FAMILIES OF COMMUTING AUTOMORPHISMS, AND A CHARACTERIZATION OF THE AFFINE SPACE

SERGE CANTAT, ANDRIY REGETA, AND JUNYI XIE

ABSTRACT. We prove that the affine space of dimension $n \geq 1$ over an uncountable algebraically closed field $k$ is determined, among connected affine varieties, by its automorphism group (viewed as an abstract group). The proof is based on a new result concerning algebraic families of pairwise commuting automorphisms.

1. INTRODUCTION

1.1. Characterization of the affine space. In this paper, $k$ is an algebraically closed field and $\mathbb{A}_k^n$ denotes the affine space of dimension $n$ over $k$.

**Theorem A.**— Let $k$ be an algebraically closed and uncountable field. Let $n$ be a positive integer. Let $X$ be a reduced, connected, affine variety over $k$. If its automorphism group $\text{Aut}(X)$ is isomorphic to $\text{Aut}(\mathbb{A}_k^n)$ as an abstract group, then $X$ is isomorphic to $\mathbb{A}_k^n$ as a variety over $k$.

Note that no assumption is made on $\dim(X)$; in particular, we do not assume $\dim(X) = n$. This theorem is our main goal. It would be great to lighten the hypotheses on $k$, but besides that the following remarks show the result is optimal:

- The affine space $\mathbb{A}_k^n$ is not determined by its automorphism group in the category of quasi-projective varieties because
  
  (1) $\text{Aut}(\mathbb{A}_k^n)$ is naturally isomorphic to $\text{Aut}(\mathbb{A}_k^n \times Z)$ for any projective variety $Z$ with $\text{Aut}(Z) = \{\text{id}\}$;
  
  (2) for every algebraically closed field $k$ there is a projective variety $Z$ over $k$ such that $\dim(Z) \geq 1$ and $\text{Aut}(Z) = \{\text{id}\}$ (one can take a general curve of genus $\geq 3$; see [11] and [12, Main Theorem]).

- The connectedness is crucial: $\text{Aut}(\mathbb{A}_k^n)$ is isomorphic to the automorphism group of the disjoint union of $\mathbb{A}_k^n$ and $Z$ if $Z$ is a variety with $\text{Aut}(Z) = \{\text{id}\}$.

1.2. Previous results. The literature contains already several theorems that may be compared to Theorem A. We refer to [2] for an interesting introduction and for the case of the complex affine plane; see [6, 7] for extension and generalisations of Déserti’s results in higher dimension. Some of those results assume $\text{Aut}(X)$ to be isomorphic to $\text{Aut}(\mathbb{A}_k^n)$ as an ind-group; this is a rather strong hypothesis. Indeed,
there are examples of affine varieties \(X\) and \(Y\) such that \(\text{Aut}(X)\) and \(\text{Aut}(Y)\) are isomorphic as abstract groups, but not isomorphic as ind-groups (see [8, Theorem 2]). In [9] the authors prove that an affine toric surface is determined by its group of automorphisms in the category of affine surfaces; unfortunately, their methods do not work in higher dimension.

1.3. Commutative families. The proof of Theorem A relies on a new result concerning families of pairwise commuting automorphisms of affine varieties. To state it, we need a few standard notions. If \(V\) is a subset of a group \(G\), we denote by \(\langle V \rangle\) the subgroup generated by \(V\), i.e. the smallest subgroup of \(G\) containing \(V\). We say that \(V\) is commutative if \(fg = gf\) for all pairs of elements \(f\) and \(g\) in \(V\) or, equivalently, if \(\langle V \rangle\) is an abelian group. In the following statement, \(\text{Aut}(X)\) is viewed as an ind-group, so that it makes sense to speak of algebraic subsets of it (see the definitions in Section 2.2).

**Theorem B.** Let \(k\) be an algebraically closed field and let \(X\) be an affine variety over \(k\). Let \(V\) be a commutative irreducible algebraic subvariety of \(\text{Aut}(X)\) containing the identity. Then \(\langle V \rangle\) is an algebraic subgroup of \(\text{Aut}(X)\).

It is crucial to assume that \(V\) contains the identity. Otherwise, a counter-example would be given by a single automorphism \(f\) of \(X\) for which the sequence \(n \mapsto \text{deg}(f^n)\) is not bounded (see Section 2.1). To get a family of positive dimension, consider the set \(V\) of automorphisms \(f_a: (x, y) \mapsto (x, axy)\) of \((\mathbb{A}^1_k \setminus \{0\})^2\), for \(a \in k \setminus \{0\}\); \(V\) is commutative and irreducible, but \(\langle V \rangle\) has infinitely many connected components (hence \(\langle V \rangle\) is not algebraic). However, even if \(V\) does not contain the identity, the subset \(V \cdot V^{-1} \subseteq \text{Aut}(X)\), is irreducible, commutative and contains the identity; if its dimension is positive, Theorem B implies that \(\text{Aut}(X)\) contains a commutative algebraic subgroup of positive dimension.

1.4. Acknowledgement. We thank Jean-Philippe Furter, Hanspeter Kraft, and Christian Urech for interesting discussions related to this article.

2. Degrees and ind-groups

2.1. Degrees and compactifications. Let \(X\) be an affine variety. Embed \(X\) in the affine space \(\mathbb{A}^N_k\) for some \(N\), and denote by \(x = (x_1, \ldots, x_N)\) the affine coordinates of \(\mathbb{A}^N_k\). Let \(f\) be an automorphism of \(X\). Then, there are \(N\) polynomial functions \(f_i \in k[x]\) such that \(f(x) = (f_1(x), \ldots, f_N(x))\) for \(x \in X\). One says that \(f\) has degree \(\leq d\) if one can choose the \(f_i\) of degree \(\leq d\); the degree \(\text{deg}(f)\) can then be defined as the minimum of these degrees \(d\). This notion depends on the embedding \(X \hookrightarrow \mathbb{A}^N_k\).

Another way to proceed is as follows. To simplify the exposition, assume that all irreducible components of \(X\) have the same dimension \(k = \text{dim}(X)\). Fix a compactification \(X_0\) of \(X\) by a projective variety and let \(\overline{X} \to X_0\) be the normalization
of $\mathcal{X}_0$. If $H$ is an ample line bundle on $\mathcal{X}$, and if $f$ is a birational transformation of $\mathcal{X}$, one defines $\deg_H(f)$ (or simply $\deg(f)$) to be the intersection number

$$\deg(f) = (f^*H) \cdot (H)^{k-1}. \quad (2.1)$$

Since $\text{Aut}(X) \subset \text{Bir}(\mathcal{X})$, we obtain a second notion of degree. It is shown in [1, 16] that these notions of degrees are compatible: if we change the embedding $X \hookrightarrow \mathbb{A}^N_k$, or the polarization $H$ of $X$, or the compactification $\overline{X}$, we get different degrees, but any two of these degree functions are always comparable, in the sense that there are positive constants satisfying

$$a \deg(f) \leq \deg'(f) \leq b \deg(f) \quad (\forall f \in \text{Aut}(X)). \quad (2.2)$$

A subset $V \subset \text{Aut}(X)$ is of bounded degree if there is a uniform upper bound $\deg(g) \leq D < +\infty$ for all $g \in V$. This notion does not depend on the choice of degree. If $V \subset \text{Aut}(X)$ is of bounded degree, then $V^{-1} = \{f^{-1} : f \in V\} \subset \text{Aut}(X)$ is of bounded degree too, but we shall not use this result (see [1] and [3] for instance).

2.2. Automorphisms of affine varieties and ind-groups. The notion of an ind-group goes back to Shafarevich, who called these objects infinite dimensional groups in [14]. We refer to [3, 5] for detailed introductions to this notion.

2.2.1. Ind-varieties. By an ind-variety we mean a set $\mathcal{V}$ together with an ascending filtration $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \ldots \subset \mathcal{V}$ such that the following is satisfied:

1. $\mathcal{V} = \bigcup_{k \in \mathbb{N}} \mathcal{V}_k$;
2. each $\mathcal{V}_k$ has the structure of an algebraic variety over $k$;
3. for all $k \in \mathbb{N}$ the inclusion $\mathcal{V}_k \subset \mathcal{V}_{k+1}$ is a closed immersion.

We refer to [3] for the notion of equivalent filtrations on ind-varieties.

A map $\Phi : \mathcal{V} \to \mathcal{W}$ between ind-varieties $\mathcal{V} = \bigcup_k \mathcal{V}_k$ and $\mathcal{W} = \bigcup_l \mathcal{W}_l$ is a morphism if for each $k$ there is $l \in \mathbb{N}$ such that $\Phi(\mathcal{V}_k) \subset \mathcal{W}_l$ and the induced map $\Phi : \mathcal{V}_k \to \mathcal{W}_l$ is a morphism of algebraic varieties. Isomorphisms of ind-varieties are defined in the usual way. An ind-variety $\mathcal{V} = \bigcup_k \mathcal{V}_k$ has a natural Zariski topology: $S \subset \mathcal{V}$ is closed (resp. open) if $S_k := S \cap \mathcal{V}_k \subset \mathcal{V}_k$ is closed (resp. open) for every $k$. A closed subset $S \subset \mathcal{V}$ inherits a natural structure of ind-variety and is called an ind-subvariety. An ind-variety $\mathcal{V}$ is said to be affine if each $\mathcal{V}_k$ is affine.

