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On dynamical degrees of birational transformations

Jérémy Blanc and Serge Cantat

Abstract

The dynamical degree $\lambda(f)$ of a birational transformation f measures the exponential growth rate of the degree of the formulae that define the n-th iterate of f. We describe some of the properties of the set of all dynamical degrees of all birational transformations of projective surfaces, and the relationship between the value of $\lambda(f)$ and the structure of the conjugacy class of f. For instance, the set of all dynamical degrees of birational transformations of the complex projective plane is a closed and well ordered set of algebraic numbers. We also survey some recent advances on this topic, both for surfaces and higher dimensional algebraic varieties.

1 Introduction

Dynamical degrees and dynamics.– Let $f\colon X \dashrightarrow X$ be a rational transformation of a projective variety X, defined over a field $\mathbf{k}$. For each codimension p between 0 and $\dim(X)$, one associates to f its p-th dynamical degree $\lambda_p(f)$ (see Section 2.1 for a definition). The *dynamical degrees* $\lambda_p(f)$ are positive real numbers that measure the complexity of the dynamics of f. They play a crucial role if one wants to describe the dynamics of f from an algebraic, an arithmetic, a topological or an ergodic viewpoint. For instance, if $\mathbf{k}$ is the field of complex numbers, then

- $\lambda_{\dim(X)}(f)$ is the toplogical degree of f;
- if one sets $\lambda(f) = \max\{\lambda_p(f)\ ;\ 0 \leq p \leq \dim(X)\}$, the neperian logarithm $\log(\lambda(f))$ provides an upper bound for the topological entropy of $f\colon X(\mathbf{C}) \dashrightarrow X(\mathbf{C})$ and is equal to it under natural assumptions, for instance when f is a regular endomorphism of X (see [1, 12]).

We refer to [16] for a recent study of the relationship between dynamical degrees and topological dynamics when $\mathbf{C}$ is replaced by an ultrametric field.

Surfaces and analogies.– The goal of [4] was to study the structure of the set of all dynamical degrees, when f runs over the group of all birational transformations $\mathsf{Bir}(X)$ and X over the collection of all projective *surfaces*. In that case, $\lambda_0(f) = 1 = \lambda_2(f)$, and we only have to study $\lambda(f) = \lambda_1(f)$. This set is now quite well understood. In particular, as we will explain later, it consists of some special algebraic integers, called Pisot or Salem numbers, it is well-ordered, and it is closed if the ground field is $\mathbf{C}$. Dynamical degrees are invariant under conjugacy, and an important feature of the results of [4] may be summarized by the following slogan: *Precise knowledge on $\lambda(f)$ provides useful information on the conjugacy class of f.* In particular, [4] provides quantitative bounds for the solutions of certain equations in $\mathsf{Bir}(X)$, like the conjugacy problem asking for a solution h of the equation $hfh^{-1} = g$.

Another motivation of [4] is to develop a dictionary between groups of birational transformations of projective surfaces and mapping class groups of higher genus, closed, orientable surfaces. The dynamical degree $\lambda(f)$ plays a role which is similar to the dilatation factor $\lambda(\varphi)$ of pseudo-Anosov mapping classes (see [4, §8]). The main results of [4] can also be compared to two theorems proved by W. Thurston. The first one describes explicitly the set of topological entropies of post-critically finite, continuous, multimodal transformations of the unit interval as the set of logarithms of "weak Perron numbers". The second describes the structure of the set of volumes of hyperbolic manifolds of dimension 3; this set is also countable, non-discrete, and well ordered subset of the real line.

