APPENDIX TO A PAPER BY GUIVARC'H AND RAUGI

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The goal of this appendix is to show that the limit set of a Zariski dense subgroup of $SL(n, \mathbf{C})$ is not locally contained in strict analytic subsets of $\mathbb{P}^{n-1}(\mathbf{C})$. The proof illustrates the use of the main results of [3] (see also the appendix of [1]): It is an easy combination of Chow's theorem concerning analytic subsets of projective varieties ([2]) and of the existence of regular elements in Zariski-dense subgroups ([3], [4]).

1. Notations.– Let T be a sub-semigroup of $SL(n, \mathbf{C})$. The flag space of $SL(n, \mathbf{C})$ is the usual flag space $Flag(n, \mathbf{C})$ of complete flags

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbf{C}^n$$

in $\mathbf{C}^n$. The Plücker embedding shows that $Flag(n, \mathbf{C})$ is a smooth projective variety of dimension $n(n-1)/2$ in a huge projective space $\mathbb{P}(\mathbf{C}^N)$: The group $SL(n, \mathbf{C})$ acts linearly on $\mathbf{C}^N$ and the induced action on the ambiant projective space $\mathbb{P}(\mathbf{C}^N)$ preserves the subvariety $Flag(n, \mathbf{C})$. We shall denote by $L_T \subset Flag(n, \mathbf{C})$ the limit set of T. This is a closed subset of $Flag(n, \mathbf{C})$.

2. Complex analytic dimension.– Let Λ be a closed subset of a smooth complex analytic space M. The complex analytic dimension of Λ at a point p is defined by

$$adim_{\mathbf{C}}(\Lambda; p) = \min\{\dim_{\mathbf{C}}(W)\}$$

where W describes the set of germs of complex analytic subsets of M that contain p in a neighborhood of p (by definition the dimension $\dim_{\mathbf{C}}(W)$ is the maximum of the dimensions of the irreducible components of W).

The map $p \mapsto adim_{\mathbf{C}}(\Lambda; p)$ is a semi-continuous function from Λ to

$$\{0, 1, 2, ..., \dim_{\mathbf{C}}(M)\}.$$

The complex analytic dimension of Λ is defined as

$$adim_{\mathbf{C}}(\Lambda) = \max_{p \in \Lambda}\{adim_{\mathbf{C}}(\Lambda; p)\}.$$

Remark. Let T be a sub-semigroup of $SL(n, \mathbf{C})$. Since $adim_{\mathbf{C}}(L_T; p)$ is a T-invariant and semi-continuous function of $p \in L_T$ and L_T is T-minimal, it is constant and equal to $adim_{\mathbf{C}}(L_T)$.

Remark. Let Λ be a closed subset of M of complex analytic dimension d, and let p be a point of Λ. By definition, there exists a neighborhood $\mathcal{U}_p$ of p and a complex analytic subset W of $\mathcal{U}_p$ of dimension d that contains $\Lambda \cap \mathcal{U}_p$. Let

W_1 be an irreducible component of W and let W_1' be the subset of points q in W_1 which are not contained in a second irreducible component of W: W_1' is an open analytic subset of W_1 and W coincides locally with W_1 around any point q of W_1'. As a consequence, if q is a point of W_1' there is a neighborhood $\mathcal{U}_q$ of q such that W_1' contains $\Lambda \cap \mathcal{U}_q$. Then

$$adim_{\mathbf{C}}(\Lambda; q) \leq \dim_{\mathbf{C}}(W_1') = \dim_{\mathbf{C}}(W_1) \leq \dim_{\mathbf{C}}(W).$$

When Λ is a limit set L_T, there exists an element γ of T such that $\gamma(p)$ is contained in $\mathcal{U}_q$. As a consequence, we can replace W by $\gamma^{-1}(W_1)$ and assume that W is irreducible.

3. Statement of the result.– The goal of this appendix is to prove the following theorem,

Theorem 1. *Let T be a sub-semigroup of $SL(n, \mathbf{C})$. If*

$$adim_{\mathbf{C}}(L_T) < \dim_{\mathbf{C}}(Flag(n, \mathbf{C}))$$

then T is not Zariski dense in $SL(n, \mathbf{C})$.

4. Preliminaries on algebraic geometry.– We now recall two important results which are proved in details in chapter 4 of Mumford's book [2].

Let M be a smooth compact complex manifold. A closed subset $X \subset M$ is $*$-analytic if X can be decomposed as

$$X = X^{(r)} \cup X^{(r-1)} \cup ... \cup X^{(0)}$$

where for all i, $X^{(i)}$ is an i dimensional complex submanifold of M and

$$\overline{X^{(i)}} \subset X^{(i)} \cup ... \cup X^{(0)}.$$

If $X^{(r)}$ is not empty, then r is the dimension of X.