We shall only consider affine ind-varieties and for simplicity we just call them ind-varieties. An ind-subvariety $S$ is an algebraic subvariety of $\mathcal{V}$ if $S \subset \mathcal{V}_k$ for some $k \in \mathbb{N}$; by definition, a constructible subset will always be a constructible subset in an algebraic subvariety of $\mathcal{V}$.

2.2.2. Ind-groups. The product of two ind-varieties is defined in the obvious way. An ind-variety $G$ is called an ind-group if the underlying set $G$ is a group such that the map $G \times G \to G$, defined by $(g, h) \mapsto gh^{-1}$, is a morphism of ind-varieties. If a subgroup $H$ of $G$ is closed for the Zariski topology, then $H$ is naturally an
ind-subgroup of \( G \); it is an **algebraic subgroup** if it is an algebraic subvariety of \( G \). A connected component of an ind-group \( G \), with a given filtration \( G_0 \subset G_1 \subset G_2 \subset \ldots \), is an increasing union of connected components \( G_i \) of \( G_i \). The **neutral component** \( G^0 \) of \( G \) is the union of the connected components of the \( G_i \) containing the neutral element \( \text{id} \in G \). We refer to [3], and in particular to Propositions 1.7.1 and 2.2.1, showing that \( G^0 \) is an ind-subgroup in \( G \) whose index is at most countable (the proof of [3] works in arbitrary characteristic).

**Theorem 2.1.** Let \( X \) be an affine variety over an algebraically closed field \( k \). Then \( \text{Aut}(X) \) has the structure of an ind-group acting “morphically” on \( X \); this means that the action \( G \times X \to X \) of \( G \) on \( X \) induces a morphism of algebraic varieties \( G_i \times X \to X \) for every \( i \in \mathbb{N} \).

In particular, if \( V \) is an algebraic subset of \( \text{Aut}(X) \), then \( V(x) \subset X \) is constructible for every \( x \in X \) by Chevalley’s theorem. The proof can be found in [5, Proposition 2.1] (see also [3, Theorems 5.1.1 and 5.2.1]): the authors assume that the field has characteristic 0, but their proof works in the general setting. To obtain a filtration, one starts with a closed embedding \( X \hookrightarrow A^N_k \), and define \( \text{Aut}(X)_d \) to be the set of automorphisms \( f \) such that \( \max \{ \deg(f), \deg(f^{-1}) \} \leq d \). For example, if \( X = A^n_k \), the ind-group filtration \( (\text{Aut}(A^n_k)_d) \) of \( A^n_k \) is defined by the following property: an automorphism \( f \) is in \( (\text{Aut}(A^n_k)_d) \) if the polynomial formulas for \( f = (f_1, \ldots, f_n) \) and for its inverse \( f^{-1} = (g_1, \ldots, g_n) \) satisfy

\[
\deg f_i \leq d \quad \text{and} \quad \deg g_i \leq d, \quad (\forall i \leq n).
\]

Note that an ind-subgroup is algebraic if and only if it is of bounded degree. Thus, we get the following basic fact.

**Proposition 2.2.** Let \( X \) be an affine variety over an algebraically closed field \( k \). Let \( V \) be an irreducible algebraic subset of \( \text{Aut}(X) \) that contains \( \text{id} \). Then \( \langle V \rangle \) is an algebraic subgroup of \( \text{Aut}(X) \), acting algebraically on \( X \), if and only if \( \langle V \rangle \) is of bounded degree.

**Proof.** If \( \langle V \rangle \) is algebraic, then it is contained in some \( \text{Aut}(X)_d \) and, as such, is of bounded degree; moreover, Theorem 2.1 implies that the action \( \langle V \rangle \times X \to X \) is algebraic. If \( \langle V \rangle \) is of bounded degree, then \( \langle V \rangle^{-1} = \langle V \rangle \) is of bounded degree too, and \( \langle V \rangle \) is contained in some \( \text{Aut}(X)_d \). The Zariski closure \( \overline{\langle V \rangle} \) of \( \langle V \rangle \) in \( \text{Aut}(X)_d \) is an algebraic subgroup of \( \text{Aut}(X) \); we are going to show that \( \overline{\langle V \rangle} = \langle V \rangle \). We note that \( \langle V \rangle \) is the increasing union of the subsets \( W = V \cdot V^{-1} \subset W \cdot W \subset \cdots \subset W^k \subset \cdots \) (note that \( W \) contains \( V \) because \( \text{id} \in V \), and by Chevalley theorem, each \( W^k \subset \overline{\langle V \rangle} \) is constructible. The \( W^k \) are irreducible, because \( V \) is irreducible; by noetherianity, there exists \( l \geq 1 \) such that \( W^l = \bigcup_{k \geq l} W^k \subset \overline{\langle V \rangle} \). Since \( \langle V \rangle \subset \bigcup_{k \geq 1} W^k \), we get \( W^l = \overline{\langle V \rangle} \); thus, there exists a Zariski dense open subset \( U \) of \( \overline{\langle V \rangle} \) which is contained in \( W^l \). Now, pick any element \( f \) in \( \overline{\langle V \rangle} \). Then \( (f \cdot U) \) and \( U \) are
two Zariski dense open subsets of \( \overline{V} \), so \((f \cdot U)\) intersects \( U \) and this implies that \( f \) is in \( U \cdot U^{-1} \subset \langle V \rangle \). So \( \overline{V} \subset \langle V \rangle \).

\[ \square \]

3. \textbf{Algebraic Varieties of Commuting Automorphisms}

Let \( k \) be an algebraically closed field. Let \( X \) be an affine variety over \( k \) of dimension \( d \). In this section, we prove Theorem B. Since \( V \) is irreducible and contains the identity, every irreducible component of \( X \) is invariant under the action of \( V \) (and \( \langle V \rangle \)); thus, we may and do assume \( X \) to be irreducible.

3.1. \textbf{Invariant fibrations, base change, and degrees.} Let \( B \) and \( Y \) be irreducible affine varieties, and let \( \pi: Y \to B \) be a dominant morphism. By definition, \( \text{Aut}_\pi(Y) \) is the group of automorphisms \( g: Y \to Y \) such that \( \pi \circ g = \pi \).

Let \( B' \) be another irreducible affine variety, and let \( \psi: B' \to B \) be a finite morphism. Pulling-back \( \pi \) by \( \psi \), we get a new variety \( Y \times_B B' = \{ (y, b') \in Y \times B' ; \pi(y) = \psi(b') \} \); the projections \( \pi_Y: Y \times_B B' \to Y \) and \( \pi': Y \times_B B' \to B' \) satisfy \( \psi \circ \pi' = \pi \circ \pi_Y \). There is a natural homomorphism

\[ \tau_\psi: \text{Aut}_\pi(Y) \hookrightarrow \text{Aut}_{\pi'}(Y \times_B B') \]

(3.1)

defined by \( \tau_\psi(g) = g \times_B \text{id}_{B'} \). For every \( g \in \text{Aut}_\pi(Y) \), we have

\[ g \circ \pi_Y = \pi_Y \circ \tau_\psi(g) \quad \text{and} \quad \pi' = \pi' \circ \tau_\psi(g). \]

If \( \tau_\psi(g) = \text{id} \) then \( g \circ \pi_Y = \pi_Y \) and \( g = \text{id} \) because \( \pi_Y \) is dominant; hence, \( \tau_\psi \) is an embedding.

**Lemma 3.1.** If \( S \) is a subset of \( \text{Aut}_\pi(Y) \), then \( S \) is of bounded degree if and only if its image \( \tau_\psi(S) \) in \( \text{Aut}_{\pi'}(Y \times_B B') \) is of bounded degree.

**Proof of Lemma 3.1.** We can suppose that \( B' \) is normal, because the normalization is a finite morphism (thus, composing it with \( \psi \) gives a finite morphism).

Let \( \overline{B} \subset \mathbb{P}^M_k \) and \( Y' \subset \mathbb{P}^N_k \) be irreducible projective varieties containing \( B \) and \( Y \) as affine open subsets. Let \( \overline{Y} \) be the Zariski closure of the graph of \( \pi \) in \( Y' \times \overline{B} \). Then \( \overline{Y} \) is an irreducible projective variety containing (a copy of) \( Y \) as an affine open subset, with a morphism \( \pi: \overline{Y} \to \overline{B} \) satisfying \( \pi|_Y = \pi \). Denote by \( \varphi: \overline{B} \to B \) the normalization of \( \overline{B} \) in \( k(B') \); it is a finite morphism. Then \( \overline{Y} \times_\overline{B} \overline{B}' \) is an irreducible projective variety containing \( Y \times_B B' \) as an affine open subset, and the projection \( \pi_{\overline{Y}}: \overline{Y} \times_\overline{B} \overline{B}' \to \overline{Y} \) is finite.