Higher dimensions.– In higher dimensions, dynamical degrees are still rather mysterious. Altogether, the set of possible dynamical degrees is countable (see [21], [7]). But recently, the existence of a birational map of $\mathbb{P}^3_{\mathbf{C}}$ (and thus of $\mathbb{P}^n_{\mathbf{C}}$ for each $n \geq 3$) with *transcendental* dynamical degree was given in [2], showing that the situation is very different from the surface case. Some features are however independent of the dimension, like the semi-continuity of the dynamical degrees [23] (see Section 4.2). We will also mention the case of polynomial automorphisms of the affine space $\mathbb{A}^m_{\mathbf{C}}$: they correspond to interesting particular birational maps of the $\mathbb{P}^m_{\mathbf{C}}$ and their dynamical degrees are more restricted (see section 5.2 below).

2 Dynamical degrees, Pisot and Salem numbers, algebraic stability

2.1 Dynamical degrees

Let X be a projective variety defined over an algebraically closed field $\mathbf{k}$. Set

$$m = \dim(X).$$

In what follows, $\mathsf{NS}(X)$ denotes the Néron-Severi group of X. Given a ring $\mathbf{A}$, $\mathsf{NS}_{\mathbf{A}}(X)$ stands for $\mathsf{NS}(X) \otimes_{\mathbf{Z}} \mathbf{A}$; hence, $\mathsf{NS}_{\mathbf{Z}}(X)$ coincides with $\mathsf{NS}(X)$.

Let f be a rational transformation of X defined over $\mathbf{k}$. It determines an endomorphism $f^*\colon \mathsf{NS}(X) \to \mathsf{NS}(X)$, and the **first dynamical degree** $\lambda_1(f)$ of f is defined as the spectral radius of the sequence of endomorphisms $(f^n)^*$, as n goes to $+\infty$. More precisely, once a norm $\parallel \cdot \parallel$ has been chosen on the real vector space $\mathsf{End}(\mathsf{NS}_{\mathbf{R}}(X))$, one defines

$$\lambda(f) = \lim_{n\to\infty} \parallel (f^n)^* \parallel^{1/n} \; ;$$

this limit exists, and does not depend on the choice of the norm. Moreover, for every ample divisor $D \subset X$

$$\lambda_1(f) = \lim_{n\to\infty} \left(D^{m-1} \cdot (f^n)^* D\right)^{1/n},$$

where $(C_1 \cdot C_2 \cdots C_m)$ denotes the intersection number between divisors or divisor classes, and $D^k = D \cdot D \cdots D$ taken k times. Similarly, the p-th dynamical degree $\lambda_p(f)$ is defined by looking at the action of f on classes of subvarieties of codimension p, and is the limit of $(D^{m-p} \cdot (f^n)^*(D^p))^{1/n}$ as n goes to $+\infty$.

The most important features of dynamical degrees are: (1) their definition does not depend on the choice of norm $\|\cdot\|$ (resp. of polarization D), (2) they are invariant under conjugacy, and (3) they form a log concave sequence, which means that

$$\lambda_p(f)^2 \leq \lambda_{p-1}(f)\lambda_{p+1}(f).$$

This follows from the Khovansky-Teissier inequalities for intersection numbers, hence to the Alexandrov-Fenchel inequalities concerning mixed volumes of convex bodies (see [15, 23, 13]).

Example 2.1 *Let $m \geq 2$ be an integer. The Néron-Severi group of $\mathbb{P}^m_{\mathbf{k}}$ coincides with the Picard group $\mathsf{Pic}(\mathbb{P}^m_{\mathbf{k}})$, has rank 1, and is generated by the class e_0 of a hyperplane:*

$$\mathsf{NS}(\mathbb{P}^m_{\mathbf{k}}) = \mathsf{Pic}(\mathbb{P}^m_{\mathbf{k}}) = \mathbf{Z}e_0.$$

Fix a choice of homogeneous coordinates $[x_0 : \cdots : x_m]$ on the projective space $\mathbb{P}^m_{\mathbf{k}}$. Let f be an element of $\mathsf{Cr}_m(\mathbf{k}) = \mathsf{Bir}(\mathbb{P}^m_{\mathbf{k}})$. One can then find homogeneous polynomials $P_0, \ldots, P_m \in \mathbf{k}[x_0, \ldots, x_m]$ of the same degree d, and without common factor of positive degree, such that

$$f([x_0 : \cdots : x_m]) = [P_0(x_0, \ldots, x_m) : \cdots : P_m(x_0, \ldots, x_m)].$$

