N_f. Since $\langle [C] | v_f^+ \rangle = \langle [C_-] | v_f^+ \rangle$ and $\langle v_f^+ | v_f^- \rangle = 1$, we have $[C_+] = \langle [C] | v_f^- \rangle v_f^+$. When n goes to $+\infty$, the sequence $(f^*)^n [C]_N$ is bounded and $(f^n)^*[C]_-$ goes to 0. Thus,

$$\frac{1}{\lambda(f)^n}(f^n)^*[C] \to \langle [C] | v_f^- \rangle v_f^+.$$

In particular, the sequence of currents $(f^n)^*\{C\}/\lambda(f)^n$ has bounded mass, and all limits of convergent subsequences are currents with cohomology class $\langle [C] | v_f^- \rangle v_f^+$. Since $\langle [C] | v_f^- \rangle T_f^+$ is the unique closed positive current with cohomology class $\langle [C] | v_f^- \rangle v_f^+$, the sequence $(f^*)^n\{C\}/\lambda(f)^n$ converges towards $\langle [C] | v_f^- \rangle T_f^+$.

The same proof applies to Kähler forms κ. □

5.2.1. *The measure μ_f.* Since T_f^+ and T_f^- have continuous potentials, we can multiply them: This defines a probability measure

$$\mu_f = T_f^+ \wedge T_f^-;$$

the total mass of μ_f is 1 because our choice for the cohomology classes v_f^+ and v_f^- implies

$$\langle [T_f^+] | [T_f^-] \rangle = \langle v_f^+ | v_f^- \rangle = 1.$$

The probability measure μ_f is f-invariant because T_f^+ is multiplied by $\lambda(f)$ while T_f^- is divided by the same quantity.

Note that μ_f is uniquely determined by the diffeomorphism f and the f-invariant complex structure on X: Both T_f^+ and T_f^- are uniquely determined by the equation $f^* T_f^\pm = \lambda(f)^\pm T_f^\pm$ up to scalar multiplication, so that the product $\mu_f = T_f^+ \wedge T_f^-$ is uniquely determined once one imposes $\mu_f(X) = 1$.

Remark 5.5. Since T_f^+ and T_f^- have Hölder continuous potentials, one can show that the Hausdorff dimension of μ is strictly positive. We refer to [67] for a discussion of this topic for endomorphisms of projective spaces.

5.3. **The measure μ_f is mixing.** The following statement is deeper than Corollary 5.4. It applies, for example, when S is the current of integration over a disk Δ contained in a stable manifold of f (see §6.1.2 below).

Theorem 5.6 (Bedford-Smillie, Fornaess-Sibony, see [11, 34, 70])**.** *Let f be a loxodromic automorphism of a connected compact Kähler surface X. Let S be a positive current and $\psi\colon X \to \mathbf{R}_+$ be a smooth function which vanishes in a neighborhood of the support of ∂S. Then*

$$\frac{1}{\lambda(f)^n}(f^n)^*(\psi S)$$

converges towards

$$(T_f^-|\psi S)T_f^+$$

in the weak topology as n goes to $+\infty$.

The number $(T_f^-|\psi S)$ is the total mass of the positive measure $T_f^- \wedge (\psi S)$. One drawback of this statement resides in the difficulty to decide whether $T_f^- \wedge (\psi S)$ is not zero, but there is at least one interesting and easily accessible corollary.

Corollary 5.7 (see [11, 34, 70])**.** *If f is a loxodromic automorphism of a connected compact Kähler surface, the measure μ_f is mixing (hence ergodic).*

Ergodicity means that all f-invariant measurable subsets have measure 0 or 1. The mixing property is stronger, and says that $\mu_f(f^n(A) \cap B)$ converges towards $\mu_f(A)\mu_f(B)$ as n goes to ∞ for all pairs (A,B) of measurable subsets of X. Equivalently, μ_f is mixing if and only if

$$\int_X (\phi \circ f^n)\,\psi\, d\mu_f \to \int_X \phi\, d\mu_f \int_X \psi\, d\mu_f$$

for all pairs of smooth (resp. smooth and non-negative) functions (ϕ,ψ) on X.

To prove the Corollary, start with two smooth functions ϕ and ψ with non-negative values. By definition of μ_f we have

$$\begin{aligned}
\int_X (\phi \circ f^n)\,\psi\, d\mu_f &= (T_f^+ \wedge T_f^- \,|\, (\phi \circ f^n)\psi) \\
&= ((\phi T_f^+) \wedge T_f^- \,|\, (\psi \circ f^{-n})) \\
&= \left(\frac{1}{\lambda(f)^n}(f^n)^*(\phi T_f^+) \wedge T_f^- \,|\, \psi\right)
\end{aligned}$$

because $f^*T_f^- = \lambda(f)^{-1}T_f^-$. From Theorem 5.6, we obtain

$$\frac{1}{\lambda(f)^n}(f^n)^*(\phi T_f^+) \to cT_f^+, \quad \text{with } c = \int_X \phi \, d\mu_f.$$

Since products of currents are compatible with weak convergence, the sequence $\int_X (\phi \circ f^n)\psi \, d\mu_f$ converges towards

$$c\,(T_f^+ \wedge T_f^- \,|\, \psi) = \int_X \phi \, d\mu_f \int_X \psi \, d\mu_f,$$

as desired.

6. Entire curves, stable manifolds, and laminarity

6.1. **Entire curves and stable manifolds.**

6.1.1. *Entire curves.* By definition, an **entire curve** on X is a non-constant holomorphic map $\xi : \mathbf{C} \to X$. Fix such a curve, and a Kähler form κ on X. For each real number $r \geq 0$, denote by $\mathbb{D}_r \subset \mathbf{C}$ the open disk of radius r, and denote by $A(r)$ and $L(r)$ the area and perimeter of $\xi(\mathbb{D}_r)$:

$$A(r) = \int_{t=0}^{r}\int_{\theta=0}^{2\pi} \| \xi'(te^{i\theta}) \|_\kappa^2 \, t dt d\theta \tag{6.1}$$

$$L(r) = \int_{\theta=0}^{2\pi} \| \xi'(te^{i\theta}) \|_\kappa \, t d\theta, \tag{6.2}$$

where $\| \xi'(te^{i\theta}) \|_\kappa$ is the norm of the velocity vector $\xi'(te^{i\theta})$ with respect to the Kähler metric defined by κ. By Cauchy-Schwartz inequality, one gets Ahlfors inequality

$$L(r)^2 \leq 2\pi r \frac{dA}{dr}(r).$$

It implies that the infimum limit of the ratio $L(r)/A(r)$ vanishes. As a consequence, there are sequences of radii (r_n), going to $+\infty$ with n, such that

$$\frac{1}{A(r_n)}\{\xi(\mathbb{D}_{r_n})\}$$

converges toward a closed positive current.

Another useful family of currents is defined by

$$N(r) = \frac{1}{T(r)}\int_0^r \{\xi(\mathbb{D}_t)\}\frac{dt}{t}$$

with

$$T(r) = \int_0^r A(t)\frac{dt}{t}.$$

Then, Ahlfors and Jensen inequalities, together with Nakai-Moishezon Theorem, imply the following result.

Theorem 6.1 (Positivity of Ahlfors currents, see [22, 34, 47, 106])**.** *Let X be a compact Kähler surface with a Kähler form κ. Let $\xi : \mathbf{C} \to X$ be an entire curve. There exist sequences of radii (r_n) going to ∞ such that $(N(r_n))$ converges towards a closed positive current. Let T be such a current.*

(1) *If $\xi(\mathbf{C})$ is contained in a compact curve E, its normalization has genus* 0 *or* 1 *(E may be singular), and T is equal to the current $\langle \kappa | [E] \rangle^{-1}\{E\}$.*
(2) *If $A(r)$ is bounded, then $\xi(\mathbf{C})$ is contained in a rational curve E.*

If $\xi(\mathbf{C})$ is not contained in a compact curve, then

(3) *$[T]$ intersects all classes of curves non-negatively, i.e. $\langle [T] | [C] \rangle \geq 0$ for all curves $C \subset X$;*
(4) *$[T]$ is in the nef cone and $\langle [T] \,|\, [T] \rangle \geq 0$.*

6.1.2. *Stable manifolds.* Let now p be a saddle periodic point of the loxodromic automorphism f, and let k be its period (there is an infinite number of such points by Theorem 4.12). Locally, f^k is continuously linearizable: There is an open subset U of X containing p, and a local homeomorphism $\psi : U \to V$ onto a ball $V \subset \mathbf{C}^2$ such that $\psi(p) = (0,0)$ and

$$\psi \circ (f^k) \circ \psi^{-1}(x,y) = (\alpha x, \beta y)$$

where α, β are the eigenvalues of $D(f^k)_p$, $|\alpha| < 1$ and $|\beta| > 1$. Thus, locally, the set of points q near p such that $f^{kn}(q)$ stays in U and converges towards p as n goes to $+\infty$ is the image of the horizontal axis by ψ^{-1}. This set is the **local stable manifold** $W^s_{loc}(p)$ of p.

