CONSTRAINTS ON AUTOMORPHISM GROUPS OF HIGHER DIMENSIONAL MANIFOLDS

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Abstract. In this note, we prove, for instance, that the automorphism group of a rational manifold X which is obtained from $\mathbb{P}^k(\mathbf{C})$ by a finite sequence of blow-ups along smooth centers of dimension at most r with $k > 2r+2$ has finite image in $GL(H^*(X,\mathbf{Z}))$. In particular, every holomorphic automorphism $f : X \to X$ has zero topological entropy.

1. Introduction

1.1. **Dimensions of indeterminacy loci.** Recall that a rational map admitting a rational inverse is called birational. Birational transformations are, in general, not defined everywhere. The domain of definition of a birational map $f\colon M \to N$ is the largest Zariski-open subset on which f is locally a well defined morphism. Its complement is the indeterminacy set $\mathrm{Ind}(f)$; its codimension is always larger than, or equal to, 2. The following statement shows that the dimension of $\mathrm{Ind}(f)$ and $\mathrm{Ind}(f^{-1})$ can not be too small simultaneously unless f is an automorphism. The proof of this result follows a nice argument of Nessim Sibony concerning the degrees of regular automorphisms of the complex space $\mathbf{C}^k$ (see [Sib99]) ; this idea was explained to us by an anonymous referee (compare [BC12]). It may be considered as an extension of a theorem due to Matsusaka and Mumford (see [MaMu64], and [KSC04], Exercise 5.6).

Theorem 1.1. *Let $\mathbf{k}$ be a field. Let M be a smooth connected projective variety defined over $\mathbf{k}$. Let f be a birational transformation of M. Assume that the following two properties are satisfied.*

(i) *the Picard number of M is equal to 1;*
(ii) *the indeterminacy sets of f and its inverse satisfy*

$$\dim(\mathrm{Ind}(f)) + \dim(\mathrm{Ind}(f^{-1})) < \dim(M) - 2.$$

Then f is an automorphism of M.

Moreover, $\mathsf{Aut}(M)$ is an algebraic group because the Picard number of M is equal to 1. As explained below, this statement provides a direct proof of the following corollary, which was our initial motivation.

Corollary 1.2. *Let M_0 be a smooth, connected, projective variety with Picard number 1. Let m be a positive integer, and $\pi_i\colon M_{i+1} \to M_i$, $i = 0, \ldots, m-1$, be a sequence of blow-ups of smooth irreducible subvarieties of dimension at most r. If $\dim(M_0) > 2r+2$ then the number of connected components of $\mathsf{Aut}(M_m)$ is finite;*

Date: May 28, 2013.
2000 *Mathematics Subject Classification.* 37F10, 32H50, 14E09.
Key words and phrases. automorphism, rational manifold, topological entropy.

moreover, the projection $\pi\colon M_m \to M_0$ *conjugates* $\mathsf{Aut}(M_m)$ *to a subgroup of the algebraic group* $\mathsf{Aut}(M_0)$.

For instance, if M_0 is the projective space (respectively a cubic hypersurface of $\mathbb{P}^4_{\mathbf{k}}$) and if one modifies M_0 by a finite sequence of blow-ups of points, then $\mathsf{Aut}(M_0)$ is isomorphic to a linear algebraic subgroup of $\mathrm{PGL}_4(\mathbf{k})$ (respectively is finite). This provides a sharp (and strong) answer to a question of Eric Bedford. In Section 3, we provide a second, simpler proof of this last statement.

Remark 1.3. The initial question of E. Bedford concerned the existence of automorphisms of compact Kähler manifolds with positive topological entropy in dimension > 2. This link with dynamical systems is described, for instance, in [Can11]. If a compact complex surface S admits an automorphism with positive entropy, then S is Kähler and is obtained from the projective plane $\mathbb{P}^2(\mathbf{C})$, a torus, a $K3$ surface or an Enriques surface, by a finite sequence of blow-ups (see [Can01, Can99] and [Nag61]). Examples of automorphisms with positive entropy are easily constructed on tori, K3 surfaces, or Enriques surfaces. Examples of automorphisms with positive entropy on rational surfaces are given in [BK06, BK10, McM07]; these examples are obtained from birational transformations f of the plane by a finite sequence of blow-ups that resolves all indeterminacies of f and its iterates simultaneously. These results suggest to look for birational transformations of $\mathbb{P}^n_{\mathbf{C}}$, $n \geq 3$, that can be lifted to automorphisms with a nice dynamical behavior after a finite sequence of blow-ups; the above result shows that at least one center of the blow-ups must have dimension $\geq n/2 - 1$.

Remark 1.4. Recently, Tuyen Truong obtained results which are similar to Corollary 1.2, but with hypothesis on the Hodge structure and nef classes of M_0 that replace our strong hypothesis on the Picard number (see [Tru12, Tru13]).

