# BERS AND HÉNON, PAINLEVÉ AND SCHRÖDINGER

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ABSTRACT. In this paper, we pursue the study of the holomorphic dynamics of mapping class groups on 2-dimensional character varieties, also called trace-maps dynamics in the literature, as initiated in [44] (see also [20]). We shall show that the dynamics of pseudo-Anosov mapping classes resembles in many ways the dynamics of Hénon mappings, and then apply this idea to answer open questions concerning (1) the geometry of discrete and faithful representations of free groups into $\mathsf{SL}(2, \mathbf{C})$, (2) the dynamics of Painlevé sixth equations, and (3) the spectrum of certain discrete Schrödinger operators.

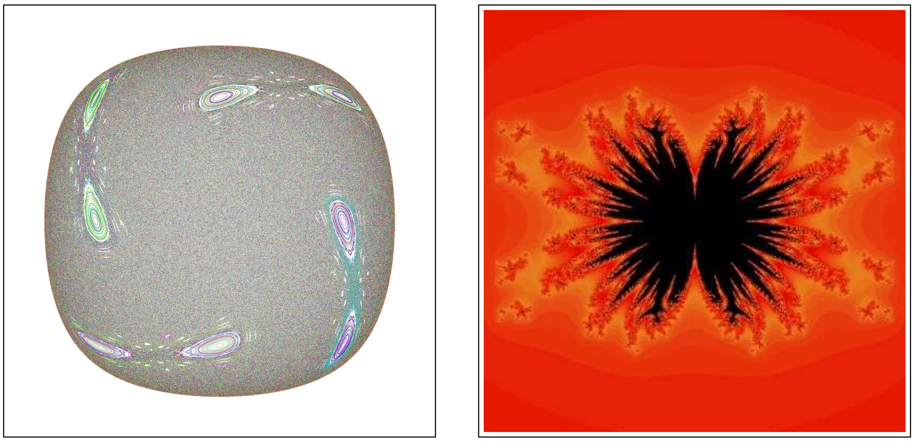

FIGURE 1. Dynamics on character surfaces. Left: Dynamics of an automorphism on the real part of a cubic surface (the surface is $S_{(0,0,0,2)}$, see below). Right: A slice of the set of complex points with bounded orbit (this is a slice through the origin for the Markov surface $S_{(0,0,0,0)}$).

CONTENTS



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## 1. INTRODUCTION

1.1. **Character variety and dynamics.** Let $\mathbb{T}_1$ be the once punctured torus. Its fundamental group is isomorphic to the free group $F_2 = \langle \alpha, \beta \,|\, \emptyset \rangle$, the commutator of $\alpha$ and $\beta$ corresponding to a simple loop around the puncture. Since any representation $\rho : F_2 \to \mathsf{SL}(2, \mathbf{C})$ is uniquely determined by $\rho(\alpha)$ and $\rho(\beta)$, the set $\mathsf{Rep}(\mathbb{T}_1)$ of representations of $\pi_1(\mathbb{T}_1)$ into $\mathsf{SL}(2, \mathbf{C})$ is isomorphic to $\mathsf{SL}(2, \mathbf{C}) \times \mathsf{SL}(2, \mathbf{C})$. The group $\mathsf{SL}(2, \mathbf{C})$ acts on this set by conjugation, preserving the three traces

$$x = \mathsf{tr}(\rho(\alpha)), \quad y = \mathsf{tr}(\rho(\beta)), \quad z = \mathsf{tr}(\rho(\alpha\beta)).$$

It turns out that the map $\chi : \mathsf{Rep}(\mathbb{T}_1) \to \mathbf{C}^3$, defined by $\chi(\rho) = (x, y, z)$, realizes an isomorphism between the algebraic quotient $\mathsf{Rep}(\mathbb{T}_1) /\!\!/ \mathsf{SL}(2, C)$, where $\mathsf{SL}(2, \mathbf{C})$ acts by conjugation, and the complex affine space $\mathbf{C}^3$. This quotient will be referred to as the *character variety of the once punctured torus.*

The automorphism group $\mathsf{Aut}(F_2)$ acts by composition on $\mathsf{Rep}(\mathbb{T}_1)$, and induces an action of the mapping class group

$$\mathsf{MCG}^*(\mathbb{T}_1) = \mathsf{Out}(F_2) = \mathsf{GL}(2, \mathbf{Z})$$

on the character variety $\mathbf{C}^3$ by polynomial diffeomorphisms. Since the conjugacy class of the commutator $[\alpha, \beta]$ is invariant under $\mathsf{Out}(F_2)$, this action preserves the level sets of the polynomial function $\mathsf{tr}(\rho[\alpha, \beta]) = x^2 + y^2 + z^2 - xyz - 2$. As a consequence, for each complex number $D$, we get a morphism from $\mathsf{MCG}^*(\mathbb{T}_1)$ to the group $\mathsf{Aut}(S_D)$ of polynomial diffeomorphisms of the surface $S_D$, defined by

$$x^2 + y^2 + z^2 = xyz + D.$$

The goal of this paper is to describe the dynamics of all mapping classes $f \in \mathsf{MCG}^*(\mathbb{T}_1)$ both on the complex surfaces $S_D(\mathbf{C})$ and on the real surfaces $S_D(\mathbf{R})$ when $D$ is a real number. More generally, *we shall study the dynamics of mapping classes on the character variety of the* 4*-punctured sphere,* but we restrict ourselves to the simpler case of the punctured torus in the introduction.

1.2. **Hénon type dynamics.** Let us fix an element $f$ of the mapping class group $\mathsf{MCG}^*(\mathbb{T}_1)$, that we view simultaneously as a matrix $M_f$ in $\mathsf{GL}(2,\mathbf{Z}) = \mathsf{Out}(F_2)$ or as a polynomial automorphism, still denoted $f$, of the affine space $\chi(\mathbb{T}_1) = \mathbf{C}^3$ preserving the family of cubic surfaces $S_D$. Let $\lambda(f)$ be the spectral radius of $M_f$, so that $f$ is pseudo-Anosov if and only if $\lambda(f) > 1$.

In [44, 16, 20], it is proved that the topological entropy of $f : S_D(\mathbf{C}) \to S_D(\mathbf{C})$ is equal to $\log(\lambda(f))$ for all choices of $D$. The dynamics of mapping classes with zero entropy is described in details in [36, 20]. In section 3, we shall show that the dynamics of pseudo-Anosov classes resembles the dynamics of Hénon automorphisms of the complex plane: All techniques from holomorphic dynamics that have been developed for Hénon automorphisms can be applied to understand the dynamics of mapping classes on the character surfaces $S_D(\mathbf{C})$ (a precise list of results is given in section 3.3).

This principle provides new tools to study the dynamics of mapping class groups on charater varieties. As a consequence, we shall get a positive answer to three different questions : The first one concerns quasi-fuchsian groups and the geometry of the quasi-fuchsian set, the second one concerns the spectrum of certain discrete Schrödinger operators, while the third question is related to Painlevé sixth equation.

1.3. **Quasi-Fuchsian spaces and a question of Goldman and Dumas.** First, we answer positively a question of Goldman and Dumas (see problem 3.5 in [38]), that we now describe.

When the parameter $D$ is equal to 2, the trace of $\rho[\alpha,\beta]$ vanishes, so that the representations $\rho$ with $\chi(\rho) \in S_2(\mathbf{C})$ send the commutator $[\alpha,\beta]$ to an element of order 4 in $\mathsf{SL}(2,\mathbf{C})$. This means that the surface $S_2$ indeed corresponds to representations of the group

$$G = \langle \alpha, \beta \,|\, [\alpha,\beta]^4 \rangle.$$

Let $\mathsf{DF}$ be the subset of $S_2(\mathbf{C})$ corresponding to discrete and faithful representations of $G$. Some of these representations are fuchsian: They come from the existence of hyperbolic metrics on $\mathbb{T}_1$ with an orbifold point of angle $\pi$ at the puncture. The interior of $\mathsf{DF}$ corresponds to quasi-fuchsian deformations of those fuchsian representations (see for example [50]).

Let us now consider the set of conjugacy classes of representations $\rho : G \to \mathsf{SU}(2)$. This set coincides with the unique compact connected component of $S_2(\mathbf{R})$ and is homeomorphic to a sphere $\mathbb{S}^2$. Typical representations into $\mathsf{SU}(2)$ have a dense image and, in this respect, are quite different from discrete faithful representations into $\mathsf{SL}(2,\mathbf{C})$.

The following theorem shows that orbits of the mapping class group may contain both types of representations in their closure.

**Theorem 1.1.** *Let $G$ be the finitely presented group $\langle \alpha,\beta \,|\, [\alpha,\beta]^4\rangle$. There exists a representation $\rho : G \to \mathsf{SL}(2,\mathbf{C})$, such that the closure of the orbit of its conjugacy class $\chi(\rho)$ under the action of $\mathsf{Out}(F_2)$ contains both*

- *the conjugacy class of at least one discrete and faithful representation $\rho' : G \to \mathsf{SL}(2,\mathbf{C})$,*
- *the whole set of conjugacy classes of $\mathsf{SU}(2)$-representations of $G$.*

This result answers positively and precisely the question raised by Dumas and Goldman. It also sheds light on questions raised by Bowditch (see [13], corollary 5.6 and the discussion thereafter). The strategy of proof is quite general and leads to many other examples; one of them is given in §4.3. The representations $\rho'$ which we choose for the proof are very special: They correspond to certain discrete representations provided by Thurston's hyperbolization theorem for mapping tori with pseudo-Anosov monodromy. The same idea may be used to describe DF in dynamical terms (see section 4). To sum up, *holomorphic dynamics turns out to be useful to understand the quasi-fuchsian locus and its Bers parameterization.*

1.4. **Real dynamics, discrete Schrödinger operators, and Painlevé VI equation.** The fact that the dynamics of mapping classes is similar to the dynamics of Hénon automorphisms will prove useful to study the real dynamics of mapping classes, i.e. the dynamics on the real part $S_D(\mathbf{R})$ when $D$ is a real number. The following theorem, which is the main result of section 5, answers a conjecture popularized by Kadanoff twenty five years ago (see [46], p. 1872). We refer to papers of Casdagli and Roberts for a nice mathematical introduction to the subject (see [21] and [54] and references therein).

**Theorem 1.2.** *Let $D$ be a real number. If $f \in \mathsf{MCG}^*(\mathbb{T}_1)$ is a pseudo-Anosov mapping class, the topological entropy of $f : S_D(\mathbf{R}) \to S_D(\mathbf{R})$ is bounded from above by $\log(\lambda(f))$, and the five following properties are equivalent*

- *the topological entropy of $f : S_D(\mathbf{R}) \to S_D(\mathbf{R})$ is equal to $\log(\lambda(f))$;*
- *all periodic points of $f : S_D(\mathbf{C}) \to S_D(\mathbf{C})$ are contained in $S_D(\mathbf{R})$;*
- *the topological entropy of $f : S_D(\mathbf{R}) \to S_D(\mathbf{R})$ is positive and the dynamics of $f$ on the set $K(f,\mathbf{R}) = \{m \in S_D(\mathbf{R}) \,|\, (f^n(m))_{n\in\mathbf{Z}}$ is bounded $\}$ is uniformly hyperbolic;*

- *the surface $S_D(\mathbf{R})$ is connected;*
- *the real parameter D is greater than or equal to* 4.

The main point is the fact that the dynamics is uniformly hyperbolic when $D \geq 4$. In particular, uniform hyperbolicity occurs simultaneously for all pseudo-Anosov mapping classes. Casdagli had a similar result for one explicit mapping class (linked to Fibonacci substitutions) when $D > 260$, and Damanik and Gorodetski recently extended it to the case where $D$ is close to 4 (see [21, 25]).

TABLE 1. Dynamics of pseudo-Anosov classes on $S_D(\mathbf{R})$

| values of parameter | real part of $S_D$ | dynamics on $K(f,\mathbf{R})$ |
|---|---|---|
| $D < 0$ | four disks | $K(f,\mathbf{R}) = \emptyset$ |
| $D = 0$ | four disks and a point | $K(f,\mathbf{R}) = \{(0,0,0)\}$ |
| $0 < D < 4$ | four disks and a sphere | non uniformly hyperbolic |
| $D = 4$ | the Cayley cubic | uniformly hyperbolic |
| $D > 4$ | a connected surface | uniformly hyperbolic |

As we shall explain in section 6, this may be used to study the spectrum of discrete Schrödinger operators, the potential of which is generated by a primitive substitution: We shall show that the *Hausdorff dimension of the spectrum of such operators is positive but strictly less than* 1 (see §6 for precise results).

This gives also examples of Painlevé VI equations with nice and rich monodromy (see §7), thereby answering a question of Iwasaki and Uehara in [43].

1.5. **Organization of the paper.** As mentioned above, we shall study the dynamics of the mapping class group of the four punctured sphere on its character variety; this includes the case of the once punctured torus as a particular case. Section 2 summarizes known useful results, fixes the notations, and describes the dynamics of mapping classes at infinity. Section 3 establishes a dictionary between the Hénon case and the case of character varieties, listing important consequences regarding the dynamics of mapping classes. This is applied in section 4 to study the quasi-fuchsian space. Section 5 describes the dynamics of mapping classes on the real algebraic surfaces $S_D(\mathbf{R})$, for $D \in \mathbf{R}$. This is certainly the most involved part of this paper. It requires a translation of most known facts for Hénon automorphisms to the case of character varieties, and a study of one parameter families of real polynomial automorphisms with maximal entropy. The proof of theorem 1.2, which is given in sections 5.2 and 5.3, could also be used in the study of families of Hénon mappings. We then apply theorem 1.2 to Schrödinger operators and Painlevé VI equations (§6 and 7).

1.6. **Acknowledgement.** This paper greatly benefited from discussions with Frank Loray, with whom I collaborated on a closely related article (see [20]). I also want to thank Eric Bedford, Cliff Earle, Bill Goldman, Katsunori Iwasaki, Robert MacKay, Yair Minsky, John Smillie, Takato Uehara and Karen Vogtmann for illuminating talks and useful discussions. Most of the content of this paper has been written while I was visiting Cornell University in 2006/2007, and part of it was already described during a conference of the ACI "Systèmes Dynamiques Polynomiaux" in 2004: I thank both institutions for their support.

## 2. THE CHARACTER VARIETY OF THE FOUR PUNCTURED SPHERE

This section summarizes known results concerning the character variety of a four punctured sphere and the action of its mapping class group on this algebraic variety. Most of these results can be found in [10], [44], and [20].

2.1. **The sphere minus four points.** Let $\mathbb{S}^2_4$ be the four punctured sphere. Its fundamental group is isomorphic to a free group of rank 3,

$$\pi_1(\mathbb{S}^2_4) = \langle \alpha, \beta, \gamma, \delta \,|\, \alpha\beta\gamma\delta = 1 \rangle,$$

where the four homotopy classes $\alpha$, $\beta$, $\gamma$, and $\delta$ correspond to loops around the puncture. Let $\mathsf{Rep}(\mathbb{S}^2_4)$ be the set of representations of $\pi_1(\mathbb{S}^2_4)$ into $\mathsf{SL}(2,\mathbf{C})$. Let us associate the 7 following traces to any element $\rho$ of $\mathsf{Rep}(\mathbb{S}^2_4)$,

$$\begin{array}{l} a = \mathsf{tr}(\rho(\alpha)) \quad ; \quad b = \mathsf{tr}(\rho(\beta)) \quad ; \quad c = \mathsf{tr}(\rho(\gamma)) \quad ; \quad d = \mathsf{tr}(\rho(\delta)) \\ x = \mathsf{tr}(\rho(\alpha\beta)) \quad ; \quad y = \mathsf{tr}(\rho(\beta\gamma)) \quad ; \quad z = \mathsf{tr}(\rho(\gamma\alpha)). \end{array}$$

The polynomial map $\chi : \mathsf{Rep}(\mathbb{S}^2_4) \to \mathbf{C}^7$ defined by $\chi(\rho) = (a,b,c,d,x,y,z)$ is invariant under conjugation, by which we mean that $\chi(\rho') = \chi(\rho)$ if $\rho'$ is conjugate to $\rho$ by an element of $\mathsf{SL}(2,\mathbf{C})$, and it turns out that the algebra of polynomial functions on $\mathsf{Rep}(\mathbb{S}^2_4)$ which are invariant under conjugation is generated by the components of $\chi$. Moreover, the components of $\chi$ satisfy the quartic equation

$$x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D, \tag{2.1}$$

in which the variables $A$, $B$, $C$, and $D$ are given by

$$\begin{aligned} A = ab + cd, \quad B = ad + bc, \quad C = ac + bd, \\ \text{and} \quad D = 4 - a^2 - b^2 - c^2 - d^2 - abcd. \end{aligned} \tag{2.2}$$

In other words, the algebraic quotient $\chi(\mathbb{S}^2_4) := \mathsf{Rep}(\mathbb{S}^2_4) /\!\!/ \mathsf{SL}(2, \mathbf{C})$ of $\mathsf{Rep}(\mathbb{S}^2_4)$ by the action of $\mathsf{SL}(2, \mathbf{C})$ by conjugation is isomorphic to the six-dimensional quartic hypersurface of $\mathbf{C}^7$ defined by equation (2.1).

The affine algebraic variety $\chi(\mathbb{S}^2_4)$ is called the *character variety of* $\mathbb{S}^2_4$. For each choice of four complex parameters $A$, $B$, $C$, and $D$, $S_{(A,B,C,D)}$ (or $S$ if there is no obvious possible confusion) will denote the cubic surface of $\mathbf{C}^3$ defined by the equation (2.1). The family of surfaces $S_{(A,B,C,D)}$, with $A$, $B$, $C$, and $D$ describing $\mathbf{C}$, will be denoted by $\mathsf{Fam}$.

2.2. **Automorphisms and the modular group $\Gamma_2^*$.** The (extended) mapping class group of $\mathbb{S}^2_4$ acts on $\chi(\mathbb{S}^2_4)$ by polynomial automorphisms: This defines a morphism

$$\begin{cases} \mathsf{Out}(\pi_1(\mathbb{S}^2_4)) & \to & \mathsf{Aut}(\chi(\mathbb{S}^2_4)) \\ \Phi & \mapsto & f_\Phi \end{cases}$$

such that $f_\Phi(\chi(\rho)) = \chi(\rho \circ \Phi^{-1})$ for any representation $\rho$.

The group $\mathsf{Out}(\pi_1(\mathbb{S}^2_4))$ contains a copy of $\mathsf{PGL}(2, \mathbf{Z})$ which is obtained as follows. Let $\mathbb{T} = \mathbf{R}^2/\mathbf{Z}^2$ be a torus and $\sigma$ be the involution of $\mathbb{T}$ defined by $\sigma(x,y) = (-x,-y)$. The fixed point set of $\sigma$ is the 2-torsion subgroup of $\mathbb{T}$. The quotient $\mathbb{T}/\sigma$ is homeomorphic to the sphere, $\mathbb{S}^2$, and the quotient map $\pi : \mathbb{T} \to \mathbb{T}/\sigma = \mathbb{S}^2$ has four ramification points, corresponding to the four fixed points of $\sigma$. The group $\mathsf{GL}(2, \mathbf{Z})$ acts linearly on $\mathbb{T}$ and commutes with $\sigma$. This yields an action of $\mathsf{PGL}(2, \mathbf{Z})$ on the sphere $\mathbb{S}^2$, which permutes the ramification points of $\pi$. Taking these four ramification points as the punctures of $\mathbb{S}^2_4$, we get a morphism

$$\mathsf{PGL}(2, \mathbf{Z}) \to \mathsf{MCG}^*(\mathbb{S}^2_4),$$

that turns out to be injective, with finite index image (see [11, 20]). As a consequence, $\mathsf{PGL}(2, \mathbf{Z})$ acts by polynomial transformations on $\chi(\mathbb{S}^2_4)$.

