THE GEOMETRIC BOGOMOLOV CONJECTURE

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ABSTRACT. We prove the geometric Bogomolov conjecture over a function field of characteristic zero.

1. INTRODUCTION

1.1. The geometric Bogomolov conjecture.

1.1.1. *Abelian varieties and heights.* Let $\mathbf{k}$ be an algebraically closed field. Let B be an irreducible normal projective variety over $\mathbf{k}$ of dimension $d_B \geq 1$. Let $K := \mathbf{k}(B)$ be the function field of B. Let A be an abelian variety defined over K of dimension g. Fix an ample line bundle M on B, and a symmetric ample line bundle L on A.

Denote by $\hat{h} : A(\overline{K}) \to [0, +\infty)$ the canonical height on A with respect to L and M where $\overline{K}$ is an algebraic closure of K (see Section 3.1). For any irreducible subvariety X of $A_{\overline{K}}$ and any $\varepsilon > 0$, we set

$$X_\varepsilon := \{x \in X(\overline{K}) | \hat{h}(x) < \varepsilon\}. \tag{1.1}$$

Set $A_{\overline{K}} = A \otimes_K \overline{K}$, and denote by $(A^{\overline{K}/\mathbf{k}}, \mathsf{tr})$ the $\overline{K}/\mathbf{k}$-trace of $A_{\overline{K}}$: it is the final object of the category of pairs (C, f), where C is an abelian variety over $\mathbf{k}$ and f is a morphism from $C \otimes_{\mathbf{k}} \overline{K}$ to $A_{\overline{K}}$ (see [15, §7] or [4, §6]). If $\operatorname{char} \mathbf{k} = 0$, tr is a closed immersion and $A^{\overline{K}/\mathbf{k}} \otimes_{\mathbf{k}} \overline{K}$ can be naturally viewed as an abelian subvariety of $A_{\overline{K}}$. By definition, a **torsion coset** of A is a translate $a + C$ of an abelian subvariety $C \subset A$ by a torsion point a. An irreducible subvariety X of $A_{\overline{K}}$ is said to be **special** if

$$X = \mathsf{tr}(Y \otimes_{\mathbf{k}} \overline{K}) + T \tag{1.2}$$

for some torsion coset T of $A_{\overline{K}}$ and some subvariety Y of $A^{\overline{K}/\mathbf{k}}$. When X is special, X_ε is Zariski dense in X for all $\varepsilon > 0$ ([16, Theorem 5.4, Chapter 6]).

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1.1.2. *Bogomolov conjecture.* The following conjecture was proposed by Yamaki [24, Conjecture 0.3], but particular instances of it were studied earlier by Gubler in [12]. It is an analog over function fields of the Bogomolov conjecture which was proved by Ullmo [21] and Zhang [29].

Geometric Bogomolov Conjecture.– *Let X be an irreducible subvariety of $A_{\overline{K}}$. If X is not special there exists $\varepsilon > 0$ such that X_ε is not Zariski dense in X.*

The aim of this paper is to prove the geometric Bogomolov conjecture over a function field of characteristic zero.

Theorem A. *Assume that* $\mathbf{k}$ *is an algebraically closed field of characteristic* 0*. Let X be an irreducible subvariety of $A_{\overline{K}}$. If X is not special then there exists* $\varepsilon > 0$ *such that X_ε is not Zariski dense in X.*

1.1.3. *Historical note.* Gubler proved the geometric Bogomolov conjecture in [12] when A is totally degenerate at some place of K. Then, Yamaki reduced the conjecture to the case of abelian varieties with good reduction everywhere and trivial trace (see [26]). He also settled the conjecture when $\dim(X)$ or $\operatorname{codim}(X)$ is equal to 1 (see [27], and [22, 23] for previous works on curves). These important contributions of Gubler and Yamaki work in arbitrary characteristic.

In characteristic 0, Cinkir had proved the geometric Bogomolov conjecture when X is a curve of arbitrary genus (see [3], and [7] when the genus is small). Recently, the second and the third-named authors [8] proved the conjecture in the case $\operatorname{char}\mathbf{k} = 0$ and $\dim B = 1$. This last reference, as well as the present article, make use of the Betti map and its monodromy: the idea comes from [14], in which the third-named author gave a new proof of the conjecture in characteristic 0 when A is the power of an elliptic curve and $\dim B = 1$.

1.2. An overview of the proof of Theorem A.

1.2.1. *Notation.* We keep the notation of Section 1.1.1, with $\mathbf{k}$ an algebraically closed field of characteristic 0.

We now construct a model of A that is sufficient for our purpose. Since the symmetric line bundle L is ample we can replace it by some positive power to assume it be very ample, and then we use L to embed A into $\mathbb{P}^N_{\mathbf{k}(B)}$ for some $N > 0$. The Zariski closure $\mathcal{A}$ of A inside $\mathbb{P}^N \times B$ is an irreducible projective variety. We write $\pi : \mathcal{A} \to B$ for the projection. The pullback $\mathcal{L}'$ of $O(1)$ on $\mathbb{P}^N \times B$ to $\mathcal{A}$ is very ample relative to B. But $\mathcal{L}'$ may fail to be ample on $\mathcal{A}$. To remedy this we use instead $\mathcal{L} = \mathcal{L}' \otimes \pi^* M^{\otimes k}$ which is ample for all $k \geq 1$ large enough by Proposition 13.65 [9]. The restriction of $\mathcal{L}$ to A still equals

L. Finally, replacing $\mathcal{A}$ by its normalization, we assume that $\mathcal{A}$ is normal ($\mathcal{L}$ remains ample on the normalization).

We may also assume that M is very ample, and we fix an embedding of B in a projective space such that the restriction of $O(1)$ to B coincides with M. For $b \in B$, we set $\mathcal{A}_b = \pi^{-1}(b)$. We denote by $e : B \dashrightarrow \mathcal{A}$ the zero section and by $[n]$ the multiplication by n on A; it defines a rational mapping $\mathcal{A} \dashrightarrow \mathcal{A}$. Fix a Zariski dense open subset B^o of B such that B^o is smooth and $\pi|_{\pi^{-1}(B^o)}$ is smooth; then, set $\mathcal{A}^o := \pi^{-1}(B^o)$.

Let X be a geometrically irreducible subvariety of A such that X_ε is Zariski dense in X for every $\varepsilon > 0$. We denote by $\mathcal{X}$ its Zariski closure in $\mathcal{A}$, by $\mathcal{X}^o$ its Zariski closure in $\mathcal{A}^o$, and by $\mathcal{X}^{o,reg}$ the regular locus of $\mathcal{X}^o$. Our goal is to show that X is special.

1.2.2. *Complex numbers.* We will see below in Remark 3.2 that it suffices to prove Theorem A in the case $\mathbf{k} = \mathbf{C}$. For the rest of the paper, except if explicitly stated otherwise (in § 3.1 and 3.2), we will assume that B and M are defined over $\mathbf{C}$ and A, X, and L are defined over $\mathbf{C}(B)$. Since M is the restriction of $O(1)$ (in some fixed embedding of B in a projective space), its Chern class is represented by the restriction of the Fubini-Study form to B; we denote by ν this Kähler form.

1.2.3. *The main ingredients.* One of the main ideas of this paper is to consider the Betti foliation (see Section 2.1). It is a $\mathcal{C}^\infty$-smooth foliation of $\mathcal{A}^o$ by holomorphic leaves, which is transverse to π. Every torsion point of A gives local sections of $\pi|_{\pi^{-1}(B^o)}$: these sections are local leaves of the Betti foliation, and this property characterizes it.

To prove Theorem A, the **first step** is to show that $\mathcal{X}^o$ is invariant under the foliation when small points are dense in X. In other words, at every smooth point $x \in \mathcal{X}^o$, the tangent space to the Betti foliation is contained in $T_x\mathcal{X}^o$. For this, we introduce a semi-positive closed $(1,1)$-form ω on $\mathcal{A}^o$ which is canonically associated to L and vanishes along the foliation. An inequality of Gubler implies that the canonical height $\hat{h}(X)$ of X is 0 when small points are dense in X; Theorem B asserts that the condition $\hat{h}(X) = 0$ translates into

$$\int_{\mathcal{X}^o} \omega^{\dim X+1} \wedge (\pi^*\nu)^{m-1} = 0 \tag{1.3}$$

where ν is any Kähler form on the base B^o. From the construction of ω, we deduce that $\mathcal{X}$ is invariant under the Betti foliation.

The first step implies that the fibers of $\pi|_{\mathcal{X}^o}$ are invariant under the action of the holonomy of the Betti foliation; the **second step** shows that a subvariety of a fiber $\mathcal{A}_b$ which is invariant under the holonomy is the sum of a torsion coset and a subset of $A^{\overline{K}/\mathbf{k}}$. The conclusion easily follows from these two main steps.

The second step already appeared in [14] and [8], but the final argument was based on Pila-Zannier's counting strategy. Here, we import ideas from dynamical systems, and in particular a result of Muchnik [20].

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2. THE BETTI FOLIATION AND THE BETTI FORM

In this section, $\mathbf{k} = \mathbf{C}$. We define a foliation and a closed $(1,1)$-form on $\mathcal{A}^o$. This form, which is naturally associated to the line bundle L, was introduced by Mok in [18, pp. 374] to study the Mordell-Weil group over function fields; a recent paper of André, Corvaja and Zannier also studied this foliation to prove the density of torsion points on sections of certain abelian schemes with maximal variation (see [1, Theorem 2.3.2]).

2.1. **The local Betti maps.** Let b be a point of B^o, and $U \subseteq B^o(\mathbf{C})$ be a connected and simply connected open neighbourhood of b in the euclidean topology. Fix a basis of $H_1(\mathcal{A}_b;\mathbf{Z})$ and extend it by continuity to all fibers above U.

Consider the Lie algebra of $\mathcal{A}_c$, for $c \in U$: it may be identified to the tangent space $T_{e(c)}\mathcal{A}_c$, where e denotes the zero section. The family of these vector spaces determines a complex vector bundle of dimension g over U. If U is small enough, we can trivialize this bundle, and we obtain g holomorphic vector fields $(\theta_j)_{1\leq j\leq g}$ on $\pi^{-1}(U)$ which are tangent to the fibers of π and trivialize their tangent bundle. Integrating these vector fields gives a holomorphic action of the additive group $\mathbf{C}^g$ on $\pi^{-1}(U)$ whose orbits are the fibers of π. Then, the stabilizer of $e(c)$, for c in U, is a lattice Λ_c in $\mathbf{C}^g$ and $\mathcal{A}_c = \mathbf{C}^g/\Lambda_c$. The continuous choice of a basis for $H_1(\mathcal{A}_c;\mathbf{Z})$, $c \in U$, gives a choice of basis of the $\mathbf{Z}$-module $\Lambda_c \subset \mathbf{C}^g$ that depends holomorphically on c. Now, using this basis to identify Λ_c with $\mathbf{Z}^{2g}$ and $\mathbf{C}^g$ with $\mathbf{R}^{2g}$, we see that there is a real analytic diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times \mathbf{R}^{2g}/\mathbf{Z}^{2g}$ such that

(1) $\pi_1 \circ \phi_U = \pi$, where $\pi_1 : U \times \mathbf{R}^{2g}/\mathbf{Z}^{2g} \to U$ is the first projection;
(2) for every $c \in U$, the map $\phi_U|_{\mathcal{A}_c} : \mathcal{A}_c \to \pi_1^{-1}(c)$ is an isomorphism of real Lie groups that maps the basis of $H_1(\mathcal{A}_c;\mathbf{Z})$ to the canonical basis of $\mathbf{Z}^{2g}$.