Acknowledgement

We received many interesting comments after we had posted a first version of this work on the web (see [BC12]). In particular, we thank Igor Dolgachev, Mattias Jonsson, Tuyen Truong for their valuable comments, and Brian Lehmann for stimulating correspondence. An anonymous referee explained to us how the main ideas of [Sib99], Prop. 2.3.2 and Rem. 2.3.3, could be applied in our setting to obtain Theorem 1.1 for $M = \mathbb{P}^k(\mathbf{C})$, and then recover the first results of [BC12]. We would like to express our gratitude to him/her.

2. Dimensions of Indeterminacy loci

In this section, we prove Theorem 1.1 under a slightly more general assumption. Indeed, we replace assumption (i) with the following assumption

(i') There exists an ample line bundle L such that $f^*(L) \cong L^{\otimes d}$ for some $d > 1$.

This property is implied by (i). Indeed, if M has Picard number 1, the torsion-free part of the Néron-Severi group of M is isomorphic to $\mathbf{Z}$, and is generated by the class $[H]$ of an ample divisor H. Thus, $[f^*H]$ must be a multiple of $[H]$.

In what follows, we assume that f satisfies property (i') and property (ii). Replacing H by a large enough multiple, we may and do assume that H is very ample. Thus, the complete linear system $|H|$ provides an embedding of M into some projective space $\mathbb{P}^n_{\mathbf{k}}$, and we identify M with its image in $\mathbb{P}^n_{\mathbf{k}}$. With such a convention, members of $|H|$ correspond to hyperplane sections of M.

2.1. **Degrees.** Denote by k the dimension of M, and by $\deg(M)$ its degree, i.e. the number of intersections of M with a generic subspace of dimension $n-k$.

If H_1, ..., H_k are hyperplane sections of M, and if $f^*(H_1)$ denotes the total transform of H_1 under the action of f, one defines the degree of f by the following intersection of divisors of M

$$\deg(f) = \frac{1}{\deg(M)} f^*(H_1) \cdot H_2 \cdots H_k.$$

Since M has Picard number 1, we know that divisor class $[f^*(H_1)]$ is proportional to $[H]$. Our definition of $\deg(f)$ implies that $f^*[H_1] = \deg(f)[H_1]$. As a consequence,

$$f^*(H_1) \cdot f^*(H_2) \cdots f^*(H_j) \cdot H_{j+1} \cdots H_k = \deg(f)^j \deg(M)$$

for all $0 \leq j \leq k$.

2.2. **Degree bounds.** Assume that the sum of the dimension of $\mathrm{Ind}(f)$ and of $\mathrm{Ind}(f^{-1})$ is at most $k-3$. Then there exist at least two integers $l \geq 1$ such that

$$\begin{aligned} \dim(\mathrm{Ind}(f)) &\leq k-l-1; \\ \dim(\mathrm{Ind}(f^{-1})) &\leq l-1. \end{aligned}$$

Let H_1, ..., H_l and H'_1, ..., H'_{k-l} be generic hyperplane sections of M; by Bertini's theorem,

(a) H_1, ..., H_l intersect transversally the algebraic variety $\mathrm{Ind}(f^{-1})$ (in particular, $H_1 \cap \ldots \cap H_l$ does not intersect $\mathrm{Ind}(f^{-1})$ because $\dim(\mathrm{Ind}(f^{-1})) < l$);

(b) H'_1, ..., H'_{k-l} intersect transversally the algebraic variety $\mathrm{Ind}(f)$ (in particular, $H'_1 \cap \ldots \cap H'_{k-l}$ does not intersect $\mathrm{Ind}(f)$ because $\dim(\mathrm{Ind}(f)) < k-l$).

For $j \leq l$, consider the variety $V_j = f^*(H_1 \cap \ldots \cap H_j)$: In the complement of $\mathrm{Ind}(f)$, V_j is smooth, of dimension $k-j$; since $j \leq l$ and $\dim(\mathrm{Ind}(f)) < k-l$, V_j extends in a unique way as a subvariety of dimension $k-j$ in M. The varieties V_j are reduced and irreducible.

Since each H_i, $1 \leq i \leq l$, intersects $\mathrm{Ind}(f^{-1})$ transversally, $f^*(H_i)$ is an irreducible hypersurface (it does not contain any component of the exceptional locus of f). Thus

$$\begin{aligned} V_j &= f^*(H_1 \cap \ldots \cap H_j) \\ &= f^*(H_1) \cap \ldots \cap f^*(H_j) \end{aligned}$$

is the intersection of j hypersurfaces of the same degree; for $j = l$ one gets

$$\deg(f)^l \deg(M) = f^*(H_1 \cap \ldots \cap H_l) \cdot (H'_1 \cap \cdots \cap H'_{k-l}).$$

More precisely, since the H'_i are generic, this intersection is transversal and $V_j \cdot (H'_1 \cap \ldots \cap H'_{k-l})$ is made of $\deg(f)^l \deg(M)$ points, all of them with multiplicity 1, all of them in the complement of $\mathrm{Ind}(f)$ (see property (b) above).

