

RATIONAL SURFACES WITH A LARGE GROUP OF AUTOMORPHISMS

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ABSTRACT. We classify rational surfaces X for which the image of the automorphisms group $\text{Aut}(X)$ in the group of linear transformations of the Picard group $\text{Pic}(X)$ is the largest possible. This answers a question raised by Arthur Coble in 1928, and can be rephrased in terms of periodic orbits of birational actions of infinite Coxeter groups.

CONTENTS

1. Introduction	1
2. Halphen surfaces	6
3. Coble surfaces	16
4. Gizatullin's Theorem and Cremona special points set of nine points	28
5. The general case	31
6. Cremona special sets of $n \geq 10$ points lying on a cubic curve	41
7. Non algebraically closed fields and other surfaces	49
References	53

1. INTRODUCTION

Let \mathbb{K} be an algebraically closed field, and X be a projective surface defined over \mathbb{K} . The group of automorphisms $\text{Aut}(X)$ acts on the Néron-Severi group of X . This action preserves the intersection form and the canonical class K_X , and therefore provides a morphism from $\text{Aut}(X)$ to the group of integral isometries $\text{O}(K_X^\perp)$ of the orthogonal complement K_X^\perp . When X is rational, the image satisfies further constraints: It is contained in an explicit Coxeter subgroup W_X of $\text{O}(K_X^\perp)$, and W_X has infinite index in $\text{O}(K_X^\perp)$ as soon as the rank $\rho(X)$ of the Néron-Severi group of X exceeds 11.

A natural problem is to describe all projective surfaces X for which $\text{Aut}(X)$ is infinite and its image in this orthogonal or Coxeter group is of finite index. When \mathbb{K} is the field of complex numbers, the problem asks for a classification

of complex projective surfaces with maximal possible groups of isotopy classes of holomorphic diffeomorphisms.

We solve this problem when X is a rational surface. It turns out that this is the most interesting and difficult case. In Section 7.2 we briefly discuss other types of surfaces; one can treat them by more or less standard arguments.

1.1. Automorphisms of rational surfaces. Let X be a rational surface defined over \mathbb{K} . The Néron-Severi group of X coincides with the Picard group $\text{Pic}(X)$; its rank $\rho(X)$ is the Picard number of X . We denote by $\text{Aut}(X)^*$ the image of $\text{Aut}(X)$ in the orthogonal group $\text{O}(\text{Pic}(X))$. There are two alternative possibilities for $\text{Aut}(X)$ to be infinite.

The first occurs when the kernel $\text{Aut}(X)^0$ of the action of $\text{Aut}(X)$ on $\text{Pic}(X)$ is infinite. In this case, $\text{Aut}(X)^0$ is a linear algebraic group of positive dimension and $\text{Aut}(X)^*$ is a finite group (see [25]). All such examples are easy to describe because the surface X is obtained from a minimal rational surface by a sequence of $\text{Aut}(X)^0$ -equivariant blow-ups. Toric surfaces provide examples of this kind.

In the second case, the group $\text{Aut}(X)^*$ is infinite, $\text{Aut}(X)^0$ is finite, and then X is obtained from the projective plane \mathbb{P}^2 by blowing up a sequence of points p_1, \dots, p_n , with $n \geq 9$ (see [35]). The existence of such an infinite group $\text{Aut}(X)^*$ imposes drastic constraints on the points set $\mathcal{P} = \{p_1, \dots, p_n\}$ and leads to nice geometric properties of this set. There are classical examples of this kind as well as very recent constructions (see [20], [3], [31], [40]). Our goal is to classify points sets \mathcal{P} for which the group $\text{Aut}(X)^*$ is the largest possible, in a sense which we now make more precise.

1.2. The hyperbolic lattice. Let $\mathbb{Z}^{1,n}$ denote the standard odd unimodular lattice of signature $(1, n)$. It is generated by an orthogonal basis $(\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n)$ with

$$\mathbf{e}_0^2 = 1, \quad \text{and} \quad \mathbf{e}_i^2 = -1 \quad \text{for} \quad i \geq 1.$$

The orthogonal complement of the vector

$$k_n = -3\mathbf{e}_0 + (\mathbf{e}_1 + \dots + \mathbf{e}_n)$$

is a sublattice $\mathbb{E}_n \subset \mathbb{Z}^{1,n}$. A basis of \mathbb{E}_n is formed by the vectors

$$\boldsymbol{\alpha}_0 = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \quad \text{and} \quad \boldsymbol{\alpha}_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \quad i = 1, \dots, n-1.$$

The intersection matrix $(\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_j)$ is equal to $\Gamma_n - 2I_n$, where Γ_n is the incidence matrix of the graph $T_{2,3,n-3}$ from Figure 1. In particular, each class $\boldsymbol{\alpha}_i$ has self-intersection -2 and determines an involutive isometry of $\mathbb{Z}^{1,n}$ by

$$\mathbf{s}_i : x \mapsto x + (x \cdot \boldsymbol{\alpha}_i)\boldsymbol{\alpha}_i.$$

By definition, these involutions generate the Coxeter (or Weyl) group W_n .

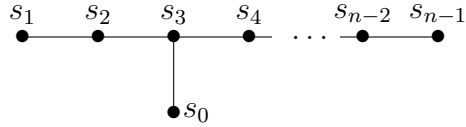


FIGURE 1. Coxeter-Dynkin diagram of type $T_{2,3,n-3}$

1.3. Automorphisms and Coxeter groups. From now on, X is a rational surface for which $\text{Aut}(X)^*$ is infinite. We write X as the blow-up of \mathbb{P}^2 at n points p_1, \dots, p_n with $n \geq 9$; some of them can be infinitely near points and, by convention, $j \geq i$ if p_j is infinitely near p_i . We now describe known constraints on the structure of the group $\text{Aut}(X)^*$.

A basis (e_0, \dots, e_n) of $\text{Pic}(X)$ is obtained by taking for e_0 the class of the total transform of a line in \mathbb{P}^2 , and for e_i , $1 \leq i \leq n$, the class of the total transform of the exceptional divisor obtained by blowing up p_i ; in particular, the Picard number $\rho(X)$ is equal to $n + 1$. This basis is orthogonal with respect to the intersection form: $e_0^2 = 1$, $e_i^2 = -1$ for $i \geq 1$, and $e_i \cdot e_j = 0$ if $i \neq j$. We call such a basis of $\text{Pic}(X)$ a **geometric basis**. A geometric basis makes $\text{Pic}(X)$ isometric to the lattice $\mathbb{Z}^{1,n}$, by an isometry which maps e_i to \mathbf{e}_i . Under this isomorphism, the canonical class

$$K_X = -3e_0 + e_1 + \dots + e_n$$

is mapped to the element $k_n \in \mathbb{Z}^{1,n}$. We denote by (α_i) the basis of K_X^\perp corresponding to $(\mathbf{\alpha}_i)$ under this isomorphism, i.e.

$$(1) \quad \alpha_0 = e_0 - e_1 - e_2 - e_3, \alpha_1 = e_1 - e_2, \dots, \alpha_{n-1} = e_{n-1} - e_n,$$

and by s_i the involutive isometry of $\text{Pic}(X)$ which is conjugate to s_i . By definition, the group W_X is the group of isometries of $\text{Pic}(X)$ generated by these n involutions; thus, W_X is isomorphic to the Coxeter group W_n . It is known that the group W_X does not depend on a choice of a geometric basis (see [17], Theorem 5.2, page 27).

The group $\text{Aut}(X)^*$ acts by isometries on $\text{Pic}(X)$ and preserves the canonical class. According to Kantor-Nagata's theorem (see [27], Theorem XXXIII, [35], p. 283, or [17], Theorem 5.2), the group $\text{Aut}(X)^*$ is contained in W_X . Thus, we get a series of inclusions

$$\text{Aut}(X)^* \subset W_X \subset \text{O}(\text{Pic}(X); K_X) \subset \text{O}(\text{Pic}(X)),$$

where $\text{O}(\text{Pic}(X))$ is the orthogonal group of $\text{Pic}(X)$ with respect to the intersection form, and $\text{O}(\text{Pic}(X); K_X)$ is the stabilizer of the canonical class K_X .

1.4. Cremona special points sets. When $n \leq 8$, W_n is a finite group. We say that the points set $\mathcal{P} := \{p_1, \dots, p_n\}$ is **Cremona special** if $n \geq 9$ and the group $\text{Aut}(X)^*$ has finite index in W_X .¹ A rational surface obtained by blowing up a Cremona special set will be called Cremona special. In this sense, Cremona special surfaces with fixed Picard number are rational surfaces with largest possible discrete automorphism groups among all rational surfaces with the same rank of the Picard group.

Our goal is to classify Cremona special points sets, a problem that is already mentioned by Arthur Coble in his book [10], p. 278 (see also [20] and [25]).

Two kinds of such sets are known since the beginning of the last century. They are general Halphen sets of 9 points and general Coble sets of 10 points (see [10]). Brian Harbourne showed in [24] that, in characteristic $p > 0$, and for any integer $n \geq 9$, a general set of n nonsingular points on an irreducible cuspidal cubic curve is Cremona special; for this, he employed the fact that all such points are p -torsion points in the group law on the set of nonsingular points on the cubic curve. When $n = 9$, Harbourne sets are particular cases of Halphen sets.

We discuss the geometry of Halphen and Coble points sets in Sections 2 and 3, prove that the general ones are indeed Cremona special, and describe precisely what “general” means in this context. By definition, the points set is **unnodal** if its blow-up does not contain smooth rational curves with self-intersection equal to -2 (also called nodal or (-2) -curves). This terminology is borrowed from the theory of Enriques surfaces, where it is known that the isomorphism classes of unnodal surfaces form an open subset in the moduli space. A surface obtained by blowing up an unnodal set is called unnodal. We show that unnodal sets form an open Zariski subset in the variety of points sets defining Halphen and Coble surfaces, and that Cremona special Halphen and Coble surfaces are exactly the unnodal ones.

Harbourne examples are defined at the beginning of Section 5 and are discussed in Section 6.1.

Then, our main result shows that the examples constructed by Halphen, Coble, and Harbourne exhaust all possibilities of Cremona special points sets. As a corollary, if a points set \mathcal{P} is Cremona special, then it is unnodal.

Main Theorem *Let \mathbb{K} be an algebraically closed field. Let \mathcal{P} be a Cremona special points set in $\mathbb{P}_{\mathbb{K}}^2$. Then \mathcal{P} is unnodal and one of the following cases occurs*

- $n = 9$ and \mathcal{P} is a Halphen set;

¹Coble called such subsets *special*, so we somewhat deviate from his terminology.

- $n = 10$ and \mathcal{P} is a Coble set;
- $n \geq 10$, $\text{char}(\mathbb{K}) > 0$, and \mathcal{P} is a Harbourne set.

Conversely, any such unnodal set is Cremona special.

As a corollary, if a rational surface X is Cremona special, then $-K_X$ or $-2K_X$ is effective.

Remark 1.1. a.– As explained in Section 7, there is a stronger version of this theorem which does not assume that \mathbb{K} is algebraically closed, but this requires a careful definition of Cremona special points sets. Non rational surfaces are dealt with in Section 7.2.

1.1. b.– When W_X is infinite, it is Zariski dense in the real algebraic group $O(K_X^\perp \otimes \mathbb{R})$. Thus, a natural question is the following. If X is a rational surface and $\text{Aut}(X)^*$ is infinite and Zariski dense in $O(K_X^\perp \otimes \mathbb{R})$, does it follow that X is Cremona special? In other words, is it possible to generalize our Main Theorem under the weaker assumption that $\text{Aut}(X)^*$ is infinite and is Zariski dense in $O(K_X^\perp \otimes \mathbb{R})$? Since W_9 contains a finite index, free abelian group of rank 8, every Zariski dense subgroup of W_9 has finite index. Thus, the problem concerns rational surfaces with Picard number at least 10.

1.1. c.– There is a notion of Cremona special points sets in projective spaces of higher dimension and in their Cartesian products. Interesting examples of such sets are known (see [10], [20], [19]). Unfortunately, the methods of this paper are specific to dimension 2 and do not extend to the higher-dimensional case.

1.5. An action of W_n on points sets and its periodic orbits. Consider the variety $(\mathbb{P}^2)^n$ and the diagonal action of PGL_3 on it. Consider the GIT-quotient \mathbb{P}_n^2 of the action. It turns out that the group W_n acts on \mathbb{P}_n^2 by birational transformations; this **Cremona action** is described in chapter VI of [20].

Let Γ be a subgroup of W_n . Let (p_1, \dots, p_n) be an ordered stable set of distinct points representing a point $\mathfrak{p} \in \mathbb{P}_n^2$. Let $X_{\mathfrak{p}}$ be the surface obtained by blowing up the projective plane at p_1, \dots, p_n ; its isomorphism class depends only on \mathfrak{p} . The group $\text{Pic}(X_{\mathfrak{p}})$ is isomorphic to $\mathbb{Z}^{1,n}$, with an isomorphism depending only on \mathfrak{p} ; we fix such an isomorphism, and the corresponding isomorphism between W_X and W_n .

If \mathfrak{p} is contained in the domain of definition of γ and $\gamma(\mathfrak{p}) = \mathfrak{p}$ for all γ in Γ , then there exists a subgroup $\Gamma' \subset \text{Aut}(X_{\mathfrak{p}})$ such that the action of Γ' on $\text{Pic}(X_{\mathfrak{p}})$ and the identification $W_n \sim W_{X_{\mathfrak{p}}}$ provide an isomorphism $\Gamma' \rightarrow \Gamma$. In other words, *points $\mathfrak{p} \in \mathbb{P}_n^2$ which are fixed by the group Γ correspond to rational surfaces on which Γ is represented by a subgroup of $\text{Aut}(X_{\mathfrak{p}})$.*

Thus, our Main Theorem *classifies periodic orbits of the group W_n , for $n \geq 9$* , i.e. for infinite Coxeter groups W_n . This provides a dynamical interpretation of the Main Theorem in terms of birational actions of Coxeter groups.

Remark 1.2. There are other nice examples of algebraic dynamical systems for which periodic orbits are related to the construction of interesting geometric objects.

One of them is given by Thurston's pull back map. If $F : S^2 \rightarrow S^2$ is a (topological) orientation preserving branched covering map of the sphere S^2 with a finite post-critical set² \mathcal{P}_F of cardinality n , one can ask whether F is equivalent to a holomorphic endomorphism f of the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, in the sense that

$$F = \phi \circ f \circ \psi$$

where ϕ and ψ are homeomorphisms which are isotopic relatively to \mathcal{P}_F . The map σ_F defined by Thurston acts on the Teichmüller space of S^2 with n -marked points; fixed points of σ_F correspond to holomorphic structures on the sphere for which F is realized by an endomorphism f . This situation is similar to the one studied here, with σ_F in place of W_n and the Teichmüller space replacing \mathcal{P}_n^2 . We refer to [22] for a precise description of Thurston's construction.

Another similar situation, with the mapping class group of a surface Σ (in place of W_n) acting on the character variety of the fundamental group $\pi_1(\Sigma)$ (in place of \mathcal{P}_n^2) is related to hyperbolic structures on three dimensional manifolds, and to algebraic solutions of Painlevé sixth equation (see [7] and references therein).

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2. HALPHEN SURFACES

In this section, we describe Halphen surfaces, Halphen pencils of genus 1 curves, and their associated points sets. We then show that unnodal Halphen points sets are Cremona special. Most results in this section are known to experts, but may be hard to find in the literature, and will be used in the following sections.

²The post-critical set is the union of the images of the set of critical points of f under positive iterations of f .

We assume some familiarity with the theory of elliptic fibrations over fields of arbitrary characteristic and refer to [12], Chapter V, for this topic.

2.1. Halphen surfaces of index m . By definition, a $(-n)$ -**curve** on a smooth projective surface X is a smooth rational curve with self-intersection $-n$. The genus formula shows the following.

Lemma 2.1. *Let X be a smooth projective surface. Let n and l be positive integers.*

- (1) *Assume $-K_X$ is nef. If E is a $(-n)$ -curve, then $n = 1$, or $n = 2$ and $E \cdot K_X = 0$.*
- (2) *Assume that the linear system $|-lK_X|$ contains a reduced, irreducible curve C with $C^2 < 0$. If E is a $(-n)$ -curve, then $n = 1$, or $n = 2$ and $E \cdot C = 0$, or $E = C$.*

A smooth rational projective surface X is a **Halphen surface** if there exists an integer $m > 0$ such that the linear system $|-mK_X|$ is of dimension 1, has no fixed component, and has no base point. The **index** of a Halphen surface is the smallest possible value for such a positive integer m .

Let X be a Halphen surface of index m . Then $K_X^2 = 0$ and, by the genus formula, the linear system $|-mK_X|$ defines a genus 1 fibration $f : X \rightarrow \mathbb{P}^1$, which is elliptic or quasi-elliptic if $\text{char}(\mathbb{K}) = 2$ or 3 . This fibration is **relatively minimal** in the sense that there is no (-1) -curve contained in a fiber.

Proposition 2.2. *Let X be a smooth projective rational surface. Let m be a positive integer. The following four properties are equivalent:*

- (i) *X is a Halphen surface of index m ;*
- (ii) *$|-K_X|$ is nef and contains a curve F_0 such that $\mathcal{O}_{F_0}(F_0)$ is of order m in $\text{Pic}(F_0)$;*
- (iii) *there exists a relatively minimal elliptic or quasi-elliptic fibration $f : X \rightarrow \mathbb{P}^1$; it has no multiple fibers when $m = 1$ and a unique multiple fiber, of multiplicity m , when $m > 1$;*
- (iv) *there exists an irreducible pencil of curves of degree $3m$ with 9 base points of multiplicity m in \mathbb{P}^2 , such that X is the blow-up of the 9 base points and $|-mK_X|$ is the proper transform of this pencil (the base points set may contain infinitely near points).*

In the proof of (iii) \Rightarrow (iv) below, the classification of minimal rational surfaces is used. Recall that a minimal rational surface is isomorphic to \mathbb{P}^2 or to

one of the Segre-Hirzebruch surfaces $\mathbf{F}_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-n))$,³ with $n \geq 0$ and $n \neq 1$. If $n = 0$, the surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and, if $n = 1$, the surface is not minimal since it is isomorphic to the blow-up of \mathbb{P}^2 at one point. For all $n \geq 1$ there is a unique irreducible curve on \mathbf{F}_n with negative self-intersection (equal to $-n$). It is defined by a section of the \mathbb{P}^1 -bundle $\mathbf{F}_n \rightarrow \mathbb{P}^1$ corresponding to the surjection $\mathcal{O} \oplus \mathcal{O}(-n) \rightarrow \mathcal{O}(-n)$ ([26], §V.2).

Proof of Proposition 2.2. Under assumption (i), Riemann-Roch formula on a rational surface and Serre's Duality

$$h^0(D) + h^0(K_X - D) = h^1(D) + \frac{1}{2}D \cdot (D - K_X) + 1$$

and $K_X^2 = 0$ imply that $h^0(-K_X) > 0$. Let F_0 be an element of the linear system $|-K_X|$.

We now prove (i) \Leftrightarrow (ii). The exact sequence

$$(2) \quad 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(nF_0) \rightarrow \mathcal{O}_{F_0}(nF_0) \rightarrow 0$$

together with $h^1(X, \mathcal{O}_X) = 0$, because X is rational, show that

$$h^0(\mathcal{O}_X(nF_0)) = 1 + h^0(\mathcal{O}_{F_0}(nF_0)).$$

Since F_0 is a nef divisor and $F_0^2 = 0$, the restriction of $\mathcal{O}_X(nF_0)$ to each irreducible component of F_0 is an invertible sheaf of degree zero. The curve F_0 is of arithmetic genus 1, so we can apply the Riemann-Roch Theorem on F_0 (see [33], Lecture 11) to conclude that $h^0(\mathcal{O}_{F_0}(nF_0)) > 0$ if and only if $\mathcal{O}_{F_0}(nF_0) \cong \mathcal{O}_{F_0}$, if and only if $h^0(\mathcal{O}_{F_0}(nF_0)) = 1$.

This shows that the index m can be characterized by the property

$$m = \min\{n : h^0(\mathcal{O}_X(-nK_X)) = 2\} = \min\{n : h^0(\mathcal{O}_{F_0}(nF_0)) > 0\},$$

and the equivalence (i) \Leftrightarrow (ii) follows from this characterization.

