MORPHISMS BETWEEN CREMONA GROUPS, AND CHARACTERIZATION OF RATIONAL VARIETIES

SERGE CANTAT

ABSTRACT. We classify all (abstract) homomorphisms from $\mathsf{PGL}_{r+1}(\mathbf{C})$ to the group $\mathsf{Bir}(M)$ of birational transformations of a complex projective variety M, provided $r \geq \dim_{\mathbf{C}}(M)$. As a byproduct, we show that (1) $\mathsf{Bir}(\mathbb{P}^n_{\mathbf{C}})$ is isomorphic, as an abstract group, to $\mathsf{Bir}(\mathbb{P}^m_{\mathbf{C}})$ if and only if $n = m$ and (2) M is rational if and only if $\mathsf{PGL}_{\dim(M)+1}(\mathbf{C})$ embeds as a subgroup of $\mathsf{Bir}(M)$.

CONTENTS

1. INTRODUCTION

1.1. **Algebraic transformations.** Let M be a complex projective variety. Two natural groups of transformations are associated to M. The first is the group $\mathsf{Aut}(M)$ of automorphisms of M; with the topology of uniform convergence, this group is a complex Lie group (see [6]). More precisely, the connected component $\mathsf{Aut}(M)^0$ containing the identity Id_M is a connected, complex, algebraic group, while the discrete part $\mathsf{Aut}(M)^\sharp = \mathsf{Aut}(M)/\mathsf{Aut}(M)^0$ may have infinitely many elements.

The second group is the group $\mathsf{Bir}(M)$ of all birational transformations of M. In most cases, $\mathsf{Bir}(M)$ coincides with $\mathsf{Aut}(M)$ and is finite, but for some peculiar varieties, like the projective space $\mathbb{P}^n_{\mathbf{C}}$, $n > 1$, $\mathsf{Bir}(M)$ has infinite dimension.

Our goal in this article is to initiate the study of abstract morphisms from linear groups to groups of birational transformations $\mathsf{Bir}(M)$. We treat one example in details which, as a byproduct, shows that

Date: 2012.

- an irreducible variety M of dimension n is rational if, and only if $\mathsf{Bir}(M)$ is isomorphic to $\mathsf{Bir}(\mathbb{P}^n_{\mathbf{C}})$ as an abstract group;
- the Cremona groups $\mathsf{Bir}(\mathbb{P}^n_{\mathbf{C}})$ and $\mathsf{Bir}(\mathbb{P}^m_{\mathbf{C}})$ are not isomorphic if $n \neq m$.

1.2. **Field automorphisms.** Let $\mathsf{Aut}(\mathbb{P}^n_{\mathbf{C}})$ be the group of automorphisms of the complex projective space $\mathbb{P}^n_{\mathbf{C}}$. Once a system of homogeneous coordinates $[x_0 : x_1 : \ldots : x_n]$ is fixed, the group $\mathsf{Aut}(\mathbb{P}^n_{\mathbf{C}})$ can be identified to the group of projective transformations $\mathsf{PGL}_{n+1}(\mathbf{C})$.

Let $\mathsf{Aut}_{\mathbf{Q}}(\mathbf{C})$ be the group of automorphisms of the field $(\mathbf{C}, +, \cdot)$. The semi-direct product $\mathsf{Aut}_{\mathbf{Q}}(\mathbf{C}) \ltimes \mathsf{Aut}(\mathbb{P}^n_{\mathbf{C}})$ acts on the set $\mathbb{P}^n(\mathbf{C})$. To describe this action, let us use our system of homogeneous coordinates. The group $\mathsf{Aut}_{\mathbf{Q}}(\mathbf{C})$ acts diagonally on $\mathbf{C}^{n+1}$ and therefore on $\mathbb{P}^n(\mathbf{C})$: If β is an element of $\mathsf{Aut}_{\mathbf{Q}}(\mathbf{C})$, then

$$\beta([x_0 : \ldots : x_n]) = [\beta(x_0) : \ldots : \beta(x_n)].$$

It acts also on $\mathsf{PGL}_{n+1}(\mathbf{C})$, changing a matrix $B = [b_{ij}]$ into ${}^{\beta}B = [\beta(b_{ij})]$. This provides an action $g \mapsto {}^{\beta}g$ of $\mathsf{Aut}_{\mathbf{Q}}(\mathbf{C})$ on $\mathsf{Aut}(\mathbb{P}^n_{\mathbf{C}})$ such that

$${}^{\beta}g([x_0 : \ldots : x_n]) = (\beta \circ g \circ \beta^{-1})([x_0 : \ldots : x_n]),$$

and therefore an action of $\mathsf{Aut}_{\mathbf{Q}}(\mathbf{C}) \ltimes \mathsf{Aut}(\mathbb{P}^n_{\mathbf{C}})$ on $\mathbb{P}^n(\mathbf{C})$.

In a similar way, if M is a projective variety which is defined over a field $K \subset \mathbf{C}$, the group $\mathsf{Aut}_K(\mathbf{C})$ of automorphisms of the field extension $\mathbf{C}/K$ acts on $M(\mathbf{C})$ and on both $\mathsf{Aut}(M)$ and $\mathsf{Bir}(M)$, in such a way that

$${}^{\beta}g(m) = (\beta \circ g \circ \beta^{-1})(m)$$

for all β in $\mathsf{Aut}_K(\mathbf{C})$, all g in $\mathsf{Bir}(M)$, and all points m in $M(\mathbf{C})$ for which both sides of this equation are well defined. As a consequence, $\mathsf{Aut}_K(\mathbf{C})$ acts by automorphisms on the group $\mathsf{Bir}(M)$. In the case of the projective space, this provides a faithful morphism from $\mathsf{Aut}_{\mathbf{Q}}(\mathbf{C})$ to the group of outer automorphisms of the group $\mathsf{Bir}(\mathbb{P}^n_{\mathbf{C}})$.

1.3. **Abstract morphisms.** To state our main results, note that, given a field morphism $\alpha\colon \mathbf{C} \to \mathbf{C}$, the construction described in the previous paragraph provides an injective morphism $g \mapsto {}^{\alpha}g$ from $\mathsf{Aut}(\mathbb{P}^n_{\mathbf{C}})$ to $\mathsf{Aut}(\mathbb{P}^n_{\mathbf{C}})$. For example, if one writes $\mathbf{C}$ as the algebraic closure of a purely transcendental extension $\mathbf{Q}(x_i, i \in I)$ of the field of rational numbers, and if $\varphi\colon I \to I$ is an injective map, then there exists a field morphism $\alpha\colon \mathbf{C} \to \mathbf{C}$ which maps x_i to $x_{\varphi(i)}$; such a morphism is surjective if and only if φ is onto. In this way, one gets injective, non-surjective, morphisms $\mathsf{Aut}(\mathbb{P}^n_{\mathbf{C}}) \to \mathsf{Aut}(\mathbb{P}^n_{\mathbf{C}})$.

Given g in $\mathsf{PGL}_{n+1}(\mathbf{C})$, we denote by ${}^{t}g$ the linear transpose of g. The map

$$g \mapsto g^{\vee} := ({}^{t}g)^{-1}$$

determines an exterior automorphism of the group $\mathsf{Aut}(\mathbb{P}^n_{\mathbf{C}})$. It is nothing else than the natural morphism given by projective duality and it represents the only exterior and algebraic automorphism of the group $\mathsf{Aut}(\mathbb{P}^n_{\mathbf{C}})$ (see [15]).

Theorem A. *Let M be a smooth, connected, complex projective variety, and let n be its dimension. Let r be a positive integer and let $\rho : \mathsf{Aut}(\mathbb{P}^r(\mathbf{C})) \to \mathsf{Bir}(M)$ be an injective morphism of groups. Then $n \geq r$, and if $n = r$ there exists a field morphism $\alpha\colon \mathbf{C} \to \mathbf{C}$, and a birational mapping $\psi\colon M \dashrightarrow \mathbb{P}^n(\mathbf{C})$ such that either*

$$\psi \circ \rho(g) \circ \psi^{-1} = {}^{\alpha}g, \quad \forall g \in \mathsf{Aut}(\mathbb{P}^n(\mathbf{C}))$$

or

$$\psi \circ \rho(g) \circ \psi^{-1} = ({}^{\alpha}g)^{\vee}, \quad \forall g \in \mathsf{Aut}(\mathbb{P}^n(\mathbf{C}));$$

in particular, M is rational. Moreover, α is an automorphism of $\mathbf{C}$ if ρ is an isomorphism.

The following two results are direct corollaries of Theorem A. The first shows that the Cremona groups $\mathsf{Bir}(\mathbb{P}^n_{\mathbf{C}})$, $n \geq 1$, are pairwise non isomorphic, thereby solving an open problem for $n \geq 4$ (see §1.4.1 below).

Theorem B. *Let n and m be natural integers. The group $\mathsf{Bir}(\mathbb{P}^n_{\mathbf{C}})$ embeds into $\mathsf{Bir}(\mathbb{P}^m_{\mathbf{C}})$ if and only if $n \leq m$. In particular, $\mathsf{Bir}(\mathbb{P}^n_{\mathbf{C}})$ is isomorphic to $\mathsf{Bir}(\mathbb{P}^m_{\mathbf{C}})$ if and only if $n = m$.*

The second characterizes rational varieties M by the structure of $\mathsf{Bir}(M)$, as an abstract group.

