BIRRATIONAL CONJUGACIES BETWEEN ENDMORPHISMS ON THE PROJECTIVE PLANE

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1. The statement. – Let $k$ be an algebraically closed field of characteristic 0. If $f_1$ and $f_2$ are two endomorphisms of a projective surface $X$ over $k$ and $f_1$ is conjugate to $f_2$ by a birational transformation of $X$, then $f_1$ and $f_2$ have the same topological degree. When $X$ is the projective plane $\mathbb{P}_k^2$, $f_1$ (resp. $f_2$) is given by homogeneous formulas of the same degree $d$ without common factor, and $d$ is called the degree, or algebraic degree of $f_1$; in that case the topological degree is $d^2$, so, $f_1$ and $f_2$ have the same degree $d$ if they are conjugate.

Theorem A. Let $k$ be an algebraically closed field of characteristic 0. Let $f_1$ and $f_2$ be dominant endomorphisms of $\mathbb{P}_k^2$ over $k$. Let $h : \mathbb{P}_k^2 \to \mathbb{P}_k^2$ be a birational map such that $h \circ f_1 = f_2 \circ h$. If the degree $d$ of $f_1$ is $\geq 2$, there exists an isomorphism $h' : \mathbb{P}_k^2 \to \mathbb{P}_k^2$ such that $h' \circ f_1 = f_2 \circ h'$.

Moreover, $h$ itself is in $\text{Aut}(\mathbb{P}_k^2)$, except may be if $f_1$ is conjugate by an element of $\text{Aut}(\mathbb{P}_k^2)$ to

1. the composition of $g_d : [x : y : z] \mapsto [x^d : y^d : z^d]$ and a permutation of the coordinates,
2. or the endomorphism $(x, y) \mapsto (x^d, y^d + \sum_{j=2}^d a_j y^{d-j})$ of the open subset $\mathbb{A}_k^2 \setminus \{0\} \times \mathbb{A}_k^2 \subset \mathbb{P}_k^2$, for some coefficients $a_j \in k$.

Theorem A is proved in Sections 2 to 6. A counter-example is given in Section 7 when $\text{char}(k) \neq 0$. The case $d = 1$ is covered by [1]; in particular, there are automorphisms $f_1, f_2 \in \text{Aut}(\mathbb{P}_k^2)$ which are conjugate by some birational transformation but not by an automorphism.

Example 1. When $f_1 = f_2$ is the composition of $g_d$ and a permutation of the coordinates and $h$ is the Cremona involution $[x : y : z] \mapsto [x^{-1} : y^{-1} : z^{-1}]$, we have $h \circ f_1 = f_2 \circ h$.

Example 2. When

\[ f_1(x, y) = (x^d, y^d + \sum_{j=2}^d a_j y^{d-j}) \quad \text{and} \quad f_2(x, y) = (x^d, y^d + \sum_{j=2}^d a_j (B/A)^j x^j y^{d-j}) \]

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with \( a_j \in \mathbb{k} \) then \( h(x,y) = (Ax, Bxy) \) conjugates \( f_1 \) to \( f_2 \) if \( A \) and \( B \) are roots of unity of order dividing \( d - 1 \), and \( \deg(h) = 2 \). On the other hand, \( h'[x : y : z] = [Az/B : y : x] \) is an automorphism of \( \mathbb{P}^2 \) that conjugates \( f_1 \) to \( f_2 \).

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2. **The exceptional locus.** – If \( h : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \) is a birational map, we denote by \( \text{Ind}(h) \) its **indeterminacy locus** (a finite subset of \( \mathbb{P}^2(\mathbb{k}) \)), and by \( \text{Exc}(h) \) its **exceptional set**, i.e. the union of the curves contracted by \( h \) (a finite union of irreducible curves). Let \( U_h = \mathbb{P}^2 \setminus \text{Exc}(h) \) be the complement of \( \text{Exc}(h) \); it is a Zariski dense open subset of \( \mathbb{P}^2 \). If \( C \subseteq \mathbb{P}^2 \) is a curve, we denote by \( h_c(C) \) the **strict transform** of \( C \), i.e. the Zariski closure of \( h(C \setminus \text{Ind}(f)) \).