(i) \Rightarrow (iii) The pencil $|-mK_X|$ defines a morphism $f : X \rightarrow \mathbb{P}^1$ with general fiber of arithmetic genus 1. The generic fiber X_η is an irreducible curve of arithmetic genus 1 over the field $\mathbb{K}(\eta)$ of rational functions on the curve \mathbb{P}^1 . Since X is smooth, X_η is a regular curve over $\mathbb{K}(\eta)$. It is known that it is smooth if $\text{char}(\mathbb{K}) \neq 2, 3$, so that in this case f is an elliptic fibration (see [12], Proposition 5.5.1). If it is not smooth, then a general fiber of f is an irreducible cuspidal curve, so that f is a quasi-elliptic fibration. As explained above (just before Proposition 2.2) the fibration f is relatively minimal.

Since X is a rational non-minimal surface, there exists a (-1) -curve E on X satisfying $E \cdot K_X = -1$. This shows that K_X is a primitive divisor class,

³Here we adopt Grothendieck's definition of the projective bundle associated to a locally free sheaf.

i.e. K_X is not a multiple of any other divisor class. If $m = 1$, this implies that there are no multiple fibers. If $m > 1$, this implies that the multiplicity n of any multiple fiber nD divides m . Since $|mF_0|$ is a multiple fiber of multiplicity m , the class of the divisor $\frac{m}{n}F_0 - D$ is a torsion element in the Picard group of X . Since X is a rational surface, this class must be trivial, and we conclude that f has a unique multiple fiber, namely mF_0 .

(iii) \Rightarrow (iv) Since the fibration f is relatively minimal, the canonical class is proportional to the class of the fibers of f (see [1], §V.12); in particular, $K_X^2 = 0$ and $-K_X$ is nef.

Let $\pi : X \rightarrow Y$ be a birational morphism to a minimal ruled surface. Suppose that Y is not isomorphic to \mathbb{P}^2 ; then Y is isomorphic to a surface \mathbf{F}_n , $n \neq 1$. Let E_0 be the section of Y with $E_0^2 = -n$ and let E be its proper transform on X . We have $E^2 \leq -n$ and $E^2 = -n$ if and only if π is an isomorphism in an open neighborhood of E_0 . Since $-K_X$ is nef, Lemma 2.1 shows that $n = 0$ or $n = 2$. Assume $n = 2$. Then π is an isomorphism over E_0 , hence it factors through a birational morphism $\pi : X \rightarrow X_1$, where X_1 is the blow-up of \mathbf{F}_2 at a point $x \notin E_0$. Let $X_1 \rightarrow \mathbf{F}_1$ be the blow-down of the fiber of the ruling $\mathbf{F}_2 \rightarrow \mathbb{P}^1$ passing through x . Then we obtain a birational morphism $X \rightarrow \mathbf{F}_1 \rightarrow \mathbb{P}^2$. Assume now that $n = 0$. The morphism factors through $X \rightarrow X_2$, where X_2 is the blow-up of a point y on \mathbf{F}_0 . Then we compose $X \rightarrow X_2$ with the birational morphism $X_2 \rightarrow \mathbb{P}^2$ which is given by the stereographic projection of \mathbf{F}_0 onto \mathbb{P}^2 from the point y . As a consequence, changing Y and π , we can always assume that $Y = \mathbb{P}^2$.

Since $K_X^2 = 0$, the morphism $\pi : X \rightarrow \mathbb{P}^2$ is the blow-up of 9 points p_1, \dots, p_9 , where some of them may be infinitely near. Since $-K_X$ is nef, any smooth rational curve has self-intersection ≥ -2 . This implies that the set of points $\{p_1, \dots, p_9\}$ can be written in the form

$$(3) \quad \{p_1^{(1)}, p_1^{(2)}, \dots, p_1^{(a_1)}; \dots; p_k^{(1)}, p_k^{(2)}, \dots, p_k^{(a_k)}\},$$

where the $p_j^{(1)}$ are points in \mathbb{P}^2 , and $p_j^{(b+1)}$ is infinitely near, of the first order, to the previous point $p_j^{(b)}$ for $j = 1, \dots, k$ and $b = 1, \dots, a_j - 1$. Equivalently, the exceptional curve

$$E_j = \pi^{-1}(p_j^{(1)})$$

is a chain of (-2) -curves of length $(a_j - 1)$ with one more (-1) -curve at the end of the chain.

The formula for the canonical class of the blow-up of a nonsingular surface at a closed point shows that

$$(4) \quad K_X = -3e_0 + e_1 + \dots + e_9,$$

where $e_0 = c_1(\pi^*(\mathcal{O}_{\mathbb{P}^2}(1)))$ and e_j is the divisor class of E_j , $j = 1, \dots, 9$. This implies that

$$|-mK_X| = |3me_0 - m(e_1 + \dots + e_9)|,$$

hence the image of the pencil $|-mK_X|$ in the plane is the linear system of curves of degree $3m$ with singular points of multiplicity m at $p_i^{(1)}$, $1 \leq j \leq k$.

(iv) \Rightarrow (i) Let X be the blow-up of the base points of the pencil. The proper transform of the pencil on X is the linear system $|3me_0 - m(e_1 + \dots + e_9)|$. The formula for the canonical class on X shows that this system is equal to $|-mK_X|$. Since the pencil is irreducible, $|-mK_X|$ is a pencil with no fixed component and no base point, so X is a Halphen surface. \square

Remark 2.3. The proof of the proposition shows that the multiplicity m of the multiple fiber mF_0 of the genus one fibration is equal to the order of $\mathcal{O}_{F_0}(F_0)$ in $\text{Pic}(F_0)$. This property characterizes non-wild fibers of elliptic fibrations (see [12], Proposition 5.1.5). It is a consequence of the vanishing of $H^1(X, \mathcal{O}_X)$. It always holds if the multiplicity is prime to the characteristic.

2.2. Halphen pencils of index m . The following lemma is well-known and its proof is left to the reader.

Lemma 2.4. *Let $\phi : S' \rightarrow S$ be the blow-up of a point x on a smooth projective surface S and let C' be the proper transform of a curve passing through x with multiplicity 1. Then $\mathcal{O}_{C'}(C' + E) \cong (\phi|_{C'})^*\mathcal{O}_C(C)$, where $E = \phi^{-1}(x)$ is the exceptional divisor.*

In the plane \mathbb{P}^2 , an irreducible pencil of elliptic curves of degree $3m$ with 9 base points of multiplicity m is called a **Halphen pencil of index m** . If C_0 is a cubic curve through the base points, then C_0 is the image of a curve $F_0 \in |-K_X|$; such a curve is unique if $m > 1$ and moves in the pencil if $m = 1$.

The classification of fibers of genus 1 fibrations shows that $\mathcal{O}_{F_0}(F_0) \not\cong \mathcal{O}_{F_0}$ implies that F_0 is a reduced divisor of type I_m in Kodaira's notation, unless $\text{char}(\mathbb{K})$ divides m (see [12], Proposition 5.1.8). We further assume that F_0 is irreducible if $m > 1$; this will be enough for our applications. Thus F_0 is a smooth or nodal curve, unless the characteristic of \mathbb{K} divided m in which case it could be a cuspidal curve. Under this assumption, the restriction of π to F_0 is an isomorphism $F_0 \cong C_0$; in particular, no base point $p_i^{(j)}$ is a singular point of C_0 .

In the notation of Equation (3), consider the divisor class in $\text{Pic}(C_0)$ given by

$$\mathfrak{d} = 3\mathfrak{h} - a_1p_1^{(1)} - \dots - a_kp_k^{(1)},$$

where \mathfrak{h} is the intersection of C_0 with a line in the plane. Since $\mathcal{O}_{C_0}(C_0) \cong \mathcal{O}_{C_0}(3\mathfrak{h})$, Lemma 2.4 gives

$$\mathcal{O}_{F_0}(F_0) \cong \mathcal{O}_{F_0}(3e_0 - e_1 - \cdots - e_9) \cong (\pi|_{F_0})^*(\mathcal{O}_{C_0}(\mathfrak{d})).$$

This implies that $\mathcal{O}_{F_0}(F_0)$ is of order m in $\text{Pic}(F_0)$ if and only if \mathfrak{d} is of order m in $\text{Pic}(C_0)$. If we choose the group law \oplus on the set $C_0^\#$ of regular points of C_0 with a nonsingular inflection point o as the zero point, then the latter condition is equivalent to

$$(5) \quad a_1 p_1^{(1)} \oplus \cdots \oplus a_k p_k^{(1)} = \epsilon_m$$

where ϵ_m is a point of order m in the group $(C_0^\#, \oplus)$.

This provides a way to construct Halphen pencils and the corresponding Halphen surfaces (under our assumptions that F_0 is irreducible). Start with an irreducible plane cubic C_0 , and choose k points $p_1^{(1)}, \dots, p_k^{(1)}$ in $C_0^\#$ satisfying Equation (5) with $a_1 + \dots + a_k = 9$. Then blow up the points $p_1^{(1)}, \dots, p_k^{(1)}$ together with infinitely near points $p_j^{(i)}, i = 2, \dots, a_j$, to arrive at a rational surface $\pi : X \rightarrow \mathbb{P}^2$. Then $|-K_X| = |F_0|$, where F_0 is the proper transform of C_0 . Since the $p_i^{(1)}$ satisfy Equation (5), $\mathcal{O}_{F_0}(F_0) \cong (\pi|_{F_0})^* \mathcal{O}_{C_0}(\epsilon_m)$ is of order m in $\text{Pic}(F_0)$. Since F_0 is irreducible, $|-mK_X| = |mF_0|$ is nef. Consequently, Proposition 2.2 shows that X is a Halphen surface.

2.3. Unnodal Halphen surfaces. By Lemma 2.1, Halphen surfaces contain no $(-n)$ -curves with $n \geq 3$. Recall that a Halphen surface is **unnodal** if it has no (-2) -curves. Since a (-2) -curve R satisfies $R \cdot K_X = 0$, it must be an irreducible component of a fiber of the genus 1 fibration $f : X \rightarrow \mathbb{P}^1$. Conversely, all reducible fibers of f contain (-2) -curves, because f is a relatively minimal elliptic fibration. Thus X is unnodal if and only if all members of the pencil $|-mK_X|$ are irreducible.

In this case all the curves E_i are (-1) -curves; in particular, there are no infinitely near points in the Halphen set. Also, in this case, the morphism $f : X \rightarrow \mathbb{P}^1$ is an elliptic fibration because any quasi-elliptic fibration on a rational surface has a reducible fiber whose irreducible components are (-2) -curves (this follows easily from [12], Proposition 5.1.6, see the proof of Theorem 5.6.3). The fibers of f are irreducible curves of arithmetic genus 1. This shows that a Halphen surface is unnodal if and only if it arises from a Halphen pencil with irreducible members.

Proposition 2.5. *Let X be a Halphen surface of index m . Then X is unnodal if and only if the following conditions are satisfied.*

- (i) *There is no infinitely near point in the Halphen set $\mathcal{P} = \{p_1, \dots, p_9\}$;*

(ii) *the divisor classes*

$$\begin{aligned} & -dK_X + e_i - e_j, \quad i \neq j, \quad 0 \leq 2d \leq m, \\ & -dK_X \pm (e_0 - e_i - e_j - e_k), \quad i < j < k, \quad 0 \leq 2(3d \pm 1) \leq 3m, \end{aligned}$$

are not effective.

Remark 2.6. Since $K_X = -3e_0 + \sum e_i$, the two types of divisor classes in condition (ii) are equal to

$$\begin{aligned} & 3de_0 - d(e_1 + \cdots + e_9) + e_i - e_j, \quad i \neq j, \\ & 3de_0 - d(e_1 + \cdots + e_9) \pm (e_0 - e_i - e_j - e_k), \quad i < j < k. \end{aligned}$$

Example 2.7. When $m = 1$, the inequalities $2d \leq m$ and $2(3d \pm 1) \leq 3m$ lead to $d = 0$ and the conditions are respectively redundant with (i), or exclude triples of collinear points in the set $\{p_1, \dots, p_9\}$.

When $m = 2$, the inequality $2d \leq m$ reads $d = 0$ or 1 , and we have to exclude a cubic through 8 points with a double point at one of them. The inequality $2(3d \pm 1) \leq 3m$ gives rise to curves of degree 1 (for $d = 0$) and degree 2 (for $d = 1$): We have to exclude triples of collinear points, and sets of six points on a conic.

Proof of Proposition 2.5. Suppose the conditions are satisfied. Since no class $e_i - e_j$ is effective, the exceptional curves E_i are (-1) -curves. Thus the morphism $\pi : X \rightarrow \mathbb{P}^2$ is the blow-up of 9 points, none of which is infinitely near another. This implies that π does not contract any component of a member of $|-mK_X|$. Let R be a (-2) -curve on X . It must be an irreducible component of a member of $|-mK_X|$, hence $\bar{R} = \pi(R)$ is an irreducible component of a curve of degree $3m$. Taking a complementary component, we may assume that

$$(6) \quad 2 \deg(\bar{R}) \leq 3m.$$

The divisor class $r = [R]$ belongs to K_X^\perp and satisfies $r^2 = -2$. Since $(r + dK_X)^2 = -2$, we can change r into $r' = r + dK_X$ in such a way that $r' = d'e_0 - k'_1e_1 - \cdots - k'_9e_9$ with $|d'| \leq 1$. All such vectors r' can be listed:

$$(7) \quad r' = e_i - e_j, \quad i \neq j, \quad \text{or} \quad r' = \pm(e_0 - e_i - e_j - e_k), \quad i < j < k.$$

Thus

$$\begin{aligned} r &= -dK_X + e_i - e_j, \quad \text{or} \\ r &= -dK_X \pm (e_0 - e_i - e_j - e_k), \quad i < j < k. \end{aligned}$$

Since $\deg \bar{R} = r \cdot e_0 > 0$ and $-K_X \cdot e_0 = 3$, the curve \bar{R} is of degree $3d$ in the first case, and of degree $3d \pm 1$ in the second case. Thus, inequality (6) shows

that $2d \leq m$ (resp. $2(3d \pm 1) \leq 3m$). From (ii), we deduce that \bar{R} and R do not exist, and that X is general.

Conversely, if one of these divisor classes is effective, then X contains a (-2) -curve. This proves the proposition. \square

Remark 2.8. Let $f : X \rightarrow \mathbb{P}^1$ be a Halphen elliptic surface of index $m > 1$. The generic fiber X_η is a genus one curve over the field $\mathbb{K}(\eta)$ which has no rational point over this field. The Jacobian variety $\text{Jac}(X_\eta)$ of X_η (equal to the connected component of the identity of the Picard scheme of X_η over η) is an abelian variety of dimension 1 over $\mathbb{K}(\eta)$. Applying the theory of relative minimal models one can construct an elliptic surface $j : J \rightarrow \mathbb{P}^1$ with generic fibers J_η isomorphic to $\text{Jac}(X_\eta)$. It is called the Jacobian elliptic surface of $X \rightarrow \mathbb{P}^1$. In our case, this elliptic surface is a Halphen surface of index 1. The curve X_η is a torsor over J_η . Its class in the group of isomorphism classes of torsors over J_η is uniquely determined by a choice of a closed point $y \in \mathbb{P}^1$ and an element α of order m in $\text{Pic}(J_y)$. The fiber $X_y = mF_y$ is the unique multiple fiber of f , the curves F_y and J_y can be canonically identified and the isomorphism class of $\mathcal{O}_{F_y}(F_y)$ coincides with α . Since Halphen surfaces of index 1 are parameterized by an open subset of the Grassmannian $G(2, 10)$ of pencils of plane cubic curves, their moduli space is an irreducible variety of dimension 8. The construction of the Jacobian surface shows that the moduli space of Halphen surfaces of index $m > 1$ is a fibration over the moduli space of Halphen surfaces of index 1 with one-dimensional fibers. It is expected to be an irreducible variety of dimension 9.

Remark 2.9. It follows from Proposition 2.5 that unnodal Halphen sets of given index form a proper Zariski open subset in the set of all Halphen sets of this index. So, one can say that unnodal Halphen sets or the corresponding Halphen surfaces are **general** in the sense of moduli.

2.4. Automorphisms of a Halphen surface. We now discuss a result of Coble (see a modern proof in [23]) which describes the automorphism group of an unnodal Halphen surface X and its image in the group W_X .

Let X be a Halphen surface of index m . Since the group $\text{Aut}(X)$ preserves the canonical class K_X , it preserves the linear system $| -mK_X |$ and permutes the fibers of the Halphen fibration $f : X \rightarrow \mathbb{P}^1$. As explained in the Introduction, see Section 1.3, we identify $\text{Pic}(X)$ with $\mathbb{Z}^{1,9}$ and W_X with the Coxeter group W_9 . The lattice $\mathbb{E}_9 \cong K_X^\perp$ is isomorphic to the root lattice of affine type E_8 ; the radical of \mathbb{E}_9 is generated by the vector k_9 and the lattice $\mathbb{E}_8 \cong \mathbb{E}_9 / \mathbb{Z}k_9$ is isomorphic to the root lattice of finite type E_8 . Consequently, the Weyl group

W_9 is isomorphic to the affine Weyl group of type E_8 , and fits in the extension

$$(8) \quad 0 \rightarrow \mathbb{E}_8 \xrightarrow{\iota} W_9 \rightarrow W_8 \rightarrow 1,$$

where $\iota : \mathbb{E}_8 \rightarrow W_9$ is defined by the formula

$$(9) \quad \iota(w)(v) = v + (v, k_9)w - \left((w, v) + \frac{1}{2}(v, k_9)(w, w) \right) k_9,$$

and W_8 is a finite group of order $2^7 \cdot 3^3 \cdot 5 \cdot 8!$.⁴

The following theorem shows that the size of $\text{Aut}(X)$ depends on the existence of reducible fibers for f (see [23] for a more precise statement). We identify the lattice \mathbb{E}_8 with $K_X^\perp/\mathbb{Z}K_X$ and the map ι with the homomorphism $K_X^\perp/\mathbb{Z}K_X \rightarrow W_X$ defined by $\iota(D)(A) = A - (A \cdot D)K_X$.

Theorem 2.10. *Let X be a Halphen surface of index m . If X is unnodal, then $\text{Aut}(X)^*$ contains a subgroup G whose image in W_9 is equal to $\iota(m\mathbb{E}_8) \subset \iota(\mathbb{E}_8)$; in particular, an unnodal Halphen set is Cremona special. If X is not unnodal then the index of $\text{Aut}(X)^*$ in W_X is infinite, and thus X is not Cremona special.*

Proof. Let $f : X \rightarrow \mathbb{P}^1$ be the elliptic fibration defined by $|-mK_X|$ and X_η be its generic fiber. Assume X is unnodal. Then X_η is a smooth elliptic curve over the field $\mathbb{K}(\eta)$ of rational functions on \mathbb{P}^1 (see the beginning of Section 2.3). The closure in X of a closed point $\zeta \in X_\eta$ of degree $d(\zeta) = [\mathbb{K}(\zeta) : \mathbb{K}(\eta)]$ is an irreducible curve $\bar{\zeta}$ such that the restriction of π to the curve is a finite cover of degree $d(\zeta)$. Conversely, any irreducible curve C of relative degree d (a d -multi-section) intersects X_η at a closed point of degree d . In the case of an unnodal Halphen surface, any exceptional curve E_i is of relative degree m and the degree of any closed point on X_η is a multiple of m . In particular, X_η has a structure of an abelian variety only if $m = 1$.

Let $J_\eta = \text{Pic}_{X_\eta/\eta}^0$ be the Jacobian variety of X_η . Now it is an abelian curve which acts on $X_\eta \cong \text{Pic}_{X_\eta/\eta}^1$ by translations. In particular, $J_\eta(\eta)$ acts by η -automorphisms on X_η . Explicitly, the action is defined by the formula

$$(10) \quad t_{\mathfrak{a}}(x) = x' \sim x + \mathfrak{a}, \quad \mathfrak{a} \in J_\eta(\eta),$$

where x is a closed point of X_η over the algebraic closure of $\mathbb{K}(\eta)$. Since \mathfrak{a} is defined over $\mathbb{K}(\eta)$, this guarantees that the action is defined over η .