Let \( L \) be an ample line bundle on \( \overline{Y} \). Since \( \pi_{\overline{Y}} \) is finite, the line bundle \( H := \pi_{\overline{Y}}^*L \) is an ample line bundle on the projective variety \( \overline{Y} \times_\overline{B} \overline{B}' \). For every \( g \in \text{Aut}_\pi(Y) \), \( \tau_\psi(g) \) is an automorphism of \( Y \times_B B' \); it can be considered as a birational transformation of \( \overline{Y} \times_\overline{B} \overline{B}' \), and we get

\[ \tau_\psi(g)^*H \cdot (H)^{\dim(Y)-1} = (\pi_{\overline{Y}})_*((\tau_\psi(g)^*H) \cdot (H)^{\dim(Y)-1})) \]

(3.3)

\[ = \deg_{top}(\psi) \times \left((g^*L) \cdot (L)^{\dim(Y)-1}\right). \]

(3.4)
where $\deg_{\text{top}}(\psi) = \deg_{\text{top}}(\varphi)$ is the degree of the finite map $\psi: B' \to B$. Thus, the degree $\deg_{G}(g)$ for $g \in S$ is bounded by some constant $D_{Y}$ if and only if $\deg_{H}(\iota_{\psi}(g))$ is bounded by $\deg_{\text{top}}(\psi)D_{Y}$. 

Let us come back to the example $f(x, y) = (x, xy)$ from Section 1.3. This is an automorphism of the multiplicative group $\mathbb{G}_{m} \times \mathbb{G}_{m}$ that preserves the projection onto the first factor. The degrees of the iterates $f^{m}(x, y) = (x, x^{m}y)$ are not bounded, but on every fiber $\{x = x_{0}\}$, the restriction of $f^{m}$ is the linear map $y \mapsto (x_{0})^{m}y$, of constant degree 1. More generally, if $x \in B \mapsto A(x)$ is a regular map with values in $\text{GL}_{N}(k)$, then $g: (x, y) \mapsto (x, A(x)y)$ is a regular automorphism of $B \times \mathbb{A}^{N}_{k}$ and, in most cases, we observe the same phenomenon: the degrees of the restrictions $g^{n}|_{\{(x_{0})\} \times \mathbb{A}^{N}_{k}}$ are bounded, but the degrees of $g^{n}$ are not. The next proposition provides a converse result. To state it, we make use of the following notation. Let $B$ be an irreducible affine variety, and let $O(B)$ be the $k$-algebra of its regular functions. By definition, $\mathbb{A}^{N}_{B}$ denotes the affine space $\text{Spec}O(B)[x_{1}, \ldots, x_{n}]$ over the ring $O(B)$ and $\text{Aut}_{B}(\mathbb{A}^{N}_{B})$ denotes the group of $O(B)$-automorphisms of $\mathbb{A}^{N}_{B}$. Similarly, $\text{GL}_{N}(O(B))$ is the linear group over the ring $O(B)$. The inclusion $\text{GL}_{N}(O(B)) \subset \text{Aut}_{B}(\mathbb{A}^{N}_{B})$ is an embedding of ind-groups.

If $X$ is an affine variety over $k$ with a morphism $\pi: X \to B$, we denote by $\eta$ the generic point of $B$ and $X_{\eta}$ the generic fiber of $\pi$. If $G$ is a subgroup of $\text{Aut}_{\pi}(X)$, then its restriction to $X_{\eta}$ may have bounded degree even if $G$ is not a subgroup of $\text{Aut}(X)$ of bounded degree: this is shown by the previous example.

**Proposition 3.2.** Let $X$ be an irreducible and normal affine variety over $k$ with a dominant morphism $\pi: X \to B$ to an irreducible affine variety $B$ over $k$. Let $\eta$ be the generic point of $B$ and $X_{\eta}$ the generic fiber of $\pi$. Let $G$ be a subgroup of $\text{Aut}_{\pi}(X)$, whose restriction to $X_{\eta}$ is of bounded degree. Then there exists

(a) a nonempty affine open subset $B'$ of $B$,

(b) an embedding $\tau: X_{B'} := \pi^{-1}(B') \hookrightarrow \mathbb{A}^{N}_{B'}$ over $B'$ for some $N \geq 1$,

(c) and an embedding $\rho: G \hookrightarrow \text{GL}_{N}(O(B')) \subset \text{Aut}_{B'}(\mathbb{A}^{N}_{B'})$,

such that $\tau \circ g = \rho(g) \circ \tau$ for every $g \in G$.

**Notation.** For $f \in \text{Aut}(X)$ and $\xi \in O(X)$ (resp. in $k(X)$), we denote by $f^{*}\xi$ the function $\xi \circ f$. The field of constant functions is identified to $k \subset O(X)$.

**Proof of Proposition 3.2.** Shrinking $B$, we assume $B$ to be normal.

Pick any closed embedding $X \hookrightarrow \mathbb{A}^{l}_{B} \subset \mathbb{P}^{l}_{B}$ over $B$. Let $X'$ be the Zariski closure of $X$ in $\mathbb{P}^{l}_{B}$. Let $\overline{X}$ be the normalization of $X'$ with the structure morphism $\overline{\pi}: \overline{X} \to B$; thus, $\overline{\pi}: \overline{X} \to B$ is a normal and projective scheme over $B$ containing $X$ as a Zariski open subset. Moreover, $D := \overline{X} \setminus X$ is an effective divisor of $\overline{X}$. Denote by $\overline{X}_{\eta}$ the generic fiber of $\overline{\pi}$ and by $D_{\eta}$ the generic fiber of $\overline{\pi}|_{D}$. Shrinking $B$ again if...
necessary, we may assume that all irreducible components of \( D \) meet the generic fiber, i.e. \( D = \overline{D_0} \).

Write \( X = \text{Spec} A \). Let \( M \) be a finite dimensional subspace of \( A \) such that \( 1 \in M \) and \( A \) is generated by \( M \) as a \( k \)-algebra. Since the action of \( G \) on \( X_\eta \) is of bounded degree, there exists \( m \geq 0 \) such that the divisor

\[
(3.5) \quad (\text{Div}(g^*v) + mD)|_{X_\eta}
\]
is effective for every \( v \in M \) and \( g \in G \). Now, consider \( \text{Div}(g^*v) + mD \) as a divisor of \( X \) and write \( \text{Div}(g^*v) + mD = D_1 - D_2 \) where \( D_1 \) and \( D_2 \) are effective and have no common irreducible component. Since \( g \in \text{Aut}_\pi(X) \), we get \( g^*v \in A \) and \( D_2 \cap X = \emptyset \). Moreover, \( D_2 \cap \overline{X_\eta} = \emptyset \). So \( D_2 \) is contained in \( X \setminus X \), but then we deduce that \( D_2 \) is empty because \( \overline{X} \setminus X \) is covered by \( D \) and \( D = \overline{D_0} \).

Observe that \( H^0(\overline{X}, mD) \) is a finite \( O(B) \)-module. Denote by \( N \) the \( G \)-invariant \( O(B) \)-submodule of \( A \) generated by the \( g^*v \), for \( g \in G \) and \( v \in M \). Since \( N \subseteq H^0(\overline{X}, mD) \), it is a finitely generated \( O(B) \)-module. Let \( \tau \) be the morphism \( X \hookrightarrow \text{Spec} O(B)[N] \) over \( B \) induced by the inclusion \( N \subseteq A \). Let \( \rho \) be the morphism sending \( g \) to the endomorphism

\[
(3.6) \quad \rho(g) \in \text{GL}_B(\text{Hom}(N, B)) \subseteq \text{Aut}_B(\text{Spec} O(B)[N])
\]
developed by \( \rho(g)(w) = g^*w \) for all \( w \in N \); then \( \tau \circ g = \rho(g) \circ \tau \) for every \( g \in G \). \( \square \)

3.2. Orbits. If \( S \) is a subset of \( \text{Aut}(X) \) and \( x \) is a point of \( X \) the \( S \)-orbit of \( x \) is the subset \( S(x) = \{ f(x); \ f \in S \} \). Let \( V \) be an irreducible algebraic subvariety of \( \text{Aut}(X) \) containing \( \text{id} \). Set \( W = V \cdot V^{-1} \). Then, \( W \) is constructible, and the group \( \langle V \rangle \) is the union of the sets

\[
(3.7) \quad W^k = \{ f_1 \circ \cdots \circ f_k; \ f_j \in W \ \text{for all} \ j \}.
\]
Since \( W \) contains \( \text{id} \), the \( W^k \) form a non-decreasing sequence

\[
(3.8) \quad W^0 = \{ \text{id} \} \subset W \subset W^2 \subset \cdots \subset W^k \subset \cdots
\]
of constructible subsets of \( \text{Aut}(X) \); their closures are irreducible, because so is \( V \). In particular, \( k \mapsto \dim(W^k) \) is non-decreasing. The \( W^k \)-orbit of a point \( x \in X \) is the image of \( W^k \times \{ x \} \) by the morphism \( \text{Aut}(X) \times X \to X \) defining the action on \( X \): applying Chevalley’s theorem one more time, \( W^k(x) \) is a constructible subset of \( X \).

