The Cremona group

Serge Cantat*

Abstract. We survey a few results concerning groups of birational transformations. The emphasis is on the Cremona group in two variables and methods coming from geometric group theory.

2010 Mathematics Subject Classification. Primary 14E07 ; Secondary 20E32, 32M05, 37F99.

Keywords. Birational geometry, Cremona group, geometric group theory, holomorphic dynamical systems, Tits alternative.

1. An introduction based on examples

1.1. Cremona groups and groups of birational transformations. Let $\mathbf{k}$ be a field and n be a positive integer. The Cremona group $\mathrm{Cr}_n(\mathbf{k})$ is the group of $\mathbf{k}$-automorphisms of $\mathbf{k}(X_1,\ldots,X_n)$, the $\mathbf{k}$-algebra of rational functions in n independent variables. Given n rational functions $F_i \in \mathbf{k}(X_1,\ldots,X_n)$ there is a unique endomorphism of this algebra that maps X_i onto F_i. This endomorphism is an automorphism of $\mathbf{k}(X_1,\ldots,X_n)$ if, and only if the rational transformation

$$f(X_1,\ldots,X_n) = (F_1,\ldots,F_n)$$

is a birational transformation of the affine space $\mathbb{A}^n_{\mathbf{k}}$, *i.e.* an element of the group of birational transformations $\mathrm{Bir}(\mathbb{A}^n_{\mathbf{k}})$. This correspondence identifies $\mathrm{Cr}_n(\mathbf{k})$ with the group $\mathrm{Bir}(\mathbb{A}^n_{\mathbf{k}})$.

Compactify $\mathbb{A}^n_{\mathbf{k}}$ into the projective space $\mathbb{P}^n_{\mathbf{k}}$, and denote by $[x_1 : \ldots : x_{n+1}]$ a system of homogeneous coordinates with $X_i = x_i/x_{n+1}$. Every birational transformation of the affine space corresponds to a unique birational transformation of the projective space, and vice versa. Geometrically, one restricts elements of $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ to the Zariski open subset $\mathbb{A}^n_{\mathbf{k}}$ (resp. one extends elements of $\mathrm{Bir}(\mathbb{A}^n_{\mathbf{k}})$ to the compactification $\mathbb{P}^n_{\mathbf{k}}$). In terms of formulas, a rational transformation f of $\mathbb{A}^n_{\mathbf{k}}$ which is defined by rational fractions F_i, as above, gives rise to a rational transformation of the projective space which is defined by homogeneous polynomials f_i in the x_i: To obtain the f_i one just needs to homogenize the F_i and to multiply them by the lowest common multiple of their denominators. For instance, the birational transformation

$$h(X_1,X_2) = (X_1/X_2, X_2 + 17)$$

*The author is grateful to the CNRS, Université Rennes 1, the École Normale Supérieure de Paris, and the Fondation Del Duca for their support. He expresses his warmest thanks to Jérémy Blanc, Xavier Caruso, Thomas Delzant, Julie Déserti, Igor Dolgachev, Stéphane Lamy, Christian Urech, for interesting discussions on the topics covered in this survey. He is also grateful to Ivan Cheltsov, Yuri Prokhorov, and Susanna Zimmermann for interesting comments and references.

of $\mathbb{A}^2_{\mathbf{k}}$ corresponds to the birational transformation

$$h[x_1 : x_2 : x_3] = [x_1x_3 : (x_2 + 17x_3)x_2 : x_3x_2].$$

To sum up, one gets three incarnations of the same group,

$$\mathrm{Cr}_n(\mathbf{k}) = \mathrm{Bir}(\mathbb{A}^n_{\mathbf{k}}) = \mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}}). \tag{1}$$

Moreover, every birational transformation f of $\mathbb{P}^n_{\mathbf{k}}$ can be written as

$$f[x_1 : \ldots : x_{n+1}] = [f_1 : \ldots : f_{n+1}] \tag{2}$$

where the f_i are homogeneous polynomials in the variables x_i, of the same degree d, and without common factor of positive degree. This degree d is the **degree** of f. Birational transformations of degree 1 are linear projective transformations: They form the subgroup

$$\mathsf{PGL}_{n+1}(\mathbf{k}) = \mathrm{Aut}(\mathbb{P}^n_{\mathbf{k}}) \subset \mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}}) \tag{3}$$

of automorphisms of the projective space.

More generally, two groups of transformations are naturally associated to any given variety Y: The group $\mathrm{Aut}(Y)$ of its (regular) automorphisms, and the group $\mathrm{Bir}(Y)$ of its birational transformations. If M is a complex manifold, one can consider its group of holomorphic diffeomorphisms and its group of bi-meromorphic transformations. They coincide with the aforementionned groups $\mathrm{Aut}(M)$ and $\mathrm{Bir}(M)$ when M is the complex manifold determined by a (smooth) complex projective variety.

1.2. Examples, indeterminacy points, and dynamics. The group of automorphisms of $\mathbb{P}^n_{\mathbf{k}}$ is the group $\mathsf{PGL}_{n+1}(\mathbf{k})$ of linear projective transformations. In dimension 1, $\mathrm{Cr}_1(\mathbf{k})$ is equal to $\mathsf{PGL}_2(\mathbf{k})$, because a rational transformation $f(X_1) \in \mathbf{k}(X_1)$ is invertible if and only if its degree is equal to 1.

1.2.1. Monomial transformations. The multiplicative group $\mathbb{G}^n_m$ of dimension n can be identified to the Zariski open subset $(\mathbb{A}^1_{\mathbf{k}} \setminus \{0\})^n$ of $\mathbb{P}^n_{\mathbf{k}}$. Thus, $\mathrm{Cr}_n(\mathbf{k})$ contains the group of all algebraic automorphisms of the group $\mathbb{G}^n_m$ *i.e.* the group of **monomial** transformations $\mathsf{GL}_n(\mathbf{Z})$.

A first example is given by the monomial transformation of the plane $(X_1, X_2) \mapsto (1/X_1, 1/X_2)$. It is denoted by σ_2 in what follows; it can be written as

$$\sigma_2[x_1 : x_2 : x_3] = [x_2x_3 : x_3x_1 : x_1x_2] \tag{4}$$

in homogeneous coordinates, and is therefore an involution of degree 2. By definition, σ_2 is the **standard quadratic involution**.

A second example is given by $a(X_1, X_2) = (X_1^2X_2, X_1X_2)$. If $\mathbf{k}$ is the field of complex numbers $\mathbf{C}$, this transformation a preserves the 2-dimensional real torus

$$T := \{(X_1, X_2) \in \mathbf{C}^*;\ |X_1| = |X_2| = 1\}$$

and induces a diffeomorphism of T. This torus is uniformized by the plane $\mathbf{R}^2$, with covering map $(t_1, t_2) \mapsto (\exp(2\pi\sqrt{-1}t_1), \exp(2\pi\sqrt{-1}t_2))$, and the birational transformation a

is covered by the linear transformation $A(t_1,t_2) = (2t_1+t_2, t_1+t_2)$ of $\mathbf{R}^2$. The dynamics of a is quite rich, as explained in [27]. The linear transformation A has two eigenvalues,

$$\lambda_A = \frac{3+\sqrt{5}}{2}, \quad \frac{1}{\lambda_A} = \frac{3-\sqrt{5}}{2},$$

with $\lambda_A > 1$, and the affine lines which are parallel to the eigenline for λ_A (resp. for λ_A^{-1}) give rise to a linear foliation of the torus T whose leaves are uniformly expanded under the dynamics of a (resp. uniformly contracted). Periodic points of $a_{|T}: T \to T$ correspond to rational points $(t_1,t_2) \in \mathbf{Q}\times\mathbf{Q}$ and form a dense subset of T; on the other hand, there are points whose orbit is dense in T, and points whose orbit is dense in a Cantor subset of T. The action of a on T preserves the Lebesgue measure and acts ergodically with respect to it.

1.2.2. Indeterminacy points. Birational transformations may have indeterminacy points. The set of indeterminacy points of a birational transformation of a smooth projective variety Y is a Zariski closed subset of co-dimension ≥ 2, and is therefore a finite set when $\dim(Y) = 2$. For example, σ_2 is not defined at the three points $[1:0:0]$, $[0:1:0]$, and $[0:0:1]$.

Consider the involution of the projective space which is defined by

$$\sigma_3[x_1:x_2:x_3:x_4] = \left[\frac{1}{x_1} : \frac{1}{x_2} : \frac{1}{x_3} : \frac{1}{x_4}\right] = [x_2x_3x_4 : x_1x_3x_4 : x_1x_2x_4 : x_1x_2x_3].$$

Let Δ denote the tetrahedron with faces $\{x_i = 0\}$, $1 \leq i \leq 4$, and vertices $[1:0:0:0]$, ..., $[0:0:0:1]$. The transformation σ_3 blows down each face of Δ on the opposite vertex. Blow up these four vertices, to get a new projective variety Y together with a birational morphism $\pi\colon Y \to \mathbb{P}^3_{\mathbf{k}}$. Then, σ_3 lifts to a birational transformation $\hat{\sigma}_3 = \pi^{-1}\circ\sigma_3\circ\pi$ of Y; this birational transformation does not contract any hypersurface but it has indeterminacies along the strict transforms of the edges $L_{ij} = \{x_i = x_j = 0\}$, $i \neq j$, of the tetrahedron Δ.

Now, fix a field $\mathbf{k}$ of characteristic 0, and consider the birational transformation of the plane which is defined by $g(X_1,X_2) = (X_1+1, X_1X_2+1)$. The line $\{X_1 = 0\}$ is contracted to the point $(1,1)$. The forward orbit of this point is the sequence $g^n(1,1) = (n, y_n)$ with $y_{n+1} = ny_n + 1$; since y_n grows faster than $(n-1)!$, one easily checks that this orbit $(g^n(1,1))_{n\geq 0}$ is Zariski dense. ([1]) Thus, the indeterminacy points of the iterates of g form a Zariski dense set. Similarly, each vertical line $\{X_1 = -m\}$, $m \in \mathbf{Z}_+$, is contracted by some iterate g^m of g. With these remarks in mind, one can show that there is no birational mapping $\pi\colon X \dashrightarrow \mathbb{P}^2_{\mathbf{k}}$ such $\pi\circ g\circ\pi^{-1}$ becomes a regular automorphisms of (a non-empty Zariski open subset of) X. See also Remark 5.5 for other examples of this type.

1.2.3. Hénon mappings. The group $\mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}})$ of polynomial automorphisms of the affine space $\mathbb{A}^n_{\mathbf{k}}$ is contained in the Cremona group $\mathrm{Cr}_n(\mathbf{k})$. In particular, all transformations

$$(X_1,\ldots,X_n) \mapsto (X_1 + P(X_2,\ldots,X_n), X_2, \ldots, X_n),$$

[1] Another argument works as follows. Assume that this orbit is contained in a curve C, and fix an irreducible component D of C. The strict transform of D under the action of g intersects D infinitely many times, and must therefore coïncide with D. One checks that this is impossible by writing down an equation for D.

with P in $\mathbf{k}[X_2,\ldots,X_n]$, are contained in $\mathrm{Cr}_n(\mathbf{k})$. This shows that $\mathrm{Cr}_n(\mathbf{k})$ is "infinite dimensional" when $n \geq 2$.

A striking example of automorphism is furnished by the Hénon mapping

$$h_{a,c}(X_1,X_2) = (X_2 + X_1^2 + c, aX_1), \tag{5}$$

for $a \in \mathbf{k}^*$ and $c \in \mathbf{k}$. When $a = 0$, $h_{a,c}$ is not invertible: The plane is mapped into the line $\{X_2 = 0\}$ and, on this line, $h_{0,c}$ maps X_1 to $X_1^2 + c$. The dynamics of $h_{0,c}$ on this line coincides with the dynamics of the upmost studied transformation $z \mapsto z^2 + c$, which, for $\mathbf{k} = \mathbf{C}$, provides interesting examples of Julia sets (see [114]). For $a \in \mathbf{C}^*$, the main features of the dynamics of $h_{0,c}$ survive in the dynamical properties of the automorphism $h_{a,c}\colon \mathbb{A}^2_{\mathbf{C}} \to \mathbb{A}^2_{\mathbf{C}}$, such as positive topological entropy and the existence of infinitely many periodic points [8].

1.3. Subgroups of Cremona groups. Birational transformations are simple objects, since they are determined by a finite set of data, namely the coefficients of the homogeneous polynomials defining them. On the other hand, they may exhibit very rich dynamical behaviors, as shown by the previous examples. Another illustration of the beauty of $\mathrm{Cr}_n(\mathbf{k})$ comes from the study of its subgroups.

1.3.1. Mapping class groups. Let Γ be a group which is generated by a finite number of elements γ_i, $1 \leq i \leq k$. Consider the space R_Γ of all homomorphisms from Γ to $\mathsf{SL}_2(\mathbf{k})$: It is an algebraic variety over $\mathbf{k}$ of dimension at most $3k$. The group $\mathsf{SL}_2(\mathbf{k})$ acts on R_Γ by conjugacy; the quotient space $R_\Gamma /\!/ \mathsf{SL}_2(\mathbf{k})$, in the sense of geometric invariant theory, is an algebraic variety. The group of all automorphisms of Γ acts on R_Γ by pre-composition. This determines an action of the outer automorphism group $\mathrm{Out}(\Gamma)$ by regular tranformations on $R_\Gamma /\!/ \mathsf{SL}_2(\mathbf{k})$. ($\mathrm{Out}(\Gamma)$ is the quotient of $\mathrm{Aut}(\Gamma)$ by the subgroup of all inner automorphisms.)

There are examples for which this construction provides an embedding of $\mathrm{Out}(\Gamma)$ in the group of automorphisms of $R_\Gamma /\!/ \mathsf{SL}_2(\mathbf{k})$. Fundamental groups of closed orientable surfaces of genus $g \geq 3$ or free groups $\mathbb{F}_g$ with $g \geq 2$ provide such examples. Thus, the mapping class groups $\mathrm{Mod}(g)$ and the outer automorphism groups $\mathrm{Out}(\mathbb{F}_g)$ embed into groups of birational transformations [108, 3].

1.3.2. Analytic diffeomorphisms of the plane. Consider the group $\mathrm{Bir}^\infty(\mathbb{P}^2_{\mathbf{R}})$ of all elements f of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{R}})$ such that f and f^{-1} have no real indeterminacy point: Over $\mathbf{C}$, indeterminacy points come in complex conjugate pairs. Based on the work of Lukackiī, Kollár and Mangolte observed that $\mathrm{Bir}^\infty(\mathbb{P}^2_{\mathbf{R}})$ determines a dense subgroup in the group of diffeomorphisms of $\mathbb{P}^2(\mathbf{R})$ of class $\mathcal{C}^\infty$ (see [98] for stronger results). A similar result holds if we replace the projective plane by other rational surfaces, for instance by the sphere $\mathbb{S}^2_{\mathbf{R}}$. This implies that all dynamical features that can be observed for diffeomorphisms of $\mathbb{P}^2(\mathbf{R})$ (resp. of $\mathbb{S}^2(\mathbf{R})$) and are stable under small perturbations are realized in the dynamics of birational transformations. For instance, there are elements $f \in \mathrm{Bir}^\infty(\mathbb{P}^2_{\mathbf{R}})$ with a horse-shoe in $\mathbb{P}^2(\mathbf{R})$ (see [97], Chapter 2.5.c for the definition of horse-shoes, and Chapter 18.2 for their stability). And there are elements of $\mathrm{Bir}^\infty(\mathbb{S}^2_{\mathbf{R}})$ which are not conjugate to a linear projective transformation in $\mathrm{Bir}(\mathbb{S}^2_{\mathbf{R}})$ but exhibit a simple, north-south dynamics: There is one repulsive fixed point, one attracting fixed point, and all orbits in the complement of the two fixed points go from the first to the second as time flows from $-\infty$ to $+\infty$ (see [97], Chapter 1.6).

1.3.3. Groups of birational transformations. One says that a group Γ is linear if there is a field $\mathbf{k}$, a positive integer n, and an embedding of Γ into $\mathsf{GL}_n(\mathbf{k})$. Similarly, we shall say that Γ is a **group of birational transformations over the field k** if there is a projective variety $Y_{\mathbf{k}}$, and an embedding of Γ into $\mathrm{Bir}(Y_{\mathbf{k}})$. The following properties are obvious.

(1) Linear groups are groups of birational transformations.

(2) The product of two groups of birational transformations over $\mathbf{k}$ is a group of birational transformations over $\mathbf{k}$.

(3) Any subgroup of a group of birational transformations is also a group of birational transformations.

In certain cases, one may want to specify further properties: If Γ acts faithfully by birational transformations on a variety of dimension d over a field of characteristic p, we shall say that Γ is a group of birational transformations in dimension at most d in characteristic p. For instance,

(4) Every finite group is a group of birational transformations in dimension 1 and characteristic 0. (see [85], Theorem 6')

(5) The mapping class group $\mathrm{Mod}(g)$ of a closed, orientable surface of genus $g \geq 3$ and the group $\mathsf{Out}(\mathbb{F}_g)$ are groups of birational transformations in dimension $\leq 6g$, but $\mathrm{Out}(\mathbb{F}_g)$ is not linear if $g \geq 4$ (see [3, 78, 107]).

1.4. Aims and scope. This survey is organized in three main chapters. The leitmotiv is to compare groups of birational transformations, for instance Cremona groups, to classical Lie groups and to groups of diffeomorphisms of smooth compact manifolds.

We first look at the groups $\mathrm{Bir}(X)$ as (infinite dimensional) analogues of algebraic groups (see Sections 2 to 3). Then, we focus on recent results on groups of birational transformations of surfaces, with an emphasis on the most interesting example $\mathrm{Cr}_2(\mathbf{k})$ (see Sections 4 to 7). The last chapters review several open problems concerning groups of birational transformations in dimension > 2.

There are several geometrical aspects of the theory which are not described at all, including classical features such as the geometry of homaloidal nets and the Noether-Fano inequality, as well as more recent developments like the Sarkisov program and the geometry of birationally rigid varieties. The lectures notes [70] and the books [68, 99, 49] are good introductions to these topics. Dynamical properties of birational transformations are also not discussed; this would require a much longer report [37, 89].

Contents

–I–
Algebraic subgroups and generators

2. Algebraic subgroups of $\mathrm{Cr}_n(\mathbf{k})$

In this first part, the main emphasis is on the Zariski topology of the Cremona group and the structure of its algebraic subgroups. We compare $\mathrm{Cr}_n(\mathbf{k})$ to linear algebraic groups: If $\mathrm{Cr}_n(\mathbf{k})$ were such a group, what kind of linear group would it be ?

2.1. Zariski topology (see [20, 131]). Let B be an irreducible algebraic variety. A **family** of birational transformations of $\mathbb{P}^n_{\mathbf{k}}$ parametrized by B is, by definition, a birational transformation f of $B \times \mathbb{P}^n_{\mathbf{k}}$ such that (i) f determines an isomorphism between two open subsets $\mathcal{U}$ and $\mathcal{V}$ of $B \times \mathbb{P}^n_{\mathbf{k}}$ such that the first projection maps both $\mathcal{U}$ and $\mathcal{V}$ surjectively onto B, and (ii)

$$f(b,x) = (b, p_2(f(b,x)))$$

where p_2 is the second projection; thus, each $f_b := p_2(f(b,\cdot))$ is a birational transformation of $\mathbb{P}^n_{\mathbf{k}}$. The map $b \mapsto f_b$ is called a **morphism** from the parameter space B to the Cremona group $\mathrm{Cr}_n(\mathbf{k})$.

Then, one says that a subset S of $\mathrm{Cr}_n(\mathbf{k})$ is **closed** if its preimage is closed for the Zariski topology under every morphism $B \to \mathrm{Cr}_n(\mathbf{k})$. This defines a topology on $\mathrm{Cr}_n(\mathbf{k})$ wich is called the Zariski topology. Right and left translations

$$g \mapsto g \circ h, \quad g \mapsto h \circ g$$

and the inverse map

$$g \mapsto g^{-1}$$

are homeomorphisms of $\mathrm{Cr}_n(\mathbf{k})$ with respect to the Zariski topology.

Define $\mathrm{Cr}_n(\mathbf{k};d) \subset \mathrm{Cr}_n(\mathbf{k})$ to be the subset of all birational transformations of degree d: An element of $\mathrm{Cr}_n(\mathbf{k};d)$ is defined by homogeneous formulas of degree d in the variables $[x_0 : \ldots : x_n]$ without common factor of positive degree. Let $\mathrm{Cr}_n(\mathbf{k};\leq d)$ be the union of these sets for all degrees $d' \leq d$. Consider the projective space $\mathrm{Pol}_n(\mathbf{k};d)$ of dimension

$$r(n,d) = (n+1)\begin{pmatrix} n+d \\ d \end{pmatrix} - 1$$

whose elements are given by $(n+1)$-tuples of homogeneous polynomial functions $h_i(x_0,\ldots,x_n)$ of degree d, modulo multiplication by a non-zero common scalar factor; denote by $\mathrm{Form}_n(\mathbf{k};d)$ the subset of $\mathrm{Pol}_n(\mathbf{k};d)$ made of formulas for birational maps, i.e. n-tuples of polynomial functions $(h_i)_{0\leq i\leq n}$ such that

$$[x_0 : \ldots : x_n] \mapsto [h_0 : \ldots : h_n]$$

is a birational transformation of the projective space (of degree $\leq d$). The set $\mathrm{Form}_n(\mathbf{k};d)$ is locally closed in $\mathrm{Pol}_n(\mathbf{k};d)$ for the Zariski topology, and there is a natural projection $\pi_n \colon \mathrm{Form}_n(\mathbf{k};d) \to \mathrm{Cr}_n(\mathbf{k};\leq d)$. One can then show that

- If $f \colon B \to \mathrm{Cr}_n(\mathbf{k})$ is a morphism, its image is contained in $\mathrm{Cr}_n(\mathbf{k};\leq d)$ for some degree d and it can be locally lifted, on affine open subsets $B_i \subset B$, to morphisms $B \to \mathrm{Form}_n(\mathbf{k};d_i')$ for some $d_i' \geq d$;
- a subset S of $\mathrm{Cr}_n(\mathbf{k})$ is closed if and only if its π_n-preimage in $\mathrm{Form}_n(\mathbf{k};d)$ is closed for all $d \geq 1$;
- for every $d \geq 1$, $\mathrm{Cr}_n(\mathbf{k};\leq d)$ is closed in $\mathrm{Cr}_n(\mathbf{k})$;
- the projection $\mathrm{Form}_n(\mathbf{k};d) \to \mathrm{Cr}_n(\mathbf{k};\leq d)$ is surjective, continuous, and closed for every $d \geq 1$ (it is a topological quotient mapping);
- the Zariski topology on the Cremona group is the inductive limit topology of the topologies of $\mathrm{Cr}_n(\mathbf{k};\leq d)$.

These properties are described in [20]. The following example shows that morphisms into $\mathrm{Cr}_n(\mathbf{k};\leq d)$ do not always lift to morphisms into $\mathrm{Form}_n(\mathbf{k};d)$ when the degree of the formulas varies with the parameter.