Theorem C. *Let M be an irreducible complex projective variety of dimension n. The following properties are equivalent:*

(a) *M is rational;*
(b) *$\mathsf{Bir}(M)$ is isomorphic, as an abstract group, to $\mathsf{Bir}(\mathbb{P}^n_{\mathbf{C}})$;*
(c) *there is a non-trivial morphism from $\mathsf{PGL}_{n+1}(\mathbf{C})$ to $\mathsf{Bir}(M)$.*

Remark 1.1. In fact, if $\mathbf{K}$ is an uncountable subfield of $\mathbf{C}$, and if $\mathsf{PGL}_{n+1}(\mathbf{K})$ embeds into $\mathsf{Bir}(M)$ with $n = \dim_{\mathbf{C}}(M)$, then the complex variety M is rational; for instance, one can take $\mathbf{K} = \mathbf{R}$. This statement follows easily from the proof of Theorem A.

1.4. Two related results.

1.4.1. *Finite subgroups.* Let $\mathbf{k}$ be an algebraically closed field. The group of diagonal matrices in $\mathsf{PGL}_{n+1}(\mathbf{k})$ is a multiplicative group of rank n; hence, it contains a copy of the finite abelian groups $(\mathbf{Z}/p\mathbf{Z})^n$ for all prime integers $p \neq \mathsf{char}(\mathbf{k})$.

Given any prime integer $p \geq 5$ with $p \neq \mathsf{char}(\mathbf{k})$, Beauville proves that the abelian group $(\mathbf{Z}/p\mathbf{Z})^3$ does not embed into $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$; this implies that $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$

is not isomorphic to $\mathsf{Bir}(\mathbb{P}^m_{\mathbf{k}'})$, whatever the choice of $m \neq 2$ and of the algebraically closed field $\mathbf{k}'$ (see [2]).

In [31], Prokhorov proves that, for any prime integer $p \geq 17$ and any field $\mathbf{k}$ of characteristic 0, the abelian group $(\mathbf{Z}/p\mathbf{Z})^4$ does not embed into $\mathsf{Bir}(\mathbb{P}^3_{\mathbf{C}})$. Again, this implies that $\mathsf{Bir}(\mathbb{P}^3_{\mathbf{k}})$ is not isomorphic to $\mathsf{Bir}(\mathbb{P}^m_{\mathbf{k}'})$ if $m \neq 3$ and $\mathbf{k}'$ is an algebraically closed field.

Unfortunately, the methods used in the work of Beauville and Prokhorov are not available in dimension $n \geq 4$ and it was not known, up to now, whether two distinct Cremona groups $\mathsf{Bir}(\mathbb{P}^n_{\mathbf{C}})$ and $\mathsf{Bir}(\mathbb{P}^m_{\mathbf{C}})$, with n and m larger than 3, could be isomorphic. For instance, it is not yet known whether there does exist a finite group without any faithful embedding into $\mathsf{Bir}(\mathbb{P}^4_{\mathbf{C}})$ (it is expected that $\mathsf{PGL}_6(\mathbf{Z}/p\mathbf{Z})$ and even $(\mathbf{Z}/p\mathbf{Z})^5$ do not embed in $\mathsf{Bir}(\mathbb{P}^4_{\mathbf{C}})$ if p is a large prime integer).

1.4.2. *Classical groups and groups of diffeomorphisms.* Theorems B and C should be compared to well known statements concerning morphisms between classical Lie groups – results which we shall use in Section 4.2.2 – as well as morphisms between groups of diffeomorphisms of compact manifolds.

For instance, Filipkiewicz proved the following result in [18]: *Let V and W be two compact manifolds and let $\rho\colon \mathsf{Diff}^k(V) \to \mathsf{Diff}^l(W)$ be an isomorphism between their groups of diffeomorphisms of class C^k and C^l; then $k = l$, and there is a diffeomorphism $\Phi\colon V \to W$ of class C^k such that ρ is the conjugation by Φ.* This shows that the algebraic structure of $\mathsf{Diff}^k(V)$ determines V. Moreover, the existence of an embedding of $\mathsf{Diff}^\infty(V)$ into $\mathsf{Diff}^\infty(W)$ forces the inequality $\dim(V) \leq \dim(W)$, but this result has been obtained only very recently (see [19, 26, 30, 29]).

Our main results follow the same principle but, in the context of groups of birational transformations, one must require that one of the varieties is rational: in general, $\mathsf{Bir}(M)$ is too small to distinguish the birational type of M; for instance, if M has general type, $\mathsf{Bir}(M)$ is finite; more specifically, if M is a generic curve of genus $g \geq 3$ then $\mathsf{Bir}(M) = \{\mathrm{Id}_M\}$. Another issue concerning Theorem A is the existence of isomorphisms coming from automorphisms of the field $\mathbf{C}$; this kind of isomorphisms do not exist in Filipkiewicz's context.

1.5. **Strategy and relevant references.** When $n = 2$, Theorems B and C are due to Julie Déserti (see [13]). The strategy that leads to the proof of Theorem A is similar to Déserti's argument but it also requires several new ideas which can be traced back to, at least, two distinct sources:

1.– Weil's regularization Theorem (see [34]), a result that transforms a group of birational transformations of M with uniformly bounded degrees into a group of automorphisms of a new variety M' by a birational change of variables

$\Psi\colon M' \dashrightarrow M$. This is described in Section 2.3, with interesting complements that can be found in the articles [12] by Demazure, [32, 33] by Umemura, and [25, 35] by Huckleberry and Zaitsev.

2.– The work of Epstein and Thurston on nilpotent Lie subalgebras in the Lie algebra of smooth vector fields of a compact manifold; see [17], as well as [20] for similar ideas in the context of groups of analytic diffeomorphisms.

We work over the field of complex numbers because it is algebraically closed, it is not countable and it has characteristic zero. These properties are all used in some way during the proof, but the crucial point is that $\mathbf{C}$ is not countable. We could also prove Theorem A for groups of bimeromorphic transformations of compact Kähler manifolds; we stick to the case of projective varieties because one of the key steps is Weil's regularization Theorem, the proof of which is not accessible in the literature for Kähler manifolds; on the other hand, we write the proof, as much as we can, in the language of complex differential geometry.

1.6. **Acknowledgement.** Thanks to Jérémy Blanc, Julie Déserti, Adrien Dubouloz, Vincent Guirardel, and Stéphane Lamy for interesting discussions on this topic. The author is also grateful to an anonymous referee for his suggestions that led to a correction and an improvement of the exposition.

2. BIRATIONAL ACTIONS, DEGREES, AND REGULARIZATION

In this section, we collect several basic facts regarding groups of birational transformations and then describe Weil's regularization Theorem.

2.1. **Degrees and volumes.** Let M be a smooth, irreducible, complex projective variety; denote its dimension by n. Let κ be a Kähler form on M, fixed once and for all.

2.1.1. *Kähler metrics.* If k is a positive integer, denote by $\pi_i : M^k \to M$ the projection onto the i-th factor: $\pi_i(x_1, x_2, \ldots, x_k) = x_i$. The manifold M^k is then endowed with the Kähler form $\sum_{i=1}^k \pi_i^* \kappa$. Volumes of submanifolds of M^k are computed with respect to the Kähler metric determined by this Kähler form.

2.1.2. *Graphs.* Each birational transformation f of M is determined by two Zariski dense open subsets U and V of M, and a regular isomorphism $f\colon U \to V$. The largest open subset U on which f is regular is the **domain of definition** $\mathrm{Dom}(f)$; its complement is the **indeterminacy locus** $\mathrm{Ind}(f)$; the codimension of $\mathrm{Ind}(f)$ is ≥ 2.

To each birational transformation $f\colon M \dashrightarrow M$, one associates its graph

$$\Gamma_f \subset M \times M,$$

defined as the Zariski closure of the set $\{(x, f(x)) \in M\times M\,;\, x\in \mathrm{Dom}(f)\}$. By construction, Γ_f is an irreducible subvariety of $M\times M$ of dimension n. Both projections $\pi_1, \pi_2\colon M\times M\to M$ restrict to birational morphisms $\pi_i\colon \Gamma_f\to M$, a fact which characterizes the set of graphs of birational transformations.

Example 2.1. The graph Γ_f can be singular, as in the case of the monomial transformation of the plane defined by $f(x,y) = (y^3/x^2, y/x)$ in affine coordinates.

2.1.3. *Degrees.* The **total degree** (or **degree** for short) $\mathsf{tdeg}(f)$ of a birational transformation f is defined as the volume of Γ_f with respect to the fixed metric defined on $M\times M$ in §2.1.1 (confer [7]); hence,

$$\mathsf{tdeg}(f) = \int_{\Gamma_f} (\pi_1^*\kappa + \pi_2^*\kappa)^n = \int_{\mathrm{Dom}(f)} (\kappa + f^*\kappa)^n.$$

If L is a very ample line bundle on M, and κ is the pull-back of the Fubini-Study metric by the natural embedding of M in $\mathbb{P}(H^0(M,L)^\vee)$, then $\mathsf{tdeg}(f)$ is the degree of the graph of f with respect to the polarization $\pi_1^*(L)\otimes\pi_2^*(L)$.

Lemma 2.2 (cf. [16] Lemma 4, and [22])**.** *There exists a constant c_M, which depends only on M and κ, such that*

$$\mathsf{tdeg}(f\circ g) \leq c_M\, \mathsf{tdeg}(f)\, \mathsf{tdeg}(g)$$

for all f and g in $\mathsf{Bir}(M)$.

