FOUR QUESTIONS, ONE PARTIAL ANSWER

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In this note, I list four problems that I haven't been able to solve and I propose a partial solution for one of them.

A.– Simple subgroups of algebraic groups

The group $\mathsf{PSL}_2(\mathbf{C})$ is a simple group. If $\mathbf{L} \subset \mathbf{C}$ is a subfield, then $\mathsf{PSL}_2(\mathbf{L})$ is also simple; the smallest simple subgroup one constructs in this way is $\mathsf{PSL}_2(\mathbf{Q})$.

Question A.– *Does there exist a proper infinite simple subgroup of* $\mathsf{PSL}_2(\mathbf{Q})$ *?*

Of course, this question is just the simplest instance of a more general one: Let m be a positive integer, and $\mathbf{k}$ be a field; what are the infinite simple subgroups of $\mathsf{SL}_m(\mathbf{k})$?

According to a theorem of Selberg, finitely generated subgroups of $\mathsf{SL}_m(\mathbf{k})$ are residually finite; thus every simple finitely generated subgroup is finite. Question A is also related to the following problem, due to J. McKay and J.-P. Serre ([1]). Consider a subgroup H of $\mathsf{SL}_2(\mathbf{Q})$ that is dense in the Zariski topology and has no finite quotient. Is H equal to $\mathsf{SL}_2(\mathbf{Q})$?

B.– Minimal diffeomorphisms

Let X be a compact topological space and $f\colon X \to X$ be a homeomorphism of X. One says that f is minimal, or that the action of f on X is minimal, if every orbit of f in X is dense. Equivalently, every closed and f-invariant subset of X is either empty or equal to X. A totally irrational translation on the torus $\mathbf{R}^m/\mathbf{Z}^m$ is minimal, and this is certainly not the only example ([2]).

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[1]Question 15.57 in "Unsolved problems in group theory", the Kourovka Notebook No 18, edited by V. D. Mazurov, E. I. Khukhro, arXiv:1401.0300

[2]For instance, the horocyclic flow and the skew products described by Furstenberg in "Strict Ergodicity and Transformation of the Torus", Amer. J. Math., Vol. 83, No. 4 (Oct., 1961), pp. 573-601.

Question B.– *Does there exist a closed manifold M such that* $\mathrm{Diff}(M)$ *contains a non abelian free group all of whose non-trivial elements act minimally on M ?*

In other words, we are looking for a manifold M, and two diffeomorphisms f and g of M such that (i) f and g generate a free group F of rank 2 and (ii) every element h in $F \setminus \{\mathrm{id}_M\}$ acts minimally on M. I didn't specify the regularity of the diffeomorphisms; one may want to construct examples with f and g in the group of homeomorphisms of M, or rule out the existence of such examples for real analytic diffeomorphisms. I simply don't know what to expect.

The first example to look at is the circle. But a group of minimal homeomorphisms acts freely, and Hölder's theorem implies that such a group of homeomorphisms of the circle is abelian.

C.– Residual finiteness for groups of algebraic transformations

Let $\mathbf{k}$ be a field. Let $\mathbb{A}^m_{\mathbf{k}}$ be the affine space of dimension m, over $\mathbf{k}$. The group of automorphisms $\mathsf{Aut}(\mathbb{A}^m_{\mathbf{k}})$ is, by definition, the group of polynomial transformations of $\mathbb{A}^m_{\mathbf{k}}$ with a polynomial inverse. If Γ is a finitely generated subgroup of $\mathsf{Aut}(\mathbb{A}^m_{\mathbf{k}})$, then Γ is residually finite; this means that there are enough homomorphisms from Γ to finite groups to decide whether two elements in Γ are distinct: $\forall \gamma \in \Gamma$, there is a finite group G and a homomorphism $\rho\colon \Gamma \to G$ such that $\rho(\Gamma) \neq 1_G$. This result has been proved by Bass and Lubotzky ([3]).

A rational transformation f of $\mathbb{A}^m_{\mathbf{k}}$ is defined in affine coordinates $(x_1, \ldots, x_m)$ by m rational fractions $f_i \in \mathbf{k}(x_1, \ldots, x_m)$:

$$f(x_1, \ldots, x_m) = (f_1, \ldots, f_m).$$

It is birational if there is a rational transformation g such that $f \circ g = g \circ f = \mathrm{id}_{\mathbb{A}^m_{\mathbf{k}}}$.

Question C.– *Let Γ be a group of birational transformations of the complex affine space $\mathbb{A}^m_{\mathbf{C}}$. Suppose that Γ is finitely generated. Is Γ residually finite ?*

If, moreover, Γ has Kazhdan property (T), then yes, Γ is residually finite ([4]). But the general case is still open, even in dimension 2.

See also the survey paper "Construction in Elliptic Dynamics", by Katok and Fayad, Ergodic Theory Dynam. Systems, 24 (2004), p. 1477–1520.

[3]H. Bass, A. Lubotzky: "Automorphisms of groups and of schemes of finite type", Israel Journal of Mathematics, Vol. 44, No. 1, 1983

[4]S. Cantat, Junyi Xie : "Algebraic actions of discrete groups: The p-adic method", preprint

D.– Improving our mathematical activity

New technologies offer us a wealth of opportunities *and duties* to improve our mathematical activity. More people should have free access to our knowledge, projects and ideas. New means should be developed to share what we love.

What are the main publishers doing and offering ? I like to visit mathematical libraries; I attended panel discussions and read survey articles on the evolution of the scientific and economical models of our main publishers. But the best feeling it creates in me is similar to the disappointment of gradually losing an old friend who does not share the same moral standards as you anymore. And my library simply can't sustain it. And I don't do mathematics to be part of a process that I strongly disapprove.

Question D.– *What should the mathematical community do to improve its capacity to share mathematics ?*

There are many aspects in this question, and our publication process is only a tiny part of it. But this is one aspect that we know well and may improve.

A partial answer : The Annales Henri Lebesgue

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