FAMILIES OF COMMUTING AUTOMORPHISMS, AND A CHARACTERIZATION OF THE AFFINE SPACE

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ABSTRACT. We prove that the affine space of dimension $n \geq 1$ over an uncountable algebraically closed field $\mathbf{k}$ is determined, among connected affine varieties, by its automorphism group (viewed as an abstract group). The proof is based on a new result concerning algebraic families of pairwise commuting automorphisms.

1. INTRODUCTION

1.1. **Characterization of the affine space.** In this paper, $\mathbf{k}$ is an algebraically closed field and $\mathbb{A}^n_{\mathbf{k}}$ denotes the affine space of dimension n over $\mathbf{k}$.

Theorem A.– *Let* $\mathbf{k}$ *be an algebraically closed and uncountable field. Let n be a positive integer. Let X be a reduced, connected, affine variety over* $\mathbf{k}$. *If its automorphism group* $\mathsf{Aut}(X)$ *is isomorphic to* $\mathsf{Aut}(\mathbb{A}^n_{\mathbf{k}})$ *as an abstract group, then X is isomorphic to* $\mathbb{A}^n_{\mathbf{k}}$ *as a variety over* $\mathbf{k}$.

Note that no assumption is made on $\dim(X)$; in particular, we do not assume $\dim(X) = n$. This theorem is our main goal. It would be great to lighten the hypotheses on $\mathbf{k}$, but besides that the following remarks show the result is optimal:

• The affine space $\mathbb{A}^n_{\mathbf{k}}$ is not determined by its automorphism group in the category of quasi-projective varieties because

(1) $\mathsf{Aut}(\mathbb{A}^n_{\mathbf{k}})$ is naturally isomorphic to $\mathsf{Aut}(\mathbb{A}^n_{\mathbf{k}} \times Z)$ for any projective variety Z with $\mathsf{Aut}(Z) = \{\mathsf{id}\}$;

(2) for every algebraically closed field $\mathbf{k}$ there is a projective variety Z over $\mathbf{k}$ such that $\dim(Z) \geq 1$ and $\mathsf{Aut}(Z) = \{\mathsf{id}\}$ (one can take a general curve of genus ≥ 3; see [12] and [13, Main Theorem]).

• The connectedness is crucial: $\mathsf{Aut}(\mathbb{A}^n_{\mathbf{k}})$ is isomorphic to the automorphism group of the disjoint union of $\mathbb{A}^n_{\mathbf{k}}$ and Z if Z is a variety with $\mathsf{Aut}(Z) = \{\mathsf{id}\}$.

1.2. **Previous results.** The literature contains already several theorems that may be compared to Theorem A. We refer to [2] for an interesting introduction and for the case of the complex affine plane; see [7, 8] for extensionsand generalisations of Déserti's results in higher dimension. Some of those results assume $\mathsf{Aut}(X)$ to be isomorphic to $\mathsf{Aut}(\mathbb{A}^n_{\mathbf{k}})$ as an ind-group; this is a rather strong hypothesis. Indeed,

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there are examples of affine varieties X and Y such that $\mathsf{Aut}(X)$ and $\mathsf{Aut}(Y)$ are isomorphic as abstract groups, but not isomorphic as ind-groups (see [9, Theorem 2]). In [10] the authors prove that an affine toric surface is determined by its group of automorphisms in the category of affine surfaces; unfortunately, their methods do not work in higher dimension.

1.3. **Commutative families.** The proof of Theorem A relies on a new result concerning families of pairwise commuting automorphisms of affine varieties. To state it, we need a few standard notions. If V is a subset of a group G, we denote by $\langle V\rangle$ the subgroup generated by V, i.e. the smallest subgroup of G containing V. We say that V is **commutative** if $fg = gf$ for all pairs or equivalently, if $\langle V\rangle$ is an abelian group. In the following statement, $\mathsf{Aut}(X)$ is viewed as an ind-group, so that it makes sense to speak of algebraic subsets of it (see the definitions in Section 2.2).

Theorem B.– *Let* $\mathbf{k}$ *be an algebraically closed field and let* X *be an affine variety over* $\mathbf{k}$*. Let* V *be a commutative irreducible algebraic subvariety of* $\mathsf{Aut}(X)$ *containing the identity. Then* $\langle V\rangle$ *is an algebraic subgroup of* $\mathsf{Aut}(X)$.

It is crucial to assume that V contains the identity. Otherwise, a counter-example would be given by a single automorphism f of X for which the sequence $n \mapsto \deg(f^n)$ is not bounded (see Section 2.1). To get a family of positive dimension, consider the set V of automorphisms $f_a\colon (x,y)\mapsto (x,axy)$ of $(\mathbb{A}^1_{\mathbf{k}}\setminus\{0\})^2$, for $a\in \mathbf{k}\setminus\{0\}$; V is commutative and irreducible, but $\langle V\rangle$ has infinitely many connected components (hence $\langle V\rangle$ is not algebraic). However, if V satisfies the hypotheses of Theorem B except that it does not contain the identity, the subset $V\cdot V^{-1}\subseteq \mathsf{Aut}(X)$ is irreducible, commutative and contains the identity; if its dimension is positive, Theorem B implies that $\mathsf{Aut}(X)$ contains a commutative algebraic subgroup of positive dimension.

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2. Degrees and ind-groups

2.1. **Degrees and compactifications.** Let X be an affine variety. Embed X in the affine space $\mathbb{A}^N_{\mathbf{k}}$ for some N, and denote by $\mathbf{x} = (x_1,\ldots,x_N)$ the affine coordinates of $\mathbb{A}^N_{\mathbf{k}}$. Let f be an automorphism of X. Then, there are N polynomial functions $f_i\in\mathbf{k}[\mathbf{x}]$ such that $f(\mathbf{x}) = (f_1(\mathbf{x}),\ldots,f_N(\mathbf{x}))$ for $\mathbf{x}\in X$. One says that f has degree $\leq d$ if one can choose the f_i of degree $\leq d$; the degree $\deg(f)$ can then be defined as the minimum of these degrees d. This notion depends on the embedding $X\hookrightarrow \mathbb{A}^N_{\mathbf{k}}$.

Another way to proceed is as follows. To simplify the exposition, assume that all irreducible components of X have the same dimension $k = \dim(X)$. Fix a compactification $\overline{X}_0$ of X by a projective variety and let $\overline{X} \to \overline{X}_0$ be the normalization of $\overline{X}_0$. If H is an ample line bundle on $\overline{X}$, and if f is a birational transformation of $\overline{X}$, one defines $\deg_H(f)$ (or simply $\deg(f)$) to be the intersection number

$$\deg(f) = (f^*H)\cdot(H)^{k-1}. \tag{2.1}$$

Since $\mathsf{Aut}(X) \subset \mathsf{Bir}(\overline{X})$, we obtain a second notion of degree. It is shown in [1, 17] (see also § 6 below) that these notions of degrees are compatible: if we change the embedding $X \hookrightarrow \mathbb{A}^N_{\mathbf{k}}$, or the polarization H of $\overline{X}$, or the compactification $\overline{X}$, we get different degrees, but any two of these degree functions are always comparable, in the sense that there are positive constants satisfying

$$a\deg(f) \leq \deg'(f) \leq b\deg(f) \quad (\forall f \in \mathsf{Aut}(X)). \tag{2.2}$$

A subset $V \subset \mathsf{Aut}(X)$ is of **bounded degree** if there is a uniform upper bound $\deg(g) \leq D < +\infty$ for all $g \in V$. This notion does not depend on the choice of degree. If $V \subset \mathsf{Aut}(X)$ is of bounded degree, then $V^{-1} = \{f^{-1}\,;\, f \in V\} \subset \mathsf{Aut}(X)$ is of bounded degree too (see [1] and [3] for instance); we shall not use this result.

2.2. **Automorphisms of affine varieties and ind-groups.** The notion of an ind-group goes back to Shafarevich, who called these objects infinite dimensional groups in [15]. We refer to [3, 6] for detailed introductions to this notion.

2.2.1. *Ind-varieties.* By an **ind-variety** we mean a set $\mathcal{V}$ together with an ascending filtration $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \ldots \subset \mathcal{V}$ such that the following is satisfied:

(1) $\mathcal{V} = \bigcup_{k\in\mathbf{N}} \mathcal{V}_k$;

(2) each $\mathcal{V}_k$ has the structure of an algebraic variety over $\mathbf{k}$;

(3) for all $k \in \mathbf{N}$ the inclusion $\mathcal{V}_k \subset \mathcal{V}_{k+1}$ is a closed immersion.

We refer to [3] for the notion of equivalent filtrations on ind-varieties.

A map $\Phi : \mathcal{V} \to \mathcal{W}$ between ind-varieties $\mathcal{V} = \bigcup_k \mathcal{V}_k$ and $\mathcal{W} = \bigcup_l \mathcal{W}_l$ is a **morphism** if for each $k \in \mathbf{N}$ there is $l \in \mathbf{N}$ such that $\Phi(\mathcal{V}_k) \subset \mathcal{W}_l$ and the induced map $\Phi\colon\, \mathcal{V}_k \to \mathcal{V}_l$ is a morphism of algebraic varieties. Isomorphisms of ind-varieties are defined in the usual way. An ind-variety $\mathcal{V} = \bigcup_k \mathcal{V}_k$ has a natural Zariski topology: $S \subset \mathcal{V}$ is **closed** (resp. **open**) if $S_k := S \cap \mathcal{V}_k \subset \mathcal{V}_k$ is closed (resp. open) for every k. A closed subset $S \subset \mathcal{V}$ inherits a natural structure of ind-variety and is called an **ind-subvariety**. An ind-variety $\mathcal{V}$ is said to be affine if each $\mathcal{V}_k$ is affine. We shall only consider affine ind-varieties and for simplicity we just call them ind-varieties. An ind-subvariety S is an **algebraic subvariety** of $\mathcal{V}$ if $S \subset \mathcal{V}_k$ for some $k \in \mathbf{N}$; by definition, a **constructible subset** will always be a constructible subset in an algebraic subvariety of $\mathcal{V}$.

2.2.2. *Ind-groups.* The product of two ind-varieties is defined in the obvious way. An ind-variety $\mathcal{G}$ is called an **ind-group** if the underlying set $\mathcal{G}$ is a group and the map $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$, defined by $(g,h) \mapsto gh^{-1}$, is a morphism of ind-varieties. If a subgroup H of $\mathcal{G}$ is closed for the Zariski topology, then H is naturally an ind-subgroup of $\mathcal{G}$; it is an **algebraic subgroup** if it is an algebraic subvariety of $\mathcal{G}$. A connected component of an ind-group $\mathcal{G}$, with a given filtration $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \dots$, is an increasing union of connected components $\mathcal{G}_i^c$ of $\mathcal{G}_i$. The **neutral component** $\mathcal{G}^\circ$ of $\mathcal{G}$ is the union of the connected components of the $\mathcal{G}_i$ containing the neutral element $\mathsf{id} \in \mathcal{G}$. We refer to [3], and in particular to Propositions 1.7.1 and 2.2.1, showing that *$\mathcal{G}^\circ$ is an ind-subgroup in $\mathcal{G}$ whose index is at most countable* (the proof of [3] works in arbitrary characteristic).