Similarly, one defines the subvarieties $V'_j = f_*(H'_1 \cap \ldots H'_j)$ with $j \leq k-l$; as above, these subvarieties have dimension $k-j$, are smooth in the complement of $\mathrm{Ind}(f^{-1})$, and uniquely extend to varieties of dimension $k-j$ through $\mathrm{Ind}(f^{-1})$. Each of them is equal to the intersection of the j irreducible divisors $f_*(H_i)$, $1 \leq i \leq j$. Hence,

$$(H_1 \cap \ldots \cap H_l) \cdot V'_{k-l} = \deg(f^{-1})^{k-l} \deg(M).$$

If one applies the transformation $f \colon M \setminus \mathrm{Ind}(f) \to M$ to V_l and to $(H'_1 \cap \cdots \cap H'_{k-l})$, one deduces that $\deg(f)^l \deg(M) \leq \deg(f^{-1})^{k-l} \deg(M)$, because all points of intersection of V_l with $(H'_1 \cap \ldots \cap H'_{k-l})$ are contained in the complement of $\mathrm{Ind}(f)$. Applied to f^{-1}, the same argument provides the opposite inequality. Thus,

$$\deg(f)^l = \deg(f^{-1})^{k-l}$$

Since there are at least two distinct values of l for which this equation is satisfied, one concludes that

$$\deg(f) = \deg(f^{-1}) = 1.$$

As a consequence, f has degree 1 if it satisfies assumptions (i') and (ii), .

2.3. **From birational transformations to automorphisms.** To conclude the proof of Theorem 1.1, one applies the following lemma.

Lemma 2.1. *Let M be a smooth projective variety and f a birational transformation of M. If there exists an ample divisor H such that f^*H and $f_*(H)$ are numerically equivalent to H, then f is an automorphism.*

Proof. Taking multiples, we assume that H is very ample. Consider the graph Z of f in $M \times M$, together with its two natural projections π_1 and π_2 onto M.

The complete linear system $|H|$ is mapped by f^* to a linear system $|H'|$ with the same numerical class, and vice versa if one applies f^{-1} to $|H'|$. Thus, $|H'|$ is also a complete linear system, of the same dimension. Both of them are very ample (but they may differ if the dimension of $\mathrm{Pic}^0(\mathrm{M})$ is positive).

Assume that π_2 contracts a curve C to a point q. Take a generic member H_0 of $|H|$: It does not intersect q, and $\pi_2^*(H_0)$ does not intersect C. The projection $(\pi_1)_*(\pi_2^*(H_0))$ is equal to $f^*(H_0)$; since f^* maps the complete linear system $|H|$ to the complete linear system $|H'|$ and H_0 is generic, we may assume that $f^*(H_0)$ is a generic member of $|H'|$. As such, it does not intersect the finite set $\pi_1(C) \cap \mathrm{Ind}(f)$. Thus, there is no fiber of π_1 that intersects simultaneously C and $(\pi_2)^*(H_0)$, and $(\pi_1)_*(\pi_2^*(H_0))$ does not intersect C. This contradicts the fact that $f^*(H_0)$ is ample. □

2.4. **Conclusion, and Kähler manifolds.** Under the assumptions of Theorem 1.1, Section 2.2 shows that f^*H is numerically equivalent to H. Lemma 2.1 implies that f is an automorphism. This concludes the proof of Theorem 1.1.

This proof is inspired by an argument of Sibony in [Sib99] (see Proposition 2.3.2 and Remark 2.3.3); which makes use of complex analysis: the theory of closed positive current, and intersection theory. With this viewpoint, one gets the following statement.

Theorem 2.2. *Let M be a compact Kähler manifold and f a bi-meromorphic transformation of M. Assume that*

(i) *there exists a Kähler form ω such that the cohomology class of $f^*\omega$ is proportional to the cohomology class of ω;*
(ii) *the indeterminacy locus of f and its inverse satisfy*

$$\dim(\mathrm{Ind}(f)) + \dim(\mathrm{Ind}(f^{-1})) < \dim(M) - 2.$$

Then f is an automorphism of M that fixes the cohomology class of ω.

Moreover, Lieberman's theorem (see [Lie78]) implies that a positive iterate f^m of f is contained in the connected component of the identity of the complex Lie group $\mathsf{Aut}(M)$.

