FAMILIES OF COMMUTING AUTOMORPHISMS, AND A CHARACTERIZATION OF THE AFFINE SPACE

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ABSTRACT. We prove that the affine space of dimension \( n \geq 1 \) over an uncountable algebraically closed field \( k \) is determined, among connected affine varieties, by its automorphism group (viewed as an abstract group). The proof is based on a new result concerning algebraic families of pairwise commuting automorphisms.

1. INTRODUCTION

1.1. Characterization of the affine space. In this paper, \( k \) is an algebraically closed field and \( \mathbb{A}^n_k \) denotes the affine space of dimension \( n \) over \( k \).

**Theorem A.**— Let \( k \) be an algebraically closed and uncountable field. Let \( n \) be a positive integer. Let \( X \) be a reduced, connected, affine variety over \( k \). If its automorphism group \( \text{Aut}(X) \) is isomorphic to \( \text{Aut}(\mathbb{A}^n_k) \) as an abstract group, then \( X \) is isomorphic to \( \mathbb{A}^n_k \) as a variety over \( k \).

Note that no assumption is made on \( \text{dim}(X) \); in particular, we do not assume \( \text{dim}(X) = n \). This theorem is our main goal. It would be great to lighten the hypotheses on \( k \), but besides that the following remarks show the result is optimal:

- The affine space \( \mathbb{A}^n_k \) is not determined by its automorphism group in the category of quasi-projective varieties because
  1. \( \text{Aut}(\mathbb{A}^n_k) \) is naturally isomorphic to \( \text{Aut}(\mathbb{A}^n_k \times Z) \) for any projective variety \( Z \) with \( \text{Aut}(Z) = \{\text{id}\} \);
  2. for every algebraically closed field \( k \) there is a projective variety \( Z \) over \( k \) such that \( \text{dim}(Z) \geq 1 \) and \( \text{Aut}(Z) = \{\text{id}\} \) (one can take a general curve of genus \( \geq 3 \); see [12] and [13, Main Theorem]).

- The connectedness is crucial: \( \text{Aut}(\mathbb{A}^n_k) \) is isomorphic to the automorphism group of the disjoint union of \( \mathbb{A}^n_k \) and \( Z \) if \( Z \) is a variety with \( \text{Aut}(Z) = \{\text{id}\} \).

1.2. Previous results. The literature contains already several theorems that may be compared to Theorem A. We refer to [2] for an interesting introduction and for the case of the complex affine plane; see [7, 8] for extensions and generalisations of Dé Ethi’s results in higher dimension. Some of those results assume \( \text{Aut}(X) \) to be isomorphic to \( \text{Aut}(\mathbb{A}^n_k) \) as an ind-group; this is a rather strong hypothesis. Indeed,
there are examples of affine varieties $X$ and $Y$ such that $\text{Aut}(X)$ and $\text{Aut}(Y)$ are isomorphic as abstract groups, but not isomorphic as ind-groups (see [9, Theorem 2]). In [10] the authors prove that an affine toric surface is determined by its group of automorphisms in the category of affine surfaces; unfortunately, their methods do not work in higher dimension.

1.3. Commutative families. The proof of Theorem A relies on a new result concerning families of pairwise commuting automorphisms of affine varieties. To state it, we need a few standard notions. If $V$ is a subset of a group $G$, we denote by $\langle V \rangle$ the subgroup generated by $V$, i.e. the smallest subgroup of $G$ containing $V$. We say that $V$ is commutative if $fg = gf$ for all pairs or equivalently, if $\langle V \rangle$ is an abelian group. In the following statement, $\text{Aut}(X)$ is viewed as an ind-group, so that it makes sense to speak of algebraic subsets of it (see the definitions in Section 2.2).

**Theorem B.**– Let $k$ be an algebraically closed field and let $X$ be an affine variety over $k$. Let $V$ be a commutative irreducible algebraic subvariety of $\text{Aut}(X)$ containing the identity. Then $\langle V \rangle$ is an algebraic subgroup of $\text{Aut}(X)$.

It is crucial to assume that $V$ contains the identity. Otherwise, a counter-example would be given by a single automorphism $f$ of $X$ for which the sequence $n \mapsto \deg(f^n)$ is not bounded (see Section 2.1). To get a family of positive dimension, consider the set $V$ of automorphisms $f_a: (x, y) \mapsto (x, axy)$ of $(A^1_k \setminus \{0\})^2$, for $a \in k \setminus \{0\}$; $V$ is commutative and irreducible, but $\langle V \rangle$ has infinitely many connected components (hence $\langle V \rangle$ is not algebraic). However, if $V$ satisfies the hypotheses of Theorem B except that it does not contain the identity, the subset $V \cdot V^{-1} \subseteq \text{Aut}(X)$ is irreducible, commutative and contains the identity; if its dimension is positive, Theorem B implies that $\text{Aut}(X)$ contains a commutative algebraic subgroup of positive dimension.

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2. Degrees and ind-groups

2.1. Degrees and compactifications. Let $X$ be an affine variety. Embed $X$ in the affine space $A^N_k$ for some $N$, and denote by $x = (x_1, \ldots, x_N)$ the affine coordinates of $A^N_k$. Let $f$ be an automorphism of $X$. Then, there are $N$ polynomial functions $f_i \in k[x]$ such that $f(x) = (f_1(x), \ldots, f_N(x))$ for $x \in X$. One says that $f$ has degree $\leq d$ if one can choose the $f_i$ of degree $\leq d$; the degree $\deg(f)$ can then be defined as the minimum of these degrees $d$. This notion depends on the embedding $X \hookrightarrow A^N_k$. 
Another way to proceed is as follows. To simplify the exposition, assume that all irreducible components of $X$ have the same dimension $k = \dim(X)$. Fix a compactification $\overline{X}_0$ of $X$ by a projective variety and let $\overline{X} \to \overline{X}_0$ be the normalization of $\overline{X}_0$. If $H$ is an ample line bundle on $\overline{X}$, and if $f$ is a birational transformation of $\overline{X}$, one defines $\deg_H(f)$ (or simply $\deg(f)$) to be the intersection number

$$ (2.1) \quad \deg(f) = (f^*H) \cdot (H)^{k-1}. $$

Since $\text{Aut}(X) \subset \text{Bir}(\overline{X})$, we obtain a second notion of degree. It is shown in [1, 17] (see also § 6 below) that these notions of degrees are compatible: if we change the embedding $X \hookrightarrow \mathbb{A}^N_k$, or the polarization $H$ of $\overline{X}$, or the compactification $\overline{X}$, we get different degrees, but any two of these degree functions are always comparable, in the sense that there are positive constants satisfying

$$ (2.2) \quad a \deg(f) \leq \deg'(f) \leq b \deg(f) \quad (\forall f \in \text{Aut}(X)). $$

A subset $V \subset \text{Aut}(X)$ is of bounded degree if there is a uniform upper bound $\deg(g) \leq D < +\infty$ for all $g \in V$. This notion does not depend on the choice of degree. If $V \subset \text{Aut}(X)$ is of bounded degree, then $V^{-1} = \{f^{-1} : f \in V\} \subset \text{Aut}(X)$ is of bounded degree too (see [1] and [3] for instance); we shall not use this result.

2.2. Automorphisms of affine varieties and ind-groups. The notion of an ind-group goes back to Shafarevich, who called these objects infinite dimensional groups in [15]. We refer to [3, 6] for detailed introductions to this notion.

2.2.1. Ind-varieties. By an ind-variety we mean a set $V'$ together with an ascending filtration $V'_0 \subset V'_1 \subset V'_2 \subset \ldots \subset V'$ such that the following is satisfied:

1. $V' = \bigcup_{k \in \mathbb{N}} V'_k$;
2. each $V'_k$ has the structure of an algebraic variety over $k$;
3. for all $k \in \mathbb{N}$ the inclusion $V'_k \subset V'_{k+1}$ is a closed immersion.

We refer to [3] for the notion of equivalent filtrations on ind-varieties.

A map $\Phi : V' \to W'$ between ind-varieties $V' = \bigcup_{k \in \mathbb{N}} V'_k$ and $W' = \bigcup_{l \in \mathbb{N}} W'_l$ is a morphism if for each $k \in \mathbb{N}$ there is $l \in \mathbb{N}$ such that $\Phi(V'_k) \subset W'_l$ and the induced map $\Phi : V'_k \to W'_l$ is a morphism of algebraic varieties. Isomorphisms of ind-varieties are defined in the usual way. An ind-variety $V' = \bigcup_{k \in \mathbb{N}} V'_k$ has a natural Zariski topology: $S \subset V'$ is closed (resp. open) if $S_k := S \cap V'_k \subset V'_k$ is closed (resp. open) for every $k$. A closed subset $S \subset V'$ inherits a natural structure of ind-variety and is called an ind-subvariety. An ind-variety $V'$ is said to be affine if each $V'_k$ is affine. We shall only consider affine ind-varieties and for simplicity we just call them ind-varieties. An ind-subvariety $S$ is an algebraic subvariety of $V'$ if $S \subset V'_k$ for some $k \in \mathbb{N}$; by definition, a constructible subset will always be a constructible subset in an algebraic subvariety of $V'$. 

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2.2.2. Ind-groups. The product of two ind-varieties is defined in the obvious way. An ind-varety \( G \) is called an ind-group if the underlying set \( G \) is a group and the map \( G \times G \rightarrow G \), defined by \( (g, h) \mapsto gh^{-1} \), is a morphism of ind-varieties. If a subgroup \( H \) of \( G \) is closed for the Zariski topology, then \( H \) is naturally an ind-subgroup of \( G \); it is an algebraic subgroup if it is an algebraic subvariety of \( G \). A connected component of an ind-group \( G \), with a given filtration \( G_0 \subset G_1 \subset G_2 \subset \ldots \), is an increasing union of connected components \( G_i^c \) of \( G_i \). The neutral component \( G^0 \) of \( G \) is the union of the connected components of the \( G_i \) containing the neutral element \( \text{id} \in G \). We refer to [3], and in particular to Propositions 1.7.1 and 2.2.1, showing that \( G^0 \) is an ind-subgroup in \( G \) whose index is at most countable (the proof of [3] works in arbitrary characteristic).

**Theorem 2.1.** Let \( X \) be an affine variety over an algebraically closed field \( k \). Then \( \text{Aut}(X) \) has the structure of an ind-group acting “morphically” on \( X \); this means that the action \( \text{Aut}(X) \times X \rightarrow X \) of \( \text{Aut}(X) \) on \( X \) induces a morphism of algebraic varieties \( \text{Aut}(X) \times X \rightarrow X \) for every \( i \in \mathbb{N} \).

