KHOVANSKY-TEISSIER INEQUALITIES, MONOMIAL MAPS, AND DYNAMICAL DEGREES

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ABSTRACT. Beside the Khovansky-Teissier inequalities, there are no additional (log-linear) inequalities: this can already be observed on dynamical degrees.

1. LOG CONCAVITY OF DYNAMICAL DEGREES

Let us recall a few basic facts on dynamical degrees. See [1] for a good reference.

Let $f\colon \mathbb{P}^m_{\mathbf{K}} \dashrightarrow \mathbb{P}^m_{\mathbf{K}}$ be a birational transformation of the projective space of dimension m, over a field $\mathbf{K}$. For each integer $k \in \{0, \ldots, m\}$, the k-th dynamical degree $\lambda_k(f)$ is defined by

$$\lambda_k(f) = \lim_{n \to +\infty} \Big(((f^n)^*(H^k)) \cdot (H^{m-k}) \Big)^{1/n} \tag{1.1}$$

where H^k and H^{m-k} are projective subspaces of $\mathbb{P}^m_{\mathbf{K}}$ of codimensions k and $m-k$ respectively, $E \cdot F$ denotes the intersection number between algebraic subsets, and $(f^n)^*E$ is the pull back of E. The limit is well defined, and does not depend on the projective subspaces H^k and H^{m-k}.

It turns out that the intersection products between ample divisors satisfy certain convexity properties, known as Khovansky-Teissier inequalities. These inequalities imply that the dynamical degrees are log-concave. In other words, if we set

$$\ell_k(f) = \log(\lambda_k(f)) \tag{1.2}$$

(where log is the neperian logarithm), then

$$2\ell_k \geq \ell_{k-1} + \ell_{k+1} \qquad (\forall 1 \leq k \leq m-1). \tag{1.3}$$

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[1] Dang, Nguyen-Bac Degrees of iterates of rational maps on normal projective varieties. Proc. Lond. Math. Soc. (3) 121 (2020), no. 5, 1268–1310

Since $\ell_0(f) = 0 = \ell_m(f)$ for all $f \in \mathsf{Bir}(\mathbb{P}^m_{\mathbf{K}})$, we deduce that $(\ell_1(f), \ldots, \ell_{m-1}(f))$ is equal to $(0, \ldots, 0)$ if and only if $\ell_k(f) = 0$ for some $1 \leq k \leq m-1$.

2. Monomial maps

Let $\mathbb{G}$ denote the multiplicative group (thus $\mathbb{G}(\mathbf{K}) = \mathbf{K}^\times$). Let $A = (a_{i,j})$ be an $m \times m$ matrix with integer coefficients and determinant ± 1. Then, A determines an automorphism of the multiplicative group $(\mathbb{G})^m$, defined by

$$f_A(x_1, \ldots, x_m) = (x_1^{a_{1,1}} \cdots x_m^{a_{1,m}}, \ldots, x_1^{a_{m,1}} \cdots x_m^{a_{m,m}}) \tag{2.1}$$

Since $\mathbb{G}^m$ is a dense open subset of $\mathbb{P}^m$, f_A induces a birational transformation of $\mathbb{P}^m_{\mathbf{K}}$, for any field $\mathbf{K}$. Favre and Wulcan, and Jan-Li Lin, gave a formula for the dynamical degrees of f_A. Let α_1, …, α_m be the eigenvalues of A, computed in $\mathbf{C}$, counted with multiplicities. Set $\rho_i = |\alpha_i|$, and assume that

$$\rho_1 \geq \rho_2 \geq \cdots \rho_m. \tag{2.2}$$

Note that $\rho_1 \cdots \rho_m = 1$ because $\det(A) = \pm 1$. Then,

$$\lambda_k(f_A) = \rho_1 \cdots \rho_k \tag{2.3}$$

for every $1 \leq k \leq m-1$. In other words

$$\ell_k(f_A) = \sum_{j=1}^{k} \log(\rho_j). \tag{2.4}$$

Note that, *modulo the linear change of variables given by* (2.4), *the Khovansky-Teissier inequalities are equivalent to* $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_m$ *and* $\rho_1 \cdots \rho_m = 1$.

3. The dynamical shape

In dimension $m = 1$, there is a unique dynamical degree and $\lambda_1(f) = 1$ for all f because $\mathsf{Bir}(\mathbb{P}^1_{\mathbf{K}}) = \mathsf{Aut}(\mathbb{P}^1_{\mathbf{K}}) = \mathsf{PGL}_2(\mathbf{K})$. Now, suppose $m \geq 2$.

Consider the subset $L_{\mathbf{K}}(m)$ of $\mathbf{R}_+^{m-1}$ defined by

$$L_{\mathbf{K}}(m) = \{(\ell_1(f), \ldots, \ell_{m-1}(f)) \; ; \; f \in \mathsf{Bir}(\mathbb{P}^m_{\mathbf{K}})\}. \tag{3.1}$$

It is contained in the convex cone defined by the inequalities (1.2) and the positivity of the ℓ_i. Modulo positive homotheties, $\mathbf{R}_+^{m-1}$ becomes a simplex $\Delta(m) = (\mathbf{R}_+^{m-1})/\mathbf{R}_+^*$. Now, if we project $L_{\mathbf{K}}(m)$ into $\Delta(m)$ and take the closure of this projection, we get a compact subset

$$\Lambda_{\mathbf{K}}(m) \subset (\mathbf{R}_+^{m-1})/\mathbf{R}_+^*. \tag{3.2}$$

This subset is contained in the closed convex subset $\Theta(m) \subset \Delta(m)$ determined by the Khovansky-Teissier inequalities.