For b in U, denote by $i_b : \mathbf{R}^{2g}/\mathbf{Z}^{2g} \to U \times \mathbf{R}^{2g}/\mathbf{Z}^{2g}$ the inclusion $y \mapsto (b,y)$. The **Betti map** is the C^∞-projection $\beta_U^b : \pi^{-1}(U) \to \mathcal{A}_b$ defined by

$$\beta_U^b := (\phi_U|_{\mathcal{A}_b})^{-1} \circ i_b \circ \pi_2 \circ \phi_U \tag{2.1}$$

where $\pi_2 : U \times \mathbf{R}^{2g}/\mathbf{Z}^{2g} \to \mathbf{R}^{2g}/\mathbf{Z}^{2g}$ is the projection to the second factor. Changing the basis of $H_1(\mathcal{A}_b;\mathbf{Z})$, we obtain another trivialization ϕ'_U that is given by post-composing ϕ_U with a constant linear transformation

$$(b,z) \in U \times \mathbf{R}^{2g}/\mathbf{Z}^{2g} \mapsto (b, h(z)) \tag{2.2}$$

for some element h of the group $\mathsf{GL}_{2g}(\mathbf{Z})$; thus, β_U^b does not depend on ϕ_U.

Note that β_U^b is the identity on $\mathcal{A}_b$. In general, β_U^b is not holomorphic. However, for every $p \in \mathcal{A}_b$, $(\beta_U^b)^{-1}(p)$ is a complex submanifold of $\mathcal{A}^o$. To see this, pick a torsion point of A, of order r. Its Zariski closure in $\mathcal{A}$ gives a multisection of π, and above U the connected components of this multisection are fibers of β_U^b: indeed, on such a component the values of β_U^b are contained in the finite set $(\frac{1}{r}\mathbf{Z}^{2g})/\mathbf{Z}^{2g}$. Thus, a dense set of fibers are complex submanifolds. By continuity of the complex structure $\mathrm{J} \in \mathsf{End}(T\mathcal{A})$ and of the tangent spaces $x \in \pi^{-1}(U) \mapsto T_x((\beta_U^b)^{-1}(\beta_U^b(x)))$, all fibers are complex submanifolds.

2.2. **The Betti foliation.** The local Betti maps determine a natural foliation $\mathcal{F}$ on $\mathcal{A}^o$: for every point $p \in \pi^{-1}(U)$, the local leaf $\mathcal{F}_{U,p}$ through p is the fiber $(\beta_U^{\pi(p)})^{-1}(p)$. We call $\mathcal{F}$ the **Betti foliation**. The leaves of $\mathcal{F}$ are holomorphic, in the following sense: for every $p \in \mathcal{A}^o$, the local leaf $\mathcal{F}_{U,p}$ is a complex submanifold of $\pi^{-1}(U) \subset \mathcal{A}^o$. But a global leaf $\mathcal{F}_p$ can be dense in $\mathcal{A}^o$ for the euclidean topology. Moreover, $\mathcal{F}$ is everywhere transverse to the fibers of π, and $\pi|_{\mathcal{F}_p} \colon \mathcal{F}_p \to B^o$ is a regular holomorphic covering for every point p (it may have finite or infinite degree, and this may depend on p).

Remark 2.1. Assume that the family $\pi : \mathcal{A}^o \to B^o$ is trivial, *i.e.* $\mathcal{A}^o = B^o \times A_{\mathbf{C}}$ where $A_{\mathbf{C}}$ is an abelian variety over $\mathbf{C}$ and π is the first projection. Then, the leaves of $\mathcal{F}$ are exactly the fibers of the second projection.

Remark 2.2. The foliation $\mathcal{F}$ is characterized as follows. Let q be a torsion point of $\mathcal{A}_b$; it determines a multisection of the fibration π, obtained by analytic continuation of q as a torsion point in nearby fibers of π. This multisection coincides with the leaf $\mathcal{F}_q$. There is a unique foliation of $\mathcal{A}^o$ which is everywhere transverse to π and whose set of leaves contains all those multisections.

Remark 2.3. One can also think about $\mathcal{F}$ dynamically. The endomorphism $[n]$ determines a rational transformation of the model $\mathcal{A}$ and induces a regular transformation of $\mathcal{A}^o$. It preserves $\mathcal{F}$, mapping leaves to leaves. Preperiodic leaves correspond to preperiodic points of $[n]$ in the fiber $\mathcal{A}_b$; they are exactly the leaves given by the torsion points of A.

2.3. **Holonomy versus monodromy.** Let γ be a loop in B^o, based at some point b. Following the trivialization of $H_1(\mathcal{A}_b;\mathbf{Z})$ along the loop $\gamma(t)$, $t \in [0,1]$, we obtain a second basis of $H_1(\mathcal{A}_b;\mathbf{Z})$ when $t = 1$. The change of basis is an element $\mathbf{Mon}(\gamma)$ of the group $\mathsf{GL}(H_1(\mathcal{A}_b;\mathbf{Z})) \simeq \mathsf{GL}_{2g}(\mathbf{Z})$, called the monodromy along γ. Note that $\mathbf{Mon}(\gamma)$ gives a linear transformation of $H_1(\mathcal{A}_b;\mathbf{R}) \simeq \mathbf{R}^{2g}$

that preserves the lattice $H_1(\mathcal{A}_b;\mathbf{Z}) \simeq \mathbf{Z}^{2g}$, hence also a (linear) diffeomorphism of the torus $\mathbf{R}^{2g}/\mathbf{Z}^{2g}$ (*i.e.* of $\mathcal{A}_b$). By definition, the image of **Mon** in $\mathsf{GL}_{2g}(\mathbf{Z})$ (resp. in $\mathsf{GL}(H_1(\mathcal{A}_b;\mathbf{Z}))$) is the **monodromy group** of $\mathcal{A}^o \to B^o$.

Now, let x be a point of $\mathcal{A}_b$. Since $\pi\colon \mathcal{F}_x \to B^o$ is an unramified cover, γ lifts to a unique path $\hat{\gamma}_x\colon [0,1] \to \mathcal{A}$ such that $\pi \circ \hat{\gamma}_x = \gamma$ and $\hat{\gamma}_x(t) \in \mathcal{F}_x$ for all t. By definition, the point $\hat{\gamma}_x(1)$ is the image of x by the holonomy $\mathbf{Hol}(\gamma)$: this construction defines a representation of the fundamental group $\pi_1(B,b)$ in the diffeomorphism group $\mathsf{Diff}^\infty(\mathcal{A}_b)$. By construction of the Betti map, we have

$$\mathbf{Hol}(\gamma) = \mathbf{Mon}(\gamma) \tag{2.3}$$

as $\mathcal{C}^\infty$-diffeomorphisms of $\mathcal{A}_b \simeq \mathbf{R}^{2g}/\mathbf{Z}^{2g}$.

2.4. **The Betti form.** For $b \in B^o$, there exists a unique smooth $(1,1)$-form $\omega_b \in c_1(\mathcal{L}|_{\mathcal{A}_b})$ on $\mathcal{A}_b$ which is invariant under translations. If we write $\mathcal{A}_b = \mathbf{C}^g/\Lambda$ and denote by $z_1,\dots,z_g$ the standard coordinates of $\mathbf{C}^g$, then

$$\omega_b = \sum_{1\le i,j\le g} a_{i,j}\, dz_i \wedge d\bar{z}_j \tag{2.4}$$

for some complex numbers $a_{i,j}$. This form ω_b is positive since $\mathcal{L}|_{\mathcal{A}_b}$ is ample. This form is classically known as the *harmonic*, or *Riemann* form associated to $c_1(\mathcal{L}|_{\mathcal{A}_b})$.

Now, we define a smooth 2-form ω on $\mathcal{A}^o$. Let p be a point of $\mathcal{A}^o$. First, define $P_p\colon T_p\mathcal{A}^o \to T_p\mathcal{A}_{\pi(p)}$ to be the projection onto the first factor in

$$T_p\mathcal{A}^o = T_p\mathcal{A}_{\pi(p)} \oplus T_p\mathcal{F}. \tag{2.5}$$

Since the tangent spaces $T_p\mathcal{F}$ and $T_p\mathcal{A}_{\pi(p)}$ are complex subspaces of $T_p\mathcal{A}^o$, the map P_p is a complex linear map. Then, for v_1 and $v_2 \in T_p\mathcal{A}^o$ we set

$$\omega(v_1,v_2) := \omega_{\pi(p)}(P_p(v_1),P_p(v_2)). \tag{2.6}$$

We call ω the **Betti form**. By construction, $\omega|_{\mathcal{A}_b} = \omega_b$ for every b. Since ω_b is of type $(1,1)$ and P_p is $\mathbf{C}$-linear, ω is an antisymmetric form of type $(1,1)$. Since ω_b is positive, ω is semi-positive.

Let U and ϕ_U be as in Section 2.1. Let y_i, $i = 1,\dots,2g$, denote the standard coordinates of $\mathbf{R}^{2g}$. Then there are real numbers $b_{i,j}$ such that

$$(\phi_U^{-1})^*\omega = \sum_{1\le i<j\le 2g} b_{i,j}\, dy_i \wedge dy_j. \tag{2.7}$$

The $b_{i,j}$ are constant: they do not depend on the point $b \in U$. Indeed, the $b_{i,j}$ are the coordinates of the cohomology class $c_1(\mathcal{L}|_{\mathcal{A}_b})$ in a fixed basis of $H^2(\mathcal{A}_b;\mathbf{Z})$. It follows that $d((\phi_U^{-1})^*\omega) = 0$ and that ω is closed. Moreover, $[n]^*\omega = n^2\omega$. Thus, we get the following lemma.

Lemma 2.4. *The Betti form* ω *is a real analytic, closed, and semi-positive* $(1,1)$*-form on* $\mathcal{A}^o$ *such that* $\omega|_{\mathcal{A}_b} = \omega_b$ *for every point* $b \in B^o$*. In particular, the cohomology class of* $\omega|_{\mathcal{A}_b}$ *coincides with* $c_1(\mathcal{L}|_{\mathcal{A}_b})$ *for every* $b \in B^o$*.*

3. THE CANONICAL HEIGHT AND THE BETTI FORM

In Sections 3.1 and 3.2, $\mathbf{k}$ is any algebraically closed field of characteristic zero, and we use an inequality of Gubler and Zhang to reduce the proof to the case $\mathbf{k} = \mathbf{C}$. Then, Section 3.3 shows how to translate the density of small points in X into an invariance with respect to the Betti foliation.