Example 2.1. A formula like $f = [x_0R : x_1R : x_2R]$, for R a homogeneous polynomial of degree $d-1$ is a non-reduced expression for the identity map; thus, the map from formulas to actual birational transformations contracts sets of positive dimension (here, a projective space onto the point $\{id_{\mathbb{P}^2_{\mathbf{k}}}\}$). A family of birational transformations of $\mathbb{P}^2_{\mathbf{k}}$ of degree d which depends on a parameter c may degenerate (for certain values c_i of c) onto a non-reduced expression of this type. Assume that the parameter c varies on a smooth curve D, that the general member of the family has degree d, and that for two distinct points c_1 and c_2 the formulas become of type $[x_0R_{c_i} : x_1R_{c_i} : x_2R_{c_i}]$ with R_{c_1} and R_{c_2} two homogeneous polynomials which are not proportional. Glue the two points c_1 and c_2 to obtain a nodal curve C, the normalization of which is D. Then, for every point of C, one gets a well defined birational transformation of the plane parametrized by C; but there is no globally defined morphism from C to the space of homogeneous formulas of degree d that determines globally these birational transformations: The two branches of C through its singularity would lead to two distinct expressions at the singular point, one for R_{c_1}, one for R_{c_2}.

An explicit example is described in [20], with $C \subset \mathbb{A}^2_{\mathbf{k}}$ the nodal plane cubic $a^3 + b^3 = abc$ and

$$f_{a,b,c}[x_0 : x_1 : x_2] = [x_0P : x_1Q : x_2P],$$

where $P = ax_2^2 + cx_0x_2 + bx_0^2$ and $Q = ax_2^2 + (b+c)x_0x_2 + (a+b)x_0^2$. The family of transformations $f_{a,b,c}$ is globally defined by formulas of degree 3; but each element $f_{a,b,c}$ has degree ≤ 2 and there is no global parametrization by homogeneous formulas of degree 2. More precisely,

$$abP = (a^2x_2 + b^2x_0)(bx_2 + ax_0)$$

$$abQ = (a^2x_2 + b(a+b)x_0)(bx_2 + ax_0)$$

so that we can factor out the linear term $(bx_2 + ax_0)$. Thus, $f_{a,b,c}$ is a morphism from C to $\mathrm{Cr}_2(\mathbf{k})$ which lifts to a morphism into $\mathrm{Form}_2(\mathbf{k};3)$, but each $f_{a,b,c}$ is in fact a birational map of degree ≤ 2 (the degree is indeed equal to 2 if $[a : b : c] \neq [0 : 0 : 1]$). On the other hand, there is no regular lift to the space of formulas $\mathrm{Form}_2(\mathbf{k};2)$. ([2])

In some sense, the following example is even worse; it shows that there is no structure of algebraic variety on $\mathrm{Cr}_n(\mathbf{k};\leq d)$ (see [20]). The sets $\mathrm{Cr}_n(\mathbf{k};d)$ behave well, but the sets $\mathrm{Cr}_n(\mathbf{k};\leq d)$ don't.

Example 2.2. Consider the variety V that one obtains by removing $p = [0 : 1 : 0]$ and $q = [0 : 0 : 1]$ from the plane $\mathbb{P}^2_{\mathbf{k}}$. Use homogeneous coordinates $[a : b : c]$ for this parameter space $V \subset \mathbb{P}^2_{\mathbf{k}}$. Note that V contains the line $L = \{b = c\}$ (the two points p and q are not on this line). Now, consider the family $g = g_{a,b,c}$ of birational transformations defined by

$$g[x_0 : x_1 : x_2] = [x_0(ax_2 + cx_0) : x_1(ax_2 + bx_0) : x_2(ax_2 + cx_0)],$$

i.e.

$$g(x,y) = \left(\frac{ay+b}{ay+c}x, y\right)$$

in affine coordinates. One gets a family of birational transformations of degree 2, except that all points of L are mapped to the identity (one factors out the linear term $(ax_2 + bx_0)$). Thus, as a map from V to $\mathrm{Cr}_2(\mathbf{k};\leq 2)$, it contracts L to a point, but it is not constant on the $\{b = c + \varepsilon a\}$, $\varepsilon \neq 0$. This prevents $\mathrm{Cr}_2(\mathbf{k};\leq 2)$ to be a bona fide algebraic variety!

[2] Such a lift would be given by

$$\overline{f}_{a,b,c}(x_0,x_1,x_2) = (x_0(a^2x_2 + b^2x_0), x_1(a^2x_2 + b(a+b)x_0), x_2(a^2x_2 + b^2x_0))$$

modulo multiplication by a function of (a,b,c), but this expression does not correspond to a birational map when (a,b,c) is the singular point $(0,0,0)$ of C. Details are given in [20].

2.2. Algebraic subgroups (see [20, 57]). An algebraic subgroup of the Cremona group is a subgroup $G < \mathrm{Cr}_n(\mathbf{k})$ which is the image of an algebraic group K by a homomorphism ρ such that $\rho\colon K \to \mathrm{Cr}_n(\mathbf{k})$ is a morphism with respect to the Zariski topology. In particular, any algebraic group has bounded degree: It is contained in $\mathrm{Cr}_n(\mathbf{k}; \leq d)$ for some d.

Let G be a subgroup of $\mathrm{Cr}_n(\mathbf{k})$, closed for the Zariski topology, and of bounded degree. One can then prove that there is an algebraic group K and a morphism $\rho\colon K \to \mathrm{Cr}_n(\mathbf{k})$ such that ρ is a group homomorphism and ρ is a homeomorphism from K onto its image G for the Zariski topology; moreover, morphisms $B \to \mathrm{Cr}_n(\mathbf{k})$ with values in G correspond to algebraic morphisms into the algebraic variety K via ρ. Thus, *algebraic subgroups correspond exactly to closed subgroups of bounded degree.*

By a theorem of Weil, every subgroup G of bounded degree in $\mathrm{Cr}_n(\mathbf{k})$ can be regularized: There is a projective variety X and a birational mapping $\pi\colon X \dashrightarrow \mathbb{P}^n_{\mathbf{k}}$ such that $G_X := \pi^{-1}G\pi$ is contained in the group of regular automorphisms $\mathrm{Aut}(X)$ (see [38] for a description of Weil theorem and references). Moreover, the identity component $\mathrm{Aut}(X)^0$ is a linear algebraic group (because X is rational), and the intersection $G_X \cap \mathrm{Aut}(X)^0$ has finite index in G_X (see [103]).

Thus, algebraic subgroups of $\mathrm{Cr}_n(\mathbf{k})$ correspond to algebraic groups of automorphisms of rational varieties $X \dashrightarrow \mathbb{P}^n_{\mathbf{k}}$.

2.3. Algebraic tori, rank, and an infinite Weyl group.

2.3.1. Linear subgroups. The Cremona group in one variable coincides with the group of linear projective transformations $\mathsf{PGL}_2(\mathbf{k})$, and is an algebraic group of dimension 3.

The Cremona group $\mathrm{Cr}_2(\mathbf{k})$ contains two important algebraic subgroups. The first one is the group $\mathsf{PGL}_3(\mathbf{k})$ of automorphisms of $\mathbb{P}^2_{\mathbf{k}}$. The second is obtained as follows. Start with the surface $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$, considered as a smooth quadric in $\mathbb{P}^3_{\mathbf{k}}$; its automorphism group contains $\mathsf{PGL}_2(\mathbf{k}) \times \mathsf{PGL}_2(\mathbf{k})$. By stereographic projection, the quadric is birationally equivalent to the plane, so that $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ contains also a copy of $\mathsf{PGL}_2(\mathbf{k}) \times \mathsf{PGL}_2(\mathbf{k})$.

More generally, if $V = G/P$ is a homogeneous variety of dimension n, where G is a semi-simple algebraic group and P is a parabolic subgroup of G, then V is rational; once a birational map $\pi\colon V \dashrightarrow \mathbb{P}^n_{\mathbf{k}}$ is given, $\pi G \pi^{-1}$ determines an algebraic subgroup of $\mathrm{Cr}_n(\mathbf{k})$.

Example 2.3. An important subgroup of $\mathrm{Cr}_2(\mathbf{k})$ which is not algebraic is the **Jonquières group**[3] $\mathrm{Jonq}_2(\mathbf{k})$, of all transformations of $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ that permute the fibers of the projection onto the first factor. It is isomorphic to the semi-direct product $\mathsf{PGL}_2(\mathbf{k}) \ltimes \mathsf{PGL}_2(\mathbf{k}(x))$; for example, it contains all transformations $(X_1, X_2) \mapsto (aX_1, Q(X_1)X_2)$ with a in $\mathbf{k}^*$ and Q in $\mathbf{k}(X_1) \setminus \{0\}$, so that its "dimension" is infinite.

2.3.2. Rank and Weyl group. Let $\mathbf{k}$ be a field. Let S be a connected semi-simple algebraic group defined over $\mathbf{k}$. The group S acts on its Lie algebra $\mathfrak{s}$ by the adjoint representation; the $\mathbf{k}$-rank of S is the maximal dimension $\dim_{\mathbf{k}}(A)$ of a connected algebraic subgroup A of S which is diagonalizable over $\mathbf{k}$ in $\mathsf{GL}(\mathfrak{s})$. Such a maximal diagonalizable subgroup is called a **maximal torus**. For example, the $\mathbf{R}$-rank of $\mathsf{SL}_n(\mathbf{R})$ is $n-1$, and

[3] or the "de Jonquières" group

diagonal matrices form a maximal torus. If $\mathbf{k} = \mathbf{C}$ and the rank of S is equal to r, the centralizer of a typical element $g \in \mathsf{S}$ has dimension r. Thus, the value of the rank reflects the commutation properties inside S.

Theorem 2.4 (Enriques, Demazure, [73, 57]). *Let* $\mathbf{k}$ *be an algebraically closed field, and* $\mathbb{G}_m$ *be the multiplicative group over* $\mathbf{k}$*. Let* r *be an integer. If* $\mathbb{G}_m^r$ *embeds as an algebraic subgroup in* $\mathrm{Cr}_n(\mathbf{k})$*, then* $r \leq n$ *and, if* $r = n$*, the embedding is conjugate to an embedding into the group of diagonal matrices in* $\mathsf{PGL}_{n+1}(\mathbf{k})$.

In other words, in $\mathrm{Cr}_n(\mathbf{k})$ the group of diagonal matrices Δ_n plays the role of a maximal torus (more precisely, a torus of maximal dimension, see Remark 2.5 below). The normalizer of Δ_n in $\mathrm{Cr}_n(\mathbf{k})$ is the semi-direct product of Δ_n with the group of monomial transformations $\mathsf{GL}_n(\mathbf{Z})$, thus

> $\mathrm{Cr}_n(\mathbf{k})$ *looks like a group of rank* n *with maximal torus equal to the diagonal group* Δ_n *and an infinite Weyl group isomorphic to* $\mathsf{GL}_n(\mathbf{Z})$.

This property is reflected by the structure of its finite subgroups, as we shall see below. Nevertheless, for $n = 2$, we shall explain in Section 4 that $\mathrm{Cr}_2(\mathbf{k})$ is better understood as a group of rank 1, and I expect similar rank $n-1$ phenomena for all dimensions $n \geq 2$.

Remark 2.5. Theorem 2.4 is a bit misleading. If maximal tori are defined in terms of dimension, then maximal tori in $\mathrm{Cr}_n(\mathbf{C})$ have dimension n and are all conjugate to the diagonal group. On the other hand, for $n \geq 5$, $\mathrm{Cr}_n(\mathbf{C})$ contains tori of dimension $n-3$ which are not contained in higher dimensional algebraic tori, and are therefore "maximal" in terms of inclusion; since they are maximal, they are not conjugate to a subgroup of $\mathsf{PGL}_{n+1}(\mathbf{C})$. This phenomenon has been discovered by Popov; we refer to [14, 122, 121] for a study of maximal algebraic groups in $\mathrm{Cr}_n(\mathbf{C})$ or $\mathrm{Aut}(\mathbb{A}^n_{\mathbf{C}})$.

2.4. Finite subgroups. The Cremona group $\mathrm{Cr}_1(\mathbf{k})$ is isomorphic to $\mathsf{PGL}_2(\mathbf{k})$. Thus, if G is a finite subgroup of $\mathrm{Cr}_1(\mathbf{k})$ whose order is prime to the characteristic of $\mathbf{k}$, then G is cyclic, dihedral, or isomorphic to $\mathfrak{A}_4$, $\mathfrak{S}_4$, or $\mathfrak{A}_5$; if $\mathbf{k}$ is algebraically closed, each of these groups occurs in $\mathrm{Cr}_1(\mathbf{k})$ in a unique way modulo conjugacy. (here, $\mathfrak{A}_m$ and $\mathfrak{S}_m$ stand for the alternating group and the symmetric group on m symbols).

One of the rich and well understood chapters on $\mathrm{Cr}_2(\mathbf{k})$ concerns the study of its finite subgroups. While there is still a lot to do regarding fields of positive characteristic and conjugacy classes of finite groups, there is now a list of all possible finite groups and maximal algebraic subgroups that can be realized in $\mathrm{Cr}_2(\mathbf{C})$. We refer to [131, 69, 16, 14] for details and references, to [124] for finite simple subgroups of $\mathrm{Cr}_3(\mathbf{C})$, and to [5] for applications to the notion of essential dimension. In what follows, we only emphasize a few results.

2.4.1. Rank, and p-elementary subgroups. A finitary version of Theorem 2.4 has been observed by Beauville in [4] for $n = 2$.

Theorem 2.6. *Let* $\mathbf{k}$ *be an algebraically closed field. Let* $p \geq 5$ *be a prime number with* $p \neq char(\mathbf{k})$*. Assume that the abelian group* $(\mathbf{Z}/p\mathbf{Z})^r$ *embeds into* $\mathrm{Cr}_2(\mathbf{k})$*. Then* $r \leq 2$ *and, if* $r = 2$*, the image of* $(\mathbf{Z}/p\mathbf{Z})^r$ *is conjugate to a subgroup of the group of diagonal matrices of* $\mathsf{PGL}_3(\mathbf{k})$.

Similarly, Prokhorov proved that the rank r of any p-elementary abelian group $(\mathbf{Z}/p\mathbf{Z})^r$ of $\mathrm{Cr}_3(\mathbf{C})$ is bounded from above by 3 if $p \geq 17$ (see [123, 125]). One may ask whether there exists a function $n \mapsto p(n) \in \mathbf{Z}_+$ such that $p \leq p(n)$ if p is prime and $(\mathbf{Z}/p\mathbf{Z})^{n+1}$ embeds in $\mathrm{Cr}_n(\mathbf{C})$. In [130], Serre asks much more precise questions concerning the structure of finite subgroups of $\mathrm{Cr}_n(\mathbf{k})$. One of them concerns the Jordan property: Does every finite subgroup G of $\mathrm{Cr}_n(\mathbf{C})$ contain an abelian subgroup of rank $\leq n$ whose index in G is bounded by a constant $j(n)$ depending only on the dimension n ? These questions were answered positively by Prokhorov and Shramov, assuming the so-called Borisov-Alexeev-Borisov conjecture on the boundedness of families of Fano varieties with terminal singularities (see [126, 127]). Amazingly, a recent preprint of Birkar delivers a proof of this conjecture (see [13]).

2.4.2. Finite simple subgroups (see [71, 136]). There is one, and only one simple subgroup in $\mathrm{Cr}_1(\mathbf{C})$, namely $\mathfrak{A}_5$, the symmetry group of the icosaehdron.

Theorem 2.7. *If G is a finite, simple, non-abelian subgroup of $\mathrm{Cr}_2(\mathbf{C})$, then G is isomorphic to one of the groups $\mathsf{PSL}_2(\mathbf{F}_7)$, $\mathfrak{A}_5$, and $\mathfrak{A}_6$.*

- *There are two conjugacy classes of subgroups isomorphic to $\mathsf{PSL}_2(\mathbf{F}_7)$. First, $\mathsf{PSL}_2(\mathbf{F}_7)$ embeds in $\mathsf{PGL}_3(\mathbf{C})$, preserving the smooth quartic curve $x_0^3x_1 + x_1^3x_2 + x_2^3x_0 = 0$; then, it also embeds as a group of automorphisms of the double cover of the plane, ramified along the same quartic curve.*
- *There are three embeddings of $\mathfrak{A}_5$ in $\mathrm{Cr}_2(\mathbf{C})$ up to conjugacy. One in $\mathsf{PGL}_2(\mathbf{C})$, one in $\mathsf{PGL}_3(\mathbf{C})$, and one in the group of automorphisms of the del Pezzo surface which is obtained by blowing up $\mathbb{P}^2_{\mathbf{C}}$ at the points $[1:0:0]$, $[0:1:0]$, $[0:0:1]$, and $[1:1:1]$.*
- *There is a unique copy of $\mathfrak{A}_6$ up to conjugacy, given by a linear projective action on $\mathbb{P}^2_{\mathbf{C}}$ that preserves the curve*

$$10x_0^3x_1^3 + 9x_2x_0^5 + 9x_2x_1^5 + 27x_2^6 = 45x_0^2x_1^2x_2^2 + 135x_0x_1x_2^4.$$

Note that, given an embedding $\iota\colon G \to \mathrm{Cr}_2(\mathbf{C})$, one can twist it by an automorphism φ of G. When G is isomorphic to $\mathsf{PSL}_2(\mathbf{F}_7)$ or $\mathfrak{A}_6$, ι is conjugate to $\iota \circ \varphi$ in $\mathrm{Cr}_2(\mathbf{C})$ if and only if φ is an inner automorphism of G; thus, there are 4 distinct embeddings of $\mathfrak{A}_6$ (resp. $\mathsf{PSL}_2(\mathbf{F}_7)$) in $\mathrm{Cr}_2(\mathbf{C})$ up to conjugacy. On the other hand, ι is always conjugate to $\iota \circ \varphi$ when $G = \mathfrak{A}_5$; thus, $\mathfrak{A}_5$ has exactly three embeddings in $\mathrm{Cr}_2(\mathbf{C})$ up to conjugacy.

Remark 2.8. If G is a finite subgroup of $\mathrm{Cr}_2(\mathbf{k})$ and the characteristic p of the field $\mathbf{k}$ does not divide the order of G, then G "lifts" in characteristic zero; but there are new examples of simple subgroups of $\mathrm{Cr}_2(\mathbf{k})$ if we allow p to divide $|G|$ (see [69] for a classification).

There is also a classification, due to Prokhorov [124], of finite simple subgroups of $\mathrm{Cr}_3(\mathbf{C})$ up to isomorphism, but a complete list of their conjugacy classes is not available yet. Besides $\mathfrak{A}_5$, $\mathfrak{A}_6$, and $\mathsf{PSL}_2(\mathbf{F}_7)$, there are three new players: $\mathfrak{A}_7$, $\mathsf{PSL}_2(\mathbf{F}_8)$, and $\mathsf{PSP}_4(\mathbf{F}_3)$, with respective orders 2520, 504, 25920. See [45, 46, 47] for the study of their conjugacy classes in $\mathrm{Cr}_3(\mathbf{C})$.

2.5. Closed normal subgroups. Let us assume, for simplicity, that $\mathbf{k}$ is algebraically closed. In dimension $n = 1$, the Cremona group $\mathsf{PGL}_2(\mathbf{k})$ is a simple group. As we shall see in Section 7, $\mathrm{Cr}_2(\mathbf{k})$ is not simple, and contains many normal subgroups. But J. Blanc and S. Zimmerman observed that $\mathrm{Cr}_n(\mathbf{k})$ behaves as a simple group if one restricts our study to closed, normal subgroups.

Theorem 2.9 ([15, 22]). *Let* $\mathbf{k}$ *be an algebraically closed field. Every non-trivial normal subgroup of* $\mathrm{Cr}_n(\mathbf{k})$ *which is closed for the Zariski topology coincides with* $\mathrm{Cr}_n(\mathbf{k})$.

This result explains why there is no construction from algebraic geometry that produces interesting normal subgroups in $\mathrm{Cr}_n(\mathbf{k})$.

Assume now that $\mathbf{k}$ is a local field; this means that $\mathbf{k}$ is a locally compact topological field with respect to a non-discrete topology. The examples are $\mathbf{R}$, $\mathbf{C}$, and finite extensions of $\mathbf{Q}_p$ and $\mathbf{F}_q((t))$. (Here, $\mathbf{Q}_p$ is the field of p-adic numbers and $\mathbf{F}_q$ is a finite field with q elements) Then, there exists a group-topology on $\mathrm{Cr}_n(\mathbf{k})$ that extends the "transcendental, euclidean" topology of $\mathsf{PGL}_{n+1}(\mathbf{k})$ (see [20]). Blanc and Zimmermann also prove that every normal subgroup that is closed for this topology is either trivial or equal to $\mathrm{Cr}_n(\mathbf{k})$ (see [22]).

3. Generating sets and relations

3.1. Dimension 2. Recall from Example 2.3 that the Jonquières group $\mathrm{Jonq}_2(\mathbf{k})$ is the group of birational transformations of $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ that permute the fibers of the first projection; we may identify it to the group of birational transformations of $\mathbb{P}^2_{\mathbf{k}}$ preserving the pencil of lines through the point $[1:0:0]$.

The first main result on $\mathrm{Cr}_2(\mathbf{k})$ is due to Noether and Castelnuovo [116, 43]. It exhibits two sets of generators for $\mathrm{Cr}_2(\mathbf{k})$.

Theorem 3.1 (Noether, Castelnuovo). *Let* $\mathbf{k}$ *be an algebraically closed field. The group* $\mathrm{Cr}_2(\mathbf{k})$ *is generated by* $\mathsf{PGL}_3(\mathbf{k})$ *and the standard quadratic involution* σ_2. *It is also generated by* $\mathrm{Jonq}_2(\mathbf{k})$ *and the involution* $\eta(X_1, X_2) = (X_2, X_1)$.

Identify $\mathrm{Jonq}_2(\mathbf{k})$ to the group of birational transformations of $\mathbb{P}^2_{\mathbf{k}}$ that preserve the pencil of lines through the point $[1:0:0]$, and η to the involution $[x_1 : x_2 : x_3] \mapsto [x_2 : x_1 : x_3]$. With such a choice, η is in $\mathsf{PGL}_3(\mathbf{k})$ and σ_2 is in $\mathrm{Jonq}_2(\mathbf{k})$. Then, $\mathrm{Cr}_2(\mathbf{k})$ *is the amalgamated product of* $\mathrm{Jonq}_2(\mathbf{k})$ *and* $\mathsf{PGL}_3(\mathbf{k})$ *along their intersection, divided by one more relation, namely* $\sigma \circ \eta = \eta \circ \sigma$ (see [17, 95] and [83, 84] for former presentations of $\mathrm{Cr}_2(\mathbf{k})$). Thus, one knows a presentation of $\mathrm{Cr}_2(\mathbf{k})$ by generators and relations.

Example 3.2. Let $\mathbf{k}$ be an algebraically closed field. Consider the set of generators of $\mathrm{Cr}_2(\mathbf{k})$ given by σ_2 and the group of automorphisms $\mathsf{PGL}_3(\mathbf{k})$ of $\mathbb{P}^2_{\mathbf{k}}$. The following relations are satisfied

- $\sigma_2 \circ \tau = \tau \circ \sigma_2$ for every permutation τ of the three coordinates x_i;
- $\sigma_2 \circ a = a^{-1} \circ \sigma_2$ for every diagonal automorphism $a[x_0 : x_1 : x_2] = [ux_0 : vx_1 : wx_2]$.