The **stable manifold** $W^s(p)$ is the set of points q in X such that $f^{kn}(q)$ converges towards p as n goes to $+\infty$. This set coincides with the increasing union of all $f^{-kn}(W^s_{loc}(p))$, $n \geq 1$. Unstable and local unstable manifolds are defined similarly.

By the Stable Manifold Theorem, $W^s_{loc}(p)$ is a smooth holomorphic curve which is tangent to the eigenspace with eigenvalue α at p. Hence, every stable manifold $W^s(p)$ is the holomorphic image of a Riemann surface which is homeomorphic to the plane $\mathbf{R}^2$. By construction, f induces an automorphism of this Riemann surface which fixes p and acts as a contraction around p. This implies that $W^s(p)$ is not isomorphic to the unit disk, because all automorphisms of $\mathbb{D}$ are isometries with respect to the Poincaré metric. From Riemann uniformization Theorem, we deduce that $W^s(p)$ is parametrized by an entire curve $\xi_p^s : \mathbf{C} \to X$ such that ξ_p^s is injective, $\xi_p^s(0) = p$ and $\xi_p^s(\mathbf{C}) = W^s(p)$. The map ξ_p^s is unique up to composition with similitudes $z \mapsto \gamma z$. This implies that

$$f^k \circ \xi_p^s(z) = \xi_p^s(\alpha z).$$

Theorem 6.2 (see [10, 11] and [34]). *Let f be a loxodromic automorphism of a compact Kähler surface X. Let p be a saddle periodic point of f of period k, and $\xi_p^s : \mathbf{C} \to X$ be a parametrization of its stable manifold. If $\xi_p^s(\mathbf{C})$ has finite area, it is contained in a periodic rational curve of period k. If $\xi_p^s(\mathbf{C})$ has infinite area, then all closed limits of sequences $N_{\xi_p^s}(r_n)$ coincide with a positive multiple of T_f^+.*

An important feature of this Theorem is that T_f^+ can be recovered from every saddle periodic point (once periodic curves have been contracted). This will lead to a strong relationship between saddle periodic points and the invariant probability measure μ_f.

Remark 6.3. a.– Let $\varphi : \mathbb{D} \to \mathbf{R}^+$ be a smooth function with compact support such that $\varphi(0) > 0$. Denote by $\{(\xi_p^s)_*(\varphi\mathbb{D})\}$ the current that is defined by

$$(\{(\xi_p^s)_*(\varphi\mathbb{D})\}|\omega) = \int_{\mathbb{D}} \varphi\, (\xi_p^s)^*\omega.$$

Theorem 5.6 implies that the sequence

$$\frac{1}{\lambda(f)^{kn}} (f^{kn})^* \{(\xi_p^s)_*(\varphi\mathbb{D})\}$$

converges towards a multiple of T_f^+; by Theorem 6.2, the limit can not be zero, and is therefore a positive multiple of T_f^+. Similarly, T_f^- is an Ahlfors current for unstable manifolds of saddle periodic points of f. Since $T_f^+ \wedge T_f^-$ does not vanish, this suggest that the stable and unstable manifolds of saddle periodic points p and q always intersect, except when p or q is contained in a periodic curve; this fact is proved in Section 6.4.2.

b.– One can also deduce the following asymptotic behavior from Theorem 6.2, in which $A(r)$ denotes the area of $\xi_p^s(\mathbb{D}_r)$,

$$\limsup_{r\to\infty} \frac{\log(A(r))}{\log(r)} = \frac{\log(\lambda(f))}{|\log(\alpha^{1/k})|}.$$

This means that the growth rate of ξ_p^s is the ratio between the topological entropy of f and the Lyapunov exponent of f at p (see [35]).

Proof of the first assertion of Theorem 6.2. Replacing f by f^k, we assume that p is a fixed point. Let T be a closed current for which exists a sequence of radii (r_n) such that $N_{\xi_p^s}(r_n)$ converges towards T. Since $f^*\{\xi_p^s(\mathbb{D}_t)\} = \{\mathbb{D}_{t/\alpha}\} \geq \{\mathbb{D}_t\}$ for all $t \geq 0$, we get

$$f^* N_{\xi_p^s}(r_n) = \frac{1}{T_{\xi_p^s}(r_n)} \int_{t=0}^{r_n} f^*\{\xi_p^s(\mathbb{D}_t)\} \frac{dt}{t} \geq N_{\xi_p^s}(r_n)$$

and $f^*T \geq T$. This implies that

$$\langle (f^m)^*[T] | [\kappa] \rangle \geq \langle [T] | [\kappa] \rangle$$

for all Kähler classes $[\kappa]$ and all positive integers m. Since f^* preserves intersections

$$\frac{1}{\lambda(f)^m} \langle [T] | [\kappa] \rangle = \langle (f^m)^*[T] | \frac{1}{\lambda(f)^m} (f^m)^*[\kappa] \rangle \geq \langle [T] | \frac{1}{\lambda(f)^m} (f^m)^*[\kappa] \rangle$$

and, taking limits on both sides, $0 \geq \langle [T] | [T_f^+] \rangle$. Moreover, Theorem 6.1-(4) implies $\langle [T] | [T] \rangle \geq 0$. By Hodge index Theorem, $[T]$ is proportional to $[T_f^+]$ and, by Theorem 5.3, T is a positive multiple of T_f^+. □

6.2. **Laminarity.** Since T_f^+ is obtained as Ahlfors currents for stable manifolds, *i.e.* for injective entire curves $\xi_p^s \colon \mathbf{C} \to X$, one expects that the structure of T_f^+ retains some information from the injectivity of the entire curves ξ_p^s (see Figure 1, right, and the close-up in Figure 5). This leads to the theory of laminar currents.

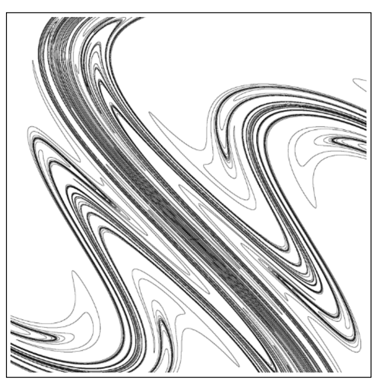

FIGURE 5. Laminar structure

6.2.1. *Uniformly laminar currents.* Let Γ be a family of disjoint horizontal graphs in the bidisk $\mathbb{D}\times\mathbb{D}$: Each element of Γ intersects the vertical disk $\{0\}\times\mathbb{D}$ in a unique point; if a is such a point of intersection and Γ_a is the graph of the family Γ containing a, there exists a holomorphic mapping $\varphi_a : \mathbb{D}\to\mathbb{D}$ such that $\Gamma_a = \{(x,\varphi_a(x));\, x\in\mathbb{D}\}$.

Remark 6.4. Let A be the set of intersection-points a of the graphs $\Gamma_a \in \Gamma$ with the vertical disk $\{0\}\times\mathbb{D}$. Then, Γ determines a holomorphic motion of the set A parametrized by the disk $\mathbb{D}$ (see [61]): A moves along the graphs from its initial position in $\{0\}\times\mathbb{D}$ to nearby disks $\{z\}\times\mathbb{D}$ by $a=\varphi_a(0)\in A\mapsto\varphi_a(z)$. By the so-called Λ-Lemma, (i) this motion extends to a motion of its closure $\overline{A}$ and (ii) the motion from $\{0\}\times\mathbb{D}$ to $\{z\}\times\mathbb{D}$ is Hölder continuous.

FIGURE 6. Uniform lamination. A family of horizontal graphs in a bidisk.

Given a finite positive measure ν on $\{0\}\times\mathbb{D}$ (or on $\overline{A}$), one obtains a measured family of disjoint graphs (Γ,ν); it determines a closed positive current $T_{\Gamma,\nu}$ in $\mathbb{D}\times\mathbb{D}$, which is defined by

$$(T_{\Gamma,\nu}|\omega) = \int_{a\in\mathbb{D}}\int_{\Gamma_a}\omega\, d\nu(a) = \int_{a\in\mathbb{D}}\int_{\mathbb{D}}\varphi_a^*\omega\, d\nu(a).$$

One says that $T_{\Gamma,\nu}$ is the current of integration over (Γ,ν).

By definition, a **flow box** Γ of a complex surface Z is a closed set of disjoint horizontal graphs in a bidisk $\mathbb{D}\times\mathbb{D}\simeq U\subset Z$; a **measured flow box** is a flow box Γ, together with a transverse measure ν. Thus, every measured flow box (Γ,ν) defines a current of integration $T_{|\Gamma,\nu}$ on U. A current T on the surface Z is **uniformly laminar** if T is locally given by integration over a measured flow box. If T is uniformly laminar, those flow boxes can be glued together to define a lamination of the support of T.