**Proposition 3.** If \( h \) is a birational transformation of the projective plane, then
1. \( \text{Ind}(h) \subseteq \text{Exc}(h) \),
2. \( h|_{U_h}(U_h) = U_h^{-1} \), and
3. \( h|_{U_h} : U_h \to U_h^{-1} \) is an isomorphism.

**Proof.** There is a smooth projective surface \( X \) and two birational morphisms \( \pi_1, \pi_2 : X \to \mathbb{P}^2 \) such that \( h = \pi_2 \circ \pi_1^{-1} \); we choose \( X \) minimal, in the sense that there is no \((-1)\)-curve \( C \) of \( X \) which is contracted by both \( \pi_1 \) and \( \pi_2 \) ([8]).

Pick a point \( p \in \text{Ind}(h) \). The divisor \( \pi_1^{-1}(p) \) is a tree of rational curves of negative self-intersections, with at least one \((-1)\)-curve. If \( p \notin \text{Exc}(h) \), any curve contracted by \( \pi_2 \) that intersects \( \pi_1^{-1}(p) \) is in fact contained in \( \pi_1^{-1}(p) \). But \( \pi_2 \) may be decomposed as a succession of contractions of \((-1)\)-curves; since it does not contract any \((-1)\)-curve in \( \pi_1^{-1}(p) \), we deduce that \( \pi_2 \) is a local isomorphism along \( \pi_1^{-1}(p) \). This contradicts the minimality of \( \mathbb{P}^2 \), hence \( \text{Ind}(h) \subseteq \text{Exc}(h) \). Thus \( h|_{U_h} : U_h \to \mathbb{P}^2 \) is regular. Since \( U_h \cap \text{Exc}(h) = \emptyset \), \( h|_{U_h} \) is an open immersion, \( h^{-1} \) is well defined on \( h|_{U_h}(U_h) \), and \( h^{-1} \) is an open immersion on \( h|_{U_h}(U_h) \). It follows that \( h|_{U_h}(U_h) \subseteq U_{h^{-1}} \). The same argument shows that \( h^{-1}|_{U_{h^{-1}}} : U_{h^{-1}} \to \mathbb{P}^2 \) is well defined and its image is in \( U_h \). Since \( h^{-1}|_{U_{h^{-1}}} \circ h|_{U_h} = \text{id} \) and \( h|_{U_h} \circ h^{-1}|_{U_{h^{-1}}} = \text{id} \), this concludes the proof.

Let \( f_1 \) and \( f_2 \) be dominant endomorphisms of \( \mathbb{P}^2 \). Let \( h : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \) be a birational map such that \( f_1 = h^{-1} \circ f_2 \circ h \). Let \( d \) be the common (algebraic) degree of \( f_1 \) and \( f_2 \). Recall that an algebraic subset \( D \) of \( \mathbb{P}^2 \) is **totally invariant** under the action of the endomorphism \( g \) if \( g^{-1}(C) = C \) (then \( g(C) = C \), and if \( \deg(g) \geq 2 \), \( g \) ramifies along \( C \)).

**Lemma 4.** The exceptional set of \( h \) is totally invariant under the action of \( f_1 \):
\[
f_1^{-1}(\text{Exc}(h)) = \text{Exc}(h).
\]
Proof. Since \( h \circ f_1 = f_2 \circ h \), the strict transform of \( f_1^{-1}(\text{Exc}(h)) \) by \( f_2 \circ h \) is a finite set, but every dominant endomorphism of \( \mathbb{P}^2_k \) is a finite map, so the strict transform of \( f_1^{-1}(\text{Exc}(h)) \) by \( h \) is already a finite set. This means that \( f_1^{-1}(\text{Exc}(h)) \) is contained in \( \text{Exc}(h) \); this implies \( f_1(\text{Exc}(E)) \subset E \) and then \( f_1^{-1}(\text{Exc}(h)) = \text{Exc}(h) = f_1(\text{Exc}(h)) \) because \( f_1 \) is onto. \( \square \)

Lemma 5. If \( d \geq 2 \) then \( \text{Exc}(h) \) and \( \text{Exc}(h^{-1}) \) are two isomorphic configurations of lines, and this configuration falls in the following list:

(P0) the empty set;
(P1) one line in \( \mathbb{P}^2 \);
(P2) two lines in \( \mathbb{P}^2 \);
(P3) three lines in \( \mathbb{P}^2 \) in general position.