- If h is the linear projective transformation $h[x_1 : x_2 : x_3] = [x_1, x_1 - x_2, x_1 - x_3]$, then $(h \circ \sigma_2)^3$ is the identity (see [83]).

The first and second list of relations occur in the semi-direct product of the group $\mathsf{GL}_2(\mathbf{Z})$ of monomial transformations and the diagonal group $\mathbb{G}_m(\mathbf{k}) \times \mathbb{G}_m(\mathbf{k})$ (i.e. in the normalizer of the maximal torus).

Remark 3.3. Similarly, Jung's theorem asserts that the group of polynomial automorphisms of the affine plane is the free product of two of its subgroups, amalgamated along their intersection (see [101] for example); the two subgroups are the group of affine transformations, and the group of elementary shears $(x,y) \mapsto (ax, by + p(x))$, with $p \in \mathbf{k}[x]$. Note that this result holds for every field $\mathbf{k}$, algebraically closed or not. This is related to the following geometric fact: If h is a polynomial automorphism of the affine plane, then h^{-1} has at most one indeterminacy point in $\mathbb{P}^2(\overline{\mathbf{k}})$, this point is the image of a general point of the line at infinity under the action of h and, as such, is contained in $\mathbb{P}^2(\mathbf{k})$; thus, the first blow-up that is required to resolve the indeterminacy point is defined over $\mathbf{k}$.

Elementary shears are examples of Jonquières transformations, preserving the pencil of vertical lines $x = c^{st}$; one feature of these shears is that there degrees remain bounded under iteration: If $g(x,y) = (ax, by + p(x))$ and $p(x)$ has degree d, then all iterates g^n are shears of degree at most d. This is not typical among Jonquières transformations (see Section 4.2).

3.2. Dimension ≥ 3. In dimension 2, the indeterminacy locus of a birational transformation is a finite set, and the curves that appear by blow-up are smooth rational curves. This simple picture changes dramatically in higher dimension: As we shall see below, for every smooth irreducible curve C, there is a birational transformation g of $\mathbb{P}^3_{\mathbf{k}}$ and a surface $X \subset \mathbb{P}^3_{\mathbf{k}}$ such that (i) X is birationally equivalent to $C \times \mathbb{P}^1_{\mathbf{k}}$ and (ii) g contracts X onto a subset of codimension ≥ 2. This new feature leads to the following result (see [118]).

Theorem 3.4 (Hudson, Pan). *Let $n \geq 3$ be a natural integer. Let $\mathbf{k}$ be an algebraically closed field. To generate $\mathrm{Cr}_n(\mathbf{k})$, one needs as many algebraic families of generators, as families of smooth hypersurfaces of $\mathbb{P}^{n-1}_{\mathbf{k}}$ of degree $\geq n+2$; one cannot generate the Cremona group by generators of bounded degree.*

Obviously, this is loosely stated, and we only present a sketch of the proof (see [118, 34] for details). Let $[x] = [x_0 : \ldots : x_{n-1}]$ be homogeneous coordinates for $\mathbb{P}^{n-1}_{\mathbf{k}}$ and $[y_0 : y_1]$ be homogeneous coordinates for $\mathbb{P}^1_{\mathbf{k}}$. Let Y be an irreducible hypersurface of degree d in $\mathbb{P}^{n-1}_{\mathbf{k}}$, which is not the plane $x_0 = 0$, and let h be a reduced homogeneous equation for Y. Define a birational transformation f_Y of $\mathbb{P}^{n-1}_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ by

$$f_Y([x],[y_0 : y_1]) = ([x],[y_0 x_0^d : h(x_0, \ldots, x_{n-1}) y_1]).$$

The transformation f_Y preserves the projection onto the first factor $\mathbb{P}^{n-1}_{\mathbf{k}}$, and acts by linear projective transformations on the general fibers $\mathbb{P}^1_{\mathbf{k}}$; more precisely, on the fiber over $[x]$, f_Y is the projective linear transformation which is determined by the 2 by 2 matrix

$$\begin{pmatrix} x_0^d & 0 \\ 0 & h(x_0, \ldots, x_{n-1}) \end{pmatrix}.$$

This matrix is invertible if and only if $x_0 \neq 0$ and $h(x) \neq 0$, and f_Y contracts the hypersurface $Y \times \mathbb{P}^1_{\mathbf{k}}$ to the codimension 2 subset $Y \times \{[1:0]\}$. Thus, given any irreducible hypersurface Y in $\mathbb{P}^{n-1}_{\mathbf{k}}$, one can construct a birational transformation of $\mathbb{P}^n_{\mathbf{k}}$ that contracts a hypersurface which is birationally equivalent to $Y \times \mathbb{P}^1_{\mathbf{k}}$.

On the other hand, one easily checks the following: Let g_1, ..., g_m be birational transformations of the projective space $\mathbb{P}^n_{\mathbf{k}}$, and let g be the composition $g = g_m \circ g_{m-1} \circ \ldots \circ g_1$. Let X be an irreducible hypersurface of $\mathbb{P}^n_{\mathbf{k}}$. If X is g-exceptional (i.e. g contracts X), then there is an index i, with $1 \leq i \leq m$, and a g_i-exceptional hypersurface X_i such that X is birationally equivalent to X_i. More precisely, for some index i, $g_{i-1} \circ \ldots \circ g_1$ realizes a birational isomorphism from X to X_i, and then g_i contracts X_i.

Thus, to generate $\mathrm{Cr}_n(\mathbf{k})$, one needs at least as many families of generators as families of hypersurfaces $Y \subset \mathbb{P}^{n-1}_{\mathbf{k}}$ modulo the equivalence relation "$Y \simeq Y'$ if and only if $Y \times \mathbb{P}^1_{\mathbf{k}}$ is birationally equivalent to $Y' \times \mathbb{P}^1_{\mathbf{k}}$". But, if Y and Y' are general hypersurfaces of degree $\geq n+2$, then Y and Y' have general type, and the relation $Y \simeq Y'$ implies that Y and Y' are isomorphic.

Remark 3.5. Given f in the Cremona group $\mathrm{Cr}_3(\mathbf{k})$, consider the set of irreducible components $\{X_i\}_{1 \leq i \leq m}$ of the union of the exceptional loci of f and of its inverse f^{-1}. Each X_i is birationally equivalent to a product $\mathbb{P}^1_{\mathbf{k}} \times C_i$, where C_i is a smooth irreducible curve. Define $g(X_i)$ as the genus of C_i, and the **genus** of f as the maximum of the $g(X_i)$, $1 \leq i \leq m$. Then, the subset of $\mathrm{Cr}_3(\mathbf{k})$ of all birational transformations f of genus at most g_0 is a subgroup of $\mathrm{Cr}_3(\mathbf{k})$: In this way, one obtains a filtration of the Cremona group by an increasing sequence of proper subgroups. See [79, 102] for related ideas and complements.

3.3. Fields which are not algebraically closed. Now, consider the case $n = 2$, but with a field which is not algebraically closed; for simplicity, take $\mathbf{k} = \mathbf{Q}$, the field of rational numbers. Given f in $\mathrm{Cr}_2(\mathbf{Q})$, the indeterminacy locus $\mathrm{Ind}(f)$ of f is a finite subset of $\mathbb{P}^2(\overline{\mathbf{Q}})$, where $\overline{\mathbf{Q}}$ is a fixed algebraic closure of $\mathbf{Q}$. Fix a number field $\mathbf{K}$, and consider the set of all $f \in \mathrm{Cr}_2(\mathbf{Q})$ such that each base point of f and f^{-1} (including infinitesimally closed points) is defined over $\mathbf{K}$; for instance, if $p \in \mathbb{P}^2(\mathbf{C})$ is an indeterminacy point of f^{-1}, then $p = [a_0 : a_1 : a_2]$ with a_i in $\mathbf{K}$. This set is a subgroup of $\mathrm{Cr}_2(\mathbf{Q})$; in this way, we get an inductive net of subgroups of $\mathrm{Cr}_2(\mathbf{Q})$. This construction is similar to the filtration obtained in Remark 3.5 (the degree of the extension $\mathbf{K}/\mathbf{Q}$ plays the same role as the genus).

More generally, fix a field $\mathbf{k}$ together with an algebraic closure $\overline{\mathbf{k}}$ of $\mathbf{k}$; denote by $\mathbf{k}_0$ the smallest subfield of $\mathbf{k}$ (either $\mathbf{Q}$ or $\mathbf{F}_p$). To an element f of $\mathrm{Cr}_2(\mathbf{k})$, one can associate the field $\mathbf{k}_f$: The smallest field $\mathbf{k}_0 \subset \mathbf{k}_f \subset \overline{\mathbf{k}}$ on which f, f^{-1} and all their base points are defined. With this definition, $\mathbf{k}_f$ may be smaller than $\mathbf{k}$. Then, the field $\mathbf{k}_{f \circ g}$ is contained in the extension generated by $\mathbf{k}_f$ and $\mathbf{k}_g$. Thus, $\mathbf{k}_f$ provides a measure for the arithmetic complexity of f, and this measure behaves sub-multiplicatively.([4])

Proposition 3.6. *Let* $\mathbf{k}$ *be a field. The Cremona group* $\mathrm{Cr}_2(\mathbf{k})$ *is not finitely generated.*

Proof. Let $\mathcal{F}$ be a finite subset of $\mathrm{Cr}_2(\mathbf{k})$. Let $\mathbf{k}_{\mathcal{F}} \subset \overline{\mathbf{k}}$ be the extension of $\mathbf{k}_0$ which is generated by the fields $\mathbf{k}_f$, $f \in \mathcal{F}$. Let G be the subgroup of $\mathrm{Cr}_2(\mathbf{k})$ generated by $\mathcal{F}$. Then

[4] This sub-section follows from a discussion with Jérémy Blanc and Christian Urech, during which Blanc explained the proof of Proposition 3.6.

$\mathbf{k}_g \subset \mathbf{k}_{\mathcal{F}}$ for all elements g of G. Let $q(x)$ be an element of $\mathbf{k}[x]$ of degree d, and consider the Jonquières transformation g_q which is defined by

$$g_q[x_0 : x_1 : x_2] = [x_0 x_2^d : q(x_0/x_2)x_1 x_2^d : x_2^{d+1}].$$

Then each root α_i of q gives rise to an indeterminacy point $[\alpha_i : 0 : 1]$ of g_q^{-1}. Thus, if g_q belongs to the group G then all roots of q are contained in $\mathbf{k}_{\mathcal{F}}$. If g_q is in G for every q, then $\mathbf{k}_{\mathcal{F}}$ is finitely generated and algebraically closed. No such field exists. □

Generating sets and relations for the group $\mathrm{Cr}_2(\mathbf{R})$ have been found in [21, 94, 128, 140]. For instance, both $\mathrm{Cr}_2(\mathbf{R})$ and $\mathrm{Bir}^\infty(\mathbb{P}^2(\mathbf{R}))$ are generated by subsets of $\mathrm{Cr}_2(\mathbf{R}; \leq 5)$; one can even provide presentations of $\mathrm{Cr}_2(\mathbf{R})$ by generators and relations.

In [140], Zimmermann describes a striking application of this circle of ideas. She generates $\mathrm{Cr}_2(\mathbf{R})$ by $\mathsf{PGL}_3(\mathbf{R})$, the group of Jonquières transformations $\mathrm{Jonq}_2(\mathbf{R})$, and a twisted form of it, namely the group $\mathrm{Jonq}_2^\pi(\mathbf{R})$ of birational transformations of the plane that permute the fibers of the rational function

$$\pi[x_1 : x_2 : x_3] = \frac{x_2^2 + (x_1 + x_3)^2}{x_2^2 + (x_1 - x_3)^2}.$$

This group $\mathrm{Jonq}_2^\pi(\mathbf{R})$ is isomorphic to the semi-direct product $A \ltimes B$ of the groups $A = \mathbf{R}_+^* \rtimes \mathbf{Z}/2\mathbf{Z}$ and $B = \mathsf{SO}(x^2 + y^2 - tz^2; \mathbf{R}(t))$. The elements of B preserve each fiber of π, acting as rotations along these circles, with an angle of rotation that depends on the circle. The elements of A permute the circles, the value of the projection π being changed into $\alpha\pi$ or α/π for some $\alpha \in \mathbf{R}_+^*$. The spinor norm provides a homomorphism from B to the group $\mathbf{R}(t)^*/(\mathbf{R}(t)^*)^2$. We may identify $\mathbf{R}(t)^*/(\mathbf{R}(t)^*)^2$ with the set of polynomial functions $g \in \mathbf{R}[t]$ with only simple roots; and to such a function g, we associate the function

$$\xi(g) \colon [0, \pi] \to \mathbf{Z}/2\mathbf{Z}$$

which is defined as follows: for each angle $\theta \in [0, \pi]$, $\xi(g)(\theta)$ is the number (modulo 2) of roots of g with argument equal to θ (i.e. $z = |z|e^{\theta\sqrt{-1}}$). It turns out that the map $g \mapsto \xi(g)$ extends to a homomorphism from $\mathrm{Jonq}_2^\pi(\mathbf{R})$ to the additive group $\oplus_{[0,\pi]}\mathbf{Z}/2\mathbf{Z}$ of functions $[0, \pi] \to \mathbf{Z}/2\mathbf{Z}$ with finite support. With her explicit presentation of $\mathrm{Cr}_2(\mathbf{R})$, Zimmermann shows that this homomorphism extends to an epimorphism $\mathrm{Cr}_2(\mathbf{R}) \to \oplus_{[0,\pi]}\mathbf{Z}/2\mathbf{Z}$, and then she gets the following result.

Theorem 3.7 (Zimmermann). *The derived subgroup of* $\mathrm{Cr}_2(\mathbf{R})$ *coincides with the normal closure of* $\mathsf{PGL}_3(\mathbf{R})$ *in* $\mathrm{Cr}_2(\mathbf{R})$ *and is a proper subgroup of* $\mathrm{Cr}_2(\mathbf{R})$*, the abelianization of* $\mathrm{Cr}_2(\mathbf{R})$ *being isomorphic to the additive group* $\oplus_{[0,\pi]}\mathbf{Z}/2\mathbf{Z}$ *of functions* $f \colon [0, \pi] \to \mathbf{Z}/2\mathbf{Z}$ *with finite support.*

We refer to §7 for a different construction of normal subgroups in $\mathrm{Cr}_2(\mathbf{k})$.

–II–
Dimension 2 and hyperbolic geometry

In the forthcoming sections, namely § 4 to 7, we focus on groups of birational transformations of surfaces. The most interesting case is the Cremona group $\mathrm{Cr}_2(\mathbf{k})$ or, what is the same, groups of birational transformations of rational surfaces. Indeed, if X is a projective surface with non-negative Kodaira dimension, then X has a unique minimal model X_0, and $\mathrm{Bir}(X)$ coincides with $\mathrm{Aut}(X_0)$; if the Kodaira dimension of X is negative and X is not rational, then X is ruled in a unique way, and $\mathrm{Bir}(X)$ preserves this ruling. As a consequence, the focus is on the group $\mathrm{Cr}_2(\mathbf{k})$.

4. An infinite dimensional hyperbolic space

Most recent results on $\mathrm{Cr}_2(\mathbf{k})$ are better understood if one explains how $\mathrm{Cr}_2(\mathbf{k})$ acts by isometries on an infinite dimensional hyperbolic space $\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$. This construction is due to Manin and Zariski, but it had not been used much until recently.

Example 4.1. The standard quadratic involution σ_2 maps lines to conics. Thus, it acts by multiplication by 2 on the Picard group of the plane $\mathbb{P}^2_{\mathbf{k}}$ (or on the homology group $H_2(\mathbb{P}^2(\mathbf{C}),\mathbf{Z})$ if $\mathbf{k}=\mathbf{C}$). Since σ_2 is an involution, the action of σ_2^2 on that group is the identity, not multiplication by 4. This shows that $\mathrm{Cr}_2(\mathbf{k})$ does not "act" on the Picard group. The forthcoming construction overcomes this difficulty by blowing up all possible indeterminacy points.

4.1. The Picard-Manin space.

4.1.1. General construction. Let X be a smooth, irreducible, projective surface. The Picard group $\mathrm{Pic}(X)$ is the quotient of the abelian group of divisors by the subgroup of principal divisors [91]. The intersection between curves determines a quadratic form on $\mathrm{Pic}(X)$, the so-called **intersection form**

$$(C,D)\mapsto C\cdot D. \tag{6}$$

The quotient of $\mathrm{Pic}(X)$ by the subgroup of divisors E such that $E\cdot D=0$ for all divisor classes D is the Néron-Severi group $\mathrm{NS}(X)$. It is a free abelian group and its rank, the Picard number $\rho(X)$, is finite; when $\mathbf{k}=\mathbf{C}$, $\mathrm{NS}(X)$ can be identified to $H^{1,1}(X;\mathbf{R})\cap H^2(X;\mathbf{Z})$. The Hodge index Theorem asserts that the signature of the intersection form is equal to $(1,\rho(X)-1)$ on $\mathrm{NS}(X)$.

If $\pi\colon X'\to X$ is a birational morphism, the pull-back map π^* is an injective homomorphism from $\mathrm{NS}(X)$ to $\mathrm{NS}(X')$ that preserves the intersection form; $\mathrm{NS}(X')$ decomposes as the orthogonal sum of $\pi^*\mathrm{NS}(X)$ and the subspace generated by classes of curves contracted by π, on which the intersection form is negative definite.

If $\pi_1: X_1\to X$ and $\pi_2: X_2\to X$ are two birational morphisms, there is a third birational morphism $\pi_3\colon X_3\to X$ that "covers" π_1 and π_2, meaning that $\pi_3\circ\pi_1^{-1}$ and $\pi_3\circ\pi_2^{-1}$ are morphisms; informally, one can obtain X_3 from X by blowing-up all points that are blown-up either by π_1 or by π_2 (blowing up more points, one gets several choices for X_3).

One can therefore define the direct limit of the groups $\mathrm{NS}(X')$, where $\pi\colon X' \to X$ runs over the set of all birational morphisms onto X. This limit

$$\mathcal{Z}(X) := \lim_{\pi\colon X' \to X} \mathrm{NS}(X') \tag{7}$$

is the **Picard-Manin space** of X. It is an infinite dimensional free abelian group. The intersection forms on $\mathrm{NS}(X')$ determine a quadratic form on $\mathcal{Z}(X)$, the signature of which is equal to $(1, \infty)$. By construction, $\mathrm{NS}(X)$ embeds naturally as a proper subspace of $\mathcal{Z}(X)$, and the intersection form is negative definite on the infinite dimensional space $\mathrm{NS}(X)^{\perp}$.

Example 4.2. The group $\mathrm{Pic}(\mathbb{P}^2_{\mathbf{k}})$ is generated by the class $\mathbf{e}_0$ of a line. Blow-up one point q_1 of the plane, to get a morphism $\pi_1 : X_1 \to \mathbb{P}^2_{\mathbf{k}}$. Then, $\mathrm{Pic}(X_1)$ is a free abelian group of rank 2, generated by the class $\mathbf{e}_1$ of the exceptional divisor E_{q_1}, and by the pull-back of $\mathbf{e}_0$ under π_1 (still denoted $\mathbf{e}_0$ in what follows). After n blow-ups $X_i \to X_{i-1}$ of points $q_i \in X_{i-1}$ one obtains

$$\mathrm{Pic}(X_n) = \mathbf{Z}\mathbf{e}_0 \oplus \mathbf{Z}\mathbf{e}_1 \oplus \ldots \oplus \mathbf{Z}\mathbf{e}_n \tag{8}$$

where $\mathbf{e}_0$ (resp. $\mathbf{e}_i$) is the class of the total transform of a line (resp. of the exceptional divisor E_{q_i}) by the composite morphism $X_n \to \mathbb{P}^2_{\mathbf{k}}$ (resp. $X_n \to X_i$). The direct sum decomposition (8) is orthogonal with respect to the intersection form. More precisely,

$$\mathbf{e}_0 \cdot \mathbf{e}_0 = 1, \quad \mathbf{e}_i \cdot \mathbf{e}_i = -1 \ \forall\, 1 \leq i \leq n, \quad \text{and} \quad \mathbf{e}_i \cdot \mathbf{e}_j = 0 \ \forall\, 0 \leq i \neq j \leq n. \tag{9}$$

In particular, $\mathrm{Pic}(X) = \mathrm{NS}(X)$ for rational surfaces. Taking limits, one sees that the Picard-Manin space $\mathcal{Z}(\mathbb{P}^2_{\mathbf{k}})$ is a direct sum $\mathcal{Z}(\mathbb{P}^2_{\mathbf{k}}) = \mathbf{Z}\mathbf{e}_0 \oplus \bigoplus_q \mathbf{Z}\mathbf{e}_q$ where q runs over all possible points that can be blown-up (including infinitely near points). More precisely, q runs over the so-called bubble space $\mathcal{B}(X)$ of X (see [109, 68, 18]).

4.1.2. Minkowski spaces. This paragraph is a parenthesis on the geometry of Minkowski spaces and their isometries.

Standard Minkowski spaces.– Let $\mathcal{H}$ be a real Hilbert space of dimension $m+1$ (m can be infinite). Fix a unit vector $\mathbf{e}_0$ of $\mathcal{H}$ and a Hilbert basis $(\mathbf{e}_i)_{i\in I}$ of the orthogonal complement of $\mathbf{e}_0$. Define a new scalar product on $\mathcal{H}$ by

$$\langle u | u' \rangle_m = a_0 a'_0 - \sum_{i\in I} a_i a'_i \tag{10}$$

for every pair $u = a_0\mathbf{e}_0 + \sum_i a_i \mathbf{e}_i$, $u' = a'_0\mathbf{e}_0 + \sum_i a'_i \mathbf{e}_i$ of vectors. In other words, we just change the sign of the scalar product on $\mathbf{e}_0^{\perp}$. Define $\mathbb{H}_m$ to be the connected component of the hyperboloid

$$\{u \in \mathcal{H} |\ \langle u | u \rangle_m = 1\} \tag{11}$$

that contains $\mathbf{e}_0$, and let dist_m be the distance on $\mathbb{H}_m$ defined by (see [11, 92])

$$\cosh(\mathrm{dist}_m(u, u')) = \langle u | u' \rangle_m. \tag{12}$$

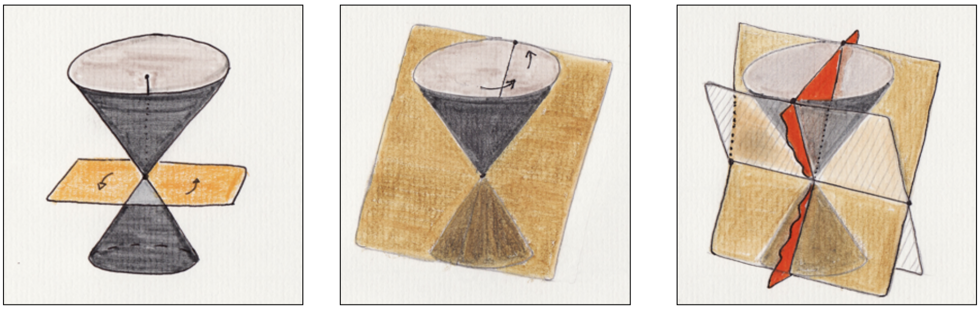

Figure 1. Three types of isometries (from left to right): Elliptic, parabolic, and loxodromic. Elliptic isometries preserve a point in $\mathbb{H}_m$ and act as a rotation on the orthogonal complement. Parabolic isometries fix an isotropic vector v; the orthogonal complement of $\mathbf{R}v$ contains it, and is tangent to the isotropic cone. Loxodromic isometries dilate an isotropic line, contract another one, and act as a rotation on the intersection of the planes tangent to the isotropic cone along those lines (see also Figure 2 below).