3.1. **The canonical height.** Recall that $K = \mathbf{k}(B)$. Let X be any subvariety of $A_{\overline{K}}$, and let K' be a finite field extension of K over which X is defined: there exists a subvariety X' of $A_{K'}$ such that $X = X' \otimes_{K'} \overline{K}$. Let $\rho' : B' \to B$ be the normalization of B in K'. Let $\mathcal{A}$ be the model of A constructed at the beginning of Section 1.2.1; $\mathcal{A}$ is normal and $\mathcal{L}$ is an ample line bundle on $\mathcal{A}$. Set $\mathcal{A}' := \mathcal{A} \times_B B'$ and denote by $\rho : \mathcal{A}' \to \mathcal{A}$ the projection to the first factor; then, denote by $\mathcal{X}'$ the Zariski closure of X' in $\mathcal{A}'$. The **naive height** of X associated to the model $\pi \colon \mathcal{A} \to B$ and the line bundles $\mathcal{L}$ and M is defined by the intersection number

$$h(X) = \frac{1}{[K' : K]} \left(\mathcal{X}' \cdot c_1(\rho^* \mathcal{L})^{d_X+1} \cdot \rho^* \pi^* (c_1(M))^{d_B-1} \right) \tag{3.1}$$

where $d_X = \dim X$ and $d_B = \dim B$. It depends on the model $\mathcal{A}$ and the extension $\mathcal{L}$ of L to $\mathcal{A}$ but it does not depend on the choice of K'.

The **canonical height** is the limit

$$\hat{h}(X) = \lim_{n \to +\infty} \frac{h([n]_* X)}{n^{2(d_X+1)}} = \lim_{n \to +\infty} \frac{\deg([n]|_X) h([n]X)}{n^{2(d_X+1)}}. \tag{3.2}$$

It depends on L but not on the model $(\mathcal{A}, \mathcal{L})$; see Gubler's work [12, Theorem 3.6] and [11, Theorem 11.18].

To simplify the notation, we suppose now that $K' = K$, so ρ is the identity and $B' = B$, $\mathcal{A}' = \mathcal{A}$, $\mathcal{X}' = \mathcal{X}$. Suppose that $\mathbf{k}'$ is an algebraically closed subfield of $\mathbf{k}$ such that B and M are the base change to $\mathbf{k}$ of a variety $B_{\mathbf{k}'}$ and a line bundle $M_{\mathbf{k}'}$ defined over $\mathbf{k}'$. Suppose furthermore, that A, X, and L are the base change of an abelian variety, a subvariety, and a line bundle which are defined over $\mathbf{k}'(B_{\mathbf{k}'})$. We get models $\mathcal{A}_{\mathbf{k}'}$ and $\mathcal{X}_{\mathbf{k}'}$ now defined over $\mathbf{k}'$. Intersection numbers as in Equation (3.1) are invariant under extending the field of constants. And so the limit in Equation (3.2) is unchanged, that is, $\hat{h}(X) = \hat{h}(X_{\mathbf{k}'})$. In particular,

$$\hat{h}(X) = 0 \quad \text{if and only if} \quad \hat{h}(X_{\mathbf{k}'}) = 0. \tag{3.3}$$

3.2. **Gubler-Zhang inequality.** By definition, the **essential minimum** $\mathrm{ess}(X)$ of a subvariety $X \subset A$ is the real number

$$\mathrm{ess}(X) = \sup_{Y} \inf_{x \in X(\overline{K}) \setminus Y} \hat{h}(x), \tag{3.4}$$

where Y runs through all proper Zariski closed subsets of X. The following inequality is due to Gubler (see [12, Lemma 4.1]); it is an analogue of Zhang's inequality [28, Theorem 1.10] that concerns the number field case:

$$0 \leq \frac{\hat{h}(X)}{(d_X + 1)\deg_L(X)} \leq \mathrm{ess}(X). \tag{3.5}$$

We refer to it as the Gubler-Zhang inequality. The converse inequality $\mathrm{ess}(X) \leq \hat{h}(X)/\deg_L(X)$ also holds, but we shall not use it in this article.

Definition 3.1. We say that X is **small**, if X_ε is Zariski dense in X for all $\varepsilon > 0$.

Clearly, X is small if and only if $\mathrm{ess}(X) = 0$. The Gubler-Zhang inequality shows that $\hat{h}(X) = 0$ if X is small (and this is in fact an equivalence).

Remark 3.2. We now explain why it suffices to prove Theorem A for $\mathbf{k} = \mathbf{C}$. Suppose X is small, so that $\hat{h}(X) = 0$ by the Gubler-Zhang inequality. There exists an algebraically closed subfield $\mathbf{k}' \subset \mathbf{k}$ of finite transcendence degree over $\mathbf{Q}$ such that B (resp. M) comes from a variety (resp. a line bundle on it) defined over $\mathbf{k}'$ via base change, and A, L, and X come from an abelian variety, a line bundle, and a subvariety defined over its function field. Now $\mathbf{k}'$ can be embedded into $\mathbf{C}$. So we get a variety $B_{\mathbf{C}}$ over $\mathbf{C}$, an abelian variety $A_{\mathbf{C}(B)}$ with a subvariety $X_{\mathbf{C}(B)} \subset A_{\mathbf{C}(B)}$, both over $\mathbf{C}(B)$, and their corresponding line bundles. Applying Equation (3.3) twice gives $\hat{h}(X_{\mathbf{C}(B)}) = 0$. Moreover, if $X_{\mathbf{C}(B)}$ is special, then so is the original X. This explains why we may take $\mathbf{k} = \mathbf{C}$.

Proposition 3.3. *Let $g : A \to A'$ be a morphism of abelian varieties over K, and let $a \in A(K)$ be a torsion point. Let X be a geometrically irreducible subvariety of A over K.*

(1) *If X is small, then $g(X)$ is small.*
(2) *If g is an isogeny and $g(X)$ is small, then X is small.*
(3) *X is small if and only if $a + X$ is small.*

Proof. Assertions (1) and (2) follow from [25, Proposition 2.6.]. To prove the third one fix an integer $n \geq 1$ such that $na = 0$. By assertions (1) and (2), $a + X$ is small if and only if $[n](a + X) = [n](X)$ is small, if and only if X is small. □

3.3. **Smallness and the Betti form.** Now we assume $\mathbf{k} = \mathbf{C}$ and we reformulate the canonical height in differential geometric terms.

Recall the setup of Equation (3.1) assuming, for simplicity, that X is already defined over K. Pick a Kähler form α in $c_1(\mathcal{L})$ (such a form exists because we choose $\mathcal{L}$ ample). For every $n \geq 1$, there exists an irreducible smooth projective scheme $\pi_n : \mathcal{A}_n \to B$ over B, extending $\pi|_{\mathcal{A}^o} : \mathcal{A}^o \to B^o$, such that the rational map $[n] : \mathcal{A} \dashrightarrow \mathcal{A}$ lifts to a morphism $f_n : \mathcal{A}_n \to \mathcal{A}$ over B. Write $\mathcal{L}_n := f_n^* \mathcal{L}$ and $\alpha_n := f_n^* \alpha$; in particular $\mathcal{A}_1$ is a smooth model of $\mathcal{A}$ and $\alpha_1 = \alpha$ on $\mathcal{A}^o$. Denote by $\mathcal{X}_n$ the Zariski closure of $\mathcal{X}^o$ in $\mathcal{A}_n$. Since the Kähler form ν introduced in Section 1.2.1 represents the class $c_1(M)$, the projection formula gives

$$
\begin{aligned}
\hat{h}(X) &= \lim_{n\to\infty} n^{-2(d_X+1)} (\mathcal{X}_n \cdot c_1(\mathcal{L}_n)^{d_X+1} \cdot c_1(\pi_n^* M)^{d_B-1}) \\
&= \lim_{n\to\infty} n^{-2(d_X+1)} \int_{\mathcal{X}_n} \alpha_n^{d_X+1} \wedge (\pi_n^* \nu)^{d_B-1} \\
&= \lim_{n\to\infty} n^{-2(d_X+1)} \int_{\mathcal{X}^o} ([n]^* \alpha)^{d_X+1} \wedge (\pi^* \nu)^{d_B-1} \qquad (3.6)
\end{aligned}
$$

because the integral on $\mathcal{X}_n$ is equal to the integral on the dense Zariski open subset $\mathcal{X}^o$ (and even on the regular locus $\mathcal{X}^{o,reg}$).

Here is the key relationship between the canonical height and the Betti form.

Theorem B. *Let X be a geometrically irreducible subvariety of A over $\overline{K}$. If $\hat{h}(X) = 0$, then*

$$
\int_{\mathcal{X}^o} \omega^{d_X+1} \wedge (\pi^* \nu)^{d_B-1} = 0,
$$

with ω the Betti form associated to L and ν the Kähler form on B representing the class $c_1(M)$.

Proof. We may assume that X is defined over K. Since $\hat{h}(X) = 0$, Equation (3.6) shows that

$$
0 = \lim_{n\to\infty} n^{-2(d_X+1)} \int_{\mathcal{X}^o} ([n]^* \alpha)^{d_X+1} \wedge (\pi^* \nu)^{d_B-1}. \qquad (3.7)
$$

Let $U \subset B^o$ be any relatively compact open subset of B^o in the euclidean topology. There exists a constant $C_U > 0$ such that $C_U \alpha - \omega$ is semi-positive on $\pi^{-1}(U)$. Since $[n] : \mathcal{A}^o \to \mathcal{A}^o$ is regular, the $(1,1)$-form $n^{-2}[n]^*(C_U \alpha - \omega) = C_U n^{-2}[n]^* \alpha - \omega$ is semi-positive. Since ω and ν are semi-positive, we get

$$
0 \leq \int_{\pi^{-1}(U)\cap\mathcal{X}^o} \omega^{d_X+1} \wedge (\pi^* \nu)^{d_B-1} \leq \left(\frac{C_U}{n^2}\right)^{d_X+1} \int_{\mathcal{X}^o} ([n]^* \alpha)^{d_X+1} \wedge (\pi^* \nu)^{d_B-1}
$$

for all $n \geq 1$. Letting n go to $+\infty$, Equation (3.7) gives

$$
\int_{\pi^{-1}(U)\cap\mathcal{X}^o} \omega^{d_X+1} \wedge (\pi^* \nu)^{d_B-1} = 0. \qquad (3.8)
$$

Since this holds for all relatively compact subsets U of B^o, the theorem is proved. □