Theorem 2.1. *Let X be an affine variety over an algebraically closed field $\mathbf{k}$. Then $\mathsf{Aut}(X)$ has the structure of an ind-group acting "morphically" on X; this means that the action $\mathcal{G} \times X \to X$ of $\mathcal{G}$ on X induces a morphism of algebraic varieties $\mathcal{G}_i \times X \to X$ for every $i \in \mathbf{N}$.*

In particular, if V is an algebraic subset of $\mathsf{Aut}(X)$, then $V(x) \subset X$ is constructible for every $x \in X$ by Chevalley's theorem. The proof can be found in [6, Proposition 2.1] (see also [3], Theorems 5.1.1 and 5.2.1): the authors assume that the field has characteristic 0, but their proof works in the general setting. To obtain a filtration, one starts with a closed embedding $X \hookrightarrow \mathbb{A}^N_{\mathbf{k}}$, and define $\mathsf{Aut}(X)_d$ to be the set of automorphisms f such that $\max\{\deg(f), \deg(f^{-1})\} \leq d$. For example, if $X = \mathbb{A}^n_{\mathbf{k}}$, the ind-group filtration $(\mathsf{Aut}(\mathbb{A}^n_{\mathbf{k}})_d)$ of $\mathsf{Aut}(\mathbb{A}^n_{\mathbf{k}})$ is defined by the following property: an automorphism f is in $(\mathsf{Aut}(\mathbb{A}^n_{\mathbf{k}})_d)$ if the polynomial formulas for $f = (f_1, \dots, f_n)$ and for its inverse $f^{-1} = (g_1, \dots, g_n)$ satisfy

$$\deg f_i \leq d \ \text{ and } \ \deg g_i \leq d, \ \ (\forall i \leq n). \tag{2.3}$$

Note that an ind-subgroup is algebraic if and only if it is of bounded degree. Thus, we get the following basic fact.

Proposition 2.2. *Let X be an affine variety over an algebraically closed field $\mathbf{k}$. Let V be an irreducible algebraic subset of $\mathsf{Aut}(X)$ that contains id. Then $\langle V \rangle$ is an algebraic subgroup of $\mathsf{Aut}(X)$, acting algebraically on X, if and only if $\langle V \rangle$ is of bounded degree.*

Example 2.3. Let $g \in \mathsf{SU}_2(\mathbf{C})$ be an irrational rotation, and set $V = \{g\} \subset \mathsf{Aut}(\mathbb{A}^2_{\mathbf{C}})$. Then $\langle V \rangle$ is not an algebraic group, but it is Zariski dense in an abelian algebraic subgroup of $\mathsf{GL}_2(\mathbf{C}) \subset \mathsf{Aut}(\mathbb{A}^2_{\mathbf{C}})$. This shows that $\mathsf{id} \in V$ is a necessary hypothesis.

Proof. If $\langle V \rangle$ is algebraic, then it is contained in some $\mathsf{Aut}(X)_d$ and, as such, is of bounded degree; moreover, Theorem 2.1 implies that the action $\langle V \rangle \times X \to X$ is algebraic. If $\langle V \rangle$ is of bounded degree, then $\langle V \rangle^{-1} = \langle V \rangle$ is of bounded degree too, and $\langle V\rangle$ is contained in some $\mathsf{Aut}(X)_d$. The Zariski closure $\overline{\langle V\rangle}$ of $\langle V\rangle$ in $\mathsf{Aut}(X)_d$ is an algebraic subgroup of $\mathsf{Aut}(X)$; we are going to show that $\overline{\langle V\rangle} = \langle V\rangle$. Set $W = V\cdot V^{-1}$, and note that W contains V because $\mathsf{id}\in V$. By definition, $\langle V\rangle$ is the increasing union of the subsets $W\subset W\cdot W\subset\cdots\subset W^k\subset\cdots$, and by Chevalley theorem, each $W^k\subset\overline{\langle V\rangle}$ is constructible. The $\overline{W^k}$ are irreducible, because V is irreducible, and their dimensions are bounded by the dimension of $\mathsf{Aut}(X)_d$; so, there exists $\ell\geq 1$ such that $\overline{W^\ell} = \cup_{k\geq 1}\overline{W^k}\subseteq\overline{\langle V\rangle}$. Since $\langle V\rangle\subseteq\cup_{k\geq 1}\overline{W^k}$, we get $\overline{W^\ell} = \overline{\langle V\rangle}$; thus, there exists a Zariski dense open subset U of $\overline{\langle V\rangle}$ which is contained in W^ℓ. Now, pick any f in $\overline{\langle V\rangle}$. Then $(f\cdot U)$ and U are two Zariski dense open subsets of $\overline{\langle V\rangle}$, so $(f\cdot U)$ intersects U and this implies that f is in $U\cdot U^{-1}\subset\langle V\rangle$. So $\overline{\langle V\rangle}\subset\langle V\rangle$. □

3. ALGEBRAIC VARIETIES OF COMMUTING AUTOMORPHSIMS

Let $\mathbf{k}$ be an algebraically closed field. Let X be an affine variety over $\mathbf{k}$ of dimension d. In this section, we prove Theorem B. Since $V\subset\mathsf{Aut}(X)$ is irreducible and contains the identity, every irreducible component of X is invariant under the action of V (and of $\langle V\rangle$); thus, we may and do assume X to be irreducible.

3.1. **Invariant fibrations, base change, and degrees.** Let B and Y be affine varieties and assume that B is irreducible. Let $\pi\colon Y\to B$ be a dominant morphism. By definition, $\mathsf{Aut}_\pi(Y)$ is the group of automorphisms $g\colon Y\to Y$ such that $\pi\circ g = \pi$.

Let B' be another irreducible affine variety, and let $\psi\colon B'\to B$ be a quasi-finite and dominant morphism. Pulling-back π by ψ, we get a new affine variety $Y\times_B B' = \{(y,b')\in Y\times B';\ \pi(y)=\psi(b')\}$; the projections $\pi_Y\colon Y\times_B B'\to Y$ and $\pi'\colon Y\times_B B'\to B'$ satisfy $\psi\circ\pi' = \pi\circ\pi_Y$. There is a natural homomorphism

$$\iota_\psi\colon \mathsf{Aut}_\pi(Y)\to\mathsf{Aut}_{\pi'}(Y\times_B B') \tag{3.1}$$

defined by $\iota_\psi(g) = g\times_B \mathsf{id}_{B'}$. For every $g\in\mathsf{Aut}_\pi(Y)$, we have

$$g\circ\pi_Y = \pi_Y\circ\iota_\psi(g)\quad\text{ and }\quad \pi' = \pi'\circ\iota_\psi(g). \tag{3.2}$$

If $\iota_\psi(g) = \mathsf{id}$ then $g\circ\pi_Y = \pi_Y$ and $g = \mathsf{id}$ because π_Y is dominant; hence, ι_ψ *is an embedding*. Since π_Y is dominant and generically finite, the next lemma follows from Proposition 6.3.

Lemma 3.1. *If S is a subset of $\mathsf{Aut}_\pi(Y)$, then S is of bounded degree if and only if its image $\iota_\psi(S)$ in $\mathsf{Aut}_{\pi'}(Y\times_B B')$ is of bounded degree.*

Let us come back to the example $f(x,y) = (x,xy)$ from Section 1.3. This is an automorphism of the multiplicative group $\mathbb{G}_m\times\mathbb{G}_m$ that preserves the projection onto the first factor. The degrees of the iterates $f^n(x,y) = (x,x^ny)$ are not bounded, but on every fiber $\{x = x_0\}$, the restriction of f^n is the linear map $y\mapsto(x_0)^n y$, of constant degree 1. More generally, if $x\in B\mapsto A(x)$ is a regular map with values in $\mathsf{GL}_N(\mathbf{k})$, then $g\colon (x,y)\mapsto (x,A(x)(y))$ is a regular automorphism of $B\times\mathbb{A}^N_{\mathbf{k}}$ and, in most cases, we observe the same phenomenon: the degrees of the restrictions $g^n_{|\{x_0\}\times\mathbb{A}^N_{\mathbf{k}}}$ are bounded, but the degrees of g^n are not.

If X is an affine variety over $\mathbf{k}$ with a morphism $\pi : X\to B$, we denote by η the generic point of B and X_η the generic fiber of π. If G is a subgroup of $\mathsf{Aut}_\pi(X)$, then its restriction to X_η may have bounded degree even if G is not a subgroup of $\mathsf{Aut}(X)$ of bounded degree: this is shown by the previous example.

The next proposition provides a converse result. To state it, we make use of the following notation. Let B be an irreducible affine variety, and let $O(B)$ be the $\mathbf{k}$-algebra of its regular functions. By definition, $\mathbb{A}^N_B$ denotes the affine space $\mathrm{Spec}\, O(B)[x_1,\ldots,x_N]$ over the ring $O(B)$ and $\mathsf{Aut}_B(\mathbb{A}^N_B)$ denotes the group of **$O(B)$-automorphisms** of $\mathbb{A}^N_B$. Similarly, $\mathsf{GL}_N(O(B))$ is the linear group over the ring $O(B)$. The inclusion $\mathsf{GL}_N(O(B))\subset\mathsf{Aut}_B(\mathbb{A}^N_B)$ is an embedding of ind-groups.

Proposition 3.2. *Let X be an irreducible and normal affine variety over $\mathbf{k}$ with a dominant morphism $\pi : X\to B$ to an irreducible affine variety B over $\mathbf{k}$. Let η be the generic point of B and X_η the generic fiber of π. Let G be a subgroup of $\mathsf{Aut}_\pi(X)$ whose restriction to X_η is of bounded degree. Then there exists*

(a) *a nonempty affine open subset B' of B,*
(b) *an embedding $\tau : X_{B'} := \pi^{-1}(B')\hookrightarrow\mathbb{A}^r_{B'}$ over B' for some $r\geq 1$,*
(c) *and an embedding $\rho : G\hookrightarrow\mathsf{GL}_r(O(B'))\subseteq\mathsf{Aut}_{B'}(\mathbb{A}^r_{B'})$,*

such that $\tau\circ g=\rho(g)\circ\tau$ for every $g\in G$.

Notation.– For $f\in\mathsf{Aut}(X)$ and $\xi\in O(X)$ (resp. in $\mathbf{k}(X)$), we denote by $f^*\xi$ the function $\xi\circ f$. The field of constant functions is identified with $\mathbf{k}\subset O(X)$.

Proof of Proposition 3.2. Shrinking B, we assume B to be normal.