Theorem 2 ([2], page 60)**.** *A closed subset X of a complex manifold M is $*$-analytic if and only is X is a complex analytic subset of M.*

Theorem 3 (Chow's theorem, see [2], page 61)**.** *Any analytic subset of a projective algebraic variety is a projective algebraic variety.*

In what follows, these theorems will be used with $Flag(n, \mathbf{C})$ or $\mathbb{P}^k(\mathbf{C})$ in the role of M.

5. Preliminaries on linear transformations.– If γ is an invertible linear transformation of $\mathbf{C}^N$ such that its eigenvalues α_1, ..., α_N satisfy

$$|\alpha_1| > |\alpha_2| > ... > |\alpha_N|,$$

we will say that γ is regular (see [3] for a more precise vocabulary).

Let γ be a regular element in $GL(N, \mathbf{C})$. For each eigenvalue α of γ, $L(\alpha)$ will denote the corresponding eigenline. The linear mapping γ acts on the projective space $\mathbb{P}(\mathbf{C}^N)$ and each line $L(\alpha)$ gives rise to a fixed point of γ. We will use the following notations:

- The attracting fixed point of γ corresponds to the line $L(\alpha_N)$; it will be denoted by $p(\gamma)$;
- the hyperplane that contains the fixed points $L(\alpha_i)$, $i < N$, will be denoted by $H(\gamma)$;
- the basin of attraction of $p(\gamma)$ is the complement of $H(\gamma)$ in $\mathbb{P}(\mathbf{C}^N)$.

Let $\mathcal{U} \subset \mathbb{P}(\mathbf{C}^N)$ be a neighborhood of $p(\gamma)$ such that $\gamma^{-1}(\mathcal{U})$ contains strictly $\overline{\mathcal{U}}$. Let W be an analytic subset of $\mathcal{U}$ which is invariant under the action of γ, by which we mean

$$\gamma(W) \subset W, \quad \gamma^{-1}(W) \cap \mathcal{U} = W.$$

Let B be a small open ball centered at p, and contained in $\mathcal{U}$. If the radius of B is small enough, and if we replace γ by a sufficiently high iterate γ^m, we may assume that

$$\overline{B} \subset \gamma^{-1}(B) \subset \gamma^{-2}(\overline{B}) \subset \mathcal{U}.$$

Let V be the intersection of W with the shell $\gamma^{-1}(B) \setminus B$. If we denote by X the union $\cup_{m \geq 0} \gamma^{-m}(W)$, the γ-invariance of W implies that X is the disjoint union

$$X = (W \cap B) \cup V \cup \gamma^{-1}(V) \cup ... \cup \gamma^{-m}(V) \cup ....$$

Inside each open shell $\gamma^{-m}(\gamma^{-2}(B) \setminus \overline{B})$, X coincides with the analytic subset $\gamma^{-m}(W \cap (\gamma^{-2}(B) \setminus \overline{B}))$. Since the increasing union $\cup_{m \geq 0} \gamma^{-m}(B)$ fills the basin of $p(\gamma)$, this shows that X is an analytic subset of $\mathbb{P}(\mathbf{C}^N) \setminus H(\gamma)$.

6. Dimension 2.– Let us now prove the theorem in the easiest non trivial case. In this paragraph, we suppose that $n = 3$ and we study the action of a sub-semigroup $T \subset SL(3, \mathbf{C})$ on $\mathbb{P}^2(\mathbf{C})$ (instead of the 3-dimensional flag space $Flag(3, \mathbf{C})$). We keep the notation L_T for the limit set of T in $\mathbb{P}^2(\mathbf{C})$.

If the complex analytic dimension of L_T is 0, then L_T is a discrete and closed subset of $\mathbb{P}^2(\mathbf{C})$. As a consequence, L_T is a finite T-invariant set. This shows that T is not Zarisiki dense.

Let us now suppose that the complex analytic dimension of L_T is 1 and that T is Zariski dense in $SL(3, \mathbf{C})$. By [3] there exists a regular element γ in T. Its attracting fixed point $p = p(\gamma)$ is contained in L_T. Let W be a germ of analytic submanifold of dimension 1 containing L_T in a neighborhood $\mathcal{U}$ of p. By the remark at the end of paragraph (2), we can assume that W is irreducible. If W is not γ-invariant, then L_T is locally contained in the locally finite set $\gamma(W) \cap W$ and this contradicts $adim_{\mathbf{C}}(L_T) = 1$. Then, as a germ of analytic subset around p, W is γ-invariant.

The set

$$X = \bigcup_{m \geq 0} \gamma^{-m}(W)$$

is an increasing union of analytic subsets of dimension 1 in $\mathbb{P}^2(\mathbf{C})$.