Let $\Gamma_2^*$ be the subgroup of $\mathsf{PGL}(2, \mathbf{Z})$ whose elements coincide with the identity modulo 2. This group coincides with the stabilizer of the fixed points of $\sigma$, so that $\Gamma_2^*$ acts on $\mathbb{S}^2_4$ and fixes its four punctures. Consequently, $\Gamma_2^*$ acts polynomially on $\chi(\mathbb{S}^2_4)$ and preserves the fibers of the projection

$$(a,b,c,d,x,y,z) \mapsto (a,b,c,d).$$

From this we obtain, for any choice of four complex parameters $(A,B,C,D)$, a morphism from $\Gamma_2^*$ to the group $\mathsf{Aut}(S_{(A,B,C,D)})$ of polynomial diffeomorphisms of the surface $S_{(A,B,C,D)}$.

**Theorem 2.1** (Èl'-Huti [31], see theorem 3.1 in [20] **).** *For any choice of* $A$, $B$, $C$, *and* $D$, *the morphism* $\Gamma_2^* \to \mathsf{Aut}(S_{(A,B,C,D)})$ *is injective and the index of its image is bounded from above by* 24. *For a generic choice of the parameters, this morphism is an isomorphism.*

To sum up, $\Gamma_2^*$ is a finite index subgroup of $\mathsf{MCG}^*(\mathbb{S}_4^2)$, its action on $\chi(\mathbb{S}_2^4)$ preserves the family of cubic surfaces $\mathsf{Fam}$, and, for all choices of parameters $(A,B,C,D)$, $\Gamma_2^*$ determines a finite index subgroup of $\mathsf{Aut}(S_{(A,B,C,D)})$. We shall therefore restrict our study to the dynamics of $\Gamma_2^*$ on those surfaces.

2.3. **Area form.** The area form $\Omega$, which is globally defined by the formulas

$$\Omega = \frac{dx\wedge dy}{2z+xy-C} = \frac{dy\wedge dz}{2x+yz-A} = \frac{dz\wedge dx}{2y+zx-B}$$

on $S\setminus \mathsf{Sing}(S)$, is almost invariant under the action of $\Gamma_2^*$, by which we mean that $f^*\Omega = \pm\Omega$ for any $f$ in $\Gamma_2^*$ (see [20]). In particular, the dynamics of mapping classes on each surface $S$ is conservative.

**Remark 2.2.** The cubic surfaces $S$, together with the action of $\Gamma_2^*$, are degenerate limits of K3 surfaces together with an (almost) area preserving action of $\Gamma_2^*$. We refer to [18] for actions on K3 surfaces.

2.4. **Compactification and automorphisms.** Let $S$ be any member of the family $\mathsf{Fam}$. The closure $\overline{S}$ of $S$ in $\mathbb{P}^3(\mathbf{C})$ is given by the cubic homogeneous equation $w(x^2+y^2+z^2)+xyz = w^2(Ax+By+Cz)+Dw^3$.

As a consequence, one easily proves that the trace of $\overline{S}$ at infinity does not depend on the parameters and coincides with the triangle $\Delta$ given by the equations

$$xyz = 0, \quad w = 0,$$

and, moreover, that the surface $\overline{S}$ is smooth in a neighborhood of $\Delta$ (all singularities of $\overline{S}$, if there are such, are contained in $S$). By definition, the three sides of $\Delta$ are the lines $D_x = \{x=0, w=0\}$, $D_y = \{y=0, w=0\}$ and $D_z = \{z=0, w=0\}$; the vertices are $v_x = [1:0:0:0]$, $v_y = [0:1:0:0]$ and $v_z = [0:0:1:0]$; the "middle points" of the sides are respectively $m_x = [0:1:1:0]$, $m_y = [1:0:1:0]$, and $m_z = [1:1:0:0]$.

Since the equation defining $S$ is of degree 2 with respect to the $x$ variable, each point $(x,y,z)$ of $S$ gives rise to a unique second point $(x',y,z)$. This procedure determines a holomorphic involution of $S$, namely

$$s_x(x,y,z) = (A - yz - x, y, z).$$

Geometrically, the involution $s_x$ corresponds to the following: If $m$ is a point of $\overline{S}$, the projective line which joins $m$ and the vertex $v_x$ of the triangle $\Delta$ intersects $\overline{S}$ on a third point; this point is $s_x(m)$. The same construction provides two more involutions $s_y$ and $s_z$, and therefore a subgroup

$$\mathcal{A} = \langle s_x, s_y, s_z \rangle$$

of the group $\mathsf{Aut}(S)$ of polynomial automorphisms of the surface $S$. Section 2 of [20] (see also [44]) shows that the group $\mathcal{A}$ coincides with the image of $\Gamma_2^*$ into $\mathsf{Aut}(S)$, that is obtained by the action of $\Gamma_2^* \subset \mathsf{MCG}^*(\mathbb{S}_4^2)$ on the character variety $\chi(\mathbb{S}_4^2)$. More precisely, $s_x$, $s_y$, and $s_z$ correspond respectively to the automorphisms determined by the following elements of $\Gamma_2^*$

$$r_x = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}, \quad r_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad r_z = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}.$$

In particular, Èl'-Huti's theorem shows that there are no non trivial relations between the three involutions $s_x$, $s_y$ and $s_z$, so that $\mathcal{A}$ is isomorphic to the free product of three copies of $\mathbf{Z}/2\mathbf{Z}$.

Since the action of $\Gamma_2^*$ and $\mathcal{A}$ coincide, we shall focus on the dynamics of $\Gamma_2^* = \mathcal{A}$ on the surfaces $S \in \mathsf{Fam}$.

2.5. **Notations and remarks.** The conjugacy class of a representation $\rho$ will be denoted $[\rho]$. In general, this conjugacy class is uniquely determined by its image $\chi(\rho)$ in the character variety $\chi(\mathbb{S}_4^2)$, and we shall identify $\chi(\rho)$ to $[\rho]$ (note, however, that $\chi(\rho)$ does not determine $[\rho]$ when the representation is reducible).

Automorphisms of surfaces $S_{(A,B,C,D)}$ will be denoted by standard letters, like $f$, $g$, $h$, ... ; the group $\mathcal{A}$ will be identified to its various realizations as subgroups of $\mathsf{Aut}(S_{(A,B,C,D)})$, where $(A,B,C,D)$ describes $\mathbf{C}^4$. If $M$ is an element of $\Gamma_2^*$, the automorphism associated to $M$ is denoted $f_M$; this provides an isomorphism between $\Gamma_2^*$ and each realization of $\mathcal{A}$. If $f$ is an automorphism of $S_{(A,B,C,D)}$ which is contained in $\mathcal{A}$, $M_f$ will denote the unique element of $\Gamma_2^*$ which corresponds to $f$. If $\Phi \in \mathsf{MCG}^*(\mathbb{S}_4^2)$ is a mapping class, the associated automorphism of the character variety will be denoted by $f_\Phi$.

The character surfaces $S_D$ that appeared in the introduction in the case of the once punctured torus are isomorphic to $S_{(0,0,0,D)}$ by a simultaneous change of signs of the variables $(x,y,z)$. As a consequence, the study of the dynamics on all character surfaces $S \in \mathsf{Fam}$ includes the case of the once punctured torus.

2.6. **Dynamics at infinity.** The group $\mathcal{A}$ also acts by birational transformations on the compactification $\overline{S}$ of $S$ in $\mathbb{P}^3(\mathbf{C})$. In this section, we describe the dynamics at infinity, i.e. on the triangle $\Delta$.

If $f$ is an element of $\mathcal{A}$, the birational transformation of $\overline{S}$ defined by $f$ is not everywhere defined. The set of its indeterminacy points is denoted by $\mathrm{Ind}(f)$; $f$ is said to be *algebraically stable* if, for all $n \geq 0$, $f^n$ does not contract any curve onto $\mathrm{Ind}(f)$ (see [56, 26] for this notion).

The group $\Gamma_2^*$ acts by isometries on the Poincaré half plane $\mathbb{H}$. Let $j_x$, $j_y$ and $j_z$ be the three points on the boundary of $\mathbb{H}$ with coordinates $0$, $-1$, and $\infty$ respectively. The three generators $r_x$, $r_y$, and $r_z$ of $\Gamma_2^*$ (see 2.4) are the reflections of $\mathbb{H}$ around the three geodesics which join respectively $j_y$ to $j_z$, $j_z$ to $j_x$, and $j_x$ to $j_y$. As a consequence, $\Gamma_2^*$ coincides with the group of symmetries of the tesselation of $\mathbb{H}$ by ideal triangles, one of which has vertices $j_x$, $j_y$ and $j_z$. This picture will be useful to describe the action of $\mathcal{A}$ on $\Delta$ (see [20], section 3, and references therein for a detailed picture).

First, one easily shows that the involution $s_x$ acts on the triangle $\Delta$ in the following way: The image of the side $D_x$ is the vertex $v_x$ and the vertex $v_x$ is blown up onto the side $D_x$ ; the sides $D_y$ and $D_z$ are invariant and $s_x$ permutes the vertices and fixes the middle points $m_y$ and $m_z$ of each of these sides. An analogous statement holds of course for $s_y$ and $s_z$. In particular, the action of $\mathcal{A}$ at infinity does not depend on the set of parameters $(A,B,C,D)$.

Beside the three involutions $s_x$, $s_y$ and $s_z$, three new elements of $\mathcal{A}$ play a particular role. These elements are

$$g_x = s_z \circ s_y, \quad g_y = s_x \circ s_z, \quad \text{and} \quad g_z = s_y \circ s_x.$$

They correspond to Dehn twists in the mapping class group, i.e. to parabolic elements of $\Gamma_2^*$. Each of them preserves one of the coordinate variables $x$, $y$ or $z$ respectively. The action of $g_x$ on $\Delta$ is the following: $g_x$ contracts both $D_y$ and $D_z \setminus \{v_y\}$ on $v_z$, and preserves $D_x$; its inverse contracts $D_y$ and $D_z \setminus \{v_z\}$ on $v_y$. In particular $\mathrm{Ind}(g_x) = v_y$ and $\mathrm{Ind}(g_x^{-1}) = v_z$. The action of $g_y$ and $g_z$ are similar, up to a permutation of the coordinates.

Let $f$ be any element of $\mathcal{A} \setminus \{\mathrm{Id}\}$ and $M_f$ be the corresponding element of $\Gamma_2^*$. If $M_f$ is elliptic, $f$ is conjugate to $s_x$, $s_y$ or $s_z$. If $M_f$ is parabolic, $f$ is conjugate to an iterate of $g_x$, $g_y$ or $g_z$ (see [20], proposition 3.2). In both cases, the action of $f$ on $\Delta$ has just been described.

If $M_f$ is hyperbolic, the isometry $M_f$ of $\mathbb{H}$ has two fixed points at infinity, an attracting fixed point $\omega(f)$ and a repulsive fixed point $\alpha(f)$, and the action of $f$ on $\Delta$ can be described as follows: The three sides of $\Delta$ are blown down on the vertex $v_x$ (resp. $v_y$ resp. $v_z$) if $\omega(f)$ is contained in the interval $[j_y, j_z]$ (resp. $[j_z, j_x]$, resp. $[j_x, j_y]$); the unique indeterminacy point of $f$ is $v_x$ (resp. $v_y$ resp. $v_z$) if $\alpha(f)$ is contained in $[j_y, j_z]$ (resp. $[j_z, j_x]$, resp. $[j_x, j_y]$). In particular $\mathrm{Ind}(f)$ coincides with $\mathrm{Ind}(f^{-1})$ if and only if $\alpha(f)$ and $\omega(f)$ are in the same connected component of $\partial\mathbb{H} \setminus \{j_x, j_y, j_z\}$; Up to a conjugacy in the group $\Gamma_2^*$, we can always assume that $\alpha(f)$ and $\omega(f)$ are in different components. As a consequence, we get the following result (see [20], section 4).

**Proposition 2.3.** *Let $S$ be any member of the family* $\mathsf{Fam}$. *Let $f$ be an element of $\mathcal{A}$. Assume that the element $M_f$ of $\Gamma_2^*$ that corresponds to $f$ is hyperbolic.*

- *The birational transformation $f : \overline{S} \to \overline{S}$ is algebraically stable if, and only if $f$ is a cyclically reduced composition of the three involutions $s_x$, $s_y$ and $s_z$ (in which each involution appears at least once). In particular, any hyperbolic element $f$ of $\mathcal{A}$ is conjugate to an algebraically stable element of $\mathcal{A}$.*
- *If $f$ is algebraically stable, $f^n$ contracts the whole triangle $\Delta \setminus Ind(f)$ onto $Ind(f^{-1})$ as soon as $n$ is a positive integer.*

2.7. **Topological entropy and types of automorphisms.** An element $f$ of $\mathcal{A}$ will be termed *elliptic, parabolic or hyperbolic,* according to the type of the isometry $M_f \in \Gamma_2^*$. By theorem B of [20] (see also [44] for another formula), the *topological entropy* $h_{top}(f)$ of $f : S_{(A,B,C,D)}(\mathbf{C}) \to S_{(A,B,C,D)}(\mathbf{C})$ does not depend on the parameters $(A,B,C,D)$ and is equal to the logarithm of the spectral radius $\lambda(f)$ of $M_f$:

$$h_{top(f)} = \log(\lambda(f)). \tag{2.3}$$

In particular, pseudo-Anosov mapping classes are exactly those with positive entropy on the character surfaces $S_{(A,B,C,D)}(\mathbf{C})$. As explained in the previous section, up to conjugacy, Dehn twists correspond to powers of $g_x$, $g_y$ or $g_z$, while finite order mapping classes correspond to $s_x$, $s_y$ or $s_z$.

**Remark 2.4.** This should be compared to the description of the group of polynomial automorphisms of $\mathbf{C}^2$. If $h$ is an element of $\mathsf{Aut}(\mathbf{C}^2)$, either the topological entropy is equal to $\log(d(h))$, where $d(h) \geq 2$ is an integer, or a conjugate $g \circ h \circ g^{-1}$ preserves the pencil of lines $y = c^{ste}$ (see §3.2 for references).

2.8. **The Cayley cubic.** The surface $S_{(0,0,0,4)}$ will play a central role in this paper. This surface is the unique element of $\mathsf{Fam}$ with four singularities, and is therefore the unique element of $\mathsf{Fam}$ that is isomorphic to the Cayley cubic (see [20]). We shall call it "the Cayley cubic" and denote it by $S_C$. This surface already appeared to be a crucial example in both [19] and [20].

This surface is isomorphic to the quotient of $\mathbf{C}^* \times \mathbf{C}^*$ by the involution $\eta(x,y) = (x^{-1}, y^{-1})$. The map

$$\pi_C(u,v) = -\left(u + \frac{1}{u}, v + \frac{1}{v}, uv + \frac{1}{uv}\right)$$

gives an explicit isomorphism between $(\mathbf{C}^* \times \mathbf{C}^*)/\eta$ and $S_C$: Fixed points of $\eta$, as $(-1,1)$, correspond to singular points of $S_C$. Multiplication of the coordinates by $-1$ then gives an isomorphism onto $S_4$ (which will also be refered to as "the" Cayley cubic).

The group $\mathsf{GL}(2,\mathbf{Z})$ acts on $\mathbf{C}^* \times \mathbf{C}^*$ by *monomial transformations*: If $M = (m_{ij})$ is an element of $\mathsf{GL}(2,\mathbf{Z})$, and if $(u,v)$ is a point of $\mathbf{C}^* \times \mathbf{C}^*$, then

$$(u,v)^M = (u^{m_{11}} v^{m_{12}}, u^{m_{21}} v^{m_{22}}).$$

This action commutes with $\eta$, so that $\mathsf{PGL}(2,\mathbf{Z})$ acts on the quotient $S_C$. The induced action coincides with the action of $\Gamma_2^* \subset \mathsf{MCG}(\mathbb{S}^2_4)$ on the character surface corresponding to parameters $(a,b,c,d) = (0,0,0,0)$ or $(2,2,2,-2)$, up to permutation of $a$, $b$, $c$, and $d$ and multiplication by $-1$ (see §2.1 for the significance of $a$, $b$, $c$, and $d$, and [20] for details). Changing signs of coordinates, we get the surface $S_4$, that is one of the character surfaces for the once punctured torus: It corresponds to reducible representations of $\pi_1(\mathbb{T}_1)$ (with $\mathsf{tr}(\rho[\alpha,\beta]) = 2$). Of course, the monomial action of $\mathsf{PGL}(2,\mathbf{Z})$ on $S_4$ coincides with the action of the mapping class group of $\mathbb{T}_1$ on the character surface $S_4$.

The product $\mathbf{C}^* \times \mathbf{C}^*$ retracts by deformation onto the 2-dimensional real torus $\mathbb{S}^1 \times \mathbb{S}^1$. The monomial action of $\mathsf{GL}(2,\mathbf{Z})$ preserves this torus: It acts "linearly" on this torus if we use the parameterization $u = e^{2i\pi s}$, $v = e^{2i\pi t}$. After deleting the four singularities of $S_C$, the real part $S_C(\mathbf{R})$ has five components, and the closure of the unique bounded component is the image of $\mathbb{S}^1 \times \mathbb{S}^1$ by $\pi_C$. The closure of the four unbounded components are images of the four subsets $\mathbf{R}^+ \times \mathbf{R}^+$, $\mathbf{R}^+ \times \mathbf{R}^-$, $\mathbf{R}^- \times \mathbf{R}^+$, and $\mathbf{R}^- \times \mathbf{R}^-$ of $\mathbf{C}^* \times \mathbf{C}^*$.

2.9. **Topology of the real part.** Benedetto and Goldman studied the various topologies that can occur for $S(\mathbf{R})$ (see [10]). Good examples to keep in mind are small deformations of the Cayley cubics (one can deform each singular point independently). There are two main results that we shall use repetitively in section 5. We state them in the case of smooth surfaces ; singular one are limits of smooth surfaces.

The first one characterizes connectedness. Using $(a,b,c,d)$ parameters (see section 2.1), $S(\mathbf{R})$ is connected if and only $(i)$ none of the parameters $a$, $b$, $c$, and $d$ is contained in the interval $(-2,2)$ and $(ii)$ the product $abcd$ is negative. In that case, the surface $S(\mathbf{R})$ is homeomorphic to a sphere minus four punctures. These conditions on $(a,b,c,d)$ define eight arcwise connected subsets of $\mathbf{R}^4$, that contain respectively the 8 points $(2\varepsilon_1, 2\varepsilon_2, 2\varepsilon_3, 2\varepsilon_4)$, with $\varepsilon_i = \pm 1$ and $\Pi\varepsilon_i = -1$. All these points correspond to the same surface $S_{(0,0,0,4)}$, *i.e.* to the Cayley cubic $S_C$. As a consequence, any connected surface $S(\mathbf{R})$ can be smoothly deformed to the Cayley cubic $S_C$ inside $\mathsf{Fam}$.