Pick any closed embedding $X\hookrightarrow\mathbb{A}^\ell_B\subseteq\mathbb{P}^\ell_B$ over B. Let X' be the Zariski closure of X in $\mathbb{P}^\ell_B$. Let $\overline{X}$ be the normalization of X', with the structure morphism $\overline{\pi} : \overline{X}\to B$; thus, $\overline{\pi} : \overline{X}\to B$ is a normal and projective scheme over B containing X as a Zariski open subset. By Proposition 3.1 in [4, Chap. II], $D := \overline{X}\setminus X$ is an effective Weil divisor of $\overline{X}$. Denote by $\overline{X}_\eta$ the generic fiber of $\overline{\pi}$ and by D_η the generic fiber of $\overline{\pi}|_D$. Shrinking B again if necessary, we may assume that all irreducible components of D meet the generic fiber, i.e. $D=\overline{D_\eta}$.

Write $X=\mathrm{Spec} A$, where $A=O(X)$. Let M be a finite dimensional subspace of A such that $1\in M$ and A is generated by M as a $\mathbf{k}$-algebra. Since the action of G on X_η is of bounded degree, there exists $m\geq 0$ such that the divisor

$$(\mathrm{Div}(g^*v)+mD)|_{\overline{X}_\eta} \tag{3.3}$$

is effective for every $v\in M$ and $g\in G$. Now, consider $\mathrm{Div}(g^*v)+mD$ as a divisor of $\overline{X}$ and write $\mathrm{Div}(g^*v)+mD=D_1-D_2$ where D_1 and D_2 are effective and have no common irreducible component. Since $g\in\mathsf{Aut}_\pi(X)$, we get $g^*v\in A$ and $D_2\cap X=$

$\emptyset$. Moreover, $D_2 \cap \overline{X}_\eta = \emptyset$. So D_2 is contained in $\overline{X} \setminus X$, but then we deduce that D_2 is empty because $\overline{X} \setminus X$ is covered by D and $D = \overline{D_\eta}$.

Observe that $H^0(\overline{X}, mD)$ is a finitely generated $O(B)$-module. Denote by N the G-invariant $O(B)$-submodule of A generated by the g^*v, for $g \in G$ and $v \in M$. Since $N \subseteq H^0(\overline{X}, mD)$, N is a finitely generated $O(B)$-module. Let r be the dimension of the $\mathbf{k}(B)$-vector space $N \otimes_{O(B)} \mathbf{k}(B)$. Fix a basis $(w_1, \ldots, w_r)$ of this space made of elements $w_i \in N$. After shrinking B, we may assume that N is a free $O(B)$-module generated by $w_1, \ldots, w_r$. Let W be a free $O(B)$-module of rank r with a basis $(z_1, \ldots, z_r)$; thus, $W = \oplus_{i=1}^r O(B) z_i$ and

$$\operatorname{Spec} O(B)[W] = \operatorname{Spec} O(B)[z_1, \ldots, z_r] = \mathbb{A}^r_{O(B)}. \tag{3.4}$$

Let $\tau_W^* : W \to N$ be the isomorphism of modules defined by $\tau_W^*(z_i) = w_i$. The action of G on N induces a representation $\rho \colon G \to \mathsf{GL}_B(W)$ such that $\tau_W^* \circ \rho(g) = g^* \circ \tau_W^*$.

Using the basis (z_i), we obtain a homomorphism $\rho \colon G \to \mathsf{GL}_r(O(B))$. Let τ be the morphism $X \hookrightarrow \operatorname{Spec} O(B)[W] = \mathbb{A}^r_{O(B)}$ over B induced by $\tau_W^* : W \to N \subseteq A$. The group $\mathsf{GL}_r(O(B))$ can naturally be identified to a subgroup of $\mathsf{Aut}_B(\mathbb{A}^r_{O(B)})$, and then $\tau \circ g = \rho(g) \circ \tau$ for every $g \in G$. $\square$

3.2. **Orbits.** If S is a subset of $\mathsf{Aut}(X)$ and x is a point of X the S-**orbit** of x is the subset $S(x) = \{f(x);\ f \in S\}$. Let V be an irreducible algebraic subvariety of $\mathsf{Aut}(X)$ containing id. Set $W = V \cdot V^{-1}$; it is a constructible subset of $\mathsf{Aut}(X)$ containing V (for $\mathsf{id} \in V$). Then, the group $\langle V \rangle$ is the union of the sets

$$W^k = \{f_1 \circ \cdots \circ f_k;\ f_j \in W \text{ for all } j\}. \tag{3.5}$$

Since W contains id, the W^k form a non-decreasing sequence

$$W^0 = \{\mathsf{id}\} \subset W \subset W^2 \subset \cdots \subset W^k \subset \cdots \tag{3.6}$$

of constructible subsets of $\mathsf{Aut}(X)$; their closures are irreducible, because so is V. In particular, $k \mapsto \dim(W^k)$ is non-decreasing.

The W^k-orbit of a point $x \in X$ is the image of $W^k \times \{x\}$ by the morphism $\mathsf{Aut}(X) \times X \to X$ defining the action on X: applying Chevalley's theorem one more time, $W^k(x)$ is a constructible subset of X. If $U \subset X$ is open, its W^k-orbit $W^k(U)$ is open too; thus, $\langle W \rangle(U) = \cup_{k \geq 0} W^k(U)$ is open in X.

An increasing union of irreducible constructible sets needs not be stationary: the sequence of subsets of $\mathbb{A}^2_{\mathbf{C}}$ defined by $Z_k = \left(\mathbb{A}^2_{\mathbf{C}} \setminus \{y = 0\}\right) \cup_{j=1}^k \{(j, 0)\}$ provides such an example. However, we shall see in the next proposition that the $W^k(x)$ are better behaved.

Let π_1 and π_2 be the projections from $X \times X$ to the first and second factor, respectively. Let Δ_X be the diagonal in $X \times X$; if Y is a subvariety of X, set

$$\Delta_Y = \pi_1^{-1}(Y) \cap \Delta_X = \{(y, y) \in X \times X;\ y \in Y\} \subset X \times X. \tag{3.7}$$

Consider the morphism $\Phi\colon \mathsf{Aut}(X)\times X\to X\times X$ defined by

$$\Phi(g,x)=(x,g(x)), \tag{3.8}$$

and set $\Gamma_i=\Phi(W^i\times X)$ for $i\in\mathbf{Z}_{>0}$. The family $(\Gamma_i)_{i\in\mathbf{N}}$ forms a non-decreasing sequence of constructible sets; we denote by Γ_∞ their union. Then, consider the action of $\mathsf{Aut}(X)$ on $X\times X$ given by $g\cdot(x,y)=(x,g(y))$. By construction, $\Gamma_i=W^i\cdot\Delta_X$ and $\Gamma_\infty=\langle W\rangle\cdot\Delta_X$; similarly $W^i\cdot\Delta_Y=\Gamma_i\cap\pi_1^{-1}(Y)$ and $\langle W\rangle\cdot\Delta_Y=\Gamma_\infty\cap\pi_1^{-1}(Y)$ for every subvariety $Y\subset X$.

Lemma 3.3. *The subset Γ_∞ of $X\times X$ is constructible.*

Proof. Let us prove, by an induction on $\dim(Y)$, that $\pi_1^{-1}(Y)\cap\Gamma_\infty$ is constructible for every irreducible subvariety $Y\subseteq X$. By convention, set $\dim Y=-1$ when $Y=\emptyset$. So, the case $\dim Y=-1$ is trivial. Now assume that $\dim Y\geq 0$ and that the result holds in dimension $<\dim(Y)$. Set $Z_Y=\overline{\langle W\rangle\cdot\Delta_Y}$; this set is invariant under the action of $\langle W\rangle$ on $X\times X$. Since $\overline{W^i\cdot\Delta_Y}$ is irreducible and increasing for each $i\geq 0$, there is $m\geq 0$, such that

$$Z_Y=\overline{\langle W\rangle\cdot\Delta_Y}=\overline{W^i\cdot\Delta_Y}\quad(\forall i\geq m). \tag{3.9}$$

Then there is a dense open subset U_Y of Z_Y which is contained in $W^m\cdot\Delta_Y$, hence in $\langle W\rangle\cdot\Delta_Y$. Shrinking U_Y if necessary, we may assume that $\pi_1(U_Y)$ is open in Y. Then $Y\setminus\pi_1(U_Y)$ is a closed subset of X, the irreducible component of which have dimension $<\dim Y$. By the induction hypothesis, $\pi_1^{-1}(Y\setminus\pi_1(U_Y))\cap\Gamma_\infty$ is constructible. We also know that $\pi_1^{-1}(\pi_1(U_Y))\cap\Gamma_\infty=\langle W\rangle\cdot U_Y$ is an open subset of Z_Y. Thus, $\pi_1^{-1}(Y)\cap\Gamma_\infty=(\pi_1^{-1}(Y\setminus\pi_1(U_Y))\cap\Gamma_\infty)\cup(\pi_1^{-1}(\pi_1(U_Y))\cap\Gamma_\infty)$ is constructible. □

Proposition 3.4. *The orbits $W^k(x)$ satisfy the following properties.*

(1) *The function $k\in\mathbf{Z}_{>0}\mapsto\dim(W^k(x))$ is non-decreasing.*
(2) *The function $x\in X\mapsto\dim(W^k(x))$ is lower semi-continuous in the Zariski topology: the subsets $\{x\in X\,;\,\dim(W^k(x))\leq n\}$ are Zariski closed for all pairs (n,k) of integers.*
(3) *The integers*

$$s(x):=\max_{k\geq 0}\{\dim(W^k(x))\}\quad \textit{and}\quad s_X:=\max_{x\in X}\{s(x)\}$$

are bounded from above by $\dim(X)$.
(4) *There is a Zariski dense open subset $\mathcal{U}$ of X and an integer k_0 such that* $\dim(W^k(x))=s_X$ *for all $k\geq k_0$ and all $x\in\mathcal{U}$.*
(5) *There is an integer $\ell\geq 0$, such that for every x in X, $W^\ell(x)=\langle W\rangle(x)$ and $W^\ell(x)$ is an open subset of $\overline{\langle W\rangle(x)}$.*

Proof. The first assertion follows from the inclusions (3.6), and the third one is obvious. Since the action $(f,x)\in W^k\times X\mapsto f(x)\in X$ is a morphism, the second and fourth assertions follow from Chevalley's constructibility result and the semi-continuity of the dimension of the fibers (see [5, II, Exercise 3.19] and [16, Section I.6.3, Corollary] respectively). By Lemma 3.3, Γ_∞ is constructible. Since it is the increasing union of the constructible subsets Γ_i, there is an integer ℓ such that $\Gamma_\infty = \Gamma_i$ for $i \geq \ell$. Then, $W^\ell(x) = \langle W\rangle(x)$ because $W^i(x) = \pi_2(\Gamma_i \cap \pi_1^{-1}\{x\})$ and $\langle W\rangle(x) = \pi_2(\Gamma_\infty \cap \pi_1^{-1}\{x\})$. Now, the constructible set $W^\ell(x)$ contains a dense open subset U of $\overline{\langle W\rangle(x)}$; since $\langle W\rangle$ acts transitively on $W^\ell(x)$, $W^\ell(x) = \langle W\rangle(U)$ is open in $\overline{\langle W\rangle(x)}$. □

3.3. **Open orbits.** Let us assume in this paragraph that $s_X = \dim X$: there is an orbit $W^k(x_0)$ which is open and dense and coincides with $\langle W\rangle(x_0)$. We fix such a pair (k, x_0). Let f be an element of $\langle W\rangle$. Since $f(x_0)$ is in the set $W^k(x_0)$, there is an element g of W^k such that $g(x_0) = f(x_0)$, i.e. $g^{-1}\circ f(x_0) = x_0$. By commutativity, $(g^{-1}\circ f)(h(x_0)) = h(x_0)$ for every h in W^k, and this shows that $g^{-1}\circ f = \mathsf{id}$ because $W^k(x_0)$ is dense in X. Thus, $\langle W\rangle$ coincides with W^k, and $\langle W\rangle = \langle V\rangle$ is an irreducible algebraic subgroup of the ind-group $\mathsf{Aut}(X)$.

