BIRATIONAL CONJUGACIES BETWEEN ENDOMORPHISMS ON THE PROJECTIVE PLANE

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1. The statement. – Let $\mathbf{k}$ be an algebraically closed field of characteristic 0. If f_1 and f_2 are two endomorphisms of a projective surface X over $\mathbf{k}$ and f_1 is conjugate to f_2 by a birational transformation of X, then f_1 and f_2 have the same topological degree. When X is the projective plane $\mathbb{P}^2_{\mathbf{k}}$, f_1 (resp. f_2) is given by homogeneous formulas of the same degree d without common factor, and d is called the degree, or algebraic degree of f_1; in that case the topological degree is d^2, so, f_1 and f_2 have the same degree d if they are conjugate.

Theorem A. *Let $\mathbf{k}$ be an algebraically closed field of characteristic 0. Let f_1 and f_2 be dominant endomorphisms of $\mathbb{P}^2_{\mathbf{k}}$ over $\mathbf{k}$. Let $h : \mathbb{P}^2_{\mathbf{k}} \dashrightarrow \mathbb{P}^2_{\mathbf{k}}$ be a birational map such that $h \circ f_1 = f_2 \circ h$. If the degree d of f_1 is ≥ 2, there exists an isomorphism $h' : \mathbb{P}^2_{\mathbf{k}} \to \mathbb{P}^2_{\mathbf{k}}$ such that $h' \circ f_1 = f_2 \circ h'$.*

Moreover, h itself is in $\mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}})$, except may be if f_1 is conjugate by an element of $\mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}})$ to

1. *the composition of $g_d : [x : y : z] \mapsto [x^d : y^d : z^d]$ and a permutation of the coordinates,*
2. *or the endomorphism $(x,y) \mapsto (x^d, y^d + \sum_{j=2}^d a_j y^{d-j})$ of the open subset $\mathbb{A}^1_{\mathbf{k}} \setminus \{0\} \times \mathbb{A}^1_{\mathbf{k}} \subset \mathbb{P}^2_{\mathbf{k}}$, for some coefficients $a_j \in \mathbf{k}$.*

Theorem A is proved in Sections 2 to 6. A counter-example is given in Section 7 when $\mathrm{char}(\mathbf{k}) \neq 0$. The case $d = 1$ is covered by [1]; in particular, there are automorphisms $f_1, f_2 \in \mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}})$ which are conjugate by some birational transformation but not by an automorphism.

Example 1. When $f_1 = f_2$ is the composition of g_d and a permutation of the coordinates and h is the Cremona involution $[x : y : z] \mapsto [x^{-1} : y^{-1} : z^{-1}]$, we have $h \circ f_1 = f_2 \circ h$.

Example 2. When

$$f_1(x,y) = (x^d, y^d + \sum_{j=2}^d a_j y^{d-j}) \text{ and } f_2(x,y) = (x^d, y^d + \sum_{j=2}^d a_j (B/A)^j x^j y^{d-j})$$

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with $a_j \in \mathbf{k}$ then $h(x,y) = (Ax, Bxy)$ conjugates f_1 to f_2 if A and B are roots of unity of order dividing $d-1$, and $\deg(h) = 2$. On the other hand, $h'[x:y:z] = [Az/B : y : x]$ is an automorphism of $\mathbb{P}^2$ that conjugates f_1 to f_2.

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2. The exceptional locus. – If $h : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a birational map, we denote by $\mathrm{Ind}(h)$ its **indeterminacy locus** (a finite subset of $\mathbb{P}^2(\mathbf{k})$), and by $\mathrm{Exc}(h)$ its **exceptional set**, i.e. the union of the curves contracted by h (a finite union of irreducible curves). Let $U_h = \mathbb{P}^2_{\mathbf{k}} \setminus \mathrm{Exc}(h)$ be the complement of $\mathrm{Exc}(h)$; it is a Zariski dense open subset of $\mathbb{P}^2_{\mathbf{k}}$. If $C \subset \mathbb{P}^2_{\mathbf{k}}$ is a curve, we denote by $h_\circ(C)$ the **strict transform** of C, i.e. the Zariski closure of $h(C \setminus \mathrm{Ind}(f))$.