Example 6.5. Let $X = \mathbf{C}^2/\Lambda$ be a complex torus. Let α be a non-zero holomorphic 1-form on X; such a form is induced by a constant 1-form $adx + bdy$ on $\mathbf{C}^2$, and the kernel of α determines a holomorphic foliation $\mathcal{F}_\alpha$ on X, whose leaves are projections of parallel lines $ax + by = \mathsf{C}^{\mathsf{ste}}$. Let T be the current $\{\alpha \wedge \overline{\alpha}\}$. Then T is uniformly laminar: Locally, T is given by integration over disks in the leaves of the foliation $\mathcal{F}_\alpha$ with respect to Lebesgue measure on the transversal.

6.2.2. *Laminar currents (see* [10, 34, 66]*).* A positive current T on a complex surface Z is **laminar** if there is an increasing sequence of open subsets $\Omega_i \subset Z$, $i \in \mathbf{N}$, and an increasing sequence of currents T_i supported on Ω_i such that

(i) for each i, $\parallel T \parallel (\partial \Omega_i) = 0$, *i.e.* the boundary of Ω_i does not support any mass of T;
(ii) each T_i is uniformly laminar in its domain of definition Ω_i;
(iii) the sequence of currents $(T_i)_{i \geq 1}$ weakly converges towards T.

Equivalently, T is laminar if there is a family of disjoint measured flow boxes (Γ_i, ν_i) such that

$$T = \sum_i T_{|\Gamma, \nu_i}.$$

Thus, every laminar current has a representation

$$T = \int_{\mathcal{A}} \{\Delta_a\} \, d\mu(a) \tag{6.3}$$

as a current of integration over a measured family of disjoint disks (each disk Δ_a is the image of $\mathbb{D}$ by an injective holomorphic map $\varphi_a : \mathbb{D} \to \mathbf{C}$ which extends to a neighborhood of $\mathbb{D}$ in $\mathbf{C}$). In general, those disks Δ_a can not be glued together into a lamination, as the following example shows.

Example 6.6. Let p be a point of the projective plane $\mathbb{P}^2(\mathbf{C})$. Identify the set of lines through p with the projective line $\mathbb{P}^1(\mathbf{C})$; each line L_x, $x \in \mathbb{P}^1(\mathbf{C})$, determines a current of integration $\{L_x\}$. Let ν_p be a probability measure on this set of lines. Then

$$T_p = \int_{\mathbb{P}^1(\mathbf{C})} \{L_x\} \, d\nu_p(x)$$

is a laminar current. Let $q \in \mathbb{P}^2(\mathbf{C})$ be a second point and ν_q be a probability measure on the space of lines through q. This provides a second laminar current T_q. Suppose that (i) the supports of ν_p and ν_q have zero Lebesgue measure, but (ii) both ν_p and ν_q are given by continuous potentials (*i.e.* $d\nu =$

$dd^c(u)$ locally on $\mathbb{P}^1(\mathbf{C})$, with u continuous). Then $T_p + T_q$ is laminar, has continuous potentials, and $T_p \wedge T_q$ is a well-defined positive (non-zero) measure (see [63] for details). This example shows that the constitutive disks of $T_p + T_q$ can not be glued together to form a lamination of its support.

6.3. The current T_f^+ is laminar.

Theorem 6.7 (de Thélin, [44])**.** *Let Ω be a bounded open subset of $\mathbf{C}^2$. Let (C_n) be a sequence of curves, defined in neighborhoods of Ω, and let (d_n) be a sequence of positive real numbers such that $(1/d_n)\{C_n\}$ converges towards a closed positive current T on Ω. If $\mathsf{genus}(C_n) \leq \mathsf{C}^{\mathsf{ste}} d_n$, the current T is laminar.*

Remark 6.8. Let κ be a Kähler form on $\mathbf{C}^2$, and let $Area(C_n)$ denote the area of the curve C_n with respect to κ. If T is not zero, then

$$\frac{Area(C_n)}{d_n} = \frac{\langle \{C_n\} | \kappa \rangle}{d_n} \to \langle T | \kappa \rangle$$

so that d_n is asymptotically proportional to, and can be replaced by, $Area(C_n)$.

Being laminar is a local property. To explain how the proof of Theorem 6.7 starts, one can therefore choose a linear projection $\pi \colon \mathbf{C}^2 \to \mathbf{C}$ and assume that Ω is a bidisk $\mathbb{D} \times \mathbb{D}$ on which π coincides with the projection on the first factor. Let $r_m = 2^{-m}$. For each $m \geq 1$, tesselate the complex line $\mathbf{C}$ into the open squares of size r_m defined by

$$Q_m(i,j) = r_m Q_0 + r_m(i,j), \quad (i,j) \in \mathbf{Z}^2,$$

where $Q_0 = \{(x,y) \in \mathbf{R}^2 = \mathbf{C};\, |x| < 1, |y| < 1\}$. This induces a tessellation $\mathcal{Q}_m$ of the unit disk $\mathbb{D}$ into pieces $Q_m(i,j) \cap \mathbb{D}$. Fix an index n and consider the curve C_n. For each element Q of $\mathcal{Q}_m$, organize the connected components D of $\pi^{-1}(Q) \cap C_n$ in two families:

- D is a good component if $\pi \colon D \to Q$ is an isomorphism,
- D is a bad component otherwise (eg. $\pi : D \to Q$ has degree > 1).

Denote by $\mathcal{G}_{n,m}$ the set of all good components when Q runs over all tiles of the tessellation $\mathcal{Q}_m$, and define the currents

$$\{\mathcal{G}_{n,m}\} = \sum_{D \in \mathcal{G}_{n,m}} \{D\}.$$

By construction, each $\{\mathcal{G}_{n,m}\}/d_n$ is uniformly laminar on an open subset Ω_m of Ω (Ω_m is the union of the $\pi^{-1}(Q_m(i,j) \cap \mathbb{D})$) . Moreover, when m is fixed,

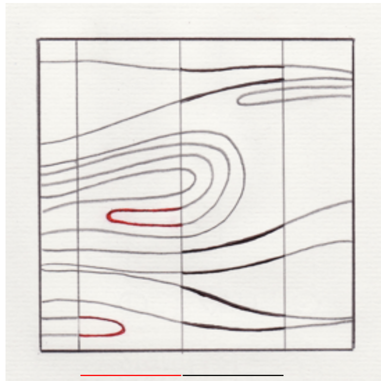

FIGURE 7. Good and bad components. Two bad components drawn above the center left tile (red) and five good components above the center right tile (black).

these currents are laminar with respect to the same bidisks and projections; this implies that the limit of these currents as n goes to infinity, m being fixed, is uniformly laminar on Ω_m. For all (n,m) the total mass of $\{\mathcal{G}_{n,m}\}$ is bounded from above by d_n. If the total area of bad components becomes small with respect to d_n when n and m become large, this will show that T is laminar. The control of bad components is precisely what de Thélin obtains, using ideas from Ahlfors and Nevanlinna theory, under the assumption $\mathsf{genus}(C_n) = O(d_n)$ and for a generic projection π (see [44]).

In our context, one can apply this strategy to

$$C_n = f^{-n}(C), \quad d_n = \lambda(f)^n, \quad T = T_f^+,$$

where C is as in Corollary 5.4. For those examples, and with carefully chosen bidisks $\Omega \subset X$ and projections $\pi : \Omega \to \mathbb{D}$, one obtains the estimate (see [62])

$$0 \leq (T - \frac{1}{d_n}\{\mathcal{G}_{n,m}\}|\kappa) \leq \mathsf{C}^{\mathsf{ste}} r_m^2.$$

Hence, T is laminar, with an explicit rate of convergence of order $O(r_m^2)$ in the proof. Such currents are said to be strongly approximated by algebraic curves (see [63] and [66], Prop. 5.1); if T is such a current – *i.e.* closed, positive, laminar and strongly approximated by algebraic curves – and T does not charge any analytic set, we say that T is a **good laminar current**.

Theorem 6.9 (Bedford-Lyubich-Smillie, see [3, 34, 62])**.** *Let f be a loxodromic automorphism of a compact Kähler surface X. Both T_f^+ and T_f^- are laminar, and are good laminar currents when X is projective.*

Note, however, that the currents T_f^+ and T_f^- are rarely uniformly laminar, as Figure 5 suggests (see [34], §7, and [38]).