In particular, if \( V \) is an algebraic subset of \( \text{Aut}(X) \), then \( V(x) \subset X \) is constructible for every \( x \in X \) by Chevalley’s theorem. The proof can be found in [6, Proposition 2.1] (see also [3], Theorems 5.1.1 and 5.2.1): the authors assume that the field has characteristic 0, but their proof works in the general setting. To obtain a filtration, one starts with a closed embedding \( X \hookrightarrow \mathbb{A}_k^N \), and define \( \text{Aut}(X)_d \) to be the set of automorphisms \( f \) such that \( \max \{ \deg(f), \deg(f^{-1}) \} \leq d \). For example, if \( X = \mathbb{A}_k^n \), the ind-group filtration \( (\text{Aut}(\mathbb{A}_k^n))_d \) of \( \text{Aut}(\mathbb{A}_k^n) \) is defined by the following property: an automorphism \( f \) is in \( (\text{Aut}(\mathbb{A}_k^n))_d \) if the polynomial formulas for \( f = (f_1, \ldots, f_n) \) and for its inverse \( f^{-1} = (g_1, \ldots, g_n) \) satisfy

\[
\deg f_i \leq d \quad \text{and} \quad \deg g_i \leq d, \quad (\forall i \leq n).
\]

Note that an ind-subgroup is algebraic if and only if it is of bounded degree. Thus, we get the following basic fact.

**Proposition 2.2.** Let \( X \) be an affine variety over an algebraically closed field \( k \). Let \( V \) be an irreducible algebraic subset of \( \text{Aut}(X) \) that contains \( \text{id} \). Then \( \langle V \rangle \) is an algebraic subgroup of \( \text{Aut}(X) \), acting algebraically on \( X \), if and only if \( \langle V \rangle \) is of bounded degree.

**Example 2.3.** Let \( g \in \text{SU}_2(\mathbb{C}) \) be an irrational rotation, and set \( V = \{ g \} \subset \text{Aut}(\mathbb{A}_k^2) \). Then \( \langle V \rangle \) is not an algebraic group, but it is Zariski dense in an abelian algebraic subgroup of \( \text{GL}_2(\mathbb{C}) \subset \text{Aut}(\mathbb{A}_k^2) \). This shows that \( \text{id} \in V \) is a necessary hypothesis.

**Proof.** If \( \langle V \rangle \) is algebraic, then it is contained in some \( \text{Aut}(X)_d \) and, as such, is of bounded degree; moreover, Theorem 2.1 implies that the action \( \langle V \rangle \times X \rightarrow X \) is algebraic. If \( \langle V \rangle \) is of bounded degree, then \( \langle V \rangle^{-1} = \langle V \rangle \) is of bounded degree too,
and \( \langle V \rangle \) is contained in some \( \text{Aut}(X)_d \). The Zariski closure \( \overline{\langle V \rangle} \) of \( \langle V \rangle \) in \( \text{Aut}(X)_d \) is an algebraic subgroup of \( \text{Aut}(X) \); we are going to show that \( \overline{\langle V \rangle} = \langle V \rangle \). Set \( W = V \cdot V^{-1} \), and note that \( W \) contains \( V \) because \( \text{id} \in V \). By definition, \( \langle V \rangle \) is the increasing union of the subsets \( W \subset W \cdot W \subset \cdots \subset W^k \subset \cdots \), and by Chevalley theorem, each \( W^k \subset \overline{\langle V \rangle} \) is constructible. The \( W^k \) are irreducible, because \( V \) is irreducible, and their dimensions are bounded by the dimension of \( W \); thus, there exists \( \ell \geq 1 \) such that \( W^\ell = \bigcup_{k \geq 1} W^k \subseteq \overline{\langle V \rangle} \). Since \( \langle V \rangle \subseteq \bigcup_{k \geq 1} W^k \), we get \( W^\ell = \overline{\langle V \rangle} \); therefore, there exists a Zariski dense open subset \( U \) of \( \langle V \rangle \) which is contained in \( W^\ell \).

Now, pick any \( f \) in \( \overline{\langle V \rangle} \). Then \( (f \cdot U) \) and \( U \) are two Zariski dense open subsets of \( \overline{\langle V \rangle} \), so \( (f \cdot U) \) intersects \( U \) and this implies that \( f \) is in \( U \cdot U^{-1} \subseteq \langle V \rangle \). So \( \overline{\langle V \rangle} \subseteq \langle V \rangle \). \[ \square \]

3. ALGEBRAIC VARIETIES OF COMMUTING AUTOMORPHISMS

Let \( k \) be an algebraically closed field. Let \( X \) be an affine variety over \( k \) of dimension \( d \). In this section, we prove Theorem B. Since \( V \subset \text{Aut}(X) \) is irreducible and contains the identity, every irreducible component of \( X \) is invariant under the action of \( V \) (and of \( \langle V \rangle \)); thus, we may and do assume \( X \) to be irreducible.

3.1. Invariant fibrations, base change, and degrees. Let \( B \) and \( Y \) be affine varieties and assume that \( B \) is irreducible. Let \( \pi: Y \to B \) be a dominant morphism. By definition, \( \text{Aut}_\pi(Y) \) is the group of automorphisms \( g: Y \to Y \) such that \( \pi \circ g = \pi \).

Let \( B' \) be another irreducible affine variety, and let \( \psi: B' \to B \) be a quasi-finite and dominant morphism. Pulling-back \( \pi \) by \( \psi \), we get a new affine variety \( Y \times_B B' = \{(y,b') \in Y \times B'; \pi(y) = \psi(b')\} \); the projections \( \pi_Y: Y \times_B B' \to Y \) and \( \pi': Y \times_B B' \to B' \) satisfy \( \psi \circ \pi' = \pi \circ \pi_Y \). There is a natural homomorphism

\[
(3.1) \quad t_\psi: \text{Aut}_\pi(Y) \to \text{Aut}_{\pi'}(Y \times_B B')
\]

defined by \( t_\psi(g) = g \times_B \text{id}_{B'} \). For every \( g \in \text{Aut}_\pi(Y) \), we have

\[
(3.2) \quad g \circ \pi_Y = \pi_Y \circ t_\psi(g) \quad \text{and} \quad \pi' = \pi' \circ t_\psi(g).
\]

If \( t_\psi(g) = \text{id} \) then \( g \circ \pi_Y = \pi_Y \) and \( g = \text{id} \) because \( \pi_Y \) is dominant; hence, \( t_\psi \) is an embedding. Since \( \pi_Y \) is dominant and generically finite, the next lemma follows from Proposition 6.3.

**Lemma 3.1.** If \( S \) is a subset of \( \text{Aut}_\pi(Y) \), then \( S \) is of bounded degree if and only if its image \( t_\psi(S) \) in \( \text{Aut}_{\pi'}(Y \times_B B') \) is of bounded degree.

Let us come back to the example \( f(x,y) = (x,xy) \) from Section 1.3. This is an automorphism of the multiplicative group \( \mathbb{G}_m \times \mathbb{G}_m \) that preserves the projection onto the first factor. The degrees of the iterates \( f^n(x,y) = (x,x^ny) \) are not bounded, but on every fiber \( \{x = x_0\} \), the restriction of \( f^n \) is the linear map \( y \mapsto (x_0)^ny \), of constant degree 1. More generally, if \( x \in B \mapsto A(x) \) is a regular map with values
in $\text{GL}_N(k)$, then $g: (x, y) \mapsto (x, A(x)(y))$ is a regular automorphism of $B \times \mathbb{A}_k^N$ and, in most cases, we observe the same phenomenon: the degrees of the restrictions $g^n((b) \times \mathbb{A}_k^N)$ are bounded, but the degrees of $g^n$ are not.

If $X$ is an affine variety over $k$ with a morphism $\pi: X \to B$, we denote by $\eta$ the generic point of $B$ and $X_\eta$ the generic fiber of $\pi$. If $G$ is a subgroup of $\text{Aut}_\pi(X)$, then its restriction to $X_\eta$ may have bounded degree even if $G$ is not a subgroup of $\text{Aut}(X)$ of bounded degree: this is shown by the previous example.

The next proposition provides a converse result. To state it, we make use of the following notation. Let $B$ be an irreducible affine variety, and let $O(B)$ be the $k$-algebra of its regular functions. By definition, $\mathbb{A}_B^N$ denotes the affine space $\text{Spec}O(B)[x_1, \ldots, x_N]$ over the ring $O(B)$ and $\text{Aut}_B(\mathbb{A}_B^N)$ denotes the group of $O(B)$-automorphisms of $\mathbb{A}_B^N$. Similarly, $\text{GL}_N(O(B))$ is the linear group over the ring $O(B)$. The inclusion $\text{GL}_N(O(B)) \subset \text{Aut}_B(\mathbb{A}_B^N)$ is an embedding of ind-groups.

**Proposition 3.2.** Let $X$ be an irreducible and normal affine variety over $k$ with a dominant morphism $\pi: X \to B$ to an irreducible affine variety $B$ over $k$. Let $\eta$ be the generic point of $B$ and $X_\eta$ the generic fiber of $\pi$. Let $G$ be a subgroup of $\text{Aut}_\pi(X)$ whose restriction to $X_\eta$ is of bounded degree. Then there exists

(a) a nonempty affine open subset $B'$ of $B$,

(b) an embedding $\tau: X_{B'} := \pi^{-1}(B') \to \mathbb{A}_{B'}^r$ over $B'$ for some $r \geq 1$,

(c) and an embedding $\rho: G \hookrightarrow \text{GL}_r(O(B')) \subset \text{Aut}_B(\mathbb{A}_{B'}^r)$,

such that $\tau \circ g = \rho(g) \circ \tau$ for every $g \in G$.

**Notation.** For $f \in \text{Aut}(X)$ and $\xi \in O(X)$ (resp. in $k(X)$), we denote by $f^*\xi$ the function $\xi \circ f$. The field of constant functions is identified with $k \subset O(X)$.

**Proof of Proposition 3.2.** Shrinking $B$, we assume $B$ to be normal.

Pick any closed embedding $X \hookrightarrow \mathbb{A}_B^\ell \subset \mathbb{P}_B^d$ over $B$. Let $X'$ be the Zariski closure of $X$ in $\mathbb{P}_B^d$. Let $\overline{X}$ be the normalization of $X'$, with the structure morphism $\overline{\pi}: \overline{X} \to B$; thus, $\overline{\pi}: \overline{X} \to B$ is a normal and projective scheme over $B$ containing $X$ as a Zariski open subset. By Proposition 3.1 in [4, Chap. III], $D := \overline{X} \setminus X$ is an effective Weil divisor of $\overline{X}$. Denote by $\overline{X}_\eta$ the generic fiber of $\overline{\pi}$ and by $D_\eta$ the generic fiber of $\overline{\pi}|_D$. Shrinking $B$ again if necessary, we may assume that all irreducible components of $D$ meet the generic fiber, i.e. $D = D_\eta$.