Proposition 3. *If h is a birational transformation of the projective plane, then (1) $Ind(h) \subseteq Exc(h)$, (2) $h|_{U_h}(U_h) = U_{h^{-1}}$, and (3) $h|_{U_h} : U_h \to U_{h^{-1}}$ is an isomorphism.*

Proof. There is a smooth projective surface X and two birational morphisms $\pi_1, \pi_2 : X \to \mathbb{P}^2$ such that $h = \pi_2 \circ \pi_1^{-1}$; we choose X minimal, in the sense that there is no (-1)-curve C of X which is contracted by both π_1 and π_2 ([8]).

Pick a point $p \in \mathrm{Ind}(h)$. The divisor $\pi_1^{-1}(p)$ is a tree of rational curves of negative self-intersections, with at least one (-1)-curve. If $p \notin \mathrm{Exc}(h)$, any curve contracted by π_2 that intersects $\pi_1^{-1}(p)$ is in fact contained in $\pi_1^{-1}(p)$. But π_2 may be decomposed as a succession of contractions of (-1)-curves: since it does not contract any (-1)-curve in $\pi_1^{-1}(p)$, we deduce that π_2 is a local isomorphism along $\pi_1^{-1}(p)$. This contradicts the minimality of $\mathbb{P}^2_{\mathbf{k}}$, hence $\mathrm{Ind}(h) \subset \mathrm{Exc}(h)$. Thus $h|_{U_h} : U_h \to \mathbb{P}^2$ is regular. Since $U_h \cap \mathrm{Exc}(h) = \emptyset$, $h|_{U_h}$ is an open immersion, h^{-1} is well defined on $h|_{U_h}(U_h)$, and h^{-1} is an open immersion on $h|_{U_h}(U_h)$. It follows that $h|_{U_h}(U_h) \subseteq U_{h^{-1}}$. The same argument shows that $h^{-1}|_{U_{h^{-1}}} : U_{h^{-1}} \to \mathbb{P}^2$ is well defined and its image is in U_h. Since $h^{-1}|_{U_{h^{-1}}} \circ h|_{U_h} = \mathrm{id}$ and $h|_{U_h} \circ h^{-1}|_{U_{h^{-1}}} = \mathrm{id}$; this concludes the proof. □

Let f_1 and f_2 be dominant endomorphisms of $\mathbb{P}^2_{\mathbf{k}}$. Let $h : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map such that $f_1 = h^{-1} \circ f_2 \circ h$. Let d be the common (algebraic) degree of f_1 and f_2. Recall that an algebraic subset D of $\mathbb{P}^2_{\mathbf{k}}$ is **totally invariant** under the action of the endomorphism g if $g^{-1}(C) = C$ (then $g(C) = C$, and if $\deg(g) \geq 2$, g ramifies along C).

Lemma 4. *The exceptional set of h is totally invariant under the action of f_1: $f_1^{-1}(Exc(h)) = Exc(h)$.*

Proof. Since $h \circ f_1 = f_2 \circ h$, the strict transform of $f_1^{-1}(\mathrm{Exc}(h))$ by $f_2 \circ h$ is a finite set, but every dominant endomorphism of $\mathbb{P}^2_{\mathbf{k}}$ is a finite map, so the strict transform of $f_1^{-1}(\mathrm{Exc}(h))$ by h is already a finite set. This means that $f_1^{-1}(\mathrm{Exc}(h))$ is contained in $\mathrm{Exc}(h)$; this implies $f_1(\mathrm{Exc}(E)) \subset E$ and then $f_1^{-1}(\mathrm{Exc}(h)) = \mathrm{Exc}(h) = f_1(\mathrm{Exc}(h))$ because f_1 is onto. □

Lemma 5. *If $d \geq 2$ then $Exc(h)$ and $Exc(h^{-1})$ are two isomorphic configurations of lines, and this configuration falls in the following list:*

(P0) *the empty set;*
(P1) *one line in $\mathbb{P}^2$;*
(P2) *two lines in $\mathbb{P}^2$;*
(P3) *three lines in $\mathbb{P}^2$ in general position.*

Proof. Assume $\mathrm{Exc}(h)$ is not empty; then, by Lemma 4, the curve $\mathrm{Exc}(h)$ is totally invariant under f_1. According to [6, §4] and [4, Proposition 2], $\mathrm{Exc}(h)$ is one of the three curves listed in (P1) to (P3).