Thus, Theorem B is proved in case $s_X = \dim X$. The proof when $s_X < \dim X$ occupies the next section, and is achieved in § 3.4.4.

3.4. **No dense orbit.** Assume now that there is no dense orbit; in other words, $s_X < \dim(X)$. Fix an integer $\ell > 0$ and a W-invariant open subset $\mathcal{U} \subset X$ such that

$$s(x) = s_X \ \text{ and } W^\ell(x) = \langle W\rangle(x) \tag{3.10}$$

for every $x \in \mathcal{U}$ (see Proposition 3.4, assertions (4) and (5)).

3.4.1. *A fibration.* Let C be an irreducible algebraic subvariety of X of codimension s_X that intersects the general orbit $W^\ell(x)$ transversally (in k points). As in § 3.2, denote by $\pi_1 : X \times X \to X$ the projection to the first factor. The morphism

$$\pi' := (\pi_1)|_{(X\times C)\cap\Gamma_\ell} : (X \times C) \cap \Gamma_\ell \to X \tag{3.11}$$

is generically finite of degree k. So there is a non-empty open subset $\mathcal{V}$ of $\mathcal{U}$ such that $\pi'|_{\pi'^{-1}(\mathcal{V})} : \pi'^{-1}(\mathcal{V}) \to \mathcal{V}$ is finite étale. Observe that for every $g \in \langle W\rangle$, $g(\mathcal{V})$ is open in $\mathcal{U}$ and $\pi'|_{\pi'^{-1}(g(\mathcal{V}))}$ is finite étale of degree k. Set $Y := \langle W\rangle(\mathcal{V})$; it is open in $\mathcal{U}$ and satisfies

(i) for each $x \in Y$ the intersection of C and $W^\ell(x)$ is transverse and contains exactly k points;
(ii) Y is W-invariant.

To each point $x \in Y$, we associate the intersection $C \cap W^\ell(x)$, viewed as a point in the space $C^{[k]}$ of cycles of length k and dimension 0 in C. This gives a dominant morphism

$$\pi\colon Y \to B \tag{3.12}$$

where, by definition, B is the irreducible variety $B = \overline{\pi(Y)} \subset C^{[k]}$. The group $\langle W\rangle$ is now contained in $\mathsf{Aut}_\pi(Y)$. Shrinking B and Y accordingly, we may assume that B is normal. Let η be the generic point of B.

The fiber $\pi^{-1}(b)$ of $b \in B$, we denote by Y_b. By construction, for every $b \in B(\mathbf{k})$, Y_b is an orbit of $\langle W\rangle$; and Section 3.3 shows that Y_b is isomorphic to the image $\langle W\rangle_b$ of $\langle W\rangle$ in $\mathsf{Aut}(Y_b)$: this group $\langle W\rangle_b$ coincides with the image of W^ℓ in $\mathsf{Aut}(Y_b)$ and the action of $\langle W\rangle$ on Y_b corresponds to the action of $\langle W\rangle_b$ on itself by translation. Thus, Section 3.3 implies the following properties

(1) every fiber, in particular the generic fiber, of π is geometrically irreducible;
(2) the generic fiber of π is normal and affine, shrinking B (and Y accordingly) again, we may assume B and Y to be normal and affine;
(3) the action of $\langle W\rangle$ on the generic fiber Y_η has bounded degree.

3.4.2. *Reduction to $Y = U_B \times_B (\mathbb{G}_{m,B}^s)$.* In this section, the variety Y will be modified, so as to reduce our study to the case when Y is an abelian group scheme over B. Note that B and Y will be modified several times in this paragraph, keeping the same names.

By Proposition 3.2, after shrinking B, there exist an embedding $\tau : Y \hookrightarrow \mathbb{A}_B^N$ for some $N \geq 0$ and a homomorphism $\rho : \langle W\rangle \hookrightarrow \mathsf{GL}_N(O(B)) \subseteq \mathsf{Aut}_B(\mathbb{A}_B^N)$ such that

$$\tau \circ g = \rho(g) \circ \tau \quad (\forall g \in \langle W\rangle). \tag{3.13}$$

Via τ, we view Y as a B-subscheme of $\mathbb{A}_B^N$, and via ρ we view $\langle W\rangle$ in $\mathsf{GL}_N(O(B))$. Consider the inclusion of $\mathsf{GL}_N(O(B))$ into $\mathsf{GL}_N(\mathbf{k}(B))$, and compose it with the embedding of W into $\mathsf{GL}_N(O(B))$. Denote by $\langle W\rangle_\eta$ the Zariski closure of $\langle W\rangle$ in $\mathsf{GL}_N(\mathbf{k}(B), Y_\eta) \subseteq \mathsf{Aut}(Y_\eta)$, where $\mathsf{GL}_N(\mathbf{k}(B), Y_\eta)$ is the subgroup of $\mathsf{GL}_N(\mathbf{k}(B))$ which preserves Y_η. There is a natural inclusion of sets $W \hookrightarrow W \otimes_{\mathbf{k}} \mathbf{k}(B)$: a point x of W, viewed as a morphism $x\colon \operatorname{Spec}\mathbf{k}(x) \to W$, is mapped to the point

$$x^B : \operatorname{Spec}\mathbf{k}(x)(B \otimes_{\mathbf{k}} \mathbf{k}(x)) = \operatorname{Spec}\operatorname{Frac}(\mathbf{k}(x) \otimes_{\mathbf{k}} \mathbf{k}(B)) \to W \otimes_{\mathbf{k}} \mathbf{k}(B), \tag{3.14}$$

where $\mathbf{k}(x)(B \otimes_{\mathbf{k}} \mathbf{k}(x))$ is the function field of $B \otimes_{\mathbf{k}} \mathbf{k}(x)$ which is the variety over the field $\mathbf{k}(x)$; note that $\mathbf{k}$ being algebraically closed, $B \otimes_{\mathbf{k}} \mathbf{k}(x)$ is irreducible over $\mathbf{k}(x)$ and $\mathbf{k}(x) \otimes_{\mathbf{k}} \mathbf{k}(B)$ is an integral domain. The image of this inclusion is Zariski dense in $W \otimes_{\mathbf{k}} \mathbf{k}(B)$. The morphism $W \hookrightarrow \mathsf{GL}_N(\mathbf{k}(B), Y_\eta)$ naturally extends to a morphism $W \otimes_{\mathbf{k}} \mathbf{k}(B) \hookrightarrow \mathsf{GL}_N(\mathbf{k}(B), Y_\eta)$. It follows that $\langle W\rangle_\eta$ is the Zariski closure of $\langle W \otimes_{\mathbf{k}} \mathbf{k}(B)\rangle$ in $\mathsf{GL}_N(\mathbf{k}(B), Y_\eta)$.

Since $W \otimes_{\mathbf{k}} \mathbf{k}(B)$ is geometrically irreducible, $\langle W\rangle_\eta$ is a geometrically irreducible commutative linear algebraic group over $\mathbf{k}(B)$. As a consequence ([11], Chap. 16.b), there exists a finite extension L of $\mathbf{k}(B)$ and an integer $s \geq 0$ such that

$$\langle W\rangle_\eta \otimes_{\mathbf{k}(B)} L \simeq U_L \times \mathbb{G}_{m,L}^s \tag{3.15}$$

where U_L is a unipotent commutative linear algebraic group over L.

Let $\psi : B' \to B$ be the normalization of B in L. We obtain a new fibration $\pi' \colon Y \times_B B' \to B'$, together with an embedding ι_ψ of $\mathsf{Aut}_\pi(Y)$ in $\mathsf{Aut}_{\pi'}(Y \times_B B')$; by Lemma 3.1, the subgroup $\langle W \rangle$ has bounded degree if and only if its image $\iota_\psi \langle W \rangle$ has bounded degree too. Because the generic fiber of π is geometrically irreducible, $Y \times_B B'$ is irreducible. After such a base change, we may assume that $\langle W \rangle_\eta \simeq U_\eta \times \mathbb{G}^s_{m,\mathbf{k}(B)}$, where U_η corresponds to the group U_L of Equation (3.15). Replacing (this new) B by an affine open subset, and shrinking Y accordingly, we may assume that $Y = U_B \times_B (\mathbb{G}^s_{m,B})$, where U_B is an integral unipotent commutative algebraic group scheme over B, and

$$W \subseteq U_B(B) \times \mathbb{G}^s_{m,B}(B) \subseteq \mathsf{Aut}_\pi(Y) \tag{3.16}$$

acts on Y by translation; here $U_B(B)$ and $\mathbb{G}^s_{m,B}(B)$ denote the ind-varieties of sections of the structure morphisms $U_B \to B$ and $\mathbb{G}^s_{m,B} \to B$ respectively.

Remark 3.5. A section $\sigma \colon B \to U_B$ defines an automorphism of $U_B \simeq B \times_B U_B$ by $\phi(\sigma \times_B \mathsf{id}_{U_B})$, where $\phi \colon U_B \times U_B \to U_B$ is the multiplication morphism of U_B; it defines in the same way an element of $\mathsf{Aut}_\pi(Y)$. Similarly $\mathbb{G}^s_{m,B}(B)$ embeds into $\mathsf{Aut}_\pi(Y)$, so $U_B(B) \times \mathbb{G}^s_{m,B}(B) \subseteq \mathsf{Aut}_\pi(Y)$, and this is the meaning of (3.16).

Remark 3.6. Both $U_B(B) \times \mathbb{G}^s_{m,B}(B)$ and $\mathsf{Aut}_\pi(Y)$ are ind-varieties over $\mathbf{k}$ and the inclusions in (3.16) are morphisms between ind-varieties.

Now, to prove Theorem B, we only need to show that W is contained in an algebraic subgroup of $U_B(B) \times \mathbb{G}^s_{m,B}(B)$.