Proof. Assume \( \text{Exc}(h) \) is not empty; then, by Lemma 4, the curve \( \text{Exc}(h) \) is totally invariant under \( f_1 \). According to [6, §4] and [4, Proposition 2], \( \text{Exc}(h) \) is one of the three curves listed in (P1) to (P3).

Changing \( h \) into \( h^{-1} \) and permuting the role of \( f_1 \) and \( f_2 \), we see that \( \text{Exc}(h^{-1}) \) is also a configuration of type (Pi) for some \( i \). Proposition 3 shows that \( U_h \simeq U_{h^{-1}} \). Since the four possibilities (Pi) correspond to pairwise non-isomorphic complements, we deduce that \( \text{Exc}(h) \) and \( \text{Exc}(h^{-1}) \) have the same type. \( \square \)

Remark 6. One can also refer to [7] to prove this lemma. Indeed, \( f_1 \) induces a map from the set of irreducible components of \( \text{Exc}(h) \) into itself, and since \( f_1 \) is onto, this map is a permutation; the same applies to \( f_2 \). Thus, replacing \( f_1 \) and \( f_2 \) by \( f_1^m \) and \( f_2^m \) for some suitable \( m \geq 1 \), we may assume that \( f_1(C) = C \) for every irreducible component \( C \) of \( \text{Exc}(h) \). Since \( f_1 \) is finite, \( \text{Exc}(h) \) has only finitely many irreducible components, and \( f_1(\text{Exc}(h)) = \text{Exc}(h) \), we obtain \( f_1^{-1}(C) = C \) for every component. Since \( f_1 \) acts by multiplication by \( d \) on \( \text{Pic}(\mathbb{P}^2_k) \), the ramification index of \( f_1 \) along \( C \) is \( d > 1 \), and the main theorem of [7] implies that \( C \) is a line.

Remark 7. Totally invariant hypersurfaces of endomorphisms of \( \mathbb{P}^3 \) are unions of hyperplanes, at most four of them (we refer to [9] for a proof and important additional references, notably the work of J.-M. Hwang, N. Nakayama and D.-Q. Zhang). So, an analog of Lemma 5 holds in dimension 3 too; but our proof in case (P1), see § 4 below, does not apply in dimension 3, at least not directly. (Note that [2] contains an important gap, since its main result is based on a wrong lemma from [3]).

3. Normal forms. – Two configurations of the same type (Pi) are equivalent under the action of \( \text{Aut}(\mathbb{P}^2_k) = \text{PGL}_3(k) \). If we change \( h \) into \( A \circ h \circ B \) for some well chosen pair of automorphisms \( (A,B) \), or equivalently if we change \( f_1 \) into
Theorem A is proved. and that exactly one of the following situation occurs (see also [6]):

(P0).– Exc(h) = Exc(h⁻¹) = ∅.– Then h is an automorphism of \( \mathbb{P}^2_k \) and Theorem A is proved.

(P1).– Exc(h) = Exc(h⁻¹) = \{ z = 0 \}.– Then h induces an automorphism of \( A^2_k \) and \( f_1 \) and \( f_2 \) restrict to endomorphisms of \( A^2_k = \mathbb{P}^2_k \setminus \{ z = 0 \} \) (that extend to endomorphisms of \( \mathbb{P}^2_k \)).