2.5. **Proof of Corollary 1.2.** Since M_m is obtained from M_0 by a sequence of blow-ups of centers of dimension $< \dim(M_m)/2-1$, all automorphisms f of M_m are conjugate, through the obvious birational morphism $\pi\colon M_m \to M_0$, to birational transformations of M_0 that satisfy

$$\dim(\mathrm{Ind}(f)) < \dim(M_0)/2 - 1 \text{ and } \dim(\mathrm{Ind}(f^{-1})) < \dim(M_0)/2 - 1.$$

Thus, by Theorem 1.1 π conjugates $\mathsf{Aut}(M)$ to a subgroup of $\mathsf{Aut}(M_0)$. Moreover, given any polarization of M_0 by a very ample class, all elements of $\mathsf{Aut}(M_0)$ have degree 1 with respect to this polarization. Hence, $\mathsf{Aut}(M_0)$ is an algebraic group, and the kernel of the action of $\mathsf{Aut}(M_0)$ on $\mathrm{Pic}^0(\mathrm{M}_0)$ is a linear algebraic group; if $\mathrm{Pic}^0(\mathrm{M}_0)$ is trivial, there is a projective embedding of $\Theta\colon M_0 \to \mathbb{P}^n_{\mathbf{k}}$ that conjugates $\mathsf{Aut}(M_0)$ to the group of linear projective transformations $G \subset \mathrm{PGL}_{n+1}(\mathbf{k})$ that preserve $\Theta(M)$.

3. Constraints on automorphisms from the structure of the intersection form

Let X be a smooth projective variety of dimension k over a field $\mathbf{k}$. Denote by NS(X) the Néron-Severi group of X, *i.e.* the group of classes of divisors for the numerical equivalence relation. We consider the multi-linear forms

$$Q_d\colon \mathrm{NS(X)^d} \to \mathbf{Z}$$

which are defined by

$$Q_d(u_1, u_2, \ldots, u_d) = u_1 \cdot u_2 \cdots u_d \cdot K_X^{k-d}.$$

These forms are invariant under $\mathsf{Aut}(X)^*$ and we shall derive new constraints on the size of $\mathsf{Aut}(X)^*$ from this invariance.

Theorem 3.1. *Let X be a smooth projective variety of dimension $k \geq 3$, defined over a field $\mathbf{k}$. Let d be an integer that satisfies $3 \leq d \leq k$. If the projective variety*

$$W_d(X) := \{u \in \mathbb{P}(\mathrm{NS(X)} \otimes_{\mathbf{Z}} \mathbf{C}) |\ \mathrm{Q_d(u, u, \ldots, u) = 0}\}$$

is smooth, then $\mathsf{Aut}(X)^*$ is finite.

Proof. The group $\mathsf{Aut}(X)^*$ acts by linear projective transformations on the projective space $\mathbb{P}(\mathrm{NS(X)} \otimes_{\mathbf{Z}} \mathbf{C})$ and preserves the smooth hypersurface W_d. Since $d \geq 3$ it follows from [MaMo64] that the group of linear projective transformations preserving a smooth hypersurface of degree d is finite. Hence, there is a finite index subgroup A of $\mathsf{Aut}(X)^*$ which is contained in the center of GL(NS(X)); since the later is a finite group of homotheties, this finishes the proof. □

As a corollary, let us state the following one, already obtained in the previous sections:

Corollary 3.2. *Let X be a smooth projective variety of dimension $k \geq 3$. Assume that there exists a birational morphism $\pi : X \to V$ such that*

- *the Picard number of V is equal to 1*
- *π^{-1} is the blow-up of l distinct points of V.*

Then $\mathsf{Aut}(X)^*$ is a finite group.

Proof. We identify NS(V) with $\mathbf{Z}e_0$ where e_0 is the class of an ample divisor. Let $a := e_0^k$. Since X is obtained from V by blowing up l distinct points $p_1, \ldots, p_l$ we have

$$\mathrm{NS(X)} = \mathbf{Z}\mathrm{e}_0 + \bigoplus_{1 \leq \mathrm{i} \leq \mathrm{l}} \mathbf{Z}\mathrm{e}_\mathrm{i}$$

where e_i is the class of the exceptional divisor $E_i := \pi^{-1}(p_i)$. Then the form Q_k is given by

$$Q_k(u) = a(X_0)^k + (-1)^{k+1} \sum_{i=1}^{l} (X_i)^k$$

where $u = X_0 e_0 + \sum_i X_i e_i$ and $[X_0 : \ldots X_l]$ denotes the homogeneous coordinates on $\mathbb{P}(\mathrm{NS(X)} \otimes_{\mathbf{Z}} \mathbf{C})$. Hence, the projective variety defined by Q_k in $\mathbb{P}(\mathrm{NS(X)} \otimes_{\mathbf{Z}} \mathbf{C})$ is smooth and $\mathsf{Aut}(X)^*$ is finite. □

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