**Proposition 3.3.** The orbits \( W^k(x) \) satisfy the following properties.

1. The function \( k \in \mathbb{Z}_{\geq 0} \mapsto \dim(W^k(x)) \) is non-decreasing.
2. The function \( x \in X \mapsto \dim(W^k(x)) \) is semi-continuous in the Zariski topology: the subsets \( \{ x \in X; \ \dim(W^k(x)) \leq n \} \) are Zariski closed for all pairs \((n,k)\) of integers.
3. The integers

\[
s(x) := \max_{k \geq 0} \{ \dim(W^k(x)) \} \quad \text{and} \quad s_x := \max_{x \in X} \{ s(x) \}
\]
are bounded from above by \( \dim(X) \).

(4) There is a Zariski dense open subset \( \mathcal{U} \) of \( X \) and an integer \( k_0 \) such that \( \dim(W^k(x)) = s_X \) for all \( k \geq k_0 \) and all \( x \in \mathcal{U} \).

(5) For every \( x \) in \( X \), \( W^k(x) = \langle W \rangle(x) \) if \( k \) is large enough.

**Proof.** The first assertion follows from the inclusions (3.8), and the third one is obvious. The map \( W^k \times X \to X \) given by the action \( (f,x) \mapsto f(x) \) is a morphism. The second and fourth assertion follow from Chevalley’s constructibility result and the semi-continuity of the dimension of the fibers (see [4, Exercise 3.19] and [15, Section I.6.3, Theorem 7] respectively). The fifth property follows from noetheri-

3.3. **Open orbits.** Let us assume in this paragraph that \( s_X = \dim X \): there is an orbit \( W^k(x_0) \) which is open and dense and coincides with \( \langle W \rangle(x_0) \). We fix such a pair \((k,x_0)\).

Let \( f \) be an element of \( \langle W \rangle \). Since the point \( f(x_0) \) is in the set \( W^k(x_0) \), there is an element \( g \) of \( W^k \) such that \( g(x_0) = f(x_0) \), i.e. \( g^{-1} \circ f(x_0) = x_0 \). By commutativity, \( (g^{-1} \circ f)(h(x_0)) = h(x_0) \) for every \( h \) in \( W^k \), and this shows that \( g^{-1} \circ f = id \) because \( W^k(x_0) \) is dense in \( X \). Thus, \( \langle W \rangle \) coincides with \( W^k \), and \( \langle W \rangle = \langle V \rangle \) is an irreducible algebraic subgroup of the ind-group \( Aut(X) \).

Thus, Theorem B is proved in case \( s_X = \dim X \). The proof when \( s_X < \dim X \) occupies the next section, and is achieved in § 3.4.4.

3.4. **No dense orbit.** Assume now that there is no dense orbit; in other words, \( s_X < \dim(X) \). Fix an integer \( \ell > 0 \) and a \( W \)-invariant open subset \( \mathcal{U} \subset X \) such that

\[
(3.9) \quad s(x) = s_X \quad \text{and} \quad W^\ell(x) = \langle W \rangle(x)
\]

for every \( x \in \mathcal{U} \) (see Proposition 3.3, assertions (4) and (5)).

3.4.1. **A fibration.** Let \( C \) be an irreducible algebraic subvariety of \( X \) of codimension \( s_X \) that intersects the general orbit \( W^\ell(x) \) transversally (in a finite number of points). There exists an integer \( k > 0 \) and a dense open subset \( Y \subset \mathcal{U} \) such that the following conditions are satisfied:

(i) for each \( x \in Y \) the intersection of \( C \) and \( W^\ell(x) \) is transverse and contains exactly \( k \) points;

(ii) \( Y \) is \( W \)-invariant.

To each point \( x \in Y \), we associate the intersection \( C \cap W^\ell(x) \), viewed as a point in the space \( C^{[k]} \) of cycles of length \( k \) and dimension 0 in \( C \). This gives a dominant morphism

\[
(3.10) \quad \pi: Y \to B
\]
for some irreducible variety \( B = \pi(Y) \subset C^k \). The group \( \langle W \rangle \) is now contained in \( \text{Aut}_\pi(Y) \). Shrinking \( B \), we may assume that it is normal. Let \( \eta \) be the generic point of \( B \).

The fiber \( \pi^{-1}(b) \) of \( b \in B \), we denote by \( Y_b \). By construction, for every \( b \in B(k) \), \( Y_b \) is an orbit of \( W \); and Section 3.3 shows that \( Y_b \) is isomorphic to the image \( \langle W \rangle_b \) of \( \langle W \rangle \) in \( \text{Aut}(Y_b) \): this group \( \langle W \rangle_b \) coincides with the image of \( W' \) in \( \text{Aut}(Y_b) \) and the action of \( W' \) on \( Y_b \) corresponds to the action of \( \langle W \rangle_b \) on itself by translation. Thus, Section 3.3 implies the following properties

1. the generic fiber of \( \pi \) is normal and, shrinking \( B \) again, we may assume \( Y \) to be normal;
2. the action of \( \langle W \rangle \) on the generic fiber \( Y_\eta \) has bounded degree.

3.4.2. Reduction to \( Y = U_B \times_B (\mathbb{G}_m^r) \). In this section, the variety \( Y \) will be modified, so as to reduce our study to the case when \( Y \) is an abelian group scheme over \( B \).

By Proposition 3.2, after shrinking \( B \), there exists an embedding \( \tau : Y \hookrightarrow \mathbb{A}^N_B \) for some \( N \geq 0 \) and a homomorphism \( \rho : \langle W \rangle \hookrightarrow \text{GL}_N(O(B)) \subset \text{Aut}_B(\mathbb{A}^N_B) \) such that

\[
\tau \circ g = \rho(g) \circ \tau \quad (\forall g \in \langle W \rangle). 
\]

Via \( \tau \), we view \( Y \) as a \( B \)-subscheme of \( \mathbb{A}^N_B \). Denote by \( \langle W \rangle_\eta \) the Zariski closure of \( \langle W \rangle \) in \( \text{GL}_N(k(B), Y_\eta) \subset \text{Aut}(Y_\eta) \), where \( \text{GL}_N(k(B), Y_\eta) \) is the subgroup of \( \text{GL}_N(k(B)) \) which preserves \( Y_\eta \).

Let us consider the inclusion of \( \text{GL}_N(O(B)) \) into \( \text{GL}_N(k(B)) \), and compose it with the embedding of \( W \) into \( \text{GL}_N(O(B)) \). There is a natural inclusion of sets \( W \hookrightarrow W \otimes_k k(B) \): a point \( x \) of \( W \), viewed as a morphism \( x : \text{Spec} k(x) \rightarrow W \), is mapped to the point

\[
x^B : \text{Spec} k(x) (B \otimes_k k(B)) = \text{Spec} k(x) \otimes_k k(B) \rightarrow W \otimes_k k(B).
\]

The image of this inclusion is Zariski dense in \( W \otimes_k k(B) \). The morphism \( W \hookrightarrow \text{GL}_N(k(B), Y_\eta) \) naturally extends to a morphism \( W \otimes_k k(B) \hookrightarrow \text{GL}_N(k(B), Y_\eta) \). It follows that \( \langle W \rangle_\eta \) is the Zariski closure of \( \langle W \otimes_k k(B) \rangle \) in \( \text{GL}_N(k(B), Y_\eta) \). Since \( W \otimes_k k(B) \) is geometrically irreducible, \( \langle W \rangle_\eta \) is a geometrically irreducible commutative linear algebraic group over \( k(B) \). As a consequence ([10], Chap. 16.b), there exists a finite extension \( L \) of \( k(B) \) and an integer \( s \geq 0 \) such that

\[
\langle W \rangle_\eta \otimes_{k(B)} L \simeq U_L \times \mathbb{G}_m^s_L
\]

where \( U_L \) is a unipotent commutative linear algebraic group over \( L \).

Let \( \psi : B' \rightarrow B \) be the normalization of \( B \) in \( L \). We obtain a new fibration \( \pi' : Y \times_B B' \rightarrow B' \), together with an embedding \( \iota_\psi \) of \( \text{Aut}(Y) \) in \( \text{Aut}_\pi(Y \times_B B') \); by Lemma 3.1, the subgroup \( \langle W \rangle \) has bounded degree if and only if its image \( \iota_\psi(W) \) has bounded degree too. After such a base change, we may assume that
Replacing \( B \) on this basic remark, with two extra difficulties: the structure of where \( U \) acts on \( Y \) subtle in positive characteristic (see [13], §VII.2); instead of iterating one element \( n \) dependently of \( na \) where each \( a \) \nomenclature{\( W \)}{acts on \( Y \) by translation; here \( U(B) \) and \( \mathbb{G}_{m,B}^s(B) \) denote the ind-varieties of sections of the structure morphisms \( U_B \to B \) and \( \mathbb{G}_{m,B}^s \to B \) respectively.

Remark 3.4. A section \( \sigma : B \to U_B \) defines an automorphism of \( U_B \simeq B \times B U_B \) by \( \delta(\sigma \times _B \id) \), where \( \delta : U_B \times U_B \to U_B \) is the multiplication morphism of \( U_B \); it defines in the same way an element of \( \Aut_\pi(Y) \). Similarly \( \mathbb{G}_{m,B}^s(B) \) embeds into \( \Aut_\pi(Y) \), so \( U_B(B) \times \mathbb{G}_{m,B}^s(B) \subseteq \Aut_\pi(Y) \), and this is the meaning of (3.14).