It follows from the theory of relative minimal models that the action of $J_\eta(\eta)$ on X_η extends to a biregular action of X/\mathbb{P}^1 . Thus we obtain an injective morphism

$$J_\eta(\eta) \hookrightarrow \text{Aut}(X).$$

⁴It is an exercise to check that $\iota(w + w') = \iota(w) \circ \iota(w')$.

By taking the closure of a divisor on X_η , we have a natural surjective restriction homomorphism

$$\mathrm{tr} : K_X^\perp \rightarrow J_\eta(\eta).$$

Its kernel is equal to the subgroup $\mathrm{Pic}_{\mathrm{fib}}(X)$ of $\mathrm{Pic}(X)$ generated by irreducible components of fibers. If X is unnodal, $\mathrm{Pic}_{\mathrm{fib}}(X) = \mathbb{Z}K_X$, hence $J_\eta(\eta) \cong \mathbb{Z}^8$. This proves that $\mathrm{Aut}(X)$ contains a subgroup isomorphic to \mathbb{Z}^8 .

It remains to find out how this abelian group $J_\eta(\eta)$ acts on $\mathrm{Pic}(X)$. Any effective divisor D on X is a sum of irreducible components of fibers and the rest, which we call the **horizontal part**. The restriction of the fibration $f : X \rightarrow \mathbb{P}^1$ to the horizontal part is a finite cover of degree equal to

$$\deg_f(D) = -mD \cdot K_X.$$

We say that an effective divisor is **separable** if its horizontal part is a separable cover of \mathbb{P}^1 . If $\mathrm{char}(\mathbb{K})$ does not divide m , any effective divisor is separable.

For any divisor class D on X let $\mathrm{tr}(D)$ be its scheme-theoretical intersection with the generic fiber X_η . If D is separable and irreducible, then $\mathrm{tr}(D)$ is a closed point of X_η which is equal to the sum of $\deg_f(D)$ closed points over the algebraic closure of $\mathbb{K}(\eta)$. Thus $t_{\mathfrak{a}}(\mathrm{tr}(D)) = \mathrm{tr}(D) + \deg_f(D)\mathfrak{a}$. Let A be an element of K_X^\perp such that $\mathrm{tr}(A) = \mathfrak{a}$. Then

$$t_{\mathfrak{a}}(D) \sim D - m(D \cdot K_X)A \pmod{\mathrm{Pic}_{\mathrm{fib}}(X)}.$$

At this point, we assume that X is unnodal, so that

$$t_{\mathfrak{a}}(D) \sim D - m(D \cdot K_X)A + \lambda K_X$$

for some integer λ . Intersecting both sides with D and $D' = t_{\mathfrak{a}}(D)$, and using that $D^2 = D'^2$, we obtain $\lambda = \frac{m}{2}(D \cdot A + D' \cdot A)$. Intersecting both sides with A , we obtain $D' \cdot A = D \cdot A - m(K \cdot D)A^2$. Combining the two formulas, we get

$$(11) \quad t_{\mathfrak{a}}(D) \sim D - m(D \cdot K_X)A + [m(D \cdot A) - \frac{m^2}{2}(D \cdot K_X)A^2]K_X.$$

(cf. [23], Proposition 9, where the sign in front of m^2 must be changed). This should be compared with formula (9). Since $\mathrm{Pic}(X)$ is generated by separable effective divisors (e.g. by the divisor classes e_0, \dots, e_9), this determines the action of $J_\eta(\eta)$ on $\mathrm{Pic}(X)$. Restricting to K_X^\perp , we obtain

$$t_{\mathfrak{a}}(D) \sim D + m(D \cdot A)K_X.$$

So, the image of $K_X^\perp/\mathbb{Z}K_X$ in $\mathrm{Aut}(X)$ acts on K_X^\perp as the subgroup $\iota(mK_X^\perp)$ of W_X . This proves the first assertion of the proposition.

Assume now that X is not unnodal. The set of (-2) -curves on X is not empty, and coincides with the set of irreducible components of reducible fibers of f . The group $\text{Aut}(X)$ permutes the elements of this finite set. Thus, a finite index subgroup of $\text{Aut}(X)$ fixes all the divisor classes of these (-2) -curves. The subgroup of all elements of $\mathbb{E}_8 \subset W_9$ that fix such a class has infinite index in \mathbb{E}_8 ; more precisely, the rank of this free abelian group is at most $8 - k$, where k is the dimension of the subspace which is spanned by classes of (-2) -curves. This implies that $\text{Aut}(X)$ has infinite index in W_9 , and therefore that X is not Cremona special. \square

Remark 2.11. Consider the variety $\widetilde{\text{Halp}}(m)^{\text{gen}} \subset (\mathbb{P}^2)^9$ of ordered unnodal Halphen sets (p_1, \dots, p_9) . It follows from the Hilbert-Mumford numerical criterion of stability that the GIT quotient $\widetilde{\text{Halp}}^{\text{gen}} // \text{SL}(3)$ exists and parameterizes the orbits of unnodal Halphen ordered sets of index m . The group W_9 acts on this space regularly by means of the Coble action (see [20]). Since two Halphen surfaces are isomorphic if and only if the corresponding Halphen sets are projectively equivalent, we obtain that the subgroup $m\mathbb{E}_8$ acts trivially on this variety, hence the quotient group $G = W_9/\mathbb{E}_8 \cong (\mathbb{E}_8/m\mathbb{E}_8) \rtimes W_8$ acts on the orbit space.

Since all automorphisms of a Halphen surface preserve the elliptic pencil, we have a natural homomorphism $\rho : \text{Aut}(X) \rightarrow \text{Aut}(\mathbb{P}^1)$ whose image is a finite subgroup preserving the set of points corresponding to singular non-multiple fibers. If $m > 1$, it must also fix the multiple fiber, so the group is a cyclic group which has orbits of cardinality ≤ 12 . The kernel of ρ is a subgroup of $\text{Aut}(X_\eta)$ which is a finite extension of J_η by a cyclic group of order dividing 24 (6 if $\text{char}(\mathbb{K}) \neq 2, 3$). If $m \leq 2$ the group of order 2 is always present. It corresponds to the automorphism $x \mapsto -x$ of the generic fiber if $m = 1$ and the double cover $X_\eta \rightarrow \mathbb{P}_\eta^1$ given by a 2-section if $m = 2$. For X general enough, we have $\text{Aut}(X) \cong \mathbb{Z}^8 \rtimes (\mathbb{Z}/2\mathbb{Z})$ if $m \leq 2$ and $\text{Aut}(X) \cong \mathbb{Z}^8$ otherwise. The image of the generator of $(\mathbb{Z}/2\mathbb{Z})$ in $\text{Aut}(X)^*$ is equal to an element of W_9 which is mapped to the center of the group $W_8 = W_9/\iota(\mathbb{E}_8)$.

3. COBLE SURFACES

The construction of Cremona special points sets with 10 points is due to Coble (see [9], [10]). A **Coble surface** is a rational smooth surface X such that the linear system $| -K_X |$ is empty, but $| -2K_X |$ is not. The classification of such surfaces can be found in [21]. In what follows, we only need the special case, where we additionally assume that $K_X^2 = -1$ and $| -2K_X |$ consists of an irreducible curve C . So, in this paper, a Coble surface is always assumed to

be such a surface. A **Coble set** is a points set \mathcal{P} such that the blow-up of \mathcal{P} is a Coble surface.

In this section, we study Coble surfaces, and show that unnodal Coble surfaces are Cremona special. We provide a proof, which works in any characteristic. Most arguments and constructions of this section are used in Section 5 where we prove the Main Theorem.

3.1. From Coble to Halphen surfaces. Let X be a Coble surface, and C be an irreducible curve in the linear system $|-2K_X|$. By definition, the arithmetic genus $p_a(C)$ satisfies

$$p_a(C) = 1 + \frac{1}{2}(C^2 + C \cdot K_X) = 1 + K_X^2 = 0.$$

Thus C is a smooth rational curve with self-intersection $C^2 = 4K_X^2 = -4$.

Proposition 3.1. *Let X be a Coble surface and $\pi_E : X \rightarrow Y$ be the blowing down of a (-1) -curve E . Then*

- Y is a Halphen surface of index 2;
- C is the proper transform of the fiber F containing $y_0 = \pi_E(E)$;
- the fiber F is irreducible, and y_0 is its unique singular point.

Conversely, the blow-up of a singular point of an irreducible non-multiple fiber of a Halphen surface of index 2 is a Coble surface.

Proof. Since $K_X^2 = -1$, we get $K_Y^2 = 0$. Let y_0 be the image of E by π_E . Let F be the image of C ; since C is irreducible, so is F . Since $C \cdot E = -2K_X \cdot E = 2$, the curve F is singular at y_0 . From $\pi_E^*(F) = C + 2E$, we deduce that F is an irreducible curve in the linear system $|-2K_Y|$; in particular, $F^2 = 0$ and $-K_Y$ is nef.

By Riemann-Roch, $h^0(-K_Y) > 0$. If $h^0(-K_Y) > 1$, we can find a curve G from $|-K_Y|$ passing through y_0 , and thus $\pi^*(G) - E$ is effective and $|-K_X| \neq \emptyset$, contradicting the definition of a Coble surface. Thus $h^0(-K_Y) = 1$ and the unique effective divisor F_0 in $|-K_Y|$ does not pass through y_0 . This implies that $|-2K_Y|$ contains two linearly independent divisors, namely $2F_0$ and F , hence $h^0(-2K_Y) \geq 2$. Since $K_Y^2 = 0$, $2F_0$ and F are disjoint, and the linear system $|-2K_Y|$ has no fixed components; since it contains the reduced and irreducible curve F , this pencil is irreducible. Thus Y is a Halphen surface of index 2.

Conversely, let X be obtained from a Halphen surface Y of index 2 as indicated in the assertion of the proposition. Since the irreducible fiber F belongs to $|-2K_Y|$ and its singular point is blown up, the linear system $|-2K_X|$ is not empty and contains the proper transform of F , which is a smooth rational

curve. Moreover, $|-K_X|$ is empty because $|-K_Y|$ consists of the unique multiple fiber F_0 of Y and F_0 does not pass through the point which we blow up. So, X is a Coble surface. \square

Let $\pi : X \rightarrow \mathbb{P}^2$ be the composition of $\pi_E : X \rightarrow Y$ and the blow-down morphism $\pi' : Y \rightarrow \mathbb{P}^2$ which is described in Proposition 2.2. The image F of C in Y belongs to $|-2K_Y| = |6e_0 - 2(e_1 + \dots + e_9)|$, hence its image in the plane is an irreducible plane curve of degree 6 with 10 singular points p_1, \dots, p_{10} , maybe with some infinitely near points. This set $\{p_1, \dots, p_{10}\}$ is a Coble set of 10 points. It contains a Halphen set of index 2; the remaining tenth point corresponds to the singular point of F . Conversely, starting from a set of 10 singular points p_1, \dots, p_{10} of an irreducible curve of degree 6, we choose a point p_i such that no other point is infinitely near it. The remaining set of 9 points is a Halphen points set of index 2.

3.2. Unnodal Coble surfaces. By Lemma 2.1, a Coble surface X has no $(-n)$ -curves with $n \geq 3$ except the unique curve in $|-2K_X|$ (with self-intersection -4). If X is unnodal and $\pi_E : X \rightarrow Y$ is the blowing down map of a (-1) -curve E , then Y is an unnodal Halphen surface of index 2. Otherwise the pre-image, in X , of a component of a reducible fiber would define a (-2) -curve or a (-3) curve if $\pi_E(E)$ is a singular point of a reducible fiber of the elliptic pencil on Y . However, the converse is not generally true; one needs to impose more conditions on the points set to ensure that X is unnodal.

Theorem 3.2. *A Coble surface is unnodal if and only if it is obtained from a Coble set $\{p_1, \dots, p_{10}\}$ satisfying the following 496 conditions:*

- (i) *no points among the ten points are infinitely near;*
- (ii) *no three points are collinear;*
- (iii) *no six points lie on a conic;*
- (iv) *no plane cubic passes through 8 points with one of them being a singular point of the cubic;*
- (v) *no plane quartic curve passes through the 10 points with one of them being a triple point.*

This result is due to A. Coble [9], (10). A modern proof was sketched in [11], Remark 4.7. We supply the full details here.

Proof of Theorem 3.2. It is known (and is easy to check) that the lattice $\mathbb{E}_{10} \cong K_X^\perp$ is a unimodular even lattice of signature $(1, 9)$, hence isomorphic to the orthogonal sum $\mathbb{E}_8 \perp \mathbb{H}$, where \mathbb{H} is the hyperbolic plane defined by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Explicitly, the \mathbb{E}_8 -part is generated by the vectors $\alpha_i, i = 0, \dots, 7$.

The sublattice generated by \mathbb{E}_8 and α_8 is isomorphic to \mathbb{E}_9 . The radical of this copy of \mathbb{E}_9 and the vector α_9 generate the \mathbb{H} -part. Suppose conditions (i)-(v) are satisfied. Since there are no infinitely near points, X is obtained from \mathbb{P}^2 by blowing up 10 distinct points p_i . We denote by E_i the corresponding 10 exceptional divisors, and denote by $(e_0, e_1, \dots, e_{10})$ the standard basis of $\text{Pic}(X)$ (see Section 1.3).

Let

$$f_i = 3e_0 - (e_1 + \dots + e_{10}) + e_i, \quad i = 1, \dots, 10.$$

We have $f_i \cdot f_j = 1 - \delta_{ij}$. Let Y_i be the Halphen surface of index 2 obtained from X by blowing down the curve E_i . Then f_i is the divisor class of the proper transform of the half-fiber of the elliptic fibration on Y_i . By Proposition 2.5, the first four conditions guarantee that the Halphen surfaces Y_i are unnodal. The pencil $|2f_i|$ is equal to the pre-image of the elliptic pencil on Y_i ; it contains only one reducible fiber, namely $C + 2E_i$.

We have

$$f_1 + \dots + f_{10} = 30e_0 - 9(e_1 + \dots + e_{10}) = 3\Delta,$$

with

$$\Delta = 10e_0 - 3(e_1 + \dots + e_{10}) = -3K_X + e_0, \quad \Delta^2 = 10, \quad \Delta \cdot K_X = 0.$$

The linear system $|\Delta|$ is the proper transform of the linear system of plane curves of degree 10 passing through the points p_i with multiplicities ≥ 3 . By counting constants, or applying Riemann-Roch, $\dim |\Delta| \geq 5$. Since the divisor classes f_i are represented by irreducible curves, the divisor class Δ is nef and big. An irreducible curve R on X with $R \cdot \Delta = 0$ satisfies $R \cdot f_i = 0$, for all $i = 1, \dots, 10$, hence must coincide with the curve C . Thus $|\Delta|$ defines a morphism

$$\phi : X \rightarrow \mathbb{P}^{\dim |\Delta|}$$

that contracts C onto a point z .

Consider the restriction of the linear system $|\Delta|$ to a general member F_i of $|2f_i|$. It is a linear series of degree 6. Since

$$\Delta - F_i \sim \Delta - 2f_i = 4e_0 - (e_1 + \dots + e_{10}) - 2e_i,$$

by condition (v), this divisor class can not be effective. This shows that the restriction of ϕ to F_i is given by a complete linear system of degree 6. Thus, $\phi|_{F_i}$ is an isomorphism onto a normal elliptic curve of degree 6 spanning $\mathbb{P}^{\dim |\Delta|}$, $\dim |\Delta| = 5$ and ϕ is a birational isomorphism onto a surface of degree 10. This surface has a quotient singularity of type $\frac{1}{4}(1, 1)$ at $z = \phi(C)$.

The remaining arguments follow the proof of Theorem 4.4 from [11], almost verbatim. Let R be a (-2) -curve and $m_0e_0 - m_1e_1 - \dots - m_{10}e_{10}$ be its divisor class. We have $m_0 = R \cdot e_0 \geq 0$ and $m_i = R \cdot E_i \geq 0$. Since $\Delta = -3K_X + e_0$, we have $\Delta \cdot R = e_0 \cdot R = m_0$. Suppose first that $m_0 \leq 4$. Then listing all solutions of the diophantine equations

$$\begin{aligned} 3m_0 &= m_1 + \dots + m_{10}, \\ m_0^2 + 2 &= m_1^2 + \dots + m_{10}^2 \end{aligned}$$

with $m_0 \leq 4$, we find that R is contained in an exceptional curve coming from an infinitely near point, or is equal to the proper transform of one of the curves corresponding to conditions (ii) to (v), or belongs to the class $4e_0 - (e_1 + \dots + e_{10}) - e_i - e_j - e_k + e_s$ with four distinct indices i, j, k, s . All cases except the last one are prohibited by the assumptions of the theorem. In the last case, the curve R does not intersect the exceptional curve E_s ; hence R is coming from a (-2) -curve on the Halphen surface Y_s obtained by blowing down E_s . It follows from Proposition 2.5, that in this case there exists a conic passing through the six points p_a with $a \neq i, j, k, s$. This is prohibited by condition (iii). Thus, there is no (-2) -curve R with $m_0 \leq 4$.

So, assume now that $m_0 > 4$ for any (-2) -curve R . Repeating the argument of the proof of Theorem 4.4 from [11], $R \cdot w(\Delta) > 4$ for all $w \in W_X$. Taking w to be the reflection with respect to R , we obtain $R \cdot w(\Delta) = w(R) \cdot \Delta = -R \cdot \Delta < 0$, a contradiction. \square

Remark 3.3. Theorem 3.2 shows that unnodal Coble sets form a Zariski open subset in the set of all Coble sets. In particular, an unnodal Coble surface is a general Coble surface in sense of moduli.

3.3. Effective divisors on unnodal Coble and Halphen surfaces. Recall that an element of \mathbb{E}_n of norm -2 is called a **root**. The lattice is spanned by (simple) roots α_i (cf. Section 1.2). A **real root** is a root which belongs to the W_n -orbit of one (or any) of these simple roots. It is known that all roots are real if and only if $n \leq 10$ (see, for example, [20], Remark 5, p. 79). Let X be a rational surface obtained by blowing up a points set \mathcal{P} with $|\mathcal{P}| = n$. Using an isomorphism $\mathbb{E}_n \rightarrow K_X^\perp$ defined by a choice of a geometric basis on X , we can transfer these definitions to elements of the lattice K_X^\perp ; since W_X acts transitively on the set of geometric bases, the definitions do not depend on the choice of a geometric basis.

We say that a root α in K_X^\perp is **effective**, if it can be represented by the divisor class of an effective divisor. Clearly, the divisor class of a (-2) -curve is an example of an effective root.

Later on we shall use the following result due to M. Nagata [35] which applies to Coble and Halphen surfaces. For the sake of completeness, we include a proof.

Lemma 3.4. *Let $X \rightarrow \mathbb{P}^2$ be the blow-up of $n \geq 9$ points such that $| -mK_X |$ contains an irreducible curve for some $m > 0$. Suppose that X has no (-2) -curves. Then,*

- (0) *for any effective divisor D and any $w \in W_X$, the class $w([D])$ is effective;*
- (i) *for any (-1) -curve E and any $w \in W_X$, the linear system $|w(E)|$ consists of a unique (-1) -curve;*
- (ii) *for any simple root α_i and any $w \in W_X$, the linear system $|w(\alpha_i)|$ is empty;*
- (iii) *for any primitive isotropic effective divisor class f and any $w \in W_X$, the divisor class $w(f)$ is a primitive isotropic effective divisor class.*

Proof. Let C be an element of $| -mK_X |$. For any irreducible curve $Z \neq C$, we have $Z \cdot C = -mZ \cdot K_X \geq 0$. By adjunction formula, $Z^2 \geq -2$; since X does not contain (-2) -curves, all irreducible curves $Z \neq C$ satisfy $Z^2 \geq -1$. Moreover, $Z^2 = -1$ implies that Z is a (-1) -curve.

