BIRATIONAL AUTOMORPHISM GROUPS AND THE MOVABLE CONE THEOREM FOR CALABI-YAU MANIFOLDS OF WEHLER TYPE VIA UNIVERSAL COXETER GROUPS

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Dedicated to Professor Yujiro Kawamata on the occasion of his sixtieth birthday.

ABSTRACT. Thanks to the theory of Coxeter groups, we produce the first family of Calabi-Yau manifolds X of arbitrary dimension n, for which $\mathrm{Bir}(X)$ is infinite and the Kawamata-Morrison movable cone conjecture is satisfied. For this family, the movable cone is explicitly described; it's fractal nature is related to limit sets of Kleinian groups and to the Apollonian Gasket. Then, we produce explicit examples of (biregular) automorphisms with positive entropy on some Calaby-Yau varieties.

1. INTRODUCTION

1.1. **Coxeter groups.** Coxeter groups (see eg. [Hum], [Vi]) play a fundamental role in group theory. Among all Coxeter groups generated by N involutions, the **universal Coxeter group** of rank N

$$\mathrm{UC}(N) := \underbrace{\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z} * \cdots * \mathbf{Z}/2\mathbf{Z}}_{N},$$

where $\mathbf{Z}/2\mathbf{Z}$ is the cyclic group of order 2, is the most basic one : there is no non-trivial relation between its N natural generators, hence every Coxeter group is a quotient of some $\mathrm{UC}(N)$. The group $\mathrm{UC}(1)$ coincides with $\mathbf{Z}/2\mathbf{Z}$, and $\mathrm{UC}(2)$ is isomorphic to the infinite dihedral group; in particular, $\mathrm{UC}(2)$ is almost abelian in the sense that it contains a cyclic abelian subgroup $\mathbf{Z}$ as an index 2 subgroup. For $N \geq 3$, $\mathrm{UC}(N)$ contains the free group $\mathbf{Z} * \mathbf{Z}$.

Given a Coxeter diagram with N vertices and its associated Coxeter group W, one can construct a real vector space V of dimension N, together with a quadratic form b, and a linear representation $W \to \mathrm{GL}(V)$ that preserves b (see §2). We shall refer to this representation $W \to \mathrm{GL}(V)$ as the **geometric representation** of W. The group W preserves a convex cone $\mathsf{T} \subset V$, called the **Tits cone**, that contains an explicit sub-cone $\mathsf{D} \subset \mathsf{T}$ which is a fundamental domain for the action of W on T. A precise description of the geometric representation and the invariant cone for $W = \mathrm{UC}(N)$ is given in Section 2.2. In particular,

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for $N \geq 3$, the quadratic form b is non-degenerate and has signature $(1, N-1)$, and the fundamental domain D is a cone over a simplex.

1.2. **Wehler varieties.** Coxeter groups also appear in the theory of algebraic surfaces, mostly as hyperbolic reflection groups acting on Néron-Severi groups (see for instance [Ni], [Bor], [CaDo], [Do], [Mc2], [To]).

Let n be a positive integer. Let X be a smooth complex hypersurface of $(\mathbb{P}^1)^{n+1}$. The adjunction formula implies that the Kodaira dimension of X vanishes if and only if X has multi-degree $(2,2,\ldots,2)$, if and only if the canonical bundle of X is trivial. These smooth hypersurfaces $X \subset (\mathbb{P}^1)^{n+1}$ of degree $(2,2,\ldots,2)$ will be called **Wehler varieties** in what follows.

Let X be such a Wehler variety. The $n+1$ projections $p_i\colon X \to (\mathbb{P}^1)^n$ which are obtained by forgetting one coordinate are ramified coverings of degree 2; thus, for each index i, there is a birational transformation

$$\iota_i\colon X \dashrightarrow X$$

that permutes the two points in the fibers of p_i. This provides a morphism

$$\Psi\colon \mathrm{UC}(n+1) \to \mathrm{Bir}(X)$$

into the group of birational transformations of X.

Remark 1.1. Wehler surfaces and their variants appear in the study of complex dynamics and arithmetic dynamics as handy, concrete examples ([Ca], [Mc1], [Sil]).

1.2.1. *Curves and surfaces.* In dimension $n = 1$, the hypersurface X is an elliptic curve $\mathbf{C}/\Lambda$, and the image of the dihedral group $\mathrm{UC}(2)$ is contained in the semi-direct product of $\mathbf{Z}/2\mathbf{Z}$, acting by $z \mapsto -z$ on X, and $\mathbf{C}/\Lambda$, acting by translations.

For $n = 2$, X is a surface of type K3. Since the group of birational transformations of a K3 surface coincides with its group of (biregular) automorphisms, the image of $\mathrm{UC}(3)$ is contained in $\mathrm{Aut}(X)$. From a paper of Wehler, specifically from the last three lines in [We], one can deduce the following properties.

- The morphism $\Psi\colon \mathrm{UC}(3) \to \mathrm{Aut}(X)$ is injective;
- if X is generic, the dimension of its Néron-Severi group $\mathrm{NS}(X)$ is equal to 3 and the linear representation of $\mathrm{Aut}(X)$ in $\mathrm{GL}(\mathrm{NS}(X) \otimes \mathbf{R})$ is conjugate to the geometric representation of $\mathrm{UC}(3)$;
- if X is generic, the ample cone coincides with one connected component of the set of vectors $u \in \mathrm{NS}(X) \otimes \mathbf{R}$ with self-intersection $u \cdot u > 0$.

See Section 3.4 for the proof of a slightly more general result. For simplicity, we refer to these statements as Wehler's theorem, even if they are not contained in Wehler's paper.

1.2.2. *Higher dimensional Wehler varieties.* The aim of this article is to extend the results of the previous section, in the birational and biregular ways, for Wehler varieties X of higher dimension. For $n \geq 3$, X is a **Calabi-Yau manifold** of dimension n. By definition, this means that X is simply connected, there is no holomorphic k-form on X for $0 < k < n$, and there is a nowhere vanishing holomorphic n-form ω_X; hence,

$$H^0(X, \Omega_X^k) = 0, \ \forall k \in \{1, \ldots, n-1\}, \quad \text{and} \quad H^0(X, \Omega_X^d) = \mathbf{C}\omega_X.$$

Projective K3 surfaces are Calabi-Yau manifolds of dimension 2.

Before stating our main results, we need to recall the definition of various cones in $\mathrm{NS}(X) \otimes \mathbf{R}$ and to describe the movable cone conjecture of Morrison and Kawamata.

1.3. **The movable cone conjecture.**

The precise formulation of the Kawamata-Morrison movable cone conjecture may be found in [Ka2] (see Conjecture (1.12)). One can also consult [Mo] for the biregular version of the conjecture and relation with mirror symmetry, as well as [To] for a more general version of these conjectures. In this short section, we introduce the main players and describe the conjectures.

1.3.1. *Cones (see* [Ka2]*).* Let M be a complex projective manifold. An integral divisor D on M is **effective** if the complete linear system $|D|$ is not empty; it is **Q-effective** if there exists a positive integer m such that mD is effective; and it is **movable** if $|D|$ has no fixed components, so that the base locus of $|D|$ has codimension at least 2.

The Néron-Severi group $\mathrm{NS}(X)$ is the group of classes of divisors modulo numerical equivalence; given a ring $\mathbf{A}$, for example $\mathbf{A} \in \{\mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}\}$, we denote by $\mathrm{NS}(X; \mathbf{A})$ the group $\mathrm{NS}(X) \otimes_{\mathbf{Z}} \mathbf{A}$.

The **effective cone** $\mathcal{B}^e(M) \subset \mathrm{NS}(X; \mathbf{R})$ is the convex cone generated by the classes of effective divisors. The **big cone** $\mathcal{B}(M)$ (resp. the **ample cone** $\mathrm{Amp}(X)$) is the convex cone generated by the classes of big (resp. ample) divisors. The **nef cone** $\overline{\mathrm{Amp}}(M)$ is the closure of the ample cone, and $\mathrm{Amp}(M)$ is the interior of the nef cone. Note that $\mathcal{B}(M) \subset \mathcal{B}^e(M)$ and the **pseudo-effective cone** $\overline{\mathcal{B}}(M)$ is the closure of both $\mathcal{B}(M)$ and $\mathcal{B}^e(M)$.

The **movable cone** $\overline{\mathcal{M}}(M)$ is the closure of the convex cone generated by the classes of movable divisors. The **movable effective cone** $\mathcal{M}^e(M)$ is defined to be $\overline{\mathcal{M}}(M) \cap \mathcal{B}^e(M)$.

Both the nef cone and the movable cone involve taking closures. So, *a priori*, $\overline{\mathcal{M}}(M)$ is not necessarily a subset of $\mathcal{B}^e(M)$; similarly, $\overline{\mathrm{Amp}}(M)$ is not necessarily a subset of $\mathcal{B}^e(M)$ nor of $\mathcal{M}^e(M)$, while it is always true that $\overline{\mathrm{Amp}}(M) \subset \overline{\mathcal{M}}(M)$.

1.3.2. *The cone conjecture.* Assume that M is a Calabi-Yau manifold. All birational transformations of M are isomorphisms in codimension one (see §3.2.2 below); in other words, the group of birational transformations coincides with the group of **pseudo-automorphisms**. Thus, if g is an element of $\mathrm{Bir}(M)$ and D is movable (resp. effective, $\mathbf{Q}$-effective), then $g^*(D)$ is again movable (resp. effective, $\mathbf{Q}$-effective). As a consequence, $\mathrm{Bir}(M)$ naturally acts on the three cones $\overline{\mathcal{M}}(M)$, $\mathcal{B}^e(M)$ and $\mathcal{M}^e(M)$.

The abstract version of the Morrison-Kawamata movable cone conjecture is the following:

Conjecture 1.2 (Morrison and Kawamata)**.** The action of $\mathrm{Bir}(M)$ on the movable effective cone $\mathcal{M}^e(M)$ has a finite rational polyhedral cone Δ as a fundamental domain. Here Δ is called a fundamental domain if

$$g^*(\Delta^\circ)\cap\Delta^\circ = \emptyset\,,\ \forall g\in \mathrm{Bir}(X) \text{ with } g^*\neq \mathrm{Id}\,,$$

where Δ° is the interior of Δ, and

$$\mathrm{Bir}(M)\cdot\Delta = \mathcal{M}^e(M)\,.$$

This conjecture holds for log K3 surfaces ([To]) and abelian varieties ([Ka2], [PS]), and its relative version has been verified for fibered Calabi-Yau threefolds ([Ka2]). The conjecture is also satisfied for several interesting examples of Calabi-Yau threefolds: See for instance [GM] for the biregular situation and [Fr] for the birational version. A version of the conjecture is proved in [Mar] for compact hyperkähler manifolds.

1.4. **The movable cone conjecture for Wehler varieties.** Our first main result is summarized in the following statement.

Theorem 1.3. *Let $n\geq 3$ be an integer. Let X be a generic hypersurface of multi-degree $(2,\dots,2)$ in $(\mathbb{P}^1)^{n+1}$. Then,*

1. *the automorphism group $\mathrm{Aut}(X)$ is trivial, i.e. $\mathrm{Aut}(X)=\{\mathrm{Id}_X\}$;*
2. *The morphism Ψ that maps each generator t_j of $\mathrm{UC}(n+1)$ to the involution ι_j of X is an isomorphism $\Psi\colon \mathrm{UC}(n+1)\to \mathrm{Bir}(X)$;*
3. *X satisfies Conjecture 1.2: The cone $\overline{\mathrm{Amp}}(X)$ is a fundamental domain for the action of $\mathrm{Bir}(X)$ on the movable effective cone $\mathcal{M}^e(X)$; this fundamental domain is a cone over a simplex.*

More precisely, there is a linear conjugacy between the (dual of the) geometric representation of $\mathrm{UC}(n+1)$ and the representation of $\mathrm{Bir}(X)$ on $\mathrm{NS}(X)$ that maps equivariantly the Tits cone T to the movable cone and the fundamental cone D to the nef cone $\overline{\mathrm{Amp}}(M)$.

The conjugacy described in the second and third assertions enables us to describe more precisely the geometry of the movable cone. First, one can list explicitly all rational points on the boundary of the movable cone and prove that *every rational point on the boundary of the (closure of the) movable cone is* *movable*, a result that is not predicted by Kawamata's conjecture. Second, one can *draw pictures of this cone*, and show how it is related to Kleinian groups and the Appollonian Gasket. In particular, we shall see that *the boundary of the movable cone has a fractal nature when* $n \geq 3$; it is not smooth, nor polyhedral.