The second result describes bounded components: If $S(\mathbf{R})$ has a bounded connected component, then this component is unique, it is homeomorphic to a sphere, and $S(\mathbf{R})$ has four unbounded components, each of which is homeomorphic to an open disk.

**Remark 2.5.** The surface $S$ is singular if and only if one of the two following conditions occur (see [10], [42]): $(i)$ at least one of the parameters $a$, $b$, $c$, or $d$ equals $\pm 2$ ; $(ii)$ there is a reducible representation $\rho$ of $\pi_1(\mathbb{S}^2_4)$ with boundary traces $(a,b,c,d)$. This latter case occurs exactly when $\Delta(a,b,c,d) = 0$, where $\Delta$ is the polynomial

$$(2(a^2+b^2+c^2+d^2) - abcd - 16)^2 - (4-a^2)(4-b^2)(4-c^2)(4-d^2).$$

## 3. Elements with positive entropy

In this section, we describe the dynamics of hyperbolic elements in the group $\mathcal{A}$ on any complex surface $S_{(A,B,C,D)}(\mathbf{C})$ of our family $\mathsf{Fam}$.

Let $f$ be a hyperbolic element of $\mathcal{A}$. After conjugation by an element $h$ of $\mathcal{A}$, we can assume that $f$ is algebraically stable; in our context, this property means that, for any element $S$ of $\mathsf{Fam}$, the indeterminacy set of the birational transformation $\overline{f} : \overline{S} \dashrightarrow \overline{S}$ and the indeterminacy set of $\overline{f}^{-1}$ are two distinct vertices of the triangle at infinity $\Delta$ (see §2.6). In what follows, we shall assume that $f$ is algebraically stable and denote $\mathrm{Ind}(f^{-1})$ by $v_+$ and $\mathrm{Ind}(f)$ by $v_-$.

### 3.1. Attracting basin of $\mathbf{Ind}(f^{-1})$.

The birational transformation $\overline{f}$ is holomorphic in a neighborhood of $v_+$ and contracts $\Delta \setminus \{v_-\}$ on $v_+$. In particular, $\overline{f}$ contracts the two sides of $\Delta$ that contain $v_+$ on the vertex $v_+$. Using the terminology of [33], $\overline{f}$ determines a rigid, reducible, contracting germ near $v_+$.

**Theorem 3.1.** *If $f$ is an algebraically stable hyperbolic element of $\mathcal{A}$, there exist an element $N_f$ of $\mathsf{GL}(2,\mathbf{Z})$ with non negative entries which is conjugate to $M_f$ in $\mathsf{PGL}(2,\mathbf{Z})$, a neighborhood $\mathcal{U}$ of $v_+$ in $\overline{S}$, and a holomorphic diffeomorphism $\Psi_f^+ : \mathbb{D}\times\mathbb{D} \to \mathcal{U}$ such that $\Psi_f^+(0,0) = v_+$ and*

$$\Psi_f^+((u,v)^{N_f}) = f(\Psi_f^+(u,v))$$

*for all $(u,v)$ in the bidisk $\mathbb{D}\times\mathbb{D}$ (see §2.8 for monomial transformations).*

*Proof.* Let $\mathcal{U}$ be a small bidisk around $v_+$, in which the two sides of $\Delta$ correspond to the two coordinate axis. The fundamental group of $\mathcal{U}\setminus\Delta$ is isomorphic to $(\mathbf{Z}^2,+)$, with generators winding exactly once along the first (resp. the second) axis. The map $\overline{f}$ induces an endomorphism $N_f$ of this group. To prove that $N_f$ is conjugate to $\pm M_f$ in $\mathsf{GL}(2,\mathbf{Z})$, one argues as follows. First, in the case of the Cayley cubic,

$$\pi_C : \mathbf{C}^*\times\mathbf{C}^* \to \overline{S_C}\setminus\Delta$$

is a 2 to 1 covering, $\mathbf{C}^*\times\mathbf{C}^*$ retracts by deformation on the torus $\mathbb{S}^1\times\mathbb{S}^1$, and the action of $f$ on the fundamental group of $\mathcal{U}\setminus\Delta$ is therefore covered by the action of $M_f$ on $\pi_1(\mathbb{S}^1\times\mathbb{S}^1) = \mathbf{Z}\times\mathbf{Z}$. This implies that $N_f$ is conjugate to $M_f$ in $\mathsf{PGL}(2,Z)$. Since the general case is obtained from the Cayley case by a smooth deformation, this is true for any set of parameters $(A,B,C,D)$. Being conjugate to $\pm M_f$, the matrix $N_f$ is invertible.

Since $f$ is a rigid and reducible contracting germ near $v_+$, and since $N_f$ is invertible, a theorem of Dloussky and Favre asserts that $f$ is locally conjugate to the monomial transformation that $N_f$ determines (see class 6 of the classification, Table II, and page 483 in [33]). In particular, $f$ being a local contraction, $N_f$ has non negative entries, and the square of $N_f$ has positive entries.

The fact that the conjugacy $\Psi_f$ is defined on the whole bidisk will be part of the next proposition. □

Let $s(f)$ be the slope of the eigenline of the linear planar transformation $N_f$, which corresponds to the eigenvalue $1/\lambda(f)$; $s(f)$ is a negative real number. The basin of attraction of the origin for the monomial transformation $N_f$ is

$$\overline{\Omega(N_f)} = \{(u,v)\in\mathbf{C}^2 \mid \quad |v| < |u|^{s(f)}\}.$$

In particular, this basin contains the full bidisk. We shall denote by $\Omega(N_f)$ the intersection of $\overline{\Omega(N_f)}$ with $\mathbf{C}^*\times\mathbf{C}^*$. Similar notations will be used for the basin of attraction $\overline{\Omega(v_+)}$ of the point $v_+$ for $\overline{f}$ in $\overline{S}$, and for its intersection $\Omega(v_+)$ with $S$.

**Proposition 3.2.** *The conjugacy* $\Psi_f^+$ *extends to a biholomorphism between* $\Omega(N_f)$ *and* $\Omega(v_+)$.

*Proof.* Since the monomial transformation $N_f$ is contracting and $f : S \to S$ is invertible, we can extend $\Psi_f^+$ to $\Omega(N_f) \cap (\mathbf{C}^* \times \mathbf{C}^*)$ by the functional equation

$$\Psi_f^+(u,v) = f^{-n}(\Psi_f^+((u,v)^{N_f^n})),$$

where $n$ is large enough for $(u,v)^{N_f^n}$ to be in the initial domain of definition of $\Psi_f^+$. The map $\Psi_f^+ : \Omega(N_f) \cap (\mathbf{C}^* \times \mathbf{C}^*) \to S$ is a local diffeomorphism, the image of which coincides with the basin of attraction of $v_+$ in $S$. It remains to prove that the map $\Psi_f^+$ is injective. Assume that $\Psi_f^+(u_1,v_1) = \Psi_f^+(u_2,v_2)$. Then $f^n(\Psi_f^+(u_1,v_1)) = f^n(\Psi_f^+(u_2,v_2))$, and $\Psi_f^+((u_1,v_1)^{N_f^n}) = \Psi_f^+((u_1,v_1)^{N_f^n})$, for any $n$. Since $\Psi_f^+$ is injective in a neighborhood of the origin, and since the monomial transformation $N_f$ is also injective, one gets $(u_1,v_1) = (u_2,v_2)$. $\square$

In what follows, $\|\,.\,\|$ will denote the usual euclidean norm in $\mathbf{C}^3$.

**Corollary 3.3.** *Let* $f$ *be an algebraically stable hyperbolic element of* $\mathcal{A}$. *If* $m$ *is a point of* $S$ *with an unbounded forward orbit, then* $f^n(m)$ *goes to* $Ind(f^{-1})$ *when* $n$ *goes to* $+\infty$ *and*

$$\log \|f^n(m)\| \sim \lambda(f)^n.$$

*Proof.* First we apply the previous results to the study of $f^{-1}$ and its basin of attraction near $v_-$. Let us fix a small ball $B$ around $v_-$ in the surface $\overline{S}$. If $B$ is small enough, then $B$ is contained in the basin of attraction of $f^{-1}$: The orbit of a point $m_0 \in B$ by $f^{-1}$ stays in $B$ and converges towards $v_-$. Since $f$ contracts $\Delta \setminus \{v_-\}$ on $v_+$, there is a neighborhood $\mathcal{V} \subset \overline{S}$ of $\Delta \setminus B$ which is contained in the basin of attraction of $v_+$. Let $m$ be a point with unbounded orbit. Since $\mathcal{V} \cup B$ is a neighborhood of $\Delta$, the sequence $(f^n(m))$ will visit $\mathcal{V} \cup B$ infinitely many times. Let $n_1$ be the first positive time for which $f^{n_1}(m)$ is contained in $\mathcal{V} \cup B$. Let $n_2$ be the first time after $n_1$ such that $f^{n_2}(m)$ escapes $B$. Then $f^n(m)$ never comes back in $B$ for $n > n_2$. Pick a $n > n_2$ such that $f^n(m)$ is contained in $\mathcal{V} \cup B$. Then $f^n(m)$ is in $\mathcal{V}$ and therefore in the basin of $v_+$. This implies that the sequence $f^n(m)$ converges towards $v_+$. In order to study the growth of $\|f^n(m)\|$ in a neighborhood of $v_+$, we apply the conjugacy $\Psi_f^+$: What we now need to control is the growth of $\|(u,v)^{N_f^n}\|^{-1}$, and the result is an easy exercise using exponential coordinates $(u,v) = (e^s, e^t)$, in $\mathbb{D}^* \times \mathbb{D}^*$. $\square$

**Corollary 3.4** (see lemma 16 in [44]). *If $f$ is a hyperbolic element of $\mathcal{A}$ and $A, B, C,$ and $D$ are four complex numbers, $f$ does not preserve any algebraic curve in $S_{(A,B,C,D)}$.*

*Proof (see also lemma 16 in* [44]*).* Let us assume the existence of a set of parameters $(A,B,C,D)$ and of an $f$-invariant algebraic curve $E \subset S_{(A,B,C,D)}$. Let $\overline{E}$ be the Zariski-closure of $E$ in $\overline{S}_{(A,B,C,D)}(\mathbf{C})$; $f$ induces an automorphism $\overline{f}$ of the compact Riemann surface $\overline{E}$. Since $\mathbf{C}^3$ does not contain any 1-dimensional compact complex subvariety, $\overline{E}$ contains points at infinity. These points must coincide with $v_+$ and/or $v_-$. In particular, the restriction of $f$ to $\overline{E}$ has at least one superattracting (or superrepulsive) fixed point. This is a contradiction with the fact that $\overline{f} : \overline{E} \to \overline{E}$ is an automorphism. □

3.2. **Bounded orbits and Julia sets.** Let us consider the case of a polynomial diffeomorphism $h$ of the affine plane $\mathbf{C}^2$ with positive topological entropy (an automorphism of Hénon type). After conjugation by an element of $\mathsf{Aut}[\mathbf{C}^2]$, we may assume that $h$ is algebraically stable in $\mathbb{P}^2(\mathbf{C})$. In that case, the dynamics of $h$ at infinity also exhibits two attracting fixed points, one for $h$, $w_+$, and one for $h^{-1}$, $w_-$, but there are three differences with the dynamics of hyperbolic elements of $\mathcal{A}$: The exponential escape growth rate is an integer $d(h)$ (while $\lambda(f)$ is an irrational quadratic integer), the model to which $h$ is conjugate near $w_+$ is not invertible, and the conjugacy $\Psi_h$ is a covering map of infinite degree between the basins of attraction. We refer the reader to [41], [33] and [40] for an extensive study of this situation. Beside these differences, we shall see that the dynamics of hyperbolic elements of $\mathcal{A}$ is similar to the dynamics of Hénon automorphisms. In analogy with the Hénon case, let us introduce the following definitions:

- $K^+(f)$ is the set of bounded forward orbits. This is also the set of points $m \in S$, for which $(f^n(m))$ does not converge to $v_+$ when $n$ goes to $+\infty$.
  $K^-(f)$ is the set of bounded backward orbits, and $K(f) = K^-(f) \cap K^-(f)$.
- $J^+(f)$ is the boundary of $K^+(f)$, $J^-(f)$ is the boundary of $K^-(f)$, and $J(f)$ is the subset of $\partial K(f)$ defined by $J(f) = J^-(f) \cap J^+(f)$. The set $J(f)$ will be called *the Julia set of $f$*.
- $J^*(f)$ is the closure of the set of saddle periodic points of $f$ (see below).

3.3. **Green functions and dynamics.** We define the Green functions of $f$ by

$$G_f^+(m) = \lim_{n\to+\infty} \frac{1}{\lambda(f)^n} \log^+ \| f^n(m)\|, \tag{3.1}$$

$$G_f^-(m) = \lim_{n\to+\infty} \frac{1}{\lambda(f)^n} \log^+ \| f^{-n}(m)\|. \tag{3.2}$$

By proposition 3.2 and its corollary, both functions are well defined and the zero set of $G_f^\pm$ coincides with $K^\pm(f)$. Moreover, the convergence is uniform on compact subsets of $S$. Since $\log^+ \| \,.\, \|$ is a pluri-subharmonic function, $G_f^+$ (resp. $G_f^-$) is pluri-subharmonic and is pluri-harmonic on the complement of $K^+(f)$ (resp. $K^-(f)$) (see [6, 34, 56] for the details of the proof). These functions satisfy the invariance properties

$$G_f^+ \circ f = \lambda(f) G_f^+ \quad \text{and} \quad G_f^- \circ f = \lambda(f)^{-1} G_f^- \tag{3.3}$$

The following results have been proved for Hénon mappings; we list them with appropriate references, in which the reader can find a proof which applies to our context (see also [18], [3], [30], [56] for similar contexts).

(1) $G_f^+$ and $G_f^-$ are Hölder continuous (see [27], sections 2.2, 2.3). The currents

$$T_f^+ = dd^c G_f^+ \quad \text{and} \quad T_f^- = dd^c G_f^- \tag{3.4}$$

are closed and positive, and $f^* T_f^\pm = \lambda(f)^\pm T_f^\pm$. By [6], section 3, the support of $T_f^+$ is $J^+(f)$, the support of $T_f^-$ is $J^-(f)$ (see also [56]).

(2) Since the potentials $G_f^+$ and $G_f^-$ are continuous, the product

$$\mu_f = T^+ \wedge T^- \tag{3.5}$$

is a well defined positive measure, and is $f$-invariant. Multiplying $G_f^+$ and $G_f^-$ by positive constants, we can, and we shall assume that $\mu_f$ is a probability measure. (see [6], section 3)

(3) The topological entropy of $f$ is $\log(\lambda(f))$ and the measure $\mu_f$ is the unique $f$-invariant probability measure with maximal entropy. (see [5], section 3, and [3, 28] for more general results)

(4) If $m$ is a saddle periodic point of $f$, its unstable (resp. stable) manifold $W^u(m)$ (resp. $W^s(m)$) is parameterized by $\mathbf{C}$. Let $\xi : \mathbf{C} \to S$ be such a parameterization of $W^u(m)$ with $\xi(0) = m$. Let $\mathbb{D} \subset \mathbf{C}$ be the unit disk, and let $\chi$ be a smooth non negative function on $\xi(\mathbb{D})$, with $\chi(m) > 0$ and $\chi = 0$ in a neighborhood of $\xi(\partial\mathbb{D})$. Let $[\xi(\mathbb{D})]$ be the

current of integration on $\xi(\mathbb{D})$. The sequence of currents

$$\frac{1}{\lambda(f)^n} f_*^{-n}(\chi.[\xi(\mathbb{D})])$$

weakly converges toward a positive multiple of $T_f^-$. Unstable (resp. stable) manifolds are dense in the support $J^-(f)$ (resp. $J^+(f)$) of $T^-(f)$ (resp. $T^+(f)$) (see [7], sections 2 and 3, [34])

(5) By corollary 3.4, periodic points of $f$ are isolated. The number of periodic points of period $N$ grows like $\lambda(f)^N$. Most of them are hyperbolic saddle points: If $\mathcal{P}(f,N)$ denotes either the set of periodic points with period $N$ or the set of periodic saddle points of period $N$, then

$$\frac{1}{|\mathcal{P}(f,N)|} \sum_{m\in\mathcal{P}(f,N)} \delta_m \to \mu_f$$

where the convergence is a weak convergence in the space of probability measures on compact subsets of $S$. (see [5], [4], and [30])

(6) The support $J^*(f)$ of $\mu_f$ simultaneously coincides with the Shilov boundary of $K(f)$ and with the closure of periodic saddle points of $f$. In particular, any periodic saddle point of $f$ is in the support of $\mu_f$. If $p$ and $q$ are periodic saddle points, then $J^*(f)$ coincides with the closure of $W^u(p)\cap W^s(q)$. (see [5] and [4])

(7) Since $f$ is area preserving (see §2.3), the interior of $K(f)$, $K^+(f)$ and $K^-(f)$ coincide. In particular, the interior of $K^+(f)$ is a bounded open subset of $S(\mathbf{C})$. (see lemma 5.5 of [6])

## 4. THE QUASI-FUCHSIAN LOCUS AND ITS COMPLEMENT

In this section, we shall mostly restrict the study to the case of the once punctured torus with a cusp, and provide hints for more general statements. We therefore consider the family $S_D$ and use notations from section 1.1.

### 4.1. **Quasi-fuchsian space and Bers' parameterization.**

Let $\mathbb{T}_1$ be a once punctured torus. Let $\mathsf{Teich}(\mathbb{T}_1)$ be the Teichmüller space of complete hyperbolic metrics on $\mathbb{T}_1$ with finite area $2\pi$, or equivalently with a cusp at the puncture: $\mathsf{Teich}(\mathbb{T}_1)$ is isomorphic, and will be identified, to the upper half plane $\mathbb{H}^+$. The dynamics of $\mathsf{MCG}(\mathbb{T}_1)$ on $\mathsf{Teich}(\mathbb{T}_1)$ is conjugate to the usual action of $\mathsf{PSL}(2,\mathbf{Z})$ on $\mathbb{H}^+$.

Any point in the Teichmüller space gives rise to a representation $\overline{\rho}: F_2 \to \mathsf{PSL}(2,\mathbf{R})$ that can be lifted to four distinct representations into $\mathsf{SL}(2,\mathbf{R})$. The cusp condition gives rise to the same equation $\mathsf{tr}(\rho[\alpha,\beta]) = -2$ for any of these four representations. This provides four embeddings of the Teichmüller space into the surface $S_0(\mathbf{R})$: The four images are the four unbounded components of $S_0(\mathbf{R})$, each of which is diffeomorphic to $\mathbb{H}^+$; apart from these four components, $S_0(\mathbf{R})$ contains an isolated singularity at the origin. This singular point corresponds to the conjugacy class of the representation $\rho_q$, defined by

$$\rho_q(\alpha) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \rho_q(\beta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.1}$$

Its image coincides with the quaternionic group of order eight. The mapping class group of the torus acts on $S_0(\mathbf{R})$, preserves the origin and the connected component

$$S_0^+(\mathbf{R}) = S_0(\mathbf{R}) \cap (\mathbf{R}^+)^3,$$

and permutes the remaining three components.

