Linearity and Nonlinearity of Groups of Polynomial Automorphisms of the Plane

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Abstract. A group Γ is called *linear over a ring*, respectively *linear over a field*, if there is an embedding $\Gamma \subset \mathrm{GL}(n, R)$, resp. $\Gamma \subset \mathrm{GL}(n, L)$, for some positive integer n and some commutative ring R, resp. some field L.

Various authors have shown that the automorphism groups of algebraic varieties share many properties with linear groups, e.g. the Tits alternative holds in $\mathrm{Aut}\,\mathbb{A}^2_K$ (S. Lamy 2000). In this paper, we will investigate the following related

Question: which subgroups of $\mathrm{Aut}\,\mathbb{A}^2_K$ *are indeed linear or not?*

where $\mathrm{Aut}\,\mathbb{A}^2_K$ is the group of polynomial automorphisms of the affine plane $\mathbb{A}^2_K$ over a field K. Roughly speaking, the answer is that $\mathrm{Aut}\,\mathbb{A}^2_K$ contains large linear subgroups and small ones which are not.

In order to be more specific, let us consider the following subgroups

$$\mathrm{Aut}_0\,\mathbb{A}^2_K = \{\phi \in \mathrm{Aut}\,\mathbb{A}^2_K \mid \phi(\mathbf{0}) = \mathbf{0})\},$$
$$\mathrm{SAut}_0\,\mathbb{A}^2_K = \{\phi \in \mathrm{Aut}_0\,\mathbb{A}^2_K \mid \mathrm{Jac}(\phi) = 1\}, \text{ and}$$
$$\mathrm{Aut}_{U(K)}\,\mathbb{A}^2_K = \{\phi \in \mathrm{Aut}_0\,\mathbb{A}^2_K \mid \mathrm{d}\phi|_{\mathbf{0}} \in U(K)\}.$$

where $\mathrm{Jac}(\phi) := \det \mathrm{d}\phi$ is the jacobian of ϕ and where $U(K)$ is the group of linear transformations $(x, y) \mapsto (x, y+ax)$ for some $a \in K$. *Informally speaking* $\mathrm{Aut}_0\,\mathbb{A}^2_K$ is a subgroup of codimension 2. , $\mathrm{SAut}_0\,\mathbb{A}^2_K$ has codimension 3 and $\mathrm{Aut}_{U(K)}\,\mathbb{A}^2_K$ has codimension 5. Anyhow, they are large subgroups.

It was known that the Cremona group $Cr_2(\mathbb{C})$ and $\mathrm{Aut}\,\mathbb{A}^2_{\mathbb{C}}$ are not linear over a field, (S. Cantat, Y. Cornulier). For the large subgroups of $\mathrm{Aut}_0\,\mathbb{A}^2_K$, the linearity results are much more contrasted, as it is shown by the following

Theorem A. *(A.1) Whenever K is infinite, the group* $\mathrm{SAut}_0\,\mathbb{A}^2_K$ *is not linear, even over a ring.*

(A.2) However, there is an embedding $\mathrm{Aut}_{U(K)}\,\mathbb{A}^2_K \subset \mathrm{SL}(2, K(t)$.

For a finite field K, the index $[\operatorname{Aut}\mathbb{A}^2_K : \operatorname{Aut}_{U(K)}\mathbb{A}^2_K]$ is finite. Therefore

Corollary. *If K is finite, the group $\operatorname{Aut}\mathbb{A}^2_K$ is linear over the field $K(t)$.*

The existence of large linear subgroups in $\operatorname{Aut}\mathbb{A}^n_K$ is specific to the dimension 2. The case $n = 3$ is enough to show this. For a finite-codimensional ideal $\mathbf{m}$ of $K[x, y, z]$, let $\operatorname{Aut}_{\mathbf{m}}\mathbb{A}^3_K$ be the group of all automorphisms

$$(x, y, z) \mapsto (x + f, y + g, z + h),$$

where f, g and h belongs to $\mathbf{m}$. Equivalently, ϕ fixes some infinitesimal neighborhood of a finite subset in $\mathbb{A}^3_K$. However, even if K is finite, we have

Theorem B. *The group $\operatorname{Aut}_{\mathbf{m}}\mathbb{A}^3_K$ is not linear, even over a ring.*

We will now turn our attention to the small subgroups, namely the finitely generated (f.g. in the sequel) subgroups of $\operatorname{Aut}\mathbb{A}^2_K$. Let $\Gamma \subset \operatorname{Aut}\mathbb{A}^2_{\mathbb{Q}}$ be

$$\Gamma = \langle S, T\rangle, \text{ where } S(x, y) = (y, 2x) \text{ and } T(x, y) = (x, y + x^2).$$

Theorem C. *Let K be a field of characteristic zero.*

(C.1) The subgroup $\Gamma \subset \operatorname{Aut}_0\mathbb{A}^2_K$ is not linear, even over a ring.

(C.2) Any f.g. subgroup of $\operatorname{SAut}_0\mathbb{A}^2_K$ is linear over $K(t)$.

It turns out that the group Γ, which is presented by

$$\langle \sigma, \tau \mid \sigma^2\tau\sigma^{-2} = \tau^2\rangle,$$

appears in a paper of C.Drutu and M. Sapir as the first example of a 1-related group which is residually finite but not linear over a field.

Assume now that K is infinite. For $S \subset \mathrm{SL}(2, K)$, set

$$\operatorname{Aut}_S\mathbb{A}^2_K := \{\phi \in \operatorname{Aut}_0\mathbb{A}^2_K \mid \ d\phi|_{\mathbf{0}} \in S\},$$

Theorem A.2 suggests to ask

For a given subgroup $S \subset \mathrm{SL}(2, K)$, is the group $\operatorname{Aut}_S\mathbb{A}^2_K$ linear?

Three criteria provide an almost complete answer. Some examples of application of the criteria are

Example A. *Let q be a quadratic form on K^2 and $S = \mathrm{SO}(q)$.*

If q is anisotropic, $\operatorname{Aut}_S\mathbb{A}^2_K$ is linear over a field extension of K.

Otherwise $\operatorname{Aut}_S\mathbb{A}^2_K$ is not linear, even over a ring.

Example B. *For some lattices $S \subset \mathrm{SL}(2, \mathbb{R})$, $\operatorname{Aut}_S\mathbb{A}^2_{\mathbb{C}}$ is linear over $\mathbb{C}$.*

For any lattice $S \subset \mathrm{SL}(2, \mathbb{C})$, $\operatorname{Aut}_S\mathbb{A}^2_{\mathbb{C}}$ is not linear, even over a ring.