If W is locally contained in one of the two lines joining p to the other fixed points of γ, then $\overline{X}$ coincides with this line. If not, we can reduce the neighborhood $\mathcal{U}$ in order to assume that the germ W intersects these two lines only at p. Then, the adherence of X is the union of X and the repulsive fixed point q of γ. The previous paragraph shows that X is analytic in the complement of the line $H(\gamma)$. This shows that $\overline{X}$ is a $*$-analytic subset of $\mathbb{P}^2(\mathbf{C})$ with $\overline{X}^{(1)} = X$ and $\overline{X}^{(0)} = q$. Chow's theorem now says that X is in fact an algebraic subset of dimension 1 in $\mathbb{P}^2(\mathbf{C})$.

As a consequence, p is contained in an algebraic curve $\overline{X}$ such that $\overline{X} \cap L_T$ is a neighborhood of p in L_T. Let y be a point of L_T which is not contained in $H(\gamma)$. Then y is in the basin of attraction of p and $\gamma^m(y)$ is in $\mathcal{U}$ if m is sufficiently large. This implies that y is a point of X. As a consequence, L_T is contained in the 1-dimensional algebraic set $X \cup H(\gamma)$, the Zariski closure of L_T is a one dimensional T-invariant subset of $\mathbb{P}^2(\mathbf{C})$, and T is not Zariski dense in $SL(3, \mathbf{C})$.

Remark In the general case below, we reproduce the same argument, but with the help of several regular elements γ_i of T.

7. General case.– The flag space $Flag(n, \mathbf{C})$ is a complex projective subvariety of some projective space $\mathbb{P}(\mathbf{C}^N)$. The group $SL(n, \mathbf{C})$ acts linearly on this projective space and the action preserves the image of $Flag(n, \mathbf{C})$. We assume that the sub-semigroup T is Zariski-dense and that the complex analytic dimension of the limit set L_T of T inside $Flag(n, \mathbf{C})$ is smaller than the dimension of $Flag(n, \mathbf{C})$.

Let γ be a regular element of T (see [3] or [4]). Let W be a germ of irreducible analytic subset of $Flag(n, \mathbf{C})$ of dimension $adim_{\mathbf{C}}(L_T)$ containing L_T in a neighborhood $\mathcal{U}$ of the attracting fixed point $p(\gamma)$. This set W is γ-invariant, because otherwise $\gamma(W) \cap W$ is a germ of complex analytic space that contains locally L_T around p, the dimension of which is strictly less than $adim_{\mathbf{C}}(L_T)$. As shown in paragraph (5) above, the increasing union

$$X = \bigcup_{m \geq 0} \gamma^{-m}(W)$$

is an analytic subset of $\mathbb{P}(\mathbf{C}^N) \setminus H(\gamma)$.

Now, since T is Zariski-dense, the set of regular elements in T form a Zariski-dense subset of $SL(n, \mathbf{C})$ (see [3]), and we can choose $k+1$ regular elements $\gamma_0 = \gamma$, γ_1, ..., γ_k of T such that

- the intersection of the hyperplanes $H(\gamma_i)$, $i = 0, ..., k$, is empty;
- the attracting fixed points $p(\gamma_i)$ are not contained in $H(\gamma)$.

For each index i, the point $p(\gamma_i)$ is contained in L_T; if m is large enough, $\gamma^m(p(\gamma_i))$ is in $\mathcal{U}$, and therefore in W. This shows that, near $p(\gamma_i)$, X is an irreducible analytic set that locally contains L_T. Since its dimension is $adim_{\mathbf{C}}(L_T)$, X is locally γ_i-invariant around $p(\gamma_i)$. Let X_i be the increasing union

$$X_i = \bigcup_{m \geq 0} \gamma_i^m(W_i),$$

where W_i denotes the intersection of X with a small neighborhood of $p(\gamma_i)$. The sets X_i are analytic in $\mathbb{P}(\mathbf{C}^N) \setminus H(\gamma_i)$ and they coincide with X near $p(\gamma_i)$. By analyticity these local equality propagates, and we get successively

- X_i coincides with X on $\mathbb{P}(\mathbf{C}^N) \setminus (H(\gamma) \cup H(\gamma_i))$ $(0 \leq i \leq k)$;
- X_i and X_j coincide inside $\mathbb{P}(\mathbf{C}^N) \setminus (H(\gamma_i) \cup H(\gamma_j))$ $(0 \leq i, j \leq k)$.

This shows that we can define a *global* analytic subset $\overline{X}$ by

$$\overline{X} = X_i \quad \text{in} \quad \mathbb{P}(\mathbf{C}^N) \setminus H(\gamma_i).$$

The set $\overline{X}$ is an analytic subset of the projective space and, by Chow's theorem, $\overline{X}$ is an algebraic subvariety of $Flag(n, \mathbf{C})$ of dimension $adim_{\mathbf{C}}(L_T)$. Since L_T is contained in W around $p(\gamma)$ and is γ-invariant, this shows that the Zariski-closure of L_T is a strict subvariety of $Flag(n, \mathbf{C})$. Since this subvariety is T-invariant and the action of $SL(n, \mathbf{C})$ on the flag variety is transitive, we obtain a contradiction with the Zariski-density of T.

References

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