Changing κ into $c_M^{1/n}\kappa$, one may, and do, assume that

$$\mathsf{tdeg}(f\circ g) \leq \mathsf{tdeg}(f)\, \mathsf{tdeg}(g). \tag{2.1}$$

Similarly, if $\psi\colon M' \dashrightarrow M$ is a birational transformation, and κ', κ are Kähler metrics on M' and M, there exists a constant c_ψ such that

$$\mathsf{tdeg}(\psi^{-1}\circ f\circ \psi) \leq c_\psi\, \mathsf{tdeg}(f) \tag{2.2}$$

for all f in $\mathsf{Bir}(M)$.

2.1.4. *Groups with bounded degrees.* Let $d\geq 1$ be a natural integer. The subset $\mathsf{Bir}_d(M)$ of $\mathsf{Bir}(M)$ is defined as

$$\mathsf{Bir}_d(M) = \{f\in \mathsf{Bir}(M)\,;\quad \mathsf{tdeg}(f)\leq d\}.$$

A subgroup G of $\mathsf{Bir}(M)$ has **bounded degree** if it is contained in $\mathsf{Bir}_d(M)$, for some $d\in \mathbf{N}^*$.

2.2. **Components of** $\mathsf{Bir}(M)$ **(see** [23, 25, 35]**).** We summarize a few facts that are proved in [23] in the language of Hilbert schemes; they may be replaced by Douady spaces if one wants to work on compact Kähler manifolds (see [6] for a panorama and references to the literature). For complex projective manifolds M, the Hilbert scheme and Douady space coincide (in the sense that the associated analytic spaces are isomorphic).

On our way, we introduce notation that will be useful to the proof of Theorem A. As above, M is a smooth and irreducible complex projective variety of dimension n.

2.2.1. *Components.* The set $\mathsf{Bir}(M)$ is contained in the Douady space (resp. Hilbert scheme) parametrizing complex analytic subsets of $M\times M$ of dimension n; more precisely $\mathsf{Bir}(M)$ is identified to the subset of irreducible subvarieties $\Gamma\subset M\times M$ of dimension n such that both projections $\pi_1,\pi_2\colon\Gamma\to M$ have degree 1. The Douady space of n-dimensional subvarieties of $M\times M$ of volume at most d is made of finitely many components W_j; each W_j is compact. The intersections of $\mathsf{Bir}_d(M)$ with these components are denoted by $\mathsf{Bir}_d^j(M)$; hence each $\mathsf{Bir}_d^j(M)$ is a subset of some component W_j of the Douady space. We shall call the sets $\mathsf{Bir}_d^j(M)$ the **components** of $\mathsf{Bir}(M)$.[1]

Given a subvariety V of $M\times M$ of dimension n, one can compute the degrees $\deg_1(V)$ and $\deg_2(V)$ of the natural projections $\pi_{1|V}$ and $\pi_{2|V}\colon V\to M$. These degrees define two functions on the Douady space (resp. Hilbert scheme). If W is a component of the Douady space, the function $\deg_1(\cdot)$ and $\deg_2(\cdot)$ are constant on W. Thus, if W contains a graph of a birational transformation f, then $\deg_1=\deg_2=1$ on W (see for instance, [24], §III.9, or [1] Chap. IV). Moreover, the subset of elements V of W corresponding to irreducible subvarieties of $M\times M$ is open; for Hilbert schemes, this statement is contained in Theorem 12.2 of [21].

Thus, as proved by Hanamura, each $\mathsf{Bir}_d^j(M)$ is an open subset in the component W_j of the Douady space that contains it (see [23], Proposition (1.7)); this endows $\mathsf{Bir}_d^j(M)$ with the structure of an analytic space.

Given two components $\mathsf{Bir}_d^j(M)$ and $\mathsf{Bir}_{d'}^{j'}(M)$, the composition $(f,g)\mapsto f\circ g$ defines a rational map from $\mathsf{Bir}_d^j(M)\times\mathsf{Bir}_{d'}^{j'}(M)$ to one of the components

[1] The notation $\mathsf{Bir}^{W_j}(M)$ instead of $\mathsf{Bir}_d^j(M)$ would be better, because the value of the degree d is already encoded in the choice of the component W_j. Our choice has two advantages: the notation is not too heavy; it keeps track of the degree bound $\leq d$ and of the fact that a component of the Douady space has been fixed.

The terminology "*component*" for the sets $\mathsf{Bir}_d^j(M)$ may be misleading: We do not mean that these subsets are irreducible components of an algebraic variety or connected components of a topological space.

$\mathsf{Bir}^k_{dd'}(M)$ (see Equation (2.1)); similarly, the action of $\mathsf{Bir}(M)$ on M defines rational mappings $\mathsf{Bir}^j_d(M) \times M \dashrightarrow M$. These assertions are proved or implicitly used in [23] (see Proposition (2.7)), and a complete proof is given in [25] (see Lemma 5.4 and Lemma 5.5) in the language of Barlet spaces.

Example 2.3 (see [10])**.** Let us describe an example in dimension 2 to illustrate the different viewpoints that one can use and explain why the composition may have indeterminacies on $\mathsf{Bir}^j_d(M) \times \mathsf{Bir}^{j'}_{d'}(M)$). Consider the set Q of quadratic birational transformations of the plane $\mathbb{P}^2_{\mathbf{C}}$, *i.e.* birational transformations $f[x : y : z] = [P : Q : R]$ defined by homogeneous polynomials of degree 2 with no common factor of positive degree. The total degree of such a transformation is equal to 4. Indeed, the class of the graph $\Gamma_f \subset \mathbb{P}^2_{\mathbf{C}} \times \mathbb{P}^2_{\mathbf{C}}$ is equal to

$$[\mathbb{P}^2(\mathbf{C})] \times \{\text{point}\} + \{\text{point}\} \times [\mathbb{P}^2(\mathbf{C})] + 2[\mathbb{P}^1(\mathbf{C})] \times [\mathbb{P}^1(\mathbf{C})]$$

where $[\mathbb{P}^1(\mathbf{C})]$ is the class of a line; thus, the volume of Γ_f with respect to the polarization $O(1)$ of $\mathbb{P}^2_{\mathbf{C}}$ is equal to 4. One can show that Q coincides with one component of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{C}})$ in the sense of Hanamura. More precisely, $\mathsf{PGL}_3(\mathbf{C}) \times \mathsf{PGL}_3(\mathbf{C})$ acts on Q by left and right composition, and there are exactly three orbits: Every element $f \in Q$ is a composition $a \circ g \circ b$ where a and b are automorphisms and g is one of the three quadratic involutions

$$\sigma[x : y : z] = [yz : zx : xy],\ \rho[x : y : z] = [xy : z^2 : yz],\ \tau[x : y : z] = [x^2 : xy : y^2 - xz].$$

The orbit of σ is an open and dense subset U_σ of Q of dimension 14. The orbits of ρ and τ have dimension 13 and 12 respectively.

Let $f = a \circ \sigma \circ b$ and $g = a' \circ \sigma \circ b'$ be elements of U_σ. The indeterminacy locus $\mathrm{Ind}(\sigma)$ is the set $\{e_1, e_2, e_3\}$ with $e_1 = [1 : 0 : 0]$, $e_2 = [0 : 1 : 0]$, and $e_3 = [0 : 0 : 1]$; its exceptional locus, $\mathrm{Exc}(\sigma)$ is the triangle of the three lines that go through pairs of indeterminacy points; each of these lines is contracted to the opposite vertex. Thus, $\mathrm{Ind}(f)$ is $b^{-1}(\mathrm{Ind}(\sigma))$ and $\mathrm{Exc}(f)$ is mapped to $a(\mathrm{Ind}(\sigma))$. Then the birational transformation $g \circ f$ is defined by homogeneous formulas of degree 4 if and only if f does not contract any curve onto an indeterminacy point of g, if and only if $a(\mathrm{Ind}(\sigma))$ is disjoint from $(b')^{-1}(\mathrm{Ind}(\sigma))$. This condition determines a Zariski open subset W of $U_\sigma \times U_\sigma$ and one easily verifies that the composition $(f, g) \mapsto g \circ f$ is a regular map from W to a component of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{C}})$. On the other hand, when f contracts a curve on an indeterminacy point of g, then $\deg(g \circ f) < 4$; this phenomenon shows that the composition does not extend as a regular map to $Q \times Q$.

Remark 2.4. We refer to [12, 4] for another structure on $\mathsf{Bir}(M)$ that differs from Hanamura's viewpoint. In both cases, $\mathsf{Bir}_d(M)$ does not have a natural structure of algebraic variety (see [4] and [23] for interesting examples).

2.2.2. *Zariski closures.* Let A be a subset of $\mathsf{Bir}(M)$. Let $\mathsf{Bir}_d^j(M)$ be a component of $\mathsf{Bir}(M)$. The **Zariski closure** $Z_d^j(A)$ is, by definition, the intersection of $\mathsf{Bir}_d^j(M)$ with the Zariski closure of $A \cap \mathsf{Bir}_d^j(M)$ in the component of the Douady space (or Hilbert scheme) that contains $\mathsf{Bir}_d^j(M)$. There are at most countably many components $\mathsf{Bir}_d^j(M)$. If A is uncountable, at least one of the $\mathsf{Bir}_d^j(M)$ intersects A on an uncountable subset. Hence, at least one of the $Z_d^j(A)$ has dimension ≥ 1.

2.3. **Weil's regularization Theorem.**

2.3.1. *Regularization.* Let M be a complex projective variety and G a subgroup of $\mathsf{Bir}(M)$. One says that G can be **regularized**, if there exist a smooth complex projective variety M' and a birational map $\Psi \colon M' \dashrightarrow M$ such that

$$\Psi^{-1} \circ G \circ \Psi \subset \mathsf{Aut}(M').$$

In other words, changing M into another birationally equivalent variety M', all indeterminacy points of all elements of G disappear simultaneously.