Write $X = \text{Spec} A$, where $A = O(X)$. Let $M$ be a finite dimensional subspace of $A$ such that $1 \in M$ and $A$ is generated by $M$ as a $k$-algebra. Since the action of $G$ on $X_\eta$ is of bounded degree, there exists $m \geq 0$ such that the divisor

$$ (\text{Div}(g^*v) + mD)|_{X_\eta} $$

is effective for every $v \in M$ and $g \in G$. Now, consider $\text{Div}(g^*v) + mD$ as a divisor of $\overline{X}$ and write $\text{Div}(g^*v) + mD = D_1 - D_2$ where $D_1$ and $D_2$ are effective and have no common irreducible component. Since $g \in \text{Aut}_\pi(X)$, we get $g^*v \in A$ and $D_2 \cap X = \emptyset$.

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\( \emptyset. \) Moreover, \( D_2 \cap \bar{X}_\eta = \emptyset. \) So \( D_2 \) is contained in \( \bar{X} \setminus X \), but then we deduce that \( D_2 \) is empty because \( \bar{X} \setminus X \) is covered by \( D \) and \( D = \bar{D}_\eta \).

Observe that \( H^0(\bar{X}, mD) \) is a finitely generated \( O(B) \)-module. Denote by \( N \) the \( G \)-invariant \( O(B) \)-submodule of \( A \) generated by the \( g^*v \), for \( g \in G \) and \( v \in M \). Since \( N \subseteq H^0(\bar{X}, mD) \), \( N \) is a finitely generated \( O(B) \)-module. Let \( r \) be the dimension of the \( k(B) \)-vector space \( N \otimes_{O(B)} k(B) \). Fix a basis \( (w_1, \ldots, w_r) \) of this space made of elements \( w_i \in N \). After shrinking \( B \), we may assume that \( N \) is a free \( O(B) \)-module generated by \( w_1, \ldots, w_r \). Let \( W \) be a free \( O(B) \)-module of rank \( r \) with a basis \( (z_1, \ldots, z_r) \); thus, \( W = \oplus_{i=1}^r O(B)z_i \) and

\[
\text{Spec } O(B)[W] = \text{Spec } O(B)[z_1, \ldots, z_r] = \mathbb{A}^r_{O(B)}.
\]

\( \text{Let } \tau_W : W \to N \text{ be the isomorphism of modules defined by } \tau_W(z_i) = w_i. \) The action of \( G \) on \( N \) induces a representation \( \rho : G \to \text{GL}_r(W) \) such that \( \tau_W \circ \rho(g) = g^* \circ \tau_W \).

Using the basis \( (z_i) \), we obtain a homomorphism \( \rho : G \to \text{GL}_r(O(B)). \) Let \( \tau \) be the morphism \( X \hookrightarrow \text{Spec } O(B)[W] = \mathbb{A}^r_{O(B)} \) over \( B \) induced by \( \tau_W : W \to N \subseteq A. \) The group \( \text{GL}_r(O(B)) \) can naturally be identified to a subgroup of \( \text{Aut}_B(\mathbb{A}^r_{O(B)}), \) and then \( \tau \circ g = \rho(g) \circ \tau \) for every \( g \in G. \)

3.2. Orbits. If \( S \) is a subset of \( \text{Aut}(X) \) and \( x \) is a point of \( X \) the \( S \)-orbit of \( x \) is the subset \( S(x) = \{f(x); f \in S\} \). Let \( V \) be an irreducible algebraic subvariety of \( \text{Aut}(X) \) containing \( \text{id} \). Set \( W = V \cdot V^{-1}; \) it is a constructible subset of \( \text{Aut}(X) \) containing \( V \) (for \( \text{id} \in V \)). Then, the group \( \langle V \rangle \) is the union of the sets

\[
W^k = \{f_1 \circ \cdots \circ f_k; f_j \in W \text{ for all } j\}.
\]

Since \( W \) contains \( \text{id} \), the \( W^k \) form a non-decreasing sequence

\[
W^0 = \{\text{id}\} \subset W \subset W^2 \subset \cdots \subset W^k \subset \cdots
\]

of constructible subsets of \( \text{Aut}(X) \); their closures are irreducible, because so is \( V \).

In particular, \( k \mapsto \dim(W^k) \) is non-decreasing.

The \( W^k \)-orbit of a point \( x \in X \) is the image of \( W^k \times \{x\} \) by the morphism \( \text{Aut}(X) \times X \to X \) defining the action on \( X \): applying Chevalley’s theorem one more time, \( W^k(x) \) is a constructible subset of \( X \). If \( U \subseteq X \) is open, its \( W^k \)-orbit \( W^k(U) \) is open too; thus, \( \langle W \rangle(U) = \cup_{k \geq 0} W^k(U) \) is open in \( X \).

An increasing union of irreducible constructible sets needs not be stationary: the sequence of subsets of \( \mathbb{A}^2_C \) defined by \( Z_k = (\mathbb{A}^2_C \setminus \{y = 0\}) \cup \bigcup_{j=1}^k \{(j, 0)\} \) provides such an example. However, we shall see in the next proposition that the \( W^k(x) \) are better behaved.

Let \( \pi_1 \) and \( \pi_2 \) be the projections from \( X \times X \) to the first and second factor, respectively. Let \( \Delta_X \) be the diagonal in \( X \times X \); if \( Y \) is a subvariety of \( X \), set

\[
\Delta_Y = \pi_1^{-1}(Y) \cap \Delta_X = \{(y, y) \in X \times X; y \in Y\} \subset X \times X.
\]
Consider the morphism $\Phi: \text{Aut}(X) \times X \to X \times X$ defined by
\begin{equation}
(3.8) \quad \Phi(g, x) = (x, g(x)),
\end{equation}
and set $\Gamma_i = \Phi(W^i \times X)$ for $i \in \mathbb{Z}_{>0}$. The family $(\Gamma_i)_{i\in\mathbb{N}}$ forms a non-decreasing sequence of constructible sets; we denote by $\Gamma_\infty$ their union. Then, consider the action of $\text{Aut}(X)$ on $X \times X$ given by $g \cdot (x, y) = (x, g(y))$. By construction, $\Gamma_i = W^i \cdot \Delta_X$ and $\Gamma_\infty = (W) \cdot \Delta_X$; similarly $W^i \cdot \Delta_Y = \Gamma_i \cap \pi_1^{-1}(Y)$ and $(W) \cdot \Delta_Y = \Gamma_\infty \cap \pi_1^{-1}(Y)$ for every subvariety $Y \subset X$.

**Lemma 3.3.** The subset $\Gamma_\infty$ of $X \times X$ is constructible.

**Proof.** Let us prove, by an induction on $\dim(Y)$, that $\pi_1^{-1}(Y) \cap \Gamma_\infty$ is constructible for every irreducible subvariety $Y \subset X$.

So, the case $\dim Y = -1$ is trivial. Now assume that $\dim Y \geq 0$ and that the result holds in dimension $< \dim(Y)$. Set $Z_Y = (W) \cdot \Delta_Y$; this set is invariant under the action of $(W)$ on $X \times X$. Since $(W) \cdot \Delta_Y$ is irreducible and increasing for each $i \geq 0$, there is $m \geq 0$, such that
\begin{equation}
(3.9) \quad Z_Y = (W) \cdot \Delta_Y = W^i \cdot \Delta_Y \quad (\forall i \geq m).
\end{equation}

Then there is a dense open subset $U_Y$ of $Z_Y$ which is contained in $W^m \cdot \Delta_Y$, hence in $(W) \cdot \Delta_Y$. Shrinking $U_Y$ if necessary, we may assume that $\pi_1(U_Y)$ is open in $Y$. Then $Y \setminus \pi_1(U_Y)$ is a closed subset of $X$, the irreducible component of which have dimension $< \dim Y$. By the induction hypothesis, $\pi_1^{-1}(Y \setminus \pi_1(U_Y)) \cap \Gamma_\infty$ is constructible. We also know that $\pi_1^{-1}(\pi_1(U_Y)) \cap \Gamma_\infty = (W) \cdot U_Y$ is an open subset of $Z_Y$. Thus, $\pi_1^{-1}(Y \setminus \pi_1(U_Y)) \cap \Gamma_\infty \cup (\pi_1^{-1}(\pi_1(U_Y)) \cap \Gamma_\infty)$ is constructible.

**Proposition 3.4.** The orbits $W^k(x)$ satisfy the following properties.

1. The function $k \in \mathbb{Z}_{>0} \mapsto \dim(W^k(x))$ is non-decreasing.
2. The function $x \in X \mapsto \dim(W^k(x))$ is lower semi-continuous in the Zariski topology: the subsets $\{x \in X; \dim(W^k(x)) \leq n\}$ are Zariski closed for all pairs $(n,k)$ of integers.
3. The integers $s(x) := \max_{k \geq 0} \{\dim(W^k(x))\}$ and $s_X := \max_{x \in X} \{s(x)\}$ are bounded from above by $\dim(X)$.
4. There is a Zariski dense open subset $\mathcal{U}$ of $X$ and an integer $k_0$ such that $\dim(W^k(x)) = s_X$ for all $k \geq k_0$ and all $x \in \mathcal{U}$.
5. There is an integer $\ell \geq 0$, such that for every $x$ in $X$, $W^\ell(x) = (W)(x)$ and $W^\ell(x)$ is an open subset of $(W)(x)$.

**Proof.** The first assertion follows from the inclusions (3.6), and the third one is obvious. Since the action $(f, x) \in W^k \times X \mapsto f(x) \in X$ is a morphism, the second
and fourth assertions follow from Chevalley’s constructibility result and the semi-
continuity of the dimension of the fibers (see [5, II, Exercise 3.19] and [16, Section
I.6.3, Corollary] respectively). By Lemma 3.3, $\Gamma_\infty$ is constructible. Since it is
the increasing union of the constructible subsets $\Gamma_i$, there is an integer $\ell$ such that
$\Gamma_\infty = \Gamma_i$ for $i \geq \ell$. Then, $W^\ell(x) = \langle W \rangle(x)$ because $W^\ell(x) = \pi_2(\Gamma_i \cap \pi_1^{-1}\{x\})$ and
$\langle W \rangle(x) = \pi_2(\Gamma_\infty \cap \pi_1^{-1}\{x\})$. Now, the constructible set $W^\ell(x)$ contains a dense
open subset $U$ of $\langle W \rangle(x)$; since $\langle W \rangle$ acts transitively on $W^\ell(x)$, $W^\ell(x) = \langle W \rangle(U)$
is open in $\langle W \rangle(x)$. \hfill $\Box$

3.3. Open orbits. Let us assume in this paragraph that $s_X = \dim X$: there is an
orbit $W^k(x_0)$ which is open and dense and coincides with $\langle W \rangle(x_0)$. We fix such
a pair $(k, x_0)$. Let $f$ be an element of $\langle W \rangle$. Since $f(x_0)$ is in the set $W^k(x_0)$,
there is an element $g$ of $W^k$ such that $g(x_0) = f(x_0)$, i.e. $g^{-1} \circ f(x_0) = x_0$. By
commutativity, $(g^{-1} \circ f)(h(x_0)) = h(x_0)$ for every $h$ in $W^k$, and this shows that
$g^{-1} \circ f = \text{id}$ because $W^k(x_0)$ is dense in $X$. Thus, $\langle W \rangle$ coincides with $W^k$, and
$\langle W \rangle = \langle V \rangle$ is an irreducible algebraic subgroup of the ind-group $\text{Aut}(X)$.