Let Z be a curve with $Z^2 \geq -1$ and let $de_0 - d_1e_1 - \dots - d_n e_n$ be its divisor class. We have $Z^2 = d^2 - d_1^2 - \dots - d_n^2 \geq -1$, and $d = [Z] \cdot e_0 > 0$ unless Z is one of the exceptional curves E_i . We claim that, for any $w \in W_X$,

$$(12) \quad w([Z]) \cdot e_0 > 0, \quad \text{unless } w([Z]) = e_i \text{ for some } i > 0.$$

We use induction on the length $l(w)$ of w as a word in simple reflections. Write $w = s_i w'$ with $l(w') < l(w)$, where s_i is the reflection given by the simple root α_i (see §1.3). By induction $w'([Z]) \cdot e_0 > 0$ or $w'([C]) = e_j$ for some $j > 0$. In the latter case, $s_i(e_j)$ is either equal to e_k for some index k or to $e_0 - e_a - e_b$ for some indices $a \neq b > 0$; so the assertion is true in the latter case. In the first case, the claim is obvious for $\alpha_i \neq \alpha_0$; so we assume $\alpha_i = \alpha_0$. Let $w'([Z]) = de_0 - d_1e_1 - \dots - d_n e_n$ with $d \geq 1$. Then $w([Z]) = w'([Z]) + (w'([Z]) \cdot \alpha_0)\alpha_0$ implies that $w([Z]) \cdot e_0 = 2d - d_1 - d_2 - d_3$. Assume it is negative, i.e. $1 \leq d < (d_1 + d_2 + d_3)/2$. Then

$$\begin{aligned} -1 &\leq d^2 - d_1^2 - \dots - d_n^2 < \frac{1}{4}(d_1 + d_2 + d_3)^2 - d_1^2 - \dots - d_n^2 \\ &< \frac{3}{4}(d_1^2 + d_2^2 + d_3^2) - d_1^2 - \dots - d_n^2. \end{aligned}$$

This gives

$$-4 < -d_1^2 - d_2^2 - d_3^2 - 4(d_4^2 - \dots - d_n^2),$$

hence $d_4 = \dots = d_n = 0$ and $d_1, d_2, d_3 \leq 1$; since $2 \leq 2d < d_1 + d_2 + d_3$ we obtain

$$w'([Z]) = e_0 - e_1 - e_2 - e_3 = \alpha_0.$$

This contradicts $Z^2 \geq -1$ and proves the claim.

(0) Let $[D]$ be the class of an irreducible curve, with $[D] \notin \mathbb{Q}[C]$. For all $w \in W_X$, we get $w([D]) \cdot K_X = [D] \cdot w^{-1}(K_X) = D \cdot K_X = -\frac{1}{m}D \cdot C \leq 0$; moreover, $D^2 \geq -1$. By Riemann-Roch, $w([D])$ or $[K_X - w(D)]$ is effective. Intersecting $[K_X - w(D)]$ with e_0 , and using (12), we see that $[K_X - w(D)]$ is not effective; hence $w([D])$ is effective.

Let now D be any effective divisor. Write D as a sum of irreducible components

$$D = sC + \sum_{j=1}^k b_j Z_j$$

where $s \geq 0$, $b_j > 0$, and Z_j is an irreducible curve with $[Z_j] \notin \mathbb{Q}[C]$ for all $1 \leq j \leq k$. Then all classes $w([Z_j])$ are effective. Since $w([D]) = s[C] + \sum_j b_j w([Z_j])$, the class $w([D])$ is effective.

(i) Let E be a (-1) -curve which we identify with its divisor class. As we saw above, $w(E)$ is effective. Write an effective representative of $w(E)$ as a sum of irreducible components. Since $w(E)^2 = -1 < 0$, some of the components must have negative self-intersection. Thus we can write

$$(13) \quad w(E) \sim sC + a_1 G_1 + \dots + a_k G_k + Z,$$

where G_1, \dots, G_k are (-1) -curves, Z is a sum of irreducible curves Z_j with $Z_j^2 \geq 0$, and at least one of the coefficients s, a_1, \dots, a_k is positive. Applying w^{-1} we get

$$E \sim sC + a_1 w^{-1}(G_1) + \dots + a_k w^{-1}(G_k) + w^{-1}(Z).$$

From Property (0), we know that the classes $w^{-1}(G_i)$, $1 \leq i \leq k$, and $w^{-1}(Z)$ are effective. Since E is a (-1) -curve, we have $|E| = \{E\}$. Also $E \neq C$ since $E^2 = E \cdot K_X = -1$ but $C^2 = m^2 K_X^2$ and $C \cdot K_X = -m K_X^2$ cannot be both equal to -1 . This implies $s = 0$, $Z = 0$, $k = 1$ and $E = w^{-1}(G_1)$; thus $|w(E)| = \{G_1\}$ and Property (i) is proved.

(ii) A simple root $\alpha_i = e_i - e_{i+1}$ is effective if and only if there are infinitely near points, which is excluded since we know that X has no $(-n)$ -curves with $n \geq 2$. The simple root $\alpha_0 = e_0 - e_1 - e_2 - e_3$ is effective if and only if the points p_1, p_2, p_3 are collinear. The proper transform of the corresponding line is a (-2) -curve, hence this root is not effective. Suppose $w(\alpha_i)$ is effective for

some index $i \geq 0$ and some element w in W_X . Write an effective representative as in Equation (13):

$$w(\alpha) \sim sC + a_1G_1 + \dots + a_kG_k + Z.$$

Applying w^{-1} , and using the same argument as in (i), we get a contradiction.

(iii) By (0) we know that $f' = w(f)$ is effective. This class is isotropic because w preserves the intersection form. It is primitive because otherwise $f = w^{-1}(f')$ is not primitive. \square

3.4. Automorphisms of unnodal Coble surfaces. Here we provide a proof, valid in any characteristic, of the following theorem which is implicitly contained in [9] (expressed in terms of projective orbits of points sets, see §1.5); several steps of the proof are used in Section 5.

Theorem 3.5. *Let X be an unnodal Coble surface. Then $\text{Aut}(X)^*$ contains the subgroup of W_X which is isomorphic to*

$$W_{10}(2) := \{w \in W_{10} : w(v) - v \in 2\mathbb{E}_{10} \text{ for all } v \in \text{Pic}(X)\}$$

under the natural identification of W_X with W_{10} .

Remark 3.6. The subgroup $W_{10}(2)$ is obviously normal, and the quotient group is isomorphic to the finite orthogonal group $O^+(10, \mathbb{F}_2)$ (see [12], Theorem 2.9). We have to explain the meaning of the notation $O^+(10, \mathbb{F}_2)$. Up to conjugacy, there are only two types of non-degenerate quadratic forms over \mathbb{F}_2^{10} . Modulo 2, the intersection form on K_X^\perp is equivalent to the quadratic form

$$x_1x_2 + x_3x_4 + \dots + x_9x_{10}.$$

In other words, this quadratic form is of **even type**; the number of its isotropic vectors (including the null vector) is $2^4(2^5 + 1)$. The notation $O^+(10, \mathbb{F}_2)$ is meant to distinguish this quadratic form from the form of **odd type** $x_1x_2 + x_3x_4 + \dots + x_7x_8 + x_9x_{10} + x_9^2 + x_{10}^2$, which has only $2^4(2^5 - 1)$ isotropic vectors.

To prove Theorem 3.5, we use the notations from the proof of Theorem 3.2. Consider the elliptic fibrations $|2f_1|$ and $|2f_2|$ defined by the first two isotropic vectors among f_1, \dots, f_{10} . From Proposition 3.1, each of these elliptic fibrations comes from a Halphen surface of index 2, and each $|f_i|$ is reduced to a unique element; we denote these curves by $F_1 \in |f_1|$ and $F_2 \in |f_2|$. Let $|D| = |2F_1 + 2F_2|$.

Lemma 3.7. *The linear system $|D| = |2F_1 + 2F_2|$ has no base points and defines a morphism $\phi : S \rightarrow \mathbb{P}^4$.*

Remark 3.8. The proof of this lemma does not use the fact that $-2K_X$ is effective; it depends only on the intersection properties of f_1 , f_2 and K_X , and the fact that $|f_i| = \{F_i\}$ for some irreducible curve. This will be used in Section 5.5 (see Lemma 5.7).

Proof. Consider the following three exact sequences

$$(14) \quad \begin{aligned} 0 &\rightarrow \mathcal{O}_X(F_1) \rightarrow \mathcal{O}_X(F_1 + F_2) \rightarrow \mathcal{O}_{F_1}(F_1 + F_2) \rightarrow 0, \\ 0 &\rightarrow \mathcal{O}_X(F_1 + F_2) \rightarrow \mathcal{O}_X(2F_1 + F_2) \rightarrow \mathcal{O}_{F_1}(2F_1 + F_2) \rightarrow 0, \\ 0 &\rightarrow \mathcal{O}_X(F_1 + 2F_2) \rightarrow \mathcal{O}_X(2F_1 + 2F_2) \rightarrow \mathcal{O}_{F_1}(2F_1 + 2F_2) \rightarrow 0. \end{aligned}$$

Since $\deg \mathcal{O}_{F_1}(F_1 + F_2) = 1$ and $p_a(F_1) = 1$, we get $h^0(\mathcal{O}_{F_1}(F_1 + F_2)) = 1$; similarly, $h^0(\mathcal{O}_{F_1}(F_2)) = 1$.

Since $h^0(\mathcal{O}_X(F_1)) = 1$, Riemann-Roch formula implies $h^1(\mathcal{O}_X(F_1)) = 0$. Thus the first exact sequence shows that $h^0(\mathcal{O}_X(F_1 + F_2)) = 2$, and, by Riemann-Roch, $h^1(\mathcal{O}_X(F_1 + F_2)) = 0$. Since $\deg \mathcal{O}_{F_1}(2F_1 + F_2) = 1$, the second exact sequence gives $h^0(\mathcal{O}_X(2F_1 + F_2)) = 3$. By Riemann-Roch, $h^1(\mathcal{O}_X(2F_1 + F_2)) = 0$. Since $\deg \mathcal{O}_{F_1}(2F_1 + 2F_2) = 2$, the third exact sequence gives $h^0(\mathcal{O}_X(2F_1 + 2F_2)) = 5$, hence $\dim |D| = 4$.

Since F_1 and F_2 are irreducible and $h^0(\mathcal{O}_X(2F_1 + 2F_2)) > h^0(\mathcal{O}_X(F_i + 2F_j))$, we obtain that $|D|$ has no fixed components. Let us now assume that $|D|$ has a base point. Such a point must lie on F_1 or F_2 and, without loss of generality, we may assume that it lies on F_1 . The third exact sequence shows that the restriction map

$$H^0(\mathcal{O}_X(2F_1 + 2F_2)) \rightarrow H^0(\mathcal{O}_{F_1}(2F_1 + 2F_2))$$

is a surjective morphism onto a complete linear system of degree 2 on F_1 . Thus $|D|$ has no base point on F_1 , and $|D|$ has no base point at all. \square

Lemma 3.9. *The image of $\phi: X \rightarrow \mathbb{P}^4$ is a surface S of degree 4.*

Proof. Since the map is given by the complete linear system $|D|$, its image spans \mathbb{P}^4 and its degree divides $D^2 = 8$. So, there are only two possibilities: ϕ is 1-to-1 and its image has degree 8, and ϕ is 2-to-1 and its image has degree 4; its image can not have degree ≤ 2 because every quadric surface of \mathbb{P}^4 is contained in a hyperplane.

We know that $|F_1 + F_2|$ is an irreducible pencil. Let P be its general member. It is an irreducible curve of arithmetic genus 2. Let ω_P be its canonical sheaf. Since $\mathcal{O}_{F_i}(C) \cong \mathcal{O}_{F_i}(-2K_X) \cong \mathcal{O}_{F_i}$, the adjunction formula gives

$$\omega_P^{\otimes 2} \cong \mathcal{O}_P(2F_1 + 2F_2 + 2K_X) \cong \mathcal{O}_P(2F_1 + 2F_2).$$

The exact sequence

$$(15) \quad 0 \rightarrow \mathcal{O}_X(F_1 + F_2) \rightarrow \mathcal{O}_X(2F_1 + 2F_2) \rightarrow \mathcal{O}_P(2F_1 + 2F_2) \rightarrow 0,$$

shows that $|D|$ cuts out on P the bi-canonical linear system, hence the map $\phi|_P$ is of degree 2 onto a plane conic. In particular, the degree of $\phi : X \rightarrow \phi(X)$ cannot be equal to 1, hence it is of degree 2 onto a quartic surface S in \mathbb{P}^4 . \square

Recall that an **anti-canonical Del Pezzo surface** S of degree $d \geq 3$ is a surface of degree d in \mathbb{P}^d whose minimal resolution is isomorphic to the blow-up V of $9 - d$ points in \mathbb{P}^2 (maybe infinitely near) with $-K_V$ nef and big. Each such surface S is obtained as the image of V by the map given by the linear system $|-K_V|$. An anti-canonical Del Pezzo surface may have singularities (when $-K_V$ is not ample). They are Du Val singularities (or ADE-singularities).

It is classically known that a surface of degree 4 in \mathbb{P}^4 that spans \mathbb{P}^4 is either an anti-canonical Del Pezzo, or a cone over an elliptic curve, or a projection of a surface of degree 4 in \mathbb{P}^5 (see [34, 35] and [18]). Since X is rational, and ϕ is given by a complete linear system there is only one possibility: S is an anti-canonical Del Pezzo surface.

Lemma 3.10. *The image S of X under the map $\phi : X \rightarrow \mathbb{P}^4$ is a Del Pezzo surface of degree 4 with four ordinary double points.*

Proof. Consider the four curves F_1, F_2, E_1, E_2 , where E_i is the (-1) -curve corresponding to the points p_i , $i = 1, 2$, in the Coble set defining X . By definition, $F_i \cdot E_i = 0$, $i = 1, 2$, and $F_i \cdot E_j = 1$, $i \neq j$. Moreover, $2E_i + C \in |2F_i|$, where $C \in |-2K_X|$, $C \cdot F_i = 0$, and $C \cdot E_j = 2$, $i \neq j$.

The restriction of $|D|$ to each of the curves F_i is of degree 2, and cuts out a complete linear system of degree 2 on these curves of arithmetic genus 1; thus $\phi|_{F_i}$ is a 2-to-1 cover of F_i onto a line ℓ_i , $i = 1, 2$.

The restriction of $|D|$ to each of the curves E_i is also of degree 2. Since $F_i \cdot C = 0$ and $C^2 = -4$, the morphism ϕ contracts C onto a singular point q of S . This point is contained in $\phi(E_1) \cap \phi(E_2)$, and is a ramification point for the maps $\phi|_{E_i} : E_i \rightarrow \phi(E_i)$. Thus, the images of E_1 and E_2 are two lines ℓ_3 and ℓ_4 , that intersect at the singular point q .

We infer from the exact sequences (14) that

$$3 = h^0(D - F_1) > h^0(D - F_1 - F_2) = 2,$$

hence $\ell_1 \neq \ell_2$. Similarly, we prove that

- the four lines ℓ_i are distinct;
- the intersections $\ell_1 \cap \ell_2, \ell_2 \cap \ell_3, \ell_3 \cap \ell_4, \ell_4 \cap \ell_1$ are non-empty;
- all other intersections of two of these lines are empty.

Let Π_1 be the plane spanned by ℓ_1 and ℓ_2 and Π_2 be the plane spanned by ℓ_3 and ℓ_4 ; the plane Π_2 contains the singularity $q = \ell_3 \cap \ell_4 = \phi(C)$. Since Π_1 and Π_2 intersect in at least two points and do not coincide (for $\ell_1 \cap \ell_3 = \emptyset$) they span a hyperplane H in \mathbb{P}^4 . The pre-image of $H \cap S$ in X is the divisor $F_1 + F_2 + C + E_1 + C + E_2 \in |2F_1 + 2F_2|$.

It is known that a quartic Del Pezzo surface is equal to the base locus of a pencil of quadrics. The quadrangle of lines is equal to the base locus of the restriction of the pencil to H . It is easy to see that this pencil must be spanned by two quadrics of rank 2, i.e. the union of two planes. It follows from this that the pencil of quadrics containing S is spanned by two quadrics of rank 3. This implies that S contains 4 singular points of type A_1 (ordinary nodes) or 2 singular points of type A_1 and one singular point of type A_3 (see [12], Lemma 0.4.2). In the second case the surface does not contain a quadrangle of lines. Thus we obtain that S is a 4-nodal quartic Del Pezzo surface. Its four nodes are the vertices of the quadrangle of lines. It is known that a 4-nodal quartic Del Pezzo surface is isomorphic to the anti-canonical model of the blow-up of 5 points p_1, \dots, p_5 such that p_3 is infinitely near p_2 , p_5 is infinitely near p_4 and the points p_1, p_2, p_3 and p_1, p_4, p_5 are collinear (see [12], Proposition 0.4.3). The quadrangle of lines is formed by the images of the classes e_1, e_3, e_5 , and $e_0 - e_2 - e_4$. \square

Proof of Theorem 3.5. Let $\sigma : S' \rightarrow S$ be the blow-up of the point $q = \phi(C)$. The exceptional curve is a (-2) -curve R on S' . The morphism ϕ factors through a finite map $\phi' : X \rightarrow S'$ of degree 2. The pre-image of R on X is the curve C .

Let τ be the automorphism of X defined by the deck transformation of the finite double cover $\phi' : X \rightarrow S'$. Note that it is defined even when $\text{char}(\mathbb{K}) = 2$. Since the genus 1 fibration X is not a quasi-elliptic fibration (the image of its general member is a conic, but an elliptic curve cannot be mapped to a rational curve by an inseparable map), the map ϕ' is a separable finite morphism.

The Picard group $\text{Pic}(S') \otimes \mathbb{Q}$ of \mathbb{Q} -divisor classes on S' is of rank 3. It is generated by the classes of the curve R and the proper transforms of the lines ℓ_1 and ℓ_2 . For any divisor class A on X , we have, in $\text{Pic}(X) \otimes \mathbb{Q}$,

$$A + \tau(A) = a_1 f_1 + a_2 f_2 + a_3 K_X.$$

Suppose $A \in (\mathbb{Z}f_1 + \mathbb{Z}f_2 + \mathbb{Z}K_X)^\perp$. Since f_1, f_2 and K_X are τ -invariant, we obtain, by intersecting both sides with f_1, f_2 , and K_X , that $a_1 = a_2 = a_3 = 0$. Thus $\tau(A) = -A$. The sublattice of K_X^\perp spanned by f_1 and f_2 is isomorphic to the hyperbolic plane \mathbb{H} . Its orthogonal complement is isomorphic to the lattice

\mathbb{E}_8 . Thus we obtain

$$(16) \quad \tau^* = \text{id}_{\mathbb{H}} \oplus -\text{id}_{\mathbb{E}_8}.$$

Now let w be an element of W_X , and $f'_i = w(f_i)$, $i = 1, 2$. Since X has no (-2) -curves, we can apply Lemma 3.4 to obtain that f'_i are primitive, isotropic and effective divisor class; since w preserves both K_X and the intersection form, $f'_i \cdot K_X = 0$ and $f'_1 \cdot f'_2 = 1$. Similarly, each class $w([E_i])$, $i = 1, 2$, is represented by a unique (-1) -curve E'_i . The curve E'_i does not intersect f'_i and if one contracts E'_i and apply Proposition 3.1, one sees that f'_i is the pull back of a Halphen pencil of index 2. As above we deduce that each linear system $|2f'_i|$ is an elliptic pencil, and repeating the argument by taking (f'_1, f'_2) instead of (f_1, f_2) , we obtain an automorphism $\tau_\omega = w \circ \tau \circ w^{-1}$. This shows that $\text{Aut}(X)^*$ contains the minimal normal subgroup containing the involution $\text{id}_{\mathbb{H}} \oplus -\text{id}_{\mathbb{E}_8}$. This finishes the proof of Theorem 3.5 because this normal subgroup is isomorphic to $W_{10}(2)$ (this non-trivial result is due to A. Coble, a modern proof can be found in [12], Theorem 2.10.1). \square

Remark 3.11. One can show that, for a general Coble surface in moduli sense, the group of automorphisms is isomorphic to $W_{10}(2)$.

Remark 3.12. Assume $\text{char}(\mathbb{K}) \neq 2$. Let $p : X' \rightarrow X$ be the double cover of X branched along the curve $C \in |-2K_X|$. This is a K3-surface which admits an involution whose fixed-point locus consists of a (-2) -curve. Conversely, the quotient of a K3-surface by such an involution is a Coble surface. The Picard lattice of X' contains the pre-image of the Picard lattice of X , it is a primitive sublattice isomorphic to the lattice $\langle -2 \rangle \perp \mathbb{E}_{10}(2)$, where $\langle -2 \rangle$ is given by the matrix (-2) and $\mathbb{E}_{10}(2)$ is obtained from the lattice \mathbb{E}_{10} by multiplying the values of the quadratic form by 2.