To our best knowledge, this theorem is the first non-trivial result in which the movable cone conjecture is checked for non-trivial examples of Calabi-Yau manifolds in dimension ≥ 4. Our proof is a combination of recent important progress in the minimal model theory in higher dimension due to Birkar, Cascini, Hacon, and McKernan ([BCHM]) and to Kawamata ([Ka3]) and of classical results concerning Coxeter groups and Kleinian groups (Theorems 2.1 and 2.4 below).

1.5. **Automorphisms with positive entropy.** As already indicated by Theorem 1.3, in higher dimensional algebraic geometry, birational transformations are more natural, and in general easier to find, than regular automorphisms. Nevertheless, it is also of fundamental interest to find non-trivial regular automorphisms of higher dimensional algebraic varieties. The following result describes families of Calabi-Yau manifolds in all even dimensions ([OS], Theorem 3.1): *Let* Y *be an Enriques surface and* $\mathrm{Hilb}^n(Y)$ *be the Hilbert scheme of* n *points on* Y*, where* $n \geq 2$*. Let*

$$\pi : \widetilde{\mathrm{Hilb}^n(Y)} \to \mathrm{Hilb}^n(Y)$$

be the universal cover of $\mathrm{Hilb}^n(Y)$*. Then* π *is of degree* 2 *and* $\widetilde{\mathrm{Hilb}^n(Y)}$ *is a Calabi-Yau manifold of dimension* $2n$*.*

In Section 5, we prove that the automorphism group of $\widetilde{\mathrm{Hilb}^n(Y)}$ can be very large, and may contain element with positive topological entropy. This is, again, related to explicit Coxeter groups. For instance, we prove the following result.

Theorem 1.4. *Let* Y *be a generic Enriques surface. Then, for each* $n \geq 2$*, the biregular automorphism group of* $\widetilde{\mathrm{Hilb}^n(Y)}$ *contains a subgroup isomorphic to the universal Coxeter group* $\mathrm{UC}(3)$*, and this copy of* $\mathrm{UC}(3)$ *contains an automorphism with positive entropy.*

The Calabi-Yau manifolds of Theorems 1.3 and 1.4 are fairly concrete. We hope that these examples will also provide non-trivial handy examples for complex dynamics ([DS], [Zh]) and arithmetic dynamics ([Sil], [Kg]) in higher dimension.

2. Universal Coxeter groups and their geometric representations

In this section we collect a few preliminary facts concerning the universal Coxeter group on N generators, and describe the geometry of its Tits cone.

2.1. **Coxeter groups (see** [Hum]**).**

2.1.1. *Definitions.* Let W be a group with a finite set of generators $S = \{s_j\}_{j=1}^N$. The pair (W,S) is a **Coxeter system** if there are integers $m_{ij} \in \mathbf{Z}_+ \cup \{\infty\}$ such that

- $W = \langle s_j \in S \,|\, (s_i s_j)^{m_{ij}} = 1 \rangle$ is a presentation of W,
- $m_{jj} = 1$, i.e., $s_j^2 = 1$ for all j,
- $2 \leq m_{ij} = m_{ji} \leq \infty$ when $i \neq j$ (here $m_{ij} = \infty$ means that $s_i s_j$ is of infinite order).

A group W is called a **Coxeter group** if W contains a finite subset S such that (W,S) forms a Coxeter system.

Let $\mathrm{UC}(N)$ be the free product of N cyclic groups $\mathbf{Z}/2\mathbf{Z}$ of order 2, as in Section 1.1. The group $\mathrm{UC}(N)$ is generated by N involutions t_j, $1 \leq j \leq N$, with no non-obvious relations between them. With this set of generators, $\mathrm{UC}(N)$ is a Coxeter group with $m_{ij} = \infty$ for all $i \neq j$.

If (W,S) is a Coxeter system with $|S| = N$, there is a unique surjective homomorphism $\mathrm{UC}(N) \to W$ that maps t_j onto s_j; its kernel is the minimal normal subgroup containing $\{(t_i t_j)^{m_{ij}}\}$. In this sense, the group $\mathrm{UC}(N)$ is universal among all Coxeter groups with N generators; thus, we call $\mathrm{UC}(N)$ (resp. $(\mathrm{UC}(N), \{t_j\}_{j=1}^N)$) the **universal Coxeter group** (resp. the universal Coxeter system) **of rank** N.

2.1.2. *Geometric representation.* Let $(W, \{s_j\}_{j=1}^N)$ be a Coxeter system with $(s_i s_j)^{m_{ij}} = 1$, as in the previous paragraph. An N-dimensional real vector space $V = \oplus_{j=1}^N \mathbf{R}\alpha_j$ and a bilinear form $b(*,**)$ on V are associated to these data; the quadratic form b is defined by its values on the basis $(\alpha_j)_{j=1}^N$:

$$b(\alpha_i, \alpha_j) = -\cos\frac{\pi}{m_{ij}}\,.$$

Then, as explained in [Hum] (see the Proposition page 110), there is a well-defined linear representation $\rho : W \to \mathrm{GL}(V)$, which maps each generator s_j to the symmetry

$$\rho(s_j)\colon \lambda \in V \mapsto \lambda - 2b(\alpha_j, \lambda)\alpha_j\,.$$

The representation ρ is the **geometric representation** of the Coxeter system (W,S). The following theorem (see [Hum], Page 113, Corollary) is one of the fundamental results on Coxeter groups:

Theorem 2.1. *The geometric representation ρ of a Coxeter system (W,S) is faithful. In particular, all Coxeter groups are linear.*

2.1.3. *Tits cone.* The dual space V^* contains two natural convex cones, which we now describe. Denote the dual representation of W on V^* by ρ^*. The first cone $\mathsf{D} \subset V^*$ is the intersection of the half-spaces

$$\mathsf{D}_i^+ = \{f \in V^* |\, f(\alpha_i) \geq 0\}.$$

It is a closed convex cone over a simplex of dimension $N-1$; the facets of D are the intersections $\mathsf{D} \cap \mathsf{D}_i^+$.

Remark 2.2. Since $\rho(s_i)$ maps α_i to its opposite, the action of $\rho^*(s_i)$ on V^* exchanges D_i^+ and $-\mathsf{D}_i^+$. This is similar to the behavior of the relatively ample classes in the flopping diagram.

The second cone, called the **Tits cone**, is the union T of all images $\rho^*(w)(\mathsf{D})$, where w describes W.

Theorem 2.3 (see [Hum], §5.13, Theorem on page 126)**.** *The Tits cone $\mathsf{T} \subset V$ of a Coxeter group W is a convex cone. It is invariant under the action of W on V and D is a fundamental domain for this action.*

Given $J \subset S$, consider the subgroup W_J of W generated by the elements of J. Define $\mathsf{D}(J)$ by

$$\mathsf{D}(J) = \big(\cap_{j\in J}\{f \in V^* |\, f(\alpha_j) = 0\}\big) \cap \big(\cap_{i\notin J}\{f \in V^* |\, f(\alpha_i) > 0\}\big);$$

hence, each $\mathsf{D}(J)$ is the interior of a face of dimension $N-|J|$. Then W_J is a Coxeter group; it coincides with the stabilizer of every point of $\mathsf{D}(J)$ in W.

2.2. **The universal Coxeter group.** We now study the Coxeter group $\mathrm{UC}(N)$. The vector space of its geometric representation, the Tits cone and its fundamental domain are denoted with index N: V_N, T_N, D_N, etc.

2.2.1. *The linear representation.* With the basis $(\alpha_j)_{j=1}^N$, identify V_N to $\mathbf{R}^N$, and $\mathrm{GL}(V)$ to $\mathrm{GL}_N(\mathbf{R})$. Let $M_{N,j}$ ($1 \leq j \leq N$) be the $N \times N$ matrices with integer coefficients, defined by:

$$M_{N,j} = \begin{pmatrix} 1 & 0 & \dots & 0 & 2 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 & 2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 2 & 0 & \dots & 1 \end{pmatrix}, \tag{2.1}$$

where -1 is the (j,j)-entry. For instance,

$$M_{3,1} = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, M_{3,2} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, M_{3,3} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}.$$

Theorem 2.4. *The geometric representation* ρ *of the universal Coxeter system* $(\mathrm{UC}(N), \{t_j\}_{j=1}^N)$ *of rank* N *is given by* $\rho(t_j) = M_{N,j}^t$*, where* $M_{N,j}^t$ *is the transpose of* $M_{N,j}$*. In particular, in* $\mathrm{GL}_N(\mathbf{R})$*, we obtain*

$$\begin{aligned} \langle M_{N,j}\,,1 \le j \le N\rangle &= \langle M_{N,1}\rangle * \langle M_{N,2}\rangle * \cdots * \langle M_{N,N}\rangle \\ &= \rho(\mathrm{UC}(N)) \\ &\simeq \mathrm{UC}(N). \end{aligned}$$

Proof. By definition, $m_{jj} = 1$ and $m_{ij} = \infty$ $(i \neq j)$ for the universal Coxeter system. Hence

$$\rho(t_j)(\alpha_i) = \alpha_i + 2\alpha_j \ (i \neq j)\ ,\ \rho(t_j)(\alpha_j) = -\alpha_j\ ,$$

i.e., the matrix representation of $\rho(t_j)$ in the basis $(\alpha_j)_{j=1}^N$ is $M_{N,j}^t$. □

2.2.2. *The quadratic form.* Let b_N denote the *opposite* of the quadratic form defined in Section 2.1.2. Its matrix B_N, in the basis $(\alpha_i)_{i=1}^{i=N}$ is the integer matrix with coefficients -1 on the diagonal and $+1$ for all remaining entries.

When $N = 1$, b_N is negative definite. When $N = 2$, b_N is degenerate: $b(u,u) = (x-y)^2$ for all $u = x\alpha_1 + y\alpha_2$ in V. For $N \geq 3$, the following properties are easily verified (with $\| v \|_{euc}$ the usual euclidean norm):

(1) The vector $u_N = \sum_{i=1}^N \alpha_i$ is in the positive cone; more precisely

$$b_N(u_N, u_N) = N(N-2).$$

(2) $b_N(v,v) = -2 \| v \|_{euc}^2 = -2\sum x_i^2$ for all vectors $v = \sum_{i=1}^N x_i\alpha_i$ of the orthogonal complement $u_N^\perp = \{v = \sum_i x_i\alpha_i |\ \sum_i x_i = 0\}$.

(3) The signature of b_N is $(1, N-1)$.

In what follows, $N \geq 3$ and u_N denotes the vector $\sum_i \alpha_i$. A vector $w = au_N + v$, with v in $u_N^\perp$, is isotropic if and only if

$$N(N-2)a^2 = 2 \| v \|_{euc}^2\ .$$

Thus, if $(\beta_1, \ldots, \beta_{N-1})$ is an orthonormal basis of $u_N^\perp$ and $\beta_N = u_N$, then the isotropic cone is the cone over a round sphere; its equation is

$$(N(N-2)/2)y_n^2 = \sum_{i=1}^{N-1} y_i^2$$

for $v = \sum_j y_i\beta_j$. The vectors $\alpha_i + \alpha_j$, with $i \neq j$, are isotropic vectors.

Example 2.5. For $N = 3$, consider the basis $((0,1,1),(1,0,1),(1,1,0))$. The matrix of b_3 in this basis has coefficients 0 along the diagonal, and coefficients 1 on the six remaining entries.

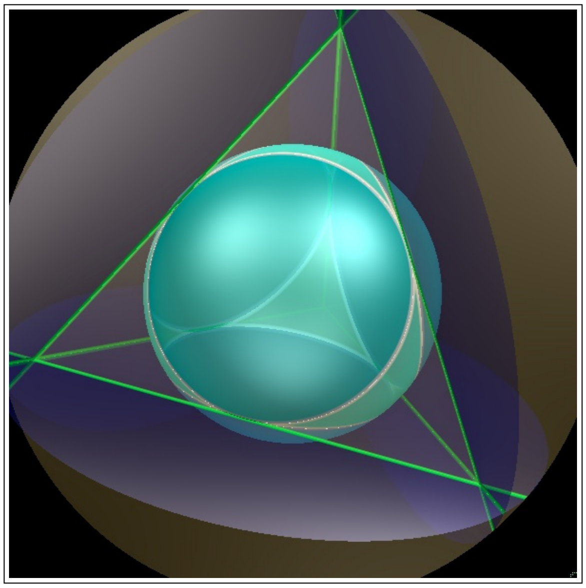

FIGURE 1. Fundamental domain D_4 (projective view): $\mathbb{P}(\mathsf{D}_4)$ is the tetrahedron; the sphere $\mathbb{S}_4$ is shown, together with its four tangency points with the edges of $\mathbb{P}(\mathsf{D}_4)$.