Thus, Theorem B is proved in case $s_X = \dim X$. The proof when $s_X < \dim X$
occupies the next section, and is achieved in § 3.4.4.

3.4. No dense orbit. Assume now that there is no dense orbit; in other words,
$s_X < \dim(X)$. Fix an integer $\ell > 0$ and a $W$-invariant open subset $\mathcal{U} \subset X$ such that
\begin{equation}
\tag{3.10}
s(x) = s_X \text{ and } W^\ell(x) = \langle W \rangle(x)
\end{equation}
for every $x \in \mathcal{U}$ (see Proposition 3.4, assertions (4) and (5)).

3.4.1. A fibration. Let $C$ be an irreducible algebraic subvariety of $X$ of codimen-
sion $s_X$ that intersects the general orbit $W^\ell(x)$ transversally (in $k$ points). As in
§ 3.2, denote by $\pi_1 : X \times X \to X$ the projection to the first factor. The morphism
\begin{equation}
\tag{3.11}
\pi' := (\pi_1)|_{(X \times C) \cap \Gamma_\ell} : (X \times C) \cap \Gamma_\ell \to X
\end{equation}
is generically finite of degree $k$. So there is a non-empty open subset $\mathcal{V}'$ of $\mathcal{U}$ such that $\pi'|_{\pi^{-1}(\mathcal{V}')}$ is finite étale. Observe that for every $g \in \langle W \rangle$, $g(\mathcal{V}')$
is open in $\mathcal{U}$ and $\pi'|_{\pi^{-1}(g(\mathcal{V}'))}$ is finite étale of degree $k$. Set $Y := \langle W \rangle(\mathcal{V}')$; it is open
in $\mathcal{U}$ and satisfies

(i) for each $x \in Y$ the intersection of $C$ and $W^\ell(x)$ is transverse and contains
exacty $k$ points;

(ii) $Y$ is $W$-invariant.

To each point $x \in Y$, we associate the intersection $C \cap W^\ell(x)$, viewed as a point
in the space $C^{[k]}$ of cycles of length $k$ and dimension 0 in $C$. This gives a dominant
morphism
\begin{equation}
\tag{3.12}
\pi : Y \to B
\end{equation}
where, by definition, $B$ is the irreducible variety $B = \overline{\pi(Y)} \subset C^{|S|}$. The group $\langle W \rangle$ is now contained in $\text{Aut}_x(Y)$. Shrinking $B$ and $Y$ accordingly, we may assume that $B$ is normal. Let $\eta$ be the generic point of $B$.

The fiber $\pi^{-1}(b)$ of $b \in B$, we denote by $Y_b$. By construction, for every $b \in B(k)$, $Y_b$ is an orbit of $\langle W \rangle$; and Section 3.3 shows that $Y_b$ is isomorphic to the image $\langle W \rangle_b$ of $\langle W \rangle$ in $\text{Aut}(Y_b)$: this group $\langle W \rangle_b$ coincides with the image of $W^f$ in $\text{Aut}(Y_b)$ and the action of $\langle W \rangle$ on $Y_b$ corresponds to the action of $\langle W \rangle_b$ on itself by translation. Thus, Section 3.3 implies the following properties

1. every fiber, in particular the generic fiber, of $\pi$ is geometrically irreducible;
2. the generic fiber of $\pi$ is normal and affine, shrinking $B$ (and $Y$ accordingly) again, we may assume $B$ and $Y$ to be normal and affine;
3. the action of $\langle W \rangle$ on the generic fiber $Y_\eta$ has bounded degree.

3.4.2. Reduction to $Y = U_B \times_B (\mathbb{G}_{m,B}^s)$. In this section, the variety $Y$ will be modified, so as to reduce our study to the case when $Y$ is an abelian group scheme over $B$. Note that $B$ and $Y$ will be modified several times in this paragraph, keeping the same names.

By Proposition 3.2, after shrinking $B$, there exist an embedding $\tau : Y \hookrightarrow \mathbb{A}^N_B$ for some $N \geq 0$ and a homomorphism $\rho : \langle W \rangle \hookrightarrow \text{GL}_N(O(B)) \subseteq \text{Aut}_B(\mathbb{A}^N_B)$ such that

$$\tau \circ g = \rho(g) \circ \tau \quad (\forall g \in \langle W \rangle).$$

Via $\tau$, we view $Y$ as a $B$-subscheme of $\mathbb{A}^N_B$, and via $\rho$ we view $\langle W \rangle$ in $\text{GL}_N(O(B))$. Consider the inclusion of $\text{GL}_N(O(B))$ into $\text{GL}_N(k(B))$, and compose it with the embedding of $W$ into $\text{GL}_N(O(B))$. Denote by $\langle W \rangle_\eta$ the Zariski closure of $\langle W \rangle$ in $\text{GL}_N(k(B), Y_\eta) \subseteq \text{Aut}(Y_\eta)$, where $\text{GL}_N(k(B), Y_\eta)$ is the subgroup of $\text{GL}_N(k(B))$ which preserves $Y_\eta$. There is a natural inclusion of sets $W \hookrightarrow W \otimes_k k(B)$: a point $x$ of $W$, viewed as a morphism $x : \text{Spec} k(x) \to W$, is mapped to the point

$$x^B : \text{Spec} k(x) (B \otimes_k k(x)) = \text{Spec} \text{Frac}(k(x) \otimes_k k(B)) \to W \otimes_k k(B),$$

where $k(x) (B \otimes_k k(x))$ is the function field of $B \otimes_k k(x)$ which is the variety over the field $k(x)$; note that $k$ being algebraically closed, $B \otimes_k k(x)$ is irreducible over $k(x)$ and $k(x) \otimes_k k(B)$ is an integral domain. The image of this inclusion is Zariski dense in $W \otimes_k k(B)$. The morphism $W \to \text{GL}_N(k(B), Y_\eta)$ naturally extends to a morphism $W \otimes_k k(B) \hookrightarrow \text{GL}_N(k(B), Y_\eta)$. It follows that $\langle W \rangle_\eta$ is the Zariski closure of $\langle W \rangle_\eta$ in $\text{GL}_N(k(B), Y_\eta)$.

Since $W \otimes_k k(B)$ is geometrically irreducible, $\langle W \rangle_\eta$ is a geometrically irreducible commutative linear algebraic group over $k(B)$. As a consequence ((111), Chap. 16.b), there exists a finite extension $L$ of $k(B)$ and an integer $s \geq 0$ such that

$$\langle W \rangle_\eta \otimes_{k(B)} L \simeq U_L \times \mathbb{G}_{m,L}^s$$

where $U_L$ is a unipotent commutative linear algebraic group over $L$. 

Let $\psi : B' \to B$ be the normalization of $B$ in $L$. We obtain a new fibration $\pi' : Y \times_B B' \to B'$, together with an embedding $\iota_\psi$ of $\Aut_\pi(Y)$ in $\Aut_\pi(Y \times_B B')$; by Lemma 3.1, the subgroup $\langle \psi \rangle$ defines in the same way an element of $\Aut U$ on this basic remark, with two extra difficulties: the structure of Section 3.4.2, that $B$ subtle in positive characteristic (see [14], §VII.2); instead of iterating one element $\Aut_\pi(Y)$ may assume that $B$ replaces (this new) $B$ by an affine open subset, and shrinking $Y$ accordingly, we may assume that $Y = U_B \times_B (\mathbb{G}^{s}_{m,B})$, where $U_B$ is an integral unipotent commutative algebraic group scheme over $B$, and

$$(3.16) \quad W \subseteq U_B(B) \times \mathbb{G}^{s}_{m,B}(B) \subseteq \Aut_\pi(Y)$$

acts on $Y$ by translation; here $U_B(B)$ and $\mathbb{G}^{s}_{m,B}(B)$ denote the ind-varieties of sections of the structure morphisms $U_B \to B$ and $\mathbb{G}^{s}_{m,B} \to B$ respectively.

**Remark 3.5.** A section $\sigma : B \to U_B$ defines an automorphism of $U_B \simeq B \times_B U_B$ by $\phi(\sigma \times_B \text{id}_{U_B})$, where $\phi : U_B \times U_B \to U_B$ is the multiplication morphism of $U_B$; it defines in the same way an element of $\Aut_\pi(Y)$. Similarly $\mathbb{G}^{s}_{m,B}(B)$ embeds into $\Aut_\pi(Y)$, so $U_B(B) \times \mathbb{G}^{s}_{m,B}(B) \subseteq \Aut_\pi(Y)$, and this is the meaning of (3.16).

**Remark 3.6.** Both $U_B(B) \times \mathbb{G}^{s}_{m,B}(B)$ and $\Aut_\pi(Y)$ are ind-varieties over $k$ and the inclusions in (3.16) are morphisms between ind-varieties.

Now, to prove Theorem B, we only need to show that $W$ is contained in an algebraic subgroup of $U_B(B) \times \mathbb{G}^{s}_{m,B}(B)$.

3.4.3. **Structure of $U_B$.** Let $B$ be a normal affine variety over the algebraically closed field $k$, and let $U_B$ be an integral, connected and unipotent algebraic group scheme over $B$ (we do not assume $U_B$ to be commutative here).

**Lemma 3.7.** The ind-group $U_B(B)$ is an increasing union of algebraic subgroups.

Before describing the proof, let us assume that $U_B$ is just an $r$-dimensional additive group $\mathbb{G}^{r}_{a,B}$. Then, each element of $U_B$ can be written

$$(3.17) \quad f = (a_1^f(z), \ldots, a_r^f(z))$$

where each $a_i^f(z)$ is an element of $O(B)$; its $n$-th power is given by $f^n = (na_1^f(z), \ldots, na_r^f(z))$. Thus, viewed as automorphisms of $Y$, the degrees of the $f^n$ are bounded independently of $n$, by (a function of) the degrees of the $a_i^f$. Our proof is a variation on this basic remark, with two extra difficulties: the structure of $U_B$ may be more subtle in positive characteristic (see [14], §VII.2); instead of iterating one element $f$, we need to control the group $U_B$ itself.