Theorem 2.5 (Weil's regularization Theorem, I)**.** *Let M be a complex projective variety. Let G be a subgroup of* $\mathsf{Bir}(M)$*. If G has bounded degree, then G can be regularized.*

The proof of this result can be found in [25, 35]. The heuristic idea is to replace G by its Zariski closure $\overline{G}$ in the components $\mathsf{Bir}_d^j(M)$, with d large enough to assure that $\mathsf{Bir}_d(M)$ contains G. Since G is Zariski dense in $\overline{G}$, the composition law on $\mathsf{Bir}(M)$ restricts to a rational map $\overline{G} \times \overline{G} \dashrightarrow \overline{G}$. Similarly, the action of G on M extends to a rational map $\overline{G} \times M \dashrightarrow M$. These mappings endow $\overline{G}$ with the structure of a pre-algebraic group acting by birational transformations on M, in the sense of [34], and Weil's original theorem can then be applied to this group.

2.3.2. *Complement.* Let M be a complex projective variety, or more generally a compact Kähler manifold. The group $\mathsf{Aut}(M)$ is a complex Lie group. Moreover, each subset

$$\mathsf{Aut}_d(M) = \mathsf{Bir}_d(M) \cap \mathsf{Aut}(M)$$

intersects only finitely many connected components of $\mathsf{Aut}(M)$. If d is larger than the volume of the diagonal Γ_{Id_M}, then $\mathsf{Aut}_d(M)$ contains the connected component of the identity $\mathsf{Aut}(M)^0$, and $\mathsf{Aut}_d(M)/\mathsf{Aut}(M)^0$ is a finite set. If G is a subgroup of $\mathsf{Aut}(M)$ that is contained in some $\mathsf{Aut}_d(M)$, then $G \cap \mathsf{Aut}(M)^0$ is a normal subgroup of G with finite index (see [28] for instance). As a consequence, when Weil's regularization Theorem is applied, one obtains the following stronger result.

Theorem 2.6 (Weil's regularization Theorem, II)**.** *Let M be a complex projective variety. Let G be a subgroup of $\mathsf{Bir}(M)$. If G has bounded degree, there exists a smooth, complex, projective variety M', and a birational map $\Psi\colon M' \dashrightarrow M$ such that*

(1) $\Psi^{-1} \circ G \circ \Psi$ *is a subgroup of* $\mathsf{Aut}(M')$*;*
(2) *there exists a normal, finite index subgroup $G_0 \subset G$ such that $\Psi^{-1} \circ G_0 \circ \Psi$ is contained in* $\mathsf{Aut}(M')^0$.

In particular, $\Psi^{-1} \circ G \circ \Psi$ is a subgroup of $\mathsf{Aut}(M')^0$ if G is simple.

3. VECTOR FIELDS AND ACTIONS OF NILPOTENT GROUPS

Let M be a smooth and irreducible complex projective variety. This section is devoted to the construction of meromorphic (or rational) vector fields and the study of the Lie algebra they generate. This is applied to the study of uncountable abelian and nilpotent subgroups of $\mathsf{Bir}(M)$.

3.1. **Construction of meromorphic vector fields.**

3.1.1. *Meromorphic vector fields.* Denote by $\Theta_m(M)$ the complex vector space of meromorphic (or rational) vector fields on M. Given $Y \in \Theta_m(M)$, we denote by $\mathrm{Dom}(Y)$ the domain of definition of Y, *i.e.* the Zariski dense open subset of M on which Y is locally regular. Since M is projective (and $n \geq 1$), $\Theta_m(M)$ is infinite dimensional: The field of meromorphic functions $\mathbf{C}(M)$ is an infinite dimensional complex vector space which acts by left multiplication on $\Theta_m(M)$.

With its Lie bracket $[.,.]$, the vector space $\Theta_m(M)$ forms a complex Lie algebra. In local coordinates $(x_i)_{1\leq i\leq k}$, the Lie bracket of two vector fields $X = \sum_i a_i(x)\partial_i$ and $Y = \sum_j b_j(x)\partial_j$ is given by

$$[X,Y](x) = \sum_j \sum_i \left(a_i(x)\frac{\partial b_j}{\partial x_i}(x) - b_i(x)\frac{\partial a_j}{\partial x_i}(x) \right) \partial_j$$

(where ∂_j stands for the vector field $\partial/\partial x_j$).

3.1.2. *Construction of vector fields.* Fix a component $\mathsf{Bir}_d^j(M)$ of $\mathsf{Bir}(M)$. Let h be a smooth point of $\mathsf{Bir}_d^j(M)$ and let v be a tangent vector to $\mathsf{Bir}_d^j(M)$ at the point h. The derivative of the action

$$\mathsf{act}\colon (h,x) \mapsto h(x)$$

in the direction v determines a meromorphic vector field X_v on M. More precisely, if h_t is a path in Z such that $h_0 = h$ and $\partial_t(h_t)_{|t=0} = v$, then

$$X_v(h(x)) = \frac{\partial}{\partial t}(h_t(x))_{|t=0}. \tag{3.1}$$

By construction, X_v does not vanish identically if $v \neq 0$. This linear injective map

$$X\colon T_h\mathsf{Bir}^j_d(M) \to \Theta_m(M) \tag{3.2}$$

provides a link between $\mathsf{Bir}^j_d(M)$ and $\Theta_m(M)$ that plays an important role in the proof of Theorem A.

Example 3.1. Consider the following family of birational transformations of $\mathbf{C}^2 \subset \mathbb{P}^2_{\mathbf{C}}$: $h_t(x,y) = ((1+t)x, ((1+t)x)^d y)$, where d is a fixed positive integer and t describes the open unit disk $\mathbb{D}$. For $t = 0$, h_0 is the monomial transformation which maps (x,y) to $(x, x^d y)$. Then

$$\frac{\partial}{\partial t}(h_t(x,y))_{|t=0} = (x, dx^d y),$$

and the corresponding vector field X_v satisfies $X_v(x, x^d y) = (x, dx^d y)$. This leads to the formula $X_v(x,y) = (x, dy)$, so that X_v has degree 1 for all $d \geq 1$.

Example 3.2. Starting with the family $h_t(x,y) = (x, (1+tx^d)y)$, with $h_0(x,y) = (x,y)$, one gets

$$\frac{\partial}{\partial_t}(h_t(x,y))_{|t=0} = (0, x^d y)$$

and $X_v(x,y) = (0, x^d y)$ has degree d.

3.2. **Uncountable abelian groups and abelian Lie algebras.**

3.2.1. *Zariski closures.* Let A be a subgroup of $\mathsf{Bir}(M)$. For all components $\mathsf{Bir}^j_d(M)$ of $\mathsf{Bir}(M)$, denote by A^j_d the intersection of A with $\mathsf{Bir}^j_d(M)$, and by A_d the union of the subsets A^j_d (for fixed d). The sets A_d form an increasing sequence of subsets of A, and

$$A_d \circ A_{d'} \subset A_{dd'} \tag{3.3}$$

for all pairs of integers (d, d'). Let $Z^j_d(A)$ be the Zariski closure of A^j_d in $\mathsf{Bir}^j_d(M)$ and $Z_d(A)$ be the disjoint union of the $Z^j_d(A)$. Since $Z^j_d(A)$ may have irreducible components with distinct dimensions, $\dim(Z^j_d(A))$ is defined as the maximum of the dimensions of its components; then, one defines

$$\dim(Z_d(A)) = \max_j \dim(Z^j_d(A)).$$

The following Lemma follows from Section 2.2.1, Equation (3.3), the Zariski density of A^j_d in $Z^j_d(A)$, and the fact that at least one A^j_d is infinite if A is not countable.

Lemma 3.3. *Let A be an uncountable subgroup of* $\mathsf{Bir}(M)$.

(1) *There exists a component* $\mathsf{Bir}^j_d(M)$ *such that* $\dim(Z^j_d(A)) \geq 1$.

(2) *The function* $d \mapsto \dim(Z_d(A))$ *is non-decreasing.*
(3) $Z_d(A) \circ Z_{d'}(A) \subset Z_{dd'}(A)$ *for all* d, $d' \geq 1$.
(4) *If* A *is abelian, then* $f \circ g = g \circ f$ *for all pairs* $(f,g) \in Z_d(A) \times Z_{d'}(A)$*; in particular,* $f \circ g = g \circ f$ *for all* f *in* $Z_d(A)$ *and all* g *in* A.

3.2.2. *Abelian Lie algebras.* We now assume that A is abelian and uncountable.