Let $\mathsf{DF} \subset S_0(\mathbf{C})$ be the set of conjugacy classes of discrete and faithful representations $\rho : F_2 \to \mathsf{SL}(2,\mathbf{C})$ with $\mathsf{tr}(\rho[\alpha,\beta]) = -2$. This set is composed of four distinct connected components, one of them, $\mathsf{DF}^+$, containing $S_0^+(\mathbf{R})$. The component $S_0^+(\mathbf{R})$ is made of conjugacy classes of fuchsian representations, and the set $\mathsf{QF}$ of their quasi-fuchsian deformations coincides with the interior of $\mathsf{DF}^+$ (see [50], and references therein).

Let $\mathbb{T}_1'$ be the once punctured torus with the opposite orientation. Bers's parameterization of the space of quasi-fuchsian representations provides a holomorphic bijection

$$\mathsf{Bers} : \mathsf{Teich}(\mathbb{T}_1) \times \mathsf{Teich}(\mathbb{T}_1') \to \mathsf{Int}(\mathsf{DF}^+).$$

We may identify $\mathsf{Teich}(\mathbb{T}_1)$ with the upper half plane $\mathbb{H}^+$ and $\mathsf{Teich}(\mathbb{T}_1')$ with the lower half plane $\mathbb{H}^-$. The group $\mathsf{PSL}(2,\mathbf{Z})$ acts on $\mathbb{P}^1(\mathbf{C})$, preserving $\mathbb{P}^1(\mathbf{R})$, $\mathbb{H}^+$, and $\mathbb{H}^-$. In particular, $\mathsf{MCG}(\mathbb{T}_1) = \mathsf{SL}(2,\mathbf{Z})$ acts diagonally on

$$\mathsf{Teich}(\mathbb{T}_1) \times \mathsf{Teich}(\mathbb{T}_1') = \mathbb{H}^+ \times \mathbb{H}^-.$$

With these identifications, the map Bers conjugates the diagonal action of $\mathsf{MCG}(\mathbb{T}_1)$ on $\mathbb{H}^+ \times \mathbb{H}^-$ with its action on the character variety: If $\Phi$ is a mapping class and $f_\Phi$ is the automorphism of $S_0$ which is determined by $\Phi$, then

$$\mathsf{Bers}(\Phi(X), \Phi(Y)) = f_\Phi(\mathsf{Bers}(X,Y))$$

for any $(X,Y)$ in $\mathbb{H}^+ \times \mathbb{H}^-$. It conjugates the action of $\mathsf{MCG}(\mathbb{T}_1)$ on the set

$$\{(z_1,z_2) \in \mathbb{H}^+ \times \mathbb{H}^- | z_1 = \overline{z_2}\}$$

with the corresponding action on $S_0^+(\mathbf{R})$. The Bers map extends up to the boundary of $\mathbb{H}^+ \times \mathbb{H}^-$ minus its diagonal (we shall call it the restricted boundary, and denote it by $\partial^*(\mathbb{H}^+ \times \mathbb{H}^-)$). Minsky proved in [49] that Bers induces a continuous bijection from $\partial^*(\mathbb{H}^+ \times \mathbb{H}^-)$ to the boundary of $\mathsf{DF}^+$.

4.2. **Mapping torus and fixed points (see** [48]**).** Let $\Phi \in \mathsf{MCG}(\mathbb{T}_1)$ be a pseudo-Anosov mapping class. Let $X_\Phi$ be the mapping torus determined by $\Phi$: The threefold $X_\Phi$ is obtained by suspension of $\mathbb{T}_1$ over the circle, with monodromy $\Phi$. Thurston's hyperbolization theorem tells us that $X_\Phi$ can be endowed with a complete hyperbolic metric of finite volume. This provides a discrete and faithful representation

$$\rho_\Phi : \pi_1(X_f) \to \mathsf{Isom}(\mathbb{H}^3) = \mathsf{PSL}(2, \mathbf{C})$$

If we restrict $\rho_\Phi$ to the fundamental group of the torus fiber of $X_\Phi$, and if we choose the appropriate lift to $\mathsf{SL}(2, \mathbf{C})$, we get a point $[\rho_\Phi]$ in $\mathsf{DF}^+ \subset S_0(\mathbf{C})$ which is fixed by the automorphism $f_\Phi$. Let $\alpha(\Phi)$ (resp. $\omega(\Phi)$) be the repulsive (resp. attracting) fixed point of $\Phi$ on the boundary of $\mathsf{Teich}(\mathbb{T}_1)$. Since $(\alpha(\Phi), \omega(\Phi))$ is in the restricted boundary, and Bers is a continuous bijective conjugacy $\partial^*(\mathbb{H}^+ \times \mathbb{H}^-)$, we have

$$\mathsf{Bers}(\alpha(\Phi), \omega(\Phi)) = [\rho_\Phi].$$

The fixed point $(\omega(\Phi), \alpha(\Phi))$ provides a second fixed point on the boundary of $\mathsf{DF}^+$: This point may be obtained by the same construction with $\Phi^{-1}$ in place of $\Phi$. In [48], McMullen proved that $[\rho_\Phi]$ is a hyperbolic fixed point of $f_\Phi$. The stable and unstable manifolds of $f_\Phi$ at $[\rho_\Phi]$ intersect $\mathsf{DF}^+$ along its boundary,

$$W^u([\rho_\Phi]) \cap \mathsf{DF}^+ = \mathsf{Bers}(\{\alpha(\Phi)\} \times \overline{\mathbb{H}^-} \setminus \{(\alpha(\Phi), \alpha(\Phi))\}), \quad (4.2)$$
$$W^s([\rho_\Phi]) \cap \mathsf{DF}^+ = \mathsf{Bers}(\overline{\mathbb{H}^+} \times \{\omega(\Phi)\} \setminus \{(\omega(\Phi), \omega(\Phi))\}). \quad (4.3)$$

In particular, the union of stable manifolds $W^s([\rho_\Phi]) \cap \mathsf{DF}^+$, where $\Phi$ describes the set of pseudo-Anosov mapping classes, form a dense subset of $\partial\mathsf{DF}^+$.

**Remark 4.1.** Each pseudo-Anosov class $\Phi$ determines an automorphism $f_\Phi$, and therefore a subset $K^+(f_\Phi)$ of $S_0(\mathbf{C})$. The complement $\Omega^+(f_\Phi)$ of $K^+(f_\Phi)$ is open: It coïncides with the bassin of attraction of $f_\Phi$ at infinity. Since the dynamics of $f_\Phi$ on $\mathsf{QF}$ is conjugate to the dynamics of $\Phi$ on $\mathsf{Teich}(\mathbb{T}_1) \times \mathsf{Teich}(\mathbb{T}_1')$, the interior of $\mathsf{DF}^+$ is contained in the intersection

$$\Omega(\mathsf{MCG}(\mathbb{T}_1)) := \bigcap_\Phi \Omega^+(f_\Phi)$$

where $\Phi$ describes the set of pseudo-Anosov classes in the mapping class group $\mathsf{MCG}(\mathbb{T}_1) = \mathsf{SL}(2,\mathbf{Z})$. Since stable manifolds are dense in the boundary of $\mathsf{DF}$, one gets the following result: *The quasi fuchsian locus* $\mathsf{Int}(\mathsf{DF}^+)$ *is a connected component of the interior of* $\Omega(\mathsf{MCG}(\mathbb{T}_1))$. Since there are four copies of $\mathsf{QF}$ in $S_0(\mathbf{C})$, this provides four connected components. The question remains to decide whether there are other connected components (see [13]).

4.3. **Two examples.** The action of the mapping class group on the complement of $\mathsf{DF}$ is not well understood yet. We refer to Goldman's list of questions [38] for interesting conjectures and to Bowditch's article [13] for important advances and a discussion of this action. We now present two interesting orbits in the complement of $\mathsf{DF}$.

**Theorem 4.2.** *Let $\Phi$ be any pseudo-Anosov mapping class and $[\rho_\Phi]$ be one of the two fixed points of $f_\Phi$ on the boundary of $\mathsf{DF}^+ \subset S_0(\mathbf{C})$. There exists a representation $\rho_0 : \pi_1(\mathbb{T}_1) \to \mathsf{SL}(2,\mathbf{C})$, with $[\rho_0] \in S_0(\mathbf{C})$, such that*

- *the sequence $(f_\Phi)^n[\rho_0]$ converges toward the discrete and faithful representation $[\rho_\Phi]$ when n goes to $+\infty$;*
- *the closure of the mapping-class group orbit of $[\rho_0]$ contains the origin $(0,0,0)$, i.e. the conjugacy class of the finite representation $\rho_q$.*

**Remark 4.3** (see [47])**.** The Kobayashi semi-distance on a complex manifold $M$ is defined as follows. Let $m$ and $m'$ be two points of $M$. Then, $\mathsf{dist}_K(m,m')$ is the infimum of the sum of the Poincaré distances $\mathsf{dist}_P(x_i,y_i)$, where the infimum is taken over all chains of holomorphic disks $\xi_i : \mathbb{D} \to M$, $k > 0$, $1 \leq i \leq k$, such that $\xi_1(x_1) = m$, $\xi_i(y_i) = \xi_{i+1}(x_{i+1})$ and $\xi_k(y_k) = m'$. This semi-distance is invariant under the group of holomorphic diffeomorphisms of $M$. Schwarz lemma implies that $\mathsf{dist}_K$ is indeed a distance when $M$ is a bounded, open, and connected subset of an affine variety.

**Remark 4.4.** According to a theorem of Bowditch (see theorem 5.5 of [13]), there exists a neighborhood $U_B$ of the origin in $S_0(\mathbf{C})$ with the property that any mapping class group orbit starting in $U_B$ contains the origin in its closure.

*Proof.* The fixed point $[\rho_\Phi]$ is hyperbolic, with a stable manifold $W^s([\rho_\Phi])$. The origin $(0,0,0)$ is the unique singular point of $S_0(\mathbf{C})$. It corresponds to the representation $[\rho_q]$ which is defined by equation (4.1). This point is fixed by $f_\Phi$, and a direct computation shows that the differential of $f_\Phi$ at the origin has finite order (order 1 or 2).

From section 3.3, the interior of $K^+(f)$ coincides with the interior of $K^-(f_\Phi)$ and is therefore an $f_\Phi$-invariant bounded open subset of $S_0(\mathbf{C})$. In particular, $\mathsf{Int}(K^+(f_\Phi))$ is Kobayashi hyperbolic, and the Kobayashi distance is $f_\Phi$ invariant. Consequently, if $[\rho_q]$ is in the interior of $(K^+(f_\Phi))$, then $f_\Phi$ is locally linearizable around the origin $[\rho_q]$. Since $(Df_\Phi)_{[\rho_q]}$ has finite order, $f_\Phi$ would have finite order too. This contradiction shows that $[\rho_q]$ is not in the interior of $K^+(f_\Phi)$.

We know that $W^s([\rho_\Phi])$ is dense in the boundary of $K^+(f_\Phi)$ (see §3.3). Since $[\rho_q]$ is in $\partial K^+(f_\Phi)$, $W^s([\rho_\Phi])$ intersects the Bowditch's neighborhood $U_B$. The previous remark shows that any point $[\rho_0]$ in $W^s([\rho_\Phi]) \cap U_B$ satisfies the properties of the theorem. □

*Proof of theorem 1.1.* Let us consider the surface $S_2(\mathbf{C})$, that corresponds to representations $\rho : G \to \mathsf{SL}(2,\mathbf{C})$, where $G = \langle \alpha, \beta | [\alpha,\beta]^4 \rangle$ (see §1.3). Its equation is $x^2+y^2+z^2 = xyz+2$. Let $\Psi$ be the mapping class

$$\Psi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Its action on $S_2(\mathbf{C})$ is given by the polynomial transformation

$$f_\Psi(x,y,z) = (z, yz-x, z(yz-x)-y).$$

The set of fixed points of $f_\Psi$ on $S_2(\mathbf{C})$ is made of four points $(x, x/(x-1), x)$, where $x$ describes the solutions of the quartic equation

$$x^4 - 3x^3 + x^2 + 4x - 2 = 0.$$

This equation is the product of $P(x) = x^2+\chi x+\chi$ and $Q(x) = x^2+(\chi-3)x+(3-\chi)$ where $\chi = (3+\sqrt{17})/2$. The roots of $P$ give rise to two complex conjugate fixed points, while the roots of $Q$ give two real fixed points. Roots of $P$ are given in [48], section 3.7, and roots of $Q$ are equal to

$$x = \frac{3-\sqrt{17}}{4} \pm \frac{\sqrt{2+2\sqrt{17}}}{4}$$

i.e. to $\approx 0.52$ and $\approx -1.1$.

As explained for example in [48], section 3.7, the surface $S_2(\mathbf{C})$ contains an $f_\Psi$-invariant open subset corresponding to quasi-fuchsian deformations of the fuchsian groups obtained from the existence of hyperbolic metrics on $\mathbb{T}_1$ with an orbifold point of angle $\pi$ at the puncture. Thurston's hyperbolization theorem provides a hyperbolic fixed point $[\rho_\Psi]$ of $f_\Psi$ on the boundary of this set: The representation $\rho_\Psi : G \to \mathsf{SL}(2,\mathbf{C})$ is discrete and faithful and comes from the existence of a hyperbolic structure on the complement of the figure eight knot, with an orbifold structure along the knot. This fixed point is one of the two complex conjugate fixed points (the second one corresponding to $\rho_{\Psi^{-1}}$).

The subset of $S_2(\mathbf{C})$ corresponding to conjugacy classes of $\mathsf{SU}(2)$-representations coincides with the unique bounded connected component of $S_2(\mathbf{R})$, and is homeomorphic to a sphere (see [35], figure 4). This component is $f_\Psi$-invariant, and the two fixed points of $f$ corresponding to roots of $Q(x)$ are located on this bounded component of $S_2(\mathbf{R})$.

The differential of $f : \mathbf{C}^3 \to \mathbf{C}^3$ at a fixed point has trace $2x^2/(x-1)$. One of its eigenvalues is equal to $1$, because $f$ preserves the rational function $x^2+y^2+z^2-xyz$. This implies that the sum of the remaining eigenvalues is $2x^2/(x-1)-1$, while their product is $1$, because $f$ is area preserving. As a consequence, the fixed point corresponding to the root $x \approx -1.08...$ is a saddle fixed point (the trace being $< -2$). Let $[\rho_{\mathsf{SU}}]$ be this fixed point, and let $W^s([\rho_{\mathsf{SU}}])$ and $W^u([\rho_\Psi])$ be the stable and unstable manifolds of $f_\Psi$ through $[\rho_{\mathsf{SU}}]$ and $[\rho_\Psi]$.

From property (6) in section 3.3, we know that $W^s([\rho_{\mathsf{SU}}])$ intersects $W^u([\rho_\Psi])$. Let $[\rho_0]$ be one of these intersection points. The $f_\Psi$-orbit of $[\rho_0]$ contains both $[\rho_\Psi]$ and $[\rho_{\mathsf{SU}}]$.

Finite orbits of $\mathsf{MCG}(\mathbb{T}_1)$ are listed in [29] and correspond to finite subgroups of $\mathsf{SU}(2)$ ; the point $[\rho_{\mathsf{SU}}]$ does not appear in the list. From this we deduce that the mapping class group orbit of $[\rho_{\mathsf{SU}}]$ is infinite and dense in the component of $\mathsf{SU}(2)$-representations (see [36], [37], or also [32, 17, 20] for related ideas). This implies that the closure of the orbit of $[\rho_{\mathsf{SU}}]$ contains both $[\rho_\Psi]$ and the $\mathsf{SU}(2)$-component of $S_2(\mathbf{R})$. □

## 5. Real Dynamics of Hyperbolic Elements

In this section, we study the dynamics of hyperbolic elements on the real surfaces $S_{(A,B,C,D)}(\mathbf{R})$ when the parameters are real numbers. The main goal of this section is to prove theorem 5.10 below, which extends, and precises, theorem 1.2.

### 5.1. **Maximal entropy.**

Let us fix a hyperbolic element $f \in \mathcal{A}$. If the parameters $(A,B,C,D)$ are real, we get two dynamical systems: The first one takes place on the complex surface $S(\mathbf{C})$ and its main stochastic properties have been listed in section 3.3; the second one is induced by the restriction of $f$ to the real part $S(\mathbf{R})$. From time to time, we shall use the notation $f_\mathbf{R}$ to denote the restriction of $f$ to $S(\mathbf{R})$. For example, we shall say that $f_\mathbf{R}$ *has maximal entropy* if the entropy of $f : S(\mathbf{R}) \to S(\mathbf{R})$ is equal to the topological entropy of $f : S(\mathbf{C}) \to S(\mathbf{C})$, *i.e.* to $\log(\lambda(f))$.

**Theorem 5.1.** *Let $f$ be a hyperbolic element of $\mathcal{A}$. If $A$, $B$, $C$, and $D$ are real parameters, the following conditions are equivalent:*

(1) *$f_{\mathbf{R}}$ has maximal entropy;*
(2) *$J^*(f)$ is contained in $S(\mathbf{R})$;*
(3) *$K(f)$ is contained in $S(\mathbf{R})$.*

*In that case, $J^*(f) = J(f) = K(f)$.*

This theorem is an easy consequence of the results of section 3.3 (see [5], section 10 for a proof). Our first goal is to prove the following result.

**Theorem 5.2.** *Let $f$ be a hyperbolic element of $\mathcal{A}$. If $(A,B,C,D)$ are real parameters such that $S_{(A,B,C,D)}(\mathbf{R})$ is connected, then $f_{\mathbf{R}}$ has maximal entropy.*

Before giving a proof of theorem 5.2, let us review a result of Bowen concerning topological lower bounds for the entropy (see [14]). Let $f$ be a homeomorphism of a marked topological space $(X,m)$, by which we mean that $m$ is a fixed point of $f$. Then, $f$ determines an automorphism $f_*$ of the fundamental group $\pi_1(X,m)$. Let us assume that $\pi_1(X,m)$ is finitely generated, and fix a finite set $\{\alpha_1,\ldots,\alpha_k\}$ of generators. The growth rate of $f_*$ is defined to be

$$\lambda(f_*) = \limsup_{n\to+\infty}\left(\frac{1}{n}\mathrm{diam}(f^n(B))\right)$$

where diam is the diameter with respect to the word metric (using the generators $\alpha_i$) and $B$ is the ball of radius 1 with respect to this metric. Bowen's theorem asserts that

$$h_{top}(f) \geq \log(\lambda(f_*))$$

as soon as $f$ is a continuous transformation of a compact manifold. Even though $S(\mathbf{R})$ is not compact, we can apply this theorem because unbounded orbits are contained in the basins of attraction of $\mathrm{Ind}(f^{-1})$ and $\mathrm{Ind}(f)$.