3.4.3. *Structure of U_B.* Let B be a normal affine variety over the algebraically closed field $\mathbf{k}$, and let U_B be an integral, connected and unipotent algebraic group scheme over B (we do not assume U_B to be commutative here).

Lemma 3.7. *The ind-group $U_B(B)$ is an increasing union of algebraic subgroups.*

Before describing the proof, let us assume that U_B is just an r-dimensional additive group $\mathbb{G}^r_{a,B}$. Then, each element of U_B can be written

$$f = (a^f_1(z), \ldots, a^f_r(z)) \tag{3.17}$$

where each $a^f_i(z)$ is an element of $O(B)$; its n-th power is given by $f^n = (na^f_1(z), \ldots, na^f_r(z))$. Thus, viewed as automorphisms of Y, the degrees of the f^n are bounded independently of n, by (a function of) the degrees of the a^f_i. Our proof is a variation on this basic remark, with two extra difficulties: the structure of U_B may be more subtle in positive characteristic (see [14], §VII.2); instead of iterating one element f, we need to controle the group U_B itself.

Proof. Denote by $\pi_U : U_B \to B$ the structure morphism. Recall, from the end of Section 3.4.2, that B is an affine variety.

The proof is by induction on the relative dimension of $\pi_U\colon U_B \to B$. If this dimension is zero, there is nothing to prove. So, we assume that the lemma holds for relative dimensions $\leq \ell$, for some $\ell \geq 0$, and we want to prove it when the relative dimension is $\ell+1$. Denote by U_η the generic fiber of π_U. Our field $\mathbf{k}$ is algebraically closed, and the group U_B is connected, so by Corollary 14.55 of [11] (see also § 14.63), there exists a finite field extension L of $\mathbf{k}(B)$ such that $U_L := U_\eta \otimes_{\mathbf{k}(B)} L$ sits in a central exact sequence

$$0 \to \mathbb{G}_{a,L} \to U_L \xrightarrow{q_L} V_L \to 0, \tag{3.18}$$

where V_L is an irreducible unipotent group of dimension ℓ and V_L is isomorphic to $\mathbb{A}^\ell_L$ as an L-variety; moreover, there is an isomorphism of L-varieties $\phi_L\colon U_L \to V_L \times \mathbb{G}_{a,L}$ such that the quotient morphism q_L is given by the projection onto the first factor. So we have a section $s_L : V_L \to U_L$ such that $q_L \circ s_L = \mathsf{id}$. The section s_L is just given by a regular function on V_L, it needs not be a homomorphism of groups. Doing the base change given by the normalization of B in L, and then shrinking the base if necessary, we may assume that B is affine and

- there is an exact sequence of group schemes over B,
$$0 \to \mathbb{G}_{a,B} \to U_B \xrightarrow{q_B} V_B \to 0,$$
where V_B is a unipotent group scheme over B of relative dimension ℓ;
- there is an isomorphism of B-schemes $V_B \simeq \mathbb{A}^\ell_B$;
- s_L extends to a section $s_B : V_B \to U_B$ over B: $q_B \circ s_B = \mathsf{id}$.

For $b \in B$, denote by U_b, V_b, q_b, s_b the specialization of U_B, V_B, q_B, s_B at b. Since U_B and V_B are abelian, we denote by $+$ the group law on these groups.The morphism of B-schemes $\beta : U_B \to V_B \times \mathbb{G}_{a,B}$ sending a point x in the fiber U_b to the point $(q_b(x), x - s_b(q_b(x)))$ of the fiber $V_b \times \mathbb{G}_{a,b}$ defines an isomorphism. We use β to transport the group law of U_B into $V_B \times \mathbb{G}_{a,B}$; this defines a law $*$ on $V_B \times \mathbb{G}_{a,B}$, given by

$$a_1 * a_2 = \beta(\beta^{-1}(a_1) + \beta^{-1}(a_2)), \tag{3.19}$$

for a_1 and a_2 in $V_B \times \mathbb{G}_{a,B}$. Denote by $O(V_B \times_B V_B)$ the function ring of the $\mathbf{k}$-variety $V_B \times_B V_B \simeq B \times \mathbb{A}^\ell \times \mathbb{A}^\ell$. We write a point in $V_B \times_B V_B$ as (b, x_1, x_2) where $x_1, x_2 \in V_B$ with the same image b in B. There is an element $F(b, x_1, x_2)(y_1, y_2)$ of $O(V_B \times_B V_B)[y_1, y_2]$ such that

$$(x_1, y_1) * (x_2, y_2) = (x_1 + x_2, F(b, x_1, x_2)(y_1, y_2)) \tag{3.20}$$

for all $b \in B$ and $(x_1, y_1), (x_2, y_2) \in V_b \times \mathbb{G}_a$. (In (3.20), $+$ is the group law in V_b.) For every fixed (x_1, y_1, x_2), the morphism $y_2 \mapsto F(b, x_1, x_2)(y_1, y_2)$ is an automorphism of the variety $\mathbb{G}_a$. Thus, we can write

$$F(b, x_1, x_2)(y_1, y_2) = C_0(b, x_1, x_2)(y_1) + C_2(b, x_1, x_2)(y_1) y_2. \tag{3.21}$$

The function $C_2(b,x_1,x_2)(y_1)$ does not vanish on $V_B \times_B V_B \times \mathbb{A}^1 \simeq B \times \mathbb{A}^{2\ell+1}$; thus, C_2 is an element of $O(B)$. By symmetry we get

$$F(b,x_1,x_2)(y_1,y_2) = C_0(b,x_1,x_2) + C_1(b)y_1 + C_2(b)y_2 \tag{3.22}$$

and

$$(x_1,y_1) * (x_2,y_2) = (x_1+x_2, C_0(b,x_1,x_2) + C_1(b)y_1 + C_2(b)y_2). \tag{3.23}$$

Now, apply this equation for $x_1 = x_2 = 0$ (the neutral element of V_B). The restriction of β to the fiber $q_b^{-1}(0)$ is $x \mapsto x - s_b(0)$, so for $(0,y_1)$ and $(0,y_2)$ in $V_b \times \mathbb{G}_a$, we obtain $(0,y_1)*(0,y_2) = (0, y_1+y_2+s_b(0))$; then $C_1 = C_2 = 1$.

We identify now the ind-varieties $U_B(B)$ and $V_B(B) \times \mathbb{G}_a(B)$. By induction, the ind-group $V_B(B)$ is an increasing union of algebraic subgroups V_i; as observed before the proof of this lemma, the ind-group $\mathbb{G}_a(B)$ is an increasing union of subgroups G_j. If S and T are elements of $V_B(B)$ and $\mathbb{G}_a(B)$ respectively, we set

$$\delta_V(S) = \min\{i\,;\, S \in V_i\}, \quad \delta_{\mathbb{G}_a}(T) = \min\{j\,;\, T \in G_j\}. \tag{3.24}$$

Each element of $U_B(B)$ is given by a section $(S,T) \in V_B(B) \times \mathbb{G}_a(B)$ and the group law in $U_B(B)$ corresponds to the law

$$(S_1,T_1) * (S_2,T_2) = (S_1+S_2, C_0(S_1,S_2) + T_1 + T_2) \tag{3.25}$$

because $C_1 = C_2 = 1$. Here $C_0 : V_B(B) \times V_B(B) \to \mathbb{G}_a(B)$ is a morphism of ind-varieties, so there is a function $\alpha\colon \mathbf{N} \to \mathbf{N}$ such that

$$\delta_{\mathbb{G}_a} C_0(S_1,S_2) \leq \alpha(\delta_V(S_1) + \delta_V(S_2)). \tag{3.26}$$

Now, note that $V_i \times G_{\alpha(2i)}$ is an algebraic subgroup of $U_B(B)$, because

$$\delta_{\mathbb{G}_a}(C_0(S_1,S_2) + T_1 + T_2) \quad \leq \quad \max\{\delta_{\mathbb{G}_a}(C_0(S_1,S_2)), \delta_{\mathbb{G}_a}(T_1), \delta_{\mathbb{G}_a}(T_2)\}. \tag{3.27}$$

Thus, $U_B(B)$ is the increasing union of the algebraic subgroups $V_i \times G_{\alpha(2i)}$. □

3.4.4. *Subgroups of $\mathbb{G}_m^s(B)$ and conclusion.*

Lemma 3.8. *If Z is an irreducible subvariety of $\mathbb{G}_m^s(B)$ containing* id, *then $\langle Z \rangle$ is an algebraic subgroup of $\mathbb{G}_m^s(B)$.*

Proof of Lemma 3.8. Pick a projective compactification $\overline{B}$ of B. After taking the normalization of $\overline{B}$, we may assume $\overline{B}$ to be normal. If h is any non-constant rational function on $\overline{B}$, denote by $\mathrm{Div}(h)$ the divisor $(h)_0 - (h)_\infty$ on $\overline{B}$.

Let $\mathbf{y} = (y_1, \ldots, y_s)$ be the standard coordinates on $\mathbb{G}_m^s$. Each element $f \in \mathbb{G}_m^s(B)$ can be written as $(b_1^f(z), \ldots, b_s^f(z))$, for some $b_j^f \in O^*(B)$. Let R be an effective divisor whose support $\mathrm{Support}(R)$ contains $\overline{B} \setminus B$. Replacing R by some large multiple, Z is contained in the subset P_R of $\mathbb{G}_m^s(B)$ made of automorphisms $f \in \mathbb{G}_m^s(B)$ such that $\mathrm{Div}(b_i^f) + R \geq 0$ and $\mathrm{Div}(1/b_i^f) + R \geq 0$ for all $i = 1, \ldots, s$. Let us study the structure of this set $P_R \subset \mathbb{G}_m^s(B)$.

Let K be the set of pairs (D_1, D_2) of effective divisors supported on $\overline{B} \setminus B$ such that D_1 and D_2 have no common irreducible component, $D_1 \leq R$, $D_2 \leq R$, and D_1 and D_2 are rationally equivalent. Then K is a finite set. For every pair $\alpha = (D_1^\alpha, D_2^\alpha) \in K$, we choose a function $h_\alpha \in O^*(Y)$ such that $\mathrm{Div}(h_\alpha) = D_1^\alpha - D_2^\alpha$; if h is another element of $O^*(Y)$ such that $\mathrm{Div}(h) = D_1^\alpha - D_2^\alpha$, then $h/h_\alpha \in \mathbf{k}^*$. By convention $\alpha = 0$ means that $\alpha = (0,0)$, and in that case we choose h_α to be the constant function 1. For every $\beta = (\alpha_1, \ldots, \alpha_s) \in K^s$, denote by P_β the set of elements $f \in \mathbb{G}_m^s(B)$ such that the $b_i^f \in O^*(B)$ satisfy $\mathrm{Div}(b_i^f) = D_1^{\alpha_i} - D_2^{\alpha_i}$ for all $i = 1, \ldots, s$. Then $P_\beta \simeq \mathbb{G}_m^s(\mathbf{k})$ is an irreducible algebraic variety over $\mathbf{k}$. Moreover, $\mathsf{id} \in P_\beta$ if and only if $\beta = 0$, and P_0 is an algebraic subgroup of $\mathbb{G}_m^s(B)$, isomorphic to $\mathbb{G}_m^s(\mathbf{k})$ as an algebraic group.