**Proof.** Denote by $\pi_U : U_B \to B$ the structure morphism. Recall, from the end of Section 3.4.2, that $B$ is an affine variety.
The proof is by induction on the relative dimension of \( \pi_U: U_B \to B \). If this dimension is zero, there is nothing to prove. So, we assume that the lemma holds for relative dimensions \( \leq \ell \), for some \( \ell \geq 0 \), and we want to prove it when the relative dimension is \( \ell + 1 \). Denote by \( U_\eta \) the generic fiber of \( \pi_U \). Our field \( \mathbf{k} \) is algebraically closed, and the group \( U_B \) is connected, so by Corollary 14.55 of [11] (see also § 14.63), there exists a finite field extension \( L \) of \( \mathbf{k}(B) \) such that \( U_L := U_\eta \otimes_{\mathbf{k}(B)} L \) sits in a central exact sequence

\[
0 \to \mathbb{G}_{a,L} \to U_L \xrightarrow{q_L} V_L \to 0,
\]

where \( V_L \) is an irreducible unipotent group of dimension \( \ell \) and \( V_L \) is isomorphic to \( \mathbb{A}_L^{\ell} \) as an \( L \)-variety; moreover, there is an isomorphism of \( L \)-varieties \( \Phi_L: U_L \to V_L \times \mathbb{G}_{a,L} \) such that the quotient morphism \( q_L \) is given by the projection onto the first factor. So we have a section \( s_L: V_L \to U_L \) such that \( q_L \circ s_L = \text{id} \). The section \( s_L \) is just given by a regular function on \( V_L \), it needs not be a homomorphism of groups. Doing the base change given by the normalization of \( B \) in \( L \), and then shrinking the base if necessary, we may assume that \( B \) is affine and

- there is an exact sequence of group schemes over \( B 
\]

\[
0 \to \mathbb{G}_{a,B} \to U_B \xrightarrow{q_B} V_B \to 0,
\]

where \( V_B \) is a unipotent group scheme over \( B \) of relative dimension \( \ell \);

- there is an isomorphism of \( B \)-schemes \( V_B \simeq \mathbb{A}_B^{\ell} \);

- \( s_L \) extends to a section \( s_B: V_B \to U_B \) over \( B 
\]

\[
q_B \circ s_B = \text{id}.
\]

For \( b \in B \), denote by \( U_b, V_b, q_b, s_b \) the specialization of \( U_B, V_B, q_B, s_B \) at \( b \). Since \( U_B \) and \( V_B \) are abelian, we denote by \( + \) the group law on these groups. The morphism of \( B \)-schemes \( \beta: U_B \to V_B \times \mathbb{G}_{a,B} \) sending a point \( x \) in the fiber \( U_b \) to the point \( (q_b(x), x - s_b(q_b(x))) \) of the fiber \( V_B \times \mathbb{G}_{a,B} \) defines an isomorphism. We use \( \beta \) to transport the group law of \( U_B \) into \( V_B \times \mathbb{G}_{a,B} \); this defines a law \( \ast \) on \( V_B \times \mathbb{G}_{a,B} \), given by

\[
(3.19) \quad a_1 \ast a_2 = \beta(\beta^{-1}(a_1) + \beta^{-1}(a_2)),
\]

for \( a_1 \) and \( a_2 \) in \( V_B \times \mathbb{G}_{a,B} \). Denote by \( O(V_B \times_B V_B) \) the function ring of the \( \mathbf{k} \)-variety \( V_B \times_B V_B \simeq B \times \mathbb{A}^\ell \times \mathbb{A}^\ell \). We write a point in \( V_B \times_B V_B \) as \( (b, x_1, x_2) \) where \( x_1, x_2 \in V_B \) with the same image \( b \) in \( B \). There is an element \( F(b, x_1, x_2)(y_1, y_2) \) of \( O(V_B \times_B V_B)[y_1, y_2] \) such that

\[
(3.20) \quad (x_1, y_1) \ast (x_2, y_2) = (x_1 + x_2, F(b, x_1, x_2)(y_1, y_2))
\]

for all \( b \in B \) and \( (x_1, y_1), (x_2, y_2) \in V_b \times \mathbb{G}_{a,B} \). (In (3.20), \( + \) is the group law in \( V_b \).) For every fixed \( (x_1, y_1, x_2) \), the morphism \( y_2 \mapsto F(b, x_1, x_2)(y_1, y_2) \) is an automorphism of the variety \( \mathbb{G}_{a,B} \). Thus, we can write

\[
(3.21) \quad F(b, x_1, x_2)(y_1, y_2) = C_0(b, x_1, x_2)(y_1) + C_2(b, x_1, x_2)(y_1)y_2.
\]
The function $C_2(b,x_1,x_2)(y_1)$ does not vanish on $V_B \times_B V_B \times A^1 \simeq B \times A^{2\ell+1}$; thus, $C_2$ is an element of $O(B)$. By symmetry we get

\begin{equation}
F(b,x_1,x_2)(y_1,y_2) = C_0(b,x_1,x_2) + C_1(b)y_1 + C_2(b)y_2
\end{equation}

and

\begin{equation}
(x_1,y_1) \ast (x_2,y_2) = (x_1 + x_2, C_0(b,x_1,x_2) + C_1(b)y_1 + C_2(b)y_2).
\end{equation}

Now, apply this equation for $x_1 = x_2 = 0$ (the neutral element of $V_B$). The restriction of $\beta$ to the fiber $q_b^{-1}(0)$ is $x \mapsto x - s_b(0)$, so for $(0,y_1)$ and $(0,y_2)$ in $V_B \times \mathbb{G}_a$, we obtain $(0,y_1) \ast (0,y_2) = (0,y_1 + y_2 + s_b(0))$; then $C_1 = C_2 = 1$.

We identify now the ind-varieties $U_B(B)$ and $V_B(B) \times \mathbb{G}_a(B)$. By induction, the ind-group $B_B$ is an increasing union of algebraic subgroups $V_i$; as observed before the proof of this lemma, the ind-group $\mathbb{G}_a(B)$ is an increasing union of subgroups $G_j$. If $S$ and $T$ are elements of $V_B(B)$ and $\mathbb{G}_a(B)$ respectively, we set

\begin{equation}
\delta_V(S) = \min\{i ; S \in V_i\}, \quad \delta_{\mathbb{G}_a}(T) = \min\{j ; T \in G_j\}.
\end{equation}

Each element of $U_B(B)$ is given by a section $(S,T) \in V_B(B) \times \mathbb{G}_a(B)$ and the group law in $U_B(B)$ corresponds to the law

\begin{equation}
(S_1,T_1) \ast (S_2,T_2) = (S_1 + S_2, C_0(S_1,S_2) + T_1 + T_2)
\end{equation}

because $C_1 = C_2 = 1$. Here $C_0 : V_B(B) \times V_B(B) \to \mathbb{G}_a(B)$ is a morphism of ind-varieties, so there is a function $\alpha : \mathbb{N} \to \mathbb{N}$ such that

\begin{equation}
\delta_{\mathbb{G}_a}C_0(S_1,S_2) \leq \alpha(\delta_V(S_1) + \delta_V(S_2)).
\end{equation}

Now, note that $V_i \times G_{\alpha(2i)}$ is an algebraic subgroup of $U_B(B)$, because

\begin{equation}
\delta_{\mathbb{G}_a}(C_0(S_1,S_2) + T_1 + T_2) \leq \max\{\delta_{\mathbb{G}_a}(C_0(S_1,S_2)), \delta_{\mathbb{G}_a}(T_1), \delta_{\mathbb{G}_a}(T_2)\}.
\end{equation}

Thus, $U_B(B)$ is the increasing union of the algebraic subgroups $V_i \times G_{\alpha(2i)}$. \qed

3.4.4. Subgroups of $\mathbb{G}_m(B)$ and conclusion.

**Lemma 3.8.** If $Z$ is an irreducible subvariety of $\mathbb{G}_m(B)$ containing $\text{id}$, then $\langle Z \rangle$ is an algebraic subgroup of $\mathbb{G}_m(B)$.

**Proof of Lemma 3.8.** Pick a projective compactification $\overline{B}$ of $B$. After taking the normalization of $\overline{B}$, we may assume $\overline{B}$ to be normal. If $h$ is any non-constant rational function on $\overline{B}$, denote by $\text{Div}(h)$ the divisor $(h)_{\infty} - (h)_{0}$ on $\overline{B}$.

Let $y = (y_1, \ldots, y_s)$ be the standard coordinates on $\mathbb{G}_m^n$. Each element $f \in \mathbb{G}_m^n(B)$ can be written as $\langle b_1^T(z), \ldots, b_s^T(z) \rangle$, for some $b_j^T \in O^*(B)$. Let $R$ be an effective divisor whose support $\text{Support}(R)$ contains $\overline{B} \setminus B$. Replacing $R$ by some large multiple, $Z$ is contained in the subset $P_R$ of $\mathbb{G}_m^n(B)$ made of automorphisms $f \in \mathbb{G}_m^n(B)$ such that $\text{Div}(b_i^T) + R \geq 0$ and $\text{Div}(1/b_i^T) + R \geq 0$ for all $i = 1, \ldots, s$. Let us study the structure of this set $P_R \subset \mathbb{G}_m^n(B)$. 
Let $K$ be the set of pairs $(D_1, D_2)$ of effective divisors supported on $\overline{B} \setminus B$ such that $D_1$ and $D_2$ have no common irreducible component, $D_1 \leq R$, $D_2 \leq R$, and $D_1$ and $D_2$ are rationally equivalent. Then $K$ is a finite set. For every pair $\alpha = (D_1^0, D_2^0) \in K$, we choose a function $h_\alpha \in O^* (Y)$ such that $\text{Div}(h_\alpha) = D_1^0 - D_2^0$; if $h$ is another element of $O^* (Y)$ such that $\text{Div}(h) = D_1^0 - D_2^0$, then $h / h_\alpha \in k^*$. By convention $\alpha = 0$ means that $\alpha = (0,0)$, and in that case we choose $h_\alpha$ to be the constant function 1. For every $\beta = (\alpha_1, \ldots, \alpha_s) \in K^s$, denote by $P_\beta$ the set of elements $f \in G_m (B)$ such that the $b_i^\beta \in O^* (B)$ satisfy $\text{Div}(b_i^\beta) = D_1^0 - D_s^0$ for all $i = 1, \ldots, s$. Then $P_\beta \cong G_m^s (k)$ is an irreducible algebraic variety over $k$. Moreover, id \in $P_\beta$ if and only if $\beta = 0$, and $P_0$ is an algebraic subgroup of $G_m (B)$, isomorphic to $G_m^s (k)$ as an algebraic group.