(P2).– Exc(h) = Exc(h⁻¹) = \{ x = 0 \} \cup \{ z = 0 \}.– Then, \( U_h \) and \( U_{h⁻¹} \) are both equal to the open set \( U := \{(x,y) \in A^2 \mid x \neq 0 \} \). Moreover,

\[
h|_U(x,y) = (Ax, Bx^my + C(x))
\]

for some regular function \( C(x) \) on \( A^1_k \setminus \{ 0 \} \) and \( m \in \mathbb{Z} \), and

\[
f_i|_U(x,y) = (x^{±d}, F_i(x,y))
\]

for some rational functions \( F_i \in k(x)[y] \) which are regular on \( (A^1_k \setminus \{ 0 \}) \times A^1 \) and have degree \( d \) (more precisely, \( f_i \) must define an endomorphism of \( \mathbb{P}^2 \) of degree \( d \)). Moreover, the signs of the exponent \( ±d \) in Equation (2) are the same for \( f_1 \) and \( f_2 \).

(P3).– Exc(h) = Exc(h⁻¹) = \{ x = 0 \} \cup \{ y = 0 \} \cup \{ z = 0 \}.– In this case, each \( f_i \) is equal to \( a_i \circ g_d \) where \( g_d([x : y : z]) = [x^d : y^d : z^d] \) and each \( a_i \) is an automorphism of \( \mathbb{P}^2_k \) acting by permutation of the coordinates, while \( h \) is an automorphism of \( (A^1 \setminus \{ 0 \}) \times (A^1 \setminus \{ 0 \}) \).

4. Endomorphisms of \( A^2_k \).– This section proves Theorem A in case (P1):

**Proposition 8.** Let \( f_1 \) and \( f_2 \) be endomorphisms of \( A^2 \) that extend to endomorphisms of \( \mathbb{P}^2 \) of degree \( d \geq 2 \). If \( h \) is an automorphism of \( A^2 \) that conjugates \( f_1 \) to \( f_2 \) then \( h \) is an affine automorphism i.e. \( \deg h = 1 \).

We follow the notation from [5] and denote by \( V_\infty \) the valuative tree of \( A^2 = \text{Spec}(k[x,y]) \) at infinity. If \( g \) is an endomorphism of \( A^2 \), we denote by \( g_\bullet \) its action on \( V_\infty \).

Set \( V_1 = \{ v \in V_\infty \mid \alpha(v) \geq 0, A(v) \leq 0 \} \), where \( \alpha \) and \( A \) are respectively the skewness and thinness function, as defined in page 216 of [5]; the set \( V_1 \) is a closed subtree of \( V_\infty \). For \( v \in V_1 \), \( v(F) \leq 0 \) for every \( F \in k[x,y] \setminus \{ 0 \} \). Then \( V_1 \) is invariant under each \( (f_i)_\bullet \), and if we set

\[
T_i = \{ v \in V_1 : (f_i)_\bullet v = v \}
\]

then \( T_2 = h_\bullet T_1 \). Since each \( f_i \) extends to an endomorphism of \( \mathbb{P}^2_k \), the valuation \( \deg \) is an element of \( T_1 \cap T_2 \). Also, in the terminology of [5], \( \lambda_2(f_i) = \cdots \)
A valuation \( v \in V_\infty \) is **monomial** of weight \((s, t)\) for the pair of polynomial functions \((P, Q) \in k[x, y]^2\) if

1. \( P \) and \( Q \) generate \( k[x, y] \) as a \( k \)-algebra,
2. if \( F \) is any non-zero element of \( k[x, y] \) and \( F = \sum_{i,j \geq 0} a_{ij} P^i Q^j \) is its decomposition as a polynomial function of \( P \) and \( Q \) then
   \[
   v(F) = -\max\{si + tj : a_{ij} \neq 0\}.
   \]

We say that \( v \) is monomial for the basis \((P, Q)\) of \( k[x, y] \), if \( v \) is monomial for \((P, Q)\) and some weight \((s, t)\). In particular, \( -\deg \) is monomial for \((x, y)\), of weight \((1, 1)\).

**Lemma 9.** If \( v \in V_1 \) is monomial for \((P, Q)\) of weight \((s, t)\), then \( s, t \geq 0 \) and \( \min\{s, t\} = \min\{-v(F) : F \in k[x, y] \setminus k\} \).