Let $d \geq 1$ be an integer such that $\dim(Z_d(A)) \geq 1$. Choose a component $\mathsf{Bir}_d^j(M)$ for which $\dim Z_d^j(A) \geq 1$, and let h be a smooth point of $Z_d^j(A)$. The map

$$X\colon\; v \in T_h Z_d^j(A) \mapsto X_v \in \Theta_m(M)$$

is linear and injective. If f is an element of A and v is an element of $T_h Z_d^j(A)$,

$$f_* X_v = X_v.$$

Indeed, writing v as the velocity vector of a path h_t at $t=0$, with t in the unit disk $\mathbb{D}$, one has

$$\begin{aligned}(f_*X_v)(f(h(x))) &= Df_x(X_v(h(x))) = \frac{\partial}{\partial t}(f \circ h_t(x))_{|t=0} \\ &= \frac{\partial}{\partial t}(h_t(f(x)))_{|t=0} = X_v(h(f(x))) \\ &= X_v(f(h(x))),\end{aligned}$$

so that $X_v(y) = (f_*X_v)(y)$ for all y in a Zariski dense open subset of M. Consequently $g_*X_v = X_v$ for all $d' \geq 1$ and g in $Z_{d'}(A)$. For $g=h$, one gets a new formula for X_v, namely

$$X_v(x) = (h^{-1})_* \frac{\partial}{\partial t}(h_t(x))_{|t=0}.$$

As a consequence, the various vector fields X_v, X_w, for $v \in T_h Z_d^j(A)$ and $w \in T_g Z_{d'}^{j'}(A)$ commute:

$$[X_v, X_w] = 0 \tag{3.4}$$

in the Lie algebra $\Theta_m(M)$. In particular, we have:

Lemma 3.4. *Let* A *be an abelian subgroup of* $\mathsf{Bir}(M)$*, and let* d *be a positive integer. For all components* $\mathsf{Bir}_d^j(M)$*, and all smooth points* h *of* $Z_d^j(A)$*, the image of*

$$X\colon\; T_h Z_d^j(A) \to \Theta_m(M)$$

is an abelian Lie subalgebra of $\Theta_m(M)$ *of dimension* $\dim(T_h(Z_d^j(A)))$*. Moreover,* $g_*Y = Y$ *for all elements* Y *in this algebra, all* $d' \geq 1$*, and all* g *in* $Z_{d'}(A)$.

Let α and β be two complex numbers. If (h_t) is a path in $Z_d^j(A)$ with $h_0 = h$ and velocity vector v, and if (g_s) is a path in $Z_{d'}^{j'}(A)$ with $g_0 = g$ and velocity vector w, then $(h_{\alpha t} \circ g_{\beta t})$ is a path in $Z_{dd'}(A)$ such that

$$\frac{\partial}{\partial t}(h_{\alpha t} \circ g_{\beta t})_{|t=0} = \alpha X_v + \beta X_w.$$

This shows that the union of all vector spaces $X(T_h Z_d^j(A))$, for all components $\mathsf{Bir}_d^j(M)$ and all $h \in Z_d^j(A)$, is an abelian sub-algebra of $\Theta_m M$. We denote this abelian algebra by $\mathfrak{a}_\infty(A)$ and call it the **Lie algebra associated to** A.

Example 3.5. Let A be the abelian group $(\mathbf{C}, +)$. This group is isomorphic to the group $(\mathbf{C}[y], +)$ of polynomial functions in one variable. In particular, A is isomorphic to the group of birational transformations of the plane of the type

$$(x, y) \mapsto (x + p(y), y),$$

where p describes $\mathbf{C}[y]$. The Lie algebra $\mathfrak{a}_\infty(A)$ is made of all vector fields $q(y)\partial_x$, with q in $\mathbf{C}[y]$.

Example 3.6. We can also embed $A = (\mathbf{C}, +)$ into $\mathsf{Bir}(\mathbb{P}^3_{\mathbf{C}})$ as follows. Let $\rho : A \to \mathbf{Z}$ be any surjective morphism. Then A acts on $\mathbb{P}^3(\mathbf{C})$ by

$$(x, y, z) \mapsto (x + p(y), y, y^{\rho(p)} z).$$

The Lie algebra $\mathfrak{a}_\infty(A)$ coincides with the set of vector fields $q(y)\partial_x$, with q in $\mathbf{C}[y]$. Moreover, there are now two types of elements f in $Z_d(A)$: If $\rho(p) = 0$, then $f^l \in \mathsf{Bir}_d(\mathbb{P}^3_{\mathbf{C}})$ for all l; if $\rho(p) \neq 0$, then $\mathsf{tdeg}(f^l)$ goes to infinity with l.

3.2.3. *Orbits.* Given m in M, define

$$\begin{aligned} V(m) &= \mathrm{Vect}_{\mathbf{C}}\{X_v(m)\,;\, X_v \in \mathfrak{a}_\infty(A),\, m \in \mathrm{Dom}(X_v)\} \\ s(m) &= \dim_{\mathbf{C}}(V(m)) \quad (\text{hence } s(m) \leq n = \dim_{\mathbf{C}}(M)) \\ s(A) &= \max(s(m)\,;\, m \in M). \end{aligned}$$

Note that $s(A)$ coincides with the value of $s(m)$ at the generic point.

The following lemmas are not required for the proof of Theorem A, but they illustrate some of the forthcoming arguments. We thus state them and only provide a hint for their proofs.

Lemma 3.7. *If* $\dim_{\mathbf{C}}(\mathfrak{a}_\infty(A)) > s(A)$*, there exists a non-constant, A-invariant, rational function* $\alpha\colon M \to \mathbf{C}$.

Proof. Let $X_1, \ldots, X_s$, $s = s(A)$, be elements of $\mathfrak{a}_\infty(A)$ such that $(X_1(m), \ldots, X_s(m))$ is a basis for $V(m)$ at the generic point of M. Since $\dim_{\mathbf{C}}(\mathfrak{a}_\infty(A)) > s$, there is an element Y of $\mathfrak{a}_\infty(A)$ which is not a linear combination of the X_i with coefficients in $\mathbf{C}$. On the other hand, by definition of s, there exist rational functions $\alpha_1, \ldots, \alpha_s \in \mathbf{C}(M)$ such that

$$Y(m) = \sum_{i=1}^{i=s} \alpha_i(m) X_i(m).$$

The α_i are uniquely determined by this relation and at least one of them is not constant. Given $f \in A$, Lemma 3.4 shows that $f_* X_i = X_i$ for all $1 \leq i \leq s$ and $f_* Y = Y$; this implies

$$\alpha_i \circ f = \alpha_i, \ \forall i \geq 1,$$

and the conclusion follows because at least one of the α_i is not constant. □

For component $\mathsf{Bir}_d^j(M)$ and each m in M, define the **orbit** of $Z_d^j(A)$ as the following subset $\mathrm{Orb}_d^j(m)$ of M,

$$\mathrm{Orb}_d^j(m) = \{h(m)\,;\, h \in Z_d^j(A) \text{ and } m \in \mathrm{Dom}(h)\}.$$

Denote its Zariski closure by $\overline{\mathrm{Orb}}_d^j(m)$. The orbit $\mathrm{Orb}_d(m)$ is defined as the union of the $\mathrm{Orb}_d^j(m)$, and $\overline{\mathrm{Orb}}_d(m)$ is the Zariski closure of $\mathrm{Orb}_d(m)$ in M. By definition, these orbits $\mathrm{Orb}_d(m)$ are tangent to the distribution of subspaces $V(m)$, $m \in M$. Thus, choosing good components to assure that the generic orbits $\mathrm{Orb}_d^j(m)$ have dimension $s(A)$, one obtains the following lemma.

Lemma 3.8. *The distribution of subspaces $V(m)$, $m \in M$, is integrable in the following sense. There exists a projective variety B of dimension $n - s(A)$, a rational map $\Psi\colon M \dashrightarrow B$, and a component $\mathsf{Bir}_d^j(M)$, such that*

(1) $\mathrm{Orb}_d^j(m)$ *has dimension $s(A) = \dim_{\mathbf{C}} V(m)$ and is tangent to $V(m)$ for generic $m \in M$;*
(2) Ψ *is constant on each irreducible component of the generic orbit* $\mathrm{Orb}_d^j(m)$*;*
(3) Ψ *is a local submersion at the generic point.*

This fibration $\Psi\colon M \dashrightarrow B$ is A-invariant: There is a morphism $\rho_B\colon A \to \mathsf{Bir}(B)$ with $\rho(f) \circ \Psi = \Psi \circ f$ for all f in A. Moreover, $\dim Z_d(\rho_B(A)) = 0$ for all $d \geq 1$, since otherwise one would be able to construct meromorphic vector fields in $\mathfrak{a}_\infty(A)$ that are transverse to the generic fibers of Ψ, in contradiction with the definition of $V(m)$ and $s(A)$. Thus, the image of ρ_B is countable (compare with Example 3.6).

3.3. **Bounding degrees of abelian groups.** The following proposition is a crucial step towards proving Theorem A.

Proposition 3.9. *Let M be a smooth, connected, complex projective variety. Let A be an abelian subgroup of $\mathsf{Bir}(M)$. If $s(A) = \dim_{\mathbf{C}}(M)$, then A has bounded degree: There exists $d \geq 1$ such that $\mathsf{Bir}_d(M)$ contains A.*

Proof. Since $s(A) = n$, one can find n elements $X_1, \ldots, X_n$ of $\mathfrak{a}_\infty(A)$ such that

$$\mathrm{Vect}_{\mathbf{C}}(X_1(m), \ldots, X_n(m)) = T_m M$$

at the generic point m of M. Each X_i is obtained from a tangent vector $v_i \in T_{h_i} Z^j_{d_i}(A)$ for some for some birational transformation h_i in one of the varieties $Z^j_{d_i}(A)$. Let h_{i,t_i}, $t_i \in \mathbb{D}$, be a path in $Z^j_{d_i}(A)$ with $h_{i,0} = h_i$ and velocity vector v_i at $t_i = 0$. Composing those paths together, one gets a subset of some $Z^l_d(A)$ with $d = d_1 \cdots d_n$ (cf. Equation (2.1)). By construction, if m_0 is generic, its orbit $\mathrm{Orb}^l_d(m_0)$ under the action of $Z^l_d(A)$ contains an open neighborhood of m_0. We now fix such a point m_0.