Observe that $P_R$ is the disjoint union $P_R = \bigsqcup_{\beta \in K^s} P_\beta$. Since id \in $Z$, $Z$ is irreducible, and $Z \subseteq P_R$, we obtain $Z \subseteq P_0$. Since $P_0$ is an algebraic subgroup of $G_m^s (B)$, $(Z)$ coincides with $(Z \cdot Z^{-1})^\ell$ for some $\ell \geq 1$, and $(Z)$ is a connected algebraic group.

**Proof of Theorem B.** By Proposition 2.2, we only need to prove that $W = \langle V \rangle$ is of bounded degree. By Lemma 3.1 $W$ is a subgroup of bounded degree if and only if $W \subseteq \text{Aut}_\pi(Y)$ is a subgroup of bounded degree. Moreover, by (3.16), $W$ is a subgroup of $U_B(B) \times G_m^s (B) \subseteq \text{Aut}_\pi(Y)$. Denote by $\pi_1 : U_B(B) \times G_m^s (B) \rightarrow U_B(B)$ the projection to the first factor and $\pi_2 : U_B(B) \times G_m^s (B) \rightarrow G_m^s (B)$ the projection to the second. By Lemma 3.7, there exists an algebraic subgroup $H_1$ of $U_B(B)$ containing $\pi_1(W)$. Since $\pi_2(W)$ is irreducible and contains id, Lemma 3.8 shows that $\pi_2(W)$ is contained in an algebraic subgroup $H_2$ of $G_m(B)$. Then $W$ is contained in the algebraic subgroup $H_1 \times H_2$ of $U_B(B) \times G_m(B)$. This concludes the proof. □

### 4. ACTIONS OF ADDITIVE GROUPS

**Theorem 4.1.** Let $k$ be an uncountable, algebraically closed field. Let $X$ be a connected affine variety over $k$. Let $G \subseteq \text{Aut}(X)$ be an algebraic subgroup isomorphic to $G_m^r$ for some $r \geq 1$. Let $H = \{ h \in \text{Aut}(X) \mid gh = hg \text{ for every } g \in G \}$ be the centralizer of $G$. If $H / G$ is at most countable then $G$ acts simply transitively on $X$, so that $X$ is isomorphic to $G$ as a $G$-variety.

This section is devoted to the proof of this result. A proof is also described in [3, §11.4] when $X$ is irreducible and the characteristic of $k$ is 0.

**Lemma 4.2.** Let $X$ be an irreducible affine variety endowed with a faithful action of $G = G_m^r$. Let $I$ be a non-zero, $G$-invariant ideal of $O(X)$. If

\[ I^G := \{ \xi \in I \mid g^* \xi = \xi \text{ for every } g \in G \} \]

is contained in the field $k$ of constant functions, then

1. every non-zero, $G$-invariant ideal $J \subseteq O(X)$ coincides with $O(X)$;
In particular, $I = O(X)$.

Proof of Lemma 4.2. Let $\xi$ be a non-zero element of $I$. To prove the first assertion, pick $\psi \in J \setminus \{0\}$, then $\xi \psi \in JJ \setminus \{0\}$. Let $V$ be the linear subspace of $O(X)$ generated by the orbit $\{g^{*}(\xi \psi) \mid g \in G\}$. Firstly, $V$ is contained in $IJ$ because $I$ and $J$ are $G$-invariant. Secondly, the dimension of $V$ is finite, for $G$ acts regularly on $X$ (see [18, §1.2]). Thus, $G$ being isomorphic to $G_{x}$, there exists a $G$-invariant vector $\varphi \in V \setminus \{0\} \subseteq JJ$. Since $I_{G} \subseteq k$, the function $\varphi$ is a constant, and $J$ must be equal to $O(X)$ because it contains $\varphi$.

To prove the second and third assertions, fix a point $x \in X$. The closure $\overline{G(x)}$ of the orbit $G(x)$ is a closed, $G$-invariant subvariety, and the same is true for $X \setminus \overline{G(x)}$.

Looking at the ideal of functions vanishing on those subvarieties we obtain $X = G(x)$. Since $G$ is abelian and acts faithfully on $X$, the stabilizer of $x$ must be trivial. Thus, $G$ acts simply transitively on $X$. \hfill $\Box$

Let $X$ be an affine variety over $k$, and let $G$ be a subgroup of $\text{Aut}(X)$ isomorphic to $G_{x}$ for some $r > 0$. Denote by $X_{1}, \ldots, X_{l}$ the irreducible components of $X$. Then each of the $X_{i}$, $i = 1, \ldots, l$, is invariant under $G$; permuting the $X_{i}$ if necessary, there exists $s \leq l$ such that the action of $G$ on $X_{s}$ is nontrivial if and only if $i \leq s$.

For every $i \leq l$, denote by $\pi_{i}: O(X) \twoheadrightarrow O(X_{i})$ the quotient map. Let $J_{i}$ be the ideal of functions $\xi \in O(X)$ vanishing on the closed subset $\bigcup_{j \neq i} X_{j}$; its projection $I_{i} := \pi_{i}(J_{i})$ is an ideal of $O(X_{i})$. Observe that $I_{i}$ is non-zero, is invariant under the action of $G$, and is contained in the ideal of $O(X_{i})$ associated to the closed subset $X_{i} \cap \bigcup_{j \neq i} X_{j}$. In particular, $I_{i} = O(X_{i})$ if and only if $X_{i}$ is a connected component of $X$. Moreover, the homomorphism $\pi_{i}|_{J_{i}}: J_{i} \rightarrow I_{i}$ is a bijection. Indeed, it is a surjective homomorphism by definition. And if $\pi_{i}|_{J_{i}}(\xi) = 0$, then $\xi|_{X_{i}} = 0$ and since $\xi \in J_{i}$, $\xi|_{X_{j}} = 0$ for all $j \neq i$; thus $\xi = 0$, so that $\ker(\pi_{i}|_{J_{i}}) = 0$.

We denote by $(\pi_{i}|_{J_{i}})^{-1}: I_{i} \rightarrow J_{i}$ the inverse of $\pi_{i}|_{J_{i}}$.

Lemma 4.3. Let $k$ be an uncountable, algebraically closed field. Let $X$ be an affine variety, and $G$ be an algebraic subgroup of $\text{Aut}(X)$ isomorphic to $G_{x}$. Let

$$H := \{h \in \text{Aut}(X) \mid gh = hg \text{ for every } g \in G\}$$

be the centralizer of $G$. If $H / G$ is at most countable then $I_{i}^{G} \subseteq k$ for every irreducible component $X_{i}$ on which $G$ acts non-trivially.

Proof of Lemma 4.3. Keeping the notation that precedes the statement of the lemma, we only need to treat the case $i = 1$. Fix an identification $G_{x} \simeq G$. Then, we get an identification $O(X)^{r} = \text{Mor}(X, G_{x}) = \text{Mor}(X, G)$ for which $(O(X)^{G})^{r}$ corresponds
to \( \text{Mor}(X, G)^G \) (here \( G \) acts on \( X \) only). We also identify \( G \) to the group of constant morphisms in \( \text{Mor}(X, G) \); then, \( G \) becomes a subgroup of the additive group \((O(X)^G)'/\). Let us modify the action of \( G \) on \( X \), as in [3, §0.7]:

**Fact 1.**– Define \( \Psi : (O(X)^G)' \to \text{End}(X) \) by \( \Psi(\xi) : x \mapsto \xi(x)(x) \) for every \( x \in X \). Then \( \Psi \) is a homomorphism of additive groups. Here we view \( \xi \in (O(X)^G)' \) as an element of \( \text{Mor}(X, G) \).

We need to prove that \( \Psi(\xi_1 + \xi_2) = \Psi(\xi_1) \circ \Psi(\xi_2) \) for every pair of elements \( \xi_1, \xi_2 \in (O(X)^G)' \). For \( x \in X \), we have

\[
(1.1) \quad \Psi(\xi_1 + \xi_2)(x) = (\xi_1(x) + \xi_2(x))(x) = \xi_1(x)((\xi_2(x)(x))).
\]

On the other hand, \( \xi_1(y) = \xi_2(x)(y) \) for every pair \( (x, y) \in X^2 \) because \( \xi_2(x) \in G \) and \( \xi_1 \) is in \( \text{Mor}(X, G)^G \). Thus \( \xi_1(x) = \xi_1(\xi_2(x)(x)) \), and the claim follows:

\[
(1.2) \quad \Psi(\xi_1 + \xi_2)(x) = \xi_1(\xi_2(x)(x))(\xi_2(x)(x)) = \Psi(\xi_1)(\Psi(\xi_2)(x)).
\]

This fact implies that \( \Psi \) is a homomorphism from the additive group \((O(X)^G)' \) to the group of automorphisms \( \text{Aut}(X) \). We note that \( \Psi|_G = \text{id} \); since \((O(X)^G)' \) is abelian, \( \Psi((O(X)^G)') \) is a subgroup of the centralizer \( H \) that contains \( G \). Since \( H/G \) is countable, so is \((O(X)^G)' / \Psi^{-1}(G) \).

Now, define \( \Phi : (I_1^G)' \to \text{Aut}(X) \) to be the composition of \( \Psi \) with the inclusion \( ((\pi_1|x_i)^{-1})' : (I_1^G)' \to (O(X)^G)' \). We obtain an inclusion

\[
(1.3) \quad (I_1^G)' / \Phi^{-1}(G) \hookrightarrow (O(X)^G)' / \Psi^{-1}(G);
\]

hence, \((I_1^G)' / \Phi^{-1}(G) \) is also countable.

**Fact 2.**– We have \( \Phi^{-1}(G) = (I_1^G)' \).

To prove it, denote by \( G_x \) the stabilizer of \( x \in X \) in \( G \), and for \( \xi \in (I_1^G)' \), set

\[
(1.4) \quad Y(\xi) := \bigcap_{x \in X} (\xi(x) + G_x).
\]

Then \( Y(\xi) \) is an affine linear subspace of \( G \simeq A'(k) = k' \), and \( \xi \in \Phi^{-1}(G) \) if and only if \( Y(\xi) \neq \emptyset \). It follows that \( \Phi^{-1}(G) \) is a linear subspace of \((I_1^G)' \). Since \( k \) is uncountable and \((I_1^G)' / \Phi^{-1}(G) \) is at most countable, we get \( \Phi^{-1}(G) = (I_1^G)' \).