2.2.3. *The Tits cone* T_N. To understand the Tits cone of $\mathsf{UC}(N)$, one can identify V_N to its dual V_N^* by the duality given by the non-degenerate quadratic form b_N; with such an identification, D_N is the set of vectors w such that $b_N(w,\alpha_i)\geq 0$ for all i, and T_N becomes a convex cone in V_N. The half-space $\mathsf{D}_{N,i}^+$ is the set of vectors w such that $b_N(w,\alpha_i)\geq 0$ and its boundary is the hyperplane $\alpha_i^\perp$. The convex cone D_N is the convex hull of its N extremal rays $\mathbf{R}_+c_j$, $1\leq j\leq N$, where

$$\begin{aligned} c_N &= -(N-2)\alpha_N+\sum_{i=1}^{N}\alpha_i \\ &= (1,1,\ldots,1,-(N-3)) \end{aligned}$$

and the $N-1$ remaining c_j are obtained from c_N by permutation of the coordinates. For all $N\geq 3$, and all indices j, one obtains

$$b_N(c_j,c_j)=-2(N-2)(N-3).$$

Example 2.6. For $N=3$, (c_1,c_2,c_3) is the isotropic basis already obtained in Example 2.5. For $N=4$, one gets the basis $(-1,1,1,1)$, $(1,-1,1,1)$, $(1,1,-1,1)$, $(1,1,1,-1)$, with $b_4(c_j,c_j)=-4$.

2.2.4. *A projective view of* D_N. To get some insight in the geometry of the Tits cone, one can draw its projection in the real projective space $\mathbb{P}(V_N)$, at least for

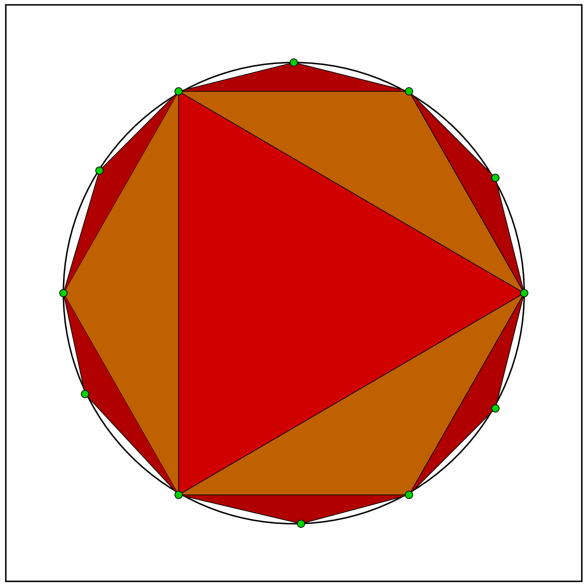

FIGURE 2. The Tits cone in dimension $N = 3$: projective view. One obtains a tiling of the disk by triangles; the vertices describe all rational points of the boundary $\mathbb{S}_3$.

small values of N. We denote the projection of a non-zero vector v in $\mathbb{P}(V_N)$ by $[v]$.

To describe $\mathbb{P}(\mathsf{T}_N)$ and $\mathbb{P}(\mathsf{D}_N)$, denote by $\mathbb{S}_N$ the projection of the isotropic cone (it is a round sphere, as described in § 2.2.2). Consider the projective line L_{ij} through the two points $[c_i]$ and $[c_j]$ in $\mathbb{P}(V_N)$; this line is the projection of the plane $\mathrm{Vect}(c_i, c_j)$. For example, with $(i, j) = (N-1, N)$, this plane is parametrized by $sc_{N-1} + tc_N$ with s and t in $\mathbf{R}$, and its intersection with the isotropic cone corresponds to parameters (s,t) such that

$$(N-3)(s^2+t^2) = 2st.$$

Hence, we get the following behaviour:

- If $N = 3$, L_{ij} intersects the sphere $\mathbb{S}_N$ transversally at $[c_i]$ and $[c_j]$.
- If $N = 4$, L_{ij} is tangent to $\mathbb{S}_N$ at $[c_i + c_j]$.
- If $N \geq 5$, L_{ij} does not intersect $\mathbb{S}_N$.

Similarly, one shows that the faces $\alpha_i^\perp \cap \alpha_j^\perp$ of D_N of codimension 2 intersect the isotropic cone on the line $\mathbf{R}(\alpha_i + \alpha_j)$: Projectively, they correspond to faces of $\mathbb{P}(\mathsf{D}_N)$ that intersect the sphere $\mathbb{S}_N$ on a unique point. The faces $\alpha_i^\perp \cap \alpha_j^\perp \cap \alpha_k^\perp$ of codimension ≥ 3 do not intersect the isotropic cone (i.e. do not intersect $\mathbb{S}_N$ in $\mathbb{P}(V_N)$).

In dimension $N = 3$, D is a triangular cone with isotropic extremal rays and $\mathbb{P}(\mathsf{D}_3)$ is a triangle (see Figure 2). In dimension $N = 4$, a projective view of D_4 is shown on Figure 1.

2.2.5. *A projective view of the Tits cone* T_3 *(see* [Mag]*).* The cone T_N is the union of all images $\rho(w)(\mathsf{D}_N)$, for w in $\mathrm{UC}(N)$. When $N=3$, the sphere $\mathbb{S}_3$ is a circle, that bounds a disk. This disk can be identified to the unit disk in $\mathbf{C}$ with its hyperbolic metric, and the group $\mathrm{UC}(3)$ to a discrete subgroup of the isometry group $\mathrm{PGL}_2(\mathbf{R})$. Up to conjugacy, $\mathrm{UC}(3)$ is the congruence subgroup

$$\{M \in \mathrm{PGL}_2(\mathbf{Z}) \mid M = \mathrm{Id}_2 \bmod (2)\}.$$

In particular, $\mathrm{UC}(3)$ is a non-uniform lattice in the Lie group $\mathrm{PGL}_2(\mathbf{R})$ (i.e. in the orthogonal group of b_3). The fundamental domain $\mathbb{P}(\mathsf{D}_3)$ is a triangle with vertices on the circle $\mathbb{S}_3$. Its orbit $\mathbb{P}(\mathsf{T}_3)$ under $\mathrm{UC}(3)$ is the union of

- the projection of the positive cone $\{w \in V \mid b(w,u) > 0 \text{ and } b(w,w) > 0\}$ in $\mathbb{P}(V)$,
- the set of rational points $[w] \in \mathbb{S}_3$ where w describes the set of isotropic vectors with integer coordinates.

All rational points of $\mathbb{S}_3$ can be mapped to one of the vertices of $\mathbb{P}(\mathsf{D}_3)$ by the action of $\mathrm{UC}(3)$.

2.2.6. *The limit set.* The group $\mathrm{UC}(N)$ preserves the quadratic form b_N and this form is non-degenerate, of signature $(1, N-1)$. Thus, after conjugacy by an element of $\mathrm{GL}_N(\mathbf{R})$, $\mathrm{UC}(N)$ becomes a discrete subgroup of $\mathrm{O}_{1,N-1}(\mathbf{R})$. The **limit set** of such a group is the minimal compact subset of the sphere $\mathbb{S}_N$ that is invariant under the action of $\mathrm{UC}(N)$; we shall denote the limit set of $\mathrm{UC}(N)$ by Λ_N. This set coincides with (see [Rat], chap. 12)

- the closure of the points $[v]$ for all vectors $v \in V_N$ which are eigenvectors of at least one element f in $\mathrm{UC}(N)$ corresponding to an eigenvalue > 1;
- the intersection of $\mathbb{S}_N$ with the closure of the orbit $\mathrm{UC}(N)[w]$, for any given $[w]$ such that $b_N(w,w) \neq 0$.

The convex hull $\mathsf{Conv}(\Lambda_N)$ of the limit set is invariant under $\mathrm{UC}(N)$, and is contained in the closed ball enclosed in $\mathbb{S}_N$. The dual of $\mathsf{Conv}(\Lambda_N)$ with respect to the quadratic form b_N is also a closed invariant convex set. One can show that this convex set coincides with the closure of $\mathbb{P}(\mathsf{T}_N)$, and corresponds to the maximal invariant and strict cone in V_N (see [B], §3.1); we shall not use this fact (see Section 4.2 for a comment).

Remark 2.7. For $N=4$, $\mathrm{O}(b_4)$ is isogeneous to $\mathrm{PGL}_2(\mathbf{C})$ and $\mathrm{UC}(4)$ is conjugate to a Kleinian group (see [Bea]).

Remark 2.8. Note that $\mathrm{UC}(3)$ is a lattice in the Lie group $\mathrm{O}(b_3)$ while, for $N \geq 4$, the fundamental domain $\mathbb{P}(\mathsf{D}_N)$ contains points of the sphere $\mathbb{S}_N$ in its interior and the Haar measure of $\mathrm{O}(b_N)/\mathrm{UC}(N)$ is infinite.

2.2.7. *A projective view of the Tits cone* T_4. Let us fix a vertex, say c_N, of D_N. Its stabilizer is the subgroup $\mathrm{UC}(N)_{\{N\}}$ of $\mathrm{UC}(N)$ (cf. § 2.1.3). It acts on $c_N^\perp$ and the orbit of $\mathsf{D}_N \cap c_N^\perp$ under the action of $\mathrm{UC}(N)_{\{N\}}$ tesselates the intersection $\mathsf{T}_N \cap c_N^\perp$; moreover, up to conjugacy by a linear map, $\mathsf{T}_N \cap c_N^\perp$, together

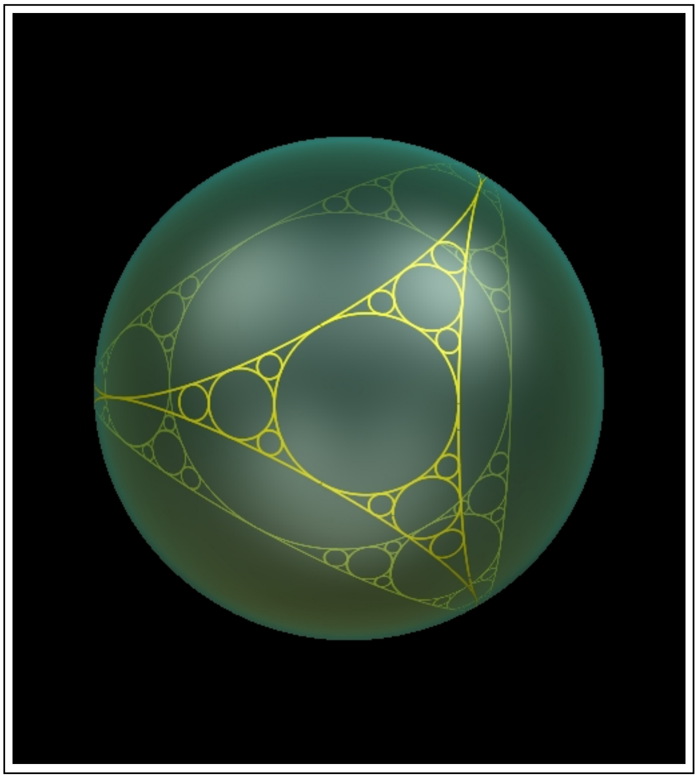

FIGURE 3. A projective view of the cone T_4. The first 56 circles of the Apollonian Gasket are represented. (a "chinese hat" should be attached to each circle)

with its action of $\mathrm{UC}(N)_{\{N\}}$, is equivalent to the Tits cone in dimension $N-1$, together with the action of $\mathrm{UC}(N-1)$. Hence, *if one looks at $\mathbb{P}(\mathsf{T}_N)$ from a vertex of $\mathbb{P}(\mathsf{D}_N)$, one obtains a cone over $\mathbb{P}(\mathsf{T}_{N-1})$.*

Let us apply this fact to the case $N=4$. Then, V_4 has dimension 4, $\mathbb{P}(V_4)$ has dimension 3, and the sphere $\mathbb{S}_4$ is a round sphere in $\mathbf{R}^3$ (once the hyperplane $[u_4^\perp]$ is at infinity). In particular, $\mathrm{UC}(4)$ acts by conformal transformations on this sphere; thus, $\mathrm{UC}(4)$ is an example of a Kleinian group, i.e. a discrete subgroup of $\mathrm{O}_{1,3}(\mathbf{R})$ (see §2.2.6).