The metric space $(\mathbb{H}_m, \mathrm{dist}_m)$ is a Riemannian, simply-connected, and complete space of dimension m with constant sectional curvature -1; these properties uniquely characterize it up to isometry. ([5])

The projection of $\mathbb{H}_m$ into the projective space $\mathbb{P}(\mathcal{H})$ is one-to-one onto its image. In homogeneous coordinates, its image is the ball $a_0^2 > \sum_i a_i^2$, and the **boundary** is the sphere obtained by projection of the isotropic cone $a_0^2 = \sum_i a_i^2$. In what follows, $\mathbb{H}_m$ is identified with its image in $\mathbb{P}(\mathcal{H})$ and its boundary is denoted by $\partial\mathbb{H}_m$; hence, boundary points correspond to isotropic lines in the space $\mathcal{H}$ (for the scalar product $\langle\cdot|\cdot\rangle_m$).

Isometries.–Denote by $\mathsf{O}_{1,m}(\mathbf{R})$ the group of linear transformations of $\mathcal{H}$ preserving the scalar product $\langle\cdot|\cdot\rangle_m$. The group of isometries $\mathrm{Isom}(\mathbb{H}_m)$ coincides with the index 2 subgroup $\mathsf{O}^+_{1,m}(\mathbf{R})$ of $\mathsf{O}(\mathcal{H})$ that preserves the chosen sheet $\mathbb{H}_m$ of the hyperboloid $\{u \in \mathcal{H} \mid \langle u|u\rangle_m = 1\}$. This group acts transitively on $\mathbb{H}_m$, and on its unit tangent bundle.

If $h \in \mathsf{O}^+_{1,m}(\mathbf{R})$ is an isometry of $\mathbb{H}_m$ and $v \in \mathcal{H}$ is an eigenvector of h with eigenvalue λ, then either $|\lambda| = 1$ or v is isotropic. Moreover, since $\mathbb{H}_m$ is homeomorphic to a ball, h has at least one eigenvector v in $\mathbb{H}_m \cup \partial\mathbb{H}_m$. Thus, there are three types of isometries [29]: **Elliptic** isometries have a fixed point u in $\mathbb{H}_m$; **parabolic** isometries have no fixed point in $\mathbb{H}_m$ but they fix a vector v in the isotropic cone; **loxodromic** (also called **hyperbolic**) isometries have an isotropic eigenvector v with eigenvalue $\lambda > 1$. They satisfy the following additional properties (see [29]).

(1) An isometry h is elliptic if and only if it fixes a point u in $\mathbb{H}_m$. Since $\langle\cdot|\cdot\rangle_m$ is negative definite on the orthogonal complement $u^\perp$, the linear transformation h fixes pointwise the line $\mathbf{R}u$ and acts by rotation on $u^\perp$ with respect to $\langle\cdot|\cdot\rangle_m$.

[5] The Riemannian structure is defined as follows. If u is an element of $\mathbb{H}_m$, the tangent space $T_u\mathbb{H}_m$ is the affine space through u that is parallel to $u^\perp$, where $u^\perp$ is the orthogonal complement of $\mathbf{R}u$ with respect to $\langle\cdot|\cdot\rangle_m$; since $\langle u|u\rangle_m = 1$, the form $\langle\cdot|\cdot\rangle_m$ is negative definite on $u^\perp$, and its opposite defines a positive scalar product on $T_u\mathbb{H}_m$; this family of scalar products determines a Riemannian metric, and the associated distance coincides with dist_m (see [11]).

(2) An isometry h is parabolic if it is not elliptic and fixes a vector v in the isotropic cone. The line $\mathbf{R}v$ is uniquely determined by the parabolic isometry h. If z is a point of $\mathbb{H}_m$, there is an increasing sequence of integers m_i such that $h^{m_i}(z)$ converges towards the boundary point v.

(3) An isometry h is loxodromic if and only if h has an eigenvector v_h^+ with eigenvalue $\lambda > 1$. Such an eigenvector is unique up to scalar multiplication, and there is another, unique, isotropic eigenline $\mathbf{R}v_h^-$ corresponding to an eigenvalue < 1; this eigenvalue is equal to $1/\lambda$. If u is an element of $\mathbb{H}_m$,

$$\frac{1}{\lambda^n}h^n(u) \longrightarrow \frac{\langle u|v_h^-\rangle_m}{\langle v_h^+|v_h^-\rangle_m}v_h^+$$

as n goes to $+\infty$, and

$$\frac{1}{\lambda^{-n}}h^n(u) \longrightarrow \frac{\langle u|v_h^+\rangle_m}{\langle v_h^+|v_h^-\rangle_m}v_h^-$$

as n goes to $-\infty$. On the orthogonal complement of $\mathbf{R}v_h^+ \oplus \mathbf{R}v_h^-$, h acts as a rotation with respect to $\langle\cdot|\cdot\rangle_m$. The boundary points determined by v_h^+ and v_h^- are the two fixed points of h in $\mathbb{H}_\infty \cup \partial\mathbb{H}_\infty$: The first one is an attracting fixed point, the second is repulsive.

Moreover, $h \in \mathrm{Isom}(\mathbb{H}_\infty)$ is loxodromic if and only if its **translation length**

$$L(h) = \inf\{\mathrm{dist}(x,h(x)) \mid x \in \mathbb{H}_\infty\} \tag{13}$$

is positive. In that case, $\lambda = \exp(L(h))$ is the largest eigenvalue of h and $\mathrm{dist}(x,h^n(x))$ grows like $nL(h)$ as n goes to $+\infty$ for every point x in $\mathbb{H}_m$. The set of points u with $L(h) = \mathrm{dist}(u,h(u))$ is the geodesic line whose endpoints are the boundary points given by v_h^+ and v_h^-: By definition, this line is called the **axis** of h.

When h is elliptic or parabolic, the translation length vanishes (there is a point u in $\mathbb{H}_m$ with $L(h) = \mathrm{dist}(u,h(u))$ if h is elliptic, but no such point exists if h is parabolic).

Remark 4.3. If h is loxodromic and preserves a geodesic subspace W of $\mathbb{H}_m$ (i.e. the intersection of $\mathbb{H}_m$ with a vector subspace of $\mathcal{H}$), then W contains the axis of W (because the attracting fixed points v_h^+ and v_h^- are automatically contained in the boundary of W). In particular, the translation length of h on $\mathbb{H}_m$ is equal to the translation length of h on W.

4.1.3. The hyperbolic space $\mathbb{H}_\infty(X)$. Let us come back to the geometry of $\mathcal{Z}(X)$, where X is a projective surface. Fix an ample class $\mathbf{e}_0$ in $\mathrm{NS}(X) \subset \mathcal{Z}(X)$. Denote by $\mathcal{Z}(X,\mathbf{R})$ and $\mathrm{NS}(X,\mathbf{R})$ the tensor products $\mathcal{Z}(X) \otimes_\mathbf{Z} \mathbf{R}$ and $\mathrm{NS}(X) \otimes_\mathbf{Z} \mathbf{R}$. Elements of $\mathcal{Z}(X,\mathbf{R})$ are finite sums $u_X + \sum_i a_i \mathbf{e}_i$ where u_X is an element of $\mathrm{NS}(X,\mathbf{R})$, each $\mathbf{e}_i$ is the class of an exceptional divisor, and the coefficients a_i are real numbers. Allowing infinite sums $\sum_i a_i \mathbf{e}_i$ with $\sum_i a_i^2 < +\infty$, one gets a new space $\mathsf{Z}(X)$, on which the intersection form extends continuously [35, 24].

The set of vectors u in $\mathsf{Z}(X)$ such that $u \cdot u = 1$ is a hyperboloïd. The subset

$$\mathbb{H}_\infty(X) = \{u \in \mathsf{Z}(X) \mid \quad u \cdot u = 1 \quad \text{and} \quad u \cdot \mathbf{e}_0 > 0\} \tag{14}$$

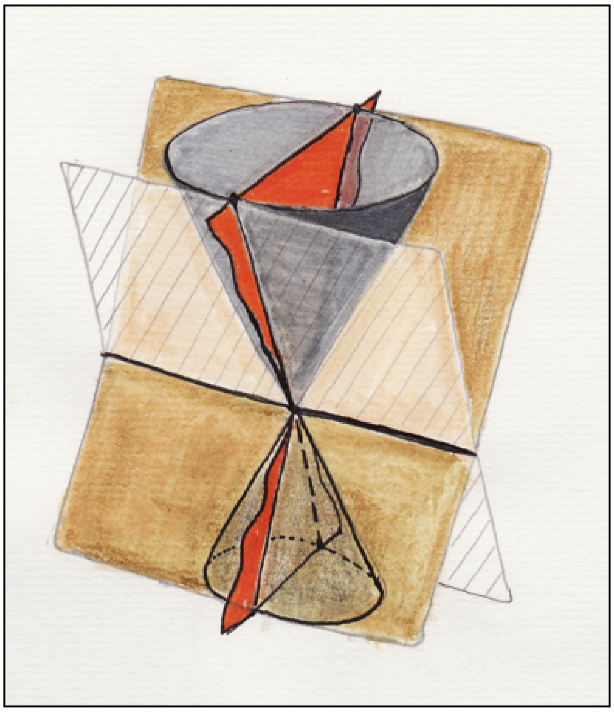

Figure 2. For a loxodromic isometry, there are two invariant isotropic lines, one corresponding to the eigenvalue $\lambda > 1$, the other to $1/\lambda$. The plane generated by these two lines cuts the hyperbolic space onto a geodesic: This geodesic is the axis of the isometry. The hyperplanes which are tangent to the isotropic cone along these eigenlines are invariant, and the action on their intersection is a rotation, preserving a negative definite quadratic form.

is the sheet of that hyperboloid containing ample classes of $\mathrm{NS}(X, \mathbf{R})$. With the distance $\mathrm{dist}(\cdot,\cdot)$ defined by

$$\cosh \mathrm{dist}(u, u') = u \cdot u', \tag{15}$$

$\mathbb{H}_\infty(X)$ is isometric to a hyperbolic space $\mathbb{H}_\infty$, as described in the previous paragraph (see [88, 11, 48]). Thus, starting with any projective surface X, one gets a natural hyperbolic space $\mathbb{H}_\infty(X) \simeq \mathbb{H}_\infty$.

We denote by $\partial\mathbb{H}_\infty(X)$ the boundary of $\mathbb{H}_\infty(X)$ (viewed as the set of lines in the isotropic cone of $\mathsf{Z}(X)$, or as a sphere in $\mathbb{P}(\mathsf{Z}(X))$). We denote by $\mathrm{Isom}(\mathsf{Z}(X))$ the group of isometries of $\mathsf{Z}(X)$ with respect to the intersection form, and by $\mathrm{Isom}(\mathbb{H}_\infty(X))$ the subgroup that preserves $\mathbb{H}_\infty(X)$.

4.1.4. Action of $\mathrm{Bir}(X)$ on $\mathcal{Z}(X)$ and $\mathbb{H}_\infty(X)$ (following Y. Manin, see [109]). Given $f \in \mathrm{Bir}(X)$, there is a birational morphism $\pi\colon X' \to X$, obtained by blowing up indeterminacy points of f, such that f lifts to a morphism $f'\colon X' \to X$ (see [91]). By pull back, the transformation f' determines an isometry $(f')^*$ from $\mathcal{Z}(X)$ to $\mathcal{Z}(X')$: Identifying $\mathcal{Z}(X)$ to $\mathcal{Z}(X')$ by π^*, we obtain an isometry f^* of $\mathcal{Z}(X)$. Since all points of X have been blown-up to define $\mathcal{Z}(X)$, birational transformations behave as regular automorphisms on $\mathcal{Z}(X)$, and one can show that the map $f \mapsto f_* = (f^{-1})^*$ is a homomorphism from $\mathrm{Bir}(X)$ to the group $\mathrm{Isom}(\mathcal{Z}(X))$; hence, after completion, $\mathrm{Bir}(X)$ acts on $\mathbb{H}_\infty(X)$ by isometries.

Theorem 4.4 (Manin, [109]). *Let X be a projective surface defined over an algebraically closed field $\mathbf{k}$. The homomorphism $f \mapsto f_*$ is an injective homomorphism from $\mathrm{Bir}(X)$ to the group of isometries of $\mathsf{Z}(X)$ with respect to its intersection form. It preserves $\mathbb{H}_\infty(X)$, acting faithfully by isometries on this hyperbolic space.*

If $\mathbf{k}$ is not algebraically closed, one embeds $\mathrm{Bir}(X_\mathbf{k})$ in $\mathrm{Bir}(X_{\overline{\mathbf{k}}})$ for some algebraic closure $\overline{\mathbf{k}}$ of $\mathbf{k}$, and the theorem applies to $\mathrm{Bir}(X_\mathbf{k})$. If $\mathbf{k}$ is countable one needs only countably many blow-ups to define $\mathcal{Z}(X_\mathbf{k})$; then $\mathbb{H}_\infty(X_\mathbf{k})$ is a hypersurface in a separable Hilbert space. A similar phenomenon occurs when one studies a countable subgroup Γ of $\mathrm{Bir}(X_\mathbf{k})$, because one only needs to blow-up the base points of the elements of Γ. On the other hand, to apply this construction for the study of $\mathrm{Cr}_2(\mathbf{C})$, one needs uncountably many blow-ups.

4.2. Types and degree growth. Since $\mathrm{Bir}(X)$ acts faithfully on $\mathbb{H}_\infty(X)$, there are three types of birational transformations: **Elliptic**, **parabolic**, and **loxodromic**, according to the type of the associated isometry of $\mathbb{H}_\infty(X)$. We now describe how each type can be characterized in algebro-geometric terms.

Let $\mathbf{h} \in \mathrm{NS}(X, \mathbf{R})$ be an ample class with self-intersection 1. Define the degree of f with respect to the polarization $\mathbf{h}$ by

$$\mathsf{deg}_\mathbf{h}(f) = f_*(\mathbf{h}) \cdot \mathbf{h} = \cosh(\mathrm{dist}(\mathbf{h}, f_*\mathbf{h})). \tag{16}$$

For instance, if f is an element of $\mathrm{Bir}(\mathbb{P}^2_\mathbf{k})$, and $\mathbf{h} = \mathbf{e}_0$ is the class of a line, then $\mathsf{deg}_\mathbf{h}(f)$ is the degree of f, as defined in §1.1. More precisely, if f has degree d, the image of a general line by f is a curve of degree d which goes through the base points q_i of f^{-1} with certain multiplicities a_i, and

$$f_*\mathbf{e}_0 = d\mathbf{e}_0 - \sum_i a_i \mathbf{e}_i$$

where $\mathbf{e}_i$ is the class corresponding to the exceptional divisor that one gets when blowing up the point q_i. Then, the intersection $f_*(\mathbf{e}_0) \cdot \mathbf{e}_0 = \mathsf{deg}_{\mathbf{e}_0}(f)$ is equal to d, because $\mathbf{e}_0 \cdot \mathbf{e}_i = 0$ for $i \neq 0$.

If the translation length $L(f_*)$ is positive, we know that the distance $\mathrm{dist}(f_*^n(x), x)$ grows like $nL(f_*)$ for every $x \in \mathbb{H}_\infty(X)$ (see Section 4.1.2). Since $\cosh(\mathrm{dist}(u, v)) = u \cdot v$, this property gives the following lemma.

Lemma 4.5. *The sequence $\mathsf{deg}_\mathbf{h}(f^n)^{1/n}$ converges towards a real number $\lambda(f) \geq 1$, called the* **dynamical degree** *of f; its logarithm $\log(\lambda(f))$ is the translation length $L(f_*)$ of the isometry f_*.*

Consequently, $\lambda(f)$ does not depend on the polarization and is invariant under conjugacy. In particular, f is loxodromic if and only if $\lambda(f) > 1$, if and only if the sequence $\mathsf{deg}_\mathbf{h}(f^n)$ grows exponentially fast.

Elliptic and parabolic transformations are also classified in terms of degree growth. Say that a sequence of real numbers $(d_n)_{n \geq 0}$ grows linearly (resp. quadratically) if $n/c \leq d_n \leq cn$ (resp. $n^2/c \leq d_n \leq cn^2$) for some $c > 0$.

Theorem 4.6 (Gizatullin, Cantat, Diller and Favre, see [82, 31, 32, 63]). *Let X be a projective surface, defined over an algebraically closed field* $\mathbf{k}$*, and* $\mathbf{h}$ *be a polarization of* X*. Let f be a birational transformation of* X*.*

- *f is elliptic if and only if the sequence* $\mathsf{deg}_{\mathbf{h}}(f^n)$ *is bounded. In this case, there exists a birational map* $\phi\colon Y \dashrightarrow X$ *and an integer* $k \geq 1$ *such that* $\phi^{-1}\circ f\circ\phi$ *is an automorphism of* Y *and* $\phi^{-1}\circ f^k\circ\phi$ *is in the connected component of the identity of the group* $\mathrm{Aut}(Y)$.
- *f is parabolic if and only if the sequence* $\mathsf{deg}_{\mathbf{h}}(f^n)$ *grows linearly or quadratically with n. If f is parabolic, there exists a birational map* $\psi\colon Y \dashrightarrow X$ *and a fibration* $\pi\colon Y \to B$ *onto a curve B such that* $\psi^{-1}\circ f\circ\psi$ *permutes the fibers of* π*. The fibration is rational if the growth is linear, and elliptic (or quasi-elliptic if char*$(\mathbf{k}) \in \{2,3\}$*) if the growth is quadratic.*
- *f is loxodromic if and only if* $\mathsf{deg}_{\mathbf{h}}(f^n)$ *grows exponentially fast with n: There is a constant* $b_{\mathbf{h}}(f) > 0$ *such that* $\mathsf{deg}_{\mathbf{h}}(f^n) = b_{\mathbf{h}}(f)\lambda(f)^n + O(1)$.

We refer to [19] for a more precise description of the degree growth in the parabolic case.

Remark 4.7. If f is parabolic, the push forward of the fibration $\pi\colon Y \to B$ by the conjugacy ψ is the unique f-invariant pencil of curves. If the characteristic of $\mathbf{k}$ is 0, this pencil is the unique f-invariant (singular) algebraic foliation on X [39].

Example 4.8. All transformations $(X,Y) \mapsto (X, Q(X)Y)$ with $Q \in \mathbf{k}(X)$ of degree ≥ 1 provide parabolic transformations of $\mathbb{P}^2_{\mathbf{k}}$ with linear degree growth.

Example 4.9. Assume $\mathbf{k} = \mathbf{C}$. Let ι be a square root of -1 (resp. a non-trivial cubic root of 1) and E be the elliptic curve $\mathbf{C}/\mathbf{Z}[\iota]$. The linear action of the group $\mathsf{GL}_2(\mathsf{Z}[\iota])$ on the complex plane $\mathbf{C}^2$ preserves the lattice $\mathbf{Z}[\iota]\times\mathbf{Z}[\iota]$; this leads to an action of $\mathsf{GL}_2(\mathsf{Z}[\iota])$ by regular automorphisms on the abelian surface $X = E\times E$. This action commutes to $m(x,y) = (\iota x, \iota y)$; this provides a homomorphism from $\mathsf{PGL}_2(\mathbf{Z}[\iota])$ to the group of automorphisms of X/m. Since X/m is a rational surface, one gets an embedding of $\mathsf{PGL}_2(\mathbf{Z}[\iota])$ into the Cremona group $\mathrm{Cr}_2(\mathbf{C})$.

Apply this construction to the linear transformation $(x,y) \mapsto (x+y, y)$ of $\mathbf{C}^2$: It determines an automorphism f of the abelian surface $X = E\times E$ (resp. a birational transformation $\overline{f}$ of X/m or $\mathbb{P}^2_{\mathbf{C}}$) such that $\mathsf{deg}_{\mathbf{h}}(f^n)$ grows quadratically. Similarly, starting with a linear transformation in $\mathsf{GL}_2(\mathbf{Z}[\iota])$ whose spectral radius is α, one gets a birational transformation of the plane whose dynamical degree is α^2. An example is given by the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Its spectral radius is the golden mean $(1+\sqrt{5})/2$. One obtains a birational transformation of the plane with dynamical degree $(3+\sqrt{5})/2$ (one easily checks that it is not conjugate to the monomial example of Section 1.2.1).

4.3. Analogy with the mapping class group of a closed, orientable surface.

4.3.1. The mapping class group. Let $g \geq 2$ be an integer, and $\mathrm{Mod}(g)$ be the mapping class group of the closed orientable surface Σ of genus g. Elements of $\mathrm{Mod}(g)$ are isotopy classes of orientation preserving homeomorphisms of Σ.

The three main examples of isotopy classes $\varphi \in \mathrm{Mod}(g)$ are represented by (1) finite order homeomorphisms, (2) Dehn (multi)-twists, and (3) pseudo-Anosov homeomorphisms. Nielsen-Thurston classification of elements $\varphi \in \mathrm{Mod}(g)$ tells us that an element which is not pseudo-Anosov has a positive iterate φ^n that preserves the homotopy class of an essential loop; one can then cut the surface along that loop to reduce the topological complexity of the pair (Σ, φ). In a finite number of steps, one ends up with a decomposition of every isotopy class φ into pieces of type (1), (2) and (3) (see [75, 37]).

The mapping class group acts isometrically on the complex of curves and on the Teichmüller space of Σ, and there is a nice analogy between those actions and the action of $\mathrm{Cr}_2(\mathbf{k})$ on $\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$.

4.3.2. Pseudo-Anosov, versus loxodromic. To a pseudo-Anosov mapping class, one associates a dilatation factor $\lambda(\varphi)$: Given any pair of non-trivial homotopy classes of simple closed curves c and c' on Σ, the intersection numbers $\varphi^n(c) \cdot c'$ grow like $c^{st}\lambda(\varphi)^n$ as n goes to $+\infty$ (here $c \cdot c'$ is the minimum number of intersection points of curves C and C' in the homotopy classes c and c'). A similar property is satisfied by every loxodromic element f of $\mathrm{Bir}(X)$: If $\mathbf{e}$ and $\mathbf{e}'$ are points on $\mathbb{H}_\infty(X)$ which are determined in $\mathrm{NS}(X)$ by

$$\mathbf{e} = C/\sqrt{C \cdot C}, \quad \mathbf{e}' = C'/\sqrt{C' \cdot C'},$$

for two curves C and C' with positive self-intersection, then $f_*^n(\mathbf{e}) \cdot \mathbf{e}'$ grows like $c^{st}\lambda(f)^n$.