Remark 6.10. Assume that X contains a curve C such that $\lambda(f)^{-n}(f^n)^*\{C\}$ converges towards T_f^+. The Néron-Severi group $NS(X,\mathbf{R})$ contains the classes $(f^n)^*[C]$, and therefore $[T_f^+]$ as well; it contains also $[T_f^-]$. Since $[T_f^+]+[T_f^-]$ has positive self-intersection, this implies that $NS(X)$ contains classes with positive self-intersection, so that X is projective. This is the reason why X is assumed to be projective in the last part of Theorem 6.9.

6.4. **Good laminar currents, and contraction properties for $T_f^\pm$.** When a current is laminar, its building disks may intersect transversally with positive probability, as in Example 6.6. Good laminar currents inherit a stronger laminar structure (see [62, 63, 64, 65]).

6.4.1. *Analytic continuation and weak lamination.* Let T be a laminar current. One says that a disk $\Delta = \varphi(\mathbb{D})$ is **subordinate** to T if there is an open set $\Omega \subset X$ containing Δ and a uniformly laminar current S on Ω such that $S \leq T$, Δ is contained in the support of S, and Δ lies inside one of the leaves of the lamination associated to S.

If Γ is a flow box, one defines $T_{|\Gamma}$ by Formula (6.3) but restricted to disks Δ_a contained in disks of Γ.

For good laminar currents T, Dujardin proved that

(1) If two disks Δ_1 and Δ_2 are subordinate to T, they are **compatible**: Their intersection is an open subset of Δ_1 and Δ_2;
(2) If Γ is any flow box, the restriction $T_{|\Gamma}$ is uniformly laminar: There is a measure ν_T such that $T_{|\Gamma}$ is the current of integration over the measured family of graphs (Γ, ν_T).

As a corollary, if Δ is subordinate to T, then all disks contained in the analytic continuation of Δ are subordinate to T. Thus, disks subordinate to T, or more generally flow boxes whose constitutive disks are subordinate to T, can be glued together in a compatible way. This provides a "weak lamination" for the support of T, and T is determined by a holonomy invariant transverse measure for this weak lamination.

Remark 6.11. Consider a current T_f^+ where f is a loxodromic automorphism. Since $\mathbf{R}^+ T_f^+$ is extremal in the convex cone of closed positive currents, the transverse invariant measure for T_f^+ is ergodic: If it decomposes into a sum of two non-trivial, holonomy invariant, transverse measures ν_1 and ν_2, then ν_1 is proportional to ν_2 (see [64]).

6.4.2. *Geometric product.* The second crucial fact concerning good laminar currents is a geometric definition of their intersection. Let T_1 and T_2 be two good laminar currents with continuous potentials. The product $T_1 \wedge T_2$ has been defined in Section 5.1.5 using Bedford-Taylor technique. One may also be tempted to represent each T_i in the form

$$T_i = \int_{\mathcal{A}_i} \{\Delta_a\}\, d\mu_i(a),$$

as in Formula (6.3), and define

$$T_1 \cap T_2 = \int_{\mathcal{A}_1} \int_{\mathcal{A}_2} \{\Delta_{a_1} \cap \Delta_{a_2}\}\, d\mu_1(a_1) d\mu_2(a_2)$$

where $\{\Delta_{a_1} \cap \Delta_{a_2}\}$ is the sum of the Dirac masses on the set $\Delta_{a_1} \cap \Delta_{a_2}$ if this set is finite, and is zero otherwise. If both T_1 and T_2 have continuous potential, one may expect ([7])

$$T_1 \wedge T_2 = T_1 \cap T_2;$$

if this equality holds, one says that the product of T_1 and T_2 is geometric. Good laminar currents with continuous potentials satisfy this Formula (see [64]). The intersection of such currents is therefore a sum of geometric intersections of uniformly laminar currents in flow boxes. Since T_f^+ and T_f^- are good laminar currents with continuous potentials, the product $T_f^+ \wedge T_f^-$ is geometric. Since T_f^+ and T_f^- are Ahlfors currents with respect to stable and unstable manifolds, one obtains the following statement (see Remark 6.3, a).

Theorem 6.12 (Bedford-Lyubich-Smillie, [10])**.** *Let f be a loxodromic automorphism of a connected compact Kähler surface. Let p and q be saddle periodic points of f. Assume that the stable manifold W_p^s and the unstable manifold W_q^u are not contained in periodic algebraic curves. Then*

- *the set of transverse intersections of W_p^s and W_q^u is dense in the support of μ_f,*
- *every intersection of W_p^s and W_q^u is contained in the support of μ_f.*

[7] If $T_1 = T_2$ is the current of integration over a compact curve C with non zero self intersection, one can not hope to define the product of T_1 with T_2 in such a simple way. But good laminar currents do not charge compact curves.

6.4.3. *Contraction properties along T_f^+.* Thanks to Section 6.4.1, we now have a well behaved notion of disks subordinate to the good laminar current T_f^+. Since T_f^+ is multiplied by $\lambda(f)$ under the action of f^*, most of those constitutive disks must be contracted by f. This is precisely what Dujardin proved in [66].

To state the result, consider a flow box Γ in some bidisk $U \simeq \mathbb{D} \times \mathbb{D}$. Denote by A the intersection of Γ with the vertical disk $\{0\} \times \mathbb{D}$, as in Section 6.2.1, and by Δ_a the unique element of Γ containing $a \in A$. Restricting T_f^+ to Γ one obtains a uniformly laminar current $T_{f|\Gamma}^+$ given by a measure ν^+ on A (see §6.4.1, Property (2)):

$$T_{f|\Gamma}^+ = \int_{a\in\Gamma} \{\Delta_a\}\, d\nu^+(a).$$

Dujardin's Theorem controls the diameter of images $f^n(\Delta_a)$ (for any riemannian metric on X):

Theorem 6.13 (Dujardin, [66])**.** *Let f be a loxodromic automorphism of a connected complex projective surface X. Let Γ be a flow box and ν^+ be the transverse measure associated to $T_{f|\Gamma}^+$. For all $\varepsilon \in (0,1)$ there is a constant $C(\varepsilon)$ and a subset A_ε of measure $\nu^+(A_\varepsilon) > \nu^+(A)(1-\varepsilon)$ such that*

$$\operatorname{diam}(f^n(\Delta_a)) \leq C(\varepsilon)\frac{n}{\lambda(f)^{n/2}}, \quad \forall\, a \in A_\varepsilon.$$

7. Fatou and Julia sets

7.1. **Definition.** Let g be an automorphism of a compact complex manifold M. A point $x \in M$ is in the **Fatou set** $\mathsf{Fat}(g)$ of g if there exists an open neighborhood U of p on which the sequence $(g^n)_{n\in\mathbf{Z}}$ forms a normal family of holomorphic mappings from U to X. Taking only positive (resp. negative) iterates one can also define the forward Fatou set $\mathsf{Fat}_+(g)$ (resp. backward Fatou set $\mathsf{Fat}_-(g)$).

Remark 7.1. Let d be the dimension of M. If p is a fixed point in $\mathsf{Fat}(g)$, then (g^n) is a normal family in a neighborhood of p, so that g is locally linearizable near p: There is a germ of holomorphic diffeomorphism $\psi : (U,p) \to (\mathbf{C}^d, 0)$ and a unitary transformation D such that

$$\psi \circ g \circ \psi^{-1} = D.$$

The linear transformation D is conjugate to Dg_p by $D\psi_p$. In the other direction, if Dg_p is a unitary transformation and g is locally conjugate to Dg_p, then

p is in the Fatou set. Section 7.3 below provides examples of this type for automorphisms of projective surfaces.

Lemma 7.2. *The Fatou set* $\mathsf{Fat}(f)$ *of a loxodromic automorphism does not intersect the supports of* T^+_f *and* T^-_f.

Proof. Let U be an open subset of the Fatou set on which $(f^n)_{n\in\mathbf{Z}}$ is a normal family. Let n_i be an increasing sequence of integers along which (f^{n_i}) converges towards a holomorphic map $g : U \to X$. Write T^+_f as the limit of the sequence of currents $\lambda(f)^{-n_i}(f^{n_i})^*\kappa$ where κ is a Kähler form on X. On U, $(f^{n_i})^*\kappa$ converges towards $g^*\kappa$. Dividing by $\lambda(f)^{n_i}$, one obtains $T^+_f = 0$ on U. The same argument applies for T^-_f. □

7.2. Kobayashi hyperbolicity and pseudo-convexity.

7.2.1. *Kobayashi pseudo-distance (see* [96]*).* Let M be a complex manifold (or, more generally, a complex analytic space). A chain of disks Ψ between two points x and y is a finite family of marked disks

$$\psi_i\colon (\mathbb{D}; z_{i,1}, z_{i,2}) \to (M; x_{i,1}, x_{i,2}), \quad 1 \leq i \leq l,$$

such that

- $\psi_i(z_{i,1}) = x_{i,1}$ and $\psi_i(z_{i,2}) = x_{i,2}$;
- $x_{1,1} = x$, $x_{i,2} = x_{i+1,1}$ for all $1 \leq i \leq l-1$, and $x_{l,2} = y$.