Viewed from the vertex $[c_4] \in \mathbb{P}(V_4)$, the convex set $\mathbb{P}(\mathsf{T}_4)$ looks like a cone over $\mathbb{P}(\mathsf{T}_3)$. More precisely, the intersection of $[c_4^\perp]$ with the sphere $\mathbb{S}_4$ is a circle, and the orbit of $\mathbb{P}(\mathsf{D}_4) \cap [c_4^\perp]$ under the stabilizer $\mathrm{UC}(4)_{\{4\}}$ tesselates the interior of this circle, as in Figure 2 (see §2.1.3). All segments that connect $[c_4]$ to a rational point of this circle are contained in $\mathbb{P}(\mathsf{T}_4)$. Thus, $\mathbb{P}(\mathsf{T}_4)$ contains a "chinese hat shell" (Calyptraea chinensis), tangentially glued to the sphere $\mathbb{S}_4$, with vertex $[c_4]$. The picture is similar in a neighborhood of $[c_1]$, $[c_2]$, and $[c_3]$.

Since $\mathrm{UC}(4)$ acts by projective linear transformations on $\mathbb{P}(V_4)$, an infinite number of smaller and smaller chinese hat shells are glued to $\mathbb{S}_4$, with vertices on the orbits of the $[c_i]$: the accumulation points of these orbits, and of the "chinese hats" attached to them, converge towards the limit set of the Kleinian group UC(4).

To get an idea of the convex set $\mathbb{P}(\mathsf{T}_4)$, one needs to describe the sequence of circles along which the shells are glued: This sequence is made of circles on the sphere $\mathbb{S}_4$, known as the Apollonian Gasket. The closure of the union of all these circles is a compact subset of $\mathbb{S}_4$ that is invariant under the action of UC(4); as such, it coincides with the limit set of the Kleinian group UC(4), *i.e.* with the unique minimal UC(4)-invariant compact subset of $\mathbb{S}_4$. Its Hausdorff dimension has been computed by McMullen (see [Mc3]):

$$\text{H.-dim(Apollonian Gasket)} = 1.305688...$$

Remark 2.9. Up to conjugacy in the group of conformal transformations of the Riemann sphere, there is a unique configuration of tangent circles with the combinatorics of the Apollonian Gasket. Thus this circle packing is “unique”.

Proposition 2.10. *The projective image of the Tits cone* T_4 *is a convex set* $\mathbb{P}(\mathsf{T}_4) \subset \mathbb{P}(V_4)$. *Its closure* $\overline{\mathbb{P}(\mathsf{T}_4)}$ *is a compact convex set. Let* $\mathsf{Ex}(\overline{\mathbb{P}(\mathsf{T}_4)})$ *be the set of extremal points of* $\overline{\mathbb{P}(\mathsf{T}_4)}$. *This set is* $\mathsf{UC}(4)$*-invariant, and the set of all its accumulation points coincides with the limit set* $\Lambda(4)$ *of* $\mathsf{UC}(4)$ *in* $\mathbb{S}_4$. *The Hausdorff dimension of* $\Lambda(4)$ *is approximately equal to* 1.305688.

In higher dimension the set $\mathbb{P}(\mathsf{T}_N)$ contains a solid cone with vertex $[c_N]$ and with basis $\mathbb{P}(\mathsf{T}_N)\cap[c_N^{\perp}]$ equivalent to $\mathbb{P}(\mathsf{T}_{N-1})$; the picture is similar around each vertex $[c_i]$, and the convex set $\mathbb{P}(\mathsf{T}_N)$ is the union of the interior of the sphere and the orbits of these chinese hat shells (see below for a precise definition). Thus, the complexity of T_N increases with N.

2.2.8. *Rational points.* This section may be skipped on a first reading; it is not needed to prove the Kawamata-Morrison conjecture in the Wehler examples, but is useful in order to provides stronger results concerning the rational points on the boundary of the movable cone. The following two propositions describe the rational points of the boundary of $\mathbb{P}(\mathsf{T}_N)$; it shows that these rational points are the obvious ones.

Proposition 2.11. *Let* $N \geq 3$ *be an integer. Let* $[v]$ *be a rational point of the boundary of the convex set* $\mathbb{P}(\mathsf{T}_N)$. *Then* $[v]$ *is of the form* $[\rho(w)(v')]$ *where*

- $[v']$ *is a rational point of the boundary of* $\mathbb{P}(\mathsf{T}_N)$*;*
- $[v']$ *is a point of* $\mathbb{P}(\mathsf{D}_N)$.

Before starting the proof, define the **chinese hat shell** of $\mathbb{P}(\mathsf{T}_N)$ with vertex $[c_N]$ as the set of points of $\mathbb{P}(\mathsf{T}_N)$ contained in the convex set generated by $[c_N]$ and $\mathbb{P}(\mathsf{T}_N)\cap[c_N^{\perp}]$; this subset of $\mathbb{P}(\mathsf{T}_N)$ is a solid cone with vertex $[c_N]$ and basis equivalent to $\mathbb{P}(\mathsf{T}_{N-1})$; it coincides with the projection of the subset

$$\{v \in V \mid v \in \mathsf{T}_N \text{ and } b_N(c_N, v) > 0\} = \{v = \sum_i a_i\alpha_i \mid v \in \mathsf{T}_N \text{ and } a_N \leq 0\}.$$

Each vertex $[c_i]$ of $\mathbb{P}(\mathsf{T}_N)$ generates a similar shell; in this way, we get N **elementary shells**, one per vertex $[c_i]$. The image of such a shell by an element γ of $\mathrm{UC}(N)$ is, by definition, the chinese hat shell with vertex $[\gamma(c_i)]$.

Remark 2.12 (see [Rat]). Let H be one of these shells; denote by s its vertex. If (γ_n) is a sequence of elements of $\mathrm{UC}(N)$ going to infinity all accumulation points of $(\gamma_n(s))$ are contained in the limit set $\Lambda_N \subset \mathbb{S}_N$ of $\mathrm{UC}(N)$. Thus, the sequence $(\gamma_n(H))$ is made of smaller and smaller shells and its accumulation points are also contained in the limit set Λ_N. Conversely, every point of Λ_N is the limit of such a sequence $(\gamma_n(H))$.

Remark 2.13. The Tits cone T_N is the orbit of D_N. Thus, an easy induction on N based on the previous remark, shows that *the boundary points of $\mathbb{P}(\mathsf{T}_N)$ are contained in the union of the shells and of the limit set Λ_N.*

Proof. We prove the statement by induction on $N \geq 3$. The case $N = 3$ has already been described previously. Let $[v]$ be a rational point. One can assume that $[v]$ is the projection of a vector $v \in V$ with integer, relatively prime, coordinates.

If $[v]$ is in a chinese hat shell, the conclusion follows from the induction hypothesis. If not, $[v]$ is in the sphere $\mathbb{S}_N$ (cf. Remark 2.13). Thus, $v = \sum a_i \alpha_i$ satisfies the equation of $\mathbb{S}_N$; this can be written

$$\sum_{i=1}^{N} a_i\,(\sigma(v) - 2a_i) = 0 \tag{2.2}$$

where $\sigma(v) = \sum_{j=1}^{N} a_j$. The condition that assures that $[v]$ is in the chinese hat shell with vertex $[c_i]$ reads $a_i \leq 0$. Thus, one can assume $a_i \geq 1$ for all indices i, because the a_i are integers. From Equation (2.2), one deduces that $\sigma(v) - 2a_i \leq -1$ for at least one index i, say for $i = 1$. Apply the involution $\rho(t_1)$. Then, the first coordinate a_1' of $\rho(t_1)(v)$ is equal to $-3a_1 + 2\sigma(v)$, while the other coordinates remain unchanged: $a_j' = a_j$ for $j \geq 2$. Hence,

$$a_1' - a_1 = -4a_1 + 2\sigma(v) = 2(-2a_1 + \sigma(v)) \leq -2,$$

so that $a_1' - a_1$ is strictly negative.

Iterating this process a finite number of times, the sum $\sigma(\cdot)$ determines a decreasing sequence of positive integers. Consequently, in a finite number of steps, one reaches the situation where a coefficient a_i is negative, which means that the orbit of $[v]$ under the action of $\mathrm{UC}(N)$ falls into one of the N elementary chinese hat shells, as required. □

Assume, now that $[v_1], \dots [v_{l+1}]$ are rational boundary points of $\mathbb{P}(\mathsf{T}_N)$. If the convex set

$$C = \mathsf{Conv}([v_1], \dots, [v_{l+1}])$$

has dimension l and is contained in the boundary of $\mathbb{P}(\mathsf{T}_N)$, we say that C is a **rational boundary flat** (of dimension l).

Proposition 2.14. *If C is a rational boundary flat of $\mathbb{P}(\mathsf{T}_N)$ of dimension l, there exists an element g of* $\mathrm{UC}(N)$ *such that $g(C)$ is contained in a boundary face of $\mathbb{P}(\mathsf{D}_N)$ of dimension l.*

Proof. We prove this proposition by induction on the dimension $N \geq 3$. When $N = 3$, the statement is equivalent to the previous proposition, because all boundary flats have dimension 0.

Assume, now, that the proposition is proved up to dimension $N-1$. Let $[u] \in C$ be a generic rational point, and let h be an element of $\mathrm{UC}(N)$ that maps $[u]$ into D_N. Since, C is in the boundary of the Tits cone, so is $h(C)$. Projects $h(C)$ into $[c_1^{\perp}]$ from the vertex $[c_1]$. The image $\pi_1(h(C))$ is in the boundary of $\mathbb{P}(\mathsf{T}_N) \cap [c_1^{\perp}]$; hence, $\pi_1(h(C))$ is a rational boundary flat of dimension l or $l-1$. Thus, there exists an element h' of $\mathrm{UC}(N)$ that stabilizes $[c_1]$ and maps $\pi_1(h(C))$ into the boundary of $\mathbb{P}(\mathsf{D}_N) \cap [c_1^{\perp}]$ ($\simeq \mathbb{P}(\mathsf{D}_{N-1})$). The convex set generated by $h'(\pi_1(h(C)))$ and $[c_1]$ is also contained in a boundary face of $\mathbb{P}(\mathsf{D}_N)$. This proves the proposition by induction. □

3. AUTOMORPHISMS, BIRATIONAL TRANSFORMATIONS, AND THE UNIVERSAL COXETER GROUP

Our goal, in this section, is to prove the first assertion of Theorem 1.3: We describe the groups of regular automorphisms and of birational transformations of generic Wehler varieties X. The group $\mathrm{Bir}(X)$ turns out to be isomorphic to $\mathrm{UC}(n+1)$ and its action on the Néron-Severi group $\mathrm{NS}(X)$ is conjugate to the geometric representation of $\mathrm{UC}(n+1)$, as soon as $\dim(X) \geq 3$.

3.1. **Calabi-Yau hypersurfaces in Fano manifolds.**

A Fano manifold is a complex projective manifold V with ample anti-canonical bundle K_V.

Theorem 3.1. *Let $n \geq 3$ be an integer and V be a Fano manifold of dimension $(n+1)$. Let M be a smooth member of the linear system $|-K_V|$. Let $\tau : M \to V$ be the natural inclusion. Then:*

(1) *M is a Calabi-Yau manifold of dimension $n \geq 3$.*
(2) *The pull-back morphism $\tau^* : \mathrm{Pic}\,(V) \to \mathrm{Pic}\,(M)$ is an isomorphism, and it induces an isomorphism of ample cones:*

$$\tau^*(\mathrm{Amp}\,(V)) = \mathrm{Amp}\,(M).$$

(3) $\mathrm{Aut}\,(M)$ *is a finite group.*

Remark 3.2. For a Calabi-Yau manifold (resp. for a Fano manifold) M, the natural cycle map $\mathrm{Pic}\,(M) \to \mathrm{NS}\,(M)$ given by $L \mapsto c_1(L)$ is an isomorphism because $h^1(O_M) = 0$. So, in what follows, we identify the Picard group $\mathrm{Pic}\,(M)$ and the Néron-Severi group $\mathrm{NS}\,(M)$.