*Proof of theorem 5.2.* Let us first study the case of the Cayley cubic $S_C$. This surface is singular, and $S_C(\mathbf{R}) \setminus \mathsf{Sing}(S_C)$ contains a unique bounded component. This component $S_C(\mathbf{R})^0$ is a sphere with four punctures and the dynamics of $\mathcal{A}$ (*i.e.* $\Gamma_2^*$) is covered by the monomial action of $\Gamma_2^*$ on the torus $\mathbb{S}^1\times\mathbb{S}^1$ in $\mathbf{C}^*\times\mathbf{C}^*$. As a consequence, for any hyperbolic element $f$ in $\Gamma_2^*$, the entropy of $f$ on $S_C(\mathbf{R})^0$ is maximal; moreover, the expanding factor $\lambda(f_*)$ coincides with the dynamical degree $\lambda(f)$, and Bowen's inequality is an equality.

If we deform the Cayley cubic in such a way that the surface $S(\mathbf{R})$ is smooth and connected, then $S(\mathbf{R})$ is homeomorphic to a four punctured sphere (the punctures are now at infinity - see §2.9), and the action of $f$ on the fundamental group of $S(\mathbf{R})$ has not been changed along the deformation. As a consequence, Bowen's inequality gives $h_{top}(f_{\mathbf{R}}) \geq \log(\lambda(f))$ and the conclusion follows from $h_{top}(f_{\mathbf{R}}) \leq h_{top}(f_{\mathbf{C}}) = \log(\lambda(f))$. This concludes the proof for smooth and connected surfaces $S(\mathbf{R})$ (see section 2.9). If $S(\mathbf{R})$ is not smooth but is connected, then $S(\mathbf{R})$ is a limit of smooth connected members of the family $\mathsf{Fam}$. By semicontinuity of topological entropy, $f_{\mathbf{R}}$ has maximal entropy (see [51]). □

**Corollary 5.3.** *Let $a$, $b$, $c$, and $d$ be four real parameters in $\mathbf{R}\setminus[-2,2]$, the product of which is negative. Let $\rho : \pi_1(\mathbb{S}_4) \to \mathsf{SL}(2,\mathbf{C})$ be a representation with boundary traces $a$, $b$, $c$, and $d$. Let $\Phi \in \mathsf{Aut}(\pi_1(\mathbb{S}_4^2))$ be a pseudo-Anosov automorphism. If $\rho\circ\Phi$ is conjugate to $\rho$, then $\rho$ is conjugate to a representation into $\mathsf{SL}(2,\mathbf{R})$.*

*Proof.* Let $S$ be the element of the family $\mathsf{Fam}$ that corresponds to the parameters $(a,b,c,d)$. The assumption on the parameters $a$, $b$, $c$, and $d$ implies that $S(\mathbf{R})$ is connected (see section 2.9), and that there is no $\mathsf{SU}(2)$-component (this is obvious if $S(\mathbf{R})$ is smooth, since $\mathsf{SU}(2)$ representations would form a compact component, and this follows from [10] in the singular case).

If $\rho\circ\Phi^{-1}$ is conjugate to $\rho$, then $\chi(\rho)$ is a fixed point of the automorphism $f_\Phi$ induced by $\Phi$ on the surface $S$. Since $S(\mathbf{R})$ is connected, $f_{\mathbf{R}}$ has maximal entropy. By theorem 5.1, all periodic points of $f$ are contained in $S(\mathbf{R})$. This implies that $\rho$ is conjugate to an $\mathsf{SL}(2,\mathbf{R})$-valued representation. □

5.2. **Maximal entropy and quasi-hyperbolicity.** Bedford and Smillie recently developped a nice theory for Hénon transformations which extends the notion of quasi-hyperbolicity, a notion that had been previously introduced for the dynamics of rational maps of one complex variable (see [53] for one variable). This theory can be applied to our context in order to study hyperbolic automorphisms with maximal entropy.

5.2.1. *Quasi-hyperbolicity.* Let $Sadd(f)$ be either the set of periodic saddle points of $f$ or the set $W^u(p)\cap W^s(q)$ where $p$ and $q$ are two periodic fixed points of $f$ (see [8] for for possible other choices concerning $Sadd(f)$). With such a choice, $Sadd(f)$ is $f$-invariant and its closure coincides with $J^*(f)$ (see §3.3, property (6)). Each point $m$ of $Sadd(f)$ has a stable manifold $W^s(m)$ and an unstable manifold $W^u(m)$, and we can find two injective immersions $\xi_m^u, \xi_m^s : \mathbf{C} \to S$ such that $\xi_m^{u/s}(0) = m$, $\xi_m^{u/s}(\mathbf{C}) = W^{u/s}(m)$, and

$$\max\{G^{+/-}(\xi_m^{u/s}(t))\mid \quad t\in\mathbb{D}\} = 1,$$

where $\mathbb{D}$ is the unit disk. The parameterization $\xi_m^u$ and $\xi_m^s$ are uniquely determined by this normalization up to a rotation of $t$. Since $Sadd(f)$ is $f$-invariant and $f$ sends the unstable manifold at $m$ to the unstable manifold at

$f(m)$, there is a non zero complex number $\lambda(m)$ such that

$$f(\xi^u_m(t)) = \xi^u_{f(m)}(\lambda(m)t).$$

The number $\lambda(m)$ depends on the choices made for $\xi^u_m$ and $\xi^u_{f(m)}$ but its modulus $|\lambda(m)|$ only depends on $m$. Since $G^+ \circ f = \lambda(f)G^+$, we obtain easily the inequality $|\lambda(m)| > 1$ for all $m \in Sadd(f)$.

We shall also need the growth function $\mathsf{gro}_m(r)$ of $G^+$ along the unstable manifold $W^u(m)$, which is defined by $\mathsf{gro}_m(r) = \max_{|t|\le r}\{G^+(\xi^u_m(t))\}$, and the uniform growth function

$$\mathsf{Gro}(r) = \sup_{m\in Sadd(f)} \{\mathsf{gro}_m(r)\}.$$

Bedford and Smillie proved in [8], section 1, that the following properties are equivalent:

(1) the family $\{\xi^u_m \,|\, m \in Sadd(f)\}$ is a normal family;
(2) $\mathsf{Gro}(r_0) < \infty$ for some $1 < r_0 < \infty$;
(3) there exists $\kappa > 1$ such that $|\lambda(m)| \ge \kappa$ for all $m$ in $Sadd(f)$;
(4) $\exists C, \beta < \infty$ such that $\mathsf{gro}_m(r) \le Cr^\beta$ for all $m$ in $S$ and $r \ge 1$.

If these properties are satisfied, $f$ is said to be *quasi-expanding*. If $f$ and $f^{-1}$ are quasi-expanding, then $f$ is said to be *quasi-hyperbolic*.

5.2.2. *Maximal entropy.* It turns out that real Hénon mappings with maximal entropy are necessarily quasi-hyperbolic (see [8], theorem 4.8 and proposition 4.9). The proof of this result can be applied word by word to our context, and gives rise to the following theorem.

**Theorem 5.4** (Bedford Smillie, [8] and [9])**.** *Let $f$ be a hyperbolic element of $\mathcal{A}$ and $S$ be an element of* $\mathsf{Fam}$ *defined by real parameters $(A,B,C,D)$. If $f_{\mathbf{R}}$ has maximal entropy, then $f$ is quasi-hyperbolic, and any periodic point $m$ of $f$ is a saddle point, with $|\lambda(m)| \ge \lambda(f)$.*

**Corollary 5.5.** *Let $f$ be a hyperbolic element of $\mathcal{A}$ and $S$ be an element of* $\mathsf{Fam}$ *defined by real parameters $(A,B,C,D)$. If $S(\mathbf{R})$ is connected, then $f_{\mathbf{R}}$ has maximal entropy and is quasi-hyperbolic.*

5.2.3. *Uniform hyperbolicity and consequences.* In a subsequent paper, Bedford and Smillie also obtain a precise obstruction to uniform hyperbolicity. Let $p \in S(\mathbf{R})$ be a saddle periodic point of $f$. The unstable manifold of $p$ in $S(\mathbf{R})$ is the intersection of $S(\mathbf{R})$ with the complex unstable manifold $W^u(p)$. This real unstable manifold is diffeomorphic to the real line $\mathbf{R}$, and $p$ disconnects it into two half lines. If one of these half unstable manifolds is contained in the complement of $K^+(f)$, one says that $p$ is *u-one-sided* or unstably one-sided; a point which is not $u$-one-sided is said to be unstably two-side. Stably (or $s$-) *one-sided* points are defined in a similar way.

**Theorem 5.6** (Bedford Smillie, [9])**.** *Let $f$ be a hyperbolic element of $\mathcal{A}$ and $S$ be an element of* Fam *defined by real parameters $(A,B,C,D)$. If $f_{\mathbf{R}}$ has maximal entropy but $K(f)$ is not a hyperbolic set for $f$, then*

- *there are periodic saddle points $p$ and $q$ (not necessarily distinct) so that $W^u(p)$ intersects $W^s(q)$ tangentially with order 2 contact ;*
- *$p$ is s-one-sided and $q$ is u-one-sided ;*
- *the restriction of $f$ to $K(f)$ is not expansive.*

**Theorem 5.7.** *Let $f$ be a hyperbolic element of $\mathcal{A}$. Let $S$ be a smooth surface in the family* Fam *which is defined by real parameters $(A,B,C,D)$. If one of the connected components of $S(\mathbf{R})$ is bounded, then the entropy of $f_{\mathbf{R}}$ is not maximal and $f$ has an infinite number of saddle periodic points in $S(\mathbf{C})\setminus S(\mathbf{R})$.*

*Proof.* Let us assume that $f$ has maximal entropy and that $S(\mathbf{R})$ has at least one bounded connected component $S(\mathbf{R})^0$. The existence of a bounded component implies that $S(\mathbf{R})$ this bounded component $S(\mathbf{R})^0$ is unique and homeomorphic to a sphere (see §2.9). Being $f$-invariant and compact, $S(\mathbf{R})^0$ is contained in $K(f)$. Since $f_{\mathbf{R}}$ has maximal entropy, $K(f)$ is contained in $S(\mathbf{R})$, has empty interior (in $S(\mathbf{C})$), and coincides with the support of $\mu_f$ (see §3.3 and theorem 5.1); in particular $\mu_f(S(\mathbf{R})^0)$ is a positive number. The ergodicity of $\mu_f$ and the $f$-invariance of $S(\mathbf{R})^0$ now imply that $S(\mathbf{R})^0$ has full $\mu_f$-measure. As a consequence, $K(f)$ coincides with $S(\mathbf{R})^0$. Since $S(\mathbf{R})^0$ is compact, there is no one-sided periodic point, and theorem 5.6 implies that $K(f)$ is a hyperbolic set. This means that the dynamics of $f$ on $S(\mathbf{R})^0$ is uniformly hyperbolic. In particular, the unstable directions of $f$ determine a continuous line field on $S(\mathbf{R})^0$, and we get a contradiction because $S(\mathbf{R})^0$ is a sphere. $\square$

**Remark 5.8.** A similar argument shows that the Julia set $K(h)$ of a Hénon automorphism $h:\mathbf{C}^2\to\mathbf{C}^2$ can not coincide with a smooth embedded 2-dimensional surface $S\subset\mathbf{C}^2$. Indeed, the restriction of $h$ to $S$ would be uniformly hyperbolic, and its entropy would be equal to $\log(d)$, where $d\in\mathbf{Z}^+$ is the dynamical degree of $h$. This implies that the Euler caracteristic of $S$ is 0 and that $h:S\to S$ is an Anosov diffeomorphism. But the topological entropy of such a diffeomorphism is not the logarithm of an integer (it is the logarithm of a quadratic integer). This provides a contradiction.

**Corollary 5.9.** *Let $D$ be a real number and $S_D$ be the element of* $\mathsf{Fam}$ *defined by the real parameters $(0,0,0,D)$. The following properties are equivalent: $(i)$ there exists a hyperbolic element $f$ in $\mathcal{A}$ such that $f : S_D(\mathbf{R}) \to S_D(\mathbf{R})$ has maximal entropy, $(ii)$ any hyperbolic element $f$ in $\mathcal{A}$ has maximal entropy on $S_D(\mathbf{R})$, and $(iii)$ $D \geq 4$.*

*Proof.* If $D > 4$, then $S(\mathbf{R})$ is connected and smooth and the result follows from theorem 5.2. If $D \leq 0$, the result follows from the fact that the action of the mapping class group on $S(\mathbf{R})$ is totally discontinuous (see [37]). If $0 < D < 4$, then $S(\mathbf{R})$ has a compact connected component $S(\mathbf{R})^0$ and the conclusion follows from the previous theorem. □

5.3. **Uniform hyperbolicity.** We now prove theorem 1.2 in the following more general form.

**Theorem 5.10.** *Let $f$ be a hyperbolic element of $\mathcal{A}$. Let $S$ be an element of* $\mathsf{Fam}$ *defined by real parameters. If $S(\mathbf{R})$ is connected, then*

- *the entropy of $f_{\mathbf{R}}$ is maximal; its value is $\log(\lambda(f))$;*
- *the set of bounded orbits of $f : S(\mathbf{C}) \to S(\mathbf{C})$ is a compact subset $K(f)$ of $S(\mathbf{R})$;*
- *the automorphism $f$ admits a unique invariant probability measure $\mu_f$ of maximal entropy, and the support of $\mu_f$ coincides with $K(f)$; periodic saddle points equidistribute toward $\mu_f$;*
- *the dynamics of $f$ on $K(f)$ is uniformly hyperbolic.*

The only property that has not been proven yet is the last one. In fact, we shall prove more than uniform hyperbolicity: Our objective includes a description of the complement of $K^+(f)$, in order to explain pictures like the one provided in figure 2. This will be achieved in section 5.4. Once again, as in the proof of theorem 5.2, the main argument is to understand perturbations of the Cayley cubic, *i.e.* perturbations of $f_{\mathbf{R}} : S_C(\mathbf{R}) \to S_C(\mathbf{R})$. The following section contains a preliminary study of its small connected real deformations.

5.3.1. *Small deformations of the Cayley cubic.* The surface $S_C$ has four conical singularities. If $s$ is one of these four points, then, locally, $S_C(\mathbf{R})$ is diffeomorphic to a quadratic cone $Q = 0$ with $Q(X,Y,Z) = X^2 + Y^2 - Z^2$ ; the singularity $s$ now coincides with the origin of $\mathbf{R}^3$.

Let $M$ be an element of the orthogonal group $\mathsf{O}(Q)$ with an eigenvalue $\lambda \in \mathbf{R}$ of absolute value $|\lambda| > 1$. The other two eigenvalues of $M$ are then

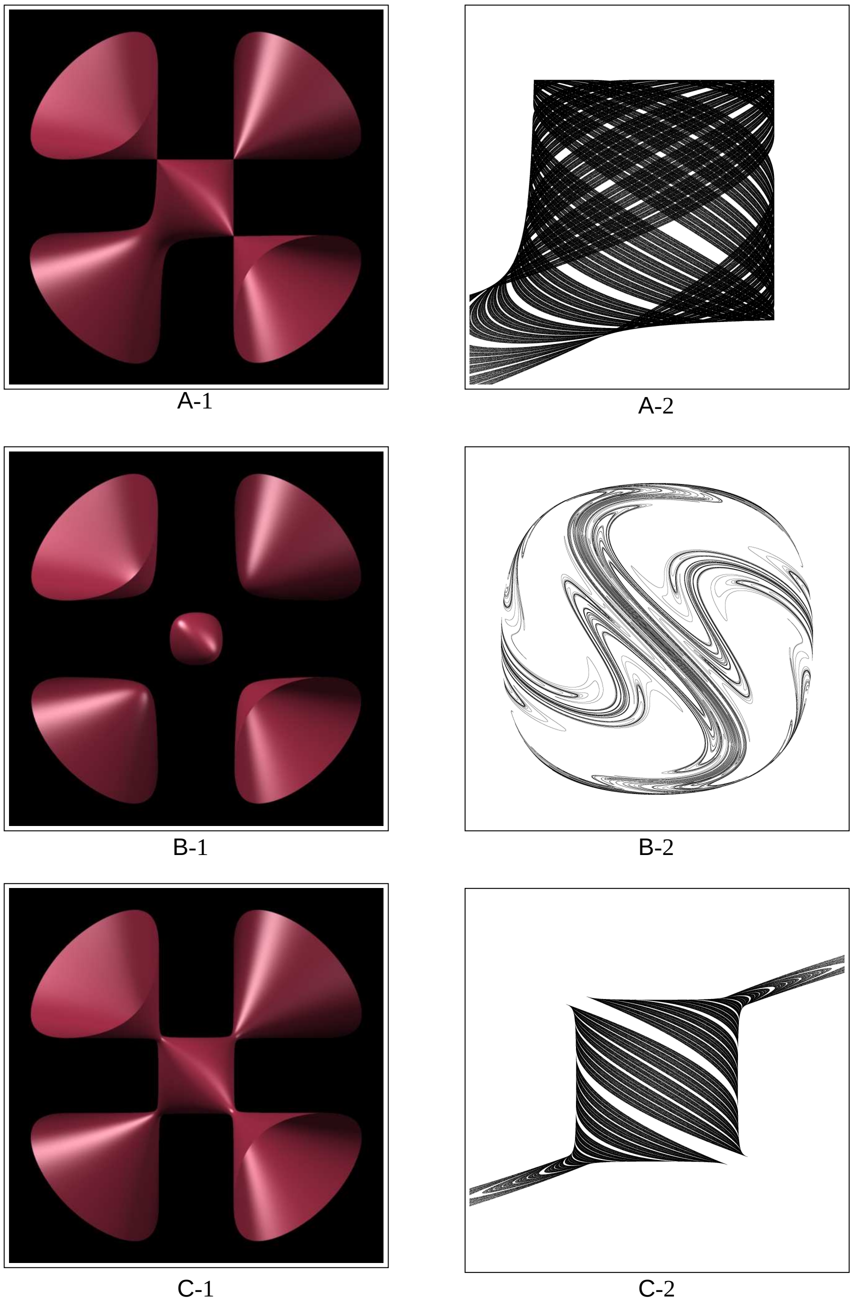


FIGURE 2. Examples of stable manifolds.

$\pm 1/\lambda$ and $\pm 1$. Let $D^+$, $D^-$ and $D^0$ be the three eigenlines corresponding to the eigenvalues $\lambda$, $\pm 1/\lambda$ and $\pm 1$.

Let $\varepsilon$ be a non negative number. Define

$$\mathcal{H}_\varepsilon := \{(X,Y,Z) \in \mathbf{R}^3; Q(X,Y,Z) = \varepsilon\};$$

when $\varepsilon = 0$, $\mathcal{H}_0$ is the quadratic cone, but when $\varepsilon > 0$, $\mathcal{H}_\varepsilon$ is a connected hyperboloid, which intersects the line $D^0$ in two opposite points $s^+(\varepsilon)$, and $s^-(\varepsilon) = -s^+(\varepsilon)$. According to the sign of the eigenvalue $\pm 1$, $M$ either fixes or permutes these two points. In any case, $s^+(\varepsilon)$ and $s^-(\varepsilon)$ are saddle periodic points for the restriction $M : \mathcal{H}_\varepsilon \to \mathcal{H}_\varepsilon$. The stable manifold of $M$ through $s^+(\varepsilon)$ (resp. $s^-(\varepsilon)$) is the line through $s^+(\varepsilon)$ (resp. $s^-(\varepsilon)$) contained in $\mathcal{H}_\varepsilon$ which is parallel to $D^-$ (see picture 3-A).