Let f be an element of A, and m be a point of $\mathrm{Dom}(f) \cap \mathrm{Orb}^l_d(m_0)$ such that $f(m)$ is also contained in $\mathrm{Orb}^l_d(m_0)$. Then there exists h in $Z^l_d(A)$ such that $h \circ f(m) = m$. Since both h and f commute to all elements g of $Z^l_d(A)$, one obtains

$$h \circ f(g(m)) = g(h \circ f(m)) = g(m)$$

for all $g \in A$ such that $\{m, h \circ f(m)\} \subset \mathrm{Dom}(g)$. Consequently, $h \circ f$ fixes pointwise all these points $g(m)$; since they form a Zariski dense subset of M, one deduces that f coincides with h^{-1}. This concludes the proof, because $h^{-1} \in \mathsf{Bir}_d(M)$ for all h in $\mathsf{Bir}_d(M)$. □

3.4. **Nilpotent groups, dimensions comparison, and degree bounds.**

3.4.1. *Bounds on derived length.* Let H be a group. We define $H^{(1)} = [H, H]$, the derived subgroup of H, generated by all commutators $aba^{-1}b^{-1}$ with a and b in H, and then inductively

$$H^{(r)} = [H^{(r-1)}, H^{(r-1)}].$$

The first integer $r \geq 1$ such that $H^{(r)}$ is trivial is called the **derived length** of H; such an r exists if and only if H is solvable. This integer is denoted by $\mathsf{dl}(H)$, and similar notations are used for Lie algebras.

Proposition 3.10 (Epstein-Thurston, [17])**.** *Let M be a connected complex manifold. Let $\mathfrak{h}$ be a nilpotent Lie subalgebra of the Lie algebra $\Theta_m(M)$. Then $\mathfrak{h}^{(r)} = 0$ if $r \geq \dim(M)$; hence*

$$\dim_{\mathbf{C}}(M) \geq \mathsf{dl}(\mathfrak{h}).$$

Remark 3.11. Note that $\mathfrak{h}$ is assumed to be nilpotent while $\mathsf{dl}(\mathfrak{h})$ is the derived length of $\mathfrak{h}$ as a solvable Lie algebra.

Proof. We prove Proposition 3.10 by induction on the dimension n of M.

Assume that $\mathfrak{h}$ has positive dimension, since otherwise the result is clear. Its center is non-trivial because $\mathfrak{h}$ is nilpotent. Let X be a non zero element in the center of $\mathfrak{h}$ and m be a point at which X is well defined and $X(m) \neq 0$. There is a local system of coordinates $(x_1, ..., x_n)$ in a neigborhood U of m such that $X = \partial_{x_n}$ in U. Since X is in the center of $\mathfrak{h}$, all elements of $\mathfrak{h}$ are of the form

$$v(x_1, ..., x_{n-1})\partial_{x_n} + \sum_{i=1}^{n-1} u_i(x_1, ..., x_{n-1})\partial_{x_i}. \tag{3.5}$$

For $n = 1$, this implies that $\mathfrak{h}$ is abelian, of dimension at most 1, so that $\mathsf{dl}(\mathfrak{h}) \leq 1 = \dim(M)$. We now assume that the result is proved up to dimension $n-1$.

Let $\pi\colon U \to \mathbf{C}^{n-1}$ be the projection $\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$. Locally, the fibers of π are the orbits of X; since X is in the center, π projects $\mathfrak{h}$ onto a nilpotent algebra $\mathfrak{h}_1$ of meromorphic vector fields on $\mathbf{C}^{n-1}$: If $Y \in \mathfrak{h}$ is defined by Equation (3.5), $\pi_* Y$ is equal to $\sum u_i(x_1, ..., x_{n-1})\partial_{x_i}$. This defines an exact sequence

$$0 \to \mathfrak{h}_0 \to \mathfrak{h} \to \mathfrak{h}_1 \to 0$$

where the kernel $\mathfrak{h}_0$ of π_* is made of vector fields of type $v(x_1, ..., x_{n-1})\partial_{x_n}$ and, as such, is abelian. The induction hypothesis implies that $\mathfrak{h}^{(n-1)} = 0$, and the conclusion follows. □

3.4.2. *Embeddings of Heisenberg groups.* Let now H_r be the group of $r \times r$ upper triangular matrices with all diagonal coefficients equal to 1; this group is nilpotent, and its derived length is the smallest integer $l(r)$ such that $2^{l(r)} > r$. If B is in H_r, we denote by $a_{i,j}(B)$ its coefficients. For all pairs of indices (i, j), with $1 \leq i < j \leq r$, we define $A_{i,j}$ as the subgroup of H_r which is made of elements B such that $a_{k,l}(B) = 0$ if $k \neq l$ and $(k,l) \neq (i, j)$. This defines a one-parameter, abelian subgroup of H_r. Moreover, $A_{1,r}$ coincides with the center of H_r.

Let us now assume that H_r embeds into the group $\mathsf{Bir}(M)$. From section 3.2.2, each $A_{i,j}$ gives rise to an abelian subalgebra $\mathfrak{a}_\infty(A_{i,j})$ of $\Theta_m(M)$.

Lemma 3.12. *The Lie algebra $\mathfrak{h}_r \subset \Theta_m(M)$ generated by the abelian algebras $\mathfrak{a}_\infty(A_{i,j})$ is nilpotent and its derived length $\mathsf{dl}(\mathfrak{h}_r)$ is equal to $l(r)$, the smallest integer l such that $2^l > r$.*

Proof. Since $[A_{i,j}, A_{k,l}]$ is equal to $A_{i,l}$ if $k = j$ and is equal to $\{\mathrm{Id}\}$ if $k \neq j$, we obtain $[\mathfrak{a}_\infty(A_{i,j}), \mathfrak{a}_\infty(A_{k,l})] \subset \mathfrak{a}_\infty(A_{i,l})$ if $k = j$ and $[\mathfrak{a}_\infty(A_{i,j}), \mathfrak{a}_\infty(A_{k,l})] = 0$ otherwise. This shows that the Lie algebra is nilpotent. If $j = k$, and B is an element of $A_{i,j}$, then B does not commute with any non trivial element of $A_{k,l}$. Thus $[\mathfrak{a}_\infty(A_{i,j}), \mathfrak{a}_\infty(A_{k,l})] \neq 0$, and the length of $\mathfrak{h}_r$ is equal to $l(r)$. □

This lemma and Proposition 3.10 provide the following bound (see [14] for $M = \mathbb{P}^2_{\mathbf{C}}$).

Corollary 3.13. *If $r > 2^{\dim_{\mathbf{C}}(M)}$, the group H_r does not embed into $\mathsf{Bir}(M)$.*

Unfortunately, this corollary does not imply Theorem A, even for small values of r and n.

4. Actions of special linear groups

In this section, we study morphisms from $\mathsf{PGL}_{r+1}(\mathbf{C})$ to $\mathsf{Bir}(M)$, where M is a complex projective variety of dimension n, and prove Theorem A.

4.1. **Bounding degrees.** Our first step is the following proposition.

Proposition 4.1. *Let M be a complex projective variety of dimension n. If $r \geq n$ and $\rho\colon \mathsf{PGL}_{r+1}(\mathbf{C}) \to \mathsf{Bir}(M)$ is a morphism, either the image of ρ is reduced to $\{\mathrm{Id}_M\}$ or ρ is injective and its image is a subgroup of bounded degree in M.*

To prove it, *we assume that ρ is not trivial.* Since $\mathsf{PGL}_{r+1}(\mathbf{C})$ is a simple group, its image G is isomorphic to $\mathsf{PGL}_{r+1}(\mathbf{C})$, and can be identified to it.

4.1.1. *A lemma.* Consider the one-parameter subgroups $A_{i,j}$, $j \neq i$, defined by

$$A_{i,j} = \left\{ \mathrm{Id} + a\delta_{i,j}\,; \quad a \in \mathbf{C} \right\}$$

where $\delta_{i,j}$ is the Kronecker matrix with all entries equal to zero, except the coefficient (i,j) which is equal to 1. The group $\mathfrak{S}_{r+1}$ of permutations of the indices $\{1,\ldots,r+1\}$ embeds into $\mathsf{PGL}_{r+1}(\mathbf{C})$ (acting on $\mathbb{P}^r(\mathbf{C})$ by permutations of the homogeneous coordinates). By conjugation, $\mathfrak{S}_{r+1}$ permutes transitively the one-parameter subgroups $A_{i,j}$.

Remark 4.2. The group $\mathsf{GL}_{r+1}(\mathbf{C})$ is generated by elementary matrices and dilatations and the proof of this result based on Gaussian elimination provides the following statement: There is an integer $k(r)$, such that every element of $\mathsf{PGL}_{r+1}(\mathbf{C})$ is a product of at most $k(r)$ elements in $\cup A_{i,j}$.

Lemma 4.3. *If one of the subgroups $A_{i,j} \subset G$ is a subgroup of bounded degree in $\mathsf{Bir}(M)$, then G is a subgroup of bounded degree in $\mathsf{Bir}(M)$.*

To prove it, note that if one of the $A_{i,j}$ has bounded degrees, then all of them do, because the $A_{i,j}$ are pairwise conjugate (cf. Equation (2.2)). This implies that all $A_{i,j}$ are contained in $\mathsf{Bir}_d(M)$ for some positive integer d. Then, Remark 4.2 and Equation (2.1) imply that G is contained in $\mathsf{Bir}_{d^{k(r)}}(M)$ for some positive integer $k(r)$.