Proof. By the adjunction formula, it follows that $O_M(K_M) \simeq O_M$. By the Lefschetz hyperplane section theorem, $\pi_1(M) \simeq \pi_1(V) = \{1\}$, because every Fano manifold is simply connected. Consider the long exact sequence which is deduced from the exact sequence of sheaves

$$0 \to \mathcal{O}_V(-K_V) \to \mathcal{O}_V \to \mathcal{O}_M \to 0$$

and apply the Kodaira vanishing theorem to $-K_V$: It follows that $h^k(\mathcal{O}_M) = 0$ for $1 \le k \le n-1$; hence $h^0(\Omega_M^k) = 0$ for $1 \le k \le n-1$ by Hodge symmetry. This proves the assertion (1).

The first part of assertion (2) follows from the Lefschetz hyperplane section theorem, because $n \geq 3$. By a result of Kollár ([Bo], Appendix), the natural map $\tau_* : \overline{\mathrm{NE}}(V) \to \overline{\mathrm{NE}}(M)$ is an isomorphism. Taking the dual cones, we obtain the second part of assertion (2).

The proof of assertion (3) is now classical (see [Wi] Page 389 for instance). By assertion (1), $T_M \simeq \Omega_M^{n-1}$; hence $h^0(T_M) = 0$. Thus $\dim \mathrm{Aut}\,(M) = 0$.

The cone $\mathrm{Amp}\,(V)$, which is the dual of $\overline{\mathrm{NE}}(V)$, is a finite rational polyhedral cone because V is a Fano manifold. As a consequence of assertion (2), $\mathrm{Amp}\,(M)$ is also a finite rational polyhedral cone. Thus, $\mathrm{Amp}\,(M)$ is the convex hull of a finite number of extremal rational rays $\mathbf{R}^+ h_i$, $1 \le i \le \ell$, each h_i being an integral primitive vector. The group $\mathrm{Aut}(M)$ preserves $\mathrm{Amp}\,(M)$ and, therefore, permutes the h_i. From this, follows that the ample class $h = \sum_{i=1}^{\ell} h_i$ is fixed by $\mathrm{Aut}\,(M)$. Let L be the line bundle with first Chern class h. Consider the embedding $\Theta_L \colon M \to \mathbb{P}(H^0(M, L^{\otimes k})^\vee)$ which is defined by a large enough multiple of L. Since $\mathrm{Aut}\,(M)$ preserves h, it acts by projective linear transformations on $\mathbb{P}(H^0(M, L^{\otimes k})^\vee)$ and the embedding Θ_L is equivariant with respect to the action of $\mathrm{Aut}(M)$ on M, on one side, and on $\mathbb{P}(H^0(M, L^{\otimes k})^\vee)$, on the other side. The image of $\mathrm{Aut}(M)$ is the closed algebraic subgroup of $\mathrm{PGL}(H^0(M, L^{\otimes k})^\vee)$ that preserves $\Theta_L(M)$. Since $\dim \mathrm{Aut}\,(M) = 0$, this algebraic group is finite, and assertion (3) follows. □

3.2. **Transformations of Wehler varieties.**

3.2.1. *Products of lines.* Let n be a positive integer. Denote

$$\begin{aligned} P(n+1) &:= (\mathbb{P}^1)^{n+1} = \mathbb{P}^1_1 \times \mathbb{P}^1_2 \times \cdots \times \mathbb{P}^1_{n+1} \\ P(n+1)_j &:= \mathbb{P}^1_1 \times \cdots \mathbb{P}^1_{j-1} \times \mathbb{P}^1_{j+1} \cdots \times \mathbb{P}^1_{n+1} \simeq P(n) \end{aligned}$$

and

$$\begin{aligned} p^j &: P(n+1) \to \mathbb{P}^1_j \simeq \mathbb{P}^1 \\ p_j &: P(n+1) \to P(n+1)_j \end{aligned}$$

the natural projections. Let H_j be the divisor class of $(p^j)^*(O_{\mathbb{P}^1}(1))$. Then $P(n+1)$ is a Fano manifold of dimension $n+1$ that satisfies

$$\begin{aligned} \mathrm{NS}(P(n+1)) &= \oplus_{j=1}^{n+1}\mathbf{Z}H_j\,, \\ -K_{P(n+1)} &= \sum_{j=1}^{n+1} 2H_j\,, \\ \mathrm{Amp}\,(P(n+1)) &= \left\{\sum a_i H_i\,|a_i \in \mathbf{R}_{>0} \text{ for all } i\right\}, . \end{aligned}$$

3.2.2. *Wehler varieties.* Let X be an element of the linear system $|-K_{P(n+1)}|$; in other words, X is a hypersurface of multi-degree $(2,2,\dots,2)$ in $P(n+1)$. More explicitly, for each index j between 1 and $n+1$, the equation of X in $P(n+1)$ can be written in the form

$$F_{j,1}x_{j,0}^2 + F_{j,2}x_{j,0}x_{j,1} + F_{j,3}x_{j,1}^2 = 0 \qquad (3.1)$$

where $[x_{j,0} : x_{j,1}]$ denotes the homogenous coordinates of $\mathbb{P}^1_j$ and the $F_{j,k}$ ($1 \le k \le 3$) are homogeneous polynomial functions of multi-degree $(2,2,\dots,2)$ on $P(n+1)_j$.

Let $\tau : X \to V$ be the natural inclusion and $h_j := \tau^* H_j$. If X is smooth, then X is a Calabi-Yau manifold of dimension $n \ge 3$, and Theorem 3.1 (1) implies that

$$\mathrm{NS}\,(X) = \oplus_{j=1}^{n+1}\mathbf{Z}h_j \ \text{ and } \ \mathrm{Amp}\,(X) = \oplus_{j=1}^{n+1}\mathbf{R}_{>0}h_j\,. \qquad (3.2)$$

Let

$$\pi_j := p_j \circ \tau : X \to P(n+1)_j \simeq P(n)\,.$$

This map is a surjective morphism of degree 2; it is finite in the complement of

$$B_j := \{F_{j,1} = F_{j,2} = F_{j,3} = 0\}\,.$$

In what follows, we assume that each B_j has codimension ≥ 3 and that X is smooth. This is satisfied for a generic choice of $X \in |-K_{P(n+1)}|$.

For $x \in B_j$, we have $\pi_j^{-1}(x) \simeq \mathbb{P}^1$. It follows that π_j contracts no divisor. For $x \notin B_j$, the set $\pi_j^{-1}(x)$ consists of 2 points, say $\{y, y'\}$: The correspondence $y \leftrightarrow y'$ defines a birational involutive transformation ι_j of X over $P(n+1)_j$. Thus, $\mathrm{Bir}(X)$ contains at least $n+1$ involutions ι_j.

The group $\mathrm{Bir}\,(X)$ naturally acts on $\mathrm{NS}\,(X)$ as a group of linear automorphisms. Indeed, since K_X is trivial, each element of $\mathrm{Bir}\,(X)$ is an isomorphism in codimension 1 (see eg. [Ka3], Page 420). When $n = 2$, X is a projective K3 surface. Since X is a minimal surface, each ι_k is a (biregular) automorphism (see eg. [BHPV], Page 99, Claim).

3.3. **The groups** $\mathrm{Aut}(X)$ **and** $\mathrm{Bir}(X)$ **in dimension** $n \ge 3$**.** In this section we prove the following strong version of the first assertions in Theorem 1.3.

Theorem 3.3. *Let $n \ge 3$ be an integer. Let $X \subset (\mathbb{P}^1)^{n+1}$ be a generic Wehler variety. Let ι_j be the $n+1$ natural birational involutions of X. Then*

(1) *In the basis* $(h_k)_{k=1}^{n+1}$ *of* $\mathrm{NS}(X)$*, the matrix of* ι_j^* *coincides with* $M_{n+1,j}$ *(see Equation* (2.1)*).*
(2) *The morphism*

$$\Psi\colon \mathrm{UC}(n+1) \to \mathrm{Bir}(X)$$

that maps the generators t_j *to the involutions* ι_j *is injective; the action of* $\Psi(\mathrm{UC}(n+1))$ *on* $\mathrm{NS}(X)$ *is conjugate to the (dual of the) geometric representation of* $\mathrm{UC}(n+1)$.
(3) *The automorphism group of X is trivial:* $\mathrm{Aut}(X) = \{\mathrm{Id}_X\}$.
(4) $\mathrm{Bir}(X)$ *coincides with the subgroup* $\langle \iota_1, \iota_2, \cdots \iota_{n+1}\rangle \simeq \mathrm{UC}(n+1)$.

Remark 3.4. As our proof shows, Assertions (1) and (2) holds whenever X is smooth and $n \geq 3$. They are also satisfied when $n = 2$, if $\mathrm{NS}(X)$ is replaced by the subspace $\mathbf{Z}h_1 \oplus \mathbf{Z}h_2 \oplus \mathbf{Z}h_3$ (this subspace is invariant under the three involutions even if $\mathrm{NS}(X)$ has dimension ≥ 4).

Assertion (4) is certainly the most difficult part of this statement, and its proof makes use of delicate recent results in algebraic geometry.

Proof of Assertions (1) and (2). By definition of ι_j, we have $\iota_j^*(h_k) = h_k$ for $k \neq j$. Write $P(n+1) = P(n+1)_j \times \mathbb{P}^1_j$, where $\mathbb{P}^1_j$ is the j-th factor of $P(n+1)$. Let $(a, [b_0 : b_1])$ be a point of $X \setminus \pi_j^{-1}(B_j)$, with $a \in P(n+1)_j$ and $[b_0 : b_1] \in \mathbb{P}^1$. Then $\iota_j(a, [b_0 : b_1])$ is the second point $(a, [c_0 : c_1])$ of X with the same projection a in $P(n+1)_j$. The relation between the roots and the coefficients of the quadratic Equation (3.1) provides the formulas:

$$\frac{c_0}{c_1} \cdot \frac{b_0}{b_1} = \frac{F_{j,3}(a)}{F_{j,1}(a)} \text{ and } \frac{c_0}{c_1} + \frac{b_0}{b_1} = -\frac{F_{j,2}(a)}{F_{j,1}(a)} .$$

Here the polynomial $F_{j,3}$ is not zero and the divisors $\mathrm{div}(F_{j,k}|X)$ ($k = 1, 2, 3$) have no common component, because X is smooth. Thus, in $\mathrm{Pic}(X) \simeq \mathrm{NS}(X)$, we obtain

$$\iota_j^*(h_j) + h_j = \sum_{k\neq j} 2h_k .$$

This proves Assertion (1).

Since $\mathrm{UC}(n+1)$ is a free product of $(n+1)$ groups of order 2, and its geometric representation is faithful, assertion (2) follows from assertion (1) and the definition of the matrices $M_{N,j}$. □

Proof of Assertion (3). Let x_j be the standard affine coordinate on $\mathbb{P}^1_j \setminus \{\infty\}$. Then $X \in |-K_{P(n+1)}|$ is determined by a polynomial function $f_X(x_1, \ldots, x_{n+1})$ of degree ≤ 2 with respect to each variable x_j.

By the third assertion in Theorem (3.1), $\mathrm{Aut}(X)$ is a finite group. Since $\mathrm{Aut}(X)$ preserves $\mathrm{Amp}(X)$, it preserves the set $\{h_j \,|\, 1 \leq j \leq n+1\}$, permuting its elements.Thus, the isomorphism $H^0(O_{P(n+1)}(H_j)) \simeq H^0(O_X(h_j))$ implies that $\mathrm{Aut}(X)$ is a subgroup of $\mathrm{Aut}(P(n+1))$:

$$\mathrm{Aut}\,(X) \subset \mathrm{Aut}\,(P(n+1)) = \mathrm{PGL}_2(\mathbf{C})^{n+1} \rtimes \mathfrak{S}_{n+1} \ ,$$

where $\mathfrak{S}_{n+1}$ denotes the group of permutations of the $(n+1)$ factors of $P(n+1)$.

The group $\mathrm{PGL}_2(\mathbf{C})^{n+1} \rtimes \mathfrak{S}_{n+1}$ acts on $|-K_{P(n+1)}|$ and $\mathrm{Aut}\,(X)$ coincides with the stabilizer of the corresponding point $X \in |-K_{P(n+1)}|$. Consider, for a generic X, the image G of the morphism

$$\mathrm{Aut}\,(X) \to \mathrm{PGL}_2(\mathbf{C})^{n+1} \rtimes \mathfrak{S}_{n+1} \to \mathfrak{S}_{n+1} \,.$$

If g is an element of G, and X is generic, there is a lift $\tilde{g}_X : X \to X$ which is induced by an automorphism of $P(n+1)$. This turns out to be impossible, by considering the actions on the inhomogeneous quadratic monomials x_j^2 in the equation f_X (for $1 \le j \le n+1$).