Let $\hat{\mathbf{R}}^3$ be the blow-up of $\mathbf{R}^3$ at the origin, let $E$ be the exceptional divisor ($E = \mathbb{P}^2(\mathbf{R}) \subset \hat{\mathbf{R}}^3$), and $\pi : \hat{\mathbf{R}}^3 \to \mathbf{R}^3$ the contraction of $E$. The linear map $M$ lifts to $\hat{M} : \hat{\mathbf{R}}^3 \to \hat{\mathbf{R}}^3$. The strict transform of the quadratic cone $\mathcal{H}_0$ is a cylinder $\mathcal{H}_0'$ that intersects $E$ along the conic curve $Q = 0$. Both $\mathcal{H}_0'$ and $E$ are invariant by $\hat{M}$, and $\hat{M} : \mathcal{H}_0' \to \mathcal{H}_0'$ has two saddle periodic points along the conic.

This conic disconnects $E$ into a disk and a Möbius band $N$. The strict transform of $D^0$ intersects $E$ in one point $\hat{s}$, which is contained in $N$. When $\varepsilon > 0$ goes to 0, the points $\pi^{-1}(s^{\pm}(\varepsilon))$ converge toward $\hat{s}$, and the family of surfaces $\pi^{-1}(\mathcal{H}_\varepsilon)$ converges toward the union of $N$ and the cylinder $\mathcal{H}_0'$, approaching $\mathcal{H}_0'$ from one side and $N$ from both sides. The point $\hat{s}$ is a saddle fixed point of $\hat{M} : E \to E$. The strict transform of $D^0$ is a neutral invariant manifold for $\hat{M} : \hat{\mathbf{R}}^3 \to \hat{\mathbf{R}}^3$ through $\hat{s}$, which intersects the surfaces $\mathcal{H}_\varepsilon$ on the saddle points $\pi^{-1}(s^{\pm}(\varepsilon))$.

Let us now come back to the Cayley cubic $S_C$. Let $f$ be a hyperbolic element of the group $\mathcal{A}$. All four singularities are saddle fixed points of $f$: Locally around each of those points $s$, $S_C(\mathbf{C})$ is a quotient of $\mathbf{C}^2$ by the involution $(u,v) \mapsto (-u,-v)$, and the map $f$ is covered by a linear map $(u,v) \mapsto (\alpha u, \beta v)$ with $|\alpha| > 1$ and $\beta\alpha = \pm 1$.

The transformation $f$ extends to an automorphism of the affine space preserving the family of cubic surfaces $S_{(0,0,0,D)}$, $D \in \mathbf{R}$. Let us denote by $f$ this automorphism and let $s$ be a singularity of $S_C$. The eigenvalues of the matrix

$$M := Df_s$$

are equal to $\lambda := \alpha^2$, $1/\lambda := \beta^2$ and $\pm 1$. By Morse lemma, the surfaces $S_{(0,0,0,D)}$ with $D = 4 + \varepsilon$ behave locally as the family of hyperboloids $\mathcal{H}_\varepsilon$. Let us use the same notation as above. Then, $f$ lifts as an automorphism $\hat{f}$ of the blow-up $\hat{\mathbf{R}}^3$ of $\mathbf{R}^3$ at the point $s$. The transformation $\hat{f}$ coincides with $\hat{M}$ along the exceptional divisor $E$.

The strict transform of the (real) Cayley cubic coincides (locally) with the cylinder $\mathcal{H}_0'$. The fixed point $s$ corresponds on this cylinder to a pair of fixed points and the conic curve $Q=0$ realizes a heteroclinic connection between this points. If we cut the cylinder along the unbounded unstable manifold of one of these two points, the cylinder becomes a strip: This is shown on picture 3-B, where stable and unstable laminations of $f: S_C(\mathbf{R}) \to S_C(\mathbf{R})$ are represented.

Since $\hat{f}$ coincides with $\hat{M}$ along $E$, $\hat{f}$ has a saddle fixed point at $\hat{s}$. The exceptional divisor $E$ is smooth and the family of surfaces $\pi^{-1}(S_{(0,0,0,4+\varepsilon)})$ determines a smooth locally trivial fibration near $\hat{s}$ (with $E$ corresponding to $\varepsilon = 0$). Saddle periodic points can be deform along smooth perturbations. As a consequence, $\hat{s}$ can be deformed into a pair of saddle periodic points $(s^+(\varepsilon), s^-(\varepsilon))$ on $S_{4+\varepsilon}$ for small $\varepsilon > 0$. The line $D^0$ is tangent to the curve $\mathcal{D}'$ which is described by this family of points.

A similar study applies for all small real deformations $S_{(A,B,C,D)}$ which are connected, and we get the following lemma.

**Lemma 5.11.** *Let $f$ be a hyperbolic element of the group $\mathcal{A}$. Let $s$ be a singularity of the Cayley cubic $S_C$. If $S_{\alpha(t)}$ is a small real and connected deformation of the Cayley cubic, then $s$ deforms as a pair of points $(s^+(t), s^-(t)) \subset S_{\alpha(t)}$ which are both saddle fixed points of $f^2 : S_{\alpha(t)} \to S_{\alpha(t)}$.*

The stable manifold of $s^{\pm}(t)$ is uniquely parameterized by an injective holomorphic map $\xi_t^s : \mathbf{C} \to S_D(\mathbf{C})$ with $\xi_t^s(0) = s^+(t)$, $|(\xi_t^s)'(0)| = 1$, and $\xi_t^s(\mathbf{R}) \subset S_D(\mathbf{R})$ (up to a possible composition of $\xi_t^s(z)$ by $z \mapsto -z$). By a coherent choice of $\xi_t^s$, one gets a continous family of holomorphic mappings.

**Remark 5.12.** For $S_D$ with $D = 4-\varepsilon$ and $\varepsilon > 0$, the surface locally looks like a hyperboloid with two sheets that doesn't intersect $D^0$: The intersection is indeed made of two complex conjugate points. This explains that we lose saddle points in the real locus, and shows that, locally, the entropy of $f: S_D(\mathbf{R}) \to S_D(\mathbf{R})$ is not maximal (for small $\varepsilon > 0$).

5.3.2. *Notations and preliminaries.* We now start the proof of theorem 5.10. In what follows, we fix a hyperbolic element $f$ of $\mathcal{A}$, and assume that $f$ preserves orientation (replace $f$ by $f^2$ if $f$ reverses orientation). We denote by $\mathcal{H}$ the space of real parameters $(A,B,C,D)$ such that $S(\mathbf{R})$ is connected. In order to prove theorem 5.10, and theorem 5.22, we shall study the dynamics of $f$ on all surfaces $S = S_{(A,B,C,D)}$ with $(A,B,C,D)$ in $\mathcal{H}$. For such surfaces, maximal entropy implies the following properties:

(1) $K(f)$ coincides with $J(f)$ and is a subset of $S(\mathbf{R})$; moreover, periodic points are hyperbolic, all of them are contained in $K(f)$, and intersections between stable and unstable manifolds are also contained in $K(f)$ (see theorem 5.4) ;
(2) the set of one-sided points is a finite subset $OS(f)$ of $J(f)$ (see [9], sections 3 and 4) ;
(3) if $m$ is a point of tangency between stable and unstable manifolds of $f$, the $\alpha$ and $\omega$-limit sets of $m$ are contained in $OS(f)$ (see theorem 2.7 of [9]) ;
(4) in the complement of $OS(f)$, stable and unstable manifolds of $f$ form two laminations of $J(f)$ (see proposition 5.3 of [8]) ;
(5) a tangency between a stable and an unstable manifold is always quadratic (see section 2 and figure 4.1 in [9], and section 5 of [8]).

**Remark 5.13.** Note that the picture provided by Bedford and Smillie's results include the fact that there is no heteroclinic connection between periodic points. This simple fact is well known, and is not related to maximal entropy in the real locus. The proof is as follows. Assume that one half of a (real) stable manifold $W^s(q)$ coincides with one half of an unstable manifold $W^u(p)$. Then the complex stable and unstable manifolds coincide because they intersect along an uncountable set. As a consequence, $W^s(q)$ can be compactified by adding the point $p$ to it, and determines a copy of $\mathbb{P}^1(\mathbf{C})$ in $S(\mathbf{C})$. Since $S(\mathbf{C}) \subset \mathbf{C}^3$ is an affine surface, and $\mathbf{C}^3$ does not contain any 1-dimensional compact subvariety, one gets a contradiction.

Note that if we resolve the singularities of the Cayley cubic by blow-ups, we create heteroclinic connections along the exceptional divisor.

5.3.3. *Deformation of periodic points and heteroclinic intersections.* For any point $(A,B,C,D)$ in $\mathcal{H}$, all periodic points of $f : S(\mathbf{C}) \to S(\mathbf{C})$ are real saddle points (property (1) above). As a consequence, we can follow all the periodic points along any deformation of the parameters $(A,B,C,D)$ in $\mathcal{H}$: If $\alpha(t)$, $t \in [0,1]$, is an arc of class $\mathcal{C}^k$ in $\mathcal{H}$, and if $p_0$ is a periodic saddle point of $f : S_{\alpha(0)} \to S_{\alpha(0)}$ of period $N$, there exists an arc $p(t)$ of class $\mathcal{C}^k$ such that

(1) for all $t$, $p(t)$ is contained in $S_{\alpha(t)}$ and $p(0) = p_0$;
(2) for all $t$, $p(t)$ is a periodic saddle point of $f : S_{\alpha(t)} \to S_{\alpha(t)}$ of period $N$ (here we also use the fact that $f$ preserves orientation; otherwise, the period could change when $p(t)$ goes through a singular point of $S_{\alpha(t)}$).

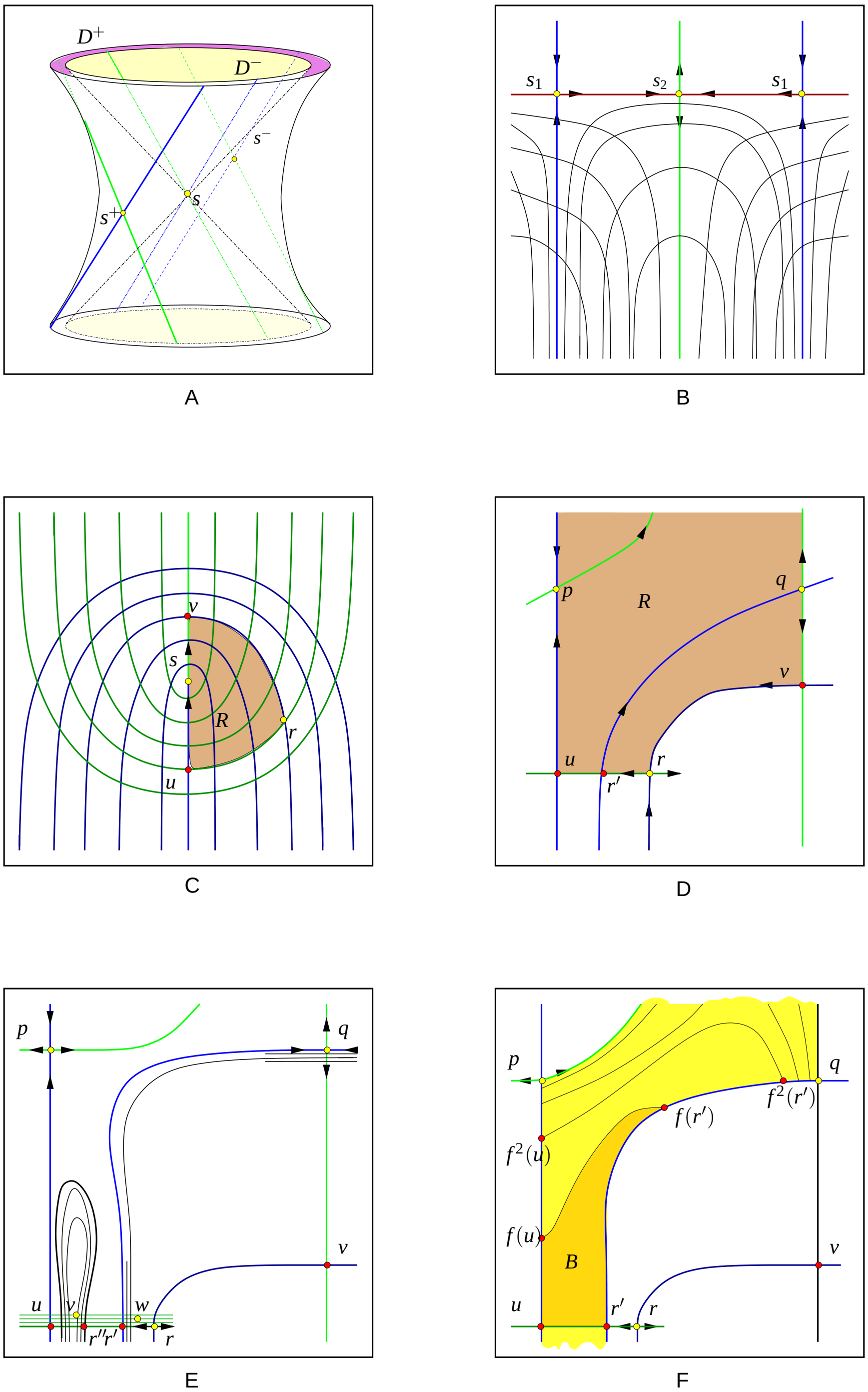


FIGURE 3. Deformation of singularities.

**Remark 5.14.** The point $p(t)$ is contained in the set $K_{\alpha(t)}(f)$ of points in $S_{\alpha(t)}$ with a bounded $f$-orbit. The family of compact sets $K_{\alpha(t)}(f)$ depends semi-continuously on $t$ ([6], lemma 3.1), so that the union $\cup_{t\in[0,1]}K_{\alpha(t)}(f)$ is contained in a fixed compact set $\mathcal{K}$. The paths $p(t)$, $t \in [0,1]$, where $p$ describes the set of periodic points of $f : S_{\alpha(0)} \to S_{\alpha(0)}$ are contained in $\mathcal{K}$.

We now explain how to follow intersection points along heteroclinic intersections. We choose two periodic points $p$ and $q$ and follow them along the deformation $\alpha(t)$, $t \in [0,1]$. We can then parameterize $W^s(p(t))$ by a continous family of holomorphic mappings

$$\xi^s_{p(t)} : \mathbf{C} \to S(\mathbf{C})$$

in such a way that $\xi^s_{p(t)}(0) = p(t)$, $|(\xi^s_{p(t)})'(0)| = 1$, and $\xi^s_{p(t)}(\mathbf{R}) \subset S(\mathbf{R})$. We parameterize $W^u_{q(t)}$ in a similar fashion by $\xi^u_{q(t)}$. We then choose one half of the stable/unstable manifolds, and assume that $\mathbf{R}^+$ is mapped onto this chosen half by $\xi^s_{p(t)}$ (resp. by $\xi^u_{q(t)}$).

Let $\Lambda_t$ be the set of parameters $(s(t),u(t))$ in $\mathbf{R}^+ \times \mathbf{R}^+$ corresponding to parameters of intersections between $W^s_{p(t)}$ and $W^u_{q(t)}$; more precisely,

$$\Lambda_t = \{(x,y) \in \mathbf{R}^+ \times \mathbf{R}^+; \quad \xi^s_{p(t)}(x) = \xi^u_{q(t)}(y)\}.$$

**Lemma 5.15.** *The set $\Lambda_t$ is a discrete subset of $\mathbf{R}^+ \times \mathbf{R}^+$. Two distinct points of $\Lambda_t$ have different first and second coordinates. For all $(x,y)$ in $\Lambda_t$, and for all $\varepsilon > 0$, the number of points in the strip $\mathbf{R}^+ \times [y-\varepsilon, y+\varepsilon]$ is infinite.*

*Proof.* Let$(x,y)$ be an element of $\Lambda_t$. Let $m = \xi^s_{p(t)}(x)$ be the intersection point corresponding to these parameters. Let $U \subset S_{\alpha(t)}(\mathbf{C})$ be a small neighborhood of $m$. Let $W^s_{loc}(m)$ (resp. $W^u_{loc}(m)$) be the connected component of $W^s_{loc}(m) \cap U$ (resp. $W^u_{loc}(m) \cap U$) containing $m$. These local stable and unstable manifolds are analytic subsets of $U$. As such, they intersect in a finite number of points. Let $I_x, I_y \subset \mathbf{R}^+$ be the intervals which are mapped on the local stable and unstable manifolds through $m$ by $\xi^s_{p(t)}$ and $\xi^u_{q(t)}$ respectively. By contruction, $I_x \times I_y$ is a neighborhood of $(x,y)$ which contains only a finite number of points of $\Lambda_t$. This shows that $\Lambda_t$ is discrete.

Let us now fix $x$. Since $\xi^s_{q(t)}$ is injective, the number of parameters $y$ such that $\xi^u_{q(t)}(y) = \xi^s_{p(t)}(x)$ is at most one. This proves the second statement.

Let now $(x,y)$ be an element of $\Lambda_t$. The intersection point $m := \xi^s_{p(t)}(x)$ is an element of $K(f)$. Let $U$ be a small neighborhood of $m$ in which $(i)$ the stable and unstable laminations of $f$ are transversal (or have at most a quadratic contact at $m$), and $(ii)$ all local stable and unstable manifolds intersect. Fix

$\varepsilon > 0$ and consider the piece of unstable manifold $W(m) = \xi^u_{q(t)}[y-\varepsilon, y+\varepsilon]$. What we have to show is that $W(m)$ intersects $W^s_{p(t)}$ infinitely many times. Shrinking $U$, one may assume that $W(m)$ coincides with $W^u_{loc}(m)$. Choose a periodic point $r$ of $f$ in $U$: The unstable manifold of $r$ intersects $W^s_{p(t)}$ at least once, and therefore infinitely many times in any neighborhood of $r$ ; all intersections points in $U$ generate intersection points between $W(m)$ and $W^s_{p(t)}$. □

Let us now *assume that p is not one-sided.* According to property (3) in §5.3.2, this implies that all intersection points of $W^s_{p(t)}$ and $W^u_{q(t)}$ are transverse. Let $(x,y)$ be any point of $\Lambda_a$, $a \in [0,1]$ and let $m \in S_{\alpha(a)}$ be the corresponding intersection point of stable/unstable manifolds. Then there exists a neighborhood $I$ of $a$ in $[0,1]$ along which $m$, and therefore $(x,y)$, can be smoothly deformed into paths $m(t)$ and $(x(t),y(t))$. In other words, all points of $\Lambda_a$ can be locally followed along the deformation $\alpha(t)$, with $t$ near $a$. Note that through such a deformation of two points $(x_1,y_1)$ and $(x_2,y_2)$, one always has $x_1(t) \neq x_2(t)$ (for $t$ in the common interval of definition).