Theorem 3.1. *For any field* $\mathbf{K}$ *and any dimension* $m \geq 2$*,*

(1) *The convex set* $\Theta(m)$ *is a simplex of dimension* $m-2$*.*
(2) *The projections of the vectors* $(\ell_1(g), \ldots, \ell_{m-1}(g))$ *for all monomial maps* $g \in \mathsf{GL}_m(\mathbf{Z})$ *form a dense subset of* $\Theta(m)$*.*
(3) *The compact subset* $\Lambda_{\mathbf{K}}(m)$ *coïncides with* $\Theta(m)$*.*

4. Proof

4.1. **Assertion 1.** By construction, $\Theta(m)$ is a compact subset of $\Delta(m)$ defined by $m-1$ inequalities. For $(u_1, \ldots, u_{m-1}) \in \Theta(m)$, the concavity inequalities give $ku_1 \geq u_k \geq \frac{m-k}{m-1} u_1$. Thus, $\Theta(m)$ does not intersect the boundary of $\Delta(m)$ (which is defined by $u_1 \cdots u_{m-1} = 0$). Thus, $\Theta(m)$ is a compact subset of the interior of $\Delta(m)$ and $\Theta(m)$ is a simplex of dimension $m-2$.

4.2. **Assertion 2.** Consider $m-1$ integers $n_1 > n_2 > \ldots > n_{m-1}$, and the polynomial function

$$P(t) = t(t-n_1)\cdots(t-n_{m-1}) + \varepsilon$$

where $\varepsilon = \pm 1$. By construction, P has integer coefficients and degree m. Assume the n_j be large. Then, there is an $\eta > 0$, small compared to 1 (in particular compared to the distance between the n_i, such that the sign of $P(t)$ changes as follows: for $t > n_{m-1} + \eta$ $P(t)$ is positive; for $t \in]n_{m-2} + \eta, n_{m-1} - \eta[$ it is negative; etc. This implies that $P(t)$ has m distinct roots, α_i with

$$\alpha_j \simeq n_j \tag{4.1}$$

for $j \leq m-1$ (up to an error of η) and $\alpha_m = \varepsilon(\alpha_1 \cdots \alpha_{m-1})^{-1}$.

Let A_P be the companion matrix of P. Then, with the notation of the second section, we have $\rho_i = \alpha_i$ for $i \leq m-1$. The dynamical degrees of the monomial map f_{A_P} are given by (2.3) and (2.4). Thus, given any vector $(u_1, \ldots, u_{m-1})$ in $\Theta(m)$, we can take a large multiple $(\ell_1, \ldots, \ell_{m-1})$ of it, make the change of variable defined by Equation (2.4) to get real numbers ρ_i, and then define n_i to be the integral part of ρ_i. This gives a polynomial P for which f_{A_P} satisfies $\lambda_k(f_{A_P}) \simeq \ell_k$, as desired.

4.3. **Assertion 3.** This is a direct consequence of the second assertion.

5. Complement

If $X_{\mathbf{K}}$ is a(n irreducible) projective variety of dimension m, one can also define the dynamical degrees $\lambda_k(f)$ for every $f \in \mathsf{Bir}(X_{\mathbf{K}})$ and $0 \leq k \leq m$; for this, H is replaced by some hyperplane section. Similarly, if M is a compact Kähler manifold, $\lambda_k(f)$ is defined for any bimeromorphic transformation f of M; here, H is replaced by a Kähler form.

The inequalities (1.2) hold in this more general context, and one can ask for a description of $\Lambda_{\mathbf{K}}(X)$ (resp. $\Lambda(M)$), with obvious generalizations. A priori, $\Lambda_{\mathbf{K}}(X)$ depends strongly on the geometry of X; for instance, the codimension of $\Lambda_{\mathbf{K}}(X)$ in $\Delta(m)$ is larger than, or equal to the Kodaira dimension of X.

Theorem 5.1. *There are complex abelian varieties X of dimension m such that $\Delta(X) = \Theta(m)$.*

Proof. Take $E = \mathbf{C}/L$ an elliptic curve, define $X = E^m$ and note that $\mathsf{GL}_m(\mathbf{Z})$ acts linearly on $X = \mathbf{C}^m/L^m$, each matrix A in $\mathsf{GL}_m(\mathbf{Z})$ inducing an automorphism $g_A \colon X \to X$. With the notation of the second section, the dynamical degrees of g_A satisfy $\ell_k(g_A) = 2\sum_{j=1}^k \rho_j$. Thus, the proof of our first theorem extends to this context. □

I don't know what to expect for the cubic hypersurface in $\mathbb{P}^{m+1}_{\mathbf{C}}$, for instance.

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