Changing h into h^{-1} and permuting the role of f_1 and f_2, we see that $\mathrm{Exc}(h^{-1})$ is also a configuration of type (Pi) for some i. Proposition 3 shows that $U_h \simeq U_{h^{-1}}$. Since the four possibilities (Pi) correspond to pairwise non-isomorphic complements, we deduce that $\mathrm{Exc}(h)$ and $\mathrm{Exc}(h^{-1})$ have the same type. □

Remark 6. One can also refer to [7] to prove this lemma. Indeed, f_1 induces a map from the set of irreducible components of $\mathrm{Exc}(h)$ into itself, and since f_1 is onto, this map is a permutation; the same applies to f_2. Thus, replacing f_1 and f_2 by f_1^m and f_2^m for some suitable $m \geq 1$, we may assume that $f_1(C) = C$ for every irreducible component C of $\mathrm{Exc}(h)$. Since f_1 is finite, $\mathrm{Exc}(h)$ has only finitely many irreducible components, and $f_1(\mathrm{Exc}(h)) = \mathrm{Exc}(h)$, we obtain $f_1^{-1}(C) = C$ for every component. Since f_1 acts by multiplication by d on $\mathrm{Pic}(\mathbb{P}^2_{\mathbf{k}})$, the ramification index of f_1 along C is $d > 1$, and the main theorem of [7] implies that C is a line.

Remark 7. Totally invariant hypersurfaces of endomorphisms of $\mathbb{P}^3$ are unions of hyperplanes, at most four of them (we refer to [9] for a proof and important additional references, notably the work of J.-M. Hwang, N. Nakayama and D.-Q. Zhang). So, an analog of Lemma 5 holds in dimension 3 too; but our proof in case (P1), see § 4 below, does not apply in dimension 3, at least not directly. (Note that [2] contains an important gap, since its main result is based on a wrong lemma from [3]).

3. Normal forms. – Two configurations of the same type (Pi) are equivalent under the action of $\mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}}) = \mathsf{PGL}_3(\mathbf{k})$. If we change h into $A \circ h \circ B$ for some well chosen pair of automorphisms (A, B), or equivalently if we change f_1 into

$B \circ f_1 \circ B^{-1}$ and f_2 into $A^{-1} \circ f_2 \circ A$, we may assume that $\mathrm{Exc}(h) = \mathrm{Exc}(h^{-1})$ and that exactly one of the following situation occurs (see also [6]):

(P0).– $\mathrm{Exc}(h) = \mathrm{Exc}(h^{-1}) = \emptyset$.– Then h is an automorphism of $\mathbb{P}^2_{\mathbf{k}}$ and Theorem A is proved.

(P1).– $\mathrm{Exc}(h) = \mathrm{Exc}(h^{-1}) = \{z = 0\}$.– Then h induces an automorphism of $\mathbb{A}^2_{\mathbf{k}}$ and f_1 and f_2 restrict to endomorphisms of $\mathbb{A}^2_{\mathbf{k}} = \mathbb{P}^2_{\mathbf{k}} \setminus \{z = 0\}$ (that extend to endomorphisms of $\mathbb{P}^2_{\mathbf{k}}$).