Corollary 3.4. *Assume that X is small. Let U and V be open subsets of B^o and $\mathcal{X}^o$ respectively (in the euclidean topology) such that U contains the closure $\overline{\pi(V)} \subset B$. If μ is any smooth real semi-positive $(1,1)$-form on U, then*

$$\int_V \omega^{d_X+1} \wedge (\pi^*\mu)^{d_B-1} = 0.$$

Proof of the Corollary. We can assume U to be a relatively compact subset of B^o. Since ω and μ are semi-positive, the integral is non-negative. Since ν is strictly positive on U, there is a constant $C > 0$ such that $C\nu - \mu$ is semi-positive. From Theorem B we get

$$0 \leq \int_V \omega^{d_X+1} \wedge (\pi^*\mu)^{d_B-1} \leq C^{d_B-1} \int_V \omega^{d_X+1} \wedge (\pi^*\nu)^{d_B-1} = 0, \tag{3.9}$$

and the conclusion follows. □

Theorem B'. *Assume that X is small. Then at every point $p \in \mathcal{X}^o$, we have $T_p\mathcal{F} \subseteq T_p\mathcal{X}^o$. In other words, $\mathcal{X}^o$ is invariant under the Betti foliation: for every $p \in \mathcal{X}^o$, the leaf $\mathcal{F}_p$ is contained in $\mathcal{X}^o$.*

Proof. We start with a simple remark. Let $P\colon \mathbf{C}^{N+1} \to \mathbf{C}^N$ be a complex linear map of rank N. Let ω_0 be a positive $(1,1)$-form on $\mathbf{C}^N$. If V is a complex linear subspace of $\mathbf{C}^{N+1}$ of dimension N, then $\ker(P) \subset V$ if and only if $P|V$ is not onto, if and only if $(P^*\omega_0^N)|V = 0$. Now, assume that B has dimension 1. Then, the integral of ω^{d_X+1} on $\mathcal{X}^o$ vanishes by Theorem B; since the form ω is semi-positive, the remark implies that the kernel of the projection P_p from Section 2.4 is contained in $T_p\mathcal{X}^o$ at every smooth point p of $\mathcal{X}^o$. This proves the proposition when $d_B = 1$.

The general case reduces to $d_B = 1$ as follows. Let U and U' be open subsets of B^o such that: (i) $\overline{U} \subset U'$ in the euclidean topology and (ii) there are complex coordinates (z_j) on U' such that $U = \{|z_j| < 1, j = 1, \ldots, d_B\}$. Set

$$\mu := i(dz_2 \wedge d\overline{z_2} + \ldots + dz_{d_B} \wedge d\overline{z_{d_B}}). \tag{3.10}$$

Note that μ^{d_B-1} is the volume form $i^{d_B-1} dz_2 \wedge d\overline{z_2} \wedge \ldots \wedge d\overline{z_{d_B}}$. It is a smooth real semi-positive $(1,1)$-form on U'. By Corollary 3.4, we have

$$\int_{\pi^{-1}(U)\cap\mathcal{X}} \omega^{d_X+1} \wedge (\pi^*\mu)^{d_B-1} = 0. \tag{3.11}$$

For $(w_2, \ldots, w_{d_B})$ in $\mathbf{C}^{d_B-1}$ with modulus $|w_j| < 1$ for all j, consider the slice

$$\mathcal{X}(w_2, \ldots, w_{d_B}) = \mathcal{X} \cap \pi^{-1}(U \cap \{z_2 = w_2, \ldots, z_{d_B} = w_{d_B}\}); \tag{3.12}$$

these slices provide a family of subsets of $\mathcal{A}$ over the one-dimensional disk $\{(z_1, w_2, \dots, w_{d_B})\ ;\ |z_1| < 1\}$. Now (3.11) can be reformulated to

$$\int_{|w_2|<1,\dots,|w_{d_B}|<1} \left(\int_{\mathcal{X}(w_2,\dots,x_{d_B})} \omega^{d_X+1} \right) (\pi^*\mu)^{d_B-1} = 0. \tag{3.13}$$

Both ω and $\pi^*\mu$ are semi-positive on $\mathcal{A}^o$, so the integral of ω^{d_X+1} over $\mathcal{X}(w_2, \dots, w_{d_B})$ vanishes for (μ^{d_B-1})-almost all $(w_2, \dots, w_{d_B})$; from the case $d_B = 1$, we deduce that, at every point p of $\mathcal{X}^o \cap \pi^{-1}U$, the intersection $T_p\mathcal{X}^o \cap T_p\mathcal{F}$ contains on a line whose projection in $T_{\pi(p)}B$ is the line $\{z_2 = \cdots = z_{d_B} = 0\}$. Doing the same for all coordinates z_i, we see that $T_p\mathcal{F}$ is contained in $T_p\mathcal{X}^o$. □

As a direct application of Theorem B' and Remark 2.1, we prove Theorem A in the isotrivival case.

Corollary 3.5. *If $A_{\overline{K}} = A^{\overline{K}/\mathbf{C}} \otimes_{\mathbf{C}} \overline{K}$ and X is small, then there exists a subvariety $Y \subseteq A^{\overline{K}/\mathbf{C}}$ such that $X \otimes_K \overline{K} = Y \otimes_{\mathbf{C}} \overline{K}$.*

Proof. Replacing K by a suitable finite extension K' and then B by its normalization in K', we may assume that $\mathcal{A}^o = B^o \times A^{\overline{K}/\mathbf{C}}$ and that $\pi\colon \mathcal{A}^o \to B$ is the projection to the first factor. By Remark 2.1, the leaves of the Betti foliation are exactly the fibers of the projection π_2 onto the second factor. Since X is small, Theorem B' shows that $\mathcal{X} = \pi_2^{-1}(Y)$, with $Y := \pi_2(\mathcal{X})$. □

4. Invariant analytic subsets of real and complex tori

Let m be a positive integer. Let $M = \mathbf{R}^m/\mathbf{Z}^m$ be the torus of dimension m and $\pi\colon \mathbf{R}^m \to M$ be the natural projection. The group $\mathsf{GL}_m(\mathbf{Z})$ acts by real analytic homomorphisms on M. In this section, we study analytic subsets of M which are invariant under the action of a subgroup $\Gamma \subset \mathsf{GL}_m(\mathbf{Z})$. The main ingredient is a result of Muchnik and of Guivarc'h and Starkov.

4.1. Zariski closure of Γ.

We denote by

$$G = \mathrm{Zar}(\Gamma)^{irr} \tag{4.1}$$

the neutral component, for the Zariski topology, of the Zariski closure of Γ in $\mathsf{GL}_m(\mathbf{R})$.

Lemma 4.1. *The group $\Gamma \cap G$ has finite index in Γ. If Γ_0 is a finite index subgroup of Γ, then $\mathrm{Zar}(\Gamma_0)^{irr} = G$.*

Proof. The index of G in $\mathrm{Zar}(\Gamma)$ is equal to the number ℓ of irreducible components of the algebraic variety $\mathrm{Zar}(\Gamma)$, and the index of $\Gamma \cap G$ is also ℓ. Now, let Γ_0 be a finite index subgroup of Γ. Then, $\Gamma_0 \cap G$ has finite index in $\Gamma \cap G$, and we can fix a finite subset $\{\alpha_1, \dots, \alpha_k\} \subset \Gamma \cap G$ such that $\Gamma \cap G = \cup_j \alpha_j(\Gamma_0 \cap G)$.

So $\mathrm{Zar}(\Gamma\cap G)\subset\cup_j\alpha_j\mathrm{Zar}(\Gamma_0\cap G)\subset G$. Because $\Gamma\cap G$ is Zariski dense in the irreducible group G we find $G=\mathrm{Zar}(\Gamma_0\cap G)$. So $G\subset\mathrm{Zar}(\Gamma_0)$ and the Lemma follows as $G=\mathrm{Zar}(\Gamma)^{irr}$. □

We shall denote by V the vector space $\mathbf{R}^m$; the lattice $\mathbf{Z}^m$ determines an integral, hence a rational structure on V. The Zariski closure $\mathrm{Zar}(\Gamma)$ is a $\mathbf{Q}$-algebraic subgroup of GL_m for this rational structure; the same is true for every subgroup of Γ. In particular, G is defined over $\mathbf{Q}$.

We shall say that Γ (or G) has **no trivial factor** if every G-invariant vector $u\in V$ is equal to 0. This notion depends only on G, not on Γ: by Lemma 4.1, this property is inherited by finite index subgroups of Γ.

4.2. **Results of Muchnik and Guivarc'h and Starkov.** From now on, we assume that G is semi-simple, in particular $\dim(G)$ is positive, and $\dim V>0$.

Assume that V is an irreducible representation of G over $\mathbf{Q}$; this means that every proper $\mathbf{Q}$-subspace of V which is G-invariant is the trivial subspace $\{0\}$. Since G is semi-simple, we can decompose V into irreducible subrepresentations W_i of G over $\mathbf{R}$ (see [17], Proposition 22.41):

$$V=W_1\oplus W_2\oplus\cdots\oplus W_s. \tag{4.2}$$

To each W_i corresponds a subgroup G_i of $\mathsf{GL}(W_i)$ given by the restriction of the action of G to W_i. Some of the groups $G_i(\mathbf{R})$ may be compact, and we denote by V_c the sum of the corresponding subspaces: V_c is the maximal G-invariant subspace of V on which $G(\mathbf{R})$ acts by a compact factor.

Lemma 4.2. *Let $W\subset V$ be a Γ-invariant subspace. Then, $W\subset V_c$ if and only if the orbit $\Gamma(w)$ of every vector $w\in W$ is a bounded subset of V.*

Proof. If $W\subset V_c$ then every orbit is bounded because $\Gamma_{|W}$ is contained in a compact subgroup of $\mathsf{GL}(W)$.

For the reverse implication, we shall use the following fact (see [5]): *Let N be a real or complex vector space. Let H be a subgroup of $\mathsf{GL}(N)$ such that all eigenvalues of all elements of H have modulus ≤ 1. Then there is a H-invariant flag $\{0\}=N^{(0)}\subset N^{(1)}\subset\cdots\subset N^{(k)}=N$ such that the action of H on $N^{(i+1)}/N^{(i)}$ is isometric for every i. If the action of H on N is irreducible, then H is contained in a compact subgroup of $\mathsf{GL}(N)$.* (see footnote ([1]) next page)

Now, assume that W is not contained in V_c. Then W contains an irreducible subrepresentation $W_0\subset W$ such that $G_0(\mathbf{R})$ (the image of G in $\mathsf{GL}(W_0)$) is not compact. The group $\Gamma_{|W_0}$ is unbounded, because otherwise its closure would be a compact group, hence it would preserve some positive definite quadratic form, $G_0(\mathbf{R})$ would also preserve this quadratic form because Γ is Zariski dense in G, and then $G_0(\mathbf{R})$ would be compact. Thus, the fact we just recalled gives an element of Γ with a (complex) eigenvalue of modulus > 1 on $W_0 \otimes \mathbf{C}$; thus, there is a vector $w \in W_0$ whose orbit is unbounded. $\square$

Lemma 4.3. *The subspace V_c is a proper subspace of V. The projection $\pi_{|V_c}\colon V_c \to M$ is injective; in other words, $V_c \cap \mathbf{Z}^m = \{0\}$. If a and a' are two distinct torsion points of M, then $a + \pi(V_c)$ does not intersect $a' + \pi(V_c)$.*

Proof. If V_c were equal to V then $G(\mathbf{R})$ would be compact, Γ would be finite, and G would be trivial (contradicting $\dim(G) > 0$).