Observe that P_R is the disjoint union $P_R = \bigsqcup_{\beta \in K^s} P_\beta$. Since $\mathsf{id} \in Z$, Z is irreducible, and $Z \subseteq P_R$, we obtain $Z \subset P_0$. Since P_0 is an algebraic subgroup of $\mathbb{G}_m^s(B)$, $\langle Z \rangle$ coincides with $(Z \cdot Z^{-1})^\ell$ for some $\ell \geq 1$, and $\langle Z \rangle$ is a connected algebraic group. □

Proof of Theorem B. By Proposition 2.2, we only need to prove that $W = \langle V \rangle$ is of bounded degree. By Lemma 3.1 W is a subgroup of bounded degree if and only if $W \subset \mathsf{Aut}_\pi(Y)$ is a subgroup of bounded degree. Moreover, by (3.16), W is a subgroup of $U_B(B) \times \mathbb{G}_m^s(B) \subset \mathsf{Aut}_\pi(Y)$. Denote by $\pi_1 : U_B(B) \times \mathbb{G}_m^s(B) \to U_B(B)$ the projection to the first factor and $\pi_2 : U_B(B) \times \mathbb{G}_m^s(B) \to \mathbb{G}_m^s(B)$ the projection to the second. By Lemma 3.7, there exists an algebraic subgroup H_1 of $U_B(B)$ containing $\pi_1(W)$. Since $\pi_2(W)$ is irreducible and contains id, Lemma 3.8 shows that $\pi_2(W)$ is contained in an algebraic subgroup H_2 of $\mathbb{G}_m(B)$. Then W is contained in the algebraic subgroup $H_1 \times H_2$ of $U_B(B) \times \mathbb{G}_m^s(B)$. This concludes the proof. □

4. Actions of additive groups

Theorem 4.1. *Let $\mathbf{k}$ be an uncountable, algebraically closed field. Let X be a connected affine variety over $\mathbf{k}$. Let $G \subset \mathsf{Aut}(X)$ be an algebraic subgroup isomorphic to $\mathbb{G}_a^r$, for some $r \geq 1$. Let $H = \{h \in \mathsf{Aut}(X) \mid gh = hg$ for every $g \in G\}$ be the centralizer of G. If H/G is at most countable then G acts simply transitively on X, so that X is isomorphic to G as a G-variety.*

This section is devoted to the proof of this result. A proof is also described in [3, §11.4] when X is irreducible and the characteristic of $\mathbf{k}$ is 0.

Lemma 4.2. *Let X be an irreducible affine variety endowed with a faithful action of $G = \mathbb{G}_a^r$. Let I be a non-zero, G-invariant ideal of $O(X)$. If*

$$I^G := \{\xi \in I \mid g^*\xi = \xi \text{ for every } g \in G\}$$

is contained in the field $\mathbf{k}$ of constant functions, then

(1) *every non-zero, G-invariant ideal $J \subset O(X)$ coincides with $O(X)$;*

(2) *G acts simply transitively on X;*

(3) *X can be identified to G, with G acting on it by translations.*

In particular, $I = O(X)$.

Proof of Lemma 4.2. Let ξ be a non-zero element of I. To prove the first assertion, pick $\psi \in J \setminus \{0\}$, then $\xi\psi \in IJ \setminus \{0\}$. Let V be the linear subspace of $O(X)$ generated by the orbit $\{g^*(\xi\psi) \mid g \in G\}$. Firstly, V is contained in IJ because I and J are G-invariant. Secondly, the dimension of V is finite, for G acts regularly on X (see [18, §1.2]). Thus, G being isomorphic to $\mathbb{G}_a^r$, there exists a G-invariant vector $\varphi \in V \setminus \{0\} \subseteq IJ$. Since $I^G \subset \mathbf{k}$, the function φ is a constant, and J must be equal to $O(X)$ because it contains φ.

To prove the second and third assertions, fix a point $x \in X$. The closure $\overline{G(x)}$ of the orbit $G(x)$ is a closed, G-invariant subvariety, and the same is true for $\overline{X \setminus G(x)}$. Looking at the ideal of functions vanishing on those subvarieties we obtain $X = G(x)$. Since G is abelian and acts faithfully on X, the stabilizer of x must be trivial. Thus, G acts simply transitively on X. □

Let X be an affine variety over $\mathbf{k}$, and let G be a subgroup of $\mathsf{Aut}(X)$ isomorphic to $\mathbb{G}_a^r$ for some $r > 0$. Denote by $X_1, \ldots, X_l$ the irreducible components of X. Then each of the X_i, $i = 1, \ldots, l$, is invariant under G; permuting the X_i if necessary, there exists $s \leq l$ such that the action of G on X_i is nontrivial if and only if $i \leq s$.

For every $i \leq l$, denote by $\pi_i \colon O(X) \twoheadrightarrow O(X_i)$ the quotient map. Let J_i be the ideal of functions $\xi \in O(X)$ vanishing on the closed subset $\cup_{j \neq i} X_j$; its projection $I_i := \pi_i(J_i)$ is an ideal of $O(X_i)$. Observe that I_i is non-zero, is invariant under the action of G, and is contained in the ideal of $O(X_i)$ associated to the closed subset $X_i \cap (\cup_{j \neq i} X_j)$. In particular, $I_i = O(X_i)$ if and only if X_i is a connected component of X. Moreover, *the homomorphism $\pi_i|_{J_i} : J_i \to I_i$ is a bijection.* Indeed, it is a surjective homomorphism by definition. And if $\pi_i|_{J_i}(\xi) = 0$, then $\xi|_{X_i} = 0$ and since $\xi \in J_i$, $\xi|_{X_j} = 0$ for all $j \neq i$; thus $\xi = 0$, so that $\mathrm{Ker}(\pi_i|_{J_i}) = 0$.

We denote by $(\pi_i|_{J_i})^{-1} \colon I_i \to J_i$ the inverse of $\pi_i|_{J_i}$.

Lemma 4.3. *Let $\mathbf{k}$ be an uncountable, algebraically closed field. Let X be an affine variety, and G be an algebraic subgroup of $\mathsf{Aut}(X)$ isomorphic to $\mathbb{G}_a^r$. Let*

$$H := \{h \in \mathsf{Aut}(X) \mid gh = hg \text{ for every } g \in G\}$$

be the centralizer of G. If H/G is at most countable then $I_i^G \subseteq \mathbf{k}$ for every irreducible component X_i on which G acts non-trivially.

Proof of Lemma 4.3. Keeping the notation that precedes the statement of the lemma, we only need to treat the case $i = 1$. Fix an identification $\mathbb{G}_a^r \simeq G$. Then, we get an identification $O(X)^r = \mathrm{Mor}(X, \mathbb{G}_a^r) = \mathrm{Mor}(X, G)$ for which $(O(X)^G)^r$ corresponds to $\mathrm{Mor}(X,G)^G$ (here G acts on X only). We also identify G to the group of constant morphisms in $\mathrm{Mor}(X,G)$; then, G becomes a subgroup of the additive group $(O(X)^G)^r$. Let us modify the action of G on X, as in [3, §0.7]:

Fact 1.– *Define* $\Psi : (O(X)^G)^r \to \mathsf{End}(X)$ *by* $\Psi(\xi)\colon x \mapsto \xi(x)(x)$ *for every* $x \in X$. *Then* Ψ *is a homomorphism of additive groups.* Here we view $\xi \in (O(X)^G)^r$ as an element of $\mathrm{Mor}(X,G)$.

We need to prove that $\Psi(\xi_1+\xi_2) = \Psi(\xi_1)\circ\Psi(\xi_2)$ for every pair of elements $\xi_1, \xi_2 \in (O(X)^G)^r$. For $x \in X$, we have

$$\Psi(\xi_1+\xi_2)(x) = (\xi_1(x)+\xi_2(x))(x) = \xi_1(x)((\xi_2(x)(x))). \tag{4.1}$$

On the other hand, $\xi_1(y) = \xi_1(\xi_2(x)(y))$ for every pair $(x,y)\in X^2$ because $\xi_2(x) \in G$ and ξ_1 is in $\mathrm{Mor}(X,G)^G$. Thus $\xi_1(x) = \xi_1(\xi_2(x)(x))$, and the claim follows:

$$\Psi(\xi_1+\xi_2)(x) = \xi_1(\xi_2(x)(x)))(\xi_2(x)(x)) = \Psi(\xi_1)(\Psi(\xi_2)(x)). \tag{4.2}$$

This fact implies that Ψ is a homomorphism from the additive group $(O(X)^G)^r$ to the group of automorphisms $\mathsf{Aut}(X)$. We note that $\Psi|_G = \mathsf{id}$; since $(O(X)^G)^r$ is abelian, $\Psi((O(X)^G)^r)$ is a subgroup of the centralizer H that contains G. Since H/G is countable, so is $(O(X)^G)^r/\Psi^{-1}(G)$.

Now, define $\Phi\colon (I_1^G)^r \to \mathsf{Aut}(X)$ to be the composition of Ψ with the inclusion $((\pi_1|_{J_1})^{-1})^r\colon (I_1^G)^r \hookrightarrow (O(X)^G)^r$. We obtain an inclusion

$$(I_1^G)^r/\Phi^{-1}(G) \hookrightarrow (O(X)^G)^r/\Psi^{-1}(G); \tag{4.3}$$

hence, $(I_1^G)^r/\Phi^{-1}(G)$ is also countable.

Fact 2.– *We have* $\Phi^{-1}(G) = (I_1^G)^r$.

To prove it, denote by G_x the stabilizer of $x \in X$ in G, and for $\xi \in (I_1^G)^r$, set

$$Y(\xi) := \bigcap_{x\in X}(\xi(x)+G_x). \tag{4.4}$$

Then $Y(\xi)$ is an affine linear subspace of $G \simeq \mathbb{A}^r(\mathbf{k}) = \mathbf{k}^r$, and $\xi \in \Phi^{-1}(G)$ if and only if $Y(\xi) \neq \emptyset$. It follows that $\Phi^{-1}(G)$ is a linear subspace of $(I_1^G)^r$. Since $\mathbf{k}$ is uncountable and $(I_1^G)^r/\Phi^{-1}(G)$ is at most countable, we get $\Phi^{-1}(G) = (I_1^G)^r$.