Thus for X generic, $\mathrm{Aut}\,(X)$ coincides with a finite subgroup of $\mathrm{PGL}_2(\mathbf{C})^{n+1}$. Let $\mathrm{Id}_X \neq g$ be an automorphism of X; g is induced by an element of $\mathrm{PGL}_2(\mathbf{C})^{n+1}$ of finite order. Then *up to conjugacy* inside $\mathrm{PGL}_2(\mathbf{C})^{n+1}$, the co-action of g can be written as $g^*(x_j) = c_j x_j$ $(1 \le j \le n+1)$, where c_j are all roots of 1 and at least one c_j, say c_1, is not 1. By construction the equation $f_X(x_j)$ is g^*-semi-invariant: $f_X \circ g = \alpha(g) f_X$ for some root of unity $\alpha(g)$. Decompose f_X into a linear combination of monomial factors

$$x_1^{k_1} x_2^{k_2} \cdots x^{k_{n+1}} \ , \ k_j = 0,1,2 \ ;$$

the possible factors satisfy the eigenvalue relation

$$c_1^{k_1} c_2^{k_2} \cdots c_{n+1}^{k_{n+1}} = \alpha(g) \,.$$

Since $c_1 \neq 1$, for each fixed choice of $(k_2, k_3, \ldots, k_{n+1})$, at most two of the three monomials

$$x_1^2 x_2^{k_2} \cdots x^{k_{n+1}} \,, \quad x_1 x_2^{k_2} \cdots x^{k_{n+1}} \,, \quad x_2^{k_2} \cdots x^{k_{n+1}}$$

satisfy the relation above. Hence the number of monomial factors in $f_X(x_j)$ is at most $2 \cdot 3^n$. Moreover, given g, the possible values of $\alpha(g)$ are finite (their number is bounded from above by the order of g).

Thus, the subset of varieties $X \in |-K_{P(n+1)}|$ with at least one automorphism $g \neq 1$ belongs to countably many subsets of dimension at most

$$2 \cdot 3^n - 1 + \dim \mathrm{PGL}_2(\mathbf{C})^{n+1} = 3^{n+1} + 3(n+1) - 3^n - 1$$

in $|-K_{P(n+1)}|$. Since $\dim |-K_{P(n+1)}| = 3^{n+1} - 1$, the codimension of these subsets is at least

$$3^n - 3(n+1) \ge 3$$

for $n \ge 2$. Thus, removing a countable union of subsets of $|-K_{P(n+1)}|$ of codimension ≥ 3, the remaining generic members X of $|-K_{P(n+1)}|$ have trivial automorphism group. □

Proof of Assertion (4). Let

$$X \overset{\overline{\pi}_j}{\to} \overline{X}_j \overset{q_j}{\to} P(n+1)_j$$

be the Stein factorization of π_j. Then, $\overline{\pi}_j$ is the small contraction corresponding to the codimension 1 face $F_j := \sum_{k \neq j} \mathbf{R}_{\geq 0} h_k$ of the nef cone $\overline{\mathrm{Amp}}(X)$. Thus, $\rho(X/\overline{X}_j) = 1$ with a $\overline{\pi}_j$-ample generator h_j. Hence $\overline{\pi}_j$ is a flopping contraction of X. Let us describe the flop of $\overline{\pi}_j$.

By definition of the Stein factorization, ι_j induces a *biregular* automorphism $\bar{\iota}_j$ of $\overline{X}_j$ that satisfies $\bar{\iota}_j \circ \overline{\pi}_j = \overline{\pi}_j \circ \iota_j$. We set

$$\overline{\pi}_j^+ := (\bar{\iota}_j)^{-1} \circ \overline{\pi}_j : X \to \overline{X}_j \, .$$

Then, $\iota \circ \overline{\pi}_j^+ = \overline{\pi}_j$ and $\iota_j^*(h_j) = -h_j + \sum_{k \neq j} 2h_k$ by (1). In particular, $\iota_j^*(h_j)$ is $\overline{\pi}_j^+$-anti-ample. Hence $\overline{\pi}_j^+ : X \to \overline{X}_j$, or by abuse of language, the associated birational transformation ι_j, is the flop of $\overline{\pi}_j : X \to \overline{X}_j$.

Recall that any flopping contraction of a Calabi-Yau manifold is given by a codimension one face of $\overline{\mathrm{Amp}}(X)$ up to automorphisms of X ([Ka2], Theorem (5.7)). Since there is no codimension one face of $\overline{\mathrm{Amp}}(X)$ other than the F_j $(1 \leq j \leq n+1)$, it follows that there is no flop other than ι_j $(1 \leq j \leq n+1)$ up to $\mathrm{Aut}(X)$. On the other hand, by a fundamental result of Kawamata ([Ka3], Theorem 1), any birational map between minimal models is decomposed into finitely many flops up to automorphisms of the source variety. Thus any $\varphi \in \mathrm{Bir}(X)$ is decomposed into a finite sequence of flops modulo automorphisms of X. Hence $\mathrm{Bir}(X)$ is generated by $\mathrm{Aut}(X)$ and ι_j $(1 \leq j \leq n+1)$. Since $\mathrm{Aut}(X) = \{\mathrm{Id}_X\}$ for X generic, assertion (4) is proved. □

Remark 3.5. In general, the minimal models of a given variety are not unique up to isomorphisms. See for instance [LO] for an example on a Calabi-Yau threefold. However, by the proof of Assertion (3), the generic Wehler variety X has no other minimal model than X itself.

3.4. **Wehler surfaces.**

3.4.1. *Statement.* Assume $n = 2$, so that X is now a smooth surface in $P(3)$. Then, X is a projective K3 surface; in particular, X is a minimal surface and each ι_k is a (biregular) automorphism (see eg. [BHPV], Page 99, Claim). Thus

$$\langle \iota_1, \iota_2, \iota_3 \rangle \subset \mathrm{Aut}(X).$$

We now prove the following result, which contains a precise formulation of § 1.2.1. To state it, we keep the same notations as in Sections 3.2 and 3.3; in particular, τ is the embedding of X in $P(3)$ and the h_j are obtained by pull-back of the classes H_j.

Theorem 3.6. *Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth Wehler surface.*

(1) *If X is smooth, then* $\Psi(\mathrm{UC}(3)) \subset \mathrm{Aut}(X)$.

(2) *If X is generic, then* $\mathrm{NS}\,(X) = \oplus_{j=1}^{3}\mathbf{Z}h_j$.

(3) *If X is generic, then* $\mathrm{Aut}\,(X) = \langle \iota_1, \iota_2, \iota_3\rangle = \langle \iota_1\rangle * \langle \iota_2\rangle * \langle \iota_3\rangle \simeq \mathrm{UC}(3)$.

Remark 3.7. The Néron-Severi group may have dimension $\rho(X) > 3$; such a jump of $\rho(X)$ can be achieved by arbitrary small deformations (see [Og]), and it may leads to very nice examples of ample cones (see [Bar]).

3.4.2. *Proof of Theorem 3.6.* The three following lemmas prove Theorem 3.6. In this section, X is a smooth Wehler surface. We start with a description of $\mathrm{NS}(X)$ and of the quadratic form defined by the intersection of divisor classes.

Lemma 3.8. *If X is generic, then* $\mathrm{NS}\,(X) = \oplus_{i=1}^{3}\mathbf{Z}h_i$. *The matrix* $((h_i \cdot h_j)_X)$ *of the intersection form on* $\mathrm{NS}(X)$ *is*

$$((h_i.h_j)_S) = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

Proof. Since X is generic, it follows from the Noether-Lefschetz theorem ([Vo], Theorem (3.33)) that $\tau^* : \mathrm{NS}\,(P(3)) \to \mathrm{NS}\,(X)$ is an isomorphism. This proves (1), and (2) follows from $(h_i.h_j)_S = (H_i.H_j.2(H_1+H_2+H_3))_{P(3)}$. □

Remark 3.9. More generally, if W is a generic element of $|-K_V|$ of a smooth Fano threefold with very ample anti-canonical divisor $-K_V$, then W is a K3 surface and $\mathrm{NS}\,(V) \simeq \mathrm{NS}\,(W)$ under the natural inclusion map.

Thus, the intersection form corresponds to the quadratic form b_3 on V_3 that is preserved by $\mathrm{UC}(3)$. The fact that b_3 has signature $(1,2)$ is an instance of Hodge index theorem (see Remark 2.5).

Once we know the intersection form, one can check that there is no effective curve with negative self-intersection on X. Indeed, if there were such a curve, one could find an irreducible curve $E \subset X$ with $E \cdot E < 0$; the genus formula would imply that E is a smooth rational curve with self-intersection -2; but the intersection form does not represent the value -2.

It is known that the ample cone is the set of vectors u in $\mathrm{NS}(X;\mathbf{R})$ such that $u \cdot u > 0$, $u \cdot E > 0$ for all effective curves, and $u \cdot u_0 > 0$ for a given ample class (for example $u_0 = \sum h_j$). Since there are no curves with negative self-intersection, one obtains:

Lemma 3.10. *If X is generic, the ample cone* $\mathrm{Amp}\,(X)$ *coincides with the positive cone*

$$\mathsf{Pos}(X) = \{u \in \mathrm{NS}(X;\mathbf{R}) \mid u \cdot u > 0 \textit{ and } u \cdot h_1 > 0\}.$$

Remark 3.11. Even though X is generic and $\tau^* : \mathrm{NS}\,(P(3)) \to \mathrm{NS}\,(X)$ is an isomorphism, the image of $\tau^* : \mathrm{Amp}\,(P(3)) \to \mathrm{Amp}\,(X)$ is strictly smaller than $\mathrm{Amp}\,(X)$, in contrast with the higher dimensional case (see Theorem (3.1)).[1]

[1]This gives an explicit, negative, answer for a question of Pr. Yoshinori Gongyo to K. Oguiso.

From the previous sections (see Remark 3.4), we know that, for all smooth Wehler surfaces,

- the subspace $N_X := \oplus_{j=1}^{3} \mathbf{Z} h_j$ is invariant under the action of $\langle \iota_1, \iota_2, \iota_3 \rangle$ on $\mathrm{NS}(X)$;
- the matrix of $(\iota_k^*)_{|N_X}$ in the basis (h_1, h_2, h_3) is equal to $M_{3,k}$, where $M_{3,k}$ is defined in § 2.2.1;
- there are no non-obvious relations between the three involutions ι_k^*, hence
$$\langle \iota_1 \rangle * \langle \iota_2 \rangle * \langle \iota_3 \rangle \simeq \mathrm{UC}(3)$$

Moreover, there is a linear isomorphism from the (dual of the) geometric representation V_3 to N_X which conjugates the action of $\mathrm{UC}(3)$ with the action of $\langle \iota_1, \iota_2, \iota_3 \rangle$, and maps D_3 to the convex cone

$$\Delta = \mathbf{R}_+ h_1 \oplus \mathbf{R}_+ h_2 \oplus \mathbf{R}_+ h_3,$$

the quadratic form b_3 to the intersection form on N_X, and the Tits cone T_3 to the positive cone $\mathsf{Pos}(X)$. Since the ample cone is invariant under the action of $\mathrm{Aut}(X)$ and contains $\mathbf{R}_+ h_1 \oplus \mathbf{R}_+ h_2 \oplus \mathbf{R}_+ h_3$, this gives another proof of Lemma 3.10.

Lemma 3.12. *If X is generic, then*

(1) *no element of* $\mathrm{Aut}(X) \setminus \{\mathrm{Id}_X\}$ *is induced by an element of* $\mathrm{Aut}(P(3))$.
(2) $\mathrm{Aut}(X) = \langle \iota_1, \iota_2, \iota_3 \rangle$.