The hyperbolic length of such a chain is

$$\mathsf{hl}(\Psi) = \sum_{i=1}^{i=l} \mathsf{dist}_{\mathbb{D}}(z_{i,1}, z_{i,2}).$$

The **Kobayashi pseudo-distance** $\mathsf{dist}_M(x,y)$ between two points x and y is the infimum of the hyperbolic length $\mathsf{hl}(\Psi)$ over all chains of disks Ψ joining x to y (see [96]). The Kobayashi pseudo-distance satisfies all axioms of a distance, except that it can take the value $+\infty$ (exactly when x and y are in two distinct connected components of M), and it may vanish for pairs of distinct points. One says that M is **Kobayashi hyperbolic** if $\mathsf{dist}_M(x,y) > 0$ for all $x \neq y$ in M.

Remark 7.3. a.– When $M = \mathbb{D}$, the Kobayahsi pseudo-distance coincides with the Poincaré metric $\mathsf{dist}_{\mathbb{D}}$. When $M = \mathbf{C}$, the Kobayashi pseudo-distance $\mathsf{dist}_{\mathbf{C}}$ vanishes identically.

b.– Holomorphic mappings between complex manifolds are distance decreasing: $\mathsf{dist}_N(f(x), f(y)) \leq \mathsf{dist}_M(x,y)$ if $f\colon M \to N$ is holomorphic. In particular, M is not hyperbolic when it contains an entire curve.

c.– When M is Kobayashi hyperbolic, the topology induced by dist_M is the same as the topology of M as a complex manifold (cf. Barth Theorem in [96]).

7.2.2. *Brody re-parametrization and hyperbolicity.* Fix a hermitian metric on the manifold M. Assume that there exists a sequence of holomorphic mappings $\psi_m \colon \mathbb{D} \to M$ such that $\| \psi_m'(0) \|$ goes to ∞ with m, where $\psi'(0)$ is the velocity vector of the curve at $z = 0$, the center of the unit disk, and $\| \psi_m'(0) \|$ is its norm with respect to the fixed hermitian metric.

Lemma 7.4 (Brody Lemma, see [96], Chap. III)**.** *There exists a sequence of real numbers r_m and automorphisms $h_m \in \mathsf{Aut}(\mathbb{D})$ such that*

(1) *r_m goes to $+\infty$ with m;*
(2) *$\varphi_m(z) = \psi_m \circ h_m(z/r_m)$ is holomorphic on $\mathbb{D}_{r_m}$, and its velocity at the origin has norm 1;*
(3) *the norm of the derivative of φ_m satisfies $\limsup_m \max_{z\in K} \| \varphi_m'(z) \| \leq 1$ for all compact subsets $K \subset \mathbf{C}$.*

Suppose now that M is compact. By (3), the sequence (φ_m) is equicontinuous. Hence, a subsequence of (φ_m) converges towards an entire curve $\varphi \colon \mathbf{C} \to M$ such that

$$\| \varphi'(0) \| = 1, \quad \text{and } \| \varphi'(z) \| \leq 1, \quad \forall z \in \mathbf{C}.$$

Such an entire curve is called a **Brody curve**. As a corollary of this construction, one gets the following Theorem.

Theorem 7.5 (Brody, see [96])**.** *Let M be a compact complex manifold, with a fixed hermitian metric $\| \cdot \|$. Then M is Kobayashi hyperbolic if and only if there is a uniform upper bound on $\| \psi'(0) \|$ for all holomorphic disks $\psi \colon \mathbb{D} \to M$, if and only if there is no Brody curve $\varphi \colon \mathbf{C} \to M$.*

7.2.3. *Fatou sets are almost hyperbolic.* We are now in a position to explain the following result.

Theorem 7.6 (Dinh-Sibony [56], Moncet [106], Ueda [119])**.** *Let f be a loxodromic automorphism of a compact Kähler surface X. The Fatou set $\mathsf{Fat}(f)$*

(1) *coincides with the complement of the supports of T_f^+ and T_f^-, i.e. $\mathsf{Fat}(f) = X \setminus \mathsf{Support}(T_f^+ + T_f^-)$,*
(2) *is Kobayashi hyperbolic modulo periodic curves,*
(3) *is pseudo-convex.*

To be more precise, the second assertion says that the Kobayashi pseudo-distance $\mathsf{dist}_{\mathsf{Fat}(f)}$ vanishes exactly along the set of algebraic periodic curves C of the following types:

- C is elliptic and contained in $\mathsf{Fat}(f)$;
- C is rational and $C \cap \mathsf{Fat}(f)$ is equal to C minus 0, 1, or 2 points.

Note, however, that I do not know of any example of a loxodromic automorphism $f : X \to X$ such that $\mathsf{Fat}(f)$ contains a smooth curve C of genus 1 (examples of Fatou sets containing rational curves are given in §7.3).

To prove this Theorem, one studies the complement

$$\Omega = X \setminus \mathsf{Support}(T_f^+ + T_f^-)$$

of the support of T_f^+ and T_f^-. This set is f-invariant and contains $\mathsf{Fat}(f)$. If it is not Kobayashi hyperbolic, there exists a Brody curve $\varphi\colon \mathbf{C} \to \overline{\Omega}$ which is obtained as limits of disks contained in $\mathsf{Fat}(f)$; this curve satisfies $\varphi^* T_f^+ = 0$ and $\varphi^* T_f^- = 0$. Let A be an Ahlfors current associated to φ, as in Section 6.1.1. Suppose that φ is not contained in a compact curve; then $\langle [A] | [A] \rangle \geq 0$, by Theorem 6.1-(4). Dinh and Sibony prove that A does not intersect T_f^+ and T_f^-, a fact which is not obvious because the supports of A, of T_f^+, and of T_f^- could very well be contained in $\overline{\Omega} \setminus \Omega$ and could have non trivial intersection. Therefore

$$\langle [A] | [T_f^+] \rangle = \langle [A] | [T_f^-] \rangle = 0,$$

By Hodge index Theorem, these equalities imply $\langle [A] | [A] \rangle < 0$, a contradiction. Thus, $\varphi(\mathbf{C})$ is contained in a compact curve C. This curve is either elliptic or rational, $A = \{C\}$, and $[A]$ does not intersect $[T_f^+]$ and $[T_f^-]$. According to Proposition 4.1, C is periodic. This shows that Ω is Kobayashi hyperbolic modulo rational or elliptic periodic curves.

7.3. **Examples.** Consider the birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given in affine and homogeneous coordinates by

$$\begin{aligned} f(x,y) &= (a+y, b+y/x) \\ f[x:y:z] &= [axz+yx : bxz+yz : zx], \end{aligned}$$

for some parameter $(a,b) \in \mathbf{C}^2$. It has three indeterminacy points, namely

$$p_1 = [1:0:0],\ p_2 = [0:1:0],\ \text{and}\ \ p_3 = [0:0:1].$$

Denote by Δ the triangle whose edges are the three coordinate axis $\{x=0\}$, $\{y=0\}$, and $\{z=0\}$. Each axis of Δ is blown down to a point by f: The first on p_2, the third on p_1, and the second axis $\{y=0\}$ on the point

$$p_4=[a:b:0].$$

Define $p_{4+m}=f^m(p_4)$, for $m\geq 1$, pick an integer $n\geq 1$, and suppose that the parameter (a,b) has been chosen in such a way that

(1) $p_j\notin\Delta$ for all $4\leq j\leq n$, (in particular, f is well defined at p_j)
(2) $p_{n+1}=p_3$.

Blowing up all points $(p_j)_{j=1}^n$, one gets a rational surface $X\to\mathbb{P}^2$ on which f lifts to a well defined automorphism $\hat{f}\colon X\to X$. As shown by Bedford and Kim and by McMullen, this leads to an infinite family of automorphisms on rational surfaces (the number of possible parameters (a,b) and the number of blow ups increase with n). The dynamical degree of $\hat{f}$ depends only on n, is equal to 1 if and only if $n\leq 9$, and is equal to the unique root $\lambda_n>1$ of the polynomial equation

$$t^{n-2}(t^3-t-1)+(t^3+t^2-1)=0$$

when $n\geq 10$. The sequence (λ_n) is increasing and converges towards the smallest Pisot number, *i.e.* the root $\lambda_P>1$ of $t^3=t+1$. When $n=10$ the dynamical degree is equal to Lehmer's number (see Section 2.4.3).