Over \mathbb{C} , it follows from the theory of periods of complex K3 surfaces that the coarse moduli space of such K3 surfaces exists and is an irreducible variety of dimension 9. In fact, it is one of the two codimension 1 irreducible components in the boundary of a compactification of the moduli space of Enriques surfaces. A Coble surface can be obtained as a degeneration of an Enriques surface when its K3-cover acquires an ordinary double point. The theory of periods also provides another proof of Theorem 3.5 (see [37], Theorem 10.1.2), and shows that the automorphism group of an unnodal Enriques surface contains a subgroup isomorphic to $W_{10}(2)$ and a general Enriques surface has its automorphism group isomorphic to $W_{10}(2)$.

Our proof extends to the case of unnodal Enriques surfaces X over a field \mathbb{K} of arbitrary characteristic. In this case any embedding of \mathbb{H} in $\text{Num}(X) \cong$

$\mathbb{H} \perp \mathbb{E}_8$ defines a separable map of degree 2 on a 4-nodal quartic surface (it is separable for the same reason as above: the surface has no quasi-elliptic fibrations). The deck transformation acts by formula (16), and we finish as in the case of Coble surfaces by invoking Theorem 2.10.1 from [12].

4. GIZATULLIN'S THEOREM AND CREMONA SPECIAL POINTS SET OF NINE POINTS

Let X denote, as above, a rational surface over \mathbb{K} ; we denote by $n + 1$ its Picard number. In this section, we prove that Cremona special points sets with $n = 9$ points are Halphen sets.

4.1. Parabolic automorphisms. Let g be an automorphism of X and g^* the linear transformation of $\text{Pic}(X)$ induced by g . Since the intersection form on $\text{Pic}(X)$ has signature $(1, n)$ and the nef cone is $\text{Aut}(X)$ -invariant, there are three possibilities for the isometry g^* (see [5], [23]).

- g^* preserves an ample class h , i.e. $g^*h = h$; in this case, a positive iterate g^k of g acts trivially on $\text{Pic}(X)$ and there is an embedding of X into a projective space $\mathbb{P}_{\mathbb{K}}^N$ such that g^k extends to a projective linear transformation of $\mathbb{P}_{\mathbb{K}}^N$.
- g^* preserves a primitive nef class h with $h^2 = 0$ but does not preserve any ample class.
- g^* does not preserve any nef class and the spectral radius (i.e. the largest possible absolute value of an eigenvalue) of the linear transformation g^* on $\text{Pic}(X) \otimes \mathbb{R}$ is larger than 1.

In the first case, one says that g (or g^*) is **elliptic**, in the second that g is **parabolic**, and in the third that g is **hyperbolic** (or loxodromic).

When g is parabolic, the class h is, up to a scalar multiple, the unique g^* -invariant class in the isotropic cone of the intersection form. In particular, the assumption that h is nef could be removed from the definition (see [23], Proposition 4, or [5]). The following fundamental theorem of M. Gizatullin describes geometric properties of parabolic automorphisms.

Theorem 4.1 (M. Gizatullin). *Let X be a rational surface over an algebraically closed field \mathbb{K} . Let $G \neq \{\text{id}_X\}$ be a group of automorphisms of X such that all elements g in $G \setminus \{\text{id}_X\}$ are parabolic.*

Then there exists a unique G -invariant primitive nef class h in $\text{Pic}(X)$ with $h^2 = h \cdot K_X = 0$ and there exists a unique positive integer m such that the linear system $|mh|$ is an irreducible pencil of curves of arithmetic genus 1.

The relative minimal model of this genus 1 fibration is a Halphen surface Y of index m .

Remark 4.2. Assume that $\mathbb{K} = \mathbb{C}$ is the field of complex numbers. Then, Gizatullin's Theorem can be strengthened: A finite index subgroup of G preserves each fiber of the genus one fibration; it acts as a group of translations along the fibers, with dense orbits in almost all fibers; if G is cyclic, periodic orbits are dense in X (see [8, 6]).

Remark 4.3. Gizatullin's Theorem is stated in [23] under the assumption that $\text{char}(\mathbb{K}) \neq 2, 3$. As the author points out himself, this assumption is made only to avoid quasi-elliptic fibrations, for which the arguments must be slightly modified but the same conclusion holds. Moreover, in our applications, quasi-elliptic fibrations are not realized.

Corollary 4.4. *Let X and G be as in Theorem 4.1. The group G descends to a subgroup of the group $\text{Aut}(Y)$. In particular, it contains a finite index free abelian subgroup G_0 , the rank of which is at most 8 and is equal to 8 if and only if Y is an unnodal Halphen surface.*

Proof. This follows from Theorems 4.1 and 2.10. □

4.2. Cremona special implies unnodal.

Proposition 4.5. *A Cremona special rational surface X does not contain (-2) -curves, i.e. it is unnodal.*

Proof. Denote by $\mathcal{P} = \{p_1, \dots, p_n\}$ a points set such that X is the blow-up of \mathbb{P}^2 at \mathcal{P} ; this provides a natural morphism $\pi : X \rightarrow \mathbb{P}^2$, and a geometric basis (e_0, \dots, e_n) of $\text{Pic}(X)$. Let \mathcal{R}_X be the set of effective real roots (see §3.3). Our first goal is to prove that \mathcal{R}_X is empty.

Let W_X^{nod} be the subgroup of W_X generated by the reflections s_α with $\alpha \in \mathcal{R}_X$. Since all reflections generating W_X^{nod} are conjugate under W_X to reflections defined by the simple roots α_i , a result of Deodhar [14] shows that W_X^{nod} is a Coxeter subgroup of the Coxeter group W_X . Its intersection with $\text{Aut}(X)^*$ is equal to $\{1\}$ ([16], Proposition 3,⁵). So, if $\text{Aut}(X)^*$ is of finite index in W_X , the group W_X^{nod} must be finite, and W_X^{nod} is a finite Coxeter group. It follows from the classification of such groups that \mathcal{R}_X is a finite set and the sublattice \mathcal{N}_X generated by this set is a negative definite sublattice of K_X^\perp ; hence $\mathcal{N}_X \otimes \mathbb{R}$ is a proper subspace of $K_X^\perp \otimes \mathbb{R}$.

⁵The assertion is stated for the subgroup generated by the reflections with respect to all (-2) -curves, but the proof is extended verbatim to our situation.

The group $\text{Aut}(X)^*$ leaves the sublattice \mathcal{N}_X invariant, and acts as a finite group on it. Identifying $K_X^\perp \otimes \mathbb{R}$ with $\mathbb{E}_n \otimes \mathbb{R}$, $\text{Aut}(X)^*$ determines a finite index subgroup of W_n that preserves the proper subspace $\mathcal{N}_X \otimes \mathbb{R}$.

Assume $n \geq 10$. We invoke a theorem of Yves Benoist and Pierre de la Harpe from [4] according to which the image of W_n in $\text{O}(\mathbb{E}_n \otimes \mathbb{R})$ is Zariski dense. Since, by assumption, $\text{Aut}(X)^*$ is a subgroup of finite index in W_n , we obtain that the image of $\text{Aut}(X)^*$ is Zariski dense either in $\text{O}(\mathbb{E}_n \otimes \mathbb{R})$ or in its connected component of the identity $\text{SO}(\mathbb{E}_n \otimes \mathbb{R})$.⁶ Thus the representation of $\text{Aut}(X^*)$ in $\mathbb{E}_n \otimes \mathbb{R}$ is irreducible, and hence $\mathcal{N}_X = \{0\}$.

When $n = 9$, the theorem proved by Benoist and de la Harpe asserts that the image of the Coxeter group W_9 is Zariski dense in the subgroup G_9 of $\text{O}(\mathbb{E}_9 \otimes \mathbb{R})$ defined by $G_9 = \{g \in \text{O}(\mathbb{E}_9 \otimes \mathbb{R}) \mid g(k_9) = k_9\}$. If N is a proper subspace of $\mathbb{E}_9 \otimes \mathbb{R}$ which is fixed by a finite index subgroup of G_9 , then $N/\mathbb{R}k_9$ is trivial. As a consequence, $\mathcal{N}_X = \{0\}$ in this case too.

Now we may assume that \mathcal{R}_X is empty, in particular, no α_i is an effective divisor class. Suppose we have a (-2) -curve R ; it represents an effective non-real root. Write $r = [R] = a_0 e_0 - \sum a_i e_i \in \mathcal{R}_X$. After permuting the elements of the basis, we may assume that $a_1 \geq \dots \geq a_n$. Intersecting with e_i , we obtain that $a_i \geq 0$. Applying Noether's inequality [20], Chapter V, Proposition 5, we get $a_0 < a_1 + a_2 + a_3$. Since $\alpha_0 = e_0 - e_1 - e_2 - e_3$ is not effective, the three points p_1, p_2, p_3 are not collinear. Let X_{123} be the surface obtained by blowing up \mathbb{P}^2 at p_1, p_2 , and p_3 . Let σ be the standard quadratic involution with fundamental points p_1, p_2, p_3 . This birational transformation of the plane lifts to an automorphism of X_{123} ; thus, the composition of π with σ is a new birational morphism from X to \mathbb{P}^2 , which provides a new geometric basis for $\text{Pic}(X)$. In terms of the first geometric basis (e_i) of $\text{Pic}(X)$, the change of basis corresponds to the linear transformation s_{α_0} . As a consequence, if r is the class of a (-2) -curve then the class r' defined by

$$r' = s_{\alpha_0}(r) = a'_0 e_0 - \sum_{i=1}^n a'_i e_i,$$

is also the class of a (-2) -curve (in the new geometric basis). Moreover

$$a'_0 = 2a_0 - a_1 - a_2 - a_3, a'_1 = a_0 - a_2 - a_3, a'_2 = a_0 - a_1 - a_3, a'_3 = a_0 - a_1 - a_2,$$

and $a'_i = a_i, i > 3$. Now, by Noether's inequality, we have $a'_0 < a_0$. Since s_{α_0} is in W_X , we can iterate this process, keeping the assumption that \mathcal{R}_X is

⁶ $\text{SO}(\mathbb{E}_n \otimes \mathbb{R})$ is the connected component containing the identity for the Zariski topology, but has two connected components as a real Lie group.

empty. We can therefore decrease a_0 until it becomes 0. In this case one of the simple roots α_i becomes effective, a contradiction. \square

4.3. Special points sets of nine points.

Theorem 4.6. *Let \mathcal{P} be a Cremona special points set with 9 points. Then \mathcal{P} is an unnodal Halphen set of some index m .*

Proof. Let X be the surface obtained by blowing up \mathcal{P} . Since $\text{Aut}(X)^*$ has finite index in W_X , this group contains also a free abelian group of rank 8 equal to a subgroup of finite index in $\iota(\mathbb{E}_8)$ (see §2.4). This group preserves the isotropic vector K_X . By Gizatullin's theorem there exists a (minimal) positive integer m such that the linear system $|-mK_X|$ is an irreducible elliptic pencil. Thus X is a Halphen surface of index m . By Proposition 4.5, X has no (-2) -curves. Thus X is an unnodal Halphen surface. \square

5. THE GENERAL CASE

The main results of this section and the next one are summarized in the following two theorems.

Theorem 5.1. *Let \mathcal{P} be a points set of $n \geq 10$ points which is not contained in a cubic curve. If \mathcal{P} is Cremona special, then \mathcal{P} is an unnodal Coble set. In particular, n is equal to 10.*

By definition, a *Harbourne set* is a set \mathcal{P} of $n \geq 9$ nonsingular points on a cuspidal cubic C_0 over a field of positive characteristic with the following property: If X denotes the surface obtained by blowing up \mathcal{P} and C denotes the strict transform of C_0 in X , then X has no (-2) -curves and the restriction morphism τ from K_X^\perp to $\text{Pic}^0(C)$ has finite image. These points set are Cremona special; this is proved by Harbourne in [24].

Theorem 5.2. *Let \mathcal{P} be a points set of $n \geq 10$ points contained in a cubic curve $C_0 \subset \mathbb{P}^2$. If \mathcal{P} is Cremona special, then \mathbb{K} has positive characteristic, C_0 is a cuspidal cubic, and \mathcal{P} is a Harbourne set.*

These results imply the Main Theorem. In this section, we prove Theorem 5.1.

5.1. A surface Y with ten Halphen pencils. Let $\mathcal{P} = \{p_1, \dots, p_n\}$ be a Cremona special set, with the convention that $j > i$ if p_j is infinitely near p_i . We blow up successively each point p_i and denote by

$$X = X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_1 \rightarrow \mathbb{P}^2$$

the sequence of blow-ups. Let $q_i : X \rightarrow X_i$ be the corresponding birational morphism and $q_i^* : \text{Pic}(X_i) \rightarrow \text{Pic}(X)$ be the canonical homomorphism of the Picard groups. Obviously $q_i^*(K_{X_i}^\perp) \subset K_X^\perp$. Moreover the image under q_i^* of the geometric basis of $\text{Pic}(X_i)$ (defined by the morphism $X_i \rightarrow \mathbb{P}^2$) coincides with the first $i + 1$ vectors of the geometric basis of X (defined by the morphism $X \rightarrow \mathbb{P}^2$). This allows one to identify W_{X_i} with a subgroup of W_X . Since $\text{Aut}(X)^*$ is of finite index in W_X , its intersection with W_{X_i} is a subgroup of finite index; this subgroup preserves the exceptional curves obtained from the blowing ups of p_j for $j > i$, and descends as a group of automorphisms of X_i . Thus all surfaces $X_i, i \geq 9$, are Cremona special, and all sets $\{p_1, \dots, p_i\}, i \geq 9$, are Cremona special sets. By Theorem 4.6, the surface X_9 is an unnodal Halphen surface. In particular, the first 9 points do not contain infinitely near points. Since the set of the first 10 points is Cremona special, Proposition 4.5 shows that the surface X_{10} has no (-2) -curves; hence p_{10} is not infinitely near p_i for $i \leq 9$ and the set $\{p_1, \dots, p_{10}\}$ contains no infinitely near point.

Let i be an index between 1 and 10 such that p_{11} is not infinitely near p_i . Consider the sequence of points $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_{11}$, and apply the same argument. We obtain that p_{11} is not infinitely near the points p_j for $1 \leq j \leq 10$. By induction on n , and permutation of the points, this proves the following lemma.

Lemma 5.3. *If \mathcal{P} is a Cremona special points set of $n \geq 9$ points, then \mathcal{P} is a proper subset of n distinct points in \mathbb{P}^2 , and all subsets of \mathcal{P} of $m \geq 9$ points are Cremona special.*

Let $Y = X_{10}$, $E_i \subset Y$ be the (-1) -curve obtained by blowing up p_i , $1 \leq i \leq 10$, and $\sigma_{E_i} : Y \rightarrow Y_i$ be the blow-down of E_i . Since Y_i is Cremona special, it is an unnodal Halphen surface of some index m_i . The pre-image of the elliptic fibration of Y_i is an elliptic pencil $|m_i f_i|$ on Y , where

$$f_i = -K_Y + e_i = \sigma_{E_i}^*(-K_{Y_i}).$$

The divisor classes f_1, \dots, f_{10} are exactly the primitive isotropic vectors in K_Y^\perp which we introduced in the proof of Theorem 3.2. We denote by $\tilde{p}_i \in Y_i$ the point $\sigma_{E_i}(E_i)$; this point is the pre-image of p_i under the natural projection $Y_i \rightarrow \mathbb{P}^2$.

5.2. Almost all indices are different from 1. Suppose that two indices m_i and m_j are equal to 1, say $m_1 = m_2 = 1$. Let $B_i \subset Y_i, i = 1, 2$, be the fiber of the Halphen fibration which contains \tilde{p}_i , and let a_i be the multiplicity of this fiber at \tilde{p}_i . Let $A_i \subset Y$ be the strict transform of B_i under the morphism σ_{E_i} . Since the fibers of the Halphen surface Y_i are irreducible, both A_i and B_i

are irreducible. Then $-K_Y$ is effective and represented by $A_i + (a_i - 1)E_i$. Since $K_Y^2 = -1$, the two curves $A_1 + (a_1 - 1)E_1$ and $A_2 + (a_2 - 1)E_2$ have a common component, and thus $A_1 = A_2$; as a consequence, $a_1 = a_2 = 1$, i.e. \tilde{p}_i is a smooth point of B_i , for $i = 1, 2$. Denote by A the curve $A_1 = A_2$. By Lemma 2.4, the normal bundle of A_i is equal to $\mathcal{O}_{A_i}(-E_i \cap A_i)$, $i = 1, 2$. Thus the points $E_1 \cap A$ and $E_2 \cap A$ are linearly equivalent on A . Since $p_1 \neq p_2$ and A is an irreducible curve of arithmetic genus 1, we get a contradiction. This proves the following lemma.

Lemma 5.4. *At least nine of the indices m_i , $1 \leq i \leq 10$, are larger than 1.*

5.3. An alternative. Effective curves with divisor class f_i are total transforms of fibers of the Halphen fibration on Y_i with multiplicity m_i . Thus, if $m_i = 1$, the class f_i is represented by an irreducible curve (the total transform of any fiber that does not contain p_i), and if $m_i \geq 2$ the class f_i is represented by an irreducible curve if and only if \tilde{p}_i is not contained in the multiple fiber of Y_i .

Lemma 5.5. *Either the set $\{p_1, \dots, p_{10}\}$ is contained in (the smooth part of) a cubic curve C and then all f_i , $1 \leq i \leq 10$, are represented by $C' + E_i$ where C' is the strict transform of C ; or $\{p_1, \dots, p_{10}\}$ is not contained in a cubic curve, and then all classes f_i are represented by at least one irreducible curve.*

Proof. If the set $\mathcal{P}_{10} = \{p_1, \dots, p_{10}\}$ is contained in a cubic curve C , no p_i is a singular point of C , because otherwise the Halphen surface Y_j , $j \neq i$, would not be special. If the set \mathcal{P}_{10} is not contained in a cubic curve, then p_i is not contained in the multiple fiber of Y_i , so that f_i is represented by an irreducible curve. \square

For the remainder of this section, we make the following assumption, where Lemma 5.5 is used to prove the equivalence.

- (A) *Each primitive isotropic class f_i is represented by an irreducible curve. Equivalently, there is no irreducible cubic curve containing the set $\{p_1, \dots, p_{10}\}$. In particular, the multiple fiber of Y_i does not contain \tilde{p}_i .*

In other words, we now assume that Y satisfies the assumption of Theorem 5.1. Note that the irreducibility of the cubic curves in Assumption (A) follows from Proposition 4.5.

5.4. All indices are different from 1. Suppose now that one index, say m_1 , is equal to 1; by Lemma 5.4, $m_i \geq 2$ for $i \geq 2$. As above, let $B_1 \subset Y_1$ be the fiber of the Halphen fibration which contains the point \tilde{p}_1 , $a_1 \leq 2$ be its multiplicity at \tilde{p}_1 , and A_1 be its strict transform in Y ; then

$$\sigma_{E_1}^*(B_1) = f_1 = A_1 + a_1 E_1.$$

Let $B_2 \subset Y_2$ be the fiber of the Halphen fibration which contains the point \tilde{p}_2 . By our assumption (A), the curve B_2 is not the multiple fiber of the Halphen surface Y_2 . Let A_2 be the strict transform of B_2 and a_2 be the multiplicity of B_2 at \tilde{p}_2 . Then

$$A_2 \sim m_2 f_2 - a_2 E_2 \sim m_2(-K_Y + E_2) - a_2 E_2 \sim -m_2 K_Y + (m_2 - a_2) E_2$$

and

$$\begin{aligned} A_1 \cdot A_2 &= (-K_Y + (1 - a_1) E_1) \cdot (-m_2 K_Y + (m_2 - a_2) E_2) \\ &= -m_2 + (m_2 - a_2) + m_2(1 - a_1) \\ &= -a_2 - (a_1 - 1)m_2; \end{aligned}$$

this number is negative because $a_i \geq 1$. Since A_1 and A_2 are irreducible curves, we obtain $A = A_1 = A_2$. Since A_1 is the strict transform of a cubic curve, this contradicts our standing assumption (A) and proves the following lemma.

Lemma 5.6. *Under assumption (A), all indices m_i are larger than 1.*

5.5. The linear system $|D| = |2f_1 + 2f_2|$. We now show that all indices m_i are indeed equal to 2; permuting the indices, it is sufficient to consider m_1 and m_2 : We already know that $m_1, m_2 > 1$ and we want to show that $m_1 = m_2 = 2$. For this purpose, we study the linear system $|2f_1 + 2f_2|$.