If $\pi_{|V_c}$ is not injective, V_c contains an element $u \neq 0$ of the lattice $\mathbf{Z}^m$. The Γ-orbit of u is contained in $V_c \cap \mathbf{Z}^m$; as a consequence, the vector subspace $W \subset V$ spanned by this orbit is defined over $\mathbf{Q}$ and is G-invariant. Since V_c is a proper subspace of V, W is a proper, G-invariant subspace defined over $\mathbf{Q}$, and this contradicts the irreducibility of the representation over $\mathbf{Q}$. This contradiction proves the second assertion.

The third assertion follows from the second: if $(a + \pi(V_c)) \cap (a' + \pi(V_c))$ were not empty, V_c would contain a non-zero element of $\pi^{-1}(a - a')$; since $\pi^{-1}(a - a') \subset \mathbf{Q}^m$, V_c would contain an element of $\mathbf{Z}^m \setminus \{0\}$. $\square$

Let z be a point of V_c and let $x = \pi(z)$ be its projection. Then the orbit $G(z)$ is compact, and $\Gamma(x)$ is contained in $\pi(G(z))$, a compact subset of M contained in $\pi(V_c)$; in particular, $\Gamma(x)$ is not dense in M. More generally, if a is a torsion point of M and $x \in a + \pi(V_c)$, then $\Gamma(x)$ is not dense in M. This shows that the two properties of the following theorem are exclusive.

Theorem 4.4 (Muchnik [20]; Guivarc'h and Starkov [13])**.** *Assume that G is semi-simple, and its representation on $\mathbf{Q}^m$ is irreducible. Let x be an element of M. Then, one of the following two exclusive properties occur*

(1) *the Γ-orbit of x is dense in M;*
(2) *there exists a torsion point $a \in M$ such that $x \in a + \pi(V_c)$.*

Remark 4.5. In the second assertion, the torsion point a is uniquely determined by x: this follows from the last assertion in Lemma 4.3.

[1] Indeed, assume that we work over $\mathbf{C}$. Pick an irreducible invariant subspace $N^{(1)}$ of dimension $d_1 \geq 1$, and consider the action of H on $N^{(1)}$. By the Burnside's theorem, H generates the vector space $\mathsf{End}(N^{(1)})$. Let $(h_i) \subset H$ be a basis of $\mathsf{End}(N^{(1)})$. The trace map $g \mapsto (\mathrm{trace}(gh_i)) \in \mathbf{C}^{d_1^2}$ is a linear isomorphism, so there is a basis (g_i) of $\mathsf{End}(N^{(1)})$ with $g = \sum_i \mathrm{trace}(gh_i) g_i$ for all $g \in \mathsf{End}(N^{(1)})$. From the hypothesis on the eigenvalues, the trace functions $h \mapsto \mathrm{trace}(hh_i)$ are bounded by d_1 on H, so the image of H in $\mathsf{GL}(N^{(1)})$ is relatively compact. When N is a real vector space, we complexify it and define $N^{(1)}$ to be the real part of the *maximal* H-invariant subspace $N_m^{(1)} \subset N \otimes \mathbf{C}$ such that the image of H in $\mathsf{GL}(N_m^{(1)})$ is relatively compact. To conclude, we repeat the same argument on $N/N^{(1)}$.

Remark 4.6. By Lemma 4.1, the hypothesis and, therefore, the conclusion of Theorem 4.4 remain unchanged if Γ is replaced by a finite index subgroup.

Remark 4.7. Theorem 4.4 will be used to describe Γ-invariant real analytic subsets $Z \subset M$. If it is infinite, such a set contains the image of a non-constant real analytic curve. The existence of such a curve is the main difficulty in Muchnik's argument, but in our situation it is given for free.

Proof of Theorem 4.4. This result is a consequence of Theorem 1.2 of [20]. Indeed, if Γ_0 is a finite index subgroup of Γ, then by Lemma 4.1 we have $\mathrm{Zar}(\Gamma_0)^{irr} = G$, so that Γ_0 does not preserve any proper, non-trivial vector subspace of V defined over $\mathbf{Q}$. This shows that Γ acts strongly irreducibly on $\mathbf{Q}^m$. If Γ were cyclic-by-finite, then Γ would contain a cyclic subgroup of finite index, and G would be abelian, contradicting its semi-simplicity. Thus, Properties (1) and (2) in Theorem 1.1 of [20] are satisfied, and we can apply Theorem 1.2 of [20]: by Lemma 4.2, it gives precisely the alternative stated in our Theorem 4.4. $\square$

Corollary 4.8. *If $F \subset M$ is a (non-empty) closed, proper, connected, and Γ-invariant subset, then F is contained in $a + \pi(V_c)$ for a unique torsion point $a \in M$. If $x \in M$ has a finite orbit under the action of Γ, then x is a torsion point.*

Proof. Let us prove the first assertion. If $x \in F$, then $\Gamma(x) \subset F$ because F is Γ-invariant. Since F is closed and proper, $\Gamma(x)$ is not dense in M. From Theorem 4.4 and Remark 4.5, there is a unique torsion point $a(x)$ such that $x \in a(x) + \pi(V_c)$. This map $x \in F \mapsto a(x)$ must be constant.

To see this, let us first assume that F is path connected. Take two points x and x' in F, and a continuous path $\tau\colon [0,1] \to F$ that connects $x = \tau(0)$ to $x' = \tau(1)$. Lifting τ to a path $\tilde{\tau}$ in V, and then projecting it to V/V_c we obtain a continuous map $[0,1] \to V/V_c$; since this map takes at most countably many values, it is constant, and there is a rational point $\tilde{a}$ in V that projects onto it. Then $a := \pi(\tilde{a})$ is a torsion point and $F \subset a + \pi(V_c)$.

To prove our main result it suffices to assume that F is path connected. If F is only assumed to be connected, a similar but more delicate argument applies, as the following lemma shows.

Lemma 4.9. *Let F be a closed and connected subset of M. Assume that every $x \in F$ is the sum of a torsion point $a(x)$ and a point $\pi(v)$ for some $v \in V_c$. Then F is contained in a unique torsion translate of $\pi(V_c)$.*

Proof. Denote by $p_c\colon V \to V/V_c$ the natural projection. The translates $b + \pi(V_c)$ form a linear foliation $\mathcal{F}_c$ of M. Locally, in small open subsets $\mathcal{U}$, this foliation is defined by the fibers of the submersion $p_{\mathcal{U}} = p_c \circ \pi^{-1}$ for some local inverse of π on $\mathcal{U}$. Say that $x \in F$ is locally transversely isolated (l.t.i. for short) if there is a small neighborhood $\mathcal{U}$ of x in M such that $F \cap \mathcal{U}$ is contained in a unique fiber of $p_{\mathcal{U}}$, *i.e.* in a unique local leaf of $\mathcal{F}_c$ in $\mathcal{U}$. If every point of F is l.t.i., the function $x \in F \mapsto a(x)$ is locally constant, and by connectedness, it is indeed constant.

Thus, we may assume that F contains at least one point which is not l.t.i.. Consider the subset $F_1 = F - F = \{x - y \mid x,\, y \in F\}$. This set is compact, connected, and is also contained in a union of torsion translates of $\pi(V_c)$. Moreover, the origin $\pi(0)$ is a point of F_1 which is not l.t.i.. Now, $F_2 = F_1 - F_1$ shares the same properties, and no point of F_2 is l.t.i.. Let $B_n \subset V_c$ be the closed ball of radius n in V_c, for some euclidean metric. Enumerate the set of torsion points by $\mathbf{N}$ and denote by a_n the n-th torsion point. Set $D_n = \cup_{k \le n}(a_k + \pi(B_n))$. This is an increasing sequence of compact subsets of M. Then, F_2 is contained in $\cup_n D_n$, and $F \cap D_n$ has empty interior in F_2 because no point of F_2 is l.t.i.. By the Baire property, we get a contradiction. $\square$

To prove the second assertion of Corollary 4.8, pick a point $x \in M$ with a finite Γ-orbit and write $x = a + \pi(z)$ for some torsion point a and some element $z \in V_c$. The orbit $\Gamma(a)$ is finite. Let G_c be the image of G in $\mathsf{GL}(V_c)$: it is a compact algebraic subgroup of $\mathsf{GL}(V_c)$, and the image Γ_c of Γ in G_c is Zariski dense. Thus, the closure of Γ_c for the euclidean topology is equal to G_c, because all closed subgroups of G_c are algebraic (see [19] §4.6). We deduce that the Γ-orbit of z is dense in $G(z) = G_c(z)$ for the euclidean topology. Since the orbit of x is finite, $G(z)$ is finite too. This implies that $G(z)$ is just one point because G is Zariski connected, and that $z = 0$ because the representation is irreducible over $\mathbf{Q}$. Thus, $z = 0$ and $x = a$. $\square$

Remark 4.10. Assume that $m = 2g$ for some $g \geq 1$ and M is in fact a complex torus $\mathbf{C}^g/\Lambda$, with $\Lambda \simeq \mathbf{Z}^{2g}$. Suppose that F is a smooth complex analytic subset of M; then F is a compact kähler manifold. The inclusion $F \to M$ factors through the Albanese torus $F \to A_F$ of F, via a morphism $A_F \to M$, and the image of A_F is the quotient of a subspace W in $\mathbf{C}^g$ by a lattice $W \cap \Lambda$ (see [10], p. 331 and 552). So, if $F \subset a + \pi(V_c)$, the subspace V_c contains a subspace $W \subset \mathbf{R}^m$ which is defined over $\mathbf{Q}$, contradicting the irreducibility assumption (Lemma 4.3). To separate clearly the arguments of complex geometry from the arguments of dynamical systems, we shall not use this type of idea before Section 4.4.

Remark 4.11. Theorem 2 of [13] is not correct, but it becomes true if the group G has no compact factor (this is implicitely assumed in [13, Proposition 1.3]).

4.3. **Invariant real analytic subsets.** Let F be an analytic (resp. subanalytic) subset of M (we refer to [2] for subanalytic sets). We say that F does not **fully generate** M if there is a proper subspace W of V and a non-empty open subset $\mathcal{U}$ of F such that $T_xF \subset W$ for every regular point x of F in $\mathcal{U}$. Otherwise, we say that F fully generates M.