Also, every pseudo-Anosov class φ is represented by a pseudo-Anosov homeomorphism $\Phi \colon \Sigma \to \Sigma$; such a homeomorphism preserves two singular foliations on Σ, one being uniformly contracted, the other uniformly dilated. Those foliations are geometric objects which, in Thurston compactification, correspond to fixed points of φ on the boundary of the Teichmüller space.

Similarly, given a loxodromic element f in $\mathrm{Cr}_2(\mathbf{C})$, the fixed points of f_* on the boundary of $\mathbb{H}_\infty(X)$ correspond to closed positive currents which are multiplied by $\lambda(f)^{\pm 1}$ under the action of f. Those currents are analogous to the invariant foliations of a pseudo-Anosov homeomorphism: They have laminar properties (a weak form of foliated structure). We refer to [6, 7, 35, 37, 62, 72, 76] for this analogy and for dynamical properties of loxodromic birational transformations.

4.3.3. Jonquières, Halphen, and Dehn twists. Recall from Remark 4.7 that a parabolic transformation f of a projective surface X preserves a unique pencil of curves on X; this pencil is birationally equivalent to a rational or a genus 1 fibration on some model X' of X. The type of the fiber is related to the degree growth of f: It is rational if the degree growth is linear, and has genus 1 if the growth is quadratic. These two types of parabolic transformations are respectively called **Jonquières twists**[6] and **Halphen twists**.

This is justified by the analogy with Dehn (multi-)twists $\varphi \in \mathrm{Mod}(g)$ and by the following two facts (they concern the case $X = \mathbb{P}^2_{\mathbf{k}}$, $f \in \mathrm{Cr}_2(\mathbf{k})$, and $\mathbf{k}$ algebraically closed):

[6] or "de Jonquières" twists

• If the degree-growth is linear, the invariant pencil of f can be transformed into a pencil of lines by an element of $\mathrm{Cr}_2(\mathbf{k})$; hence, after conjugacy, f becomes an element of the Jonquières group $\mathrm{Jonq}_2(\mathbf{k})$.

• If the degree-growth is quadratic, the f-invariant pencil can be transformed in a Halphen pencil [96, 67]. Halphen pencils are constructed as follows. Start with a smooth cubic curve $C \subset \mathbb{P}^2_{\mathbf{k}}$ and fix the group law on C with origin at an inflexion point. Choose 9 points on this curve whose sum s is a torsion point of order m for the group law on C. Then, the linear system of curves of degree $3m$ going through these 9 points with multiplicity m form a pencil of curves of genus 1. Blowing-up these 9 base points, one gets a rational surface with a Halphen fibration.

5. The Cremona group is thin

In this paragraph, we continue our description of the Cremona group $\mathrm{Cr}_2(\mathbf{k})$ as a group of isometries of an infinite dimensional group $\mathbb{H}_\infty$. One of the leitmotives is to show that this group of isometries is a thin subgroup of the group of all isometries.

5.1. Cremona isometries. Each element f of $\mathrm{Cr}_2(\mathbf{k})$ acts isometrically on $\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$.

(1) The isometry f_* preserves the "lattice" $\mathcal{Z}(\mathbb{P}^2_{\mathbf{k}})$ of $\mathsf{Z}(\mathbb{P}^2_{\mathbf{k}})$. For instance, the base point $\mathbf{e}_0$ (the class of a line in $\mathbb{P}^2_{\mathbf{k}}$) is mapped to a finite sum

$$f_*\mathbf{e}_0 = d\mathbf{e}_0 - \sum_i a_i\mathbf{e}_i$$

where each a_i is a positive integer, d is the degree of f, and the $\mathbf{e}_i$ are the classes of the exceptional divisors corresponding to the base points of f^{-1}.

More precisely, the linear system of all lines in $\mathbb{P}^2_{\mathbf{k}}$ is mapped by f to a linear system of curves of degree $d = \deg(f)$; this linear system is, by definition, the homaloidal net of f^{-1}. Its base points (including infinitely near base points), form a finite set of points q_i, with multiplicities a_i; the classes $\mathbf{e}_i$ in the previous formula are the classes $\mathbf{e}(q_i)$ of the blow-ups of the q_i. For example, the homaloidal net of the standard quadratic involution σ_2 is the net of conics through the three points $q_1 = [1:0:0]$, $q_2 = [0:1:0]$, $q_3 = [0:0:1]$. We have

$$(\sigma_2)_*\mathbf{e}_0 = 2\mathbf{e}_0 - \mathbf{e}(q_1) - \mathbf{e}(q_2) - \mathbf{e}(q_3).$$

Another invariant structure is given by the **canonical form**. Recall that the canonical class of $\mathbb{P}^2_{\mathbf{k}}$ blown up in m points $q_1, \ldots, q_m$ is equal to $-3\mathbf{e}_0 - \sum_j \mathbf{e}(q_j)$. Taking intersection products, one gets a linear form $\omega_\infty \colon \mathcal{Z}(\mathbb{P}^2_{\mathbf{k}}) \to \mathbf{Z}$, defined by

$$\omega_\infty \colon a_0\mathbf{e}_0 - \sum_j a_j\mathbf{e}_j \mapsto -3a_0 + \sum_j a_j.$$

This form does not extend to the completion $\mathsf{Z}(\mathbb{P}^2_{\mathbf{k}})$ (because there are ℓ^2 sequences which are not ℓ^1).

(2) The isometric action of $\mathrm{Cr}_2(\mathbf{k})$ on $\mathcal{Z}(\mathbb{P}^2_{\mathbf{k}})$ preserves the integral linear form ω_∞.

The following equalities, which we shall refer to as **Noether equalities**, follow from the fact that f_* is an isometry that preserves ω_∞: If $f_*\mathbf{e}_0 = d\mathbf{e}_0 - \sum_i a_i\mathbf{e}_i$, then

$$d^2 = 1 + \sum_i a_i^2 \tag{17}$$

$$3d - 3 = \sum_i a_i. \tag{18}$$

These relations impose interesting conditions on the isometries defined by birational transformations of the plane.

Lemma 5.1 (Noether inequality). *Let f be an element of $\mathrm{Cr}_2(\mathbf{k})$ of degree $d \geq 2$, and let $a_1, \ldots, a_k$ be the multiplicities of the base-points of f^{-1}.*

(3) *The following equality is satisfied.*

$$(d-1)(a_1+a_2+a_3-(d+1)) = (a_1-a_3)(d-1-a_1)+(a_2-a_3)(d-1-a_2) + \sum_{4\leq i\leq k} a_i(a_3-a_i)$$

(3') *For every pair of indices i, j with $1 \leq i < j \leq k$, we have $a_i + a_j \leq d$.*

(3") *Ordering the a_i in decreasing order, i.e. $a_1 \geq a_2 \geq a_3 \geq a_4 \geq \ldots$, we have*

$$a_1 + a_2 + a_3 \geq d + 1.$$

5.2. Noether Castelnuovo theorem. One way to state Noether-Castelnuovo theorem, is to say that $\mathrm{Cr}_2(\mathbf{k})$ is generated by the family of standard quadratic involutions, i.e. by the elements $g \circ \sigma_2 \circ g^{-1}$ with g in $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}}) = \mathsf{PGL}_3(\mathbf{k})$ (with $\mathbf{k}$ algebraically closed).

To understand the isometry $(\sigma_2)_*$, denote by q_1, q_2, and q_3 the base points of σ_2, and by X the surface which is obtained by blowing up these three points. On X, σ_2 lifts to an automorphism $\widehat{\sigma_2}$. The Néron-Severi group of X is the lattice of rank 4 generated by the classes $\mathbf{e}_0$, coming from the class of a line in $\mathbb{P}^2_{\mathbf{k}}$, and the classes $\mathbf{e}_i = \mathbf{e}(q_i)$ given by the three exceptional divisors. The action of $\widehat{\sigma_2}$ on $\mathrm{NS}(X)$ is given by

$$\begin{aligned}
(\widehat{\sigma_2})_*\mathbf{e}_0 &= 2\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 \\
(\widehat{\sigma_2})_*\mathbf{e}_1 &= \mathbf{e}_0 - \mathbf{e}_2 - \mathbf{e}_3 \\
(\widehat{\sigma_2})_*\mathbf{e}_2 &= \mathbf{e}_0 - \mathbf{e}_3 - \mathbf{e}_1 \\
(\widehat{\sigma_2})_*\mathbf{e}_3 &= \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2.
\end{aligned}$$

Thus, on $\mathrm{NS}(X)$, $(\widehat{\sigma_2})_*$ coincides with the reflexion with respect to the (-2)-class $u = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$:

$$(\widehat{\sigma_2})_*(x) = x + \langle x|u\rangle$$

for all x in $\mathrm{NS}(X)$. The class u is mapped to its opposite, and the set of fixed points is the hyperplane of vectors $x = \sum_i a_i\mathbf{e}_i$ with $a_0 = a_1 + a_2 + a_3$. Note that the class u is not effective, precisely because the three points q_1, q_2, and q_3 are not on a line.

Then, blow up all points of X (including infinitely near points) to construct a basis of $\mathsf{Z}(\mathbb{P}^2_{\mathbf{k}})$, namely

$$\mathsf{Z}(\mathbb{P}^2_{\mathbf{k}}) = \mathrm{NS}(X) \oplus \bigoplus_{p \in \mathcal{B}(X)} \mathbf{Z}\mathbf{e}(p)$$

where $\mathcal{B}(X)$ is the set of points that one blows up (see Example 4.2 and [68, 18]). The isometry $(\sigma_2)_*$ of $\mathcal{Z}(\mathbb{P}^2_{\mathbf{k}})$ acts on $\mathrm{NS}(X)$ as the reflexion $(\widehat{\sigma_2})_*$ and permutes each vector $\mathbf{e}(p)$ with $\mathbf{e}(\sigma_2(p))$. Thus, the fixed point set of $(\sigma_2)_*$ in $\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$ is quite small: It is defined by infinitely many equalities, namely $a_0 = a_1 + a_2 + a_3$, and $a_p = a_{\sigma_2(p)}$ for every $p \in \mathcal{B}(X)$.

A naive approach to the proof that standard quadratic involutions generate $\mathrm{Cr}_2(\mathbf{k})$ works as follows. Consider an element f in $\mathrm{Cr}_2(\mathbf{k})$, with $f_*(\mathbf{e}_0) = d\mathbf{e}_0 - \sum a_i \mathbf{e}(q_i)$. Assume that the multiplicities are decreasing, *i.e.* $a_i \geq a_{i+1}$, and apply Noether inequality to deduce $a_1 + a_2 + a_3 \geq d + 1$. Since $\mathbf{k}$ is algebraically closed, the base points q_i are defined over $\mathbf{k}$. Assume that the base points q_1, q_2 and q_3 are non-collinear points of $\mathbb{P}^2(\mathbf{k})$ and denote by σ a quadratic involutions with base points q_1, q_2 and q_3. Then $(\sigma \circ f)_*\mathbf{e}_0 = (2d - (a_1 + a_2 + a_3))\mathbf{e}_0 + \ldots$ and one sees that the degree $\mathbf{e}_0 \cdot (\sigma \circ f)_*\mathbf{e}_0 = 2d - (a_1 + a_2 + a_3)$ is strictly less than d. Thus, in a finite number of steps, one expect to reach a birational transformation of degree 1, i.e. an element of $\mathsf{PGL}_3(\mathbf{k})$. Of course, the difficulty arises from the fact that the dominating base points q_1, q_2 and q_3 may include infinitesimally near points.

5.3. Dynamical degrees, automorphisms, spectral gaps. Let us come back to the study of birational transformations of arbitrary projective surfaces X. If g is an automorphism of X, g already acts by isometry on $\mathrm{NS}(X, \mathbf{R})$ for the intersection form; thus, the dynamical degree $\lambda(g)$ is equal to the spectral radius of the linear transformation $g^*\colon \mathrm{NS}(X) \to \mathrm{NS}(X)$. This shows that $\lambda(g)$ is an algebraic number because g^* preserves the integral structure of $\mathrm{NS}(X)$.

Remark 5.2. As explained in the introduction of Chapter II, a projective surface with non-negative Kodaira dimension has a unique minimal model, on which every birational transformation is an automorphism. On such a surface, all dynamical degrees are algebraic integers, the degree of which is bounded from above by the Picard number of the minimal model. In fact, their degree is bounded by 24 because surfaces with positive Kodaira dimension have no automorphism with dynamical degree > 1 and minimal surfaces with vanishing Kodaira dimension have Picard number at most 24 (see [18]).

A birational transformation of a surface is **algebraically stable** if the action f_* of f on the Néron-Severi group satisfies $(f_*)^n = (f^n)_*$ for all $n \geq 1$. This property fails exactly when there is a curve E in the surface X such that f maps E to a point q (i.e. the strict transform is equal to q) and the forward orbit of q contains an indeterminacy point $q' = f^m(q)$ of f. If this occurs, one can blow up the orbit of q between q and q'; such a modification decreases the number of base points of f. Thus, in a finite number of steps, one reaches a model of X on which f becomes algebraically stable. The precise statement that one gets is the following theorem; it is proved in [63].

Theorem 5.3 (Diller-Favre, [63]). *Let* $\mathbf{k}$ *be an algebraically closed field. Let* X *be a projective surface and* f *be a birational transformation of* X*, both defined over* $\mathbf{k}$*. There exists a birational morphism* $\pi\colon Y \to X$ *such that* $f_Y := \pi^{-1} \circ f \circ \pi$ *is algebraically stable.*

For example, if $f = \sigma_2$ is the standard quadratic involution, one just needs to blow up its three indeterminacy points. If h is a Hénon automorphism of the affine plane, then h determines an algebraically stable birational transformation of $\mathbb{P}^2_{\mathbf{k}}$.

Once f is algebraically stable, the dynamical degree arises as an eigenvalue of f_* on the Néron-Severi group and, as such, is an algebraic integer.

A **Pisot number** is a real algebraic integer $\alpha > 1$, all of whose conjugates $\alpha' \neq \alpha$ have modulus < 1. A **Salem number** is a real algebraic integer $\beta > 1$ such that $1/\beta$ is a conjugate of β, all other conjugates have modulus 1, and there is at least one conjugate β' on the unit circle. With such a definition, quadratic units $\alpha > 1$ are Pisot numbers (and are not Salem numbers). The set of Pisot numbers is countable, closed, and contains accumulation points (the smallest one being the golden mean); the smallest Pisot number is the root $\lambda_P \simeq 1.3247$ of $t^3 = t + 1$. Salem numbers are not well understood yet; the smallest known Salem number is the Lehmer number $\lambda_L \simeq 1.1762$, a root of $t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1 = 0$, and the existence of Salem numbers between 1 and λ_L is an open problem.

The following result, contained in [18], is a manifestation of Hodge index theorem. Its proof depends deeply on [63, 111, 112, 113].

Theorem 5.4. *Let* X *be a projective surface, defined over an algebraically closed field* $\mathbf{k}$*. Let* f *be a birational transformation of* X *with dynamical degree* $\lambda(f) > 1$*. Then* $\lambda(f)$ *is either a Pisot number or a Salem number and*

(a) *if* $\lambda(f)$ *is a Salem number, then there exists a birational map* $\psi\colon Y \dashrightarrow X$ *which conjugates* f *to an automorphism of* Y*;*

(b) *if* f *is conjugate to an automorphism, as in* (a)*,* $\lambda(f)$ *is either a quadratic integer or a Salem number.*

Moreover, $\lambda(f) \geq \lambda_L$*, where* λ_L *is the Lehmer number and there are examples of birational transformations of the complex projective plane (resp. of some complex K3 surfaces) such that* $\lambda(f) = \lambda_L$*.*

Define the **dynamical spectrum** of the surface X by

$$\Lambda(X_{\mathbf{k}}) = \{\lambda(h) \mid h \in \mathrm{Bir}(X_{\mathbf{k}})\}$$

Theorem 5.4 implies that $\Lambda(X_{\mathbf{k}})$ is contained in the union of $\{1\}$, the set of Pisot numbers, and the set of Salem numbers. Moreover, there is a spectral gap: $\Lambda(X_{\mathbf{k}})$ does not intersect the open interval $(1, \lambda_L)$. This spectral gap corresponds to an important geometric property of the action of $\mathrm{Bir}(X)$ on the hyperbolic space $\mathbb{H}_\infty(X)$: If an element f of $\mathrm{Bir}(X)$ is loxodromic, its translation length is bounded from below by the uniform constant $\log(\lambda_L)$.

Remark 5.5. Consider a birational transformation g of the plane $\mathbb{P}^2_{\mathbf{k}}$ for which $\lambda(g)$ is a natural integer ≥ 2. The dynamical degree of the Hénon map $(X_1, X_2) \mapsto (X_2 + X_1^d, X_2)$

is equal to d. Then, g can not be regularized: There is no birational change of coordinates $X \dashrightarrow \mathbb{P}^2_{\mathbf{k}}$ which conjugates g to a regular automorphism of a projective surface X; this would contradict Assertion (b) in Theorem 5.4. We refer to [28, 44] for different arguments leading to birational transformations which are not regularizable.

5.4. Dynamical degrees, well ordered sets of algebraic numbers. Consider a loxodromic element of $\mathrm{Cr}_2(\mathbf{k})$; recall that the degree $\mathsf{deg}(f)$ can be seen as the degree of the homogeneous formulas defining $f\colon \mathbb{P}^2_{\mathbf{k}} \to \mathbb{P}^2_{\mathbf{k}}$ and as the intersection $f_*(\mathbf{e}_0)\cdot\mathbf{e}_0$.

The inequality $\lambda(f) \leq \mathsf{deg}(f)$ is always satisfied, because the sequence $\mathsf{deg}(f^n)$ is submultiplicative, and $\lambda(f)$ is the limit of $\mathsf{deg}(f^n)^{1/n}$. Moreover, $\lambda(f)$ is invariant under conjugacy: $\lambda(gfg^{-1}) = \lambda(f)$ for all $g \in \mathrm{Cr}_2(\mathbf{k})$. Thus, if one defines the **minimal degree in the conjugacy class** by

$$\mathsf{mcdeg}(f) = \min\{\mathsf{deg}(gfg^{-1}) \mid g \in \mathrm{Cr}_2(\mathbf{k})\}$$

one gets the inequality

$$\lambda(f) \leq \mathsf{mcdeg}(f)$$

for all $f \in \mathrm{Cr}_2(\mathbf{k})$.

Theorem 5.6 (see [18]). *Let $\mathbf{k}$ be an algebraically closed field. Let f be a birational transformation of the plane $\mathbb{P}^2_{\mathbf{k}}$. If $\lambda(f) \geq 10^6$ then* $\mathsf{mcdeg}(f) \leq 4700\lambda(f)^5$. *If $\lambda(f) > 1$, then* $\mathsf{mcdeg}(f) \leq \cosh(110+345\log(\lambda(f)))$.

In geometric terms, if f is a loxodromic element of $\mathrm{Cr}_2(\mathbf{k})$, one can conjugate f to f' in $\mathrm{Cr}_2(\mathbf{k})$ to get

$$L(f') \leq \mathrm{dist}(\mathbf{e}_0, f'_*\mathbf{e}_0) \leq 110 + 345\, L(f') \tag{19}$$

(where $L(f')$, the translation length of f'_*, is equal to $L(f)$ and $\log(\lambda(f))$). Let us explain the meaning of this statement. Denote by $\mathsf{Ax}(f)$ the axis of f_*: By definition, $\mathsf{Ax}(f) \subset \mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$ is the geodesic line whose endpoints are the two fixed points of f_* on the boundary $\partial\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$; it coincides with the intersection of $\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$ with the plane generated by the isotropic lines which are invariant under the action of f_* (one is multiplied by $\lambda(f)$, the other by $1/\lambda(f)$). Denote by $\mathbf{e}_f$ the projection of the base point $\mathbf{e}_0$ on $\mathsf{Ax}(f)$; the geodesic segment $[\mathbf{e}_0, \mathbf{e}_f]$ is orthogonal to $\mathsf{Ax}(f)$, and its length $\delta(f)$ is the distance from $\mathbf{e}_0$ to $\mathsf{Ax}(f)$. The isometry f_* maps $\mathbf{e}_f$ to a point of $\mathsf{Ax}(f)$ such that $\mathrm{dist}(\mathbf{e}_f, f_*\mathbf{e}_f) = L(f) = \log(\lambda(f))$. The geodesic segment $[\mathbf{e}_0, \mathbf{e}_f]$ is mapped to a geodesic segment $[f_*\mathbf{e}_0, f_*\mathbf{e}_f]$: It is orthogonal to $\mathsf{Ax}(f_*)$, and its length is equal to $\delta(f)$. Thus,

$$\mathrm{dist}(\mathbf{e}_0, f_*\mathbf{e}_0) \leq 2\delta(f) + L(f).$$

To get the Inequality (19), we just need to prove that f can be conjugate to an element f' with an explicit bound on $\delta(f')$ (that depends on $L(f)$). Noether-Castelnuovo theorem leads to a similar problem: Starting with an element f in $\mathrm{Cr}_2(\mathbf{k})$, one tries to compose it with a quadratic involution to decrease $\mathrm{dist}(f_*\mathbf{e}_0, \mathbf{e}_0)$, and then to repeat this process and decrease this distance all the way down to 0 (see Section 5.2). Here, one decreases the distance from $\mathbf{e}_0$ to the axis of f by changing f into a conjugate element of $\mathrm{Cr}_2(\mathbf{k})$.

Let us describe a consequence of Theorem 5.6. Given a sequence (f_n) of loxodromic birational transformations of the plane, one gets a sequence of algebraic numbers $\lambda(f_n)$.

Assume that this sequence is strictly decreasing, hence bounded. The sequence (f_n) can be replaced by a sequence (f'_n) such that the degree of f'_n is uniformly bounded and f_n is conjugate to f'_n for all n. Let d be a degree such that infinitely many of the f'_n have degree d: One gets a sequence (f'_n) in the algebraic set $\mathrm{Cr}_2(\mathbf{k};d)$. It has been proved by Xie (see [138]) that the function

$$g \mapsto \lambda(g)$$

is lower semi-continuous with respect to the Zariski topology on $\mathrm{Cr}_2(\mathbf{k};d)$ (resp. along any algebraic family g_t of birational transformations). The sequence $\lambda(f'_n)$ decreases with n, and the sets $\{g \in \mathrm{Cr}_2(\mathbf{k};d) | \ \lambda(g) \leq \lambda(f'_n)\}$ are Zariski closed; hence, the noetherian property implies that the sequence $\lambda(f'_n) = \lambda(f_n)$ is finite. This argument is detailed in [18], and leads to the following result.

Theorem 5.7. *Let* $\mathbf{k}$ *be an algebraically closed field. The set of dynamical degrees of all birational transformations of* $\mathbb{P}^2_{\mathbf{k}}$ *is a well ordered set: Every strictly decreasing sequence of dynamical degrees is finite. If* $\mathbf{k}$ *is uncountable, this set is closed.*

In particular, given any dynamical degree λ, there is a small interval $]\lambda, \lambda+\varepsilon]$ that does not contain any dynamical degree. A similar result holds if one replaces $\mathbb{P}^2_{\mathbf{k}}$ by all projective surfaces, all of them taken together, over arbitrary fields, because dynamical degrees are algebraic integers of degree at most 24 on non-rational surfaces (see Remark 5.2).