**Proof.** First, assume that \((P, Q) = (x, y)\). For an element \( v \) of \( V_1 \), \( v(F) \leq 0 \) for every \( F \) in \( k[x, y] \), hence \( s = -v(x) \) and \( t = -v(y) \) are non-negative; and the formula for \( \min\{s, t\} \) follows from the inequality \(-v(F) \geq \min\{s, t\}\). To get the statement for any pair \((P, Q)\), change \( v \) into \( g^{-1} \cdot v \) where \( g \) is the automorphism defined by \( g(x, y) = (P(x, y), Q(x, y)) \).

**Lemma 10.** If \( -\deg \) is monomial for \((P, Q)\), of weight \((s, t)\), then \( s = t = 1 \) and \( P \) and \( Q \) are of degree one in \( k[x, y] \).

**Proof.** By Lemma 9, we may assume that \( 1 = s \leq t \); thus, after an affine change of variables, we may assume that \( P = x \). Since \( k[x, y] \) is generated by \( x \) and \( Q \), \( Q \) takes form \( Q = ay + C(x) \) where \( a \in k^* \) and \( C \in k[x] \). If \( C \) is a constant, we conclude the proof. Now we assume \( \deg(C) \geq 1 \). Then \( t = \deg(Q) = \deg(C) \). Since \( y = a^{-1}(Q - C(x)) \) and \( -\deg \) is monomial for \((x, Q)\) of weight \((1, t)\), we get \( 1 = \deg(y) = \max\{t, \deg(C)\} = t \). It follows that \( t = \deg Q = 1 \), which concludes the proof.

**Proof of Proposition 8.** By [5, Proposition 5.3 (b), (d)], there exists \( P \) and \( Q \in k[x, y] \) such that for every \( v \in T_1 \), \( v \) is monomial for \((P, Q)\). Moreover, \( -\deg \) is in \( T_1 \cap T_2 \). By Lemma 10, \( P = x \) and \( Q = y \) after an affine change of coordinates. Since \( T_2 = h_\bullet T_1 \), for every \( v \in T_2 \), \( v \) is monomial for \((h^*x, h^*y)\). Since \( -\deg \in T_2 \), Lemma 10 implies \( \deg h^*x = \deg h^*y = 1 \) and this concludes the proof.

5. **Endomorphisms of** \((\mathbb{A}^1_k \setminus \{0\}) \times \mathbb{A}^1_k\). – We now arrive at case (P2), namely \( \text{Exc}(h) = \text{Exc}(h^{-1}) = \{x = 0\} \cup \{z = 0\} \), and keep the notations from Section 4. Our first goal is to prove that,
Lemma 11. If $h$ is not an affine automorphism of the affine plane, then after a conjugacy by an affine transformation of the plane,

- Either $f_1$ and $f_2$ are equal to $(x^d, y^d)$ and $h(x, y) = (Ax, Bx^my)$ with $A$ and $B$ two roots of unity of order dividing $d - 1$ and $m \in \mathbb{Z} \setminus \{0\}$.
- Or, up to a permutation of $f_1$ and $f_2$,

$$f_1(x, y) = (x^d, y^d + \sum_{j=2}^{d} a_j y^{d-j}) \text{ and } f_2(x, y) = (x^d, y^d + \sum_{j=2}^{d} a_j(B/A)^j x^i y^{d-j})$$

with $a_j \in \mathbb{k}$, and $h(x, y) = (Ax, Bxy)$ with $A$ and $B$ two roots of unity of order dividing $d - 1$; then $h'[x : y : z] = [Az/B : y : x]$ is an automorphism of $\mathbb{P}^2$ that conjugates $f_1$ to $f_2$.

Proof. We split the proof in two steps.

Step 1.— We assume that $f_1|_U(x, y) = (x^d, F_1(x, y))$, with $d > 0$.