Note that \( U_B(B) \times \mathbb{G}_{m,B}^s(B) \) and \( \Aut_\pi(Y) \) are ind-varieties over \( k \) and the inclusions in (3.14) are morphisms between ind-varieties.

So, now, to prove Theorem B, we only need to show that \( W \) is contained in an algebraic subgroup of \( U_B(B) \times \mathbb{G}_{m,B}^s(B) \).

3.4.3. Structure of \( U_B \).

Lemma 3.5. The ind-group \( U_B(B) \) is a union of algebraic groups.

Before describing the proof, let us assume that \( U_B \) is just an \( r \)-dimensional additive group \( \mathbb{G}_{a,B}^r \). Then, each element of \( U_B \) can be written

\[
(3.15) \quad f = (a^1(z), \ldots, a^r(z))
\]

where each \( a^j(z) \) is an element of \( \mathbb{O}(B) \); its \( n \)-th iterate is given by \( f^n = (na^1(z), \ldots, na^r(z)) \). Then, the degree of \( f^n \), viewed as an automorphism of \( Y \), is bounded, independently of \( n \), by (a function of) the degrees of the \( a^j \). Our proof is a variation on this basic remark, with two extra difficulties: the structure of \( U_B \) may be more subtle in positive characteristic (see [13], §VII.2); instead of iterating one element \( f \), we need to control the group \( U_B \) itself.

Proof. Denote by \( \pi : U_B \to B \) the structure morphism. Fix some dominant morphism \( \pi_0 : B_0 \to B \) with \( B_0 \) an affine variety. The morphism \( \iota_{\pi_0} : U_B(B) \hookrightarrow (U_B \times B B_0)(B_0) \) defined by

\[
(3.16) \quad (s : B \to U_B) \mapsto (s \times_B \id : B_0 \to U_B \times_B B_0)
\]

is an embedding of ind-groups. To prove Lemma 3.5, we may always do such a base change, so we might assume that \( B \) is affine.

We prove Lemma 3.5 by induction on the relative dimension of \( \pi : U_B \to B \). If the relative dimension is zero, there is nothing to prove. So, we assume that the lemma holds for relative dimensions at most \( \leq \ell \), with \( \ell \geq 0 \), and we want to prove it when the relative dimension is \( \ell + 1 \).
Denote by $U_\eta$ the generic fiber of $\pi$. There exists a finite field extension $L$ of $k(B)$ such that $U_L := U_\eta \otimes_{k(B)} L$ is in an exact sequence:

$$(3.17) \quad 0 \to G_{a,L} \to U_L \xrightarrow{q_L} V_L \to 0,$$

where $V_L$ is an irreducible unipotent group of dimension $\ell$ and $V_L$ is isomorphic to $A^\ell_L$ as an $L$-variety; moreover, there is an isomorphism of $L$-varieties $\phi_L: U_L \to V_L \times G_{a,L}$ such that the quotient morphism $q_L$ is given by the projection onto the first factor. So we have a section $s_L: V_L \to U_L$ such that $q_L \circ s_L = id$ (see [13]).

Doing the base change given by the normalization of $B$ in $L$, and then shrinking the base if necessary, we may assume that

- there is an exact sequence of group schemes over $B$:

$$(3.18) \quad 0 \to G_{a,B} \to U_B \xrightarrow{q_B} V_B \to 0,$$

where $V_B$ is a unipotent group scheme over $B$ of relative dimension $\ell$;
- there is an isomorphism of $B$-schemes $V_B \cong A^\ell_B$;
- $s_L$ extends to a section $s_B: V_B \to U_B$ over $B$: $q_B \circ s_B = id$.

For $b \in B$, denote by $U_b$, $V_b$, $q_b$, $s_b$ the specialization of $U_B$, $V_B$, $q_B$, $s_B$ at $b$. The morphism of $B$-schemes $\beta: U_B \to V_B \times G_{a,B}$ sending a point $x$ in the fiber $U_b$ to the point $(q_b(x), y - s_b(x))$ of the fiber $V_b \times G_{a,b}$ defines an isomorphism. We use $\beta$ to transport the group law of $U_B$ into $V_B \times G_{a,B}$; this defines a law $*$ on $V_B \times G_{a,B}$, given by

$$(3.19) \quad a_1 \ast a_2 = \beta(\beta^{-1}(a_1) + \beta^{-1}(a_2)),$$

for $a_1$ and $a_2$ in $V_B \times G_{a,B}$. Denote by $O(V_B \times B)$ the function ring of the $k$-variety $V_B \times B$. There is an element $F(b, x_1, x_2)$ of $O(V_B \times B)[y_1, y_2]$ such that

$$(3.20) \quad F(b, x_1, x_2)(y_1, y_2) = C_0(b, x_1, x_2)(y_1) + C_2(b, x_1, x_2)(y_1) y_2.$$

The function $C_2(b, x_1, x_2)(y_1)$ does not vanish on $V_B \times B$, so $C_1 \otimes B^1 \simeq A^{2\ell + 1} \times B$; thus, $C_2$ is an element of $k(B)$. By symmetry we get

$$(3.21) \quad F(b, x_1, x_2)(y_1, y_2) = C_0(b, x_1, x_2) + C_1(b) y_1 + C_2(b) y_2$$

and

$$(3.22) \quad (x_1, y_1) \ast (x_2, y_2) = (x_1 + x_2, C_0(b, x_1, x_2) + C_1(b) y_1 + C_2(b) y_2).$$

Now, apply this equation for $x_1 = x_2 = 0$ (the neutral element of $V_B$), to deduce that $C_1$ and $C_2$ are both equal to the constant function 1 on $B$. 

CHARACTERIZATION OF THE AFFINE SPACE 11
We identify now the ind-varieties $U_B(B)$ and $V_B(B) \times \mathbb{G}_a(B)$. Then, each element of $U_B(B)$ is given by a section $(S, T) \in V_B(B) \times \mathbb{G}_a(B)$; we shall define its degree to be $\deg(S, T) := \max\{\deg(S), \deg(T)\}$. And for $d \in \mathbb{N}$, we denote by $V_B(B)_d$ (resp. $\mathbb{G}_a(B)_d$) the subspace of sections of degree at most $d$ in $V_B(B)$ (resp. $\mathbb{G}_a(B)$).

The group law in $U_B(B)$ corresponds to the law

$$ (S_1, T_1) \ast (S_2, T_2) = (S_1 + S_2, C_0(S_1, S_2) + T_1 + T_2) $$

because $C_1 = C_2 = 1$; here $C_0 : V_B(B) \times V_B(B) \to \mathbb{G}_a(B)$ is a morphism of ind-varieties. There exists an increasing function $\alpha : \mathbb{N} \to \mathbb{N}$ such that

$$ \deg(C_0(S_1, S_2)) \leq \alpha(d) $$

for all sections $S_1$ and $S_2 \in V_B(B)_d$.

By the induction hypothesis, there is an increasing function $\gamma : \mathbb{N} \to \mathbb{N}$ such that the group $(V_B(B)_d)$ is an algebraic group contained in $V_B(B)_{\gamma(d)}$. It follows that

$$ U_B(B) = \bigcup_{d \geq 0} (V_B(B)_d) \times \mathbb{G}_a(B)_{\alpha(\gamma(d))}. $$

To conclude, we only need to prove that each $(V_B(B)_d) \times \mathbb{G}_a(B)_{\alpha(\gamma(d))}$ is an algebraic group. But this follows from (3.23) and (3.24) because

$$ \deg(C_0(S_1, S_2) + T_1 + T_2) \leq \max\{\deg(C_0(S_1, S_2)), \deg(T_1), \deg(T_2)\} $$

(3.26)

$$ \leq \alpha(\gamma(d)) $$

for all $(S_1, T_1)$ and $(S_2, T_2)$ in $(V_B(B)_d) \times \mathbb{G}_a(B)_{\alpha(\gamma(d))}$. \hfill \qed

3.4.4. Subgroups of $\mathbb{G}_m^s(B)$ and conclusion.

**Lemma 3.6.** If $Z$ is an irreducible subvariety of $\mathbb{G}_m^s(B)$ containing id, then $\langle Z \rangle$ is an algebraic subgroup of $\mathbb{G}_m^s(B)$.

**Proof of Lemma 3.6.** Pick a projective compactification $\overline{B}$ of $B$. After taking the normalization of $\overline{B}$, we may assume $\overline{B}$ to be normal. If $h$ is any non-constant rational function on $\overline{B}$, denote by $\text{Div}(h)$ the divisor $(h)_0 - (h)_\infty$ on $\overline{B}$.

Let $y = (y_1, \ldots, y_s)$ be the standard coordinates on $\mathbb{G}_m^s$. Each element $f \in \mathbb{G}_m^s(B)$ can be written as $(b^f_1(z), \ldots, b^f_s(z))$, for some $b^f_j \in O^*(B)$. Let $R$ be an effective divisor whose support $\text{Support}(R)$ contains $\overline{B} \setminus B$. Replacing $R$ by some large multiple, $Z$ is contained in the subset $P_R$ of $\mathbb{G}_m^s(B)$ made of automorphisms $f \in \mathbb{G}_m^s(B)$ such that $\text{Div}(b^f_j) + R \geq 0$ and $\text{Div}(1/b^f_j) + R \geq 0$ for all $i = 1, \ldots, s$. Let us study the structure of this set $P_R \subset \mathbb{G}_m^s(B)$.