Proof. The proof of (1) is the same as for Theorem 3.3. Let us prove (2). Since X is generic, $\mathrm{NS}(X) = N_X$. The image G of $\mathrm{Aut}(X)$ in $\mathrm{GL}(\mathrm{NS}(X))$. contains the group generated by the three involutions ι_j^*; as such it as finite index in the group of isometries of the lattice $\mathrm{NS}(X)$ with respect to the intersection form (see § 2.2.5). Thus, if G is larger than $\langle \iota_1^*, \iota_2^*, \iota_3^* \rangle$, there exists an element g^* of $G \setminus \{\mathrm{Id}\}$ that preserves the fundamental domain Δ. Such an element permutes the vertices of Δ. As in the proof of Theorem 3.3, one sees that g^* would be induced by an element of $\mathrm{Aut}(P(3))$, contradicting Assertion (1). Thus, G coincides with $\langle \iota_1^*, \iota_2^*, \iota_3^* \rangle$. On the other hand, if $f \in \mathrm{Aut}(X)$ acts trivially on $\mathrm{NS}(X)$ then, again, f is induced by an element of $\mathrm{Aut}(P(3))$. Thus, Assertion (2) follows from Assertion (1). □

4. The movable cone

To conclude the proof of Theorem 1.3, we need to describe the movable cone of Wehler varieties. This section provides a proof of a more explicit result, Theorem 4.1.

4.1. **Statement.** To state the main result of this section, we implicitly identify the geometric representation V_{n+1} of $\mathrm{UC}(n+1)$ to its dual V_{n+1}^*; for this, we make use of the duality offered by the non-degenerate quadratic form b_{n+1}, as in Section 2.2. This said, let $\rho\colon \mathrm{UC}(n+1) \to \mathrm{GL}(V_{n+1})$ be the geometric representation of the universal Coxeter group $\mathrm{UC}(n+1)$. Let

$$\Psi\colon \mathrm{UC}(n+1) \to \mathrm{Bir}(X)$$

be the isomorphism that maps the generators t_i of $\mathrm{UC}(n+1)$ to the generators ι_i of $\mathrm{Bir}(X)$.

Theorem 4.1. *Let $n \geq 3$ be an integer, and let $X \subset (\mathbb{P}^1)^{n+1}$ be a generic Wehler variety of dimension n.*

There is a linear isomorphism $\Phi\colon V_{n+1} \to \mathrm{NS}(X)$ such that

1. $\Phi \circ \rho(w) = \Psi(w)^* \circ \Phi$ *for all elements w of* $\mathrm{UC}(n+1)$*;*
2. *the fundamental domain D_{n+1} of $\mathrm{UC}(n+1)$ is mapped onto the nef cone $\overline{\mathrm{Amp}}(X)$ by Φ;*
3. *the Tits cone $\mathsf{T}_{n+1} \subset V$ is mapped onto the movable effective cone $\mathcal{M}^e(X)$ by Ψ.*

In particular, the nef cone is a fundamental domain for the action of $\mathrm{Bir}(X)$ *on the movable effective cone.*

Remark 4.2. [2] Since $\mathrm{Bir}(X)$ is much bigger than $\mathrm{Aut}(X)$, Theorem 4.1 implies that the movable effective cone $\mathcal{M}^e(X)$, whence the pseudo effective cone $\overline{\mathcal{B}}(X)$, is much bigger than the nef cone $\overline{\mathrm{Amp}}(X)$. On the other hand, for the ambient space $P(n+1)$, we have

$$\overline{\mathcal{B}}(P(n+1)) = \overline{\mathrm{Amp}}(P(n+1)).$$

This is a direct consequence of the fact that the intersection numbers

$$(v.H_1 \cdots H_{k-1} \cdot H_{k+1} \cdots H_{n+1})_{P(n+1)}\,,\ 1 \leq k \leq n+1\,,$$

are non-negative if $v \in \overline{\mathcal{B}}(P(n+1))$. So, under the isomorphism $\mathrm{NS}(P(n+1)) \simeq \mathrm{NS}(X)$, we have

- $\overline{\mathrm{Amp}}(P(n+1)) \simeq \overline{\mathrm{Amp}}(X)$ (see Theorem 3.1)
- $\overline{\mathcal{B}}(P(n+1)) \not\simeq \overline{\mathcal{B}}(X)$, even if X is generic.

In the rest of this section, we prove Theorem 4.1. Thus, in what follows, X is a generic Wehler variety of dimension $n \geq 3$.

4.2. **Proof.** Let Ψ be the isomorphism described in Section 4.1, and Φ the linear map which applies the cone D_{n+1} onto the nef cone $\overline{\mathrm{Amp}}(X)$, mapping each vertex c_j to h_j. With such a choice, Assertions (1) and (2) are part of Section 3.3, and we only need to prove Assertion (3), i.e. that the Tits cone T_{n+1} is mapped bijectively onto the movable effective cone $\mathcal{M}^e(X)$ by Φ.

Lemma 4.3. *Let X be a generic Wehler variety of dimension $n \geq 3$. Then*

1. $\overline{\mathrm{Amp}}(X) \subset \mathcal{M}^e(X)$*;*

[2]This remark provides an explicit negative answer to the question asked by Mr Y. Gongyo (in any dimension ≥ 3) to K. Oguiso.

(2) $g^*(\mathrm{Amp}(X)) \cap \mathrm{Amp}(X) = \emptyset$ *for all* $g \neq \mathrm{Id}_X$ *in* $\mathrm{Bir}(X)$.

Proof. Since the divisor classes h_j ($1 \leq j \leq n+1$) are free, they are movable; assertion (1) follows from the fact that the nef cone is generated, as a convex cone, by these classes. If $g \in \mathrm{Bir}(X)$ satisfies $g^*(\mathrm{Amp}(X)) \cap \mathrm{Amp}(X) \neq \emptyset$, then $g \in \mathrm{Aut}(X)$ (see [Ka2], Lemma 1.5). Thus, Assertion (2) follows from $\mathrm{Aut}(X) = \{\mathrm{Id}_X\}$. □

Since the movable effective cone is $\mathrm{Bir}(X)$-invariant, the orbit of $\overline{\mathrm{Amp}}(X)$ is contained in $\mathcal{M}^e(X)$. Hence,

$$\Phi(\mathsf{T}_{n+1}) \subset \mathcal{M}^e(X)$$

and we want to show the reverse inclusion (note that the closures of these two sets are equal).

Lemma 4.4. *Let X be a generic Wehler variety of dimension $n \geq 3$. For any given effective integral divisor class D, there is a birational transformation g of X such that $g^*(D) \in \overline{\mathrm{Amp}}(X)$. That is, D is contained in $\Phi(\mathsf{T}_{N+1})$.*

The proof is similar to the proof of Proposition 2.11. Here, one makes use of the intersection form and positivity properties of effective divisor classes, instead of the quadratic form b_N.

Remark 4.5. We do not know any geometric interpretation of the quadratic form b_{n+1} on $\mathrm{NS}(X)$ for $n \geq 3$.

Proof. Define $D_1 := D$. In $\mathrm{NS}(X)$, we can write

$$D_1 = \sum_{j=1}^{n+1} a_j(D_1) h_j \ ,$$

where the coefficients $a_j(D_1)$ are integers. Put

$$s(D_1) := \sum_{j=1}^{n+1} a_j(D_1) \ .$$

By definition, $s(D_1)$ is an integer. Since D_1 is an effective divisor class and the classes h_i are nef, it follows that

$$a_n(D_1) + a_{n+1}(D_1) = (D \cdot h_1 \cdot h_2 \cdots h_{n-1})_X \geq 0 \, .$$

For the same reason, $a_i(D_1) + a_j(D_1) \geq 0$ for all $1 \leq i \neq j \leq n+1$. Hence there is *at most* one i such that $a_i(D_1) < 0$.

Moreover the sum $s(D_1)$ is non-negative. Indeed, there is at most one negative term, say $a_1(D_1)$, in the sum defining $s(D_1)$; since

$$s(D_1) = (a_1(D_1) + a_2(D_1)) + a_3(D_1) + \cdots + a_{n+1}(D_1) \geq 0$$

and $a_1(D) + a_2(D_1) \geq 0$, this shows that $s(D_1)$ is non-negative.

If $D_1 \in \overline{\mathrm{Amp}}(X)$, then we can take $g = 1$. So, we may assume that D_1 is not in $\overline{\mathrm{Amp}}(X)$, which means that there is a unique index i with $a_i(D_1) < 0$. Then, consider the new divisor class $D_2 := \iota_i^*(D_1)$; it is effective. By definition of $a_j(\cdot)$ and Theorem 3.3 (first assertion), we have

$$\begin{aligned} D_2 &= \sum_{j=1}^{n+1} a_j(D_2)h_j \\ &= -a_i(D_1)h_i + \sum_{j\neq i}(a_j(D_1)+2a_i(D_1))h_j \ . \end{aligned}$$

Computing $s(D_2) = \sum a_j(D_2)$, the inequality $a_i(D_1) < 0$ provides

$$\begin{aligned} s(D_2) &= -a_i(D_1) + \sum_{j\neq i}(a_j(D_1)+2a_i(D_1)) \\ &= s(D_1) + (2n-1)a_i(D_1) \\ &< s(D_1) \ . \end{aligned}$$

If $a_j(D_2) \geq 0$ for all j, then $D_2 \in \overline{\mathrm{Amp}}(X)$, and we are done. Otherwise, there is j such that $a_j(D_2) < 0$. Consider then the divisor class $D_3 := \iota_j^*(D_2)$. As above, D_3 is an effective divisor class such that $s(D_3) < s(D_2)$.

We repeat this process: At each step, the sum $s(\cdot)$ decreases by at least one unit. Since $s(\cdot)$ is a positive integer for all effective divisors, the process stops, and provides an effective divisor D_k in the $\mathrm{Bir}(X)$-orbit of D_1 such that all coefficients $a_i(D_k)$ are non-negative, which means that D_k is an element of $\overline{\mathrm{Amp}}(X)$. □

Let u be an element of the movable cone $\mathcal{M}^e(X)$. As $\mathcal{M}^e(X)$ is the intersection of $\overline{\mathcal{M}}(X)$ and $\mathcal{B}^e(X)$, we can write

$$u = \sum_{i=1}^{l+1} r_i D_i$$

where each D_i is an effective divisor class and all r_i are positive real numbers. The Tits cone T_{N+1} is a convex set (see § 2.1.3) and the previous lemma shows that each D_i is in the image of T_{N+1}; since u is a convex combination of the D_i, u is an element of $\Phi(\mathsf{T}_{N+1})$. Consequently, we obtain

$$\Phi(\mathsf{T}_{n+1}) = \mathcal{M}^e(X),$$

and Theorem 4.1 is proved.

4.3. **Complement.** The following theorem follows from our description of the movable cone as (the image of) the Tits cone T_{N+1} and the description of rational boundary points of T_{N+1} obtained in Section 2.2.8. As far as we know, this kind of statement is not predicted by the original cone conjecture.

Theorem 4.6. *Let $n \geq 3$ be an integer. Let X be a generic hypersurface of multi-degree $(2,\ldots,2)$ in $(\mathbb{P}^1)^{n+1}$. Let D be rational boundary point of the movable cone $\overline{\mathcal{M}}(M)$. Then there exists a pseudo-automorphism f of X such* *that $f^*(D)$ is in the nef cone $\overline{\mathrm{Amp}}(M)$. Hence, rational boundary points of $\overline{\mathcal{M}}(M)$ are movable.*

5. Universal cover of Hilbert schemes of Enriques surfaces

In this section, we shall prove Theorem 1.4 and a few refinements.

5.1. **Enriques surfaces (see eg.** [BHPV]**, Chapter VIII).** An Enriques surface S is a compact complex surface whose universal cover $\tilde{S}$ is a K3 surface. The Enriques surfaces form a 10-dimensional family; all of them are projective and their fundamental group is isomorphic to $\mathbf{Z}/2\mathbf{Z}$.

Let S be an Enriques surface. The free part of the Néron-Severi group $\mathrm{NS}_f(S)$ is isomorphic to the lattice $U \oplus E_8(-1)$, were (cf. [BHPV], Chapter I, Section 2)

- U is the unique even unimodular lattice of signature $(1,1)$;
- $E_8(-1)$ is the unique even unimodular *negative* definite lattice of rank 8.