Since $f_2$ extends to a degree $d$ endomorphism of $\mathbb{P}^2_k$, we can write $F_1(x, y) = a_0y^d + \sum_{j=1}^{d} a_j(x)y^{d-j}$ where $a_0 \in \mathbb{k}^*$ and the $a_j \in \mathbb{k}[x]$ satisfy $\text{deg}(a_j) \leq j$ for all $j$. Changing the coordinates to $(x, by)$ with $b^d = a_0$, we assume $a_0 = 1$. We can also conjugate $f_1$ by the automorphism

$$(x, y) \mapsto \left(x, y + \frac{1}{d}a_1(x)\right)$$

and assume $a_1 = 0$. Altogether, the change of coordinates $(x, y) \mapsto (x, by + \frac{1}{d}a_1(x))$ is affine because $\text{deg}(a_1) \leq 1$, and conjugates $f_1$ to an endomorphism $(x^d, F_1(x, y))$ normalized by $F_1(x, y) = y^d + \sum_{j=2}^{d} a_j(x)y^{d-j}$ with $\text{deg}(a_j) \leq j$.

Similarly, we may assume that $F_2(x, y) = y^d + \sum_{j=2}^{d} b_j(x)y^{d-j}$ for some polynomial functions $b_j$ with $\text{deg}(b_j) \leq j$ for all $j$.

Now, with the notation used in Equation (1), the two terms of the conjugacy relation $h \circ f_1 = f_2 \circ h$ are

$$h \circ f_1 = (Ax^d, Bx^m y^d + \sum_{j=2}^{d} a_j(x)y^{d-j} + C(x^d))$$

$$f_2 \circ h = (A^d x^d, (Bx^m y + C(x))^d + \sum_{j=2}^{d} b_j(Ax)(Bx^m y + C(x))^{d-j})$$

This gives $A^{d-1} = 1$, and comparing the terms of degree $d$ in $y$ we get $B^{d-1} = 1$. Then, looking at the term of degree $d - 1$ in $y$, we obtain $C(x) = 0$. Thus $h(x, y) = (Ax, Bx^m y)$ for some roots of unity $A$ and $B$, the orders of which divide $d - 1$. Since $h$ is not an automorphism, we have

$$m \neq 0.$$
In case (b), we set $m \geq 1$. Coming back to (6) and (7), we obtain the sequence of equalities

$$b_j(Ax) = a_j(x)(Bx^m)^j$$

for all indices $j$ between 2 and $d$. On the other hand, $a_j$ and $b_j$ are elements of $k[x]$ of degree at most $j$. Since $m \geq 1$, there are only two possibilities.

(a) All $a_j$ and $b_j$ are equal to 0; then $f_1(x,y) = f_2(x,y) = (x^d,y^d)$, which concludes the proof.

(b) Some $a_j$ is different from 0 and $m = 1$. Then all coefficients $a_j$ are constant, and $b_j(x) = a_j(\frac{Bx}{A})^j$ for all indices $j = 2, \ldots, d$.

In case (b), we set $\alpha = B/A$ (a root of unity of order dividing $d - 1$), and use homogeneous coordinates to write

$$f_1[x : y : z] = [x^d : y^d + \sum_{j=2}^d a_j\alpha^j y^{d-j} : z^d]$$

$$f_2[x : y : z] = [x^d : y^d + \sum_{j=2}^d a_j\alpha^j y^{d-j} : z^d].$$

The conjugacy $h[x : y : z] = [Axz : Bxy : z^2]$ is not a linear projective automorphism of $\mathbb{P}^2$, but the automorphism defined by $[x : y : z] \mapsto [z/\alpha : y : x]$ conjugates $f_1$ to $f_2$.

**Step 2.** The only remaining case is when $f_i = (x^{-d}, F_i(x,y))$, for $i = 1, 2$, with

$$F_1(x,y) = \sum_{j=0}^d a_j(x)x^{-d}y^{d-j} \quad \text{and} \quad F_2(x,y) = \sum_{j=0}^d b_j(x)x^{-d}y^{d-j}$$

for some polynomial functions $a_j, b_j \in k[x]$ that satisfy $\deg(a_j), \deg(b_j) \leq j$ and $a_0b_0 \neq 0$. Writing the conjugacy equation $h \circ f_1 = f_2 \circ h$ and looking at the term of degree $d$ in $y$, we get the relation

$$Bx^{-md}a_0x^{-d}y^d = b_0(Ax)^{-d}(Bx^my)^d.$$

Comparing the degree in $x$ we get $-md - d = md - d$, hence $m = 0$. Moreover, $h$ conjugates $f_1^2$ to $f_2^2$; thus, by the first step, $h$ should be an affine automorphism since $m = 0$ (see Equation (8)).