This degree d does not depend on the choice of homogeneous coordinates; it is denoted by $\mathsf{deg}(f)$ and called the degree of f. On $\mathsf{Pic}(\mathbb{P}^m_{\mathbf{k}})$, f acts by multiplication by $\mathsf{deg}(f)$; thus, we have $\lambda(f) = \lim \mathsf{deg}(f^n)^{1/n}$. For instance, the **standard involution**

$$\sigma([x_0 : \cdots : x_n]) = [\frac{1}{x_0} : \cdots : \frac{1}{x_m}] = [x_1 x_2 \cdots x_m : x_0 x_2 \cdots x_m : \cdots : x_0 \cdots x_{m-1}].$$

satisfies $\mathsf{deg}(\sigma^n) = 1$ or m, according to the parity of n; hence $\lambda_1(\sigma) = 1$. Similarly, $\lambda_p(\sigma) = 1$ for every $p \leq m$.

2.2 Pisot and Salem numbers (see [3])

A **Pisot number** is an algebraic integer $\lambda \in]1, \infty[$ whose other Galois conjugates lie in the open unit disk; the set of Pisot numbers includes all integers $d \geq 2$ as well as all reciprocal quadratic integers $\lambda > 1$. A **Salem number** is an algebraic integer $\lambda \in]1, \infty[$ whose other Galois conjugates are in the closed unit disk, with at least one on the boundary; hence, the minimal polynomial of λ has at least two complex conjugate roots on the unit circle, its roots are permuted by the involution $x \mapsto 1/x$, and its degree is at least 4. We denote by Pis the set of Pisot numbers and by Sal the set of Salem numbers.

It is known that Pis is a closed subset of the real line. It is contained in the closure of Sal, and its infimum is equal to $\lambda_P \simeq 1.324717$, the unique root $\lambda_P > 1$ of the cubic equation $x^3 = x + 1$; this Pisot number is known as the **plastic number**, or **padovan number**. The smallest accumulation point of Pis is the golden mean $\lambda_G = (1+\sqrt{5})/2$; all Pisot numbers between λ_P and λ_G have been listed.

Our present knowledge of Salem numbers is much weaker. Conjecturally, the infimum of Sal is larger than 1, and should be equal to the **Lehmer number**, i.e. to the Salem number $\lambda_L \simeq 1.176280$ obtained as the unique root > 1 of the irreducible polynomial $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$.

2.3 Dynamical degrees and algebraic stability

Now, assume that X is a surface and f is birational. Then, as said above, we have $\lambda(f) = \lambda_1(f)$. By definition, f is **loxodromic** if $\lambda(f) > 1$. The **dynamical spectrum** of X is defined as the set

$$\Lambda(X) = \{\lambda(f) \mid f \in \mathsf{Bir}(X)\}.$$

If one wants to specify the field $\mathbf{k}$, one may denote the dynamical spectrum by $\Lambda(X, \mathbf{k})$.

One says that $f \in \mathsf{Bir}(X)$ is **algebraically stable** when the endomorphism f^* of the Néron-Severi group $\mathsf{NS}(X)$ satisfies

$$(f^n)^* = (f^*)^n \tag{2.1}$$

for all positive integers n. If f is algebraically stable, then f^{-1} is also algebraically stable and $\lambda(f)$ is the spectral radius of the endomorphism f^* of $\mathsf{NS}(X)$; in particular, $\lambda(f)$ is an algebraic integer. Diller and Favre proved in [11] that every birational transformation of a projective surface X is conjugate by a birational morphism $\pi : Y \to X$ to an algebraically stable transformation $\pi^{-1} \circ f \circ \pi$. From this fact and the Hodge index theorem, they obtained the following result.