6. Finitely generated subgroups

According to Sections 4.1 and 4.2, $\mathrm{Cr}_2(\mathbf{k})$ acts by isometries on an infinite dimensional hyperbolic space, and there is a powerful dictionary between the classification of isometries and the classification of birational maps in terms of degree growth and invariant fibrations. In this section, we explain how this dictionary can be used to describe the algebraic structure of $\mathrm{Cr}_2(\mathbf{k})$ and its subgroups.

6.1. Tits Alternative. A group G **satisfies Tits alternative** if the following property holds for every finitely generated subgroup Γ of G: Either Γ contains a finite index solvable subgroup or Γ contains a free non-abelian subgroup (i.e. a copy of the free group $\mathbb{F}_r$, with $r \geq 2$). Tits alternative holds for the linear groups $\mathsf{GL}_n(\mathbf{k})$ (see [134]), but not for the group of $\mathcal{C}^\infty$-diffeomorphisms of the circle $\mathbb{S}^1$ (see [26], [81]). If G satisfies Tits alternative, it does not contain groups with intermediate growth, because solvable groups have either polynomial or exponential growth.

The main technique to prove that a group contains a non-abelian free group is the ping-pong lemma. Let g_1 and g_2 be two bijections of a set S. Assume that S contains two non-empty disjoint subsets S_1 and S_2 such that $g_1^m(S_2) \subset S_1$ and $g_2^m(S_1) \subset S_2$ for all $m \in \mathbf{Z} \setminus \{0\}$. Then, according to the ping-pong lemma, the group of transformations of S generated by g_1 and g_2 is a free group on two generators [53]. The proof is as follows. If $w = w(a,b)$ is a reduced word that represents a non-trivial element in the free group $\mathbb{F}_2 = \langle a,b \rangle$, one needs to prove that $w(g_1,g_2)$ is a non-trivial transformation of S; for this, one conjugates w with a power of g_1 to assume that $w(g_1,g_2)$ starts and ends with a power of g_1; writing

$$w(g_1,g_2)=g_1^{l_n}g_2^{m_n}\ldots g_2^{m_1}g_1^{l_0},$$

one checks that $g_1^{l_0}$ maps S_2 into S_1, then $g_2^{m_1}g_1^{l_0}$ maps S_2 into S_2, ..., and w maps S_2 into S_1; this proves that $w(g_1,g_2)$ is non-trivial because S_2 is disjoint from S_1.

Now, consider a group Γ that acts on a hyperbolic space $\mathbb{H}_\infty$ and contains two loxodromic isometries h_1 and h_2 whose fixed points in $\partial\mathbb{H}_\infty$ form two disjoint pairs. Take disjoint neighborhoods $S_i\subset\overline{\mathbb{H}_\infty}$ of the fixed point sets of h_i, $i=1,2$. Then, the ping-pong lemma applies to sufficiently high powers $g_1=h_1^n$ and $g_2=h_2^n$, and produce a free subgroup of Γ.

This strategy can be used for $\mathrm{Bir}(X)$, acting by isometries on $\mathbb{H}_\infty(X)$. The difficulty resides in the study of subgroups that do not contain any ping-pong pair of loxodromic isometries; Theorem 4.6 comes in help to deal with this situation, and leads to the following result.

Theorem 6.1 ([35]). *If X is a projective surface over a field $\mathbf{k}$, the group $\mathrm{Bir}(X)$ satisfies Tits alternative.*

Moreover, solvable subgroups of $\mathrm{Bir}(X)$ which are generated by finitely many elements are well understood: Up to finite index, such a group preserves an algebraic foliation (defined over the algebraic closure of $\mathbf{k}$), or is abelian (see [36] and [61]). This is analogous to the fact that every solvable subgroup in $\mathsf{GL}_n(\mathbf{k})$ contains a finite index subgroup that preserves a full flag in $\mathbf{k}^n$ (if $\mathbf{k}$ is algebraically closed).

If M is a projective variety (resp. a compact Kähler manifold), its group of automorphisms satisfies also the Tits alternative [35, 64].([7])

Question 6.2. Does $\mathrm{Cr}_n(\mathbf{k})$ satisfy Tits alternative for all $n\geq 3$? Does Tits alternative holds for $\mathrm{Bir}(M)$, for all projective varieties M ?

Would the answer be yes, one would obtain a proof of Tits alternative for subgroups of $\mathrm{Bir}(M)$: This includes linear groups, mapping class groups of surfaces, and $\mathrm{Out}(\mathbb{F}_g)$ for all $g\geq 1$ (see §1.3.1; see [12] for Tits alternative in this context). The first open case for Question 6.2 concerns the group of polynomial automorphisms of the affine space $\mathbb{A}^3_{\mathbf{k}}$.

6.2. Rank one phenomena. As explained in §2.3, the Cremona group $\mathrm{Cr}_2(\mathbf{k})$ behaves like an algebraic group of rank 2, with a maximal torus given by the group of diagonal matrices in $\mathsf{PGL}_3(\mathbf{k})$. On the other hand, typical elements of degree $d\geq 2$ in $\mathrm{Cr}_2(\mathbf{C})$ are loxodromic (not elliptic) and, as such, cannot be conjugate to elements of this maximal torus. This suggests that $\mathrm{Cr}_2(\mathbf{k})$ has rank 1 from the point of view of its typical elements. The following statement provides a strong version of this principle.

Theorem 6.3 ([35, 18]). *Let $\mathbf{k}$ be a field. Let X be a projective surface over $\mathbf{k}$ and f be a loxodromic element of $\mathrm{Bir}(X)$. Then, the infinite cyclic subgroup of $\mathrm{Bir}(X)$ generated by f has finite index in the centralizer $\{g\in\mathrm{Bir}(X)\mid g\circ f=f\circ g\}$.*

[7]The proof is simple: The action of $\mathrm{Aut}(M)$ on the cohomology of M is a linear representation, and Tits theorem can be applied to its image; its kernel is a Lie group with finitely many components, and Tits theorem can again be applied to it. There is a mistake in the proof of Lemma 6.1 of [35]; this has been corrected in [64] and [36].

Sketch of proof for $X = \mathbb{P}^2_{\mathbf{k}}$ (see [35, 18]). If g commutes to f, the isometry g_* of $\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$ preserves the axis of f_* and its two endpoints. Consider the homomorphism which maps the centralizer of f to the group of isometries of $\mathsf{Ax}(f)$; view it as a homomorphism into the group of translations $\mathbf{R}$ of the line. Since the translation lengths are bounded from below by $\log(\lambda_L)$ and every discrete subgroup of $\mathbf{R}$ is trivial or cyclic, the image of this homomorphism is a cyclic group; its kernel K is made of elliptic elements h fixing all points of $\mathsf{Ax}(f)$ and commuting to f. Let $\mathbf{e}_f$ be the projection of $\mathbf{e}_0$ on the axis of f; then $\mathrm{dist}(h_*\mathbf{e}_0, \mathbf{e}_0) \leq 2\,\mathrm{dist}(\mathbf{e}_0, \mathbf{e}_f)$ because K fixes $\mathbf{e}_f$. Thus, the group K is a group of birational transformations of bounded degree.

Section 2.2 shows that one can conjugate K to a group of automorphisms of a rational surface Y, and that $\mathrm{Aut}(Y)^0 \cap K$ becomes a finite index subgroup of K. The Zariski closure of this group in $\mathrm{Aut}(Y)^0$ is a linear algebraic group G that commutes to f (where f is viewed as a birational transformation of Y); if this group is infinite, it contains a one-dimensional abelian group that commutes to f and whose orbits form a pencil of curves in Y: This contradicts the fact that f does not preserve any pencil of curves (such a pencil would give a fixed point of f_* in $\mathcal{Z}(\mathbb{P}^2_{\mathbf{k}})$ with non-negative self intersection, contradicting the loxodromic behaviour of f_*). $\square$

Another rank one phenomenum comes from the rigidity of rank 2 subgroups of $\mathrm{Cr}_2(\mathbf{k})$. Let G be a real, almost simple, linear algebraic group and Γ be a lattice in G, *i.e.* a discrete subgroup such that G/Γ has finite Haar volume. When the $\mathbf{R}$-rank of G is at least 2, Γ inherits its main algebraic properties from G (see [110] and Section 8.1 below). For instance, Γ has Kazhdan property (T), according to which every representation of Γ by unitary motions on a Hilbert space has a global fixed point.

Theorem 6.4 (Déserti, Cantat, [58, 35]). *Let* $\mathbf{k}$ *be an algebraically closed field and* X *be a projective surface over* $\mathbf{k}$. *Let* Γ *be a countable group with Kazhdan property (T). If* $\rho\colon \Gamma \to \mathrm{Bir}(X)$ *is a homomorphism with infinite image, then* ρ *is conjugate to a homomorphism into* $\mathsf{PGL}_3(\mathbf{k})$ *by a birational map* $\psi\colon X \dashrightarrow \mathbb{P}^2_{\mathbf{k}}$.

Sketch of proof. The first step is based on a fixed point property: If a group Γ with Kazhdan property (T) acts by isometries on a hyperbolic space $\mathbb{H}_\infty$, then Γ has a fixed point and, as a consequence, all its orbits have bounded diameter (see [54]). Apply this to the action of $\mathrm{Bir}(X)$ on $\mathbb{H}_\infty(X)$ to deduce that a subgroup of $\mathrm{Bir}(X)$ with Kazhdan property (T) has bounded degree (with respect to any given polarization of X). Consequently, there is a birational map $\pi\colon Y \dashrightarrow X$ that conjugates Γ to a subgroup Γ_Y of $\mathrm{Aut}(Y)$ such that $\mathrm{Aut}(Y)^0 \cap \Gamma_Y$ has finite index in Γ_Y. The last step is based on the classification of algebraic groups of transformations of surfaces, and the fact that every subgroup of $\mathsf{SL}_2(\mathbf{k})$ with Kazhdan property (T) is finite; this leads to the following statement, which concludes the proof: If $\mathrm{Aut}(Y)^0$ contains an infinite group with Kazhdan property (T), the surface Y must be isomorphic to the projective plane $\mathbb{P}^2_{\mathbf{k}}$ (and then Γ_Y becomes a subgroup of $\mathsf{PGL}_3(\mathbf{k})$). $\square$

In [58, 59, 60], Déserti draws several algebraic consequences of this result; for instance, she can list all abstract automorphisms of $\mathrm{Cr}_2(\mathbf{C})$

Corollary 6.5. *The group of automorphisms of* $\mathrm{Cr}_2(\mathbf{C})$ *(as an abstract group) is the semi-direct product of* $\mathrm{Cr}_2(\mathbf{C})$ *(acting by conjugacy), and the group* $\mathrm{Aut}(\mathbf{C};+,\cdot)$ *of automorphisms of the field* $\mathbf{C}$ *(acting on the coefficients of the polynomial formulas defining the elements of* $\mathrm{Cr}_2(\mathbf{C})$*).*

There are now several proofs of this result. It would be interesting to decide whether this statement holds for all algebraically closed fields $\mathbf{k}$ (in place of $\mathbf{C}$). Since all proofs depend on Noether-Castelnuovo theorem; they do not extend to higher dimension (see [38], [100, 133] for partial results).

7. Small cancellation and normal subgroups

Small cancellation theory is a technique which, starting with a presentation of a group by generators and relations, can be used to prove that the group is large. Assume that the group G is given by a finite symmetric set of generators g_i and a finite set of relations R_i, each of them being a word in the g_i. Enlarge the set of relators in order to satisfy the following property: If R is one of the relators and R ends by the letter g_i, then $g_iRg_i^{-1}$ is also an element of our finite set of relators. Under this assumption, a typical small cancellation property assumes that two relators cannot coincide (as words in the g_i) on a piece that occupies at least $1/6$ of their length; under such an assumption, the group G is large (it contains a non abelian free group). In particular, in the free group generated by the letters g_i, the normal subgroup generated by the R_i is rather small. Thus, small cancellation theory can also be seen as a mean to show that a normal subgroup is a proper subgroup.

The first application of this technique to groups of algebraic transformations is due to Danilov (see [52]). He considered the group $\mathrm{Aut}_1(\mathbb{A}^2_{\mathbf{k}})$ of polynomial automorphisms of the affine plane with jacobian determinant 1. This group is the amalgamated product of the group of special affine transformations of the plane with the group of elementary auromorphisms

$$(X_1,X_2)\mapsto(aX_1,a^{-1}X_2+p(X_1))$$

with $a\in\mathbf{k}^*$ and $p(t)\in\mathbf{k}[t]$; the amalgamation is along their intersection. As such, $\mathrm{Aut}_1(\mathbb{A}^2_{\mathbf{k}})$ acts on a tree by automorphisms (see [129]), and a version of small cancellation theory can be applied to construct many normal subgroups in $\mathrm{Aut}_1(\mathbb{A}^2_{\mathbf{k}})$. Thus, $\mathrm{Aut}_1(\mathbb{A}^2_{\mathbf{k}})$ is not simple.

Since then, small cancellation theory has made huge progresses, with more geometric, less combinatorial versions. In particular, the work of Gromov, Olshanskii and Delzant on small cancellation and hyperbolic groups led to techniques that can now be applied to the Cremona group $\mathrm{Cr}_2(\mathbf{k})$. We refer to [90] and [50] for recent geometric viewpoints on small cancellation.

7.1. Rigidity, tightness, axes. Let G be a group of isometries of $\mathbb{H}_\infty$. Let L be a geodesic line in $\mathbb{H}_\infty$. The line L is **rigid** under the action of G if every element $h\in G$ that does not move L too much fixes L. To give a precise definition, one needs to measure the deplacement of L under the action of an isometry h. Say that two geodesic lines L and L' are (ε,ℓ)-close if the diameter of the set

$$\{x \in L \mid \mathrm{dist}(x, L') \leq \varepsilon\}$$

is larger than ℓ. The precise notion of rigidity is: L is (ε,ℓ)-**rigid** if, for every $h \in G$, $h(L) = L$ if and only if $h(L)$ is (ε,ℓ)-close to L; L is rigid if it is (ε,ℓ)-rigid for some pair of positive numbers (ε,ℓ) (this pair depends on L and G). In other words, if a geodesic line L is rigid for the action of the group G, the orbit $G(L)$ forms a discrete set in the space of geodesic lines.

Fix a loxodromic element $g \in G$. Consider the stabilizer of its axis:

$$Stab(\mathsf{Ax}(g)) = \{h \in G \mid h(\mathsf{Ax}(g)) = \mathsf{Ax}(g)\}.$$

Say that g is **tight** if its axis is rigid and every element h of $Stab(\mathsf{Ax}(g))$ satisfies

$$h \circ g \circ h^{-1} = g \quad \text{or} \quad g^{-1}.$$

An element of the Cremona group $\mathrm{Cr}_2(\mathbf{k})$ is **tight** (resp. has a **rigid axis**) if if it is tight (resp. its axis is rigid) with respect to the action of $\mathrm{Cr}_2(\mathbf{k})$ by isometries on $\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$.

These notions are directly related to the study of the stabilizer of $\mathsf{Ax}(g)$, and the following examples show that this stabilizer may be large.

Example 7.1. Consider the group of monomial transformations in two variables; this group is isomorphic to $\mathsf{GL}_2(\mathbf{Z})$. To a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $\det(M) = \pm 1$ corresponds a monomial transformation f_M: In affine coordinates,

$$f_M(x,y) = (x^a y^b, x^c y^d).$$

The dynamical degree $\lambda(f_M)$ is equal to $\max\{|\alpha|,|\beta|\}$, where α and β are the eigenvalues of M. Thus, $\lambda(f_M) > 1$ if and only if $ad - bc = 1$ and $|a+d| > 2$ or $ad - bc = -1$ and $a + d \neq 0$. Assuming that f_M is loxodromic, we shall prove that f_M is not tight.

The monomial group $\mathsf{GL}_2(\mathbf{Z})$ normalizes the group of diagonal transformations: If $t(x,y) = (ux, vy)$ then

$$f_M \circ t \circ f_M^{-1}(x,y) = (u^a v^b x, u^c v^d y) = t'(x,y) \tag{20}$$

where t' is obtained from t by the monomial action of $\mathsf{GL}_2(\mathbf{Z})$ on $\mathbb{G}_m \times \mathbb{G}_m$.

The indeterminacy points of monomial transformations are contained in the vertices $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ of the triangle whose edges are the coordinate axis. Blow-up these three points to get a new surface X_1, and consider the total transform of the triangle: One gets a hexagon of rational curves in X_1. The group of monomial transformations lifts to a group of birational transformations of X_1 with indeterminacy points located on the 6 vertices of this hexagon. The group of diagonal transformations lifts to a subgroup of $\mathrm{Aut}(X_1)^0$. One can iterate this process, blowing-up the vertices of the hexagon, etc.

The limit of the Néron-Severi groups along this sequence of surfaces $X_{n+1} \to X_n$ gives a subspace of $\mathcal{Z}(\mathbb{P}^2_{\mathbf{k}})$ which is invariant under the action of the monomial group $\mathsf{GL}_2(\mathbf{Z})$ and is fixed pointwise by the diagonal group $\mathbb{G}_m(\mathbf{k}) \times \mathbb{G}_m(\mathbf{k})$. Intersect this space with $\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$ and denote by $\mathbb{H}_\infty(\text{toric})$ its metric completion: One gets a totally geodesic, infinite dimensional subspace of $\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$. The axis of every loxodromic element of $\mathsf{GL}_2(\mathbf{Z})$ is contained in $\mathbb{H}_\infty(\text{toric})$ and is fixed pointwise under the action of $\mathbb{G}_m(\mathbf{k}) \times \mathbb{G}_m(\mathbf{k})$.

On the other hand, the Equation (20) implies that $t \circ f_M \circ t^{-1} = t'' \circ f_M$ where $t''(x,y) = (u^{1-a}v^{-b}x, u^{-c}v^{1-d}y)$. Thus, f_M is not tight (as soon as $\mathbf{k}^*$ contains elements v with $v^b \neq 1$).

Example 7.2 (see [105, 132]). A similar example works for the additive group in place of the multiplicative group when the characteristic p of the field $\mathbf{k}$ is positive. For instance, the Hénon mapping $h(x,y) = (x^p - y, x)$ conjugates the translation $s(x,y) = (x+u, y+v)$ to

$$h \circ s \circ h^{-1} = (x + u^p - v, x + u).$$

The dynamical degree of h is equal to p, and h normalizes the additive group $\mathbb{G}_a(\mathbf{k}) \times \mathbb{G}_a(\mathbf{k})$ (acting by translations).

The normalizer of the additive group in $\mathrm{Cr}_2(\mathbf{k})$ coincides with the subgroup of elements $f \in \mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ which are given in affine coordinates by formulas of type

$$f(x,y) = (a(x) + b(y), c(x) + d(y))$$

with $a(t)$, $b(t)$, $c(t)$, $d(t)$ polynomial functions of type $\sum_i q_i t^{p^i}$. Another way to state the same result is as follows. Denote by A the ring of linearized polynomials in one variable, *i.e.* polynomials in the Frobenius endomorphism $z \mapsto z^p$ of $\mathbf{k}$. This is a non-commutative ring. Then, every 2 by 2 matrix with coefficients in A which is invertible over A determines an algebraic automorphism of $\mathbb{G}_a(\mathbf{k}) \times \mathbb{G}_a(\mathbf{k})$, and every algebraic automorphism of $\mathbb{G}_a(\mathbf{k}) \times \mathbb{G}_a(\mathbf{k})$ is of this type. Thus, $\mathsf{GL}_2(A)$ plays the same role as $\mathsf{GL}_2(\mathbf{Z})$ in the previous example.

Base points of elements of $\mathsf{GL}_2(A)$ are above the line at infinity of the affine plane and are all fixed by $\mathbb{G}_a(\mathbf{k}) \times \mathbb{G}_a(\mathbf{k})$. Thus, again, the group $\mathbb{G}_a(\mathbf{k}) \times \mathbb{G}_a(\mathbf{k})$ acts trivially on a hyperbolic subspace of $\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$ that contains the axis of every loxodromic element of $\mathsf{GL}_2(A)$.

Theorem 7.3 ([40, 105, 132]). *Let $\mathbf{k}$ be an algebraically closed field, and g be a loxodromic element of* $\mathrm{Cr}_2(\mathbf{k})$.

(1) *The axis of g is rigid.*

(2) *The cyclic subgroup $g^{\mathbf{Z}}$ has finite index in the stabilizer of* $\mathsf{Ax}(g)$ *in* $\mathrm{Cr}_2(\mathbf{k})$, *if and only if there exists a non-trivial iterate g^n of g which is tight.*

(3) *If the index of $g^{\mathbf{Z}}$ in $Stab(\mathsf{Ax}(g))$ is infinite, one of the following possibilities occurs*

- *g is conjugate to a monomial transformation ;*
- *g is conjugate to a polynomial automorphism of the affine plane $\mathbb{A}^2_{\mathbf{k}}$ that normalizes the group of translations $(x,y) \mapsto (x+u, y+v)$, $(u,v) \in \mathbf{k}^2$ (this case does not occur if $char(\mathbf{k}) = 0$).*

Remark 7.4. Let N be a subgroup of $\mathrm{Cr}_2(\mathbf{k})$ that contains at least one loxodromic element. Assume that there exists a short exact sequence $1 \to S \to N \to Q \to 1$ where S is infinite and contains only elliptic elements. Then N is conjugate to a subgroup of the group $\mathsf{GL}_2(\mathbf{Z}) \ltimes \mathbb{G}_m(\mathbf{k})^2$ or $\mathsf{GL}_2(A) \ltimes \mathbb{G}_a(\mathbf{k})^2$, as in the previous examples. This statement is equivalent to Property (3). (see the appendix of [56] and [132])

Remark 7.5. Tightness is equivalent to another property which appeared in the study of the mapping class group $\mathrm{Mod}(g)$ of a closed surface (see [50], and references therein). Consider a group G acting by isometries on $\mathbb{H}_\infty$ and a loxodromic element g in G. One says that g is "wpd" (for "weakly properly discontinuous") if $\forall D \geq 0$, $\forall x \in \mathbb{H}_\infty$, there exists a positive integer N such that the set

$$S(D,x;N) = \{h \in G \,|\, \mathrm{dist}(h(x),x) \leq D,\ \mathrm{dist}(h(g^N(x)), g^N(x)) \leq D\}$$

is finite. To test this property, one can fix the starting point x; for instance, one can fix x on the axis of g.

When one studies the action of the Cremona group $\mathrm{Cr}_2(\mathbf{k})$ on $\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$, the axis of every loxodromic element is rigid and the stabilizer of the axis $\mathsf{Ax}(g)$ is virtually cyclic if and only if some positive iterate of g is tight (see Theorem 7.3). It follows that for N large, the set $S(D,x;N)$ is contained in the stabilizer of the axis $\mathsf{Ax}(g)$, and *g is wpd if and only if some positive iterate g^m of g is tight.* Thus, tightness (for g^m, for some $m \neq 0$) can be replaced by the wpd property when one studies the Cremona group in 2 variables.