Proposition 4.12. *Let Γ be a subgroup of $\mathsf{GL}_m(\mathbf{Z})$. Assume that the neutral component $\mathrm{Zar}(\Gamma)^{irr} \subset \mathsf{GL}_m(\mathbf{R})$ is semi-simple, and has no trivial factor. Let F be a subanalytic and Γ-invariant subset of M. If F fully generates M, it is equal to M.*

To prove this result, we decompose the linear representation of $G = \mathrm{Zar}(\Gamma)^{irr}$ on V into a direct sum of irreducible representations over $\mathbf{Q}$ (see [17], Proposition 22.41):

$$V = V_1 \oplus \cdots \oplus V_s. \tag{4.3}$$

Since there is no trivial factor, none of the V_i is the trivial representation. For each index i, we denote by $V_{i,c}$ the compact factor of V_i. As in Lemma 4.3, the projection π is an injective map from $V_{i,c}$ onto its image in M. Set

$$M_i = V_i/(\mathbf{Z}^m \cap V_i). \tag{4.4}$$

Then, each M_i is a compact torus of dimension $\dim(V_i)$, and M is isogenous to the product of the M_i. We may, and we shall assume that M is in fact equal to this product:

$$M = M_1 \times \cdots \times M_s; \tag{4.5}$$

this assumption simplifies the exposition without any loss of generality, because the image and the pre-image of a subanalytic set by an isogeny is subanalytic too. We can also assume (see Remark 4.6) that Γ is contained in G. For every index $1 \leq i \leq s$, we denote by π_i the projection on the i-th factor M_i.

Lemma 4.13. *If F fully generates M, the projection $F_i := \pi_i(F)$ is equal to M_i for every $1 \leq i \leq s$.*

Proof. By construction, F_i is a closed, Γ-invariant subset of M_i. Fix a connected component F_i^0 of F_i; it is invariant by a finite index subgroup Γ_0 of Γ. If it were contained in a translate of $\pi(V_{i,c})$, then F would not fully generate M. The first assertion of Corollary 4.8, applied to Γ_0, implies $F_i^0 = M_i$. □

We do an induction on the number s of irreducible factors. For just one factor, this is the previous lemma. Assuming that the proposition has been proven for $s-1$ irreducible factors, we now want to prove it for s factors. To simplify the exposition, we suppose that $s = 2$, which means that M is the product of just two factors $M_1 \times M_2$. The proof will only use that $\pi_1(F) = M_1$

and F fully generates M; thus, changing M_1 into $M_1 \times \ldots \times M_{s-1}$, this proof also establishes the induction in full generality.

Let $\varphi \colon N \to F$ be a surjective analytic map, from an analytic manifold N of dimension $\dim(F)$, as in the uniformization theorem of [2]. The composition $\pi_1 \circ \varphi \colon N \to M_1$ is analytic and onto. Let C be the set of critical values of $\pi_1 \circ \varphi$. From Sard's theorem, C is a closed subanalytic subset of M_1 of dimension $< \dim(M_1)$.

The set of points $x \in M_1$ with $F_x = M_2$ is closed; if it coincides with M_1, then $F = M$. Otherwise, there is an open ball $U_0 \subset M_1$ such that F_x is a non-empty, proper and subanalytic subset of M_2 for every $x \in U_0$. Let U be an open ball contained in $U_0 \setminus C$. On $N_U := (\pi_1 \circ \varphi)^{-1}(U)$, the map $\pi_1 \circ \varphi$ is a trivial fibration: there is a diffeomorphism $\psi \colon N_U \to U \times Y$ for some analytic manifold Y such that $\pi_1 \circ \varphi$ corresponds to the first projection. The fibers F_x, for x in U, are parametrized by $\varphi \circ \psi^{-1} \colon \{x\} \times Y \to F_x$. Let Y_1, $\ldots$, Y_{J_0} be the connected components of Y. The number $J(x)$ of connected components of F_x is a semi-continuous function of $x \in U$, because the condition $\varphi \circ \psi^{-1}(\{x\} \times Y_j) \cap \varphi \circ \psi^{-1}(\{x\} \times Y_k) = \emptyset$ is open. Let J be the maximum of this function on U; changing U in a smaller ball if necessary, we may assume that (1) $J(x) = J$ for all $x \in U$, and (2) each connected component $F_{x,j}$ of F_x is the image of $\bigcup_{i \in I(j)}(\{x\} \times Y_i)$ by $\varphi \circ \psi^{-1}$ for a fixed set of indices $I(j) \subset \{1, \ldots, J\}$. In particular, $\bigcup_{x \in U} F_{x,j}$ is a connected component of $F \cap \pi_1^{-1}(U)$ and is subanalytic.

Let $x \in U$ be a torsion point. The stabilizer of x is a finite index subgroup of Γ, and we can apply Corollary 4.8 to each connected component of F_x. We deduce that there is a torsion point $a_j(x)$ such that

$$F_{x,j} \subset a_j(x) + \pi(V_{2,c}), \quad \text{and} \quad F_x \subset \bigcup_{j=1}^{J} a_j(x) + \pi(V_{2,c}). \tag{4.6}$$

Since torsion points are dense in U and $\varphi \circ \psi^{-1}$ is analytic, the inclusions (4.6) hold for every x in U, but now the $a_j(x) \in M_2$ are not torsion points anymore.

Assume temporarily that $J = 1$, so that $F_x = F_{x,1}$ is contained in $a(x) + \pi(V_{2,c})$ for some point $a(x)$ of M_2. The point $a(x)$ is not uniquely defined by this property (one can replace it by $a(x) + \pi(v)$ for any $v \in V_{2,c}$), but there is a way to choose $a(x)$ canonically. First, the action of $G(\mathbf{R})$ on $V_{2,c}$ factors through a compact subgroup of $\mathsf{GL}(V_{2,c})$, so we can fix a $G(\mathbf{R})$-invariant euclidean metric dist_2 on $V_{2,c}$. Then, any compact subset K of $V_{2,c}$ is contained in a unique ball of smallest radius for the metric dist_2; we denote by $c(K)$ and $r(K)$ the center and radius of this ball. Since J is assumed to be 1, F_x is a compact, connected, and subanalytic subset of M that is contained in $a + \pi(V_{2,c})$ for some point a. Since π is injective on $V_{2,c}$, every loop in F_x can be contracted in

M_2, and the inclusion $\iota\colon (F_x - a) \to M$ lifts to an inclusion $\tilde{\iota}\colon (F_x - a) \to V_{2,c}$; then we define the center of F_x by

$$c(x) := a + \pi_2(c(\tilde{\iota}(F_x - a))) \in M_2. \tag{4.7}$$

By construction, $c(x)$ does not depend on a, and F_x is contained in $c(x) + \pi(V_{2,c})$. When $J > 1$, this procedure gives a finite set of centers $\{c_j(x)\}_{1 \leq j \leq J}$.

Lemma 4.14. *Let $E_1 = \mathbf{R}^m$ and $E_2 = \mathbf{R}^n$ be two euclidean vector spaces. Let $B_1 \subset E_1$ be a closed ball. Let $Z \subset B_1 \times E_2$ be a relatively compact subanalytic subset such that the projection $\pi_1 : Z \to B_1$ is onto. For each x in E_1, denote by $r(x)$ and $c(x)$ the radius and center of the smallest ball containing the fiber Z_x. Then r and c are subanalytic functions of x.*

Proof. Denote by $\| \cdot \|$ the euclidean norm on E_2. Let $B_2 \subset E_2$ be a closed ball such that $Z \subset B_1 \times B_2$, let R be its radius, and let I be the interval $[0, R]$. As in [2], Remark 3.11(1), we consider the set

$$A = \{(x,y,z,t) \in B_1 \times B_2 \times Z \times I \,|\, \pi_1(z) = x, \text{ and } t < \| \pi_2(z) - y \|\}. \tag{4.8}$$

It is subanalytic, and so is its projection $\tau(A) \subset B_1 \times B_2 \times I$, where $\tau(x,y,z,t) = (x,y,t)$. This projection is the set $\{(x,y,t) \mid \exists z \in Z_x,\ t < \| z - y \|\}$. By the theorem of the complement,

$$\tau(A)^c = \{(x,y,t) \in B_1 \times B_2 \times I \mid t \geq \| z - y \| \quad \text{for every } z \in Z_x\} \tag{4.9}$$

is also subanalytic. By Remark 3.11(2) of [2], the function

$$r(x) = \min_{y \in B_2} (\min\{t \mid (x,y,t) \in \tau(A)^c\}) \tag{4.10}$$

is subanalytic. Now, consider the subanalytic set

$$C = \{(x,y,t) \in B_1 \times B_2 \times I \mid t = r(x)\} \cap \tau(A)^c. \tag{4.11}$$

Denote by $\iota : C \to B_1 \times B_2$ the projection $(x,y,t) \mapsto (x,y)$. Then $\iota(C)$ is subanalytic and it is the graph of the map $B_1 \to B_2 : x \mapsto c(x)$. It follows that $c(x)$ is a subanalytic function of x. $\square$

This lemma shows that the radius $r_j(x)$ and the center $c_j(x)$ are subanalytic functions of x for every index $j \leq J$. The uniformization theorem provides a real analytic manifold N_j and a real analytic mapping $\Phi_j = (\varphi_j, \eta_j)\colon N_j \to U \times \mathbf{R}$ such that the graph of r_j is the image of Φ, and $\varphi_j\colon N_j \to U$ is generically of rank $\dim(U) = \dim(M_1)$. By [2, Theorem 7.10] there is a proper, closed, analytic subset D_j of U with the following property: if $a \in N_j$ and $\varphi_j(a) \notin D_j$, there is a neighborhood W of a and an analytic funtion $\hat{\eta}_j$ on $\varphi_j(W)$ such that φ_j is a diffeomorphism from W to $\varphi_j(W)$ and $\eta_j = \hat{\eta}_j \circ \varphi_j$ on W. Thus, on $U \setminus D_j$, r_j is locally a smooth analytic function. A similar result holds for c_j, for some proper analytic set $D'_j \subset U$. Set $D = \cup_j (D_j \cup D' j)$. Let $\mathcal{G}$

be the subset of $\pi_1^{-1}(U \setminus D)$ given by the union of the graphs of the centers: $\mathcal{G} = \{(x,y) \in M_1 \times M_2;\, x \in U \setminus D_1,\, y = c_j(x) \text{ for some } j\}$.

Lemma 4.15. *The tangent space $z \in \mathcal{G} \mapsto T_z\mathcal{G}$ takes only finitely many values $(W_j)_{1\leq j\leq k}$; given any point $z \in \mathcal{G}$, there is a neighborhood of z in M in which $\mathcal{G}$ coincides with $z+\pi(W_j)$ for one of these subspaces.*

This lemma concludes the proof of Proposition 4.12, because if $\mathcal{G}$ is locally contained in $a+\pi(W)$ for some proper subset W of V of dimension $\dim M_1$, then F is locally contained in $a+\pi(W+V_{2,c})$, and F does not fully generate M because $\dim(W+V_{2,c}) < \dim V$.