Theorem 2.2 ([11]) *Let $\mathbf{k}$ be a field and let f be a birational transformation of a projective surface defined over $\mathbf{k}$. If $\lambda(f)$ is different from 1, then $\lambda(f)$ is a Salem or a Pisot number.*

Note that this result is false in dimension at least 3: first, there are birational transformations of threefolds (resp. dominant rational transformations of surfaces) which are not conjugate to algebraically stable ones (see [14]); second, there are regular automorphisms of threefolds (for instance abelian threefolds) for which $\lambda_1(f)$ is not in $\{1\} \cup \mathsf{Pis} \cup \mathsf{Sal}$ (see [17]); third, there are birational transformations of threefolds (resp. dominant rational transformations of surfaces) with transcendental dynamical degrees [2], as already mentioned in the Introduction.

In [4], we initiated the study of the dynamical spectrum $\Lambda(X)$, where X is a surface. By Diller-Favre Theorem, $\Lambda(X)$ splits in two parts, its Pisot part $\Lambda^P(X)$ and its Salem part $\Lambda^S(X)$. The problem is to describe the nature of these sets, which numbers can appear in each of these sets, as well as the relationship between these two sets.

Example 2.3 *When f is an algebraically stable transformation of $\mathbb{P}^2_{\mathbf{k}}$, one gets $\lambda(f) = \mathsf{deg}(f)$. For instance, the automorphism h of the affine plane defined by $h(X,Y) = (Y, X + Y^d)$ extends to a birational map of the projective plane such that $\mathsf{deg}(h^n) = d^n$ for all $n \geq 0$. In particular, $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ contains all integers $d \geq 1$, for all fields $\mathbf{k}$.*

Example 2.4 *Consider the group $\mathsf{GL}_2(\mathbf{Z})$ acting by (monomial) automorphisms of the multiplicative group $\mathbf{k}^* \times \mathbf{k}^*$: If*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an element of $\mathsf{GL}_2(\mathbf{Z})$ and (X,Y) denotes the coordinates on $\mathbf{k}^* \times \mathbf{k}^*$, the automorphism associated to A is defined by $f_A(X,Y) = (X^aY^b, X^cY^d)$. This provides an embedding of $\mathsf{GL}_2(\mathbf{Z})$ in the automorphism group $\mathsf{Aut}(\mathbf{k}^* \times \mathbf{k}^*)$, and thus in $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}}(\mathbf{k}))$. For every A in $\mathsf{GL}_2(\mathbf{Z})$, the dynamical degree of f_A is equal to the spectral radius of the matrix A, i.e. to the modulus of its unique eigenvalue λ with $|\lambda| \geq 1$; this implies that f_A is not an algebraically stable transformation of $\mathbb{P}^2_{\mathbf{k}}$ as soon as $\lambda(f_A) > 1$, because $\lambda(f_A)$ is not an integer in that case. As a byproduct of this example, the dynamical spectrum of the plane contains all reciprocal quadratic integers, i.e. all roots $\lambda > 1$ of equations $x^2 + 1 = tx$ with t in $\mathbf{Z}$.

2.4 Salem numbers and automorphisms

The dynamical degree of an automorphism of a projective surface, if different from 1, is either a quadratic number or a Salem number (see [11]). In [4] we proved a converse statement:

Theorem A [4, Theorem A] *Let $\mathbf{k}$ be an algebraically closed field. Let f be a birational transformation of a projective surface X, defined over $\mathbf{k}$. If $\lambda(f)$ is a Salem number, there exists a projective surface Y and a birational mapping $\varphi\colon Y \dashrightarrow X$ such that $\varphi^{-1} \circ f \circ \varphi$ is an automorphism of Y.*

Thus, one can decide whether a birational transformation is conjugate to an automorphism by looking at its dynamical degree, except when this degree is 1 or a quadratic integer. For the quadratic case, [4, Examples 2.2 and 2.3] show that there are quadratic integers which are simultaneously realized as dynamical degrees of automorphisms and of birational transformations that cannot be conjugate to an automorphism. See [4, Remark 2.4] for birational transformations with dynamical degree equal to 1.