Let $K$ be the set of pairs $(D_1, D_2)$ of effective divisors supported on $\overline{B} \setminus B$ such that $D_1$ and $D_2$ have no common irreducible component, $D_1 \leq R$, $D_2 \leq R$ and $D_1$ and $D_2$ are rationally equivalent. Then $K$ is a finite set. For every pair $\alpha = (D^\alpha_1, D^\alpha_2) \in K$, we choose a function $h_\alpha \in O^*(Y)$ such that $\text{Div}(h_\alpha) = D^\alpha_1 - D^\alpha_2$; if $h$ is another element of $O^*(Y)$ such that $\text{Div}(h) = D_1 - D_2$, then $h/h_\alpha \in \mathbb{k}^*$. By
convention \( \alpha = 0 \) means that \( \alpha = (0, 0) \), and in that case we choose \( h_\alpha \) to be the constant function 1. For every element \( \beta = (\alpha_1, \ldots, \alpha_s) \in K^s \), denote by \( P_\beta \) the set of elements \( f \in G_m^n(B) \) such that \( b_i^f \in \mathcal{O}^n(B) \) satisfies \( \text{Div}(b_i^f) = D_1^{a_1} - D_2^{a_2} \) for every \( i = 1, \ldots, s \). Then \( P_\beta \cong G_m^n(k) \) is an irreducible algebraic variety. Moreover, \( \id \in P_\beta \) if and only if \( \beta = 0 \), and \( P_0 \) is an algebraic subgroup of \( G_m^n(B) \).

Observe that \( P_R \) is the disjoint union \( P_R = \bigsqcup_{\beta \in K^s} P_\beta \). Since \( \id \in Z \), \( Z \) is irreducible, and \( Z \subseteq P_R \), we obtain \( W \subseteq P_0 \). Since \( P_0 \) is an algebraic subgroup of \( G_m^n(B) \), \( \langle Z \rangle \) coincides with \( Z^\ell \) for some \( \ell \geq 1 \), and \( \langle Z \rangle \) is a connected algebraic group.

\[ \square \]

Proof of Theorem B. By Proposition 2.2, we only need to prove that \( W = \langle V \rangle \) is of bounded degree. By Lemma 3.1 \( W \) is a subgroup of bounded degree if and only if \( W \subset \text{Aut}_\pi(Y) \) is a subgroup of bounded degree. Moreover, by (3.14), \( W \) is a subgroup of \( U_B(B) \times G_m^n(B) \subset \text{Aut}_\pi(Y) \). Denote by \( \pi_1 : U_B(B) \times G_m^n(B) \to U_B(B) \) the projection to the first factor and \( \pi_2 : U_B(B) \times G_m^n(B) \to G_m^n(B) \) the projection to the second. By Lemma 3.5, there exists an algebraic subgroup \( H_1 \) of \( U_B(B) \) containing \( \pi_1(W) \). Since \( \pi_2(W) \) is irreducible and contains \( \id \), Lemma 3.6 shows that \( \pi_2(W) \) is contained in an algebraic subgroup \( H_2 \) of \( G_m(B) \). Then \( W \) is contained in the algebraic subgroup \( H_1 \times H_2 \) of \( U_B(B) \times G_m^n(B) \). This concludes the proof. \[ \square \]

4. Actions of additive groups

Theorem 4.1. Let \( k \) be an uncountable, algebraically closed field. Let \( X \) be a connected affine variety over \( k \). Let \( G \subset \text{Aut}(X) \) be an algebraic subgroup isomorphic to \( G_r^r \), for some \( r \geq 1 \). Let \( H = \{ h \in \text{Aut}(X) \mid gh = hg \text{ for every } g \in G \} \) be the centralizer of \( G \). If \( H/G \) is at most countable then \( G \) acts simply transitively on \( X \), so that \( X \) is isomorphic to \( G \) as a \( G \)-variety.

This section is devoted to the proof of this result. A proof is also described in [3, §11.4] when \( X \) is irreducible and the characteristic of \( k \) is 0.

Lemma 4.2. Let \( X \) be an irreducible affine variety endowed with a faithful action of \( G = G_r^r \). Let \( I \) be a non-zero, \( G \)-invariant ideal of \( O(X) \). If

\[ I^G := \{ \xi \in I \mid g^*\xi = \xi \text{ for every } g \in G \} \]

is contained in the field \( k \) of constant functions, then

1. every non-zero, \( G \)-invariant ideal \( J \subset O(X) \) coincides with \( O(X) \);
2. \( G \) acts simply transitively on \( X \);
3. \( X \) can be identified to \( G \), with \( G \) acting on it by translations.

In particular, \( I = O(X) \).

Proof of Lemma 4.2. Let \( \xi \) be a non-zero element of \( I \). To prove the first assertion, pick \( \psi \in J \setminus \{0\} \), then \( \xi \psi \in IJ \setminus \{0\} \). Let \( V \) be the linear subspace of \( O(X) \) generated by the orbit \( \{ g^*(\xi \psi) \mid g \in G \} \). Firstly, \( V \) is contained in \( IJ \) because \( I \) and \( J \)
are $G$-invariant. Secondly, the dimension of $V$ is finite, for $G$ acts regularly on $X$ (see [17, §1.2]). Thus, $G$ being isomorphic to $\mathbb{G}_a^r$, there exists a $G$-invariant vector $\varphi \in V \setminus \{0\} \subseteq J$. Since $I^G \subseteq k$, the function $\varphi$ is a constant, and $J$ must be equal to $O(X)$ because it contains $\varphi$.

To prove the second and third assertions, fix a point $x \in X$. The closure $\overline{G(x)}$ of the orbit $G(x)$ is a closed, $G$-invariant subvariety, and the same is true for $X \setminus G(x)$. Looking at the ideal of functions vanishing on those subvarieties we obtain $X = G(x)$. Since $G$ is abelian and acts faithfully on $X$, the stabilizer of $x$ must be trivial. Thus, $G$ acts simply transitively on $X$.  

Let $X$ be an affine variety over $k$, and let $G$ be a subgroup of $\text{Aut}(X)$ isomorphic to $\mathbb{G}_a^r$ for some $r > 0$. Denote by $X_1, \ldots, X_l$ the irreducible components of $X$. Then all $X_i$, $i = 1, \ldots, l$, are invariant under $G$; permuting the $X_i$ if necessary, there exists $s \leq l$ such that the action of $G$ on $X_i$ is nontrivial if and only if $i \leq s$.

For every $i \leq l$, denote by $\pi_i : O(X) \to O(X_i)$ the quotient map. Let $J_i$ be the ideal of functions $\xi \in O(X)$ vanishing on the closed subset $\cup_{j \neq i} X_j$; its projection $I_i := \pi_i(J_i)$ is an ideal of $O(X_i)$. Observe that $I_i$ is non-zero, is invariant under the action of $G$, and is contained in the ideal of $O(X_i)$ associated to the closed subset $X_i \cap (\cup_{j \neq i} X_j)$. In particular, $I_i = O(X_i)$ if and only if $X_i$ is a connected component of $X$.

The homomorphism $\pi_i|_{J_i} : J_i \to I_i$ is a bijection. Indeed, it is a surjective homomorphism by definition. And if $\pi_i|_{J_i} (\xi) = 0$, then $\xi|_{X_i} = 0$ and since $\xi \in J_i$, $\xi|_{X_j} = 0$ for all $j \neq i$, thus $\xi = 0$, so that $\ker(\pi_i|_{J_i}) = 0$.

We denote by $(\pi_i|_{J_i})^{-1} : I_i \to J_i$ the inverse of $\pi_i|_{J_i}$.

**Lemma 4.3.** Let $k$ be an uncountable, algebraically closed field. Let $X$ be an affine variety, and $G$ be an algebraic subgroup of $\text{Aut}(X)$ isomorphic to $\mathbb{G}_a^r$. Let

$$H := \{ h \in \text{Aut}(X) \mid gh = hg \text{ for every } g \in G \}$$

be the centralizer of $G$. If $H/G$ is at most countable then $I_i^G \subseteq k$ for every irreducible component $X_i$ on which $G$ acts non-trivially.

**Proof of Lemma 4.3.** Keeping the notation that precedes the statement of the lemma, we only need to treat the case $i = 1$. Fix an identification $\mathbb{G}_a^r \simeq G$. Then, we get an identification

$$O(X) = \text{Mor}(X, \mathbb{G}_a^r) = \text{Mor}(X, G)$$

for which $(O(X)^G)'$ corresponds to $\text{Mor}(X, G)^G$ (here $G$ acts on $X$ only). We also identify $G$ to the group of constant morphisms in $\text{Mor}(X, G)$; then, $G$ becomes a subgroup of the additive group $(O(X)^G)'$. Let us modify the action of $G$ on $X$, as in [3, §0.7]:

$$O(X) = \text{Mor}(X, \mathbb{G}_a^r) = \text{Mor}(X, G)$$
Fact 1.– Define $\Psi : (O(X)^G)^r \to \text{End}(X)$ by $\Psi(\xi) : x \mapsto \xi(x)(x)$ for every $x \in X$. Then $\Psi$ is a homomorphism of additive groups.