For $n=10$, there is a parameter $(a,b)=(a,\overline{a})$ with

$$a\simeq 0.04443-0.44223\sqrt{-1}$$

that satisfies Properties (1) and (2) above. The dynamical degree of the corresponding automorphism $\hat{f}$ is equal to Lehmer's number. This automorphism has a fixed point q such that the tangent map $D\hat{f}_q$ is (conjugate to) a unitary transformation with eigenvalues α and β. Moreover, α and β are algebraic numbers, and are multiplicatively independent. Results from Diophantine approximation imply that products like $\alpha^k\beta^l$ are not well approximated by roots of 1, so that Siegel's linearization Theorem can be applied (see [101, 102]): In a neighborhood of the fixed point q, $\hat{f}$ is conjugate to its linear part $D\hat{f}_q$; this neighborhood is therefore contained in the Fatou set of $\hat{f}$.

Blowing up q, one obtains example of loxodromic automorphisms with invariant rational curves in the Fatou set. In [9], a similar construction leads to examples of Fatou components that contain several fixed points and invariant rational curves (moreover, when these curves are blown-down, the singularity is not a quotient singularity).

7.4. **Julia sets.** There are three Julia sets for each loxodromic automorphism f. The forward Julia set $\mathsf{J}^+(f)$ is the complement of the forward Fatou set $\mathsf{Fat}^+(f)$. The backward Julia set $\mathsf{J}^-(f)$ is the complement of $\mathsf{Fat}^-(f)$. The Julia set $\mathsf{J}(f)$ is the intersection $\mathsf{J}^+(f) \cap \mathsf{J}^-(f)$. The support of T_f^+ (resp. T_f^-) is contained in $J^-(f)$ (resp. $J^+(f)$); the support of μ_f is contained in $\mathsf{J}(f)$.

In the previous paragraph, examples of loxodromic automorphisms with non-empty Fatou set have been described. One can construct examples for which the Lebesgue measure of $\mathsf{J}^+(f)$ vanishes and the forward orbit of every point in $\mathsf{Fat}^+(f)$ goes to an attracting fixed points (see [102]). Beside these examples, not much is known. For instance, the following questions remain open. Does there exist a loxodromic automorphism of a projective K3 surface – for example a smooth surface of degree $(2,2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ – with non-empty Fatou set ? Does there exist a loxodromic automorphism f of a K3 surface for which μ_f is singular with respect to Lebesgue measure ? (Based on Figure 1, one may expect a positive answer to the second question). We refer to [38] for other open problems of this type.

8. The measure of maximal entropy and periodic points

In this section, two important characterizations of the measure μ_f are described. Both of them show that μ_f, a measure which is uniquely determined by the f-invariant complex structure on the manifold X, is also uniquely determined by its dynamical properties.

8.1. **Entropy, Pesin's Theory, and laminarity.**

8.1.1. *Entropy of invariant measures.* Let $(Z, \mathcal{T}, \mu)$ be a probability space, with σ-algebra $\mathcal{T}$ and probability measure μ. Let g be a measure preserving transformation of $(Z, \mathcal{T}, \mu)$. Let $\mathcal{P} = \{P_i ; 1 \leq i \leq l\}$ be a partition of Z into a finite number of measurable subsets P_i: The P_i are disjoint, have positive measure, and cover a subset of full measure in Z. One defines the entropy of $\mathcal{P}$ with respect to μ by

$$\mathsf{h}(\mathcal{P}, \mu) = -\sum_i \mu(P_i) \log(\mu(P_i)).$$

By pull back, g transforms $\mathcal{P}$ into a new partition $g^*\mathcal{P}$ of Z, whose elements are the subsets $g^{-1}(P_i)$. Iterating, we get a sequence of partitions $g^{-k}(\mathcal{P})$, and we denote by

$$\mathcal{P}_n = \mathcal{P} \vee g^{-1}(\mathcal{P}) \vee \ldots \vee g^{-n+1}(\mathcal{P})$$

the partition generated by the first n elements of this sequence. The entropy of g with respect to μ is then defined as the supremum

$$\mathsf{h}(g,\mu) = \sup_{\mathcal{P}} \left\{ \limsup_{n} \frac{1}{n} \mathsf{h}(\mathcal{P}_n, \mu) \right\}$$

over all measurable partitions of Z in a finite number of pieces (see [91]).

Assume, now, that g is a continuous transformation of a compact space Z. Let $\mathcal{T}$ be the σ-algebra of Borel subsets of Z. Then g has at least one invariant probability measure μ on $(Z, \mathcal{T})$. By the so called variational principle [91], it turns out that the supremum of $\mathsf{h}(g,\mu)$ over all g-invariant probability measures is equal to the topological entropy of g:

$$\mathsf{h}_{top}(g) = \sup_{\mu} \mathsf{h}(g,\mu).$$

Newhouse's Theorem asserts that this supremum is a maximum when g is a diffeomorphism of class $\mathcal{C}^\infty$ on a compact manifold (see [109]).

8.1.2. *Pesin Theory (see* [91]*).* Let us come back to the study of a loxodromic automorphism $f : X \to X$. Since f has positive topological entropy, there exist invariant, ergodic, probability measures ν with $\mathsf{h}(f;\nu) > 0$. The Lyapunov exponents of f with respect to such a measure ν are defined point-wise by

$$\begin{aligned} \chi^+(x) &= \limsup_{n \to +\infty} \frac{1}{n} \log \parallel D(f^n)_x \parallel \\ \chi^-(x) &= \limsup_{n \to -\infty} \frac{1}{n} \log \parallel D(f^{-n})_x \parallel . \end{aligned}$$

Since ν is ergodic, both χ^+ and χ^- are constant on a set of full measure. Ruelle's inequality implies that χ^+ is positive and χ^- is negative (both count with multiplicity two because f preserves the complex structure).

By Osseledet's Theorem, the tangent space T_xX splits ν-almost everywhere into the direct sum $E^u(x) \oplus E^s(x)$ of two lines such that $f_*E^s(x) = E^s(f(x))$, the derivative of f^n along $E^s(x)$ decreases as $\exp(-n\chi^-(x))$ with n, and the lines $E^u(x)$ satisfy similar properties for f^{-1}. By Pesin's Theory, there are stable and unstable manifolds

$$\xi_x^{s/u} : \mathbf{C} \to X$$

through ν-almost every point x. The image of the stable manifold ξ_x^s is the set of points y in X such that the distance between $f^n(x)$ and $f^n(y)$ goes to 0 with n. It is tangent to $E^s(x)$ at x; f maps ξ_x^s to $\xi_{f(x)}^s$ (with a different parametrization).

One can show (see [34]) that the sequence of currents $\lambda(f)^{-n}(f^n)^*\{\xi^s_*(\psi\mathbb{D})\}$ converges towards a *positive* multiple of T_f^+ for ν-almost every point in $\Lambda_f(\nu)$ (here ψ is any non-negative smooth function with compact support in $\mathbb{D}$ such that $\psi(0)>0$). Thus, ν-almost every point determines T_f^+ and T_f^- through its stable and unstable manifolds. Since $\mu_f = T_f^+ \wedge T_f^-$, it follows that μ_f takes every invariant measure with positive entropy "into account".

In what follows, we denote by $\Lambda_f(\nu)$ a measurable set of full measure such that every $x\in\Lambda_f(\nu)$ has non zero Lyapunov exponents and stable and unstable manifolds as above. By definition, the union Λ_f of those sets $\Lambda_f(\nu)$ where ν describes the set of invariant and ergodic probability measures with positive entropy is the set of **hyperbolic (or saddle) points**.

8.1.3. *Laminar versus dynamical structures.* Let us compare the local structure of the dynamics of f to the local structure of the currents T_f^+ and T_f^-, as given by the flow boxes from §6.4.1.

Let $U\subset X$ be a bidisk, $U\simeq\mathbb{D}\times\mathbb{D}$. Let $x\in\Lambda_f\cap U$ be a hyperbolic point. The connected component of $\xi^s_x(\mathbf{C})\cap U$ that contains x is the **local stable manifold** of x in U, and is denoted by $W^s_{loc}(x)$. A similar definition applies for local unstable manifolds. A **Pesin box** (U,K) is a pair of a bidisk $U\simeq\mathbb{D}\times\mathbb{D}$ in X, and a compact subset K of U such that

- every point x in K is a hyperbolic point and its local stable and unstable manifolds are horizontal and vertical graphs in U.
- for all pairs of distinct points (x,y) in K, $W^s_{loc}(x)\cap W^u_{loc}(y)$ is a singleton and is contained in K.

In particular, the local stable (resp. unstable) manifolds determine a lamination K^s (resp. K^u) in U. From the second property, one may identify K to the product of the transversal $A^u = K\cap W^u_{loc}(x)$ and $A^s = K\cap W^s_{loc}(y)$ (for all pairs $(x,y)\in K^2$).