(P2).– $\mathrm{Exc}(h) = \mathrm{Exc}(h^{-1}) = \{x = 0\} \cup \{z = 0\}$.– Then, U_h and $U_{h^{-1}}$ are both equal to the open set $U := \{(x,y) \in \mathbb{A}^2 \,|\, x \neq 0\}$. Moreover,

$$h|_U(x,y) = (Ax, Bx^m y + C(x)) \tag{1}$$

for some regular function $C(x)$ on $\mathbb{A}^1_{\mathbf{k}} \setminus \{0\}$ and $m \in \mathbf{Z}$, and

$$f_i|_U(x,y) = (x^{\pm d}, F_i(x,y)) \tag{2}$$

for some rational functions $F_i \in \mathbf{k}(x)[y]$ which are regular on $(\mathbb{A}^1_{\mathbf{k}} \setminus \{0\}) \times \mathbb{A}^1$ and have degree d (more precisely, f_i must define an endomorphism of $\mathbb{P}^2$ of degree d). Moreover, the signs of the exponent $\pm d$ in Equation (2) are the same for f_1 and f_2.

(P3).– $\mathrm{Exc}(h) = \mathrm{Exc}(h^{-1}) = \{x = 0\} \cup \{y = 0\} \cup \{z = 0\}$.– In this case, each f_i is equal to $a_i \circ g_d$ where $g_d([x:y:z]) = [x^d : y^d : z^d]$ and each a_i is an automorphism of $\mathbb{P}^2_{\mathbf{k}}$ acting by permutation of the coordinates, while h is an automorphism of $(\mathbb{A}^1 \setminus \{0\}) \times (\mathbb{A}^1 \setminus \{0\})$.

4. Endomorphisms of $\mathbb{A}^2_{\mathbf{k}}$. – This section proves Theorem A in case (P1):

Proposition 8. *Let f_1 and f_2 be endomorphisms of $\mathbb{A}^2$ that extend to endomorphisms of $\mathbb{P}^2$ of degree $d \geq 2$. If h is an automorphism of $\mathbb{A}^2$ that conjugates f_1 to f_2 then h is an affine automorphism i.e.* $\deg h = 1$.

We follow the notation from [5] and denote by V_∞ the valuative tree of $\mathbb{A}^2 = \mathrm{Spec}(\mathbf{k}[x,y])$ at infinity. If g is an endomorphism of $\mathbb{A}^2$, we denote by $g_\bullet$ its action on V_∞.

Set $V_1 = \{v \in V_\infty \,;\, \alpha(v) \geq 0, A(v) \leq 0\}$, where α and A are respectively the skewness and thinness function, as defined in page 216 of [5]; the set V_1 is a closed subtree of V_∞. For $v \in V_1$, $v(F) \leq 0$ for every $F \in \mathbf{k}[x,y] \setminus \{0\}$. Then V_1 is invariant under each $(f_i)_\bullet$, and if we set

$$\mathcal{T}_i = \{v \in V_1 \,;\, (f_i)_\bullet v = v\} \tag{3}$$

then $\mathcal{T}_2 = h_\bullet \mathcal{T}_1$. Since each f_i extends to an endomorphism of $\mathbb{P}^2_{\mathbf{k}}$, the valuation $-\deg$ is an element of $\mathcal{T}_1 \cap \mathcal{T}_2$. Also, in the terminology of [5], $\lambda_2(f_i) =$

$\lambda_1(f_i)^2 = d^2$ and $\deg(f_i^n) = \lambda_1^n = d^n$ for all $n \geq 1$ and for $i = 1$ and 2, because f_1 and f_2 extend to regular endomorphisms of $\mathbb{P}^2_{\mathbf{k}}$ of degree d. So by [5, Proposition 5.3 (a)], $\mathcal{T}_i$ is a single point or a closed segment.

A valuation $v \in V_\infty$ is **monomial** of weight (s,t) for the pair of polynomial functions $(P,Q) \in \mathbf{k}[x,y]^2$ if

(1) P and Q generate $\mathbf{k}[x,y]$ as a $\mathbf{k}$-algebra,
(2) if F is any non-zero element of $\mathbf{k}[x,y]$ and $F = \sum_{i,j\geq 0} a_{ij} P^i Q^j$ is its decomposition as a polynomial function of P and Q then

$$v(F) = -\max\{si + tj\ ;\ a_{i,j} \neq 0\}. \tag{4}$$

We say that v is monomial for the basis (P,Q) of $\mathbf{k}[x,y]$, if v is monomial for (P,Q) and some weight (s,t). In particular, $-\deg$ is monomial for (x,y), of weight $(1,1)$.