We need to prove that $\Psi(\xi_1 + \xi_2) = \Psi(\xi_1) \circ \Psi(\xi_2)$ for every pair of elements $\xi_1, \xi_2 \in (O(X)^G)^r$. For $x \in X$, we have

$$\Psi(\xi_1 + \xi_2)(x) = (\xi_1(x) + \xi_2(x))(x) = \xi_1(x)((\xi_2(x)(x))).$$

On the other hand, we know that $\xi_1(y) = \xi_1(\xi_2(x)(y))$ for every pair $(x,y) \in X \times X$, because $\xi_2(x) \in G$ and $\xi_1 \in (O(X)^G)^r = \text{Mor}(X,G)^G$. We obtain $\xi_1(x) = \xi_1(\xi_2(x)(x))$, and the following computation proves the claim

$$\Psi(\xi_1 + \xi_2)(x) = \xi_1(\xi_2(x)(x))((\xi_2(x)(x))) = \Psi(\xi_1)(\Psi(\xi_2)(x)).$$

This fact implies that $\Psi$ is a homomorphism from the additive group $(O(X)^G)^r$ to the group of automorphisms $\text{Aut}(X)$. We note that $\Psi|_G = \text{id}$; since $(O(X)^G)^r$ is abelian, $\Psi((O(X)^G)^r)$ is a subgroup of the centralizer $H$ that contains $G$. Since $H/G$ is countable, so is $(O(X)^G)^r/\Psi^{-1}(G)$.

Now, define $\Phi : (I_1^G)^r \to \text{Aut}(X)$ to be the composition of $\Psi$ with the inclusion $((\pi_1|_r)^{-1})' : (I_1^G)^r \hookrightarrow (O(X)^G)^r$. We obtain an inclusion

$$(I_1^G)^r/\Phi^{-1}(G) \hookrightarrow (O(X)^G)^r/\Psi^{-1}(G);$$

hence, $(I_1^G)^r/\Phi^{-1}(G)$ is also countable.

Fact 2.– We have $\Phi^{-1}(G) = (I_1^G)^r$.

To prove this equality, denote by $G_x$ the stabilizer of $x \in X$ in $G$, and for $\xi \in (I_1^G)^r$, set

$$Y(\xi) := \bigcap_{x \in X} (\xi(x) + G_x).$$

Then $Y(\xi)$ is an affine linear subspace of $G \simeq \mathbb{A}^r(k) = k^r$, and $\xi \in \Phi^{-1}(G)$ if and only if $Y(\xi) \neq \emptyset$. It follows that $\Phi^{-1}(G)$ is a linear subspace of $(I_1^G)^r$. Since $k$ is uncountable and $(I_1^G)^r/\Phi^{-1}(G)$ is at most countable, we get $\Phi^{-1}(G) = (I_1^G)^r$.

It follows that for every $\xi \in (I_1^G)^r$,

$$0 \neq Y(\xi) = \bigcap_{x \in X} (\xi(x) + G_x) \subseteq W(\xi) := \bigcap_{x \in X_1} (\xi(x) + G_x).$$

Choose $\eta \in (I_1^G)^r$ such that $\dim(W(\eta))$ is minimal, and then choose $x_1, \ldots, x_m \in X_1$, such that $W(\eta) = \bigcap_{i=1}^m (\eta(x_i) + G_{x_i})$. To conclude, we assume that $I_1^G$ contains a non-constant function $\alpha$, and then we shall modify $\eta$ to get a new function $\tau$ with $\dim(W(\tau)) < \dim(W(\eta))$, in contradiction with our choice for $\eta$. For this purpose, set $\beta = \prod_{i=1}^m (\alpha - \alpha(x_i))$. Then, choose $y \in X_1 \setminus \{x_1, \ldots, x_m\}$ such that $G_y \neq G$, $\alpha(y) \neq 0$ and $\beta(y) \neq 0$, and set

$$\gamma := \frac{\alpha \beta}{\beta(y) \alpha(y)}.$$
By construction, we get

1) $\gamma \in O(X_1)^G I_1^G \subseteq I^G$;
2) $\gamma(x_i) = 0$ for all $i = 1, \ldots, m$;
3) $\gamma(y) = 1$.

Pick $g \in W(\xi)$. The set $U$ of elements $h = (a_1, \ldots, a_r) \in G$ such that $g \notin h + \xi(y) + G_y$ is Zariski dense in $G$. Take $(a_1, \ldots, a_r) \in U$ and set

$$\tau := \xi + (a_1 \gamma, \ldots, a_r \gamma).$$

By construction, $\tau$ is an element of $(I^G)^r$; and, changing $(a_1, \ldots, a_r)$ in $U$ if necessary, we may assume that $\tau \notin k^r$. From the properties (2) and (3) above, we get $\tau(x_i) = \eta(x_i)$ for $i = 1, \ldots, m$ and $\tau(y) = \eta(y) + (a_1, \ldots, a_r)$. We have

$$W(\tau) \subseteq \left( \bigcap_{i=1}^m (\tau(x_i) + G_{x_i}) \right) \cap (\tau(y) + G_y)$$

$$= \left( \bigcap_{i=1}^m (\eta(x_i) + G_{x_i}) \right) \cap (\eta(y) + (a_1, \ldots, a_r) + G_y)$$

Since $g \in W(\eta)$ but $g \notin (\eta(y) + (a_1, \ldots, a_r) + G_y)$, we get $\dim W(\tau) \leq \dim W(\eta) \cap (\eta(y) + (a_1, \ldots, a_r) + G_y) < \dim W(\eta)$. This proves the lemma.

**Proof of Theorem 4.1.** We keep the same notation. By Lemma 4.3, $I_1^G \subseteq k$. Let $G_1 \subset \text{Aut}(X_1)$ be the restriction of $G$. There exists $m \in \{1, \ldots, r\}$, such that $G_1 \cong \mathbb{G}_a^m$. We have $I_1^G = I_1^G \subseteq k$. By Lemma 4.2, $I_1 = O(X_1)$ and $X_1 \cong G_1$ as a $G_1$-variety. Since $I_1 = O(X_1)$, $X_1$ is a connected component of $X$. Since $X$ is connected, $X = X_1$ and $G_1 = G$. This concludes the proof.

5. PROOF OF THEOREM A

In this section, we prove Theorem A. So, $k$ is an uncountable, algebraically closed field, $X$ is a connected affine algebraic variety over $k$, and $\phi : \text{Aut}(\mathbb{A}^n_k) \to \text{Aut}(X)$ is an isomorphism of (abstract) groups.

5.1. Translations and dilatations. Let $\text{Tr} \subset \text{Aut}(\mathbb{A}^n_k)$ be the group of all translations and $\text{Tr}_i$ the subgroup of translations of the $i$-th coordinate:

$$\begin{align*}
(x_1, \ldots, x_n) &\mapsto (x_1, \ldots, x_i + c, \ldots, x_n) \\
\end{align*}$$

for some $c$ in $k$. Let $D \subset \text{GL}_n(k) \subset \text{Aut}(\mathbb{A}^n_k)$ be the diagonal group (viewed as a maximal torus) and let $D_i$ be the subgroup of automorphisms

$$\begin{align*}
(x_1, \ldots, x_n) &\mapsto (x_1, \ldots, ax_i, \ldots, x_n) \\
\end{align*}$$

for some $a \in k^*$. A direct computation shows that $\text{Tr}$ (resp. $D$) coincides with its centralizer in $\text{Aut}(\mathbb{A}^n_k)$. 
Lemma 5.1. Let $G$ be a subgroup of $\text{Tr}$ whose index is at most countable. Then, the centralizer of $G$ in $\text{Aut}(\mathbb{A}^n)$ is $\text{Tr}$.

Proof of Lemma 5.1. The centralizer of $G$ contains $\text{Tr}$. Let us prove the reverse inclusion. The index of $G$ in $\text{Tr}$ being at most countable, $G$ is Zariski dense in $\text{Tr}$. Thus, if $h$ centralizes $G$, we get $hg = gh$ for all $g \in \text{Tr}$, and $h$ is in fact in the centralizer of $\text{Tr}$. Since $\text{Tr}$ coincides with its centralizer, we get $h \in \text{Tr}$. □

5.2. Closed subgroups. As in Section 2.2, we endow $\text{Aut}(X)$ with the structure of an ind-group, given by a filtration by algebraic varieties $\text{Aut}_j$ for $j \geq 1$.

Lemma 5.2. The groups $\varphi(\text{Tr})$, $\varphi(\text{Tr}_1)$, $\varphi(\text{D})$ and $\varphi(\text{D}_i)$ are closed subgroups of $\text{Aut}(X)$ for all $i = 1, \ldots, n$.