**6. Endomorphisms of $(\mathbb{A}^1_\mathbb{k} \setminus \{0\})^2$.** Denote by $[x : y : z]$ the homogeneous coordinates of $\mathbb{P}^2_\mathbb{k}$ and by $(x, y)$ the coordinates of the open subset $V := (\mathbb{A}^1_\mathbb{k} \setminus \{0\})^2$ defined by $xy \neq 0, z = 1$. We write $f_i = a_i \circ g_d$, as in case (P3) of Section 3. Since $h$ is an automorphism of $(\mathbb{A}^1_\mathbb{k} \setminus \{0\})^2$, it is the composition $t_h \circ m_h$ of a
diagonal map \( t_h(x, y) = (ux, vy) \), for some pair \((u, v) \in (k^*)^2\), and a monomial map \( m_h(x, y) = (x^ay^b, x^cy^d) \), for some matrix

\[
M_h := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}).
\] (14)

Also, note that the group \( G_3 \subset \text{Bir}(\mathbb{P}^2_k) \) of permutations of the coordinates \([x : y : z]\) corresponds to a finite subgroup \( S_3 \) of \( \text{GL}_2(\mathbb{Z}) \).

Since \( m_h \) commutes to \( g_d \) and \( g_d \circ t_h = t_h^d \circ g_d \), the conjugacy equation is equivalent to

\[
t_h \circ (m_h \circ a_1 \circ m_h^{-1}) \circ (g_d \circ m_h) = a_2 \circ t_h^d \circ (g_d \circ m_h).
\] (15)

The automorphisms \( a_1 \) and \( a_2 \) are monomial maps, induced by elements \( A_1 \) and \( A_2 \) of \( S_3 \), and Equation (15) implies that \( M_h \) conjugates \( A_1 \) to \( A_2 \) in \( \text{GL}_2(\mathbb{Z}) \); indeed, the matrices can be recovered by looking at the action on the set of units \( wx^my^p \) in \( k(V) \) (or on the fundamental group \( \pi_1(V(C)) \) if \( k = \mathbb{C} \)). There are two possibilities:

(a) either \( A_1 = A_2 = \text{Id} \), there is no constraint on \( m_h \);

(b) or \( A_1 \) and \( A_2 \) are non-trivial permutations, they are conjugate by an element \( P \in S_3 \), and \( M_h = \pm A_2^j \circ P \), for some \( j \in \mathbb{Z} \).

In both cases, \( u \) and \( v \) are roots of unity (there order is determined by \( d \) and the \( A_i \)). Let \( P \) be the monomial transformation associated to \( P \); it is a permutation of the coordinates, hence an element of \( \text{Aut}(\mathbb{P}^2_k) \). Then, \( h'(x, y) = t_h \circ p \) is an element of \( \text{Aut}(\mathbb{P}^2_k) \) that conjugates \( f_1 \) to \( f_2 \).

7. An example in positive characteristic. – Assume that \( q = p^s \) with \( s \geq 2 \). Set \( G := xy^p + (x - 1)y \). Then,

\[
f_1(x, y) = (x^q, y^q + G(x, y))
\]
defines an endomorphism of \( \mathbb{A}^2 \) that extends to an endomorphism of \( \mathbb{P}^2 \).

Consider a polynomial \( P(x) \in \mathbb{F}_q[x] \) such that \( 2 \leq \deg(P) \leq \frac{q}{p} - 1 \). Observe that \( \deg(G) < \deg(G(x, y + P(x))) < q \). Then \( g(x, y) = (x, y + P(x)) \) is an automorphism of \( \mathbb{A}^2_k \) that conjugates \( f_1 \) to

\[
f_2(x, y) := g \circ f_1 \circ g^{-1}(x, y)
\]

\[
= (x^q, y^q + P(x)^q + G(x, y + P(x)) - P(x^q))
\]

\[
= (x^q, y^q + G(x, y + P(x))).
\] (16)

As \( f_1, f_2 \) is an endomorphism of \( \mathbb{A}^2 \) that extends to a regular endomorphism of \( \mathbb{P}^2 \) (here we use the inequality \( \deg(G(x, y + P(x))) < q \)).