Lemma 9. *If $v \in V_1$ is monomial for (P,Q) of weight (s,t), then $s,t \geq 0$, and* $\min\{s,t\} = \min\{-v(F)\ ;\ F \in \mathbf{k}[x,y] \setminus \mathbf{k}\}$.

Proof. First, assume that $(P,Q) = (x,y)$. For an element v of V_1, $v(F) \leq 0$ for every F in $\mathbf{k}[x,y]$, hence $s = -v(x)$ and $t = -v(y)$ are non-negative; and the formula for $\min\{s,t\}$ follows from the inequality $-v(F) \geq \min\{s,t\}$. To get the statement for any pair (P,Q), change v into $g_\bullet^{-1} v$ where g is the automorphism defined by $g(x,y) = (P(x,y), Q(x,y))$. □

Lemma 10. *If $-\deg$ is monomial for (P,Q), of weight (s,t), then $s = t = 1$ and P and Q are of degree one in $\mathbf{k}[x,y]$.*

Proof. By Lemma 9, we may assume that $1 = s \leq t$; thus, after an affine change of variables, we may assume that $P = x$. Since $\mathbf{k}[x,y]$ is generated by x and Q, Q takes form $Q = ay + C(x)$ where $a \in \mathbf{k}^*$ and $C \in \mathbf{k}[x]$. If C is a constant, we conclude the proof. Now we assume $\deg(C) \geq 1$. Then $t = \deg(Q) = \deg(C)$. Since $y = a^{-1}(Q - C(x))$ and $-\deg$ is monomial for (x,Q) of weight $(1,t)$, we get $1 = \deg(y) = \max\{t, \deg C\} = t$. It follows that $t = \deg Q = 1$, which concludes the proof. □

Proof of Proposition 8. By [5, Proposition 5.3 (b), (d)], there exists P and $Q \in \mathbf{k}[x,y]$ such that for every $v \in \mathcal{T}_1$, v is monomial for (P,Q). Moreover, $-\deg$ is in $\mathcal{T}_1 \cap \mathcal{T}_2$. By Lemma 10, $P = x$ and $Q = y$ after an affine change of coordinates. Since $\mathcal{T}_2 = h_\bullet \mathcal{T}_1$, for every $v \in \mathcal{T}_2$, v is monomial for (h^*x, h^*y). Since $-\deg \in \mathcal{T}_2$, Lemma 10 implies $\deg h^*x = \deg h^*y = 1$ and this concludes the proof. □

5. Endomorphisms of $(\mathbb{A}^1_{\mathbf{k}} \setminus \{0\}) \times \mathbb{A}^1_{\mathbf{k}}$. – We now arrive at case (P2), namely $\mathrm{Exc}(h) = \mathrm{Exc}(h^{-1}) = \{x = 0\} \cup \{z = 0\}$, and keep the notations from Section 4. Our first goal is to prove that,

Lemma 11. *If h is not an affine automorphism of the affine plane, then after a conjugacy by an affine transformation of the plane,*

• *Either f_1 and f_2 are equal to (x^d, y^d) and $h(x,y) = (Ax, Bx^m y)$ with A and B two roots of unity of order dividing $d-1$ and $m \in \mathbf{Z} \setminus \{0\}$.*

• *Or, up to a permutation of f_1 and f_2,*

$$f_1(x,y) = (x^d, y^d + \sum_{j=2}^{d} a_j y^{d-j}) \ \text{ and } \ f_2(x,y) = (x^d, y^d + \sum_{j=2}^{d} a_j (B/A)^j x^j y^{d-j})$$

with $a_j \in \mathbf{k}$, and $h(x,y) = (Ax, Bxy)$ with A and B two roots of unity of order dividing $d-1$; then $h'[x:y:z] = [Az/B : y : x]$ is an automorphism of $\mathbb{P}^2$ that conjugates f_1 to f_2.

Proof. We split the proof in two steps.

Step 1.– We assume that $f_i|_U(x,y) = (x^d, F_i(x,y))$, with $d > 0$.