Proof. By construction, $\mathcal{G}$ is an analytic subset of $\pi_1^{-1}(U \setminus D_1)$ and it is invariant by Γ: if $z \in \mathcal{G}$ and g is an element of Γ such that $g(z) \in \pi_1^{-1}(U)$, then $g(z) \in \mathcal{G}$. For x in $U \setminus D_1$, we denote by $\mathcal{G}_x$ the finite fiber $\pi_1^{-1}(x) \cap \mathcal{G}$.

For every torsion point $x \in U \setminus D_1$, the stabilizer Γ_x of x is a finite index subgroup of Γ that preserves the finite set $\mathcal{G}_x$. By Corollary 4.8, $\mathcal{G}_x$ is a finite set of torsion points of M. In particular, torsion points are dense in $\mathcal{G}$. Fix one of these torsion points $z=(x,y) \in \mathcal{G}$, and denote by Γ_z the stabilizer of z in Γ. The tangent subspace $T_z\mathcal{G}$ is the graph of a linear morphism $\varphi_z \colon T_xM_1 \to T_yM_2$. Identifying the tangent spaces T_xM_1 and T_yM_2 with V_1 and V_2 respectively, φ_z becomes a morphism that interlaces the representations ρ_1 and ρ_2 of Γ_z on V_1 and V_2; since Γ_z is Zariski dense in G, we get

$$\rho_2(g) \circ \varphi_z = \varphi_z \circ \rho_1(g) \tag{4.12}$$

for every g in G. In other words, $\varphi_z \in \mathsf{Hom}(V_1;V_2)$ is a morphism of G-spaces. This holds for every torsion point $z \in \mathcal{G}$; by continuity of tangent spaces and density of torsion points, this holds everywhere on $\mathcal{G}$.

Since $\mathcal{G}$ is Γ-invariant, we also have

$$\varphi_{g(z)} \circ \rho_1(g) = \rho_2(g) \circ \varphi_z \tag{4.13}$$

for all $g \in \Gamma$ and $z \in \mathcal{G}$ such that $g(z) \in \pi_1^{-1}(U)$. Then, Equation (4.12) shows that $\varphi_{g(z)} = \varphi_z$, which means that the tangent space $T_z\mathcal{G}$ is constant along the orbits of Γ. Taking a point z in $\mathcal{G}$ whose first projection has a dense Γ-orbit in M_1, we see that the tangent space $w \in \mathcal{G} \mapsto T_w\mathcal{G}$ takes only finitely many values, at most $|\mathcal{G}_{\pi_1(z)}|$. Let $(W_j)_{1\leq j\leq k}$ be the list of possible tangent spaces $T_z\mathcal{G}$. Locally, near any point $z \in \mathcal{G}$, $\mathcal{G}$ coincides with $z+\pi(W_j)$ for some j. $\square$

4.4. **Complex analytic invariant subsets.** Let J be a complex structure on $V=\mathbf{R}^m$, so that M is now endowed with a structure of complex torus. Then, $m=2g$ for some integer g, $\mathbf{R}^m$ can be identified to $\mathbf{C}^g$, and $M=\mathbf{C}^g/\Lambda$ where Λ is the lattice $\mathbf{Z}^m$; to simplify the exposition, we denote by A the complex torus $\mathbf{C}^g/\Lambda$ and by M the real torus $\mathbf{R}^m/\mathbf{Z}^m$. Thus, A is just M, together with the complex structure J. Let X be an irreducible complex analytic subset of A, and let X^{reg} be its smooth locus.

Lemma 4.16. *Let W be the real subspace of V generated by the tangent spaces T_xX, for $x \in X^{reg}$. Then W is both complex and rational, and X is contained in a translate of the complex torus $\pi(W)$.*

Proof. Since X is complex analytic, its tangent space is invariant under the complex structure: $\mathrm{J}T_xX = T_xX$ for all $x \in X^{reg}$. So, the sum $W := \sum_x T_xX$ of the T_xX over all points $x \in X^{reg}$ is invariant by J and W is a complex subspace of $V \simeq \mathbf{C}^g$. Observe that if V' is any real subspace of V such that $\pi(V')$ contains some translate of X^{reg}, then $W \subseteq V'$.

Let a be a point of X^{reg}, and Y be the translate $X - a$ of X. It is an irreducible complex analytic subset of A that contains the origin 0 of A and satisfies $T_yY \subset W$ for every $y \in Y^{reg}$. Thus, Y^{reg} is contained in the projection $\pi(W) \subset A$. Set $Y^{(1)} = Y$, $Y_o^{(1)} = Y^{reg}$ and then

$$Y^{(\ell+1)} = Y^{(\ell)} - Y^{(\ell)}, \quad Y_o^{(\ell+1)} = Y_o^{(\ell)} - Y_o^{(\ell)} \tag{4.14}$$

for every integer $\ell \geq 1$. Since $Y^{(1)}$ is irreducible, and $Y^{(2)}$ is the image of $Y^{(1)} \times Y^{(1)}$ by the complex analytic map $(y_1, y_2) \mapsto y_1 - y_2$, we see that $Y^{(2)}$ is an irreducible complex analytic subset of A. Moreover $Y_o^{(2)}$ is a connected, dense, and open subset of $Y^{(2),reg}$. Observe that $Y_o^{(2)}$ is contained in $\pi(W)$ and contains $Y_o^{(1)}$ because $0 \in Y_o^{(1)}$. By induction, the sets $Y^{(\ell)}$ form an increasing sequence of irreducible complex analytic subsets of A, and $Y_o^{(\ell)}$ is a connected, dense and open subset of $Y^{(\ell),reg}$ that is contained in $\pi(W)$. By the Noether property, there is an index $\ell_0 \geq 1$ such that $Y^{(\ell)} = Y^{(\ell_0)}$ for every $\ell \geq \ell_0$. This complex analytic set is a subgroup of A, hence it is a complex subtorus. Write $Y^{(\ell_0)} = \pi(V')$ for some rational subspace V' of V. Since $Y \subset \pi(V')$, we get $W \subseteq V'$. Since $Y_o^{(\ell_0)} \subseteq \pi(W)$, we derive $V' = T_xY_o^{(\ell_0)} \subseteq W$ for every $x \in Y_o^{(\ell_0)}$. This implies $W = V'$, and shows that W is rational.

Thus, $\pi(W)$ is a complex subtorus of A. Since T_xX is contained in W for every regular point, X^{reg} is locally contained in a translate of $\pi(W)$. Since X is irreducible, X and X^{reg} are connected; thus X^{reg} is contained in a unique translate $a + \pi(W)$, and by density of X^{reg}, X is also contained in $a + \pi(W)$. □

Lemma 4.17. *Let X be an irreducible complex analytic subset of A. The following properties are equivalent:*

(i) *X is contained in a translate of a proper complex subtorus $B \subset A$;*
(ii) *X does not fully generate M;*
(iii) *there is a proper real subspace V' of V that contains T_xX for every $x \in X^{reg}$.*

Proof. Obviously (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii). Also, if (iii) is satisfied, Lemma 4.16 implies that X is contained in a complex subtorus $B = \pi(W) \subset A$ for some complex subspace W of V'; hence (iii) $\Rightarrow$ (i). To conclude, we prove that (ii) implies (iii). If X does not fully generate M, then (iii) is satisfied on some non-empty open subset $\mathcal{U}$ of X^{reg}, for some subspace V' of V. Fix a point x_0 in $\mathcal{U}$, and consider a point x in X^{reg}. Since X is irreducible, X^{reg} is path connected, there is an analytic path γ: $[0,1] \to X^{reg}$ that connects $x_0 = \gamma(0)$ to $x = \gamma(1)$. The subset of parameters $t \in [0,1]$ such that $T_{\gamma(t)}X \subset V'$ is an analytic subset of $[0,1]$ that contains the open neighborhood $\gamma^{-1}(\mathcal{U})$ of 0; thus, this set is $[0,1]$ and $T_xX \subset V'$. This means that $T_xX \subset V'$ for every regular point of X. $\square$

Theorem 4.18. *Let Γ be a subgroup of $\mathsf{GL}_m(\mathbf{Z})$. Assume that the neutral component, for the Zariski topology, of the Zariski closure of Γ in $\mathsf{GL}_m(\mathbf{R})$ is semi-simple and has no trivial factor. Let $\jmath$ be a complex structure on $M = \mathbf{R}^m/\mathbf{Z}^m$ and let X be an irreducible complex analytic subset of the complex torus $A = (M, \jmath)$. If X is Γ-invariant, it is equal to a translate of a complex subtorus $B \subset A$ by a torsion point.*

Proof. Set $W := \sum_{x \in X^{reg}} T_xX$. Lemma 4.16 shows that W is complex and rational. Since X is Γ-invariant, so is W. Its projection $B = \pi(W)$ is a complex subtorus of A such that

(1) B is Γ-invariant;
(2) B contains a translate $Y = X - a$ of X;
(3) Y fully generates B.

The group Γ acts on the quotient torus A/B and preserves the image of X, *i.e.* the image $\overline{a}$ of a. Since G has no trivial factor in V, $\overline{a}$ is a torsion point of A/B; indeed, A/B is isogeneous to a product of tori $M_i = V_i/(\mathbf{Z}^m \cap V_i)$ associated to $\mathbf{Q}$-irreducible subrepresentations, as in Equation (4.4), and Corollary 4.8 shows that the projection of $\overline{a}$ in each M_i is a torsion point. Then there exists a torsion point a' in A such that $X \subseteq a' + B$. Replacing a by a' and Γ by a finite index subgroup Γ' which fixes a', we may assume that a is torsion and $Y = X - a$ is invariant by Γ. We apply Proposition 4.12 to B, the restriction Γ_B of Γ to B, and the complex analytic subset Y: we conclude via Lemma 4.17 that Y coincides with B. Thus $X = a + B$. $\square$

5. Proof of Theorem A

Let X be an irreducible subvariety of $A_{\overline{K}}$, and assume that X_ε is dense in X for every positive ε. We want to prove that X is special.

Replacing K by a finite extension we may assume that X is defined over K. In the rest of this section we use A to denote $A_{\overline{K}}$. By Remark 3.2 we may assume $\mathbf{k} = \mathbf{C}$ and $\hat{h}(X) = 0$.

5.1. **Monodromy and invariance.** Recall that X is geometrically irreducible. After replacing B^o by a Zariski open and dense subset, we may assume that $\mathcal{X}_b$ is irreducible for all $b \in B^o$.