Once Theorem A is proved, three corollaries can be deduced from results of McMullen and the second author (see [20] and [9]). The first corollary is a **spectral gap property** for dynamical degrees:

- *There is no dynamical degree in the interval* $]1, \lambda_L[$ [4, Corollary 2.7] .

The second corollary does not seem to be related to values of dynamical degrees, but a simple proof makes use of the spectral gap. It asserts that

- *For each loxodromic element* $f \in \mathsf{Bir}(X)$, *its centralizer in* $\mathsf{Bir}(X)$ *if a finite extension of the infinite cyclic group* $\langle f \rangle$ [4, Corollary 4.7].

The third consequence is an effective and explicit bound for the optimal degree of a conjugacy (see [4, §4.4]):

- Two loxodromic elements $f, g \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ of degree $\leq d$ are conjugate if and only if they are conjugate by an element h of degree $\leq (2d)^{57}$.

3 From projective surfaces to the projective plane

Non rational surfaces are easily handled with, because a birational transformation of such a surface X either preserves a fibration by curves (in which case $\lambda(f) = 1$) or is conjugate to an automorphism by some birational change of variable $Y \dashrightarrow X$. From this, and Enriques' classification of surfaces, one obtains the following theorem.

Theorem B [4, Theorem B] *Let* $\mathbf{k}$ *be an algebraically closed field. Let* X *be a projective surface defined over* $\mathbf{k}$. *If* X *is not rational, then*

1. $\Lambda(X) = \{1\}$ *if* X *is not birationally equivalent to an abelian surface, a K3 surface, or an Enriques surface;*
2. $\Lambda(X) \setminus \{1\}$ *is made of quadratic integers and of Salem numbers of degree at most* 6 (*resp.* 22, *resp.* 10) *if* X *is an abelian surface* (*resp. a K3 surface, resp. an Enriques surface*).

The union of all dynamical spectra $\Lambda(X, \mathbf{k})$, *for all fields and all surfaces which are not geometrically rational, is a closed discrete subset of the real line.*

Remark 3.1 *When the characteristic of the field* $\mathbf{k}$ *is equal to* 0, *the degree bounds of Assertion (2) become* 4, 20, *and* 10 (*in place of* 6, 22, *and* 10).

This result, proved in [4, Section 3], shows that the most interesting case is provided by rational surfaces. Thus, in the following section, one can assume that X is birationally equivalent to the projective plane $\mathbb{P}^2_{\mathbf{k}}$; the dynamical spectrum is then equal to the set $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ of dynamical degrees of elements of the **Cremona group** $\mathsf{Cr}_2(\mathbf{k}) = \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$.

4 Degrees and conjugacy classes

4.1 Minimal degree in the conjugacy class

Given an element f of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$, define the **minimal degree** of f in its conjugacy class as the positive integer

$$\mathsf{mcdeg}(f) = \min \mathsf{deg}(g \circ f \circ g^{-1})$$

where g describes $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ (thus, $\mathsf{mcdeg}(f)$ depends on the field and may decrease after a field extension). The function mcdeg is constant on conjugacy classes, and

$$\lambda(f) \leq \mathsf{mcdeg}(f) \leq \mathsf{deg}(f)$$

for all birational transformations of the plane. One of the key technical results of [4] is the following reverse inequality:

Theorem C [4, Theorem C] *Let* $\mathbf{k}$ *be an algebraically closed field and let* f *be a birational transformation of the plane* $\mathbb{P}^2_{\mathbf{k}}$.