7.2. Normal subgroups. Let us pursue the comparison between groups of birational transformations and groups of diffeomorphisms. If M is a connected compact manifold and $\mathrm{Diff}^\infty_0(M)$ denotes the group of infinitely differentiable diffeomorphisms of M which are isotopic to the identity, then $\mathrm{Diff}^\infty_0(M)$ is a simple group: It does not contain any normal subgroup except $\{\mathrm{Id}_M\}$ and the group $\mathrm{Diff}^\infty_0(M)$ itself (see [1]). One can show that $\mathrm{Cr}_2(\mathbf{C})$ is "connected" (see [15]); hence, there is no need to rule out connected components as for diffeomorphisms. Enriques conjectured in 1894 that $\mathrm{Cr}_2(\mathbf{C})$ is a simple group (see [74]), and this is indeed true from the point of view of its algebraic subgroups (see § 2.5 and [15]). On the other hand, as an abstract group, $\mathrm{Cr}_2(\mathbf{k})$ is far from being simple:

Theorem 7.6 (Cantat and Lamy, [40], Shepherd-Barron [132], Lonjou [105]). *For every field $\mathbf{k}$, the Cremona group $\mathrm{Cr}_2(\mathbf{k})$ is not simple.*

The proof relies on ideas coming from small cancellation theory and the geometry of hyperbolic groups in the sense of Gromov, as in [55]; the idea is that, starting with a tight element g in $\mathrm{Cr}_2(\mathbf{k})$, the relations generated by the conjugates of a large iterate g^n of g satisfy a small cancellation property, so that the normal subgroup generated by g^n is a proper subgroup of $\mathrm{Cr}_2(\mathbf{k})$. We refer to [50] for a recent survey on this topic. Applied to the action of the Cremona group on the hyperbolic space $\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$, the precise result that one obtains is the following.

Theorem 7.7 ([40, 51]). *Let g be a loxodromic element of $\mathrm{Cr}_2(\mathbf{k})$. If g is a tight element of $\mathrm{Cr}_2(\mathbf{k})$ and n is large enough, the normal subgroup of $\mathrm{Cr}_2(\mathbf{k})$ generated by the n-th iterate g^n*

- *does not contain any element $h \neq id$ whose translation length is less than $L(g)$*
- *is a free group.*

As a consequence, to prove that $\mathrm{Cr}_2(\mathbf{k})$ is not simple one needs to construct a tight element in $\mathrm{Cr}_2(\mathbf{k})$ (recall that the definition of tightness depends on the group, so that being tight in $\mathrm{Cr}_2(\mathbf{k})$ is not the same as being tight in $\mathrm{Cr}_2(\mathbf{k}')$ if $\mathbf{k}'$ is an extension of $\mathbf{k}$). According to Theorem 7.3, one just needs to produce an element of $\mathrm{Cr}_2(\mathbf{k})$ which is not conjugate to a monomial transformation or to an automorphism of the affine plane that normalizes the group of translations. This has been done for algebraically closed fields in [40], for finite fields in [132], and for arbitrary fields in [105].([8]) For instance, very general elements of $\mathrm{Cr}_2(\mathbf{C};2)$ are tight; this implies that $\mathrm{Cr}_2(\mathbf{C})$ contains uncountably many distinct normal subgroups.

The examples of tight elements given by Lonjou are Hénon mappings $h(x,y) = (y, y^n - x)$, with a degree n which is not divisible by the characteristic of $\mathbf{k}$. By Déserti's theorem, the group of automorphisms of $\mathrm{Cr}_2(\mathbf{C})$ is generated by inner automorphisms and the action of $\mathrm{Aut}(\mathbf{C},+,\cdot)$. Since h is defined over $\mathbf{Z}$, the normal subgroup generated by h^m is a **characteristic subgroup** of $\mathrm{Cr}_2(\mathbf{C})$.

Corollary 7.8. *The Cremona group* $\mathrm{Cr}_2(\mathbf{C})$ *contains infinitely many characteristic subgroups.*

The same strategy is used in various contexts, as in the recent proof, by Dahmani, Guirardel and Osin, that high powers of pseudo-Anosov elements generate strict, non-trivial, normal subgroups in mapping class groups. Applied to the Cremona group, their techniques lead to the following statement.

Theorem 7.9 (Dahmani, Guirardel, and Osin, [40, 51, 105]). *Let* $\mathbf{k}$ *be a field. The Cremona group* $\mathrm{Cr}_2(\mathbf{k})$ *is sub-quotient universal: Every countable group can be embedded in a quotient group of* $\mathrm{Cr}_2(\mathbf{k})$.

Remark 7.10. Being sub-quotient universal, while surprising at first sight, is a common feature of hyperbolic groups [55, 117]. For instance, $\mathsf{SL}_2(\mathbf{Z})$ is sub-quotient universal [106]. We refer to [50] for a unified viewpoint on small cancellation theory that includes the study of mapping class groups $\mathrm{Mod}(g)$ and the Cremona groups $\mathrm{Cr}_2(\mathbf{k})$.

–III–
Higher dimensions, subgroups, and growths

Our understanding of groups of birational transformations in dimension ≥ 3 is far less satisfactory than in dimension 2. In this last part, we focus on two open problems: The first one has been solved in many cases, with a wealth of different methods, and we hope that these methods may be useful for other questions; the second one, while much simpler to describe, requires new ideas.

[8]Note that Theorem 7.3 has been proved several years after [40], so that the existence of tight elements in $\mathrm{Cr}_2(\mathbf{k})$ could not rely on it.

8. Zimmer program

8.1. Groups of diffeomorphisms. Consider a compact, connected manifold M (of class $\mathcal{C}^\infty$). Denote by $\mathrm{Diff}_0^\infty(M)$ the group of smooth diffeomorphisms of M which are isotopic to the identity. This group determines M. Indeed, Filipkiewicz proved that every "abstract" isomorphism between $\mathrm{Diff}_0^\infty(M)$ and $\mathrm{Diff}_0^\infty(M')$ is given by conjugacy with respect to a diffeomorphism $\varphi\colon M \to M'$; moreover, Hurtado proved that the existence of an embedding $\mathrm{Diff}^\infty(M) \to \mathrm{Diff}^\infty(M')$ forces the inequality $\dim(M) \leq \dim(M')$ (see [77, 1, 93], and the references of these articles).

By the work of Mather, Herman, Thurston and Epstein, the group $\mathrm{Diff}_0^\infty(M)$ is simple (see [1]). One way to understand it better is to compare it to classical, (almost) simple, real linear groups, such as $\mathsf{SL}_n(\mathbf{R})$ or $\mathsf{SO}_{p,q}(\mathbf{R})$. Starting with a classical result concerning Lie groups, one may ask to what extent such a result holds in the context of groups of diffeomorphisms.

Recall that the real rank of such a linear group is the dimension of a maximal torus, i.e. the maximal dimension of a closed subgroup which is diagonalizable over $\mathbf{R}$. The real rank $\mathsf{rk}_{\mathbf{R}}(G)$ is a good measure of the "complexity" of the group G; for instance, $\mathsf{rk}_{\mathbf{R}}(G) \leq \mathsf{rk}_{\mathbf{R}}(H)$ if G embeds in H. This is reflected by actions by diffeomorphisms: *If the simple Lie group G acts smoothly and non-trivially on M, then $\dim(M) \geq \mathsf{rk}_{\mathbf{R}}(G)$ (with equality when M is the projective space of dimension $n-1$ and G is $\mathsf{PSL}_n(\mathbf{R})$).*

Lie theory concerns the case of smooth actions of connected Lie groups; Zimmer's program proposes to pursue the comparison between Lie groups and groups of diffeomorphisms by looking at finitely generated subgroups. The following is an emblematic conjecture of this program.

Conjecture 8.1 (Zimmer conjecture). *Let G be a simple Lie group and Γ be a lattice in G. If Γ acts faithfully on a compact connected manifold M by diffeomorphisms, then* $\mathsf{rk}_{\mathbf{R}}(G) \leq \dim(M)$.

This conjecture has been proved in the case when M is the circle [80, 30], or when the lattice is not cocompact and the action is by area preserving diffeomorphisms of a compact surface [119].

8.2. Groups of algebraic transformations. Groups of automorphisms or birational transformations can be compared to groups of diffeomorphisms, like $\mathrm{Diff}_0^\infty(M)$ or linear algebraic groups, like $\mathsf{SL}_n(\mathbf{k})$. Such comparisons are useful when looking at affine (resp. projective) varieties with a large group of automorphisms (resp. birational transformations); the prototypical example is given by the affine space $\mathbb{A}^n_{\mathbf{k}}$.

Conjecture 8.2 (Zimmer conjecture for birational transformations). *Let G be a simple Lie group and Γ be a lattice in G. If Γ acts faithfully on a projective variety X by birational transformations, then* $\mathsf{rk}_{\mathbf{R}}(G) \leq \dim(X)$.

8.2.1. Regular automorphisms. The same conjecture for actions by regular automorphisms is settled in [33, 42] when one looks at automorphisms of complex projective or compact Kähler manifolds: *If Γ is a lattice in a simple Lie group G and Γ acts faithfully* *by automorphisms of a compact Kähler manifold X, then $\mathsf{rk}_{\mathbf{R}}(G) \leq \dim(X)$ and in case of equality X is the projective space $\mathbb{P}^n_{\mathbf{C}}$.*

The proof works as follows. The group $\mathrm{Aut}(X)$ is a complex Lie group; it may have infinitely many connected components, but the connected component of the identity $\mathrm{Aut}(X)^0$ is a Lie group whose Lie algebra is the algebra of holomorphic vector fields on X. The group $\mathrm{Aut}(X)$ acts on the cohomology of X, and the kernel K of this action contains $\mathrm{Aut}(X)^0$ as a finite index subgroup (Lieberman's theorem, see [103]). Let Γ be a lattice in an almost simple Lie group G, and assume that Γ embeds into $\mathrm{Aut}(X)$. Assume, moreover, that the rank of G is larger than 1, since otherwise the inequality $\dim(X) \geq \mathsf{rk}_{\mathbf{R}}(G)$ is obvious. Margulis normal subgoup theorem shows that Γ is almost simple: Every normal subgroup of Γ is finite and central, or co-finite. As a consequence, one can assume (replacing Γ by a finite index subgroup), that (1) Γ embeds into $\mathrm{Aut}(X)^0$ or that (2) the action of Γ on the cohomology of X is faithful. The super-rigidity theorem of Margulis shows that homomorphisms from Γ into Lie groups H are built from homomorphisms of G into H (see [110, 33] for precise statements). In case (1), this implies that $\mathrm{Aut}(X)^0$ contains a complex Lie group of rank $\mathsf{rk}_{\mathbf{R}}(G)$; one can then use Lie theory to conclude that $\dim(X) \geq \mathsf{rk}_{\mathbf{R}}(G)$. In case (2), one concludes that the action of Γ on the cohomology of $H^*(X;\mathbf{Z})$ comes from a linear representation of G into $\mathsf{GL}(H^*(X;\mathbf{R}))$. But this linear representation preserves the Hodge decomposition, the cup product, the Poincaré duality, etc, because it comes from the original action of Γ by automorphisms. One can then put together Hodge theory (in particular the Hodge index theorem) and the theory of linear representations of (almost) simple Lie groups to get the estimate $\dim(X) \geq \mathsf{rk}_{\mathbf{R}}(G) + 1$. (details are given in [33, 42], and similar arguments are used in [65, 139]).

It would be nice to adapt such a proof for groups of birational transformations. One way to do it is to consider the limit $\mathcal{Z}^1(M)$ of Néron-Severi groups $N^1(M')$ along all birational morphisms $M' \to M$ (and more generally the limits $\mathcal{Z}^q(M)$ of all $N^q(M')$, where $N^q(M')$ denotes the space of codimension q cycles modulo numerical equivalence). The intersection determines a multilinear pairing (see [25], Chapter 4): It provides a geometric structure on these spaces, with nice properties coming from Hodge index theorem and Khovanskii-Teyssier inequalities. In dimension 2, this leads to the construction of the rank one space $\mathbb{H}_\infty(X)$; in higher dimension, one expects phenomena of rank $\dim(M) - 1$. These properties should provide rich constraints on the action of the group $\mathrm{Bir}(M)$ on $\mathcal{Z}^1(M)$, and prevent large rank lattices from acting properly on such spaces (some kind of Mostow-Margulis rigidity for actions on those infinite dimensional spaces).

8.2.2. Birational transformations. Zimmer type problems are harder to study for groups of birational transformations or groups of automorphisms of non-complete varieties, such as the affine space. Nevertheless, a new technique emerged recently in the study of non-linear analogues of the Skolem-Mahler-Lech theorem. This classical statement says that the indices n for which a linear recursive sequence $u_{n+k} = a_1 u_{n+k-1} + \cdots a_k u_n$ vanishes form a finite union of arithmetic sequences in $\mathbf{Z}_+$. In other words, when one iterates a linear transformation B of $\mathbb{A}^k_{\mathbf{C}}$, the set of times n such that the orbit $B^n(x_0)$ of a point x_0 is contained in the hyperplane $x_k = 0$ is a finite union of arithmetic progressions. As shown by Bell and his co-authors, this statement remains true if one replaces B by a polynomial automorphism f of $\mathbb{A}^k_{\mathbf{C}}$ and the hyperplane by any algebraic subvariety of $\mathbb{A}^k_{\mathbf{C}}$ (see

[9, 10] for more general statements). These results are based on the following p-adic phenomenon, which we state only in its simpler version.

Theorem 8.3 (Bell, Poonen, see [9, 120]). *Let p be a prime number, with $p \geq 3$. Let f be a polynomial automorphism of the affine space $\mathbb{A}^n_{\mathbf{Q}_p}$ which is defined by polynomial formulas with coefficients in $\mathbf{Z}_p$. Assume that f coincides with the identity map when one reduces all coefficients modulo p. Then, there exists a p-adic analytic action $\Phi\colon \mathbf{Z}_p \times (\mathbf{Z}_p)^n \to (\mathbf{Z}_p)^n$ of the abelian group $(\mathbf{Z}_p, +)$ on the polydisk $(\mathbf{Z}_p)^n \subset \mathbb{A}^n(\mathbf{Q}_p)$ such that $\Phi(m,x) = f^m(x)$ for every $m \in \mathbf{Z}$ and every x in $(\mathbf{Z}_p)^n$.*

Here, by p-adic analytic, we mean that $\Phi(t,x)$ is given by convergent power series in the variables t and $x = (x_1, \ldots, x_n)$ whose coefficients are in $\mathbf{Z}_p$. Thus, if f is the identity map after reduction of its formulas modulo p, then the action of f on the polydisk $(\mathbf{Z}_p)^n$ is given by the flow, at time $t = 1$, of an analytic vector field. Theorem 8.3 is a tool to replace a discrete group action (like $\mathbf{Z}$, generated by f) by the action of a continuous group (like $\mathbf{Z}_p$, defined by Φ), at least locally in the p-adic topology. This result turns out to be useful when, instead of a cyclic group $\mathbf{Z}$, one studies a subgroup $\Gamma \subset \mathrm{Aut}(\mathbb{A}^n_{\mathbf{Q}_p})$ whose pro-p completion is small (for instance a p-adic Lie group).

When Γ is a subgroup of $\mathrm{Aut}(\mathbb{A}^n_{\mathbf{C}})$ which is generated by a finite number of elements $f_i \in \Gamma$, one can replace the field $\mathbf{C}$ by the field generated by the coefficients of the formulas defining the f_i; such a finitely generated fields embeds (in many ways) in p-adic fields $\mathbf{Q}_p$. Thus, p-adic methods can be used to study groups of automorphisms and birational transformations of complex algebraic varieties.

This argument turns out to be quite powerful, and leads to the following statement (see [41]).

Theorem 8.4. *Let X be an irreducible complex projective variety. Let $S < \mathsf{GL}_n$ be an almost simple linear algebraic group over the field of rational numbers $\mathbf{Q}$. Assume that $S(\mathbf{Z})$ is not co-compact. If a finite index subgroup of $S(\mathbf{Z})$ embeds into $\mathrm{Bir}(X)$, then $\dim(X) \geq \mathsf{rk}_{\mathbf{R}}(S)$. If $\dim(X) = \mathsf{rk}_{\mathbf{R}}(S) \geq 2$, then $S(\mathbf{R})$ is isogeneous to $\mathsf{SL}_{\dim(X)+1}(\mathbf{R})$.*

In other words, Zimmer conjecture holds for birational actions of lattices which are not co-compact. For instance, one can take $S = \mathsf{SL}_n$ in this theorem. Unfortunately, co-compact lattices are not handled by this theorem, and Conjecture 8.2 is still open for co-compact lattices.

8.3. Residual finiteness. In the same spirit – comparing groups of rational transformations to groups of linear transformations – the most basic question that has not been answered yet is the following one, which parodies Malcev's and Selberg's theorems.

Question 8.5. Are finitely generated subgroups of $\mathrm{Cr}_n(\mathbf{k})$ residually finite ? Does every finitely generated subgroup of $\mathrm{Cr}_n(\mathbf{k})$ contain a torsion free subgroup of finite index ?

Bass and Lubotzky obtained a positive answer to this question when $\mathrm{Cr}_n(\mathbf{k})$ is replaced by the group of regular automorphisms of an algebraic variety, for instance by $\mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}})$ (see [3]). The article [41] provides a positive answer for groups with Kazhdan property (T).

9. Growths

9.1. Degree growth. Consider a birational transformation f of a smooth projective variety X, over a field $\mathbf{k}$. Fix a polarization H of X, and defines the degree of f with respect to H by

$$\mathsf{deg}^1_H(f) = (f^*H)\cdot(H^{n-1}),$$

where $f^*(H)$ is the total transform of H, $U\cdot V$ is the intersection form, and n is the dimension of X. This degree controles the complexity of the operator f^* acting on algebraic hypersurfaces of X. Similarly, for every co-dimension $1\leq k\leq n$, one defines a degree in co-dimension k by

$$\mathsf{deg}^k_H(f) = (f^*H^k)\cdot(H^{n-k}).$$

The degrees behave submultiplicatively: There is a constant $A(X,H)>0$, which depends only on X and its polarization, such that

$$\mathsf{deg}^k_H(f\circ g)\leq A(X;H)\mathsf{deg}^k_H(f)\mathsf{deg}^k_H(g)$$

for every pair of birational transformations f,g in $\mathrm{Bir}(X)$; moreover, up to a uniform multiplicative constant, $\mathsf{deg}^k_H(\cdot)$ does not depend on H:

$$\mathsf{deg}^k_H(f)\leq A'(X,H,H')\mathsf{deg}^k_{H'}(f)$$

for all f in $\mathrm{Bir}(X)$. This has been proved by Dinh and Sibony for fields of characteristic 0, and then by Truong in positive characteristic (see [66], [135], and also [115]).

Thus, given a birational transformation f of a projective variety X, one gets $\dim(X)$ sequences

$$m\mapsto \mathsf{deg}^k_H(f^m)$$

which, up to multiplicative constants, do not depend on H and are invariant under conjugacy.

Question 9.1. What type of sequences do we get under this process ? In particular, what can be said on the growth type of $m\mapsto\mathsf{deg}^1_H(f^m)$?

One can show that there are only countably many possible sequences of the form $(\mathsf{deg}^k_H(f^m))_{m\geq 0}$ (see [23, 137]). Moreover, the sequences $(\mathsf{deg}^k_H(f^m))$ are linked together: For instance, the **dynamical degrees**

$$\lambda_k(f) = \limsup_{m\to+\infty}\left(\mathsf{deg}^k_H(f^m)\right)^{1/m}$$

determine a concave sequence $k\mapsto\log(\lambda_k(f))$ and, in particular, one of the $\lambda_k(f)$, with $k>0$, is larger than 1 if and only if all of them are (see the survey [89]).

When X is a surface, there are only 4 possibilities for the sequence $\mathsf{deg}^1_H(f^m)$: It is bounded, or it grows linearly or quadratically, or it grows exponentially fast. Moreover, the first three cases have a geometric meaning (see Theorem 4.6). Nothing like that is known in dimension ≥ 3. Does there exist a polynomial automorphism g of the affine space $\mathbb{A}^3_{\mathbf{C}}$ for which $\mathsf{deg}^1_H(g^m)$ grows like $\exp(\sqrt{m})$? Do the results of Lo Bianco in [104] hold for birational transformations of $\mathbb{P}^3_{\mathbf{C}}$?

9.2. Divisibility and distorsion. Questions related to degree growths are connected to algebraic properties of (subgroups of) $\mathrm{Bir}(X)$. An element f in a group G is **distorted** if there is a subgroup Γ of G such that (1) Γ is generated by a finite subset S, (2) f is an element of Γ, and (3) f^m can be written as a word of length $\ell(m)$ in the elements of S with $\lim(\ell(m)/m) = 0$. If a birational transformation $f\colon X \dashrightarrow X$ is distorted in $\mathrm{Bir}(X)$, then $\lambda_k(f) = 1$ for all $0 \leq k \leq \dim(X)$. It would be great to classify, or at least to get geometric constraints on distorted elements in $\mathrm{Aut}(\mathbb{A}^n_{\mathbf{C}})$ and $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{C}})$.

One says that an element f of the group G is **divisible**, if for every $m > 0$ there is an element g_m in G such that $(g_m)^m = f$. Can we classify divisible elements of $\mathrm{Aut}(\mathbb{A}^3_{\mathbf{C}})$?

9.3. Groups. Given a group Γ in $\mathrm{Bir}(X)$, one gets a degree function on Γ, namely

$$f \in \Gamma \mapsto \mathsf{deg}(f),$$

where, for simplicity, $\mathsf{deg}(f)$ stands for $\mathsf{deg}^1_H(f)$. Assume that Γ is generated by a finite symmetric set S, and denote by $D_{\Gamma,S}(m)$ the maximum of $\mathsf{deg}(f)$ for f in the ball of radius m in the Cayley graph of Γ:

$$D_{\Gamma,S}(m) = \max\{\mathsf{deg}(f) \mid \exists l \leq m, \exists s_1, \ldots, s_l \in S,\ f = s_1 \circ s_2 \circ \cdots \circ s_l\}.$$

When Γ is the cyclic group generated by $S = \{f, f^{-1}\}$, $D_{\Gamma,S}(m)$ is the maximum of $\mathsf{deg}(f^l)$ for l in between $-m$ and m. Our former questions on the sequence $(\mathsf{deg}^1_H(f^m))$ can now be stated for the sequence $(D_{\Gamma,S}(m))$. Again, there are only countably many such sequences, and one would like to know their possible growth types.

This is related to the growth type of Γ, viewed as an abstract, finitely geneated group, *i.e.* to the growth of the function

$$\mathsf{Vol}_S\colon m \mapsto \mathsf{Vol}_S(m) = \mathsf{Card}\{f \in \Gamma \mid \exists l \leq m, \exists s_1, \ldots, s_l \in S,\ f = s_1 \circ s_2 \circ \cdots \circ s_l\}$$

counting the number of elements of the ball of radius m in the Cayley graph of Γ (with respect to S). If a group contains a non abelian free group, then $\mathsf{Vol}_S(m)$ grows exponentially fast; if the growth is bounded by m^d for some $d > 0$, then Γ contains a finite index nilpotent subgroup [87]; if Γ is solvable, the growth is either polyomial or exponential. In particular, if G satisfies the Tits alternative, the growth of every finitely generated subgroup of G is either polynomial or exponential. But there are many groups with intermediate growth, in between polynomial and exponential (see [86, 53, 2] for instance).