Since f_i extends to a degree d endomorphism of $\mathbb{P}^2_{\mathbf{k}}$, we can write $F_1(x,y) = a_0 y^d + \sum_{j=1}^{d} a_j(x) y^{d-j}$ where $a_0 \in \mathbf{k}^*$ and the $a_j \in \mathbf{k}[x]$ satisfy $\deg(a_j) \leq j$ for all j. Changing the coordinates to (x, by) with $b^d = a_0$, we assume $a_0 = 1$. We can also conjugate f_1 by the automorphism

$$(x,y) \mapsto \left(x, y + \frac{1}{d} a_1(x)\right) \tag{5}$$

and assume $a_1 = 0$. Altogether, the change of coordinates $(x,y) \mapsto (x, by + \frac{1}{d} a_1(x))$ is affine because $\deg(a_1) \leq 1$, and conjugates f_1 to an endomorphism $(x^d, F_1(x,y))$ normalized by $F_1(x,y) = y^d + \sum_{j=2}^{d} a_j(x) y^{d-j}$ with $\deg(a_j) \leq j$. Similarly, we may assume that $F_2(x,y) = y^d + \sum_{j=2}^{d} b_j(x) y^{d-j}$ for some polynomial functions b_j with $\deg(b_j) \leq j$ for all j.

Now, with the notation used in Equation (1), the two terms of the conjugacy relation $h \circ f_1 = f_2 \circ h$ are

$$h \circ f_1 = (Ax^d,\ Bx^{dm}(y^d + \sum_{j=2}^{d} a_j(x) y^{d-j}) + C(x^d)) \tag{6}$$

$$f_2 \circ h = (A^d x^d,\ (Bx^m y + C(x))^d + \sum_{j=2}^{d} b_j(Ax)(Bx^m y + C(x))^{d-j}). \tag{7}$$

This gives $A^{d-1} = 1$, and comparing the terms of degree d in y we get $B^{d-1} = 1$. Then, looking at the term of degree $d-1$ in y, we obtain $C(x) = 0$. Thus $h(x,y) = (Ax, Bx^m y)$ for some roots of unity A and B, the orders of which divide $d-1$. Since h is not an automorphism, we have

$$m \neq 0. \tag{8}$$

Permuting the role of f_1 and f_2 (or changing h in its inverse), we suppose $m \geq 1$. Coming back to (6) and (7), we obtain the sequence of equalities

$$b_j(Ax) = a_j(x)(Bx^m)^j \tag{9}$$

for all indices j between 2 and d. On the other hand, a_j and b_j are elements of $\mathbf{k}[x]$ of degree at most j. Since $m \geq 1$, there are only two possibilities.

(a) All a_j and b_j are equal to 0; then $f_1(x,y) = f_2(x,y) = (x^d, y^d)$, which concludes the proof.

(b) Some a_j is different from 0 and $m = 1$. Then all coefficients a_j are constant, and $b_j(x) = a_j\left(\frac{Bx}{A}\right)^j$ for all indices $j = 2, \ldots, d$.

In case (b), we set $\alpha = B/A$ (a root of unity of order dividing $d-1$), and use homogeneous coordinates to write

$$f_1[x:y:z] = [x^d : y^d + \sum_{j=2}^{d} a_j z^j y^{d-j} : z^d] \tag{10}$$

$$f_2[x:y:z] = [x^d : y^d + \sum_{j=2}^{d} a_j \alpha^j x^j y^{d-j} : z^d]. \tag{11}$$

The conjugacy $h[x:y:z] = [Axz : Bxy : z^2]$ is not a linear projective automorphism of $\mathbb{P}^2$, but the automorphism defined by $[x:y:z] \mapsto [z/\alpha : y : x]$ conjugates f_1 to f_2.