Let $b \in B^o$ be any point. As explained in Section 2.3, the holonomy of the Betti foliation and the monodromy of the abelian scheme $\mathcal{A}^o \to B^o$ give rise to the same representation $\mathbf{Mon}\colon \pi_1(B^o;b) \to \mathsf{GL}_{2g}(\mathbf{Z})$, and we call its image $\Gamma = Im(\mathbf{Mon}) \subset \mathsf{GL}_{2g}(\mathbf{Z})$ the monodromy group.

Theorem B' from Section 3.3 implies that $\mathcal{X}^o$ is invariant under the Betti foliation $\mathcal{F}$, so $\mathcal{X}_b$ is invariant under the action of the holonomy group of $\mathcal{F}$ on $\mathcal{A}_b$. Thus, $\mathcal{X}_b$ is invariant under the monodromy group Γ on the torus $\mathcal{A}_b \simeq H_1(\mathcal{A}_b;\mathbf{R})/H_1(\mathcal{A}_b;\mathbf{Z}) \simeq \mathbf{R}^{2g}/\mathbf{Z}^{2g}$.

5.2. **Trivial trace.** We first treat the case when $A^{\overline{K}/\mathbf{C}}$ is trivial. According to [26, Theorem 1.5], this is the only case we need to treat. However we shall also treat the case of a non-trivial trace below for completeness.

To show that X is special, we shall apply Theorem 4.18 to $\mathcal{X}_b \subset \mathbf{R}^{2g}/\mathbf{Z}^{2g}$ and Γ. As in Section 4.1, let G be the neutral component of $Zar(\Gamma)^{irr} \subset \mathsf{GL}_{2g}$. The key point now is to prove that Γ satisfies the assumption of Theorem 4.18; this will follow from deep results on variations of Hodge structures:

Theorem 5.1 (Deligne, Grothendieck)**.** *If the trace $A^{\overline{K}/\mathbf{C}}$ is trivial then G is semi-simple and has no trivial factor in $H_1(\mathcal{A}_b;\mathbf{R})$.*

Proof. By Deligne's semi-simplicity theorem, the group G is semi-simple (see [6, Corollary 4.2.9]).

Set $\Gamma' = \Gamma \cap G$; it is a Zariski dense subgroup of G, and to see that G has no trivial factor, it suffices to prove that $W = H_1(\mathcal{A}_b;\mathbf{Q})^{\Gamma'}$ is $\{0\}$.

Since Γ' is a finite index subgroup of Γ, there exists a finite covering $B' \to B^o$ such that the abelian scheme $\mathcal{A}' := \mathcal{A}^o \times_{B^o} B' \to B'$ has monodromy group Γ'. Note that the geometric generic fiber of $\pi'\colon \mathcal{A}' \to B'$ is still A. Fix $b' \in B'$ lying above b. Then $H_1(\mathcal{A}'_{b'};\mathbf{Q}) = H_1(\mathcal{A}_b;\mathbf{Q})$ and hence $W = H_1(\mathcal{A}'_{b'};\mathbf{Q})^{\Gamma'}$.

The local system $R_1\pi'_*\mathbf{Q}$ satisfies that $(R_1\pi'_*\mathbf{Q})_s \cong H_1(\mathcal{A}'_s;\mathbf{Q})$ for each $s \in B'$; it is a variation of Hodge structures on B' of type $(-1,0)+(0,-1)$. Let $(R_1\pi'_*\mathbf{Q})^{const}$ be the largest locally constant subsystem of $R_1\pi'_*\mathbf{Q}$. Then we have $(R_1\pi'_*\mathbf{Q})^{const}_{b'} = H_1(\mathcal{A}'_{b'};\mathbf{Q})^{\Gamma'} = W$.

Deligne's Theorem of the Fixed Part implies that $(R_1\pi'_*\mathbf{Q})^{const}$ is a sub-variation of Hodge structures of $R_1\pi'_*\mathbf{Q}$ on B' (see [6, Corollaire 4.1.2]). It gives rise to an abelian subscheme $\mathcal{C} \to B'$ of $\mathcal{A}' \to B'$ with $H_1(\mathcal{C}_{b'};\mathbf{Q}) = (R_1\pi'_*\mathbf{Q})^{const}_{b'} = W$ by [6, Rappel 4.4.3]. Moreover [6, Corollaire 4.1.2] says that the induced Hodge structure on $(R_1\pi'_*\mathbf{Q})^{const}_s$ is independent of $s \in B'$.

Thus $\mathcal{C} \to B'$ is an isotrivial abelian scheme, namely its closed fibers are isomorphic to each other. So the geometric generic fiber of $\mathcal{C} \to B'$ is contained in $A^{\overline{K}/\mathbf{C}}$. Thus, the triviality of $A^{\overline{K}/\mathbf{C}}$ implies $W = \{0\}$. $\square$

Since G is semi-simple and $H_1(\mathcal{A}_b;\mathbf{R})^G = \{0\}$, Theorem 4.18 implies that $\mathcal{X}_b$ is the translate of an abelian subvariety of $\mathcal{A}_b$ by some torsion point $y_b \in \mathcal{A}_b$. Observe that the leaf $\mathcal{F}_{y_b}$ is a multi-section of $\mathcal{A}^o$ (see Remark 2.2). By base change, we may assume that $\mathcal{F}_{y_b}$ is a section and is the Zariski closure of a torsion point $y \in A(K)$ in $\mathcal{A}^o$. Theorem B' from Section 3.3 shows that $y \in X$, and replacing X by $X - y$ we may suppose that $0 \in X$; then, $\mathcal{X}_b$ is an abelian subvariety of $\mathcal{A}_b$ for all $b \in B^o$. It follows that $\mathcal{X}^o$ is a subscheme of the abelian scheme $\mathcal{A}^o$ over B^o which is stable under the group laws. So X is an abelian subvariety of A.

5.3. **The general case.** We do not assume anymore that $A^{\overline{K}/\mathbf{C}}$ is trivial. Set $A^t = A^{\overline{K}/\mathbf{C}} \otimes_{\mathbf{C}} K$. Replacing K by a finite extension and A by a finite cover, we assume that $A = A^t \times A^{nt}$ where A^{nt} is an abelian variety over K with trivial trace. We also choose the model $\mathcal{A}$ so that $\mathcal{A}^o = (\mathcal{A}^t)^o \times_{B^o} (\mathcal{A}^{nt})^o$ where $(\mathcal{A}^t)^o$ and $(\mathcal{A}^{nt})^o$ are the Zariski closures of A^t and A^{nt} in $\mathcal{A}^o$ respectively. Denote by $\pi^t : \mathcal{A}^o \to (\mathcal{A}^t)^o$ the projection to the first factor and $\pi^{nt} : \mathcal{A}^o \to (\mathcal{A}^{nt})^o$ the projection to the second factor. After replacing K by a further finite extension and B by its normalization, we may assume that $(\mathcal{A}^t)^o = A^{\overline{K}/\mathbf{C}} \times B^o$. Note that $\pi^t|_{\mathcal{A}^t_b} : \mathcal{A}^t_b \to A^{\overline{K}/\mathbf{C}}$ is an isomorphism for every fiber $\mathcal{A}^t_b$ with $b \in B^o$.

By Proposition 3.3(1), the geometric generic fibers of $\pi^t(\mathcal{X}^o)$ and $\pi^{nt}(\mathcal{X}^o)$ are small subvarieties of A^t and A^{nt} respectively. Corollary 3.5 shows that $\pi^t(\mathcal{X}^o) = Y \times B^o$ for some subvariety Y of $A^{\overline{K}/\mathbf{C}}$. Section 5.2 shows that the geometric generic fiber of $\pi^{nt}(\mathcal{X}^o)$ is a torsion coset $a + A'$ for some torsion point $a \in A^{nt}_{\overline{K}}(\overline{K})$ and some abelian subvariety A' of $A^{nt}_{\overline{K}}$. Replacing K by a finite extension, we may assume that a and A' are defined over K. We have $\mathcal{X}^o \subseteq \pi^t(\mathcal{X}) \times_{B^o} \pi^{nt}(\mathcal{X}) = \pi^t(\mathcal{X}) + \pi^{nt}(\mathcal{X})$ and we only need to show that $\mathcal{X}^o = \pi^t(\mathcal{X}) \times_{B^o} \pi^{nt}(\mathcal{X})$.

For every $b \in B^o$, $\mathcal{A}_b = \mathcal{A}^t_b \times \mathcal{A}^{nt}_b$. The monodromy on $\mathcal{A}_b$ is the diagonal product of the monodromies on each factor. It is trivial on the first one so, for every $x \in \mathcal{A}^t_b$, the fiber $\pi^t|_{\mathcal{A}_b}^{-1}(x) \simeq \mathcal{A}^{nt}_b$ is invariant under Γ. It follows that $\pi^t|_{\mathcal{A}_b}^{-1}(x) \cap \mathcal{X}_b$, and hence $\mathcal{W}_x = \pi^{nt}(\pi^t|_{\mathcal{A}_b}^{-1}(x) \cap \mathcal{X}_b)$, is also Γ-invariant. Each irreducible component of $\mathcal{W}_x$ is Γ_0-invariant for a finite index subgroup $\Gamma_0 \subset \Gamma$. Recall that the neutral components of $\mathrm{Zar}(\Gamma_0)$ and $\mathrm{Zar}(\Gamma)$ are equal by Lemma 4.1. Since A^{nt} has trivial trace, we can apply Theorem 4.18 to each irreducible component of $\mathcal{W}_x$ as in the trivial trace case in Section 5.2. Thus each $\mathcal{W}_x$ is a Zariski closed subset whose irreducible components are torsion cosets of the abelian variety $\mathcal{A}_b^{nt}$. The abelian variety $\mathcal{A}_b^{nt}$ has only countably many Zariski closed subsets having the property that each of the finitely many irreducible components is a torsion coset. By the Baire property there exists a Zariski dense subset $\Sigma \subset \pi^t(\mathcal{X}_b)$ such that $\mathcal{W}_x$ is independent of x for all $x \in \Sigma$. Call this finite union of torsion cosets A'.

Thus the Zariski closure of $\bigcup_{x\in\Sigma} \pi^t|_{\mathcal{A}_b}^{-1}(x) \cap \mathcal{X}_b$ is $\pi^t(\mathcal{X}_b) \times A'$ under the decomposition $\mathcal{A}_b = \mathcal{A}_b^t \times \mathcal{A}_b^{nt}$. Hence $\pi^t(\mathcal{X}_b) \times A' \subset \mathcal{X}_b$. Note that $\{x\} \times A'$ is the fiber of $\pi^t|_{\mathcal{X}_b}^{-1}(x)$ for all $x \in \Sigma$. As $\mathcal{X}_b$ is irreducible we find $\pi^t(\mathcal{X}_b) \times A' = \mathcal{X}_b$ by comparing dimensions. Then $\mathcal{X}^o = \pi^t(\mathcal{X}) \times_{B^o} \pi^{nt}(\mathcal{X})$, and this concludes the proof.

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