It follows that for every $\xi \in (I_1^G)^r$,

$$\emptyset \neq Y(\xi) = \bigcap_{x\in X}(\xi(x)+G_x) \subseteq W(\xi) := \bigcap_{x\in X_1}(\xi(x)+G_x). \tag{4.5}$$

Choose $\eta \in (I_1^G)^r$ such that $\dim(W(\eta))$ is minimal, and then choose $x_1, \ldots, x_m \in X_1$, such that $W(\eta) = \bigcap_{i=1}^m(\eta(x_i)+G_{x_i})$. To conclude, we assume that I_1^G contains a non-constant function α, and then we shall modify η to get a new function τ with $\dim(W(\tau)) < \dim(W(\eta))$, in contradiction with our choice for η. For this purpose, set $\beta = \prod_{i=1}^{m}(\alpha - \alpha(x_i))$. Then, choose $y \in X_1 \setminus \{x_1,\dots,x_m\}$ such that $G_y \neq G$, $\alpha(y) \neq 0$ and $\beta(y) \neq 0$, and set

$$\gamma := \frac{\alpha\beta}{\beta(y)\alpha(y)}. \tag{4.6}$$

By construction, we get

(1) $\gamma \in O(X_1)^G I_1^G \subseteq I_1^G$;
(2) $\gamma(x_i) = 0$ for all $i = 1,\dots,m$;
(3) $\gamma(y) = 1$.

Pick $g \in W(\eta)$. The set $U = \{h \in G\,;\, g \notin h + \xi(y) + G_y\}$ is Zariski dense in G. Take $h \in U$, write $h = (a_1,\dots,a_r)$ as an element of $G = \mathbb{G}_a^r$, and set

$$\tau := \eta + \gamma h := \eta + (a_1\gamma,\dots,a_r\gamma). \tag{4.7}$$

By construction, τ is an element of $(I_1^G)^r$; and, changing $h = (a_1,\dots,a_r)$ in U if necessary, we may assume that $\tau \notin \mathbf{k}^r$. From the properties (2) and (3) above, we get $\tau(x_i) = \eta(x_i)$ for $i = 1,\dots,m$ and $\tau(y) = \eta(y) + h$. We have

$$\begin{aligned} W(\tau) &\subseteq \left(\bigcap_{i=1}^{m}(\tau(x_i) + G_{x_i})\right) \bigcap (\tau(y) + G_y) && (4.8)\\ &= \left(\bigcap_{i=1}^{m}(\eta(x_i) + G_{x_i})\right) \bigcap (\eta(y) + h + G_y) && (4.9)\\ &= W(\eta) \cap (\eta(y) + h + G_y). && (4.10) \end{aligned}$$

Since $g \in W(\eta)$ but $g \notin (\eta(y) + h + G_y)$, we get $\dim W(\tau) \leq \dim(W(\eta) \cap (\eta(y) + h + G_y)) < \dim W(\eta)$. This proves the lemma. □

Proof of Theorem 4.1. We keep the same notation. By Lemma 4.3, $I_1^G \subseteq \mathbf{k}$. Let $G_1 \subset \mathsf{Aut}(X_1)$ be the restriction of G. There exists $m \in \{1,\dots,r\}$, such that $G_1 \simeq \mathbb{G}_a^m$. We have $I_1^{G_1} = I_1^G \subseteq \mathbf{k}$. By Lemma 4.2, $I_1 = O(X_1)$ and $X_1 \simeq G_1$ as a G_1-variety. Since $I_1 = O(X_1)$, X_1 is a connected component of X. Since X is connected, $X = X_1$ and $G_1 = G$. This concludes the proof. □

5. Proof of Theorem A

In this section, we prove Theorem A. So, $\mathbf{k}$ is an uncountable, algebraically closed field, X is a connected affine algebraic variety over $\mathbf{k}$, and $\varphi : \mathsf{Aut}(\mathbb{A}_{\mathbf{k}}^n) \to \mathsf{Aut}(X)$ is an isomorphism of (abstract) groups.

5.1. **Translations and dilatations.** Let $\mathsf{Tr} \subset \mathsf{Aut}(\mathbb{A}_{\mathbf{k}}^n)$ be the group of all translations and Tr_i the subgroup of translations of the i-th coordinate:

$$(x_1,\dots,x_n) \mapsto (x_1,\dots,x_i + c,\dots,x_n) \tag{5.1}$$

for some c in $\mathbf{k}$. Let $\mathsf{D} \subset \mathsf{GL}_n(\mathbf{k}) \subset \mathsf{Aut}(\mathbb{A}^n_{\mathbf{k}})$ be the diagonal group (viewed as a maximal torus) and let D_i be the subgroup of automorphisms

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, ax_i, \ldots, x_n) \tag{5.2}$$

for some $a \in \mathbf{k}^*$. A direct computation shows that Tr *(resp.* D*) coincides with its centralizer in* $\mathsf{Aut}(\mathbb{A}^n_{\mathbf{k}})$.

Lemma 5.1. *Let G be a subgroup of* Tr *whose index is at most countable. Then, the centralizer of G in* $\mathsf{Aut}(\mathbb{A}^n)$ *is* Tr.

Proof. The centralizer of G contains Tr. Let us prove the reverse inclusion. The index of G in Tr being at most countable, G is Zariski dense in Tr. Thus, if h centralizes G, we get $hg = gh$ for all $g \in \mathsf{Tr}$, and h is in fact in the centralizer of Tr. Since Tr coincides with its centralizer, we get $h \in \mathsf{Tr}$. □

5.2. **Closed subgroups.** As in Section 2.2, we endow $\mathsf{Aut}(X)$ with the structure of an ind-group, given by a filtration by algebraic varieties Aut_j for $j \geq 1$.

Lemma 5.2. *The groups $\varphi(\mathsf{Tr})$, $\varphi(\mathsf{Tr}_i)$, $\varphi(\mathsf{D})$ and $\varphi(\mathsf{D}_i)$ are closed subgroups of* $\mathsf{Aut}(X)$ *for all $i = 1, \ldots, n$.*

Proof. Since $\mathsf{Tr} \subset \mathsf{Aut}(\mathbb{A}^n_{\mathbf{k}})$ coincides with its centralizer, $\varphi(\mathsf{Tr}) \subset \mathsf{Aut}(X)$ coincides with its centralizer too and, as such, is a closed subgroup of $\mathsf{Aut}(X)$. The same argument applies to $\varphi(\mathsf{D}) \subset \mathsf{Aut}(X)$. To prove that $\varphi(\mathsf{Tr}_i) \subset \mathsf{Aut}(X)$ is closed we note that $\varphi(\mathsf{Tr}_i)$ is the subset of elements $f \in \varphi(\mathsf{Tr})$ that commute to every element $g \in \varphi(\mathsf{D}_j)$ for every index $j \neq i$ in $\{1, \ldots, n\}$. Analogously, $\varphi(\mathsf{D}_i) \subset \mathsf{Aut}(X)$ is a closed subgroup because an element f of D is in D_i if and only if it commutes to all elements g of Tr_j for $j \neq i$. □

5.3. **Proof of Theorem A.**

5.3.1. *Abelian groups (see* [11, 14]*).* Before starting the proof, let us recall a few important facts on abelian, affine algebraic groups. Let G be an algebraic group over the field $\mathbf{k}$, such that G is abelian, affine, and connected.

(1) If $\mathsf{char}(\mathbf{k}) = 0$, then G is isomorphic to $\mathbb{G}_a^r \times \mathbb{G}_m^s$ for some pair of integers (r, s); if G is unipotent, then $s = 0$. (see [14]), §VII.2, p.172)

When the characteristic p of $\mathbf{k}$ is positive, there are other types of of abelian groups, but criteria on the p-torsion may rigidify their structure:

(2) If $\mathsf{char}(\mathbf{k}) = p$, G is unipotent, and all elements of G have order p, then G is isomorphic to $\mathbb{G}_a^r$ for some $r \geq 0$. (see [14], §VII.2, Prop. 11, p.178)
(3) If $\mathsf{char}(\mathbf{k}) = p$, and there is no non-trivial element in G of order p^ℓ, for any $\ell \geq 0$, then G is isomorphic to $\mathbb{G}_m^s$ for some $s \geq 0$. (see [11], Theorem 16.13 and Corollary 16.15, and [14], §VII.2, p.176)

To keep examples in mind, note that all elements of $\mathsf{Tr}_1(\mathbf{k})$ have order p and $\mathsf{D}_1(\mathbf{k})$ does not contain any non-trivial element of order p^ℓ when $\mathsf{char}(\mathbf{k}) = p$.

5.3.2. *Proof of Theorem A.* Let us now prove Theorem A.

By Lemma 5.2, $\varphi(\mathsf{Tr}_1) \subset \mathsf{Aut}(X)$ is a closed subgroup; in particular, $\varphi(\mathsf{Tr}_1)$ is an ind-subgroup of $\mathsf{Aut}(X)$. Let $\varphi(\mathsf{Tr}_1)^\circ$ be the connected component of the identity of $\varphi(\mathsf{Tr}_1)$; from Section 2.2.2, we know that the index of $\varphi(\mathsf{Tr}_1)^\circ$ in $\varphi(\mathsf{Tr}_1)$ is at most countable. The ind-group $\varphi(\mathsf{Tr}_1)^\circ$ is an increasing union $\cup_i V_i$ of irreducible algebraic varieties V_i, each V_i containing the identity. Theorem B implies that each $\langle V_i \rangle$ is an irreducible algebraic subgroup of $\mathsf{Aut}(X)$. Since $\varphi(\mathsf{Tr}_1)$ does not contain elements of order $k < \infty$ with $k \wedge \mathsf{char}(\mathbf{k}) = 1$, $\langle V_i \rangle$ is unipotent; and, by Properties (1) and (2) of Section 5.3.1, $\langle V_i \rangle$ is isomorphic to $\mathbb{G}_a^{r_i}$ for some r_i. Thus

$$\varphi(\mathsf{Tr}_1)^\circ = \cup_{i \geq 0} F_i \tag{5.3}$$

where the F_i form an increasing family of unipotent algebraic subgroups of $\mathsf{Aut}(X)$, each of them isomorphic to some $\mathbb{G}_a^{r_i}$. We may assume that $\dim F_0 \geq 1$.

Similarly, $\varphi(\mathsf{D}_1)^\circ \subset \varphi(\mathsf{D}_1)$ is a subgroup of countable index and

$$\varphi(\mathsf{D}_1)^\circ = \cup_{i \geq} G_i, \tag{5.4}$$

where the G_i are increasing irreducible commutative algebraic subgroups of $\mathsf{Aut}(X)$ (we do not assert that G_i is of type $\mathbb{G}_m^{s_i}$ yet). We may assume that $\dim G_0 \geq 1$.

The group D_i acts by conjugation on Tr_i for every $i \leq n$, this action has exactly two orbits $\{\mathsf{id}\}$ and $\mathsf{Tr}_i \setminus \{\mathsf{id}\}$, and the action on $\mathsf{Tr}_i \setminus \{\mathsf{id}\}$ is free; hence, the same properties hold for the action of $\varphi(\mathsf{D}_i)$ on $\varphi(\mathsf{Tr}_i)$ by conjugation.