Let (U,K) be a Pesin box. Restricting T_f^+ to the lamination K^s, as in Section 6.4.1, one obtains a current of integration $T^+_{f|K^s}$ on K^s with respect to a transverse measure μ_K^+. Similarly, T_f^- determines a transverse measure μ_K^- for K^u. Since T_f^+ and T_f^- are Ahlfors currents with respect to stable and unstable manifolds and the constitutive disks of $T_f^\pm$ are contracted by $f^\pm$ (see Theorem 6.13), it can be shown that (see [10, 34, 66])

(1) Pesin boxes provide a complete family of flow boxes for T_f^+; in other words, T_f^+ is a sum of $T^+_{f|K^s}$ over a family of Pesin boxes;

(2) the restriction $\mu_{f|K}$ is equal to the product $\mu_K^+ \otimes \mu_K^-$ (on $K \simeq A^u \times A^s$);
(3) the Lyapunov exponents $\chi^+(\mu_f)$ and $\chi^-(\mu_f)$ satisfy

$$\chi^+(\mu_f) \geq \frac{1}{2}\log(\lambda(f)) > 0 > -\frac{1}{2}\log(\lambda(f)) \geq \chi^-(\mu_f).$$

The second property shows that μ_f has a product structure in Pesin boxes. This is a strong property, from which follows that *the dynamical system (X,μ,f) is measurably equivalent to a Bernouilli shift*. In other words, the dynamics of f with respect to μ_f is equivalent to the dynamics of a random coin flip.

8.2. **Two characterizations of μ_f.** This interplay between Pesin's Theory and the laminar structure of T_f^+ and T_f^- leads to two characterizations of the invariant measure μ_f.

Theorem 8.1 (see [10, 34, 66])**.** *Let f be a loxodromic automorphism of a compact Kähler surface X. The measure μ_f has maximal entropy, i.e.*

$$\mathsf{h}(f,\mu_f) = \mathsf{h}_{top}(f) = \log(\lambda(f)),$$

and is the unique invariant probability measure with maximal entropy.

Thus, the measure μ_f is the unique point on the compact convex set of f-invariant probability measures at which the entropy $\mathsf{h}(f,\cdot)$ is maximal. This is an implicit characterization of μ_f.

Theorem 8.2 (Bedford-Lyubich-Smillie [10, 3, 34])**.** *Let f be a loxodromic automorphism of a complex projective surface X. Let $\mathsf{Per}(f,k)$ be the set of isolated periodic points of f with period at most k. Then*

$$\frac{1}{\lambda(f)^k}\sum_{p\in \mathsf{Per}(f,k)} \delta_p$$

converges towards μ_f as k goes to $+\infty$. The same result holds if $\mathsf{Per}(f,k)$ is replaced by the set $\mathsf{Per}_{sad}(f,k)$ of saddle periodic points of period at most k.

If p is a saddle periodic point, either p is contained in the support of μ_f, or p is contained in a cycle of periodic rational curves.

Hence, μ_f is determined by the repartition of periodic points of f. This provides a simple characterization of μ_f, using only the simplest dynamical objects (periodic points) associated to $f\colon X \to X$.

8.3. **Application: Complex versus real dynamics.**

Corollary 8.3. *Let X be a smooth projective surface defined over the real numbers $\mathbf{R}$. Let f be a loxodromic automorphism of X defined over $\mathbf{R}$. The entropy of $f\colon X(\mathbf{R})\to X(\mathbf{R})$ is equal to the entropy of $f\colon X(\mathbf{C})\to X(\mathbf{C})$ if, and only if all saddle periodic points of f which are not are contained in rational periodic curves are contained in $X(\mathbf{R})$.*

Proof. The topological entropy of f on $X(\mathbf{R})$ is at most equal to the entropy on $X(\mathbf{C})$.

Assume that almost all saddle periodic points are contained in $X(\mathbf{R})$. Then $\mu_f(X(\mathbf{R})) = 1$, and, by the variational principle, the topological entropy of f on $X(\mathbf{R})$ is equal to the entropy on $X(\mathbf{C})$.

Assume now that the entropy of f on $X(\mathbf{R})$ is equal to the entropy on $X(\mathbf{C})$. Then, by Newhouse's Theorem and the uniqueness of the measure of maximal entropy, μ_f is supported on $X(\mathbf{R})$. Let p be a saddle periodic point of f. If p is not contained in a periodic curve, Theorem 6.12 shows that p is contained in the support of μ_f. Thus, p is contained in $X(\mathbf{R})$. □

There are examples of automorphisms of rational surfaces, defined over $\mathbf{R}$, for which the entropy on $X(\mathbf{C})$ and $X(\mathbf{R})$ coincide (see [9]). This phenomenon is not possible on tori, because all automorphisms are induced by complex affine transformations of the universal cover $\mathbf{C}^2$. It is an open problem to decide whether such examples exist on K3 surfaces. See [105, 106] for a discussion of this problem.

9. Complements

Since we focussed on the dynamics of automorphisms of surfaces, we didn't address several interesting questions. What about birational transformations, higher dimensions, etc ? We list a few references that may help the reader.

9.1. **Birational transformations.**

Let X be a connected compact Kähler surface, with a Kähler form κ. Denote by $\mathsf{Bir}(X)$ the group of its birational (or bimeromorphic) transformations. Let f be an element of $\mathsf{Bir}(X)$. If f is not an automorphism, the indeterminacy sets $\mathrm{Ind}(f)$ and $\mathrm{Ind}(f^{-1})$ are not empty; iterating f, it may happen that the union of indeterminacy points of iterates of f is dense in $X(\mathbf{C})$. The sequence

$$d_n = \int_X (f^n)^*\kappa\wedge\kappa$$

controls several features of the dynamics of f. One can classify birational transformations f such that the dynamical degree

$$\lambda(f) = \lim_{n\to+\infty} (d_n)^{1/n}$$

is equal to 1; up to conjugation by birational maps, the list is the same as in Theorem 2.11, with one more case: It may happen that d_n grows linearly with n, in which case f preserves a pencil of rational curves (see [74, 49]). Moreover, the group $\mathsf{Bir}(X)$ acts on an infinite dimensional hyperbolic space $\mathbf{H}_\infty$ and this classification into types can be explained in terms of elliptic, parabolic and loxodromic isometries; such a classification can then be used to study groups of birational transformations (see [30, 40]).

When f is a birational transformation with $\lambda(f) > 1$, the dynamics of f should be similar to the dynamics of loxodromic automorphisms. Unfortunately, the techniques described in the previous paragraphs do not apply directly when there are indeterminacy points. Assume, however, that

$$\sum_{n\geq 0} \frac{1}{\lambda(f)^n} \log \mathsf{dist}(f^n(\mathrm{Ind}(f^{-1})), \mathrm{Ind}(f)) \; < +\infty$$

(this is Bedford-Diller condition). Then pluri-potential theory can be applied successfully to construct good f-invariant laminar currents and the unique measure of maximal entropy; this measure describes the distribution of periodic points: See [6] and [66], as well as [50, 51, 52]. On the other hand, there are families of birational mappings f_t for which the strategy through pluri-potential analysis fails for a dense set of parameters t (see [25]).

9.2. **Higher dimension.** We did not mention any result concerning the dynamics of automorphisms on higher dimensional complex manifolds, but many results have been obtained recently. Here is a short list of relevant references

- [57] and [59] consider invariant currents and measures for automorphisms in any dimension; they prove uniqueness results, mixing properties, etc.
- [89] shows that most stable manifolds are uniformized by $\mathbf{C}^k$, where k is the complex dimension of the stable manifold; [45] provides estimates for the Lyapunov exponents.
- [59] and [55] use all these results to study naturally invariant measures similar to the measure μ_f in the case of automorphisms of surfaces, but for automorphisms of higher dimensional manifolds.

Groups of automorphisms of compact Kähler manifolds are not well understood yet, but Hodge Theory provides a powerful tool in any dimension, which can be used in a spirit similar to Section 3.2: See [54], [29], [126], and [32], for example.

9.3. **Other topics.** Similar tools can be applied to describe the dynamics of (non-invertible) endomorphisms of compact complex manifolds; see [115], and [58] for two surveys on this topic.

In the case of automorphisms of the plane $\mathbf{C}^2$ (and certain affine surfaces, see [36]), the currents T_f^+ and T_f^- have global potentials: There are Green functions G_f^+ and G_f^- such that $T_f^{\pm} = dd^c G_f^{\pm}$ on $\mathbf{C}^2$ and $G_f^{\pm} \circ f = \lambda(f)^{\pm} G_f^{\pm}$. Those invariant functions provide new tools that can be used to obtain deeper results. See [117] and [5] for two surveys. While most results described in the previous paragraphs concern the stochastic properties of the dynamics of automorphisms, there are important results concerning topological aspects of the dynamics of automorphisms of $\mathbf{C}^2$ (see [83, 84], [82], and [12, 13, 14] for example).