It follows that for every \( \xi \in (I_1^G)' \),

\[
(1.5) \quad \emptyset \neq Y(\xi) = \bigcap_{x \in X} (\xi(x) + G_x) \subseteq W(\xi) := \bigcap_{x \in X_1} (\xi(x) + G_x).
\]

Choose \( \eta \in (I_1^G)' \) such that \( \text{dim}(W(\eta)) \) is minimal, and then choose \( x_1, \ldots, x_m \in X_1 \), such that \( W(\eta) = \bigcap_{i=1}^m (\eta(x_i) + G_{x_i}) \). To conclude, we assume that \( I_1^G \) contains a non-constant function \( \alpha \), and then we shall modify \( \eta \) to get a new function \( \tau \) with \( \text{dim}(W(\tau)) < \text{dim}(W(\eta)) \), in contradiction with our choice for \( \eta \). For this purpose,
set $\beta = \prod_{i=1}^{m} (\alpha - \alpha(x_i))$. Then, choose $y \in X_1 \setminus \{x_1, \ldots, x_m\}$ such that $G_y \neq G$, $\alpha(y) \neq 0$ and $\beta(y) \neq 0$, and set

$$\gamma := \frac{\alpha \beta}{\beta(y) \alpha(y)}.$$ 

By construction, we get

1. $\gamma \in O(X_1)^G I^G \subseteq I^G$;
2. $\gamma(x_i) = 0$ for all $i = 1, \ldots, m$;
3. $\gamma(y) = 1$.

Pick $g \in W(\eta)$. The set $U = \{h \in G ; g \not\in h + \xi(y) + G_y\}$ is Zariski dense in $G$. Take $h \in U$, write $h = (a_1, \ldots, a_r)$ as an element of $G = \mathbb{G}_a^r$, and set

$$\tau := \eta + \gamma h := \eta + (a_1, \gamma, \ldots, a_r).$$

By construction, $\tau$ is an element of $(I^G)^r$; and, changing $h = (a_1, \ldots, a_r)$ in $U$ if necessary, we may assume that $\tau \not\in k^r$. From the properties (2) and (3) above, we get $\tau(x_i) = \eta(x_i)$ for $i = 1, \ldots, m$ and $\tau(y) = \eta(y) + h$. We have

$$W(\tau) \subseteq \left( \bigcap_{i=1}^{m} (\tau(x_i) + G_{x_i}) \right) \bigcap (\tau(y) + G_y)$$

$$= \left( \bigcap_{i=1}^{m} (\eta(x_i) + G_{x_i}) \right) \bigcap (\eta(y) + h + G_y)$$

$$= W(\eta) \cap (\eta(y) + h + G_y).$$

Since $g \in W(\eta)$ but $g \not\in (\eta(y) + h + G_y)$, we get $\dim W(\tau) \leq \dim (W(\eta) \cap (\eta(y) + h + G_y)) < \dim W(\eta)$. This proves the lemma.

**Proof of Theorem 4.1.** We keep the same notation. By Lemma 4.3, $I_1^G \subseteq k$. Let $G_1 \subset \text{Aut}(X_1)$ be the restriction of $G$. There exists $m \in \{1, \ldots, r\}$, such that $G_1 \simeq \mathbb{G}_a^m$. We have $I_1^{G_1} = I_1^G \subseteq k$. By Lemma 4.2, $I_1 = O(X_1)$ and $X_1 \simeq G_1$ as a $G_1$-variety. Since $I_1 = O(X_1)$, $X_1$ is a connected component of $X$. Since $X$ is connected, $X = X_1$ and $G_1 = G$. This concludes the proof.

5. PROOF OF THEOREM A

In this section, we prove Theorem A. So, $k$ is an uncountable, algebraically closed field, $X$ is a connected affine algebraic variety over $k$, and $\varphi : \text{Aut}(\mathbb{A}^n_k) \to \text{Aut}(X)$ is an isomorphism of (abstract) groups.

5.1. **Translations and dilatations.** Let $\text{Tr} \subset \text{Aut}(\mathbb{A}^n_k)$ be the group of all translations and $\text{Tr}_i$ the subgroup of translations of the $i$-th coordinate:

$$\text{Tr}_i : (x_1, \ldots, x_i) \mapsto (x_1, \ldots, x_i + c, \ldots, x_n).$$
for some $c$ in $k$. Let $D \subset \text{GL}_n(k) \subset \text{Aut}(A^n_k)$ be the diagonal group (viewed as a maximal torus) and let $D_i$ be the subgroup of automorphisms

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, ax_i, \ldots, x_n)$$

for some $a \in k^*$. A direct computation shows that $\text{Tr}$ (resp. $D$) coincides with its centralizer in $\text{Aut}(A^n_k)$.

**Lemma 5.1.** Let $G$ be a subgroup of $\text{Tr}$ whose index is at most countable. Then, the centralizer of $G$ in $\text{Aut}(A^n_k)$ is $\text{Tr}$.

**Proof.** The centralizer of $G$ contains $\text{Tr}$. Let us prove the reverse inclusion. The index of $G$ in $\text{Tr}$ being at most countable, $G$ is Zariski dense in $\text{Tr}$. Thus, if $h$ centralizes $G$, we get $hg = gh$ for all $g \in \text{Tr}$, and $h$ is in fact in the centralizer of $\text{Tr}$. Since $\text{Tr}$ coincides with its centralizer, we get $h \in \text{Tr}$. \hfill \Box

5.2. **Closed subgroups.** As in Section 2.2, we endow $\text{Aut}(X)$ with the structure of an ind-group, given by a filtration by algebraic varieties $\text{Aut}_j$ for $j \geq 1$.

**Lemma 5.2.** The groups $\varphi(\text{Tr})$, $\varphi(\text{Tr}_i)$, $\varphi(D)$ and $\varphi(D_i)$ are closed subgroups of $\text{Aut}(X)$ for all $i = 1, \ldots, n$.

**Proof.** Since $\text{Tr} \subset \text{Aut}(A^n_k)$ coincides with its centralizer, $\varphi(\text{Tr}) \subset \text{Aut}(X)$ coincides with its centralizer too and, as such, is a closed subgroup of $\text{Aut}(X)$. The same argument applies to $\varphi(D) \subset \text{Aut}(X)$. To prove that $\varphi(\text{Tr}_i) \subset \text{Aut}(X)$ is closed we note that $\varphi(\text{Tr}_i)$ is the subset of elements $f \in \varphi(\text{Tr})$ that commute to every element $g \in \varphi(D_j)$ for every index $j \neq i$ in $\{1, \ldots, n\}$. Analogously, $\varphi(D_i) \subset \text{Aut}(X)$ is a closed subgroup because an element $f$ of $D$ is in $D_i$ if and only if it commutes to all elements $g$ of $\text{Tr}_j$ for $j \neq i$. \hfill \Box

5.3. **Proof of Theorem A.**

5.3.1. **Abelian groups (see [11, 14]).** Before starting the proof, let us recall a few important facts on abelian, affine algebraic groups. Let $G$ be an algebraic group over the field $k$, such that $G$ is abelian, affine, and connected.

1. If $\text{char}(k) = 0$, then $G$ is isomorphic to $\mathbb{G}_a^r \times \mathbb{G}_m^s$ for some pair of integers $(r,s)$; if $G$ is unipotent, then $s = 0$. (see [14]), §VII.2, p.172)

When the characteristic $p$ of $k$ is positive, there are other types of of abelian groups, but criteria on the $p$-torsion may rigidify their structure:

2. If $\text{char}(k) = p$, $G$ is unipotent, and all elements of $G$ have order $p$, then $G$ is isomorphic to $\mathbb{G}_a^r$ for some $r \geq 0$. (see [14], §VII.2, Prop. 11, p.178)

3. If $\text{char}(k) = p$, and there is no non-trivial element in $G$ of order $p^\ell$, for any $\ell \geq 0$, then $G$ is isomorphic to $\mathbb{G}_m^s$ for some $s \geq 0$. (see [11], Theorem 16.13 and Corollary 16.15, and [14], §VII.2, p.176)
To keep examples in mind, note that all elements of $\text{Tr}_1(k)$ have order $p$ and $D_1(k)$ does not contain any non-trivial element of order $p^f$ when $\text{char}(k) = p$.

### 5.3.2. Proof of Theorem A

Let us now prove Theorem A.

By Lemma 5.2, $\varphi(\text{Tr}_1) \subset \text{Aut}(X)$ is a closed subgroup; in particular, $\varphi(\text{Tr}_1)$ is an ind-subgroup of $\text{Aut}(X)$. Let $\varphi(\text{Tr}_1)^\circ$ be the connected component of the identity of $\varphi(\text{Tr}_1)$; from Section 2.2.2, we know that the index of $\varphi(\text{Tr}_1)^\circ$ in $\varphi(\text{Tr}_1)$ is at most countable. The ind-group $\varphi(\text{Tr}_1)^\circ$ is an increasing union $\bigcup_i V_i$ of irreducible algebraic varieties $V_i$, each $V_i$ containing the identity. Theorem B implies that each $\langle V_i \rangle$ is an irreducible algebraic subgroup of $\text{Aut}(X)$. Since $\varphi(\text{Tr}_1)$ does not contain elements of order $k < \omega$ with $k \wedge \text{char}(k) = 1$, $\langle V_i \rangle$ is unipotent; and, by Properties (1) and (2) of Section 5.3.1, $\langle V_i \rangle$ is isomorphic to $\mathbb{G}_m^r$ for some $r_i$. Thus

$$
\varphi(\text{Tr}_1)^\circ = \bigcup_{i \geq 0} F_i
$$

where the $F_i$ form an increasing family of unipotent algebraic subgroups of $\text{Aut}(X)$, each of them isomorphic to some $\mathbb{G}_m^r$. We may assume that $\dim F_0 \geq 1$.

Similarly, $\varphi(D_1)^\circ \subset \varphi(D_1)$ is a subgroup of countable index and

$$
\varphi(D_1)^\circ = \bigcup_{i \geq 0} G_i,
$$

where the $G_i$ are increasing irreducible commutative algebraic subgroups of $\text{Aut}(X)$ (we do not assert that $G_i$ is of type $\mathbb{G}_m^r$ yet). We may assume that $\dim G_0 \geq 1$.

The group $D_i$ acts by conjugation on $\text{Tr}_i$ for every $i \leq n$, this action has exactly two orbits $\{\text{id}\}$ and $\text{Tr}_i \setminus \{\text{id}\}$, and the action on $\text{Tr}_i \setminus \{\text{id}\}$ is free; hence, the same properties hold for the action of $\varphi(D_1)$ on $\varphi(\text{Tr}_1)$ by conjugation.