Let H_i be the subgroup of $\varphi(\mathsf{Tr}_1)$ generated by all $g \circ f \circ g^{-1}$ with f in F_i and g in G_i. Theorem B shows that H_i is an irreducible algebraic subgroup of $\varphi(\mathsf{Tr}_1)$. We have $H_i \subseteq H_{i+1}$ and $g \circ H_i \circ g^{-1} = H_i$ for every $g \in G_i$.

Write $H_i = \mathbb{G}_a^l$ for some $l \geq 1$. We claim that $G_i \simeq \mathbb{G}_a^r \times \mathbb{G}_m^s$ for a pair of integers r, $s \geq 0$ with $r + s \geq 1$. This follows from Properties (1) and (2) of Section 5.3.1 because, when $\mathsf{char}(\mathbf{k}) = p > 1$, the only element in $\varphi(\mathsf{D}_1)$ of order p^ℓ, $\ell \geq 0$, is the identity element. Since the action of $\varphi(\mathsf{D}_1)$ on $\varphi(\mathsf{Tr}_1 \setminus \{0\})$ is free, the action of G_i on $F_i \setminus \{0\}$ is free, and this implies $r = 0$ (see Lemma 4.2(2)). Let q be a prime number with $q \wedge \mathsf{char}(\mathbf{k}) = 1$. Then $\mathbb{G}_m^s$ contains a copy of $(\mathbf{Z}/q\mathbf{Z})^s$, and D_1 does not contain such a subgroup if $s > 1$; so, $s = 1$, $G_i \simeq \mathbb{G}_m$ and $G_i = G_{i+1}$ for all $i \geq 0$. It follows that $\varphi(\mathsf{D}_1)^\circ \simeq \mathbb{G}_m$. Since the index of $\varphi(\mathsf{D}_1)^\circ$ in $\varphi(\mathsf{D}_1)$ is countable, there exists a countable subset $I \subseteq \varphi(\mathsf{D}_1)$ such that $\varphi(\mathsf{D}_1) = \sqcup_{h \in I} \varphi(\mathsf{D}_1)^\circ \circ h$.

Let $f \in F_i$ be a nontrivial element. Since the action of $\varphi(\mathsf{D}_1)$ on $\varphi(\mathsf{Tr}_1 \setminus \{0\})$ is transitive,

$$F_i \setminus \{0\} = \bigcup_{h \in I} \left(\left(\bigcup_{g \in \varphi(\mathsf{D}_1)^\circ} (g \circ h) \circ f \circ (g \circ h)^{-1} \right) \cap F_i \right). \tag{5.5}$$

The right hand side is a countable union of subvarieties of $F_i \setminus \{0\}$ of dimension at most one. It follows that $\dim F_i = 1$, $F_i \simeq \mathbb{G}_a$, and $\varphi(\mathsf{Tr}_1)^\circ \simeq \mathbb{G}_a$. Thus, we have

$$\varphi(\mathsf{Tr}_1)^\circ \simeq \mathbb{G}_a, \text{ and } \varphi(\mathsf{D}_1)^\circ \simeq \mathbb{G}_m. \tag{5.6}$$

Since each $\varphi(\mathsf{Tr}_i)^\circ$ is isomorphic to $\mathbb{G}_a$, $\varphi(\mathsf{Tr})^\circ$ is an n-dimensional commutative unipotent group and its index in $\varphi(\mathsf{Tr})$ is at most countable. By Lemma 5.1, the centralizer of $\varphi^{-1}(\varphi(\mathsf{Tr})^\circ)$ in $\mathsf{Aut}(\mathbb{A}^n_{\mathbf{k}})$ is Tr. It follows that the centralizer of $\varphi(\mathsf{Tr})^\circ$ in $\mathsf{Aut}(X)$ is $\varphi(\mathsf{Tr})$. Then Theorem 4.1 implies that X is isomorphic to $\mathbb{A}^n_{\mathbf{k}}$.

6. Appendix: the degree functions for rational self-maps

Here, we follow [1, 17] to prove a general version of Lemma 3.1. As above, $\mathbf{k}$ is an algebraically closed field. We first start with the case of projective varieties.

6.1. Degree functions on projective varieties.

Let X be a projective and normal variety over $\mathbf{k}$ of pure dimension $d = \dim(X)$. Let H be a big and nef divisor on X. For every dominant rational self-map f of X, and every $j = 0, \ldots, d$, set

$$\deg_{j,H} f = (f^*(H^j) \cdot H^{d-j}). \tag{6.1}$$

Pick a normal resolution of f; by this we mean a projective and normal variety Γ, a birational morphism $\pi_1 : \Gamma \to X$ and a morphism $\pi_2 : \Gamma \to X$ satisfying $f = \pi_2 \circ \pi_1^{-1}$. Then we have $\deg_{j,H} f = (\pi_2^*(H^j) \cdot \pi_1^*(H^{d-j})) > 0$, for f is dominant. Let L be another big and nef divisor. There is $c > 1$ such that $cL - H$ and $cH - L$ are big. Then we have $\deg_{j,H} f = (\pi_2^*(H^j) \cdot \pi_1^*(H^{d-j})) \leq c^d (\pi_2^*(L^j) \cdot \pi_1^*(L^{d-j})) = c^d \deg_{j,L} f$. Symetrically, we get $\deg_{j,L} f \leq (c')^d \deg_{j,H} f$ for some $c' > 1$. Thus, two big and nef divisors give rise to comparable degree functions:

$$C^{-1} \deg_{j,H}(f) \leq \deg_{j,L}(f) \leq C \deg_{j,H}(f) \qquad (\forall 0 \leq j \leq d) \tag{6.2}$$

for all rational dominant maps $f \colon X \dashrightarrow X$, and some $C > 1$.

Lemma 6.1. *Let Y be a projective and normal variety over $\mathbf{k}$ of pure dimension d. Let $\pi : Y \dashrightarrow X$ be a dominant and generically finite rational map. Let H and L be big and nef divisors, on X and Y respectively. Then there is a constant $C > 1$ such that for every $j = 0, \ldots, d$, and every pair of dominant rational self-maps $f : X \dashrightarrow X$ and $g : Y \dashrightarrow Y$ satisfying $f \circ \pi = \pi \circ g$, we have*

$$C^{-1} \deg_{j,L}(g) \leq \deg_{j,H}(f) \leq C \deg_{j,L}(g).$$

Proof. Denote by $x_1, \ldots, x_s$ the generic points of X and $y_1, \ldots, y_r$ the generic points of Y. Since π is dominant and generically finite, there is a surjective map $\sigma : \{1, \ldots, r\} \to \{1, \ldots, s\}$ such that $\pi(y_i) = x_{\sigma(i)}, i = 1, \ldots, r$. For every $i = 1, \ldots, r$, set $t_i = \deg[\mathbf{k}(x_i) : \pi^* \mathbf{k}(y_{\sigma(i)})]$ and then

$$m = \min_{i=1\ldots,s} \Big(\sum_{l \in \sigma^{-1}(i)} t_l \Big), \quad m' = \max_{i=1\ldots,s} \Big(\sum_{l \in \sigma^{-1}(i)} t_l \Big) \tag{6.3}$$

Take a resolution of π, defined by a projective and normal variety Z, a birational morphism $\pi_1 : Z \to Y$ and a morphism $\pi_2 : Z \to X$ satisfying $\pi = \pi_2 \circ \pi_1^{-1}$. Set $h := \pi_1^{-1} \circ g \circ \pi_1 : Z \dashrightarrow Z$. For each index $0 \leq j \leq d$, the projection formula gives

$$\deg_{j,L} g = \deg_{j,\pi_1^* L} h \tag{6.4}$$

$$m \deg_{j,H} f \leq \deg_{j,\pi_2^* H} h \leq m' \deg_{j,H} f. \tag{6.5}$$

Since $\pi_1^* L$ and $\pi_2^* H$ are big and nef on Z, there is a constant $C_1 > 1$ that depends only on $\pi_1^* L$ and $\pi_2^* H$ such that

$$C_1^{-1} \deg_{j,\pi_2^* H} h \leq \deg_{j,\pi_1^* L} h \leq C_1 \deg_{j,\pi_2^* H} h. \tag{6.6}$$

We conclude the proof by combining the last three equations. □

6.2. **Equivalent functions.** Let S be a set. We shall say that two functions $F, G : S \to \mathbf{R}_{\geq 0}$, are **equivalent** if there is a constant $C > 1$ such that

$$C^{-1} \max\{G, 1\} \leq \max\{F, 1\} \leq C \max\{G, 1\}, \tag{6.7}$$

where $\max\{G, 1\}$ denotes the maximum between G and 1. We denote by $[F]$ the equivalence class of F; the equivalence class $[1]$ coincides with the set of bounded functions $I \to \mathbf{R}_{\geq 0}$.

6.3. **Degree functions on varieties.** Now, let X be a variety of pure dimension d over $\mathbf{k}$. Let $\pi\colon Z \dashrightarrow X$ be a birational map such that Z is projective and normal, and let H be a big and nef divisor on Z. Then, define the degrees $\deg_{j,H} f$ of any rational dominant map $f\colon X \dashrightarrow X$ by $\deg_{j,H} f = \deg_{j,H} \pi^{-1} \circ f \circ \pi$. The previous paragraph shows that if we change the model (Z, π) or the divisor H (to H'), then we get two notions of degrees $\deg_{j,H}$ and $\deg_{j,H'}$ which are equivalent functions, in the sense of § 6.2, on the set of rational dominant self-maps of X. This justifies the following definition.

Let S be a family of dominant rational maps $f_s\colon X \dashrightarrow X$, $s \in S$. A **notion of degree** on S in codimension j is a function $\deg_j\colon S \to \mathbf{R}_{\geq 0}$ in the equivalence class $[\deg_{j,H}]$ for some normal projective model $Z \to X$ and some big and nef divisor H on Z. The equivalence class $[\deg_j]$ is unique.

Remark 6.2. Assume further that X is affine. In Section 2.1, we defined a notion of degree $f \mapsto \deg f$ (in codimension 1) on the set of automorphisms of X; this notion depends on an embeding $X \hookrightarrow \mathbb{A}^N_{\mathbf{k}}, N \geq 0$. However, its equivalence class on $\mathsf{Aut}(X)$ does not depend on the choice of such an embedding and is equal to the class $[\deg_1]$ defined in this section.

From Lemma 6.1 and the definitions, we obtain:

Proposition 6.3. *Let $\pi\colon Y \dashrightarrow X$ be a dominant and generically finite rational map between two varieties X and Y over $\mathbf{k}$, each of pure dimension d. Let S be a family* *of dominant rational maps $g_s\colon Y \dashrightarrow Y$ such that for every s in S there is a rational map $f_s\colon X \dashrightarrow X$ that satisfies $\pi \circ g_s = f_s \circ \pi$. Then, for each $j = 0, \ldots, d$, the equivalence classes of the degree functions $s \in S \mapsto \deg_j(g_s)$ and $s \in S \mapsto \deg_j(f_s)$ are equal.*

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