Let us end this section with a construction that dates back to the first half of the twentieth century (see [110]). Let g be an endomorphism of the projective space $\mathbb{P}^n$. Assume that g is defined over rational numbers: There are homogeneous polynomial functions $P_i \in \mathbf{Q}[x_0, \ldots, x_n]$, such that (i) $g[x_0 : \ldots : x_n] = [P_0 : \ldots : P_n]$ and (ii) the P_i do not have common factors of degree > 1. The degree of the polynomial functions P_i does not depend on i and is the degree of g, denoted $\deg(g)$.

If $m \in \mathbb{P}^n(\mathbf{Q})$ is a rational point, one can find coordinates $[x_0 : \ldots : x_n]$ for m such that each x_i is an integer, and the largest common factor of the x_i is 1; then, one defines the logarithmic height of m by $\mathsf{h}(m) = \max_i \log |x_i|$. One can show that the limit

$$\hat{\mathsf{h}}_g(m) = \lim_n \frac{1}{\deg(g)^n} \mathsf{h}(g^n(m))$$

exists for all rational points m of $\mathbb{P}^n$. This function $\hat{\mathsf{h}}_g$ is the canonical height of g. It contains some arithmetic and dynamical information; for instance, a point m in $\mathbb{P}^n(\mathbf{Q})$ has pre-periodic orbit if and only if $\hat{\mathsf{h}}_g(m) = 0$.

A similar construction holds for loxodromic automorphisms f. One gets two canonical heights $\hat{\mathsf{h}}_f^+$ and $\hat{\mathsf{h}}_f^-$. Roughly speaking, $\hat{\mathsf{h}}_f^+$ can be decomposed as a sum of heights $\hat{\mathsf{h}}_{f,p}^+$, one for each prime number p, and one for the archimedean place $p=\infty$, and it turns out that $\hat{\mathsf{h}}^+_{f,\infty}$ provides the local potentials for the current T_f^+. Thus, the canonical heights add up informations from the complex and the p-adic dynamics of f.

We refer to [116], [92], and [93] for canonical heights in the context of automorphisms of projective surfaces, and to [41] for a survey on canonical heights and equidistribution results (see also [28] for a short introduction).

10. Appendix: Classification of surfaces

Thanks to Enriques-Kodaira classification of surfaces (see [2]), we describe the geometry of surfaces that admit a loxodromic automorphism.

10.1. Kodaira dimension. Let X be a connected compact complex surface. Consider the canonical bundle $K_X=\mathsf{det}(T^*X)$. Its holomorphic sections are holomorphic 2-forms, and the holomorphic sections of its tensor powers $K_X^{\otimes n}$ can be expressed in local coordinates in the form $a(z_1,z_2)(dz_1\wedge dz_2)^n$ for some holomorphic function a. Fix a positive integer n, fix a point $x\in X$, and consider the evaluation map

$$ev_x\colon \Omega\in H^0(X,K_X^{\otimes n})\mapsto \Omega_x\in \mathsf{det}(T_x^*X).$$

Since $\mathsf{det}(T_x^*X)$ has dimension 1, it can be identified with $\mathbf{C}$, and this identification is unique up to a non-zero scalar multiple. Thus, either ev_x vanishes identically, or it defines an element $[ev_x]$ of $\mathbb{P}(H^0(X,K_X^{\otimes n})^*)$. If $H^0(X,K_X^{\otimes n})$ is not reduced to $\{0\}$, this construction provides a meromorphic mapping

$$\Phi_n\colon x\in X \dashrightarrow [ev_x]\in \mathbb{P}(H^0(X,K_X^{\otimes n})^*).$$

By definition, the Kodaira dimension $\mathrm{kod}(X)$ is equal to $-\infty$ if $H^0(X,K_X^{\otimes n})$ is reduced to $\{0\}$ for all $n\geq 1$, and is equal to the maximum of $\dim(\Phi_n(X))$, $n\geq 1$, otherwise.

10.2. Automorphisms. The group $\mathsf{Aut}(X)$ acts linearly by pull-back on the sections of $K_X^{\otimes n}$, and preserves the positive homogeneous function

$$\omega\in H^0(X,K_X^{\otimes n})\mapsto \int_X(\Omega\wedge\overline{\Omega})^{1/n}.$$

Hence, the image of $\mathsf{Aut}(X)$ in $\mathsf{GL}\,(H^0(X,K_X^{\otimes n})^*)$ is relatively compact and, in fact, is a finite group (see [121, 108]).

The meromorphic mapping Φ_n is equivariant with respect to the natural action of $\mathsf{Aut}(X)$ on X and its projective linear action on $\mathbb{P}(H^0(X,K_X^{\otimes n})^*)$. Hence, when $\dim(\Phi_n(X))>0$, one obtains a non-trivial factorization of the dynamics, with a projective linear action of a finite group on the image $\Phi_n(X)$.

As an easy consequence, *a connected compact complex surface with a loxodromic automorphism has Kodaira dimension* 0 *or* $-\infty$.

10.3. **Classification of surfaces with loxodromic automorphisms.** Among compact complex surfaces with $\mathrm{kod}(X) = 0$, there are three important types.

- complex tori of dimension 2, *i.e.* quotients of $\mathbf{C}^2$ by a lattice Λ;
- K3 surfaces, *i.e.* simply connected surfaces with trivial canonical bundle (or equivalently, surfaces with a non vanishing holomorphic 2-form and trivial first Betti number);
- Enriques surfaces, *i.e.* quotients of K3 surfaces by fixed point free involutions.

This does not exhaust the list of surfaces with $\mathrm{kod}(X) = 0$; there are also bi-elliptic surfaces, which are quotients of tori, Kodaira surfaces, which are not Kähler, and blow-ups of all these five types of surfaces. Surfaces of degree $(2,2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ are examples of K3 surfaces.

There are three types of surfaces with negative Kodaira dimension. The first type is given by rational surfaces, *i.e.* surfaces which are birationally equivalent to the projective plane $\mathbb{P}^2$. The second type is made of ruled surfaces $\pi : X \to B$, where the generic fibers of π are rational curves and the basis B has genus ≥ 1. The third type is given by VII_0-surfaces (those surfaces are not Kähler). Section 7.3 provides examples of rational surfaces with loxodromic automorphisms.

The following result classifies surfaces with interesting automorphisms and explains why we focussed on compact Kähler surfaces.

Theorem 10.1 (Cantat [33], Nagata [107])**.** *Let X be a connected compact complex surface. Assume that $\mathsf{Aut}(X)$ contains an automorphism f with positive topological entropy (resp. assume that X is Kähler and $\mathsf{Aut}(X)$ contains a loxodromic automorphism f). Then X is a Kähler surface, and*

- *either X is obtained from the plane $\mathbb{P}^2(\mathbf{C})$ by a finite sequence of at least ten blow-ups;*
- *or there is a holomorphic birational map $\pi : X \to X_0$ such that $\pi \circ f \circ \pi^{-1}$ is an automorphism of X_0 and X_0 is a torus, a K3 surface, or an Enriques surface (X_0 is the "minimal model of X").*

10.4. **Positive characteristic.** Let us now consider projective surfaces defined on algebraically closed field $\mathbf{k}$. Enriques-Kodaira's classification has been extended to this context by Bombieri and Mumford. New phenomena appear when both $\mathrm{kod}(Y) = 0$ and the characteristic of $\mathbf{k}$ is positive, the main cases being $\mathsf{char}(\mathbf{k}) = 2$, or 3; for example, there are K3 surfaces which are unirational in characteristic 2, a fact which is impossible for complex surfaces.

Nevertheless, with appropriate definitions ([8]), the previous Theorem remains valid: *If $\mathsf{Aut}(Y)$ contains a loxodromic automorphism f, then either Y is obtained from $\mathbb{P}^2_{\mathbf{k}}$ by a sequence of at least ten blow-ups, or there is a birational morphism $\pi : Y \to Y_0$*

[8] A K3 surface Y is a surface with Kodaira dimension $\mathrm{kod}(Y) = 0$, Betti numbers $b_1(Y) = 0$ and $b_2(Y) = 22$, and characteristic $\chi(O_Y) = 2$. An Enriques surface is a surface with Kodaira dimension $\mathrm{kod}(Y) = 0$, Betti numbers $b_1(Y) = 0$ and $b_2(Y) = 10$, and characteristic $\chi(O_Y) = 1$.

such that $\pi \circ f \circ \pi^{-1}$ is an automorphism of Y_0 and Y_0 is an abelian surface, a K3 surface, or an Enriques surface.

More interestingly, one can construct surfaces in positive characteristic with automorphisms groups which are surprisingly large compared to the case of complex surfaces (see the theory of complex multiplication for tori, [60] for an example on a K3 surface, and [31] for rational surfaces).

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