**Lemma 5.16.** *For all intersection parameters $(x,y) \in \Lambda_0$, the domain of definition of the deformation $(x(t),y(t))$ coincides with the full interval of deformation $[0,1]$.*

*Proof.* What we have to show is that there is no "explosion in finite time". In other words, we have to rule out the situation where $(x(t),y(t))$ is defined on the interval $[0,a]$, but goes to infinity as $t$ increases to $a$. We therefore assume that $x(t)$ goes to infinity as $t$ goes to $a$, and try to reach a contradiction.

Let $y(a)$ be the infimum limit of $y(t)$ as $t$ approaches $a$. Let us first assume that $y(a) < \infty$ and choose a point $(x_1(a),y_1(a))$ in $\Lambda_a$ such that (i) $y_1(a) > y(a)$ and (ii) $\Lambda_a$ contains an infinite number of points in the strip $\mathbf{R}^+ \times [y(a),y_1(a)]$ (such a point exists by the third property of the previous lemma). Let now $(x_1(t),y_1(t))$ be the local deformation of $(x_1(a),y_1(a))$ on a small interval $[a-\varepsilon,a+\varepsilon]$.

Since $x(t)$ goes to $+\infty$, and since vertical lines through points of $\Lambda_t$ never coincide, we know that $x(t) > x_1(t)$ for all $t$ in $[a-\varepsilon,a[$. Similarly, $y_1(t) > y(t)$ for $t \in [a-\varepsilon,a[$. Let $B_t$ be the rectangle with upper left corner $(x_1(t),y_1(t))$ and lower right corner $(x(t),y(t))$. This is a compact subset of $\mathbf{R}^+ \times \mathbf{R}^+$, and, since $\Lambda_t$ is discrete, it contains a finite number of points of $\Lambda_t$. Since horizontal and vertical lines through points of $\Lambda_t$ never coincide, this number of points is a constant $k$, and the strip $\mathbf{R}^+ \times [y(a),y_1(a)]$ contains at most $k$ points. This contradicts the choice of $(x_1(a),y_1(a))$.

In the case where the infimum limit of $y(t)$ is $+\infty$, the contradiction is easier to get by considering the rectangle $C_t$ with lower left corner $(0,0)$ and upper right corner $(x(t),y(t))$. As a consequence, we get a contradiction in both cases, and the lemma is proved. □

This lemma shows that we can follow intersection points between stable and unstable manifolds along any deformation $S_{\alpha(t)}$ if $\alpha([0,1]) \subset \mathcal{H}$. Note that this is true up to and including the case of the Cayley cubic (see section 5.3.1).

**Remark 5.17.** When we follow an intersection point $m(t)$ of $W^s(p(t))$ and $W^u(q(t))$ along a deformation $S_{\alpha(t)}$, the point $m(t)$ never coincides with $p(t)$ or $q(t)$ (this is a consequence of the previous lemma, or of the absence of saddle connections, see remark 5.13).

5.3.4. *One-sided points.* Let us assume that $S_{\alpha(t)}$ is a deformation of the Cayley cubic $S_{\alpha(0)} = S_C$, with $\alpha(t) \in \mathcal{H}$ for all $t \in [0,1]$. We shall say that a periodic point $w$ of $S_{\alpha(1)}$ *comes from a singular point $s$ of $S_C$* if the deformation $w(t)$ of $w(1) = w$ along $\alpha(t)$ lands at $s$ when $t = 0$. This means that $w(t)$ coincides with one of the point $s^+(t)$ or $s^-(t)$ when $t$ approaches 0 (see section 5.3.1).

**Lemma 5.18.** *If $m(t)$ is a u-two-sided (resp. s-two-sided) point of $S_{\alpha(t)}$ for some $t$, then $m(t)$ is u-two-sided (resp. s-two-sided) for all $t \in [0,1]$. One sided points come from singular points of $S_C$, and if $m$ comes from a singular point, then $m$ is both stably and unstably one-sided.*

With the notations from section 5.3.2, the previous lemma shows that the set $OS(f)$ is made of the eight points coming from the singularities of $S_C$.

*Proof.* Let $m$ be a $u$-two-sided point. Following intersection points of stable and unstable manifolds, one sees that the set of parameters $t$ for which $m(t)$ is $u$-two-sided is an open set.

Let us now assume that $m(t)$ is $u$-two sided for $t \in ]a,b[$, and let $t$ decrease to $a$. Changing $f$ into one of its iterates, once can assume that $m(t)$ is a curve of fixed points and that the multiplier $\chi(t)$ of $f$ along the unstable manifold is a positive number. Let $\chi_+$ be the maximum of $\chi(t)$ on the closed interval $[a,b]$. Let $\varepsilon > 0$ be a fixed small real number. Since $m(t)$, $t > a$, is $u$-two-sided, the set $K^+(f)$ intersects the local stable manifold $W^u_{loc}(m(t))$ on both sides. Since $K^+(f) \cap W^u(m(t))$ is $f$-invariant, $K^+(f)$ intersects $W^u_{loc}(m)$ on both sides inside the annulus of radii $\varepsilon$ and $(1+\chi_+)\varepsilon$ around $m(t)$. By semi-continuity of $K^+(f)$, this implies that $K^+(f)$ intersects $W^u_{loc}(m(a))$ on both sides, at distance in between $\varepsilon$ and $(1+\chi_+)\varepsilon$, proving that $m(a)$ is $u$-two-sided.

This shows that the set of parameters $t$ for which $m(t)$ is $u$-two-sided is both open and closed. By connectedness, a point is $u$-two-sided for one parameter if an only if it is $u$-two-sided for all parameters.

Conversely, a point $m$ is $u$-one-sided for one parameter if and only if it is $u$-one sided for all parameters. On the Cayley cubic, those points are exactly the singular points. This proves the result. □

5.3.5. *Deformation and stable manifolds.* Next steps aim at giving a description of $K(f)$ and are not absolutely necessary to prove the uniform hyperbolicity. We choose one of the singularities $s$ of the Cayley cubic, and call $p(t)$ and $q(t)$ the two periodic one-sided points which come from this singularity after perturbation (these points where previously called $s^+(t)$ and $s^-(t)$). For $t=0$, we have $p(0)=q(0)=s$.

**Remark 5.19.** We shall make use of figures 3-B, to 3-E. They represent the geometry of stable and unstable manifolds near $p$ and $q$ after deformation of the Cayley cubic. Pictures 3-B,C describe the geometry of the stable/unstable laminations of $f$ on the Cayley cubic around $s$. This lamination has a singularity at $s$. Figure 3-B is obtained after one blow-up and has been described in §5.3.1. Figure 3-C is a view of the bounded part $S_C(\mathbf{R})$. Locally around $s$, we get a disk with two singular laminations (it's a typical "pseudo-Anosov" with spines, see [1], page 243). The region $R$ on this picture is described below.

Let us study the topology of stable and unstable manifolds of $f$ on a connected deformation $S(\mathbf{R})$ of $S_C(\mathbf{R})$. From lemma 5.18, we know that $p$ and $q$ are both $u$ and $s$-one sided, half of their real stable/unstable manifolds going to infinity (see picture 3-A and B for the Cayley cubic, and D for the deformation). We fix a periodic point $r$ in $S_C$ which is close to the stable manifold of $p$: The local unstable manifold of $r$ intersects transversaly the stable manifold of $s$ at $u$ and its stable manifold intersects transversaly the unstable manifold of $s$ at $v$, as in figure 3-C. Changing $f$ in one of its iterates, we assume that $r$ is a fixed point. We shall denote by $R_0$ the region bounded by $W^s(s)$, $W^u(r)$, $W^s(r)$ and $W^u(s)$ (see figure 3-C).

Thanks to section 5.3.3, we can follow this picture along a small deformation $S_{\alpha(t)}$ between $S_C$ and $S=S_{\alpha(1)}$. The point $r$ is deformed in a path $r(t)$ of saddle fixed points, and $s$ in a pair of saddle fixed points $p(t)$, $q(t)$. The intersection point $u$ can be deformed in two ways. As a point of intersection between $W^s(p(t))$ and $W^u(r(t))$, providing a point $u(t) \in S_{\alpha(t)}$, but also as a point of intersection between $W^s(q(t))$ and $W^u(r(t))$, and we denote by $r'(t)$ this second deformation. The point $v$ can also be deformed in two ways; by convention, $v(t)$ is the deformation contained in $W^u(q(t)) \cap W^s(r(t))$ (see figure 3-D).

Let $R(t) \subset S_{\alpha(t)}(\mathbf{R})$ be the closed region which is bounded by the half of $W^s(p(t)) \setminus \{u(t)\}$ that contains $p(t)$, the segment of $W^u(r(t))$ between $u(t)$ and $r(t)$, the segment of $W^s(r(t))$ that joins $r(t)$ to $v(t)$, and the half of $W^u(q(t)) \setminus \{v(t)\}$ that contains $q(t)$, (see figure 3-C). Let $W^s_+(q(t))$ be the connected component of $W^s(q(t)) \setminus \{q(t)\}$ which enters $R(t)$: This half stable manifold is parameterized by $\xi_t : \mathbf{R}^+ \to S(\mathbf{R})$, with $\xi_t(0) = q(t)$ and $\xi_t(z) \in R$ for small positive real numbers $z$.

The closure of the stable manifold of $q(t)$ covers the set $K(f)$. As a consequence, we know that $W^s_+(q(t)) \setminus \{q(t)\}$ exits the region $R(t)$. In particular, there exists a smallest positive $z$ such that $\xi_t(z)$ is on $\partial R(t)$. Since this point coincides with $u(t)$ on the Cayley cubic, we know that it coincides with $r'(t)$ all along the (small) deformation $S_{\alpha(t)}$ (figure 3-D).

**Lemma 5.20.** *For all $t \in [0,1]$, the half stable manifold $W^s_+(q(t))$ exits $R$ through $W^u(r(t))$, in between $r(t)$ and $u(t)$, at point $r'(t)$.*

**Warning.** In what follows, we keep the same notations, but the dependance with respect to the deformation parameter $t$ is made implicit. All points and stable/unstable manifolds are indeed points and curves in $S_\alpha(t)$; when $t = 0$, $S$ is the Cayley cubic, $p(t) = q(t) = s$, and $R(t)$ degenerates to $R_0$.

5.3.6. *Stable manifolds, doubly one-sided points, and wandering strips.* Let $I$ be the closed segment $[r,u] \subset W^u(r)$.

**Lemma 5.21.** *The set $K^+(f)$ does not intersect $W^u(r)$ along $I$ in between the points $u$ and $r'$.*

*Proof.* Let us assume that this is not the case. Since $q$ is $u$-one-sided, we know that there is no stable manifolds approaching $r'$ from the left. We can therefore define $r''$ to be the unique point in $I$ which is between $u$ and $r'$, is contained in $K(f)$, and is closest to $r'$ with these properties. By assumption, $r''$ is different from $u$ (see figure 3-E).

The stable manifold through $r''$ enters $R$ and cannot intersect $W^s(q)$ and $W^s(p)$. It must therefore exit $R$ through the interval $I$, in between $r''$ and $u$ (see picture 3-E). Since $r$ is not coming from a singular point, $r$ is not one-sided, and the stable and unstable manifolds of $f$ form two transverse laminations in its neighborhood. As a consequence, there are periodic points

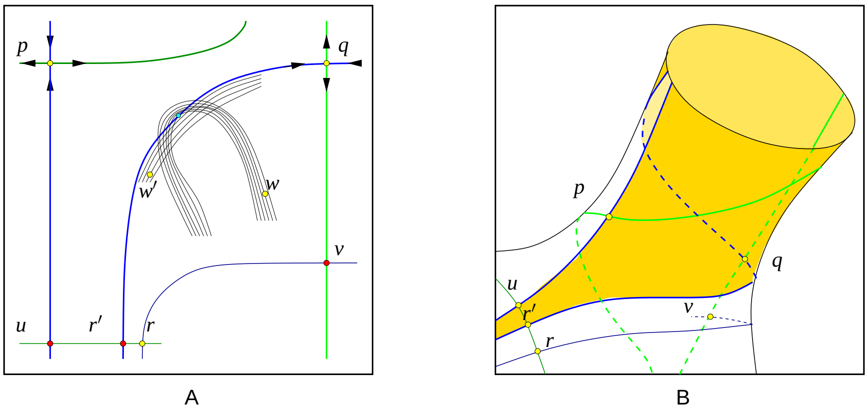


FIGURE 4. Tangency and complement of $K^+(f)$.

of $f$ in $R$ which are arbitrarily close to $r$. Let $w$ be such a point. The point $w$ being arbitrarily close to $r$, its local unstable manifold is arbitrarily close to $W^u(r)$, and we can choose $w$ in such a way that the local unstable manifold of $w$ intersects $W^s(r')$ in at least two points in $R$, as in picture 3-E.

We now choose a second periodic point $w'$ in $R$ which is close to $r''$, in such a way that the connected component of $W^s(w') \cap R$ which contains $w'$ intersects $W^u_{loc}(w)$ in two distinct points $i_1$ and $i_2$ .

The points $w$, $w'$, and the heteroclinic intersections $i_1$ and $i_2$ can then be followed up to the Cayley cubic along the deformation $S_{\alpha(t)}$ (lemma 5.16). During this deformation $w$ and $w'$ can not leave the region $R$, because two distinct periodic points can not be on the same stable/unstable manifold. As a consequence, $i_1$ (resp. $i_2$) can not exit $R$, because otherwise, for some parameter $t$ in the deformation, $i_1(t)$ would be contained in two distinct stable (resp. unstable) manifolds.

On the Cayley cubic, we then get two periodic points $w(0)$ and $w'(0)$ such that the connected component of $W^s(w') \cap R$ containing $w'$ intersects the connected component of $W^u(w) \cap R$ containing $w$ in two distinct points. This is a contradiction. □

Let $B$ be the region bounded by $[r', u]$, $W^s(p)$, $f([u, r'])$, and $W^s(q)$ (see picture 3-F). The segments $f^n[u, r']$ join the endpoints $f^n(u)$, which converge to $p$ along $W^s(p)$, to the endpoints $f^n(r')$, which converge to $q$ along $W^s(q)$. On the other end, the open segment $]r', u[$ is entirely contained in the complement of $K^+(f)$, so that all its points go to infinity when one iterates $f$ positively. This implies that points in the interior of $B$ are wandering points which are pushed away to infinity by $f$. The same is true for the images

$f^n(B)$, $n \in \mathbf{Z}$. As a consequence, the full strip

$$\bigcup_{n\in\mathbf{Z}} f^n(\mathsf{Int}(B) \cup [r', u]),$$

i.e. the strip located in between the halves of $W^{u/s}(p)$ and $W^{u/s}(q)$, is entirely contained in the complement of $K^+(f)$. (see picture 3-F, where this strip is colored) .

5.3.7. *Deformation and the geometry of $K(f)$.* We can apply the same argument to understand the geometry of stable and unstable manifolds near $p$. Part B of figure 4 summarizes our knowledge of the geometry of stable and unstable manifolds near the points $p$ and $q$ after a small deformation of the Cayley cubic: $p$ and $q$ are both $u$ and $s$-one-sided, and the colored region is contained in the complement of $K^+(f)$.

Let us now consider a large deformation $S_{\alpha(t)}$ of the Cayley cubic $S_C$. Following $p$, $u$, $r$, $v$, $q$, stable/unstable manifolds of these points, and their intersections along the deformation, we can follow the region $R$ along $\alpha(t)$. Since there is no saddle connection in $S_{\alpha(t)}$ for $t > 0$, the geometry of $R$ with respect to local stable and unstable manifolds in $R$ does not change. The results obtained above for small deformations remain therefore valid for arbitrarily large deformations $\alpha(t) \subset \mathcal{H}$ .

5.3.8. *Absence of tangency and hyperbolicity.* Let us now assume that there is at least one set of parameters $(A,B,C,D)$, for which $S(\mathbf{R})$ is connected and $f_{\mathbf{R}}$ is not uniformly hyperbolic on $K(f)$. Then, there is a tangency between the stable manifold of a $u$-one-sided periodic point $q$ and an unstable manifold (see theorem 5.6 and section 5.3.2). Iterating $f$, we can find such tangencies in arbitrarily small neighborhoods of $q$.

Since $S(\mathbf{R})$ is connected, we can deform $S$ in $S_{\alpha(t)}$ with $\alpha(t) \in \mathcal{H}$, $t \in [0,1]$, $S_{\alpha(0)} = S_C$, $S_{\alpha(1)} = S$ (see section 2.9). Since $q$ is $u$-one-sided, it comes from one of the singularities of $S_C$ (lemma 5.18). Sections 5.3.6 and 5.3.7 provide points $p(t)$, $q(t)$, $r(t)$, ..., and a region $R(t)$ in $S_{\alpha(t)}$, and describe the geometry of the stable and unstable manifolds near $q = q(1)$. Figure 4-A represents such a possible tangency (see also [9], picture 4.1 and §3 and 4).

Let $U \subset R(1) \subset S(\mathbf{R})$ be a small neighborhood of the tangency point $m$. If $a$ is a point of $K(f) \cap U$, we shall denote by $W^{s/u}_{loc}(a)$ the connected component of $W^{s/u}(a) \cap U$ that contains $a$. Since $m$ is in $K(f)$, one can find a saddle periodic point $w$ in $U$ such that $W^u_{loc}(w)$ intersects $W^s_{loc}(m)$ in two points $i_1$ and $i_2$. Then, we can find a second periodic saddle point $w'$ such that $W^s_{loc}(w')$ intersects $W^u_{loc}(w)$ in two points (see figure 4-A).

Thanks to section 5.3.3 and lemma 5.16, we can now follow the periodic saddle points $r(t)$, $v(t)$, $u(t)$, $w(t)$, $w'(t)$, and the points of intersection $i_1(t)$ and $i_2(t)$ continuously along the deformation. All of them belong to the region $R(t)$. From sections 5.3.6 and 5.3.7, the geometry of the stable and unstable manifolds of $r(t)$, $v(t)$, $q(t)$ and $p(t)$ remains unchanged along the deformation; in particular, since the periodic points $w(t)$ and $w'(t)$ cannot cross the stable or unstable manifolds of other periodic points during the deformation, they both stay in the interior of the region $R(t)$. We then get a contradiction as in the proof of lemma 5.21, §5.3.6.

Since there is no tangency, theorem 5.6 implies that the dynamics of $f$ is uniformly hyperbolic on $K(f)$. This proves theorem 5.10.