Since $m_i \geq 2$ for all i , the linear system $|f_i|$ consists of a unique element. Hence,

$$|f_1| = \{F_1\}, \quad |f_2| = \{F_2\},$$

where the divisors F_1 and F_2 are irreducible curves of arithmetic genus 1 by assumption (A). Since $F_1 \cdot F_2 = 1$, the curves F_1 and F_2 intersect transversally at one point

$$y_0 = F_1 \cap F_2.$$

Denote by $|D|$ the linear system $|2F_1 + 2F_2|$.

Lemma 5.7. *Under assumption (A), the following properties are satisfied.*

(i) *The linear system $|D| = |2F_1 + 2F_2|$ satisfies*

$$D^2 = 8, \quad \dim |D| = 4,$$

it has no fixed component, and it has no base point. It defines a morphism $\phi : Y \rightarrow \mathbb{P}^4$.

(ii) *Let $S = \phi(Y) \subset \mathbb{P}^4$ be its image. Then, on each curve F_i , $i = 1, 2$, ϕ restricts to a 2-to-1 cover onto a line $\ell_i \subset S$. The two lines ℓ_1 and ℓ_2 are different.*

- (iii) *The linear system $|F_1 + F_2|$ is an irreducible pencil with two base points $y_1 \in F_1, y_2 \in F_2$. The points $s_i = \phi(y_i), i = 0, 1, 2$, span a plane in the projective space \mathbb{P}^4 .*
- (iv) *The degree of $\phi: Y \rightarrow \mathbb{P}^4$ is equal to 2, and its image is an anti-canonical Del Pezzo surface $S \subset \mathbb{P}^4$ of degree 4; in particular, S is a normal surface.*
- (v) *The pre-image of s_0 under the map ϕ consists of exactly one point: $\phi^{-1}\{s_0\} = \{y_0\}$.*

Proof. (i) The argument is the same as in the proof of Lemma 3.7, where we used only that f_1 and f_2 were represented by irreducible curves F_1 and F_2 that do not move (see Remark 3.8).

(ii) Each curve $F_i, i = 1, 2$, is an irreducible curve of arithmetic genus 1. It follows from the third exact sequence in (14) that the restriction of $|D|$ to the curve F_i is given by a complete linear system of degree 2. Thus, for $i = 1, 2$, the curve $\phi(F_i)$ is a line $\ell_i \subset S$ and ϕ restricts to a 2-to-1 cover between F_i and ℓ_i . The same exact sequences show that $\ell_1 \neq \ell_2$.

(iii) The first exact sequence in (14) shows that $|F_1 + F_2|$ is a pencil of curves of arithmetic genus 2. Since $(F_1 + F_2) \cdot F_1 = 1$, its restriction to F_1 (resp. to F_2) has degree 1.

Let ϵ_i be the nontrivial m_i -torsion divisor class on F_i such that $\mathcal{O}_{F_i}(\epsilon_i) \cong \mathcal{O}_{F_i}(F_i)$. For $i = 1, 2$ we have $\mathcal{O}_{F_i}(F_1 + F_2) \cong \mathcal{O}_{F_i}(y_i)$, where y_i is an element of the linear system $|y_0 + \epsilon_i|$ on F_i . Clearly, $y_i \neq y_0$ and does not move because the arithmetic genus of F_i is 1. This shows that $|F_1 + F_2|$ has no fixed components and its base points are y_1 and y_2 .

The restriction of ϕ to F_i is given by the linear system $|D|$, i.e. by

$$|\mathcal{O}_{F_i}(2F_1 + 2F_2)| = |\mathcal{O}_{F_i}(2y_0 + 2\epsilon_i)|.$$

Since $2y_0 + 2\epsilon_i \sim y_i + y_0 + \epsilon_i \not\sim y_0 + y_i$, we obtain that the points y_0 and y_i are not mapped to the same point by ϕ ; thus, $s_0 = \phi(y_0)$ and $s_i = \phi(y_i), i = 1, 2$, are distinct. As a consequence, the plane Π_0 spanned by the two lines ℓ_1 and ℓ_2 is also spanned by the three points s_0, s_1 , and s_2 .

(iv) Let us now prove that the degree of ϕ is equal to 2, i.e. $\deg S = 4$. Then, as in Section 3.4 (see after Lemma 3.9), the image S of ϕ is a del Pezzo surface of degree 4.

Since the map is given by a complete linear system, and a quadric surface in \mathbb{P}^4 is contained in a hyperplane, the only other possibility is that the degree is equal to 1, i.e. $\deg S = 8$; thus, we now assume $\deg(S) = 8$ and seek a contradiction.

Let σ be a general point of Y ; in particular, σ is not a base point of $|F_1 + F_2|$ and its image $\sigma' = \phi(\sigma)$ in S is not contained in the plane Π_0 . Let $\pi : S \rightarrow S' \subset \mathbb{P}^3$ be the projection from σ' . The composition $\pi \circ \phi$ is a rational map given by the linear subsystem L of $|2F_1 + 2F_2|$ of divisors passing through the point σ . Replacing Y by its blow-up $Y' \rightarrow Y$ at the point σ , we obtain a birational morphism $\phi' : Y' \rightarrow S'$ onto a surface of degree 7.

A general member P of the pencil $|F_1 + F_2|$ is an irreducible curve of arithmetic genus 2. Consider the restriction of the map ϕ' to P ; it is given by a linear series of degree 4; would this linear series coincide with $|2K_P|$, then $\phi'|_P : P \rightarrow \phi'(P)$ would be a 2 to 1 cover onto a conic curve (see the proof of Lemma 3.9), and ϕ would have degree 2. Thus, $\phi'|_P$ is given by a linear series $|K_P + p + q|$ where $p + q \notin |K_P|$. The image $\phi'(P)$ is a plane quartic curve, the points p, q are mapped to a singular point of $\phi'(P)$, and $\phi'|_P$ provides an isomorphism from $P \setminus \{p, q\}$ to $\phi'(P) \setminus \{\phi'(p)\}$.

The unique member P_0 from $|F_1 + F_2|$ which passes through the point σ is mapped to a plane curve on S passing through σ' . This curve has degree 4, hence it is projected to a triple line ℓ'_0 on S' . Since $y_1, y_2 \in P_0$, the triple line ℓ'_0 contains the projections s'_1, s'_2 of the points s_1, s_2 . By assertion (iii), the points s_0, s_1, s_2 span a plane Π_0 in \mathbb{P}^4 . Since Π_0 does not contain the image σ' it is projected onto a plane $\Pi'_0 \subset \mathbb{P}^3$ and this plane is spanned by the images s'_0, s'_1 , and s'_2 of the three points s_0, s_1 , and s_2 . The triple line $\ell'_0 = \phi'(P_0)$ is spanned by s'_1 and s'_2 .

The image of the pencil $|F_1 + F_2|$ on S' is cut out by the pencil \mathcal{Q} of planes containing ℓ'_0 . The plane Π'_0 is a member of \mathcal{Q} and cuts out in S' a curve of degree 7: This curve is equal to the union of the three lines ℓ'_1, ℓ'_2 , and ℓ'_0 , where the first two enter with multiplicity 2 and the last one with multiplicity 3. Any other plane from \mathcal{Q} cuts out in S' the line ℓ'_0 taken with multiplicity 3 and a quartic curve $P' = \phi'(P)$ for some $P \in |F_1 + F_2|$. Let N be the closure of the set of double points of irreducible plane quartics cut out by \mathcal{Q} in S' . It is a double conic K on S' (its pre-image on Y intersects P at two points, hence intersects $2F_1 + 2F_2$ with multiplicity 4). It passes through a point on ℓ'_0 , the image of a double point of the plane quartic $\phi(P_0)$.

Let us see what else is in the singular locus of S' . Suppose Z is another irreducible curve in the singular locus of S' which is not contained in any plane from \mathcal{Q} . Then a general plane Π' from \mathcal{Q} intersects it, and hence the image of a general $P \in |F_1 + F_2|$ in Π' acquires an additional double point. This contradiction shows that Z is contained in some plane $\Pi' \in \mathcal{Q}$, certainly, different from Π'_0 . We claim that it must be a double conic. The unique alternative

possibility is that Z is a double line. Let F be its pre-image in Y . It is an irreducible component of some divisor P from $|F_1 + F_2|$, with $P = F + R$ for some effective divisor R . Since the line Z intersects only one of the points s'_1 and s'_2 , the curve F passes through only one base point of $|F_1 + F_2|$, hence we may assume that $F \cdot F_1 = 0$. Let $\sigma_{E_1} : Y \rightarrow Y_1$ be, as in Section 5.1, the blow-down of the curve E_1 . We know that Y_1 is a Halphen surface with elliptic fibration $|m_1\sigma_{E_1}(F_1)|$, the fibers of which are irreducible. We also know that $F_1 = \sigma_{E_1}^*(-K_{Y_1})$, hence $(\sigma_{E_1})_*(F) \cdot K_{Y_1} = 0$. If $\sigma_{E_1}(F)$ is not a point, it must be a fiber of the elliptic fibration $|m_1\sigma_{E_1}(F_1)|$, and it can not be the multiple fiber $\sigma_{E_1}(F_1)$ (because $Z \neq \ell'_1$). Since $F \cdot F_2 \leq 1$, this contradicts $m_1 > 1$, and shows that $\sigma_{E_1}(F)$ is a point, which means that F is the curve E_1 . In particular, $F \cdot F_2 = 1$, $F_1 \cdot R = 1$, and $F_2 \cdot R = 0$, because the self-intersection of P is 2. The same reasoning, applied to σ_{E_2} , implies that R is equal to E_2 , and provides a contradiction because $E_1 + E_2$ is not a member of $|F_1 + F_2|$.

So, we have now computed the singular locus of S' . It consists of two double lines ℓ'_1, ℓ'_2 , one triple line ℓ'_0 , the double conic K , and some number c of double conics contained in planes from the pencil \mathcal{Q} .

A general plane section of S' is a plane curve of degree 7. It has $2 + 2(c + 1)$ double points and one triple point. Its geometric genus is equal to $15 - 3 - 2 - 2(c + 1)$ which is an even number. On the other hand, the geometric genus is equal to the genus of a general curve from $|2F_1 + 2F_2|$ which is equal to 5. This contradiction proves assertion (iv).

(v) Since $\phi : Y \rightarrow S$ is a map of degree 2 onto a normal surface, the existence of the Stein decomposition of ϕ implies that the pre-image of any point consists of at most two points or contains a one-dimensional component.

We want to show that the pre-image of $\{s_0\}$ is reduced to $\{y_0\}$. It contains the union of the pre-images of s_0 under the restriction maps $F_i \rightarrow \ell_i$. Since each such map is of degree 2 and ϕ does not blow down any curve intersecting $F_1 + F_2$, we may assume that y_0 is a ramification point of $\phi|_{F_1} : F_1 \rightarrow \ell_1$ and the pre-image of $F_2 \rightarrow \ell_2$ consists of two points y_0 and y'_0 .

Let $\nu : \bar{S} \rightarrow S$ be the blow-up of the point s_0 . Suppose s_0 is a nonsingular point of S . Then the exceptional curve E of ν is a (-1) -curve on \bar{S} . The rational map $\nu^{-1} \circ \phi : S \dashrightarrow \bar{S}$ extends to a regular map $\bar{\phi} : \bar{Y} \rightarrow \bar{S}$ with disconnected exceptional locus over y_0 and y'_0 . However, the restriction of $\bar{\phi}$ over E is ramified at the point of intersection of E with the proper transform of ℓ_1 , and hence $\bar{\phi}^{-1}(E)$ can not be disconnected. This contradiction shows that s_0 is a singular point of S , hence ϕ is not étale at y_0 , hence s_0 is a ramification point of ϕ and, as such, has a unique pre-image. \square

Now we can easily deduce from this lemma that $m_1 = m_2 = 2$. We know that the maps $F_i \rightarrow \ell_i$ ramify over s_0 . Each map is given by the restriction of the linear system $|2F_1 + 2F_2|$. It is equal to the linear system $|2y_0 + 2\epsilon_i|$, where $\mathcal{O}_{F_i}(\epsilon_i) \cong \mathcal{O}_{F_i}(F_i)$ and m_i is the order of ϵ_i in $\text{Pic}^0(F_i)$. Since $2y_0 \in |2y_0 + 2\epsilon_i|$, we obtain $2\epsilon_i \sim 0$, hence $m_i \leq 2$. Since all indices m_i are larger than one, we get $m_i = 2$ for all $1 \leq i \leq 10$.

5.6. The surface Y is a Coble surface. The surface Y is obtained from Y_i by blowing up a point \tilde{p}_i ; this point is contained in a non-multiple member B_i of the Halphen pencil of Y_i . Let a_i be the multiplicity of \tilde{p}_i on B_i : It is equal to 1 if \tilde{p}_i is a nonsingular point of B_i and 2 otherwise. Let A_i be the strict transform of B_i on Y . Since $m_i = 2$, we have

$$(17) \quad A_i + a_i E_i \sim 2f_i = -2\sigma_{E_i}^*(K_{Y_i}) = -2K_Y + 2E_i.$$

If $a_i = 2$ for some index, then $A_i \in |-2K_Y|$, hence Y is a Coble surface, all A_i coincide, and their natural projection on \mathbb{P}^2 is a sextic curve with double points at p_1, p_2, \dots, p_{10} .

Suppose $a_i = 1$ for all i . Then

$$(18) \quad A_i \sim -2K_Y + E_i.$$

Let $F_i \in |f_i| = |-K_Y + E_i|$ be the pre-image of the reduced double fiber of the Halphen pencil on Y_i . Since σ_{E_i} is an isomorphism over an open neighborhood of this fiber, the normal bundle $\mathcal{O}_{F_i}(F_i)$ is of order 2 in $\text{Pic}(F_i)$. Applying (18), we obtain

$$\mathcal{O}_{F_i}(A_j - E_j) \cong \mathcal{O}_{F_i}(-2K_Y) \cong \mathcal{O}_{F_i}.$$

For $j \neq i$, $A_j \cdot F_i = E_j \cdot F_i = 1$, hence A_j and E_j intersect F_i transversally at the same point. Let $\pi : Y \rightarrow \mathbb{P}^2$ be the natural projection, i.e. the blow-down of all curves E_l , $1 \leq l \leq 10$. The image of F_i is an irreducible cubic curve C_i . By blowing down E_j , we see that $\pi(A_j)$ is tangent to C_i at p_j . Taking $k \neq i, j$, we obtain that $\pi(A_j)$ is tangent at the same point to C_k ; since A_j intersects E_j transversely, C_k and C_i are tangent at p_j . Fixing i and k and changing j , we see that C_k and C_i are tangent at the 8 points p_j with $j \neq k$, so that the two irreducible cubics C_i and C_k coincide. Hence $F_i = F_k$, while $f_i \neq f_k$: This contradiction shows that Y is a Coble surface.

Thus we have proved Theorem 5.1 when \mathcal{P} contains exactly 10 points.

5.7. The surface X coincides with the Coble surface Y . To conclude the proof of Theorem 5.1, it remains to show that $n = 10$, and thus $X = Y$ is a Coble surface, when \mathcal{P} is a Cremona special set not contained in a cubic curve.

We may assume that $X = X_{11}$, so that X is obtained by blowing up 11 points p_1, \dots, p_{11} in \mathbb{P}^2 ; by Lemma 5.3 none of them is an infinitely near point.

By assumption, \mathcal{P} is not contained in a cubic curve. Let \mathcal{P}_j , $1 \leq j \leq 11$, be the subset of \mathcal{P} obtained by removing the point p_j . Assume that three of the sets \mathcal{P}_j , say $\mathcal{P}_9, \mathcal{P}_{10}$ and \mathcal{P}_{11} are contained in cubic curves, say C_9, C_{10} and C_{11} . By Section 5.3, these three cubics are irreducible. Let $\mathcal{P}' = \{p_1, \dots, p_8\}$. Since \mathcal{P}' lies on an irreducible cubic curve, no four of its points are collinear. It follows that the linear system of cubic curves containing \mathcal{P}' is of dimension 1; let q be the ninth base point of this pencil. The curves C_9, C_{10} , and C_{11} belong to this pencil, hence any two of them intersect at q . Consequently C_9 and C_{10} contain the set \mathcal{P}' and the points q and p_{11} . It follows that $C_9 = C_{10}$ or that $q = p_{11}$. If $C_9 = C_{10}$, then \mathcal{P} is contained in this cubic curve, if $q = p_{11}$, then C_{11} contains \mathcal{P} ; in both cases, \mathcal{P} is contained in a cubic curve, a contradiction.

Thus, at most 2 of the \mathcal{P}_j are contained in cubic curves. We can therefore suppose that $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and \mathcal{P}_4 are not contained in cubic curves. The surfaces obtained by blowing up these sets are Cremona special and, as such, are unnodal Coble surfaces. In particular, each set \mathcal{P}_l , $1 \leq l \leq 4$, determines a unique curve of degree six with nodes along \mathcal{P}_l .

Let us first assume that the characteristic of the field \mathbb{K} differs from 2. Consider the Del Pezzo surface Z of degree 2 obtained by blowing up the last seven points p_5, \dots, p_{11} . We identify the elements of the set

$$\mathcal{Q} = \{p_1, p_2, p_3, p_4\}$$

with points on Z . By assumption, we have 4 curves C_i on Z in the linear system $| - 2K_Z |$ with double points at $\mathcal{Q} \setminus \{p_i\}$; each C_i is the proper transform of the sextic curve with nodes along \mathcal{P}_i , $1 \leq i \leq 4$. Consider the map $Z \rightarrow \mathbb{P}^2$ defined by the linear system $| - K_Z |$. Since $\text{char}(\mathbb{K}) \neq 2$, its branch curve is a plane quartic curve B (see [13], Chapter V.6, page 67). Each curve C_i is equal to the pre-image of a conic K_i in the plane. Since C_i is singular at three points $p_j \in \mathcal{Q} \setminus \{p_i\}$, the conic K_i is tangent to B at the images q_j of the points p_j . In particular, the conics K_i and K_j are tangent to B at two points. Consider the cubic curves $K_i + L_i$, where L_i is the tangent line to B at q_i . They all pass through q_1, q_2, q_3 , and q_4 with tangent direction L_i at q_i . Thus they generate a pencil of cubic curves with 8 base points (four are infinitely near the points q_i). The ninth base point must be the intersection point q of the lines L_i . But then three tangents of the conic K_i meet at q . This can happen only if $\text{char}(\mathbb{K}) = 2$, so that $n = 10$ and $X = Y$ when $\text{char}(\mathbb{K}) \neq 2$.

Remark 5.8. The configuration of 4 conics with each pair tangent at two points is realized in characteristic 2. Consider the 3-dimensional linear system of

conics $ax^2 + by^2 + cz^2 + dxy = 0$. Each line through the point $(0 : 0 : 1)$ is tangent to all conics in the family. Choose four general points in the plane. For each subset of three of these points find a unique conic in the family which passes through these points. Then each pair of the four conics are tangent at two points.

Note that the pencil of cubic curves which we used in this proof defines a quasi-elliptic fibration on the blow-up of the base points. It has 8 reducible fibers of type III in Kodaira's notation (two smooth rational curves tangent to each other at one point). There are no elliptic fibrations on a rational surface with such a combination of reducible fibers.

It remains to consider the case when $\text{char}(\mathbb{K}) = 2$. The difference here is that the anti-canonical linear system $| -K_Z |$ defines a separable map of degree 2 whose branch curve is a conic and the condition that the pre-image of a conic is singular is not stated in terms of the tangency to the branch locus. So we have to find another argument.