Proof. Since $\text{Tr} \subset \text{Aut}(\mathbb{A}^n)$ coincides with its centralizer, $\varphi(\text{Tr}) \subset \text{Aut}(X)$ coincides with its centralizer too and, as such, is a closed subgroup of $\text{Aut}(X)$. The same argument applies to $\varphi(\text{D}) \subset \text{Aut}(X)$. To prove that $\varphi(\text{Tr}_i) \subset \text{Aut}(X)$ is closed we note that $\varphi(\text{Tr}_i)$ is the subset of elements $f \in \varphi(\text{Tr})$ that commute to every element $g \in \varphi(\text{D}_j)$ for every index $j \neq i$ in $\{1, \ldots, n\}$. Analogously, $\varphi(\text{D}_i) \subset \text{Aut}(X)$ is a closed subgroup because an element $f$ of $\text{D}$ is in $\text{D}_i$ if and only if it commutes to all elements $g$ of $\text{Tr}_j$ for $j \neq i$. □

5.3. Proof of Theorem A.

5.3.1. Abelian groups (see [13]). Before starting the proof, let us recall a few important facts on abelian, affine algebraic groups. Let $G$ be an algebraic group over the field $k$, such that $G$ is abelian, affine, and connected.

1. If $\text{char}(k) = 0$, then $G$ is isomorphic to $\mathbb{G}_a^r \times \mathbb{G}_m^s$ for some pair of integers $(r, s)$; if $G$ is unipotent, then $s = 0$. (see [13], §VII.2, p.172)

When the characteristic $p$ of $k$ is positive, there are other types of of abelian groups, but criteria on the $p$-torsion may rigidify their structure:

2. If $\text{char}(k) = p$, $G$ is unipotent, and all elements of $G$ have order $p$, then $G$ is isomorphic to $\mathbb{G}_a^r$ for some $r \geq 0$. (see [13], §VII.2, Prop. 11, p.178)

3. If $\text{char}(k) = p$, and there is no non-trivial element in $G$ of order $p^\ell$, for any $\ell \geq 0$, then $G$ is isomorphic to $\mathbb{G}_m^s$ for some $s \geq 0$. (see [10], Theorem 16.13 and Corollary 16.15, and [13], §VII.2, p.176)

To keep examples in mind, note that all elements of $\text{Tr}_1(k)$ have order $p$ and $\text{D}_1(k)$ does not contain any non-trivial element of order $p^\ell$ when $\text{char}(k) = p$.

5.3.2. Proof of Theorem A. Let us now prove Theorem A.

By Lemma 5.2, $\varphi(\text{Tr}_1) \subset \text{Aut}(X)$ is a closed subgroup; in particular, $\varphi(\text{Tr}_1)$ is an ind-subgroup of $\text{Aut}(X)$. Let $\varphi(\text{Tr}_1)^\circ$ be the connected component of the identity of $\varphi(\text{Tr}_1)$; from Section 2.2, we know that the index of $\varphi(\text{Tr}_1)^\circ$ in $\varphi(\text{Tr}_1)$ is at
most countable. The ind-group \( \varphi(\Tr_1)^{\circ} \) is an increasing union \( \bigcup_i V_i \) of irreducible algebraic varieties \( V_i \), each \( V_i \) containing the identity. Theorem B implies that each \( \langle V_i \rangle \) is an irreducible algebraic subgroup of \( \Aut(X) \). Since \( \varphi(\Tr_1) \) does not contain any element of order \( k < \infty \) with \( k \cap \text{char}(k) = 1 \), \( \langle V_i \rangle \) is unipotent. And, by Properties (1) and (2) of Section 5.3.1, \( \langle V_i \rangle \) is isomorphic to \( G_{i}^{\rho} \) for some \( r_i \). Thus

\[
(5.3) \quad \varphi(\Tr_1)^{\circ} = \bigcup_{i \geq 0} F_i
\]

where the \( F_i \) form an increasing family of unipotent algebraic subgroups of \( \Aut(X) \), each of them isomorphic to some \( G_{i}^{\rho} \). We may assume that \( \dim F_0 \geq 1 \).

Similarly, \( \varphi(D_1)^{\circ} \subset \varphi(D_1) \) is a subgroup of countable index and

\[
(5.4) \quad \varphi(D_1)^{\circ} = \bigcup_{i \geq 0} G_i,
\]

where the \( G_i \) are increasing irreducible commutative algebraic subgroups of \( \Aut(X) \) (we do not assert that \( G_i \) is of type \( \mathbb{G}_{m}^{\nu} \) yet). We may assume that \( \dim G_0 \geq 1 \).

The group \( D_i \) acts by conjugation on \( \Tr_i \); for every \( i \leq n \), this action has exactly two orbits \( \{ \text{id} \} \) and \( \Tr_i \setminus \{ \text{id} \} \), and the action on \( \Tr_i \setminus \{ \text{id} \} \) is free; hence, the same properties hold for the action of \( \varphi(D_i) \) on \( \varphi(\Tr_i) \) by conjugation.

Let \( H_i \) be the subgroup of \( \varphi(\Tr_1) \) generated by all \( g \circ f \circ g^{-1} \) with \( f \in F_i \) and \( g \in G_i \). Theorem B shows that \( H_i \) is an irreducible algebraic subgroup of \( \varphi(\Tr_1) \). We have \( H_i \subseteq H_{i+1} \) and \( g \circ H_i \circ g^{-1} = H_i \) for every \( g \in G_i \).

Write \( H_i = G_{i}^{\rho} \) for some \( l \geq 1 \). We claim that \( G_i \simeq \mathbb{G}_{u}^{\nu} \times \mathbb{G}_{m}^{\nu} \) for a pair of integers \( r, s \geq 0 \) with \( r + s \geq 1 \). This follows from Properties (1) and (2) of Section 5.3.1 because, when \( \text{char}(k) = p > 1 \), the only element in \( \varphi(D_1) \) of order \( p^f \), \( f \geq 0 \), is the identity element. Since the action of \( \varphi(D_1) \) on \( \varphi(\Tr_1 \setminus \{ 0 \}) \) is free, the action of \( G_i \) on \( F_i \setminus \{ 0 \} \) is free, and this implies \( r = 0 \) (see Lemma 4.2(2)). Let \( q \) be a prime number with \( q \cap \text{char}(k) = 1 \). Then \( \mathbb{G}_{m}^{\nu} \) contains a copy of \( (\mathbb{Z}/q\mathbb{Z})^{\nu} \), and \( D_1 \) does not contain such a subgroup if \( s > 1 \); so, \( s = 1 \), \( G_i \simeq \mathbb{G}_{m} \) and \( G_i = G_{i+1} \) for all \( i \geq 0 \). It follows that \( \varphi(D_1)^{\circ} \simeq \mathbb{G}_{m} \). Since the index of \( \varphi(D_1)^{\circ} \) in \( \varphi(D_1) \) is countable, there exists a countable subset \( J \subseteq \varphi(D_1) \) such that \( \varphi(D_1) = \bigcup_{h \in J} \varphi(D_1)^{\circ} \circ h \). Let \( f \in F_i \) be a nontrivial element. Since the action of \( \varphi(D_1) \) on \( \varphi(\Tr_1 \setminus \{ 0 \}) \) is transitive,

\[
(5.5) \quad F_i \setminus \{ 0 \} = \bigcup_{h \in J} \left( \bigcup_{g \in \varphi(D_1)^{\circ}} (g \circ h) \circ f \circ (g \circ h)^{-1} \right) \cap F_i.
\]

The right hand side is a countable union of subvarieties of \( F_i \setminus \{ 0 \} \) of dimension at most one. It follows that \( \dim F_i = 1 \), \( F_i \simeq \mathbb{G}_{u} \), and \( \varphi(\Tr_1)^{\circ} \simeq \mathbb{G}_{u} \). Thus, we have

\[
(5.6) \quad \varphi(\Tr_1)^{\circ} \simeq \mathbb{G}_{u}, \quad \text{and} \quad \varphi(D_1)^{\circ} \simeq \mathbb{G}_{m}.
\]

Since each \( \varphi(\Tr_1)^{\circ} \) is isomorphic to \( \mathbb{G}_{u} \), \( \varphi(\Tr)^{\circ} \) is an \( n \)-dimensional commutative unipotent group and its index in \( \varphi(\Tr) \) is at most countable. By Lemma 5.1, the centralizer of \( \varphi^{-1}(\varphi(\Tr)^{\circ}) \) in \( \Aut(\mathbb{A}_{k}^{\nu}) \) is \( \Tr \). It follows that the centralizer of \( \varphi(\Tr)^{\circ} \) in \( \varphi(\Tr) \) is \( \varphi(\Tr) \). Then Theorem 4.1 implies that \( X \) is isomorphic to \( \mathbb{A}_{k}^{\nu} \).
REFERENCES


UNIVERSITÉS, CNRS, IRMAR - UMR 6625, F-35000 RENNES, FRANCE
E-mail address: serge.cantat@univ-rennes1.fr, junyi.xie@univ-rennes1.fr

INSTITUT FÜR MATHEMATIK, FRIEDRICH-SCHILLER-UNIVERSITÄT JENA,
JENA 07737, GERMANY
E-mail address: andriyregeta@gmail.com