1. *If* $\lambda(f) \geq 10^6$ *then* $\mathsf{mcdeg}(f) \leq 4700\,\lambda(f)^5$.
2. *If* $\lambda(f) > 1$, *then* $\mathsf{mcdeg}(f) \leq \cosh(18 + 345\log(\lambda(f))) \leq e^{18}\lambda(f)^{345}$.

On the other hand, there are sequences of elements $f_n \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ such that $\mathsf{mcdeg}(f_n)$ goes to $+\infty$ with n while $\lambda_1(f_n) = 1$ for all n.

4.2 Semi-continuity

Another crucial statement, obtained by Junyi Xie in [**?**], concerns the semi-continuity of the function $f \mapsto \lambda_1(f)$. To state it, one needs the following notion: an **algebraic family** of rational transformations of a projective variety X is the data of an algebraic variety T and a rational map $F \colon T \times X \dashrightarrow T \times X$ such that the indeterminacy set of F does not contain any fiber $\{t_0\} \times X$ and $F(t,x) = (t, f_t(x))$ for some family f_t of dominant rational transformations of X. Such a family is usually denoted by $(f_t)_{t\in T}$, instead of F.

Theorem 4.1 (J. Xie, [22, 23]) *If* $(f_t)_{t\in T}$ *is a family of rational transformations of a projective variety* X *and* $0 \leq p \leq \dim(X)$ *is an integer, then the function* $t \in T \mapsto \lambda_p(t) \in \mathbf{R}$ *is upper semi-continuous with respect to the Zariski topology of* T.

This result had already been proven for birational transformations of surfaces in [22], and the general statement given here is taken from the preprint [23].

4.3 Well ordered sets

The set $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ is a subset of $\mathbf{R}_+$ and, as such, is totally ordered. The following statement, which follows from Theorem C and Theorem 4.1, asserts that $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ is **well ordered**: Every non-empty subset of $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ has a minimum; equivalently, it satisfies the descending chain condition (if $(f_n)_{n\geq 0}$ is a sequence of birational transformations of $\mathbb{P}^2_{\mathbf{k}}$ and $\lambda(f_{n+1}) \leq \lambda(f_n)$ for each n, then $\lambda(f_n)$ becomes eventually constant).

Theorem D [4, Theorem D] *Let* $\mathbf{k}$ *be an algebraically closed field. The dynamical spectrum* $\Lambda(\mathbb{P}^2_{\mathbf{k}}) \subset \mathbf{R}$ *is well ordered, and it is closed if* $\mathbf{k}$ *is uncountable.*

In [4, Theorem 7.4], we also show that $\Lambda^P(\mathbb{P}^2_{\mathbf{k}})$ *is contained in the closure of* $\Lambda^S(\mathbb{P}^2_{\mathbf{k}})$ *if* $\mathbf{k}$ *is algebraically closed and of characteristic* 0. As $\Lambda(\mathbb{P}^2_{\mathbf{k}}) \subset \mathbf{R}$ is well-ordered, it corresponds to an ordinal. Recently, Anna Bot proved that

- *The ordinal of $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ is at most ω^ω, with equality when $\mathbf{k} = \mathbf{C}$* (see[8]).

From Theorems B and D, one obtains the existence of gaps in the dynamical spectrum of projective surfaces: *There are intervals of real numbers that contain infinitely many Pisot and Salem numbers, but do not contain any dynamical degree.*

Corollary 4.2 *Let Λ be the set of all dynamical degrees of birational transformations of projective surfaces, defined over any field. Then,*

(1) Λ is a well ordered subset of $\mathbf{R}_+$;
(2) if λ is an element of Λ, there is a real number $\epsilon > 0$ such that $]\lambda, \lambda + \epsilon]$ does not intersect Λ;
(3) there is a non-empty interval $]\lambda_G, \lambda_G + \epsilon]$, on the right of the golden mean, that contains infinitely many Pisot and Salem numbers but does not contain any dynamical degree.

Gaps as in the third assertion of this corollary occur infinitely often, because there are infinitely many Pisot numbers that are limits of Pisot numbers from the right.