From now on, we identify the lattices $\mathrm{NS}_f(S)$ and $U \oplus E_8(-1)$. We denote the group of isometries of $\mathrm{NS}_f(S)$ preserving the positive cone by O^+_{10}. Here, by definition, the positive cone $\mathrm{Pos}(S)$ is the connected component of $\{x \in \mathrm{NS}(S)_{\mathbf{R}} \,|\, (x^2) > 0\}$, containing the ample cone. We define

$$\mathrm{O}^+_{10}[2] = \{\varphi \in \mathrm{O}^+_{10} \,|\, \varphi = \mathrm{Id} \mod (2)\}.$$

Theorem 5.1 (see [BP], Theorem 3.4, Proposition 2.8)**.** *Let S be a generic Enriques surface. Then S does not contain any smooth rational curve and the morphism $g \mapsto g^* \in \mathrm{GL}(\mathrm{NS}_f(S))$ provides an isomorphism*

$$\mathrm{Aut}(S) \simeq \mathrm{O}^+_{10}[2].$$

Moreover, O^+_{10} is isomorphic to the Coxeter group associated to the Coxeter diagram $\mathrm{T}_{2,3,7}$, a tree with 10 vertices and three branches of length 2, 3, and 7 respectively (see [CoDo]). From now on, S is a generic Enriques surface.

5.2. **Hilbert schemes and positive entropy.** The group $\mathrm{Aut}(S)$ acts by automorphisms on the Hilbert scheme $\mathrm{Hilb}^n(S)$; this defines a morphism

$$\rho_n \colon \mathrm{Aut}(S) \to \mathrm{Aut}(\mathrm{Hilb}^n(S)).$$

Denote by $\mathcal{H}^n(S)$ the universal cover $\widetilde{\mathrm{Hilb}^n}(S)$, and by $\mathrm{A}_n(S)$ the group of all automorphisms of $\mathcal{H}^n(S)$ which are obtained by lifting elements of $\rho_n(\mathrm{Aut}(S))$. By construction, there is an exact sequence

$$1 \to \mathbf{Z}/2\mathbf{Z} \to \mathrm{A}_n(S) \to \rho_n(\mathrm{Aut}(S)) \to 1.$$

Up to conjugacy, the group $\mathrm{O}^+_{10}[2]$ is a lattice in the Lie group $\mathrm{O}_{1,9}(\mathbf{R})$; as such, it contains a non-abelian free group (we shall describe an explicit free subgroup below). Since all elements in the kernel of ρ_n have finite order, the group $\mathrm{A}_n(S)$ is commensurable to a lattice in the Lie group $\mathrm{O}_{1,9}(\mathbf{R})$ and contains a non-abelian free group. Since all holomorphic vector fields on $\mathcal{H}^n(S)$ vanish identically, one can find such a free group that acts faithfully on the Néron-Severi group of $\mathcal{H}^n(S)$.

Lemma 5.2. *Let M be a complex projective manifold. Let G be a subgroup of* $\mathrm{Aut}(M)$ *such that (i) G is a non-abelian free group and (ii) G acts faithfully on* $\mathrm{NS}(M)$. *Then there exists an element g in G such that $g^*\colon \mathrm{NS}(M) \to \mathrm{NS}(M)$ has an eigenvalue λ with $|\lambda| > 1$. In particular, the topological entropy of g is strictly positive.*

Proof. The first assertion follows from the fact that G is a subgroup of $\mathrm{GL}(\mathrm{NS}(X))$, hence of $\mathrm{GL}_m(\mathbf{Z})$, where m is the Picard number of M: If all eigenvalues of all elements g^* had modulus one, they would be roots of 1, and a finite index subgroup of G would be solvable (conjugate to a subgroup of $\mathrm{GL}_m(\mathbf{R})$ made of upper triangular matrices with coefficients 1 on the diagonal). The second follows from Yomdin's lower bound for the topological entropy.[3] □

Thus $\mathcal{H}^n(S)$ has many automorphisms with positive entropy. In the next sections, we make this statement more precise and explicit, by constructing an embedding of $\mathrm{UC}(3)$ into $\mathrm{A}_n(S)$.

5.3. **Three involutions and positive entropy.** Let e_1, e_2 be a standard basis of U, that is $U = Ze_1 \oplus \mathbf{Z}e_2$ and

$$(e_1^2)_S = (e_2^2)_S = 0, \quad (e_1 \cdot e_2)_S = 1.$$

We can, and do choose e_1 and e_2 in the closure of the positive cone $\mathsf{Pos}(S)$.

Let $v \in E_8(-1)$ be an element such that $(v^2)_S = -2$. Put $e_3 := e_1 + e_2 + v$. Then, we have

$$(e_3^2)_S = 0\,,\ (e_3 \cdot e_1)_S = (e_3 \cdot e_2)_S = 1\,.$$

So, e_2 and e_3 are also in the closure of the positive cone and the three sublattices

$$U_3 = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2\,,\ U_2 = \mathbf{Z}e_1 \oplus \mathbf{Z}e_3\,,\ U_1 = \mathbf{Z}e_2 \oplus \mathbf{Z}e_3$$

of $\mathrm{NS}_f(S)$ are isomorphic to U.

Let j be one of the indices $1, 2, 3$. The sublattice U_j determines an orthogonal decomposition $\mathrm{NS}_f(S) = U_j \oplus U_j^\perp$. Consider the isometry of $\mathrm{NS}_f(S)$ defined by

$$\iota_j^* = \mathrm{id}_{U_j} \oplus -\mathrm{id}_{U_j^\perp}$$

Then, ι_j^* is an element of $\mathrm{O}_{10}^+[2]$, and is induced by a unique automorphism ι_j of S, and ι_j is an involution (apply Theorem 5.1). Let $\langle \iota_1, \iota_2, \iota_3 \rangle$ be the subgroup of $\mathrm{Aut}(S)$ generated by ι_1, ι_2 and ι_3.

[3] An element $g \in \mathrm{Aut}(M)$ is of positive entropy if and only if the spectral radius of the action of g on $H^{1,1}(M, \mathbf{R})$ is strictly bigger than 1. The entropy $h_{top}(g)$ is equal to the spectral radius of g^* on $\oplus_p H^{p,p}(M, \mathbf{R})$. (see [Gro])

Theorem 5.3. *Let S be a generic Enriques surface. There are no non-trivial relations between the three linear transformations ι_k^*: The groups $\langle \iota_1, \iota_2, \iota_3 \rangle$, $\langle \iota_1^* \rangle * \langle \iota_2^* \rangle * \langle \iota_3^* \rangle$ and $\mathrm{UC}(3)$ are isomorphic. Moreover, the maximal eigenvalue of $\iota_1^* \iota_2^* \iota_3^*$ on $\mathrm{NS}(S)$ is equal to $9+4\sqrt{5}$. In particular, the topological entropy of the automorphism $\iota_3 \circ \iota_2 \circ \iota_1$ is positive.*

Proof. Consider the sub-lattice of $\mathrm{NS}(S)$ defined by

$$L = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2 \oplus \mathbf{Z}v = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2 \oplus \mathbf{Z}e_3.$$

By construction $\iota_1^*(e_2) = e_2$ and $\iota_1^*(e_3) = e_3$. Since $2e_2 + v$ is orthogonal to both e_2 and e_3, it follows that $2e_2 + v \in U_1^{\perp}$. Thus $\iota_1^*(2e_2 + v) = -(2e_2 + v)$ and $\iota_1^*(v) = -4e_2 - v$. One deduces easily that $\iota_1^*(e_1) = -e_1 + 2e_2 + 2e_3$. Permuting the indices, we obtain a similar formula for ι_2^* and ι_3^*. It follows that the lattice L is $\langle \iota_1, \iota_2, \iota_3 \rangle$-invariant, and the matrices of the involutions in the basis (e_1, e_2, e_3) are

$$\iota_{3|L}^* = M_{3,3} \ , \ \iota_{2|L}^* = M_{3,2}, \ , \ \iota_{1|L}^* = M_{3,1} \ ,$$

($M_{3,3}$, $M_{3,2}$ and $M_{3,1}$ are the matrices introduced in Section 2.2.1).

The remaining assertions follow from the fact that the maximal eigenvalues of the product matrix $M_{3,1}M_{3,2}M_{3,3}$ is $9+4\sqrt{5} > 1$. □

The following remark, which has been kindly communicated to us by professor Shigeru Mukai, provides a geometric explanation of the previous statement, which is related to Wehler surfaces.

Remark 5.4. The class $h = e_1 + e_2 + e_3$ determines an ample line bundle of degree 6 and the projective model of S associated to $|h|$ is a sextic surface in $\mathbb{P}^3$ singular along the 6-lines of a tetrahedron. Then the universal cover $\tilde{S}$ has a projective model of degree 12: It is a quadratic section of the Segre manifold $P(3) = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$; so, $\tilde{S}$ is a K3 surface of multi-degree $(2,2,2)$ in $P(3)$, i.e., a Wehler surface.

Equivalently, let $\pi : \tilde{S} \to S$ be the universal cover of S. The classes $\pi^* e_i$ define three different elliptic fibrations $\varphi_i : \tilde{S} \to \mathbb{P}^1$ with no reducible fiber. Hence $\varphi_1 \times \varphi_2 \times \varphi_3$ embeds $\tilde{S}$ into $P(3)$ and the image is a surface of multi-degree $(2,2,2)$.

Then, one easily shows that the action of $\mathrm{UC}(3)$ on S is covered by the natural action of $\mathrm{UC}(3)$ on the Wehler surface $\tilde{S}$ (Theorem 3.6-(2) in §6).

5.4. **Proof of Theorem 1.4.** We have a natural biregular action of the group $\mathrm{UC}(3) \simeq \langle \iota_1, \iota_2, \iota_3 \rangle$ on the Hilbert scheme $\mathrm{Hilb}^n(S)$, induced by the action on S. Each $\iota_j \in \mathrm{Aut}(\mathrm{Hilb}^n(S))$ lifts equivariantly to a biregular action $\tilde{\iota}_j \in \mathrm{Aut}(\mathcal{H}^n(S))$ on the universal cover $\mathcal{H}^n(S) = \widetilde{\mathrm{Hilb}^n(S)}$. It suffices to show that each $\tilde{\iota}_j$ is an involution (we will then have natural surjective homomorphisms $\mathrm{UC}(3) \to \langle \tilde{\iota}_1, \tilde{\iota}_2, \tilde{\iota}_3 \rangle \to \langle \iota_1, \iota_2, \iota_3 \rangle \simeq \mathrm{UC}(3)$, hence all the arrows will be isomorphic)

Lemma 5.5 (see [MN]). *Let s be a holomorphic involution of* $\mathrm{Hilb}^n(S)$. *Then, all lifts $\tilde{s}$ of s to the universal cover $\mathcal{H}^n(S)$ are involutions too.*

Proof. Let $\pi\colon \mathcal{H}^n(S) \to \mathrm{Hilb}^n(S)$ be the universal covering map and σ be the covering involution. Let $\tilde{s}$ be a lift of s, *i.e.* an automorphism of $\mathcal{H}^n(S)$ with $\pi \circ \tilde{s} = s \circ \pi$. Then $(\tilde{s})^2$ is either the identity map or σ. Assume $(\tilde{s})^2 = \sigma$ and follow [MN], Lemma (1.2), to derive a contradiction. By assumption, $\langle \tilde{s} \rangle$ is a cyclic group of order 4 and acts freely on $\mathcal{H}^n(S)$ because so does σ. By citeOS, Theorem (3.1), Lemma (3.2), the Euler characteristic of the sheaf $O^n_{\mathcal{H}}(S)$ is equal to 2. Since the group $\langle \tilde{s} \rangle$ has order 4 and acts freely, the quotient $\mathcal{H}^n(S)/\langle \tilde{\iota}_j \rangle$ is smooth and

$$\chi\left(O_{\mathcal{H}^n(S)/\langle \tilde{s} \rangle}\right) = \frac{1}{4}\chi\left(O_{\mathcal{H}^n(S)}\right) = \frac{1}{2},$$

a contradiction because this characteristic should be an integer. □

This completes the proof of Theorem 1.4 : one gets an embedding of $\mathrm{UC}(3)$ in $\mathrm{Aut}(\mathcal{H}^n(S))$, and its image contains automorphisms with positive entropy (either by Theorem 5.3 or Lemma 5.2).

5.5. **A question.** There are Enriques surfaces with $|\mathrm{Aut}(S)| < \infty$; on the other hand, Kondo shows that $|\mathrm{Aut}(\tilde{S})| = \infty$ for *every* K3 surface $\tilde{S}$ which is the universal cover of an Enriques surface (see [Kn] for these facts). Thus, it is natural to ask the following question.

Question 5.6. $|\mathrm{Aut}(\widetilde{\mathrm{Hilb}^n(S)})| = \infty$ for *every* Enriques surface?

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