Since at most 2 of the \mathcal{P}_j are contained in cubic curves, we can assume that \mathcal{P}_i are not contained in cubic curves for $1 \leq i \leq 9$; each of these nine points is special, and is therefore a Coble set. Let R_i be the proper transform of the sextic curve with double points at \mathcal{P}_i , $i = 1, \dots, 9$. The curves R_i are pairwise disjoint (-4) -curves and the divisor class of each R_i is divisible by 2. Let D be the divisor $\sum R_i$, where i runs from 1 to 9, and let \mathcal{L} be an invertible sheaf on X such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_X(D)$. This sheaf defines an inseparable double cover $\pi : Z \rightarrow X$ with branch divisor D . Recall that this means that Z is locally isomorphic to $\text{Spec } \mathcal{O}_X(U)[T]/(T^2 + \phi)$, where U is an affine open set, ϕ is a local equation of D in U , and $\pi_*(\mathcal{O}_Z) \cong \mathcal{O}_X \oplus \mathcal{L}^{-1}$. Since D is a reduced divisor, the cover has only finitely many singularities. The set of singularities supports the scheme of zeros of a section of $\Omega_X^1 \otimes \mathcal{L}^{\otimes 2}$ (see [12], Proposition 0.1.2, ⁷). The length of this 0-dimensional subscheme is equal to the second Chern class of the rank 2 locally free sheaf $\mathcal{E} = \Omega_X^1 \otimes \mathcal{L}^{\otimes 2}$. The standard formula from the theory of Chern classes gives

$$c_2(\mathcal{E}) = c_2(\Omega_X^1) + c_1(\Omega_X^1) \cdot D + D^2.$$

In our situation, we have $c_2(\Omega_X^1) = 14$ (= the l-adic Euler characteristic of X) and $c_1(\Omega_X^1) = K_X$. This gives

$$c_2(\mathcal{E}) = 14 + K_X \cdot D + D^2 = 14 + 2 \times 9 - 4 \times 9 = -4 < 0,$$

a contradiction.

⁷This is analogous to the formula for the number of singular points of a holomorphic foliation on a complex surface X defined by a section of $\Omega_X^1 \otimes \mathcal{L}^{\otimes 2}$

Thus, Theorem 5.1 is proved in all cases, including $\text{char}(\mathbb{K}) = 2$.

6. CREMONA SPECIAL SETS OF $n \geq 10$ POINTS LYING ON A CUBIC CURVE

To establish our Main Theorem, it remains to prove Theorem 5.2: Assuming that \mathcal{P} is Cremona special and lies on an irreducible cubic curve, we now prove that the set \mathcal{P} is one of Harbourne's sets in this case. Note that, being Cremona special, \mathcal{P} does not contain infinitely near point.

6.1. Torsion sets of points.

Lemma 6.1. *If \mathcal{P} is a Cremona special set of points on a cubic curve C_0 , then*

- (1) C_0 is irreducible;
- (2) there exists a positive integer s such that the divisor class of sp_i in $\text{Pic}(C_0)$ does not depend on the choice of p_i in \mathcal{P} ;
- (3) $9sp_i \in |3s\mathfrak{h}|$ for all $p_i \in \mathcal{P}$, where $\mathfrak{h} = c_1(\mathcal{O}_{C_0}(1))$, the divisor class of the intersection of C_0 with a line.

Proof of Lemma 6.1. We first assume that the number of points in \mathcal{P} is equal to 10. For all indices $i \in \{1, \dots, 10\}$, denote by \mathcal{P}_i the set $\mathcal{P} \setminus \{p_i\}$. Since \mathcal{P} is Cremona special, so is \mathcal{P}_i . By Theorem 4.6, there exists a positive integer m_i such that \mathcal{P}_i is a Halphen set of index m_i . Since \mathcal{P}_i is contained in C_0 , this implies that C_0 is irreducible (otherwise the Halphen surface is not unnodal, hence \mathcal{P}_i is not Cremona special), and that

$$m_i \left(\sum_{j=1, j \neq i}^{10} p_j \right) \sim 3m_i \mathfrak{h},$$

where \mathfrak{h} is the divisor obtained by intersecting C_0 with a line. Let s be the least common multiple of the m_i , and let Σ be the sum of the 10 points p_i . We obtain

$$s(\Sigma - p_i) \sim 3s\mathfrak{h}.$$

Hence the divisor class of sp_i does not depend on p_i . Summing up these equalities, we get $9s\Sigma \sim 30s\mathfrak{h}$, and thus $9sp_i \sim 3s\mathfrak{h}$.

If there are more than ten points in \mathcal{P} , we consider all subsets of ten points within \mathcal{P} . Properties (1), (2), and (3) hold for these subsets, and therefore also for \mathcal{P} (for some positive integer s). \square

The following definition is due to Harbourne [24].

Definition 6.2. A rational surface X with Picard number $\rho(X) \geq 11$ and with $|-K_X| = \{D\}$ for some irreducible reduced curve D is **K3-like** if the canonical restriction homomorphism

$$\mathfrak{r}: K_X^{-1} \rightarrow \text{Pic}^0(D)$$

has finite image.

We say that a points set \mathcal{P} which is contained in an irreducible reduced cubic curve C_0 is a **torsion set** if the blow-up surface X is K3-like. In that case, the strict transform C of C_0 coincides with the unique member D of the linear system $|-K_X|$.

When $\mathcal{P} \subset C_0$ is a torsion set, we denote by X the surface obtained by blowing up \mathcal{P} , and by S_τ the kernel of the restriction homomorphism $\tau: K_X^\perp \rightarrow \text{Pic}^0(C)$. The quotient group K_X^\perp/S_τ embeds as a finite subgroup in $\text{Pic}^0(C)$; we denote by $m > 1$ the smallest integer such that K_X^\perp/S_τ is contained in the m -torsion subgroup $\text{Pic}^0(C_0)[m]$.

Corollary 6.3. *A Cremona special set of $n \geq 10$ points on an irreducible cubic curve C_0 is a torsion set.*

Proof. The image of the restriction homomorphism $\tau: K_X^\perp \rightarrow \text{Pic}^0(C_0)$ is generated by the divisor classes $\tau(e_0 - e_1 - e_2 - e_3)$, and $\tau(e_i - e_j)$ for $i > j$. From Lemma 6.1, we have

$$3s\tau(e_0 - e_1 - e_2 - e_3) = \tau(3s\mathfrak{h} - 9sp_1) = 0, \quad s\tau(e_i - e_j) = \tau(sp_i - sp_j) = 0.$$

This shows that the image of the restriction homomorphism is finitely generated and is contained in the $3s$ -torsion subgroup $\text{Pic}(C_0)[3s]$ of $\text{Pic}^0(D)$. As such, $\tau(K_X^\perp)$ is finite. \square

Proposition 6.4. *Let \mathcal{P} be a torsion set of $n \geq 10$ points on an irreducible cubic curve C_0 . Then \mathcal{P} is Cremona special if and only if it is unnodal.*

Proof. By Proposition 4.5, the condition that \mathcal{P} is unnodal is necessary. By Theorem 3.2 from [24], it is also sufficient.

Let us sketch Harbourne's argument. Assume that \mathcal{P} is unnodal. Since \mathcal{P} is a torsion set, the surface X is K3-like: The kernel S_τ or τ has finite index in K_X^\perp . By definition of m , $mK_X^\perp \subset S_\tau$. Let w be an element of W_X . Let $E_i \subset X$ be the exceptional divisor obtained by blowing up p_i . The image of the class e_i of E_i by w is represented by a unique (-1) -curve $E_i(w)$ (Lemma 3.4, assertion (i)). This curve intersects C in a unique point $p_i(w)$ because $C \in |-K_X|$. Thus, w transforms the points set \mathcal{P} into a new points set $\{p_1(w), \dots, p_n(w)\}$ of the curve C_0 . This action is the same as the action of W_n on points set, described in Section 1.5 (see also [20], §VI.5). On the other hand, the image of τ being finite, there exists a finite index subgroup G of W_X such that $\{p_1(w), \dots, p_n(w)\}$ is projectively equivalent to \mathcal{P} for all w in G . From Section 1.5 (see [20], §VI), this implies that G is realized as a subgroup of $\text{Aut}(X)^*$; thus, X is special. \square

Example 6.5 (Harbourne’s examples, see Example 3.4 from [24]). Suppose that $\text{char}(\mathbb{K}) = p > 0$ and C_0 is an irreducible cuspidal curve. The group $\text{Pic}^0(C_0)$ is isomorphic to the abelian group $(\mathbb{K}, +)$.

Let $C_0^\#$ be the complement of the singular point. Choose a points set $\mathcal{P} = \{q_1, \dots, q_n\}$ of $n \geq 10$ points on $C_0^\#$, denote by X the blow-up of \mathcal{P} and consider the restriction homomorphism

$$\tau : K_X^\perp \rightarrow \text{Pic}^0(C_0) \cong \mathbb{K}.$$

Its kernel S_τ is equal to pK_X^\perp when the elements $3\mathfrak{h} - q_1 - q_2 - q_3$, $q_1 - q_2$, ..., and $q_{n-1} - q_n$ of $\text{Pic}^0(C_0)$ are linearly independent over $\mathbb{F}_p \subset \mathbb{K}$; general points set with $n \geq 10$ points satisfy this property.

Suppose, now, that $S_\tau = pK_X^\perp$. Since the divisor class of any (-2) -curve lies in the kernel of τ and, obviously, does not belong to pK_X^\perp , the points set \mathcal{P} is unnodal. By Proposition 6.4, \mathcal{P} is Cremona special.

From Proposition 6.4 we deduce that Theorem 5.2 is a consequence of the following statement.

Theorem 6.6. *Let $C_0 \subset \mathbb{P}^2$ be an irreducible cubic curve. Let $\mathcal{P} \subset C_0$ be a torsion set with $|\mathcal{P}| = 10$, and let m be the smallest integer such that the image of the restriction morphism τ is contained in $\text{Pic}^0(C_0)[m]$. If \mathcal{P} is unnodal, then C_0 is a cuspidal cubic and $\text{char}(\mathbb{K})$ divides $3m$.*

Note, in particular, that we assume in the following that the number n of blow-ups is equal to 10. Thus, \mathcal{P} is now a torsion set of ten distinct points on an irreducible cubic curve C_0 . As above, C denotes the strict transform of C_0 in the surface X . The following lemmas provide a way to decide whether \mathcal{P} is unnodal or not in terms of effective roots $\alpha \in K_X^\perp$ (see Section 3.3).

Lemma 6.7. *Let \mathcal{P} be a set of $n = 10$ points on an irreducible cubic curve C_0 . Let $\alpha \in K_X^\perp$ be an effective root. Any effective representative of α contains a (-2) -curve as one of its irreducible components.*

Proof. Let D be an effective representative of the root α . Write D as a positive sum $lC + \sum F_i$, where the F_i are irreducible components different from C . Assume that no component F_i is a (-2) -curve. Since $-K_X$ is represented by the irreducible curve C , we have $K_X \cdot F_i \leq 0$. The adjunction formula implies that any curve F_i with negative self-intersection is a (-1) -curve. Since $(\sum F_i)^2 = (D - lC)^2 = -2 - l^2 < 0$, some of the components F_i are (-1) -curves. Write $\sum F_i = \mathcal{E} + \mathcal{A}$, where \mathcal{E} is the sum of (-1) -components of D . We have

$$\mathcal{E}^2 \leq \mathcal{E}^2 + 2\mathcal{E} \cdot \mathcal{A} + \mathcal{A}^2 = \left(\sum F_i\right)^2 = -2 - l^2,$$

$$\mathcal{E} \cdot C = (D - lC - \mathcal{A}) \cdot C = l - \mathcal{A} \cdot C \leq l.$$

Write $\mathcal{E} = \sum k_j F_j$, where the curves F_j in this sum are distinct (-1) -curves. The second inequality gives $\sum k_j \leq l$ and the first one gives $\sum k_j^2 \geq 2 + l^2$. This implies $2 + l^2 \leq \sum k_j^2 \leq (\sum k_j)^2 \leq l^2$, a contradiction. \square

It follows from this lemma that X is unnodal if and only if it does not contain effective roots.

Lemma 6.8. *Let \mathcal{P} be a torsion set of $n = 10$ points on an irreducible cubic curve C_0 . The surface X contains a (-2) -curve if and only if there exists a root α in the kernel S_τ of the restriction homomorphism $\tau : K_X^\perp \rightarrow \text{Pic}^0(C)$.*

Proof. Let α be a root. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_X(K_X + \alpha) \rightarrow \mathcal{O}_X(\alpha) \rightarrow \mathcal{O}_C(\alpha) \rightarrow 0,$$

where the first non-trivial map is the multiplication by a section of $\mathcal{O}_X(-K_X)$ which vanishes along the curve C . Riemann-Roch Formula, applied to the divisor class $-\alpha$, and Serre's Duality imply

$$(19) \quad h^0(-\alpha) + h^0(K_X + \alpha) = h^1(K_X + \alpha).$$

Suppose α is the class of a (-2) -curve R . Since C is an irreducible curve and $C \cdot R = 0$, the curve R is disjoint from C . Hence $\tau(\alpha) = \mathcal{O}_C(\alpha) \cong \mathcal{O}_C$, and thus $\tau(\alpha) = 0$ in $\text{Pic}(C)$. Conversely, suppose $\mathcal{O}_C(\alpha) \cong \mathcal{O}_C$. Then $h^0(\mathcal{O}_C(\alpha)) \neq 0$, hence either $h^0(\mathcal{O}_X(\alpha)) \neq 0$, or $h^1(\mathcal{O}_X(K_X + \alpha)) \neq 0$. In the second case,

$$0 < h^1(K_X + \alpha) = h^0(K_X + \alpha) + h^0(-\alpha) \leq h^0(\alpha) + h^0(-\alpha)$$

implies that α or $-\alpha$ is effective. Thus, in both case, either α or $-\alpha$ is an effective root and Lemma 6.7 implies that X contains a (-2) -curve. \square

6.2. The reduction to a question on the arithmetic of quadratic forms. The proof of Theorem 6.6 that we now describe is rather delicate, and uses strong approximation results in a specific situation; unfortunately, we have not been able to find a simpler geometric argument.

We assume that $\mathcal{P} \subset C_0$ is a torsion set and make use of the notations introduced in the previous section. We assume that either (i) C_0 is not cuspidal or (ii) C_0 is cuspidal but $\text{char}(\mathbb{K})$ does not divide $3m$, and our goal is to show that X contains a (-2) -curve; by Lemma 6.8, all we need to prove is the existence of a root α in the kernel S_τ of the morphism τ .

Note that, under our assumptions $\text{Pic}^0(C)[m] \cong (\mathbb{Z}/m\mathbb{Z})^a$, where $a \leq 2$. Obviously, S_τ contains the sublattice mK_X^\perp . Define $V_m = S_\tau/mK_X^\perp$ and $L_m = K_X^\perp/mK_X^\perp$; V_m is a submodule of L_m over the ring $\mathbb{Z}/m\mathbb{Z}$:

$$V_m = S_\tau/mK_X^\perp \subset L_m = K_X^\perp/mK_X^\perp \cong (\mathbb{Z}/m\mathbb{Z})^{10}.$$

This submodule V_m contains a free $(\mathbb{Z}/m\mathbb{Z})$ -submodule of dimension 8 because $a \leq 2$. The morphism τ induces a morphism from L_m to $\text{Pic}^0(C)[m]$, and Theorem 6.6 becomes a consequence of the following purely arithmetic statement.

Theorem 6.9. *Let m be a positive integer. Let $L_m = \mathbb{E}_{10}/m\mathbb{E}_{10}$ and V_m be a $(\mathbb{Z}/m\mathbb{Z})$ -submodule of L_m containing a free submodule of rank 8. Then there exists a root in \mathbb{E}_{10} whose projection into L_m is contained in V_m .*

We employ the theory of quadratic forms over any commutative ring A . Let M be a finitely generated A -module. A function $q : M \rightarrow A$ is called a **quadratic form** if

- for any $x \in M$ and $a \in A$, $q(ax) = a^2q(x)$;
- $b_q(x, y) = q(x+y) - q(x) - q(y)$ is a symmetric bilinear map $M \times M \rightarrow A$.

The bilinear form b_q is called the associated symmetric bilinear form. If 2 is invertible in A , then $q = 2^{-1}b(x, x)$, and the notion of a quadratic form is equivalent to the notion of a symmetric bilinear form. A module M equipped with a quadratic form q is called a **quadratic module**.

We denote by b the bilinear form on K_X^\perp given by the intersection product, and we equip L_m with the symmetric bilinear form b_m obtained by reduction of b modulo m . When m is even, we equip L_m with the structure of a quadratic module by setting

$$q_m(x) = \frac{1}{2}b_m(x, x)$$

(recall that our lattice K_X^\perp is an even unimodular lattice). Let $O(L_m)$ be the orthogonal group of (L_m, b_m) (resp. of (L_m, q_m) when m is even).

To prove Theorem 6.9, fix a root α and consider its image $\bar{\alpha}$ in L_m . Suppose we find an element $\sigma \in O(L_m)$ such that $\sigma(\bar{\alpha}) \in V_m$. Suppose, moreover, that σ lifts to an element $w \in O(K_X^\perp)$. Then $w(\alpha)$ is a root, $w(\alpha)$ is contained in S_τ , and we are done. We now develop this strategy.

6.3. Orthogonal and Spin groups modulo p^l . Since the quadratic form b of the lattice \mathbb{E}_{10} is unimodular, there is a connected smooth group scheme SO_b over \mathbb{Z} such that $\text{SO}_b(\mathbb{Z})$ coincides with the group of isometries of the lattice \mathbb{E}_{10} with determinant 1.

The universal cover of \mathbf{SO}_b is a smooth group scheme \mathbf{Spin}_b over \mathbb{Z} ; the group $\mathbf{Spin}_b(\mathbb{Z})$ is the group $\mathrm{Spin}(\mathbb{E}_{10})$ of all invertible elements of the even part of the Clifford algebra of the quadratic \mathbb{Z} -module \mathbb{E}_{10} such that the corresponding inner automorphism leaves invariant \mathbb{E}_{10} (see [29], Chapter IV, §5).

There is an exact sequence of group schemes

$$(20) \quad 1 \rightarrow \mu_2 \rightarrow \mathbf{Spin}_b \rightarrow \mathbf{SO}_b \rightarrow 1$$

where μ_2 is the group of square roots of 1. For any commutative ring A with $\mathrm{Pic}(A) = 0$, the exact sequence defines the following exact sequence of groups

$$1 \rightarrow \mu_2(A) \rightarrow \mathbf{Spin}_b(A) \rightarrow \mathbf{SO}_b(A) \rightarrow A^*/A^{*2}$$

(see [29], Theorem 6.2.6). This will be applied to $A = \mathbb{Z}/p^k\mathbb{Z}$ for prime numbers p . In this case $|A^*/A^{*2}| = |\mu_2(A)|$; this number is equal to 2 if $p \neq 2$, and to 1, 2, or 4 if $p = 2$ and $k = 1$, $k = 2$, or $k \geq 3$ respectively. Thus the image of $\mathbf{Spin}_b(A)$ in $\mathbf{SO}_b(A)$ is of index at most 4.

The group scheme \mathbf{Spin}_b is requested in order to apply the Strong Approximation Theorem (see [28], Theorem 24.6):

Theorem 6.10 (Strong Approximation). *Let M be a unimodular indefinite integral quadratic lattice of rank ≥ 3 ; if p is a prime integer, denote by $M_p = M \otimes \mathbb{Z}_p$ its p -adic localization at p . Let p_i , $i \in I$, be a finite set of prime numbers. Then the canonical homomorphism*

$$\mathrm{Spin}(M) \rightarrow \prod_{i \in I} \mathrm{Spin}(M_{p_i})$$

has a dense image.

Note that, by the Chinese Remainder Theorem,

$$\mathrm{O}(L_m) = \prod_i \mathrm{O}(L_{p_i^{k_i}}),$$

where $m = \prod p_i^{k_i}$ is the prime factorization of m .

Since the fibers of the natural homomorphism $\mathrm{Spin}(M_{p_i}) \rightarrow \mathrm{Spin}(M/p_i^{k_i})$ are open subsets of $\mathrm{Spin}(M_{p_i})$, we obtain a commutative diagram

$$(21) \quad \begin{array}{ccc} \mathrm{Spin}(\mathbb{E}_{10}) & \longrightarrow & \mathbf{SO}_b(\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{Spin}(\mathbb{E}_{10}/m\mathbb{E}_{10}) & \longrightarrow & \mathbf{SO}(L_m), \end{array}$$

for which the first vertical arrow is surjective, by the Strong Approximation Theorem. Suppose we find a root $\alpha \in \mathbb{E}_{10}$ and an element σ in the image of the bottom horizontal arrow such that $\sigma(\bar{\alpha}) \in V_m$. Then, the commutative

diagram shows that σ can be lifted to an element $w \in \mathbf{SO}_b(\mathbb{Z})$ such that the projection of $w(\alpha)$ into L_m is contained in V_m , and therefore Theorem 6.9 is proved.