5 Open questions

5.1 Degree growth

The dynamical degrees $\lambda_p(f)$ describe the exponential growth rate of the action of $f\colon X \dashrightarrow X$ on numerical classes of divisors of codimension p. Subexponential phenomena are not yet understood, except in dimension 2. Indeed, if f is a birational transformation of a surface and $\lambda(f) = 1$, then $\parallel (f^n)^* \parallel$ is either bounded, or grows linearly, or grows quadratically with n; and in each case, f permutes the members of a pencil of curves given by some rational map $\pi\colon X \dashrightarrow C$ onto a curve (see [9]). Such a classification is missing in higher dimension, and what should be expected is not clear at all. For instance, one does not know whether a regular automorphism with $\parallel (f^n)^* \parallel$ gorwing polynomially must preserve a fibration, one does not know whether $\parallel (f^n)^* \parallel$ can grow like $\log(n)$ or $\exp(\sqrt{n})$ for a birational transformation.

5.2 Polynomial automorphisms

For each integer m and each field $\mathbf{k}$, the group $\mathsf{Aut}(\mathbb{A}^m_{\mathbf{k}})$ of polynomial automorphisms of the affine space $\mathbb{A}^m_{\mathbf{k}}$ is a natural subgroup of $\mathsf{Bir}(\mathbb{A}^m_{\mathbf{k}}) = \mathsf{Bir}(\mathbb{P}^m_{\mathbf{k}})$, whose structure is still mysterious in dimension $m \geq 3$. One may then ask what are the possible dynamical degrees of elements of $\mathsf{Aut}(\mathbb{A}^m_{\mathbf{k}})$. For instance, the following conjecture is taken from [10, Conjecture 2] and [5, Question 1.1.2]:

Conjecture 5.1 *For each integer $m \geq 2$ and each field $\mathbf{k}$, the dynamical degree of every element of $\mathsf{Aut}(\mathbb{A}^m_{\mathbf{k}})$ is an algebraic integer of degree at most $m - 1$.*

Conjecture 5.1 is only proven for $m = 2$ (see [18, Proposition 3] and [5, Corollary 2.4.3]). For $m = 3$, it has been proven for specific families of polynomial automorphisms, like quadratic automorphisms [19] or cubic and triangular automorphisms [5, 6]. For general automorphisms, the best statement obtained so far is the following result:

Theorem 5.2 [10, Corollary 3] *The dynamical degree of every element of* $\mathsf{Aut}(\mathbb{A}^3_{\mathbf{C}})$ *is an algebraic number of degree at most* 6*.*

Each of the main results described in this short introduction, for instance Theorem C and its consequences, Theorem 4.1, or Theorem 5.2, required a better understanding of the space of numerical classes of cycles in an algebraic variety X, and *in all birational models* $Y \dashrightarrow X$ *of* X; indeed, for a birational transformation f that is not algebraically stable, the study of its dynamical degrees requires to blow-up X along the indeterminacy set of all iterates of f, hence an infinite number of blow-ups. In [4], the main ingredient comes from an additional structure which is given by the intersection form and Hodge index theorem: on the inductive limit of $\mathsf{NS}(Y)$ over all birational models $\pi\colon Y \to X$ (π a birational morphism), the intersection form provides an infinite dimensional Minkowski form; then, hyperbolic geometry can be combined with algebraic geometry and dynamics to study dynamical degrees. Such a structure does not exist in dimension ≥ 3, and [10, 23] are the first important papers to provide tools towards a better understanding of these numbers $\lambda_p(f)$.

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Jérémy Blanc, Université Neuchâtel, Institut de mathématiques, Suisse *E-mail address*: `jeremy.blanc@unine.ch`

Serge Cantat, IRMAR, UMR 6625 du CNRS, Université de Rennes I, 35042 Rennes, France *E-mail address*: `cantat@univ-rennes.fr`[1]

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