Step 2.– The only remaining case is when $f_i = (x^{-d}, F_i(x,y))$, for $i = 1, 2$, with

$$F_1(x,y) = \sum_{j=0}^{d} a_j(x) x^{-d} y^{d-j} \ \text{ and } \ F_2(x,y) = \sum_{j=0}^{d} b_j(x) x^{-d} y^{d-j} \tag{12}$$

for some polynomial functions $a_j, b_j \in \mathbf{k}[x]$ that satisfy $\deg(a_j), \deg(b_j) \leq j$ and $a_0 b_0 \neq 0$. Writing the conjugacy equation $h \circ f_1 = f_2 \circ h$ and looking at the term of degree d in y, we get the relation

$$Bx^{-md} a_0 x^{-d} y^d = b_0 (Ax)^{-d} (Bx^m y)^d. \tag{13}$$

Comparing the degree in x we get $-md - d = md - d$, hence $m = 0$. Moreover, h conjugates f_1^2 to f_2^2; thus, by the first step, h should be an affine automorphism since $m = 0$ (see Equation (8)). □

6. Endomorphisms of $(\mathbb{A}^1_{\mathbf{k}} \setminus \{0\})^2$. – Denote by $[x:y:z]$ the homogeneous coordinates of $\mathbb{P}^2_{\mathbf{k}}$ and by (x,y) the coordinates of the open subset $V := (\mathbb{A}^1_{\mathbf{k}} \setminus \{0\})^2$ defined by $xy \neq 0$, $z = 1$. We write $f_i = a_i \circ g_d$ as in case (P3) of Section **3**. Since h is an automorphism of $(\mathbb{A}^1_{\mathbf{k}} \setminus \{0\})^2$, it is the composition $t_h \circ m_h$ of a diagonal map $t_h(x,y) = (ux, vy)$, for some pair $(u,v) \in (\mathbf{k}^*)^2$, and a monomial map $m_h(x,y) = (x^a y^b, x^c y^d)$, for some matrix

$$M_h := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{GL}_2(\mathbf{Z}). \tag{14}$$

Also, note that the group $\mathfrak{S}_3 \subset \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ of permutations of the coordinates $[x:y:z]$ corresponds to a finite subgroup S_3 of $\mathsf{GL}_2(\mathbf{Z})$.

Since m_h commutes to g_d and $g_d \circ t_h = t_h^d \circ g_d$, the conjugacy equation is equivalent to

$$t_h \circ (m_h \circ a_1 \circ m_h^{-1}) \circ (g_d \circ m_h) = a_2 \circ t_h^d \circ (g_d \circ m_h). \tag{15}$$

The automorphisms a_1 and a_2 are monomial maps, induced by elements A_1 and A_2 of S_3, and Equation (15) implies that *M_h conjugates A_1 to A_2 in $\mathsf{GL}_2(\mathbf{Z})$*; indeed, the matrices can be recovered by looking at the action on the set of units $wx^m y^n$ in $\mathbf{k}(V)$ (or on the fundamental group $\pi_1(V(\mathbf{C}))$ if $\mathbf{k} = \mathbf{C}$). There are two possibilities :

(a) either $A_1 = A_2 = \mathrm{Id}$, there is no constraint on m_h;
(b) or A_1 and A_2 are non-trivial permutations, they are conjugate by an element $P \in S_3$, and $M_h = \pm A_2^j \circ P$, for some $j \in \mathbf{Z}$.

In both cases, u and v are roots of unity (there order is determined by d and the A_i). Let p be the monomial transformation associated to P; it is a permutation of the coordinates, hence an element of $\mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}})$. Then, $h'(x,y) = t_h \circ p$ is an element of $\mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}})$ that conjugates f_1 to f_2.

7. An example in positive characteristic. – Assume that $q = p^s$ with $s \geq 2$. Set $G := xy^p + (x-1)y$. Then,

$$f_1(x,y) = (x^q, y^q + G(x,y))$$

defines an endomorphism of $\mathbb{A}^2$ that extends to an endomorphism of $\mathbb{P}^2$.