Question 9.2. Does $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{C}})$ contain finitely generated subgroups with intermediate growth ?

References

[1] Augustin Banyaga. *The structure of classical diffeomorphism groups*, volume 400 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1997.

[2] Laurent Bartholdi and Anna Erschler. Groups of given intermediate word growth. *Ann. Inst. Fourier (Grenoble)*, 64(5):2003–2036, 2014.

[3] Hyman Bass and Alexander Lubotzky. Automorphisms of groups and of schemes of finite type. *Israel J. Math.*, 44(1):1–22, 1983.

[4] Arnaud Beauville. p-elementary subgroups of the Cremona group. *J. Algebra*, 314(2):553–564, 2007.

[5] Arnaud Beauville. De combien de paramètres dépend l'equation générale de degré n? *Gaz. Math.*, (132):5–15, 2012.

[6] Eric Bedford and Jeffrey Diller. Energy and invariant measures for birational surface maps. *Duke Math. J.*, 128(2):331–368, 2005.

[7] Eric Bedford and Kyounghee Kim. Dynamics of rational surface automorphisms: rotation domains. *Amer. J. Math.*, 134(2):379–4050, 2012.

[8] Eric Bedford, Mikhail Lyubich, and John Smillie. Polynomial diffeomorphisms of $\mathbf{C}^2$. IV. The measure of maximal entropy and laminar currents. *Invent. Math.*, 112(1):77–125, 1993.

[9] Jason P. Bell. A generalised Skolem-Mahler-Lech theorem for affine varieties. *J. London Math. Soc. (2)*, 73(2):367–379, 2006.

[10] Jason P. Bell, Dragos Ghioca, and Thomas J. Tucker. The dynamical Mordell-Lang problem for étale maps. *Amer. J. Math.*, 132(6):1655–1675, 2010.

[11] Riccardo Benedetti and Carlo Petronio. *Lectures on hyperbolic geometry*. Universitext. Springer-Verlag, Berlin, 1992.

[12] Mladen Bestvina, Mark Feighn, and Michael Handel. The Tits alternative for $\mathrm{Out}(F_n)$. II. A Kolchin type theorem. *Ann. of Math. (2)*, 161(1):1–59, 2005.

[13] Caucher Birkar. Singularities of linear systems and boundedness of fano varieties. *preprint*, arXiv:1609.05543v1:1–33, 2016.

[14] Jérémy Blanc. Sous-groupes algébriques du groupe de Cremona. *Transform. Groups*, 14(2):249–285, 2009.

[15] Jérémy Blanc. Groupes de Cremona, connexité et simplicité. *Ann. Sci. Éc. Norm. Supér. (4)*, 43(2):357–364, 2010.

[16] Jérémy Blanc. Elements and cyclic subgroups of finite order of the Cremona group. *Comment. Math. Helv.*, 86(2):469–497, 2011.

[17] Jérémy Blanc. Simple relations in the Cremona group. *Proc. Amer. Math. Soc.*, 140(2):1495–1500, 2012.

[18] Jérémy Blanc and Serge Cantat. Dynamical degrees of birational transformations of projective surfaces. *J. Amer. Math. Soc.*, 29(2):415–471, 2016.

[19] Jérémy Blanc and Julie Déserti. Degree growth of birational maps of the plane. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 14(2):507–533, 2015.

[20] Jérémy Blanc and Jean-Philippe Furter. Topologies and structures of the Cremona groups. *Ann. of Math. (2)*, 178(3):1173–1198, 2013.

[21] Jérémy Blanc and Frédéric Mangolte. Cremona groups of real surfaces. In *Automorphisms in birational and affine geometry*, volume 79 of *Springer Proc. Math. Stat.*, pages 35–58. Springer, Cham, 2014.

[22] Jérémy Blanc and Suzanna Zimmerman. Topological simplicity of the Cremona groups. *preprint*, arXiv:1511.08907:1–11, 2015.

[23] Araceli M. Bonifant and John Erik Fornæss. Growth of degree for iterates of rational maps in several variables. *Indiana Univ. Math. J.*, 49(2):751–778, 2000.

[24] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Degree growth of meromorphic surface maps. *Duke Math. J.*, 141(3):519–538, 2008.

[25] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Valuations and plurisubharmonic singularities. *Publ. Res. Inst. Math. Sci.*, 44(2):449–494, 2008.

[26] Matthew G. Brin and Craig C. Squier. Groups of piecewise linear homeomorphisms of the real line. *Invent. Math.*, 79(3):485–498, 1985.

[27] Michael Brin and Garrett Stuck. *Introduction to dynamical systems*. Cambridge University Press, Cambridge, 2002.

[28] Marco Brunella. Minimal models of foliated algebraic surfaces. *Bull. Soc. Math. France*, 127(2):289–305, 1999.

[29] Marc Burger, Alessandra Iozzi, and Nicolas Monod. Equivariant embeddings of trees into hyperbolic spaces. *Int. Math. Res. Not.*, (22):1331–1369, 2005.

[30] Marc Burger and Nicolas Monod. Bounded cohomology of lattices in higher rank Lie groups. *J. Eur. Math. Soc. (JEMS)*, 1(2):199–235, 1999.

[31] Serge Cantat. Dynamique des automorphismes des surfaces $K3$. *Acta Math.*, 187(1):1–57, 2001.

[32] Serge Cantat. Sur la dynamique du groupe d'automorphismes des surfaces $K3$. *Transform. Groups*, 6(3):201–214, 2001.

[33] Serge Cantat. Version kählérienne d'une conjecture de Robert J. Zimmer. *Ann. Sci. École Norm. Sup. (4)*, 37(5):759–768, 2004.

[34] Serge Cantat. Generators of the Cremona group in $n > 2$ variables (after Hudson and Pan). *http://perso.univ-rennes1.fr/serge.cantat/publications.html*, pages 1–5, 2011.

[35] Serge Cantat. Sur les groupes de transformations birationnelles des surfaces. *Ann. of Math. (2)*, 174(1):299–340, 2011.

[36] Serge Cantat. Sur les groupes de transformations birationnelles des surfaces - version longue. *long version with complements*, webpage of the author, 2011.

[37] Serge Cantat. Dynamics of automorphisms of compact complex surfaces. In *Frontiers in complex dynamics*, volume 51 of *Princeton Math. Ser.*, pages 463–514. Princeton Univ. Press, Princeton, NJ, 2014.

[38] Serge Cantat. Morphisms between Cremona groups, and characterization of rational varieties. *Compos. Math.*, 150(7):1107–1124, 2014.

[39] Serge Cantat and Charles Favre. Symétries birationnelles des surfaces feuilletées. *J. Reine Angew. Math.*, 561:199–235, 2003.

[40] Serge Cantat and Stéphane Lamy. Normal subgroups in the Cremona group. *Acta Math.*, 210(1):31–94, 2013. With an appendix by Yves de Cornulier.

[41] Serge Cantat and Junyi Xie. Algebraic actions of discrete groups: the p-adic method. *see the webpages of the authors*, 2015.

[42] Serge Cantat and Abdelghani Zeghib. Holomorphic actions, Kummer examples, and Zimmer program. *Ann. Sci. Éc. Norm. Supér. (4)*, 45(3):447–489, 2012.

[43] Guido Castelnuovo. Le trasformazioni generatrici del gruppo cremoniano nel piano. *Atti della R. Acc. delle Sc. di Torino XXXVI*, (13):861–874, 1901.

[44] I. A. Chel′tsov. Regularization of birational automorphisms. *Mat. Zametki*, 76(2):286–299, 2004.

[45] Ivan Cheltsov and Constantin Shramov. Three embeddings of the Klein simple group into the Cremona group of rank three. *Transform. Groups*, 17(2):303–350, 2012.

[46] Ivan Cheltsov and Constantin Shramov. Five embeddings of one simple group. *Trans. Amer. Math. Soc.*, 366(3):1289–1331, 2014.

[47] Ivan Cheltsov and Constantin Shramov. *Cremona groups and the icosahedron*. Monographs and research notes in mathematics. Chapman and Hall, CRC Press, Basel, 2015.

[48] Pierre-Alain Cherix, Michael Cowling, Paul Jolissaint, Pierre Julg, and Alain Valette. *Groups with the Haagerup property*, volume 197 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2001. Gromov's a-T-menability.

[49] Alessio Corti and Miles Reid, editors. *Explicit birational geometry of 3-folds*, volume 281 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2000.

[50] Rémi Coulon. Théorie de la petite simplification: une approche géométrique. *Séminaire Bourbaki*, 1089:1–29, 2014.

[51] François Dahmani, Vincent Guirardel, and Denis Osin. Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. *preprint, arXiv:1111.7048*, pages 1–139, 2012.

[52] V. I. Danilov. Non-simplicity of the group of unimodular automorphisms of an affine plane. *Mat. Zametki*, 15:289–293, 1974.

[53] Pierre de la Harpe. *Topics in geometric group theory*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.

[54] Pierre de la Harpe and Alain Valette. La propriété (T) de Kazhdan pour les groupes localement compacts (avec un appendice de Marc Burger). *Astérisque*, (175):158, 1989. With an appendix by M. Burger.

[55] Thomas Delzant. Sous-groupes distingués et quotients des groupes hyperboliques. *Duke Math. J.*, 83(3):661–682, 1996.

[56] Thomas Delzant and Pierre Py. Kähler groups, real hyperbolic spaces, and the Cremona group. *Compos. Math.*, 148(1):153–184, 2012. With an appendix by S. Cantat.

[57] Michel Demazure. Sous-groupes algébriques de rang maximum du groupe de Cremona. *Ann. Sci. École Norm. Sup. (4)*, 3:507–588, 1970.

[58] Julie Déserti. Groupe de Cremona et dynamique complexe: une approche de la conjecture de Zimmer. *Int. Math. Res. Not.*, pages Art. ID 71701, 27, 2006.

[59] Julie Déserti. Sur les automorphismes du groupe de Cremona. *Compos. Math.*, 142(6):1459–1478, 2006.

[60] Julie Déserti. Le groupe de Cremona est hopfien. *C. R. Math. Acad. Sci. Paris*, 344(3):153–156, 2007.

[61] Julie Déserti. On solvable subgroups of the Cremona group. *Illinois J. Math., to appear*, arXiv:1503.02121:1–11, 2015.

[62] Jeffrey Diller. Invariant measure and Lyapunov exponents for birational maps of $\mathbf{P}^2$. *Comment. Math. Helv.*, 76(4):754–780, 2001.

[63] Jeffrey Diller and Charles Favre. Dynamics of bimeromorphic maps of surfaces. *Amer. J. Math.*, 123(6):1135–1169, 2001.

[64] Tien-Cuong Dinh. Tits alternative for automorphism groups of compact Kähler manifolds. *Acta Math. Vietnam.*, 37(4):513–529, 2012.

[65] Tien-Cuong Dinh and Nessim Sibony. Groupes commutatifs d'automorphismes d'une variété kählérienne compacte. *Duke Math. J.*, 123(2):311–328, 2004.

[66] Tien-Cuong Dinh and Nessim Sibony. Une borne supérieure pour l'entropie topologique d'une application rationnelle. *Ann. of Math. (2)*, 161(3):1637–1644, 2005.

[67] Igor V. Dolgachev. Rational surfaces with a pencil of elliptic curves (russian). *Izv. Akad. Nauk SSSR Ser. Mat.*, 30:1073–1100, 1966.

[68] Igor V. Dolgachev. *Classical ALgebraic Geometry: A Modern View*. Progress in Mathematics. Cambridge University Press, 2010.

[69] Igor V. Dolgachev. Finite subgroups of the plane Cremona group. In *Algebraic geometry in East Asia—Seoul 2008*, volume 60 of *Adv. Stud. Pure Math.*, pages 1–49. Math. Soc. Japan, Tokyo, 2010.

[70] Igor V. Dolgachev. *Lectures on Cremona transformations*. Lecture Notes, Ann Arbor-Rome. Homepage of the author, 2011.

[71] Igor V. Dolgachev and Vasily A. Iskovskikh. Finite subgroups of the plane Cremona group. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I*, volume 269 of *Progr. Math.*, pages 443–548. Birkhäuser Boston, Inc., Boston, MA, 2009.

[72] Romain Dujardin. Laminar currents and birational dynamics. *Duke Math. J.*, 131(2):219–247, 2006.

[73] Federigo Enriques. Sui gruppi continui di trasformazioni cremoniane nel piano. *Rom. Acc. L. Rend.*, (5) II_1:468–473, 1893.

[74] Federigo Enriques. Conferenze di geometria. Fondamenti di una geometria iperspaziale. *lit., Bologna*, 1894-95.

[75] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[76] Charles Favre. Points périodiques d'applications birationnelles de $\mathbf{P}^2$. *Ann. Inst. Fourier (Grenoble)*, 48(4):999–1023, 1998.

[77] Richard P. Filipkiewicz. Isomorphisms between diffeomorphism groups. *Ergodic Theory Dynamical Systems*, 2(2):159–171 (1983), 1982.

[78] Edward Formanek and Claudio Procesi. The automorphism group of a free group is not linear. *J. Algebra*, 149(2):494–499, 1992.

[79] Michael A. Frumkin. A filtration in the three-dimensional Cremona group. *Mat. Sb. (N.S.)*, 90(132):196–213, 325, 1973.

[80] Étienne Ghys. Actions de réseaux sur le cercle. *Invent. Math.*, 137(1):199–231, 1999.

[81] Étienne Ghys and Vlad Sergiescu. Sur un groupe remarquable de difféomorphismes du cercle. *Comment. Math. Helv.*, 62(2):185–239, 1987.

[82] Marat H. Gizatullin. Rational G-surfaces. *Izv. Akad. Nauk SSSR Ser. Mat.*, 44(1):110–144, 239, 1980.

[83] Marat H. Gizatullin. Defining relations for the Cremona group of the plane. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(5):909–970, 1134, 1982.

[84] Marat H. Gizatullin. On some tensor representations of the Cremona group of the projective plane. In *New trends in algebraic geometry (Warwick, 1996)*, volume 264 of *London Math. Soc. Lecture Note Ser.*, pages 111–150. Cambridge Univ. Press, Cambridge, 1999.

[85] Leon Greenberg. Maximal groups and signatures. In *Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973)*, pages 207–226. Ann. of Math. Studies, No. 79. Princeton Univ. Press, Princeton, N.J., 1974.

[86] Rotislav I. Grigorchuk. Degrees of growth of finitely generated groups and the theory of invariant means. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(5):939–985, 1984.

[87] Mikhael Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, (53):53–73, 1981.

[88] Mikhaïl Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[89] Vincent Guedj. Propriétés ergodiques des applications rationnelles. In *Panorama et Synthèses*, volume 30, pages 97–202. SMF, 2010.

[90] Vincent Guirardel. Geometric small cancellation. In *Geometric group theory*, volume 21, pages 55–89. IAS/Park City Math. Ser., 2009.

[91] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[92] John Hamal Hubbard. *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1*. Matrix Editions, Ithaca, NY, 2006. Teichmüller theory, With contributions by Adrien Douady, William Dunbar, Roland Roeder, Sylvain Bonnot, David Brown, Allen Hatcher, Chris Hruska and Sudeb Mitra, With forewords by William Thurston and Clifford Earle.

[93] Sebastian Hurtado. Continuity of discrete homomorphisms of diffeomorphism groups. *Geom. Topol.*, 19(4):2117–2154, 2015.

[94] V. A. Iskovskikh. Factorization of birational mappings of rational surfaces from the point of view of Mori theory. *Uspekhi Mat. Nauk*, 51(4(310)):3–72, 1996.

[95] Vasily A. Iskovskikh. Proof of a theorem on relations in the two-dimensional Cremona group. *Uspekhi Mat. Nauk*, 40(5(245)):255–256, 1985.

[96] Vasily A. Iskovskikh and Igor R. Shafarevich. Algebraic surfaces. In *Algebraic geometry, II*, volume 35 of *Encyclopaedia Math. Sci.*, pages 127–262. Springer, Berlin, 1996.

[97] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[98] János Kollár and Frédéric Mangolte. Cremona transformations and diffeomorphisms of surfaces. *Adv. Math.*, 222(1):44–61, 2009.

[99] Janos Kollár, Karen E. Smith, and Alessio Corti. *Rational and Nearly Rational Varieties*, volume 92 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2004.

[100] Hanspeter Kraft and Immanuel Stampfli. On automorphisms of the affine Cremona group. *Ann. Inst. Fourier (Grenoble)*, 63(3):1137–1148, 2013.

[101] Stéphane Lamy. Une preuve géométrique du théorème de Jung. *Enseign. Math. (2)*, 48(3-4):291–315, 2002.

[102] Stéphane Lamy. On the genus of birational maps between threefolds. In *Automorphisms in birational and affine geometry*, volume 79 of *Springer Proc. Math. Stat.*, pages 141–147. Springer, Cham, 2014.

[103] David I. Lieberman. Compactness of the Chow scheme: applications to automorphisms and deformations of Kähler manifolds. In *Fonctions de plusieurs variables complexes, III (Sém. François Norguet, 1975–1977)*, volume 670 of *Lecture Notes in Math.*, pages 140–186. Springer, Berlin, 1978.

[104] Federico Lo Bianco. Bornes sur les degrés dynamiques d'automorphismes de variétés kähleriennes de dimension 3. *C. R. Math. Acad. Sci. Paris*, 352(6):515–519, 2014.

[105] Anne Lonjou. Non simplicité du groupe de Cremona sur tout corps. *preprint*, arXiv:1503.03731:1–17, 2015.

[106] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.

[107] A. M. Macbeath and D. Singerman. Spaces of subgroups and Teichmüller space. *Proc. London Math. Soc. (3)*, 31(2):211–256, 1975.

[108] Wilhelm Magnus. Rings of Fricke characters and automorphism groups of free groups. *Math. Z.*, 170(1):91–103, 1980.

[109] Yuri I. Manin. *Cubic forms*, volume 4 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1986. Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel.

[110] Gregory A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991.

[111] Curtis T. McMullen. Coxeter groups, Salem numbers and the Hilbert metric. *Publ. Math. Inst. Hautes Études Sci.*, (95):151–183, 2002.

[112] Curtis T. McMullen. Dynamics on blowups of the projective plane. *Publ. Math. Inst. Hautes Études Sci.*, (105):49–89, 2007.

[113] Curtis T. McMullen. Automorphisms of projective K3 surfaces with minimum entropy. *Invent. Math.*, 203(1):179 –215, 2015.

[114] John Milnor. *Dynamics in one complex variable*, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, third edition, 2006.

[115] Dang Nguyen-Bac. Degrees of iterates of rational maps on normal projective varieties. *preprint*, pages 1–43, 2016.

[116] Max Noether. Über flächen, welche scharen rationaler curven besitzen. *Math. Ann.*, 3:161–227, 1870.

[117] Alexander Yu. Ol′shanskiĭ. SQ-universality of hyperbolic groups. *Mat. Sb.*, 186(8):119–132, 1995.

[118] Ivan Pan. Une remarque sur la génération du groupe de Cremona. *Bol. Soc. Brasil. Mat. (N.S.)*, 30(1):95–98, 1999.

[119] Leonid Polterovich. Growth of maps, distortion in groups and symplectic geometry. *Invent. Math.*, 150(3):655–686, 2002.

[120] Bjorn Poonen. p-adic interpolation of iterates. *Bull. Lond. Math. Soc.*, 46(3):525–527, 2014.

[121] V. L. Popov. Tori in the Cremona groups. *Izv. Ross. Akad. Nauk Ser. Mat.*, 77(4):103–134, 2013.

[122] Vladimir L. Popov. Some subgroups of the Cremona groups. In *Affine algebraic geometry*, pages 213–242. World Sci. Publ., Hackensack, NJ, 2013.

[123] Yuri Prokhorov. p-elementary subgroups of the Cremona group of rank 3. In *Classification of algebraic varieties*, EMS Ser. Congr. Rep., pages 327–338. Eur. Math. Soc., Zürich, 2011.

[124] Yuri Prokhorov. Simple finite subgroups of the Cremona group of rank 3. *J. Algebraic Geom.*, 21(3):563–600, 2012.

[125] Yuri Prokhorov. 2-elementary subgroups of the space Cremona group. In *Automorphisms in birational and affine geometry*, volume 79 of *Springer Proc. Math. Stat.*, pages 215–229. Springer, Cham, 2014.

[126] Yuri Prokhorov and Constantin Shramov. Jordan property for groups of birational selfmaps. *Compos. Math.*, 150(12):2054–2072, 2014.

[127] Yuri Prokhorov and Constantin Shramov. Jordan property for cremona groups. *Amer. J. Math*, 138(2), 2016.

[128] Felice Ronga and Thierry Vust. Birational diffeomorphisms of the real projective plane. *Comment. Math. Helv.*, 80(3):517–540, 2005.

[129] Jean-Pierre Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

[130] Jean-Pierre Serre. A Minkowski-style bound for the orders of the finite subgroups of the Cremona group of rank 2 over an arbitrary field. *Mosc. Math. J.*, 9(1):193–208, back matter, 2009.

[131] Jean-Pierre Serre. Le groupe de Cremona et ses sous-groupes finis. *Astérisque*, (332):Exp. No. 1000, vii, 75–100, 2010. Séminaire Bourbaki. Volume 2008/2009. Exposés 997–1011.

[132] Nicholas I. Shepherd-Barron. Some effectivity questions for plane cremona transformations. *preprint*, arXiv:1311.6608:1–23, 2013.

[133] Immanuel Stampfli. A note on automorphisms of the affine Cremona group. *Math. Res. Lett.*, 20(6):1177–1181, 2013.

[134] Jacques Tits. Free subgroups in linear groups. *J. Algebra*, 20:250–270, 1972.

[135] Tuyen Trung Truong. Relative dynamical degrees of correspondences over a field of arbitrary characteristic. *preprint*, arXiv:1501.01523:1–41, 2016.

[136] Vladimir Igorevich Tsygankov. The conjugacy classes of finite nonsolvable subgroups in the plane Cremona group. *Adv. Geom.*, 13(2):323–347, 2013.

[137] Christian Urech. Remarks on the degree growth of birational transformations. *preprint*, arXiv:1606.04822:1–12, 2016.

[138] Junyi Xie. Periodic points of birational transformations on projective surfaces. *Duke Math. J.*, 164(5):903–932, 2015.

[139] De-Qi Zhang. Algebraic varieties with automorphism groups of maximal rank. *Math. Ann.*, 355(1):131–146, 2013.

[140] Susanna Zimmermann. The abelianization of the real plane Cremona group. *arXiv:1510.08705*, pages 1–39, 2015.

Serge Cantat, CNRS, Université de Rennes 1, IRMAR, Bât. 22-23 du Campus de Beaulieu, 35042 Rennes cedex, France

E-mail: serge.cantat@univ-rennes1.fr