Let $H_i$ be the subgroup of $\varphi(\text{Tr}_1)$ generated by all $g \circ f \circ g^{-1}$ with $f \in F_i$ and $g$ in $G_i$. Theorem B shows that $H_i$ is an irreducible algebraic subgroup of $\varphi(\text{Tr}_1)$. We have $H_i \subseteq H_{i+1}$ and $g \circ H_i \circ g^{-1} = H_i$ for every $g \in G_i$.

Write $H_i = \mathbb{G}_m^l$ for some $l \geq 1$. We claim that $G_i \simeq \mathbb{G}_m^r \times \mathbb{G}_m^s$ for a pair of integers $r, s \geq 0$ with $r + s \geq 1$. This follows from Properties (1) and (2) of Section 5.3.1 because, when $\text{char}(k) = p > 1$, the only element in $\varphi(D_1)$ of order $p^f$, $f \geq 0$, is the identity element. Since the action of $\varphi(D_1)$ on $\varphi(\text{Tr}_1 \setminus \{0\})$ is free, the action of $G_i$ on $F_i \setminus \{0\}$ is free, and this implies $r = 0$ (see Lemma 4.2(2)). Let $q$ be a prime number with $q \wedge \text{char}(k) = 1$. Then $\mathbb{G}_m^s$ contains a copy of $(\mathbb{Z}/q\mathbb{Z})^s$, and $D_1$ does not contain such a subgroup if $s > 1$; so, $s = 1$, $G_i \simeq \mathbb{G}_m$ and $G_i = G_{i+1}$ for all $i \geq 0$.

It follows that $\varphi(D_1)^\circ \simeq \mathbb{G}_m$. Since the index of $\varphi(D_1)^\circ$ in $\varphi(D_1)$ is countable, there exists a countable subset $F \subseteq \varphi(D_1)$ such that $\varphi(D_1) = \bigcup_{h \in F} \varphi(D_1)^\circ \circ h$.

Let $f \in F_i$ be a nontrivial element. Since the action of $\varphi(D_1)$ on $\varphi(\text{Tr}_1 \setminus \{0\})$ is transitive,

$$
F_i \setminus \{0\} = \bigcup_{h \in F_i} \left( \bigcup_{g \in \varphi(D_1)^\circ} (g \circ h) \circ f \circ (g \circ h)^{-1} \right) \cap F_i
$$
The right hand side is a countable union of subvarieties of $F_1 \setminus \{0\}$ of dimension at most one. It follows that $\dim F_1 = 1$, $F_1 \simeq G_d$, and $\varphi(\text{Tr}_1)^\circ \simeq G_d$. Thus, we have

(5.6) \[ \varphi(\text{Tr}_1)^\circ \simeq G_d, \quad \text{and} \quad \varphi(\text{D}_1)^\circ \simeq G_m. \]

Since each $\varphi(\text{Tr}_1)^\circ$ is isomorphic to $G_d$, $\varphi(\text{Tr})^\circ$ is an $n$-dimensional commutative unipotent group and its index in $\varphi(\text{Tr})$ is at most countable. By Lemma 5.1, the centralizer of $\varphi^{-1}(\varphi(\text{Tr})^\circ)$ in $\text{Aut}(A^n_k)$ is $\text{Tr}$. It follows that the centralizer of $\varphi(\text{Tr})^\circ$ in $\text{Aut}(X)$ is $\varphi(\text{Tr})$. Then Theorem 4.1 implies that $X$ is isomorphic to $A^n_k$.

6. APPENDIX: THE DEGREE FUNCTIONS FOR RATIONAL SELF-MAPS

Here, we follow [1, 17] to prove a general version of Lemma 3.1. As above, $k$ is an algebraically closed field. We first start with the case of projective varieties.

6.1. Degree functions on projective varieties. Let $X$ be a projective and normal variety over $k$ of pure dimension $d = \dim(X)$. Let $H$ be a big and nef divisor on $X$. For every dominant rational self-map $f$ of $X$, and every $j = 0, \ldots, d$, set

(6.1) \[ \deg_{j,H} f = (f^*(H)^j \cdot H^{d-j}). \]

Pick a normal resolution of $f$; by this we mean a projective and normal variety $\Gamma$, a birational morphism $\pi_1 : \Gamma \to X$ and a morphism $\pi_2 : \Gamma \to X$ satisfying $f = \pi_2 \circ \pi_1^{-1}$. Then we have $\deg_{j,H} f = (\pi_2^*(H)^j \cdot \pi_1^*(H^{d-j})) > 0$, for $f$ is dominant. Let $L$ be another big and nef divisor. There is $c > 1$ such that $cL - H$ and $cH - L$ are big. Then we have $\deg_{L,H} f = (\pi_2^*(H)^j \cdot \pi_1^*(H^{d-j})) \leq c^d (\pi_2^*(L)^j \cdot \pi_1^*(L^{d-j})) = c^d \deg_{L,L} f$. Symetrically, we get $\deg_{L,L} f \leq (c')^d \deg_{j,H} f$ for some $c' > 1$. Thus, two big and nef divisors give rise to comparable degree functions:

(6.2) \[ C^{-1} \deg_{j,H}(f) \leq \deg_{j,L}(f) \leq C \deg_{j,H}(f), \quad (\forall 0 \leq j \leq d) \]

for all rational dominant maps $f : X \to X$, and some $C > 1$.

Lemma 6.1. Let $Y$ be a projective and normal variety over $k$ of pure dimension $d$. Let $\pi : Y \to X$ be a dominant and generically finite rational map. Let $H$ and $L$ be big and nef divisors, on $X$ and $Y$ respectively. Then there is a constant $C > 1$ such that for every $j = 0, \ldots, d$, and every pair of dominant rational self-maps $f : X \to X$ and $g : Y \to Y$ satisfying $f \circ \pi = \pi \circ g$, we have

\[ C^{-1} \deg_{j,L}(g) \leq \deg_{j,H}(f) \leq C \deg_{j,L}(g). \]

Proof. Denote by $x_1, \ldots, x_s$ the generic points of $X$ and $y_1, \ldots, y_t$ the generic points of $Y$. Since $\pi$ is dominant and generically finite, there is a surjective map $\sigma : \{1, \ldots, r\} \to \{1, \ldots, s\}$ such that $\pi(y_{i}) = x_{\sigma(i)}$, $i = 1, \ldots, r$. For every $i = 1, \ldots, r$, set $t_i = \deg k(x_i) : \pi^*k(y_{\sigma(i)})$ and then

(6.3) \[ m = \min_{i=1}^{s} \left( \sum_{l \in \sigma^{-1}(i)} t_l \right), \quad m' = \max_{i=1}^{s} \left( \sum_{l \in \sigma^{-1}(i)} t_l \right). \]
Take a resolution of $\pi$, defined by a projective and normal variety $Z$, a birational morphism $\pi_1 : Z \rightarrow Y$ and a morphism $\pi_2 : Z \rightarrow X$ satisfying $\pi = \pi_2 \circ \pi_1^{-1}$. Set $h := \pi_1^{-1} \circ g \circ \pi_1 : Z \dashrightarrow Z$. For each index $0 \leq j \leq d$, the projection formula gives

$$\deg_{j\cdot L} g = \deg_{j\cdot \pi_1^* L} h$$

(6.4)

$$m \deg_{j\cdot H} f \leq \deg_{j\cdot \pi_2^* H} h \leq m' \deg_{j\cdot H} f.$$  

(6.5)

Since $\pi_1^* L$ and $\pi_2^* H$ are big and nef on $Z$, there is a constant $C_1 > 1$ that depends only on $\pi_1^* L$ and $\pi_2^* H$ such that

$$C_1^{-1} \deg_{j\cdot \pi_2^* H} h \leq \deg_{j\cdot \pi_1^* L} h \leq C_1 \deg_{j\cdot \pi_2^* H} h.$$  

(6.6)

We conclude the proof by combining the last three equations. \hfill \square

6.2. Equivalent functions. Let $S$ be a set. We shall say that two functions $F, G : S \rightarrow \mathbb{R}_{\geq 0}$, are equivalent if there is a constant $C > 1$ such that

$$C^{-1} \max\{G, 1\} \leq \max\{F, 1\} \leq C \max\{G, 1\},$$

(6.7)

where $\max\{G, 1\}$ denotes the maximum between $G$ and 1. We denote by $[F]$ the equivalence class of $F$; the equivalence class $[1]$ coincides with the set of bounded functions $I \rightarrow \mathbb{R}_{\geq 0}$.

6.3. Degree functions on varieties. Now, let $X$ be a variety of pure dimension $d$ over $k$. Let $\pi : Z \dashrightarrow X$ be a birational map such that $Z$ is projective and normal, and let $H$ be a big and nef divisor on $Z$. Then, define the degrees $\deg_{j\cdot H} f$ of any rational dominant map $f : X \dashrightarrow X$ by $\deg_{j\cdot H} f = \deg_{j\cdot H} \pi^{-1} \circ f \circ \pi$. The previous paragraph shows that if we change the model $(Z, \pi)$ or the divisor $H$ (to $H'$), then we get two notions of degrees $\deg_{j\cdot H}$ and $\deg_{j\cdot H'}$ which are equivalent functions, in the sense of § 6.2, on the set of rational dominant self-maps of $X$. This justifies the following definition.

Let $S$ be a family of dominant rational maps $f_s : X \dashrightarrow X$, $s \in S$. A notion of degree on $S$ in codimension $j$ is a function $\deg_j : S \rightarrow \mathbb{R}_{\geq 0}$ in the equivalence class $[\deg_{j\cdot H}]$ for some normal projective model $Z \rightarrow X$ and some big and nef divisor $H$ on $Z$. The equivalence class $[\deg_j]$ is unique.

Remark 6.2. Assume further that $X$ is affine. In Section 2.1, we defined a notion of degree $f \mapsto \deg f$ (in codimension 1) on the set of automorphisms of $X$; this notion depends on an embedding $X \hookrightarrow \mathbb{A}_k^N$, $N \geq 0$. However, its equivalence class on $\text{Aut}(X)$ does not depend on the choice of such an embedding and is equal to the class $[\deg_1]$ defined in this section.

From Lemma 6.1 and the definitions, we obtain:

Proposition 6.3. Let $\pi : Y \dashrightarrow X$ be a dominant and generically finite rational map between two varieties $X$ and $Y$ over $k$, each of pure dimension $d$. Let $S$ be a family
of dominant rational maps \( g_s : Y \to Y \) such that for every \( s \) in \( S \) there is a rational map \( f_s : X \to X \) that satisfies \( \pi \circ g_s = f_s \circ \pi \). Then, for each \( j = 0, \ldots, d \), the equivalence classes of the degree functions \( s \in S \mapsto \deg_j(g_s) \) and \( s \in S \mapsto \deg_j(f_s) \) are equal.

REFERENCES


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