4.1.2. *Proof of Proposition 4.1.* To obtain Proposition 4.1, one shall prove that one $A_{i,j}$ has bounded degree, and then apply Lemma 4.3. For this, we now work with the subgroup of upper triangular matrices in $\mathsf{PGL}_{r+1}(\mathbf{C})$ with coefficients equal to 1 on the diagonal (see §3.4.2).

Preliminary remark.– The abelian groups $A_{1,j}$, $j = 2,\dots,r+1$, generate an abelian subgroup A_1 of $\mathsf{PGL}_{r+1}(\mathbf{C})$. If $s(A_1) = n$, as in Proposition 3.9, there exists a degree d such that $\mathsf{PGL}_{r+1}(\mathbf{C})$ is contained in $\mathsf{Bir}_d(M)$, and we are done.

First step: Matrices from the first row.– We now assume $s(A_1) < n$. Then, after conjugation by a permutation $\sigma \in \mathfrak{S}_{r+1}$, there is an integer $k \geq 3$ with the following property: The Lie algebras $\mathfrak{a}_\infty(A_{1,r+1})$, $\mathfrak{a}_\infty(A_{1,r})$, ..., $\mathfrak{a}_\infty(A_{1,k})$ contain meromorphic vector fields $X_{1,r+1}$, $X_{1,r}$, ..., $X_{1,k}$ such that

(1) the vector fields $X_{1,j}$, $j \geq k$, are $\mathbf{C}$-linearly independent at the generic point $m \in M$;
(2) every element $X_{1,l}$ of $\mathfrak{a}_\infty(A_{1,l})$ with $l < k$ is a linear combination of the $X_{1,j}$, $j \geq k$, with coefficients α_l^j in the field of meromorphic functions $\mathbf{C}(M)$:

$$X_{1,l}(m) = \sum_{j=k}^{j=r+1} \alpha_l^j(m) X_{1,j}(m), \quad \text{for} \quad m \in M.$$

Consider an open subset U of M on which the $X_{1,j}$, $j \geq k$, are holomorphic and everywhere $\mathbf{C}$-linearly independent (*i.e.* linearly independent at every point m of U). Since these vector fields commute, one can change U into a smaller open subset and find holomorphic coordinates $(x_{r+2-n},\dots,x_{r+1})$ on U such that

$$X_{1,j} = \partial_j, \quad \forall j \geq k \tag{4.1}$$

where ∂_j denotes the vector field $\partial/\partial x_j$. Since the $X_{1,i}$ pairwise commute, one gets

$$\partial_j \alpha_l^{j'} = 0, \quad \forall j \geq k, \quad \forall j' \geq k, \quad \forall l < k. \tag{4.2}$$

Second step: Matrices from the last column, and conclusion.– Let now $X_{k-1,r+1}$ be a non-zero element of the Lie algebra $\mathfrak{a}_\infty(A_{k-1,r+1})$. Suppose that $X_{k-1,r+1}$ is a linear combination, with coefficients β_{k-1}^j in $\mathbf{C}(M)$, of the $X_{1,j}$:

$$X_{k-1,r+1}(m) = \sum_{j=k}^{j=r+1} \beta_{k-1}^j(m) X_{1,j}(m), \quad \text{for} \quad m \in M.$$

The vector field $X_{k-1,r+1}$ commutes with the $X_{1,j}$ for all $j \geq k$ because so do the corresponding one-parameter subgroups $A_{k-1,r+1}$ and $A_{1,j}$; hence

$$\partial_j \beta_{k-1}^{j'} = 0, \quad \forall j \geq k, \quad \forall j' \geq k. \tag{4.3}$$

Equations (4.2) and (4.3) imply that

$$[X_{k-1,r+1}, X_{1,k-1}] = 0 \tag{4.4}$$

for all vector fields $X_{1,k-1}$ in $\mathfrak{a}_\infty(A_{1,k-1})$. This contradicts the fact that pairs of non-trivial elements in $A_{1,k-1}$ and $A_{k-1,r+1}$ *never commute*. Thus, our assumption leads to a contradiction, so that all non-zero elements $X_{k-1,r+1}$ in $\mathfrak{a}_\infty(A_{k-1,r+1})$ are indeed $\mathbf{C}$-linearly independent of the $X_{1,j}$, $j \geq k$, at the generic point m of M.

Fix such an element $X_{k-1,r+1} \neq 0$. Shrinking U again, and changing the system of local coordinates $(x_{r+2-n}, \dots, x_{r+1})$, one can now assume that $X_{1,j} = \partial_j$, for all $j \geq k$, as in Equation (4.1), and that

$$X_{k-1,r+1} = \partial_{k-1}. \tag{4.5}$$

We pursue the same strategy with the Lie algebra $\mathfrak{a}_\infty(A_{k-2,r+1})$, and obtain the existence of an element $X_{k-2,r+1} \neq 0$ in $\mathfrak{a}_\infty(A_{k-2,r+1})$, a non-empty open subset $U \subset M$, and a system of local coordinates $(x_{r+2-n}, \dots, x_{r+1})$ on U such that Equations (4.1) and (4.5) are satisfied and, moreover,

$$X_{k-2,r+1} = \partial_{k-2}. \tag{4.6}$$

After a finite number of steps, one obtains the following properties:

(1) $r \leq n$;
(2) if $r = n$, then the abelian group A generated by the $A_{1,j}$, $j \geq k$, and the $A_{l,r+1}$, $2 \leq l \leq k-1$, satisfies $s(A) = n$.

Then we apply Proposition 3.9 to deduce that the group A has bounded degree, and Lemma 4.3 concludes the proof of Proposition 4.1.

4.2. **Proof of Theorem A.**

4.2.1. *Regularization.* Proposition 4.1 and Weil's regularization Theorem (see §2.3), imply the following.

Corollary 4.4. *Let M be a complex projective variety. Let $r \geq 0$ be an integer and G be a subgroup of $\mathsf{Bir}(M)$ which is isomorphic to $\mathsf{PGL}_{r+1}(\mathbf{C})$. Then $r \leq n$ and if $r = n$, there exists a smooth complex projective variety M', and a birational map $\Psi\colon M' \dashrightarrow M$, such that*

$$\Psi^{-1} \circ G \circ \Psi \subset \mathsf{Aut}(M)^0.$$

4.2.2. *Automorphism groups, and conclusion.* It remains to study smooth and connected complex projective varieties M, with $\dim_{\mathbf{C}}(M) = n$, such that the group $\mathsf{PGL}_{n+1}(\mathbf{C})$ embeds into $\mathsf{Aut}(M)^0$.

Theorem 4.5. *Let M be a smooth, connected, complex projective variety. Let n be the dimension of M. If there is a non-trivial morphism $\rho\colon \mathsf{PGL}_{n+1}(\mathbf{C}) \to \mathsf{Aut}(M)^0$, then M is isomorphic to the projective space $\mathbb{P}^n_{\mathbf{C}}$.*

Proof. Let G be the image of ρ in $\mathsf{Aut}(M)^0$. The group $\mathsf{Aut}(M)^0$ is a connected complex Lie group. Let $\overline{H}$ be the Zariski closure of H in $\mathsf{Aut}(M)^0$. Since H is dense in $\overline{H}$, $\overline{H}$ is a simple Lie group. Moreover, $\overline{H}$ has rank at least n, because it contains the image, under the morphism ρ, of the diagonal subgroup of $\mathsf{PGL}_{n+1}(\mathbf{C})$, and this subgroup contains a copy of the finite abelian group $(\mathbf{Z}/p\mathbf{Z})^n$ for all prime numbers p.

To sum up, $\mathsf{Aut}(M)^0$ contains an algebraic subgroup $\overline{H}$ that is simple, of rank $\geq n$. From [8], Theorem 4.1, one knows that every compact complex manifold M which admits a faithful holomorphic action of an almost simple complex Lie group of rank $\dim_{\mathbf{C}}(M)$ is isomorphic to the projective space; this implies that M is isomorphic to $\mathbb{P}^n(\mathbf{C})$. □

Thus, in Corollary 4.4, one can replace M' by $\mathbb{P}^n(\mathbf{C})$. Theorem A is now a consequence of the following classical fact, which we state over the field of complex numbers while it holds in much greater generality.

Theorem 4.6. *Let $r \geq 0$ be a natural integer. Let $\rho\colon \mathsf{PGL}_{r+1}(\mathbf{C}) \to \mathsf{PGL}_{r+1}(\mathbf{C})$ be a non-trivial morphism of groups. Then there exists a morphism of fields $\alpha\colon \mathbf{C} \to \mathbf{C}$, and an element h of $\mathsf{PGL}_{r+1}(\mathbf{C})$ such that*

$$\rho({}^{\alpha}g) = h \circ g \circ h^{-1}, \quad \forall g \in \mathsf{Aut}(\mathbb{P}^n(\mathbf{C}))$$

or

$$\rho({}^{\alpha}g) = h \circ g^{\vee} \circ h^{-1}, \quad \forall g \in \mathsf{Aut}(\mathbb{P}^n(\mathbf{C})).$$

Such a result is not hard to prove. One technic is to reduce it to the fundamental theorem of projective geometry. A good example of this strategy is provided by Élie Cartan's proof of the continuity of homomorphisms from $\mathsf{SO}_n(\mathbf{R})$ to $\mathsf{SL}_d(\mathbf{R})$ with bounded image (see [9]); an exhaustive presentation of this method, over arbitrary fields instead of $\mathbf{C}$ or $\mathbf{R}$, can be found in Dieudonné's book [15]. Another strategy which works uniformly for all algebraic groups is proposed by Borel and Tits in [5].