Consider a polynomial $P(x) \in \mathbf{F}_q[x]$ such that $2 \leq \deg(P) \leq \frac{q}{p} - 1$. Observe that $\deg(G) < \deg(G(x, y+P(x))) < q$. Then $g(x,y) = (x, y - P(x))$ is an automorphism of $\mathbb{A}^2_{\mathbf{k}}$ that conjugates f_1 to

$$\begin{aligned} f_2(x,y) &:= g \circ f_1 \circ g^{-1}(x,y) \\ &= (x^q, y^q + P(x)^q + G(x, y+P(x)) - P(x^q)) \\ &= (x^q, y^q + G(x, y+P(x))). \end{aligned} \tag{16}$$

As f_1, f_2 is an endomorphism of $\mathbb{A}^2$ that extends to a regular endomorphism of $\mathbb{P}^2$ (here we use the inequality $\deg(G(x, y+P(x))) < q$).

Let us prove that f_1 and f_2 are not conjugate by any automorphism of $\mathbb{P}^2$. We assume that there exists $h \in \mathsf{PGL}_3(\overline{\mathbf{F}_q})$ such that $h \circ f_1 = f_2 \circ h$ and seek a contradiction. Consider the pencils of lines through the point $[0:1:0]$ in $\mathbb{P}^2$; for $a \in \mathbf{F}_q$ we denote by L_a the line $\{x = az\}$, and by L_∞ the line $\{z = 0\}$. Then

$$\{L_a \,;\, a \in \mathbf{F_q} \cup \{\infty\}\} = \{\text{lines } L \text{ such that } f_1^{-1}L = L\} \tag{17}$$

$$= \{\text{lines } L \text{ such that } f_2^{-1}L = L\}; \tag{18}$$

in other words, the lines L_a for $a \in \mathbf{F}_q \cup \{\infty\}$ are exactly the lines which are totally invariant under the action of f_1 (resp. of f_2). Since h conjugates f_1 to f_2, it permutes these lines. In particular, h fixes the point $[0:1:0]$, and if we identify $L_a \cap \mathbb{A}^2$ to $\mathbb{A}^1$ with its coordinate y by the parametrization $y \mapsto (a, y)$ then h maps L_a to another line $L_{a'}$ in an affine way: $h(a, y) = (a', \alpha y + \beta)$.

Since g conjugates f_1 to f_2 and g fixes each of the lines L_a, we know that $f_1|_{L_a}$ is conjugated to $f_2|_{L_a}$ for every $a \in \mathbf{F}_q$; for $a = \infty$, both $f_1|_{L_\infty}$ and $f_2|_{L_\infty}$ are conjugate to $y \mapsto y^q$. Moreover

- $a = \infty$ is the unique parameter such that $f_1|_{L_a}$ is conjugate to $y \mapsto y^q$ by an affine map $y \mapsto \alpha y + \beta$;
- $a = 0$ is the unique parameter such that $f_1|_{L_a}$ is conjugate to $y \mapsto y^q - y$ by an affine map;
- $a = 1$ is the unique parameter such that $f_1|_{L_a}$ is conjugate to $y \mapsto y^q + y^p$ by an affine map.

And the same properties hold for f_2. As a consequence, we obtain $h(L_\infty) = L_\infty$, $h(L_0) = L_0$ and $h(L_1) = L_1$; this means that there are coefficients $\alpha \in \overline{\mathbf{F}_q}^*$ and $\beta, \gamma \in \overline{\mathbf{F}_q}$ such that $h(x, y) = (x, \alpha y + \beta x + \gamma)$. Writing down the relation $h \circ f_1 = f_2 \circ h$ we obtain the relation

$$\alpha y^q + \alpha G(x, y) + \beta x^q + \gamma = \alpha^q y^q + \beta^q x^q + \gamma^q \tag{19}$$

$$+ G(x, \alpha y + \beta x + \gamma + P(x)). \tag{20}$$

We note that $1 < \deg G(x, y) < \deg G(x, \alpha y + \beta x + \gamma + P(x)) < q$. Compare the terms of degree q, we get $\alpha y^q + \beta x^q = \alpha^q y^q + \beta^q x^q$. It follows that

$$\alpha G(x, y) + \gamma = \gamma^q + G(x, \alpha y + \beta x + \gamma + P(x)). \tag{21}$$

Then $\deg G(x, y) = \deg G(x, \alpha y + \beta x + \gamma + P(x))$, which is a contradiction.

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