Let us prove that \( f_1 \) and \( f_2 \) are not conjugate by any automorphism of \( \mathbb{P}^2 \). We assume that there exists \( h \in \text{PGL}_3(\mathbb{F}_q) \) such that \( h \circ f_1 = f_2 \circ h \) and seek a
Consider the pencils of lines through the point $[0 : 1 : 0]$ in $\mathbb{P}^2$; for $a \in \mathbb{F}_q$ we denote by $L_a$ the line $\{x = az\}$, and by $L_\infty$ the line $\{z = 0\}$. Then
\[
\{L_a : a \in \mathbb{F}_q \cup \{\infty\}\} = \{\text{lines } L \text{ such that } f_1^{-1}L = L\} \quad (17)
\]
\[
= \{\text{lines } L \text{ such that } f_2^{-1}L = L\}; \quad (18)
\]
in other words, the lines $L_a$ for $a \in \mathbb{F}_q \cup \{\infty\}$ are exactly the lines which are totally invariant under the action of $f_1$ (resp. of $f_2$). Since $h$ conjugates $f_1$ to $f_2$, it permutes these lines. In particular, $h$ fixes the point $[0 : 1 : 0]$, and if we identify $L_a \cap \mathbb{A}^2$ to $\mathbb{A}^1$ with its coordinate $y$ by the parametrization $y \mapsto (a, y)$ then $h$ maps $L_a$ to another line $L_{a'}$ in an affine way: $h(a, y) = (a', \alpha y + \beta)$.

Since $g$ conjugates $f_1$ to $f_2$ and $g$ fixes each of the lines $L_a$, we know that $f_1|_{L_a}$ is conjugated to $f_2|_{L_a}$ for every $a \in \mathbb{F}_q$; for $a = \infty$, both $f_1|_{L_\infty}$ and $f_2|_{L_\infty}$ are conjugate to $y \mapsto y^q$. Moreover
\begin{itemize}
  \item $a = \infty$ is the unique parameter such that $f_1|_{L_a}$ is conjugate to $y \mapsto y^q$ by an affine map $y \mapsto \alpha y + \beta$;
  \item $a = 0$ is the unique parameter such that $f_1|_{L_a}$ is conjugate to $y \mapsto y^q - y$ by an affine map;
  \item $a = 1$ is the unique parameter such that $f_1|_{L_a}$ is conjugate to $y \mapsto y^q + y^p$ by an affine map.
\end{itemize}

And the same properties hold for $f_2$. As a consequence, we obtain $h(L_\infty) = L_\infty$, $h(L_0) = L_0$ and $h(L_1) = L_1$; this means that there are coefficients $\alpha \in \mathbb{F}_q^*$ and $\beta, \gamma \in \mathbb{F}_q$ such that $h(x, y) = (x, \alpha y + \beta x + \gamma)$. Writing down the relation $h \circ f_1 = f_2 \circ h$ we obtain the relation
\[
\alpha y^q + \alpha G(x, y) + \beta x^q + \gamma = \alpha' y^q + \beta' x^q + \gamma'
\]
\[
+ G(x, \alpha y + \beta x + \gamma + P(x)). \quad (19)
\]
We note that $1 < \deg G(x, y) < \deg G(x, \alpha y + \beta x + \gamma + P(x)) < q$. Compare the terms of degree $q$, we get $\alpha y^q + \beta x^q = \alpha' y^q + \beta' x^q$. It follows that
\[
\alpha G(x, y) + \gamma = \gamma' + G(x, \alpha y + \beta x + \gamma + P(x)). \quad (20)
\]
Then $\deg G(x, y) = \deg G(x, \alpha y + \beta x + \gamma + P(x))$, which is a contradiction.

REFERENCES


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