5. Questions and remarks

5.1. **Other Lie groups.** The same strategy can be applied to all simple complex Lie groups in place of $\mathsf{PGL}_{r+1}(\mathbf{C})$. This provides the following statement: *Let G be an almost simple complex Lie group, and let $r = \mathsf{rank}_{\mathbf{C}}(G)$ be the rank of G. Let M be a complex projective variety of dimension n. If there is a non-trivial morphism $G \to \mathsf{Bir}(M)$, then $r \leq n$, and if $r = n$, then G is locally isomorphic to $\mathsf{PGL}_{r+1}(\mathbf{C})$ and M is a rational variety.*

It would be more interesting to classify all possible morphisms from smaller Lie groups into $\mathsf{Bir}(M)$. For example, Blanc and Déserti classified all possible morphisms from $\mathsf{PSL}_2(\mathbf{C})$, and even from $\mathsf{PSL}_2(\mathbf{Q})$, to $\mathsf{Bir}(M)$ when $\dim_{\mathbf{C}}(M) = 2$ (see [3]).

5.2. **Automorphisms of the Cremona group.** In [13], Déserti proves that the group of all automorphisms of the Cremona group $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{C}})$ is generated by the group of field automorphisms $\mathsf{Aut}(\mathbf{C}, +, \cdot)$ and the group of interior automorphisms. For this, she makes use of the explicit set of generators given by Noether-Castelnuovo theorem. For $n \geq 3$, $\mathsf{Bir}(\mathbb{P}^3_{\mathbf{C}})$ is not generated by finitely many regularizable subgroups, and Déserti's method does not apply easily. Thus, *the problem remains open to describe the group of automorphisms of* $\mathsf{Bir}(\mathbb{P}^n_{\mathbf{C}})$ *for* $n \geq 3$.

5.3. **The cubic threefold.** Let V be a smooth cubic hypersurface of $\mathbb{P}^4_{\mathbf{C}}$. In [11], Clemens and Griffith's prove that V is not rational, a result that follows from a precise description of the intermediate jacobian variety of V. Knowing that V is not rational, Theorem C implies that the group $\mathsf{Bir}(V)$ is not isomorphic to $\mathsf{Bir}(\mathbb{P}^3_{\mathbf{C}})$. One can dream of a new proof of Clemens-Griffith's Theorem that would not require the intermediate jacobian but would show that $\mathsf{Bir}(V)$ is much smaller than $\mathsf{Bir}(\mathbb{P}^3_{\mathbf{C}})$. For instance, I suspect that the group $\mathsf{GL}_3(\mathbf{Z})$, a group that does act by linear projective transformations as well as by monomial transformations on $\mathbb{P}^3_{\mathbf{C}}$, does not act faithfully by birational transformations on V. It would be great to mix the Noether-Fano methods (see [27] for an introduction to this topic) with basic ideas from Zimmer's program, geometric group theory, or holomorphic dynamics in order to obtain such a result. This would provide a "quantitative measure" of the fact that V is not rational that is different from the information contained in the intermediate jacobian.

5.4. **Cremona representations.** Can we build a theory of representations of Cremona groups that would be similar to the theory of representations of Lie groups ? For instance, is it possible to describe all faithful representations of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{C}})$ into $\mathsf{Bir}(\mathbb{P}^n_{\mathbf{C}})$ for $n = 3, 4, \ldots$?

References

[1] Daniel Barlet. Espace analytique réduit des cycles analytiques complexes compacts d'un espace analytique complexe de dimension finie. In *Fonctions de plusieurs variables complexes, II (Sém. François Norguet, 1974–1975)*, pages 1–158. Lecture Notes in Math., Vol. 482. Springer, Berlin, 1975.

[2] Arnaud Beauville. p-elementary subgroups of the Cremona group. *J. Algebra*, 314(2):553–564, 2007.

[3] Jérémy Blanc and Julie Déserti. Degree growth of birational maps of the plane. *Ann. Sc. Norm. Super. Pisa Cl. Sci, to appear, arXiv:1109.6810*, pages 1–17, 2011.

[4] Jérémy Blanc and Jean-Philippe Furter. Topologies and structures on the Cremona group. *Ann. of Math.*, 178(3):1173–1198, 2012.
[5] Armand Borel and Jacques Tits. Homomorphismes "abstraits" de groupes algébriques simples. *Ann. of Math. (2)*, 97:499–571, 1973.
[6] F. Campana and Th. Peternell. Cycle spaces. In *Several complex variables, VII*, volume 74 of *Encyclopaedia Math. Sci.*, pages 319–349. Springer, Berlin, 1994.
[7] Serge Cantat, Antoine Chambert-Loir, and Vincent Guedj. *Quelques aspects des systèmes dynamiques polynomiaux*, volume 30 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2010.
[8] Serge Cantat and Abdelghani Zeghib. Holomorphic actions, kummer examples, and zimmer programm. *Ann. Scientifiques de l'ENS, to appear*, pages 1–50, 2012.
[9] Elie Cartan. Sur les représentations linéaires des groupes clos. *Comment. Math. Helv.*, 2(1):269–283, 1930.
[10] Dominique Cerveau and Julie Déserti. Transformations birationnelles de petit degré. *Cours Spécialisés, SMF*, (19):viii+223 pp.
[11] C. Herbert Clemens and Phillip A. Griffiths. The intermediate Jacobian of the cubic threefold. *Ann. of Math. (2)*, 95:281–356, 1972.
[12] Michel Demazure. Sous-groupes algébriques de rang maximum du groupe de Cremona. *Ann. Sci. École Norm. Sup. (4)*, 3:507–588, 1970.
[13] Julie Déserti. Sur les automorphismes du groupe de Cremona. *Compos. Math.*, 142(6):1459–1478, 2006.
[14] Julie Déserti. Sur les sous-groupes nilpotents du groupe de Cremona. *Bull. Braz. Math. Soc. (N.S.)*, 38(3):377–388, 2007.
[15] Jean A. Dieudonné. *La géométrie des groupes classiques*. Springer-Verlag, Berlin, 1971. Troisième édition, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 5.
[16] Tien-Cuong Dinh and Nessim Sibony. Une borne supérieure pour l'entropie topologique d'une application rationnelle. *Ann. of Math. (2)*, 161(3):1637–1644, 2005.
[17] D. B. A. Epstein and W. P. Thurston. Transformation groups and natural bundles. *Proc. London Math. Soc. (3)*, 38(2):219–236, 1979.
[18] R. P. Filipkiewicz. Isomorphisms between diffeomorphism groups. *Ergodic Theory Dynamical Systems*, 2(2):159–171 (1983), 1982.
[19] Étienne Ghys. Prolongements des difféomorphismes de la sphère. *Enseign. Math. (2)*, 37(1-2):45–59, 1991.
[20] Étienne Ghys. Sur les groupes engendrés par des difféomorphismes proches de l'identité. *Bol. Soc. Brasil. Mat. (N.S.)*, 24(2):137–178, 1993.
[21] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966.
[22] Vincent Guedj. Propriétés ergodiques des applications rationnelles. In *Panorama et Synthèses*, volume 30, pages 97–102. SMF, 2010.
[23] Masaki Hanamura. On the birational automorphism groups of algebraic varieties. *Compositio Math.*, 63(1):123–142, 1987.
[24] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
[25] Alan Huckleberry and Dmitri Zaitsev. Actions of groups of birationally extendible automorphisms. In *Geometric complex analysis (Hayama, 1995)*, pages 261–285. World Sci. Publ., River Edge, NJ, 1996.
[26] Sebastian Hurtado. Continuity of discrete homomorphisms of diffeomorphism groups. *preprint, arXiv:1307.4447*, pages 1–33, 2013.

[27] János Kollár, Karen E. Smith, and Alessio Corti. *Rational and nearly rational varieties*, volume 92 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2004.

[28] David I. Lieberman. Compactness of the Chow scheme: applications to automorphisms and deformations of Kähler manifolds. In *Fonctions de plusieurs variables complexes, III (Sém. François Norguet, 1975–1977)*, volume 670 of *Lecture Notes in Math.*, pages 140–186. Springer, Berlin, 1978.

[29] Kathryn Mann. Homomorphisms between diffeomorphism groups. In *preprint*, volume arXiv:1206.1196, pages 1–22. 2012.

[30] Emmanuel Militon. Éléments de distorsion de $\mathrm{Diff}_0^\infty(M)$. *Bull. Soc. Math. France*, 141(1):35–46, 2013.

[31] Yuri Prokhorov. p-elementary subgroups of the Cremona group of rank 3. In *Classification of algebraic varieties*, EMS Ser. Congr. Rep., pages 327–338. Eur. Math. Soc., Zürich, 2011.

[32] Hiroshi Umemura. Sur les sous-groupes algébriques primitifs du groupe de Cremona à trois variables. *Nagoya Math. J.*, 79:47–67, 1980.

[33] Hiroshi Umemura. Maximal algebraic subgroups of the Cremona group of three variables. Imprimitive algebraic subgroups of exceptional type. *Nagoya Math. J.*, 87:59–78, 1982.

[34] André Weil. On algebraic groups of transformations. *Amer. J. Math.*, 77:355–391, 1955.

[35] Dmitri Zaitsev. Regularization of birational group operations in the sense of Weil. *J. Lie Theory*, 5(2):207–224, 1995.

DÉPARTEMENT DE MATHÉMATIQUES ET APPLICATIONS (DMA), UMR 8553 (CNRS), ENS ULM, PARIS, FRANCE

E-mail address: cantat@dma.ens.fr, serge.cantat@univ-rennes1.fr