5.4. **Strips, bounded orbits, and Hausdorff dimension.** Let $(A,B,C,D)$ be an element of $\mathcal{H}$. Let $f$ be a hyperbolic element of $\mathcal{A}$. The surface $S(\mathbf{R})$ defined by this set of parameters is connected, and $f : S(\mathbf{R}) \to S(\mathbf{R})$ is uniformly hyperbolic on $K(f)$, so that we can apply proposition 2.1.1 of [12]: The set

$$W^s_{\mathbf{R}}(K(f)) = K^+(f) \cap S(\mathbf{R})$$

is laminated by stable manifolds of points in $K(f)$; if a point $m$ in $K^+(f)$ is on the boundary of the complement of $W^s_{\mathbf{R}}(K(f))$, then $m$ is on the stable manifold of a periodic $u$-one-sided periodic point of $f$. From section 5.3.3, we know that $f$ has exactly eight periodic one-sided points, each of them coming from a singularity of the Cayley cubic. From sections 5.3.5 and 5.3.7, the stable manifolds of the two one-sided points coming from one singularity bound a strip, as in picture 4-B. This proves the following result, which was first numerically observed by Catarino and MacKay (see [22], page 61 for example), and "explains" pictures 2-A and C.

**Theorem 5.22** (MacKay observation)**.** *If $S(\mathbf{R})$ is connected, $f$ has exactly eight one-sided fixed points $p_1$, $q_1$, $p_2$, $q_2$, $p_3$, $q_3$, $p_4$, and $q_4$. All of them come from singularities of the Cayley cubic by deformation; all of them are both u and s-one-sided. Moreover, the stable manifolds of $p_i$ and $q_i$ ($i = 1,2,3,4$) bound an open strip homeomorphic to $\mathbf{R} \times (-1,1)$, and the complement of $K^+(f) \cap S(\mathbf{R})$ coincides with the union of these four strips.*

We now study the Hausdorff dimension of the stable and unstable laminations $K^+(f)$ and $K^-(f)$ on $S_{\alpha(t)}$, where $\alpha(t)$ is an analytic path in the set $\mathcal{H}$.

**Theorem 5.23.** *Let $t \mapsto \alpha(t)$ be an analytic map from $]0,1[$ to the set of parameters $(A,B,C,D)$. Assume that $S_{\alpha(t)}$ is smooth and connected for all* *values of $t$. Then, the Hausdorff dimension of the sets $W^u_{loc}(m)\cap K^+(f)$ does not depend on $m\in K^-(f)$, and defines an analytic function of $t$ which is strictly positive and strictly less than* 1.

**Remark 5.24.** In particular, the complement of $K^+(f)\cap S(\mathbf{R})$, *i.e.* the union of the four strips, has full Lebesgue measure ; almost all orbits go to infinity under iteration of $f$. The same is true for the complement of $K^+(f)$ in $S(\mathbf{C})$.

*Proof.* By results of Hasselblatt [39], the stable and unstable distributions of $f$ are smooth, and the holonomy maps between two transversals of the stable (resp. unstable) laminations are Lipschitz continuous. In particular, the Hausdorff dimension of the sets

$$W^u_{loc}(m)\cap K^+(f)$$

does not depend on the choice of $m$ in $K(f)$ (see also [57], theorem 1). We shall denote this dimension by $H^+_t(f)$.

The map $f$ is area-preserving: As in [58], corollary 4.7, this implies that the Hausdorff dimension of the sets $W^s_{loc}(m)\cap K^-(f)$ coincides with $H^+_t(f)$.

Using Bowen-Ruelle thermodynamic formalism, as it is done in [57], theorem 2, we obtain that $H^+_t(f)$ is an analytic function of $t$. Since the function $G^+_{f|s(E)}$ is Hölder continuous this Hausdorff dimension is strictly positive.

If $H^+_t(f)$ is equal to 1, then the same is true for the Hausdorff dimension of $W^s_{loc}(m)\cap K^-(f)$ and theorem 22.1 of [52] shows that the Lebesgue measure of these sets is strictly positive. By Hasselblatt's result, the Lebesgue measure of $K(f)$ is positive, and by Bowen-Ruelle's theorem ([15], theorem 5.6), the set $K(f)$ must be an attractor of $f: S_{\alpha(t)}\to S_{\alpha(t)}$ This contradicts the fact that $K(f)$ is compact, $f$ is area preserving, and $S_{\alpha(t)}$ is not compact. □

## 6. SCHRÖDINGER OPERATORS AND PAINLEVÉ EQUATIONS

6.1. **Discrete Schrödinger operators.** Let us now apply the previous results to the study of the spectrum of certain discrete Schrödinger operators. There is a huge literature on the subject, and we refer to [23] and [24] for background results and a short bibliography.

6.1.1. *Discrete Schrödinger operators and substitutions.* Let $W^*$ be the set of finite words in the letters $a$ and $b$. Let $\iota:\{a,b\}\to W^*\setminus\{\emptyset\}$ be a substitution. In what follows, we shall assume that $\iota$ is invertible, which means that $\iota$ extends to an automorphism $\Phi_\iota$ of the free group $F_2=\langle a,b\,|\,\emptyset\rangle$, and that $\iota$ is primitive, which means that $\Phi_\iota$ is hyperbolic ; in other words, the image of $\Phi_\iota$ in $\mathsf{Out}(F_2) = \mathsf{GL}(2,\mathbf{Z})$ is a hyperbolic matrix, with two distinct eigenvalues $\lambda_+(\iota)$ and $\lambda_-(\iota)$ satisfying

$$|\lambda_+(\iota)| = |1/\lambda_-(\iota)| > 1.$$

Under these hypothesises, there is a unique infinite word $u_+$ in the two letters $a$ and $b$ such that $\iota(u_+) = u_+$.

**Example 6.1.** The Fibonacci substitution $\iota_F$, defined by $\iota_F(a) = b$ and $\iota_F(b) = ba$, provides a good and famous example of such an invertible primitive substitution. Its fixed word starts with *babbababbabbababbababbabba*...

Let $W$ be the set of bi-infinite words in $a$ and $b$ and $\tilde{T} : W \to W$ be the left shift. Let $\tilde{u}_+$ be any completion of $u_+$ on the left. We then define $\Omega$ to be the $\omega$-limit set of the $\tilde{T}$-orbit of $\tilde{u}_+$:

$$\Omega = \left\{ v \in W \mid \quad \text{there exists a sequence } n_i \to +\infty, \text{ such that } \tilde{T}^{n_i}(\tilde{u}_+) \to v \right\}.$$

Since $\iota$ is primitive, the restriction of the left shift $\tilde{T}$ to the set $\Omega$ is a minimal and uniquely ergodic homeomorphism $T : \Omega \to \Omega$. The unique $T$-invariant probability measure on $\Omega$ will be denoted by $\nu$.

**Remark 6.2.** The subshift $T : \Omega \to \Omega$ encodes the dynamics of a rotation $R_\alpha : \mathbf{R}/\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$, where $\alpha$ is a quadratic integer (see [2]).This provides a measurable conjugation between $R_\alpha$ and $T$ which sends the Lebesgue measure $dx$ to $\nu$.

Let us now fix an element $w$ in $\Omega$, and define the potential $V_w : \mathbf{Z} \to \mathbf{R}$ by $V_w(n) = 1$ if $w_n = a$ and $V_w(n) = 0$ if $w_n = b$. Let $\kappa$ be any complex number ($\kappa$ is the so called "coupling parameter"). If $(\xi(n))_{n\in\mathbf{Z}}$ is a complex valued sequence, we define

$$H_{\kappa,w}(\xi)(n) = \xi(n+1) + \xi(n-1) + \kappa V_w(n)\xi(n).$$

The discrete Schrödinger operator $H_{\kappa,w}$ induces a bounded linear operator on $l^2(\mathbf{Z})$, with norm at most $2+|\kappa|$. The adjoint of $H_{\kappa,w}$ is $H_{\overline{\kappa},w}$, so that $H_{\kappa,w}$ is self-adjoint if and only if $\kappa$ is a real number.

6.1.2. *Almost sure spectrum and Lyapunov exponent.* Since $T$ is ergodic with respect to $\nu$, there exists a subset $\Sigma_\kappa$ of $\mathbf{C}$ (of $\mathbf{R}$ if $\kappa$ is real) such that the spectrum of $H_{\kappa,w} : l^2(\mathbf{Z}) \to l^2(\mathbf{Z})$ coincides with $\Sigma_\kappa$ for $\nu$-almost all $w$ in $\Omega$. This set is the "almost sure spectrum" of the family $H_{\kappa,w}$.

To understand the spectrum of $H_{\kappa,w}$, one is led to solve the eigenvalue equation $H_{\kappa,w}(\xi) = E\xi$ ($E$ in $\mathbf{R}$ or $\mathbf{C}$). For any initial condition $(\xi(0),\xi(1))$, there is a unique solution, which is given by the recursion formula

$$\begin{pmatrix} \xi(n+1) \\ \xi(n) \end{pmatrix} = \begin{pmatrix} E-\kappa V_w(n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi(n) \\ \xi(n-1) \end{pmatrix}, \quad n \in \mathbf{Z}.$$

Let $M_{\kappa,E} : W^* \to \mathsf{SL}(2,\mathbf{C})$ be defined by

$$M_{\kappa,E}(a) = \begin{pmatrix} E-\kappa & -1 \\ 1 & 0 \end{pmatrix}, \quad M_{\kappa,E}(b) = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix},$$

and by

$$M_{\kappa,E}(u_1...u_n) = \Pi_{i=0}^{i=n-1} M_{\kappa,E}(u_{n-i})$$

for any word $u = u_1...u_n$ of length $n$. This defines a $\mathsf{SL}(2,\mathbf{C})$-valued cocyle over the dynamical system $(\Omega, T, \nu)$. Applying Osseledets' theorem, each choice of a coupling parameter $\kappa$ and an energy $E$ gives rise to a non negative Lyapunov exponent $\gamma(\kappa, E)$, such that

$$\begin{aligned} \gamma(\kappa,E) &= \lim_{n\to+\infty} \frac{1}{n} \int_\Omega \log \|M_{\kappa,E}(w_1 w_2 ... w_{n-1})\| \, d\nu(w) \\ &= \lim_{n\to+\infty} \frac{1}{n} \log \|M_{\kappa,E}(w_1 w_2 ... w_{n-1})\|, \end{aligned}$$

for $\nu$-almost all $w$ in $\Omega$. The Lyapunov function $\gamma(\kappa,E)$ is linked to the almost sure spectrum $\Sigma_\kappa$ by the following result.

**Theorem 6.3** (see [23]). *Let $\kappa$ be a real number. The almost sure spectrum $\Sigma_\kappa$ coincides with the set of energies for which the Lyapunov exponent vanishes.*

6.1.3. *Trace map dynamics, Lyapunov exponent, and Hausdorff dimension.* Let us fix the coupling parameter $\kappa$. Let $S_{4+\kappa^2}$ be the character surface $x^2 + y^2 + z^2 - xyz = 4 + \kappa^2$. The *Schrödinger curve* of $S_{4+\kappa^2}$ is the parameterized rational curve $\mathsf{s} : \mathbf{C} \to S_{4+\kappa^2}$, which is defined by $\mathsf{s}(E) = (x(E), y(E), z(E))$, with

$$\begin{aligned} (x(E), y(E), z(E)) &= (\mathsf{tr}(M_{\kappa,E}(a)), \mathsf{tr}(M_{\kappa,E}(b)), \mathsf{tr}(M_{\kappa,E}(ab))) \\ &= (E-\kappa, E, E(E-\kappa)-2). \end{aligned}$$

**Remark 6.4.** The intersection of $S_{4+\kappa^2}$ with the plane $y = x+\kappa$ is a reducible cubic curve: It is the union of $\mathsf{s}(\mathbf{C})$ with the line $\{z = 2, y = x+\kappa\}$; the involution $s_z$ permutes these two curves.

Let $f_\iota$ be the polynomial automorphism of $S_{4+\kappa^2}$ which is determined by the automorphism $(\Phi_\iota)^{-1} : F_2 \to F_2$. By definition of $f_\iota$, we have

$$(\mathsf{tr}(M_{\kappa,E}(\iota(a))), \mathsf{tr}(M_{\kappa,E}(\iota(b))), \mathsf{tr}(M_{\kappa,E}(\iota(ab)))) = f_\iota(\mathsf{s}(E)).$$

In [23], Damanik proved that $\gamma(\kappa,E)$ vanishes if and only if $\mathsf{s}(E)$ has a bounded forward $f_\iota$-orbit. In other words,

$$\Sigma_\kappa = \left\{E \in \mathbf{C} \,|\, \mathsf{s}(E) \in K^+(f_\iota)\right\}. \tag{6.1}$$

We can now apply MacKay observation, i.e. theorem 5.22, which tells us that the complement of $\mathsf{s}(\Sigma_\kappa)$ in the real Schrödinger curve is obtained by intersecting $\mathsf{s}(\mathbf{R})$ with the four strips associated to the one-sided points of $f_\iota$. This means that *gaps in the complement of the spectrum are bounded by intersection points between* $\mathsf{s}(\mathbf{R})$ *and the eight curves* $W^s(q_i)$ *and* $W^s(p_i)$, $i = 1, 2, 3,$ *and* 4*).*

**Theorem 6.5.** *The Hausdorff dimension of* $\Sigma_\kappa$, $\kappa \in \mathbf{R}$, *is a real analytic function of* $\kappa$. *Moreover,* $0 < Haus(\Sigma_\kappa) \leq 1$, $\forall \kappa \in \mathbf{R}$, *and* $Haus(\Sigma_\kappa) = 1$ *if and only if* $\kappa = 0$.

This statement confirms numerical observations that can be found, for example, in [46] and [45]; it is stronger than the fact that $\Sigma_\kappa$ has zero Lebesgue measure when $\kappa \neq 0$, a property which was proved by Kotani in the eighties (see [24]). Here, it appears as a corollary of results in dynamical systems which are due to Bowen, Pesin, and Ruelle.

*Proof.* The map $\alpha(\kappa) = (0,0,0,4+\kappa^2)$ is analytic and all surfaces $S_\alpha(t)$ are smooth and connected for $\kappa \neq 0$. We can therefore apply theorem 5.23, which tells us that the Hausdorff dimension of the sets $W^u_{loc}(m) \cap K^+(f_\iota)$ does not depend on the choice of $m$ in $K(f_\iota)$, and defines a real analytic function $H^+_\kappa(f_\iota)$ of the variable $\kappa$ such that $0 < H^+_\kappa(f_\iota) < 1, \quad \forall \kappa \neq 0$.

Let us apply this result to the spectrum $\Sigma_\kappa$. Let $t$ be an element of $\Sigma_\kappa$. Let $m$ be the point $\mathsf{s}(t)$ on the Schrödinger curve. By Damanik's theorem the image of the spectrum by $\mathsf{s}$ coincides with $K^+(f) \cap s(\mathbf{C})$ (see (6.1)). In particular, $K^+(f) \cap \mathsf{s}(\mathbf{C})$ is a compact set which contains $m$. The set $K^+(f)$ is a smooth lamination by analytic curves, and $\mathsf{s}(\mathbf{C})$ is an algebraic curve (see remark 6.4). This implies that the number of tangency points between $\mathsf{s}(\mathbf{C})$ and the lamination $K(f)$ is finite. In the complement of this finite set, $\mathsf{s}(\mathbf{C})$ is transverse to the lamination, so that locally the Hausdorff dimension of $K^+(f) \cap \mathsf{s}(\mathbf{C})$ coincides with $H^+_\kappa(f_\iota)$. Since the Hausdorff dimension is locally equal to $H^+_\kappa(f_\iota)$ in the complement of a finite set, it is globally equal to $H^+_\kappa(f_\iota)$. □

**Remark 6.6.** It would be interesting to settle a complete dictionary between dynamics of the trace map and properties of the spectrum. For example, the Green function of $f_\iota$ should coincide with the Lyapunov function $\gamma(\kappa,E)$

along the Schrödinger curve ; together with Thouless formula, this would identify the density of states $dk_\kappa$ with the measure obtained by slicing $T^+_{f_\iota}$ with the Schrödinger curve: $dk_\kappa = s^*(T^+_{f_\iota})$ (see [55] for related results and definitions).

## 7. Appendix: Monodromy of Painlevé VI equation

The sixth Painlevé equation $P_{VI} = P_{VI}(\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta)$ is the second order non linear ordinary differential equation

$$P_{VI} \quad \begin{cases} \frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right) \\ \qquad + \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left(\frac{\theta_\delta^2}{2} - \frac{\theta_\alpha^2}{2}\frac{t}{q^2} + \frac{\theta_\beta^2}{2}\frac{t-1}{(q-1)^2} + \frac{1-\theta_\gamma^2}{2}\frac{t(t-1)}{(q-t)^2}\right). \end{cases}$$

the coefficients of which depend on complex parameters $\theta = (\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta)$.

As explained in [44], the monodromy of Painlevé equation provides a representation of $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, t_0)$ into the group of analytic diffeomorphisms of the space of initial conditions $(q(t_0), q'(t_0))$ (see [44] for a precise description of this space). Via Riemann-Hilbert correspondence,

- the space of initial conditions is analytically isomorphic to (a desingularization of) $S_{(A,B,C,D)}$, with parameters

$$a = 2\cos(\pi\theta_\alpha),\ b = 2\cos(\pi\theta_\beta),\ c = 2\cos(\pi\theta_\gamma),\ d = 2\cos(\pi\theta_\delta), \quad (7.1)$$

  (see section 2.1 for the expression of $(A, B, C, D)$ in terms of $(a, b, c, d)$);
- the monodromy action on the space of initial conditions is conjugate to the action of $\Gamma_2$ on the surface $S_{(A,B,C,D)}$.

From this and sections 5.3 and 6, we deduce the following result, thereby answering a recent question raised by Iwasaki and Uehara (problem 15 of [43]).

**Theorem 7.1.** *Let $\theta_\alpha$, $\theta_\beta$, $\theta_\gamma$, and $\theta_\delta$ be parameters of Painlevé sixth equation, the real parts of which are integers with an odd sum. Let $\eta$ be any loop in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and let $f_\eta : S_{(A,B,C,D)} \to S_{(A,B,C,D)}$ be the monodromy transformation defined by $\eta$ (through Riemann-Hilbert correspondence). Either $f_\eta$ preserves a pencil of algebraic curves, or its topological entropy is positive, and then*

- *all periodic points of $f_\eta$ are contained in the real part $S_{(A,B,C,D)}(\mathbf{R})$ of the surface;*
- *the Hausdorff-dimension of the set of bounded $f_\eta$-orbits is $< 2$;*

- *the unique invariant probability measure of maximal entropy $\mu_{f_\eta}$ is supported by $S_{(A,B,C,D)}(\mathbf{R})$ and is singular with respect to the Lebesgue measure on $S_{(A,B,C,D)}(\mathbf{R})$.*

**Remark 7.2.** This theorem should be compared to Goldman's results regarding ergodic properties of the whole $\Gamma_2^*$ action with respect to the invariant area form $\Omega$ (see the definition of $\Omega$ in section 2.3). As a particular case of Goldman's theorem, the action of $\Gamma_2^*$ on $S_D(\mathbf{R})$ is ergodic with respect to $\Omega$ if, and only if $4 < D \leq 20$ (see [37]). Another interesting example is given by the Markoff surface $S_0$. In this example, the quasifuchsian space $\mathsf{QF}$ provides an open invariant subset of $S_0(\mathbf{C})$. This shows that the action of $\Gamma_2$ on $S_0(\mathbf{C})$ is not ergodic. Theorem 7.1 and these results suggest that, for most parameters, the dynamics of the monodromy of Painlevé equation is not correctly described by the invariant area form $\Omega$.

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