6.4. Orbits of \mathbf{SO}_b and \mathbf{Spin}_b modulo p . This section is a warm up for the following ones. Fix a root α in \mathbb{E}_{10} and denote by $\bar{\alpha}$ its image in L_p , where p is a prime integer.

Since L_p is a non-degenerate quadratic space over \mathbb{F}_p , the rank of any submodule of L_p on which the quadratic form vanishes identically modulo p is at most 5. Any quadratic form of rank ≥ 2 over \mathbb{F}_p represents all elements in \mathbb{F}_p^* ([39], Chapter I). Thus, b_p represents all elements of \mathbb{F}_p^* .

Let $r \in V_p$ be an element with $r^2 := b_p(r, r) = -2$. If $p \neq 2$, Witt's Theorem provides an element $\sigma \in \mathbf{O}(L_p)$ such that $\sigma(\bar{\alpha}) = r$.

If $p = 2$, we consider $L_2 = \mathbb{E}_{10}/2\mathbb{E}_{10} \cong \mathbb{F}_2^{10}$ as a quadratic space with the quadratic form $q_2(x) = \frac{1}{2}b(x, x) \pmod{2}$. Since V_2 is of dimension ≥ 8 , q_2 does not vanish identically on V_2 . There are $496 = 2^4(2^5 - 1)$ vectors in the set $q_2^{-1}(1)$ and all of them are represented by roots (see [11], Remark 4.7). Thus there are elements of V_2 which are represented by roots.

As a consequence, *there is a root α in \mathbb{E}_{10} , such that, for each prime number p , there is an element $\sigma \in \mathbf{O}(L_p)$ satisfying $\sigma(\bar{\alpha}) \in V_p$.*

We now extend this idea to the case $m = p^k$ with $k > 1$, and use this extension to prove Theorem 6.9.

6.5. Quadratic modules and orbits modulo p^k . Here, the ring is $A = \mathbb{Z}/p^k\mathbb{Z}$ and the quadratic module is $M = V_m$, equipped with the symmetric bilinear form b_{p^k} or the quadratic form q_{p^k} when $p = 2$. Since A is local, with maximal ideal $\mathfrak{m} = pA$, an element a in A is invertible if and only if a is not contained in \mathfrak{m} .

Lemma 6.11. *The quadratic module V_{p^k} represents all invertible elements of the ring $\mathbb{Z}/p^k\mathbb{Z}$ (i.e. $\forall a \in \mathbb{Z}/p^k\mathbb{Z}$, there exists $x \in V_{p^k}$ such that $q(x) = a$). The same property holds for all free quadratic submodules $M_0 \subset V_{p^k}$ of rank 8.*

Proof. We prove the lemma for V_{p^k} by induction on k . The same proof applies for submodules of V_{p^k} of rank 8.

When $k = 1$, V_p represents all non-zero elements of the field \mathbb{F}_p , as explained in Section 6.4. More precisely, for all $a \in \mathbb{F}_p^*$ there exists a pair of vectors (v, w) in V_p such that $q(v) = a \pmod{p}$ and $b_q(v, w) = 1 \pmod{p}$.

Let k be a positive integer. The induction hypothesis asserts that for all invertible elements a of $\mathbb{Z}/p^k\mathbb{Z}$ there exists a pair of vectors (v, w) in V_{p^k} such

that

$$q(v) = a \pmod{p^k} \quad \text{and} \quad b_q(v, w) = 1 \pmod{p^k}.$$

Let a be an invertible element of $\mathbb{Z}/p^{k+1}\mathbb{Z}$. Apply the induction hypothesis to find elements v and w of $V_{p^{k+1}}$ such that $q(v) = a \pmod{p^k}$ and $b_q(v, w) = 1 \pmod{p^k}$. Then $b_q(v, w)$ is invertible in $\mathbb{Z}/p^{k+1}\mathbb{Z}$ and changing w in one of its multiple we construct a vector w such that $b_q(v, w) = 1 \pmod{p^{k+1}}$. Write $q(v) = a + bp^k$ and change v into $v' = v - bp^k w$; then $q(v') = a \pmod{p^{k+1}}$. We still have $b_q(v', w) = 1 \pmod{p^k}$, so that a multiple w' of w satisfies $b_q(v', w') = 1$. The Lemma is proved by induction. \square

Now we invoke the following analog of Witt's Theorem for quadratic modules over local rings (see [28], (4.4)).

Lemma 6.12 (Witt's Theorem). *Let M be a quadratic module over a local ring A with maximal ideal \mathfrak{m} . Let F, G be free primitive submodules of M over the ring A . Any isomorphism of quadratic modules $F \rightarrow G$ extends to an automorphism of the quadratic module M .*

Here primitive means that the quotient module is free.

6.6. Proof of Theorem 6.9. We are ready to prove Theorem 6.9, hence Theorem 6.6, and Theorem 5.2.

Let m be a positive integer and $\prod_i p_i^{k_i}$ be its decomposition into prime factors. Let $\alpha \in \mathbb{E}_{10}$ be a root (for example $\alpha = e_1 - e_2$).

Let p^k be any of the factors $p_i^{k_i}$. Consider the image $\bar{\alpha}$ of α in L_{p^k} . Fix a free submodule M_0 of rank 8 in V_{p^k} . By Lemma 6.11, there is an element $v \in M_0$ with $q_{p^k}(v) = -2 \in \mathbb{Z}/p^k\mathbb{Z}$ if $p \neq 2$ and $q_{p^k}(v) = 1$ if $p = 2$. The element v generates a free primitive submodule of L_{p^k} . By Witt's Theorem, we find an element $\sigma \in \mathbf{O}(L_{p^k})$ such that $\sigma(\bar{\alpha}) = v$; in particular, $\sigma(\bar{\alpha})$ is contained in V_{p^k} . Let us show that σ can be chosen in the image of the map $\mathbf{Spin}(L_{p^k}) \rightarrow \mathbf{O}(L_{p^k})$.

Recall that the reflection s_h with respect to a vector h , for which $q(h)$ is invertible, is defined by the formula

$$s_h(x) = x - \frac{b(x, h)}{q(h)} h.$$

If h is in M_0 , s_h is an isometry of L_{p^k} that preserves M_0 . By Theorem (4.6) from [28], any isometry of M_0 (resp. L_{p^k}) is the product of reflections in elements from M_0 (resp. L_{p^k}). As explained in [28], page 39, Section 8, an element η of $\mathbf{O}(L_{p^k})$ is in the image of $\mathbf{Spin}(L_{p^k})$ if and only if η is a product

of an even number of reflections, $\eta = s_{h_1} \circ \dots \circ s_{h_{2s}}$, and its spinor norm $\text{SN}(\eta)$ is 1, i.e.

$$\text{SN}(\eta) := q(h_1) \cdots q(h_{2s}) = 1 \pmod{(\mathbb{Z}_{p^k}^*)^2}.$$

(see also [29], chapter IV, §6)

Write σ as a composition of reflections s_{h_i} , $1 \leq i \leq l$, with $q(h_i) \in \mathbb{Z}_{p^k}^*$ (l may be odd). Apply Lemma 6.11 to find vectors h_{l+1} and h_{l+2} in M_0 such that

$$q(h_{l+1}) = \prod_{i=1}^l q(h_i)^{-1} \quad \text{and} \quad q(h_{l+2}) = 1 \pmod{(\mathbb{Z}_{p^k}^*)^2}.$$

Change σ into $s_{h_{l+1}} \circ \sigma$ or $s_{h_{l+2}} \circ s_{h_{l+1}} \circ \sigma$ to obtain an isometry which is a product of an even number of reflections. After such a modification, $\text{SN}(\sigma) = 1$ and σ is in the image of $\text{Spin}(L_{p^k})$. Since M_0 is preserved by $s_{h_{l+1}}$ and $s_{h_{l+2}}$, $\sigma(\bar{\alpha})$ is contained in M_0 , and thus in V_{p^k} .

Let now $\bar{\alpha}$ denote the image of α into L_m . Since the previous argument applies to all prime factors $p_i^{k_i}$ of m , the Chinese Remainder Theorem shows the existence of an element σ in $\text{Spin}(\mathbb{E}_{10}/m\mathbb{E}_{10})$ such that $\sigma(\bar{\alpha}) \in V_m$. By the Strong Approximation Theorem, σ lifts to an element σ' in $\text{Spin}(\mathbb{E}_{10})$; then, the image w of σ' in $\text{SO}_b(\mathbb{Z})$ maps the root α onto a root $w(\alpha)$ whose projection modulo m is in V_m . This proves Theorem 6.9 and hence Theorem 6.6.

7. NON ALGEBRAICALLY CLOSED FIELDS AND OTHER SURFACES

7.1. Non algebraically closed fields. In this Section, the ground field \mathbb{K} is not necessarily algebraically closed. Let $\bar{\mathbb{K}}$ be its algebraic closure and let $\bar{X} = X \otimes_{\mathbb{K}} \bar{\mathbb{K}}$ be obtained from X by base field change. Let $W_{\bar{X}}$ denote the Coxeter subgroup of $\text{O}(\text{Pic}(X_{\bar{\mathbb{K}}}))$, as defined in Section 1.3. We have a sequence of inclusions

$$\text{Aut}(X)^* \subset \text{Aut}(\bar{X})^* \subset W_{\bar{X}} \subset \text{O}(K_{\bar{X}}^{\perp}).$$

We say that X is **Cremona special over \mathbb{K}** if $\text{Aut}(X)^*$ has finite index in $W_{\bar{X}}$ and $W_{\bar{X}}$ is infinite. The following result extends the Main Theorem to arbitrary fields \mathbb{K} .

Theorem 7.1. *If X is Cremona special over \mathbb{K} , then*

- \bar{X} is unnodal;
- X is obtained from $\mathbb{P}_{\mathbb{K}}^2$ by blowing up a finite subset of $\mathbb{P}^2(\mathbb{K})$;
- X is a Halphen, a Coble, or a Harbourne example (over \mathbb{K}).

Proof. Let $n + 1$ be the Picard number of \bar{X} . Assume that X is Cremona special over \mathbb{K} ; in particular X is obtained from $\mathbb{P}_{\mathbb{K}}^2$ by blowing up a 0-cycle

\mathcal{P} of length n defined over \mathbb{K} . Since $\text{Aut}(\bar{X})$ contains $\text{Aut}(X)$, \bar{X} is Cremona special.

From our Main Theorem, we deduce that \bar{X} is unnodal, \mathcal{P} is made of n distinct points of $\mathbb{P}^2(\bar{\mathbb{K}})$, and \bar{X} is a Halphen, a Coble, or a Harbourne example over \mathbb{K} . In the Halphen case, the genus 1 fibration on X is unique. In the Coble case, $|-2K_{\bar{X}}|$ contains a unique member (the strict transform of the sextic curve with double points along \mathcal{P}). In the Harbourne case, when the Picard number of \bar{X} is ≥ 10 , $|-K_{\bar{X}}|$ contains also a unique member (given by the proper transform of the cuspidal cubic containing \mathcal{P}). Thus, these curves and pencil are defined over \mathbb{K} .

Let us show that $\text{Pic}(X)$ has finite index in $\text{Pic}(\bar{X})$. First, assume that $n \geq 10$. Consider the subgroup $\text{Pic}(X)$ of $\text{Pic}(\bar{X})$. It contains ample classes and the canonical class K_X ; in particular, it intersects K_X^\perp on an infinite subgroup L which is $\text{Aut}(X)^*$ -invariant. But $\text{Aut}(X)^*$ has finite index in $W_{\bar{X}}$, $W_{\bar{X}}$ is Zariski dense in $\text{O}(K_X^\perp \otimes \mathbb{R})$, and the action of this orthogonal group on $K_X^\perp \otimes \mathbb{R}$ is irreducible. Thus, $L \otimes \mathbb{R}$ coincides with $K_X^\perp \otimes \mathbb{R}$. Since $\text{Pic}(X)$ also contains K_X , we deduce that $\text{Pic}(X) \otimes \mathbb{R}$ is equal to $\text{Pic}(\bar{X}) \otimes \mathbb{R}$ and that $\text{Pic}(X)$ has finite index in $\text{Pic}(\bar{X})$.

When $n = 9$, one needs a slightly different argument. Since $\text{Pic}(X)$ contains ample classes, the intersection form restricts to a form of signature $(1, m)$ on $\text{Pic}(X)$, with $m \leq 9$. In particular, the intersection form is negative definite on the orthogonal complement of $\text{Pic}(X)$, and the action of $\text{Aut}(X)$ on $\text{Pic}(X)$ has finite kernel. Since the action of $\text{Aut}(X)$ on $\text{Pic}(X)$ preserves the isotropic vector K_X and the integral structure, it contains a finite index, free abelian subgroup of rank at most $m - 1$. Since $\text{Aut}(X)^*$ has finite index in W_9 , we deduce that $m = 9$ because W_9 contains a free abelian group of rank 8. This shows that $\text{Pic}(X)$ has finite index in $\text{Pic}(\bar{X})$.

Let us now prove that all points p_i of \mathcal{P} , $1 \leq i \leq n$, are in fact defined over \mathbb{K} , i.e. that $\mathcal{P} \subset \mathbb{P}^2(\mathbb{K})$. It suffices to show that all (-1) -curves E in \bar{X} are defined over \mathbb{K} . Since $\text{Pic}(X)$ has finite index in $\text{Pic}(\bar{X})$, the divisor class of some positive multiple mE is in $\text{Pic}(X)$. Since $|mE| = \{mE\}$, we obtain that mE is defined over \mathbb{K} and hence E is defined over \mathbb{K} . \square

7.2. Other types of surfaces. We can extend the concept of Cremona special rational surfaces to other types of projective surfaces as follows. One says that a surface Y has a **large** automorphism group if $\text{Aut}(Y)^*$ is infinite and of finite index in the orthogonal group $\text{O}(K_Y^\perp) \subset \text{O}(\text{Num}(Y))$, where $\text{Num}(Y)$ is the lattice of divisor classes modulo numerical equivalence. Besides rational surfaces, other candidates of surfaces with large automorphism group are surfaces

of Kodaira dimension 0 or 1. Indeed, $\text{Aut}(Y)$ is finite if the Kodaira dimension of Y is 2, and $\text{Aut}(Y)^*$ is finite if Y is ruled but not rational.

7.2.1. Kodaira dimension 1. If the Kodaira dimension of Y is 1, some multiple of the canonical class defines an elliptic (or quasi-elliptic) fibration on Y . The Mordell-Weil group of the corresponding jacobian fibration embeds as a finite index subgroup into $\text{Aut}(Y)^*$. By Shioda-Tate formula, the rank of this group is equal to $\rho(Y) - 2$, provided there are no reducible fibres in the genus 1 fibration. Using the argument from the proof of Theorem 2.10, one can show, in this case, that $\text{Aut}(Y)^*$ is of finite index in $\text{O}(K_Y^\perp)$. Thus, $\text{Aut}(Y)^*$ has finite index in $\text{O}(K_Y^\perp)$ if and only if the canonical fibration has no reducible fiber.

7.2.2. Kodaira dimension 0. The classification of surfaces implies that minimal surfaces with Kodaira dimension 0 fall into four types: Abelian surfaces, K3 surfaces, Enriques surfaces, and bi-elliptic surfaces.

Theorem 7.2. *Let \mathbb{K} be an algebraically closed field with $\text{char}(\mathbb{K}) \neq 2$. Let Y be a projective surface over \mathbb{K} with Kodaira dimension equal to 0. If $\text{Aut}(Y)$ is large, then Y is minimal and Y is not a bi-elliptic surface. If Y is an abelian surface, a K3 surface, or an Enriques surface, then $\text{Aut}(Y)$ is large if and only if $\text{O}(\text{Num}(Y))$ is infinite and Y does not contain any smooth rational curve.*

Sketch of the proof. Recall that a $(-k)$ -curve is a smooth rational curve with self-intersection $-k$. Let Y be a projective surface with Kodaira dimension 0. The set of classes of (-1) -curves is finite, and permuted by $\text{Aut}(Y)$; thus, if $\text{Aut}(Y)$ is large, Y is minimal and $K_Y^\perp = \text{Num}(Y)$. If Y is bi-elliptic, it is easily checked that $\text{Aut}(Y)^*$ is finite.

Abelian surfaces do not contain rational curves, and smooth rational curves on K3 surfaces and Enriques surfaces are (-2) -curves. Each of them defines a reflection on $\text{Num}(Y)$. Denote by $\text{Nod}(Y)$ the set of smooth rational curves and by $\text{Ref}(Y) \subset \text{O}(\text{Num}(Y))$ the group generated by the reflections around classes of smooth rational curves.

Let Y be an abelian or K3 surface. Over the field of complex numbers, the Torelli Theorem implies that $\text{Aut}(Y)^*$ is of finite index in $\text{O}(\text{Num}(Y))$ if and only if $\text{Nod}(Y)$ is empty (this is always true for abelian surfaces). For K3 surfaces this fact has been extended recently to any characteristic $p \neq 2$ by M. Lieblich and D. Maulik (see [30]). They also extended another corollary of the Torelli Theorem: The set of orbits of smooth rational curves with respect to the automorphism group is finite. Their arguments can probably be adapted

to abelian surfaces. Thus, an abelian surface or K3 surface has large automorphism group if and only if $O(\text{Num}(Y))$ is infinite and $\text{Nod}(Y)$ is empty.⁸

Finally, the automorphism group of an Enriques surface without smooth rational curves is always large, and this is true in any characteristic. The proof and the statement are analogous to the proof of Theorem 3.5 (see [15], for $\mathbb{K} = \mathbb{C}$). \square

7.2.3. Complex surfaces and rational curves. Let us assume that Y is a complex projective surface.

- (1) If Y is an abelian surface, Y does not contain rational curves;
- (2) if Y is a K3 surface and Y does not contain any smooth rational curve, its Picard number satisfies $\rho(Y) \leq 11$;
- (3) a generic Enriques surface contains no smooth rational curve.

The first assertion is easily proved, and the third is contained in [2]. The second one was explained to the authors by V. Nikulin: *Any K3 surface with $\rho(Y) \geq 12$ contains a smooth rational curve* (see [38], Theorems 14 and 15).

To prove it, assume $\rho(Y) \geq 12$. Since $H^2(Y, \mathbb{Z})$ is an even unimodular lattice of dimension 22 and signature $(3, 19)$, the orthogonal complement $T(Y)$ of $\text{Num}(Y)$ in $H^2(Y, \mathbb{Z})$ has dimension ≤ 10 . By Theorem 1.13.1* (or 1.13.2) in [36], the primitive embedding $T(Y) \subset H^2(Y, \mathbb{Z})$ is unique up to isomorphism, because $\text{Num}(U)$ is indefinite and $\text{rank}(T(Y)) \leq 10$ (see also Corollaries 2.9 and 2.10 in [32]). On the other hand, since $\text{rank}(T(Y)) \leq 10$, Theorem 1.12.2 of [36] implies the existence of a primitive embedding

$$T(Y) \oplus \langle -2 \rangle \subset H^2(Y, \mathbb{Z})$$

where $\langle -2 \rangle$ denotes the 1-dimensional lattice generated by a vector with self-intersection -2 . Taking orthogonal complement for this second embedding, one obtains that $\text{Num}(Y)$ contains a copy of the sublattice $\langle -2 \rangle$.

This shows that K3 surfaces with large automorphism groups must satisfy $\rho(Y) \leq 11$. This is a strange coincidence with our main result that complex rational surfaces with large automorphism group must satisfy $\rho = 10$, or 11.

Corollary 7.3. *Let Y be a complex projective surface with large automorphism group. If $\text{Aut}(Y)^*$ does not contain any finite index abelian group, the Picard number of Y is at most 11.*

⁸If $\text{Aut}(Y)^*$ is infinite and Y contains smooth rational curves, $\text{Ref}(Y)$ is infinite and its elements represent different cosets of $O(\text{Num}(Y))$ modulo $\text{Aut}(Y)^*$.

Indeed, the assumption implies that $\text{Aut}(Y)^*$ is infinite but does not preserve a genus 1 fibration, since otherwise the Mordell-Weil group of the corresponding jacobian fibration would determine a finite index abelian subgroup of $\text{Aut}(Y)^*$. Thus, either Y is rational, or its Kodaira dimension vanishes. The conclusion follows from the Main Theorem, 7.2, and Nikulin's argument.

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