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CLASSICAL AND MICROLOCAL ANALYSIS OF THE X-RAY TRANSFORM ON ANOSOV MANIFOLDS

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CLASSICAL AND MICROLOCAL ANALYSIS OF
THE X-RAY TRANSFORM ON ANOSOV MANIFOLDS

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We complete the microlocal study of the geodesic x-ray transform on Riemannian manifolds with Anosov geodesic flow initiated by Guillarmou (*J. Differential Geom.* **105**:2 (2017), 177–208) and pursued by Guillarmou and Lefeuvre in (*Ann. of Math.* (2) **190**:1 (2019), 321–344). We prove new stability estimates and clarify some properties of the operator Π_m — the generalized x-ray transform. These estimates rely on a refined version of the Livšic theorem for Anosov flows, especially on a new quantitative finite-time Livšic theorem.

1. Introduction

Let $\mathcal{M}$ be a smooth closed $(n+1)$-dimensional manifold endowed with a vector field X generating a complete flow $(\varphi_t)_{t \in \mathbb{R}}$. We assume that the flow $(\varphi_t)_{t \in \mathbb{R}}$ is transitive and Anosov in the sense that there exists a continuous flow-invariant splitting

$$T_x(\mathcal{M}) = \mathbb{R}X(x) \oplus E_u(x) \oplus E_s(x), \tag{1-1}$$

where $E_s(x)$ (resp. $E_u(x)$) is the *stable* (resp. *unstable*) vector space at $x \in \mathcal{M}$, and a smooth Riemannian metric g such that

$$\begin{aligned} |d\varphi_t(x) \cdot v|_{\varphi_t(x)} &\leq C e^{-\lambda t} |v|_x \quad \text{for all } t > 0, \ v \in E_s(x), \\ |d\varphi_t(x) \cdot v|_{\varphi_t(x)} &\leq C e^{-\lambda |t|} |v|_x \quad \text{for all } t < 0, \ v \in E_u(x), \end{aligned} \tag{1-2}$$

for some uniform constants $C, \lambda > 0$. The norm, here, is $|\cdot|_x := g_x(\cdot, \cdot)^{1/2}$. The dimension of E_s (resp. E_u) is denoted by n_s (resp. n_u). As a consequence, $n + 1 = 1 + n_s + n_u$ (where the 1 stands for the *neutral* direction, that is, the direction of the flow). The case we will have in mind will be that of a geodesic flow on the unit tangent bundle of a smooth Riemannian manifold (M, g) with negative sectional curvature.

1A. X-ray transform on $\mathcal{M}$. We denote by $\mathcal{G}$ the set of closed orbits of the flow and for $f \in C^0(\mathcal{M})$, its x-ray transform If is defined by

$$\mathcal{G} \ni \gamma \mapsto If(\gamma) := \langle \delta_\gamma, f \rangle = \frac{1}{\ell(\gamma)} \int_0^{\ell(\gamma)} f(\varphi_t x) \, dt,$$

where $x \in \gamma$ and $\ell(\gamma)$ is the length of γ.

The Livšic theorem characterizes the kernel of the x-ray transform for a hyperbolic flow: the latter is reduced to the coboundaries, which are the functions of the form $f = Xu$, where u is a function defined

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on $\mathcal{M}$ whose regularity is prescribed by that of f. This result was initially proved in [Livšic 1972] in Hölder regularity: if $f \in C^\alpha(\mathcal{M})$ is such that $If = 0$, then there exists $u \in C^\alpha(\mathcal{M})$, differentiable in the flow direction, such that $f = Xu$ and u is unique up to an additive constant. There is also a version of the Livšic theorem in smooth regularity due to de la Llave, Marco, and Moriyón [1986]. Much more recently, Guillarmou [2017, Corollary 2.8] proved a version of the Livšic theorem in Sobolev regularity which implies the theorem of [de la Llave et al. 1986].

It is also rather natural to expect other versions of the Livšic theorem to hold. For instance, if we modify the condition $If = 0$ by $If \geq 0$, is it true that $f \geq Xu$ for some well-chosen function u (positive Livšic theorem)? And if $\|If\|_{\ell^\infty} := \sup_{\gamma \in \mathcal{G}} |If(\gamma)| \leq \varepsilon$, can one write $f = Xu + h$, where some norm of h is controlled by a power of ε (approximate Livšic theorem)? Eventually, what can be said if $If(\gamma) = 0$ for all closed orbits γ of length $\leq L$ (finite Livšic theorem)?

The positive Livšic theorem for Anosov flows was proved in [Lopes and Thieullen 2005] with an explicit control of a Hölder norm of the coboundary Xu in terms of a norm of f.

Theorem 1.1 (Lopes–Thieullen). *Let $0 < \alpha \leq 1$. There exist $0 < \beta \leq \alpha$, $C > 0$ such that, for all functions $f \in C^\alpha(\mathcal{M})$, there exist $u \in C^\beta(\mathcal{M})$, differentiable in the flow-direction with $Xu \in C^\beta(\mathcal{M})$ and $h \in C^\beta(\mathcal{M})$ such that $f = Xu + h + m(f)$, with $h \geq 0$ and $m(f) = \inf_{\gamma \in \mathcal{G}} If(\gamma)$. Moreover, $\|Xu\|_{C^\beta} \leq C\|f\|_{C^\alpha}$.*

In this article, we prove a finite approximate version of the Livšic theorem, as follows.

Theorem 1.2. *Let $0 < \alpha \leq 1$. There exist $0 < \beta \leq \alpha$ and τ, $C > 0$ with the following property. Let $\varepsilon > 0$. Consider a function $f \in C^\alpha(\mathcal{M})$ with $\|f\|_{C^\alpha(\mathcal{M})} \leq 1$ such that $|If(\gamma)| \leq \varepsilon$ for all γ with $\ell(\gamma) \leq \varepsilon^{-1/2}$. Then there exist $u \in C^\beta(\mathcal{M})$ differentiable in the flow-direction with $Xu \in C^\beta(\mathcal{M})$ and $h \in C^\beta(\mathcal{M})$ such that $f = Xu + h$. Moreover, $\|u\|_{C^\beta} \leq C$ and $\|h\|_{C^\beta} \leq C\varepsilon^\tau$.*

We note that a rather similar result has already been obtained by S. Katok [1990] in the particular case of a contact Anosov flow on a 3-manifold.

The assumptions of Theorem 1.2 hold in particular if $\|If\|_{\ell^\infty} = \sup_{\gamma \in \mathcal{G}} |If(\gamma)| \leq \varepsilon$. Under the assumptions of the theorem (only mentioning the closed orbits of length at most $\varepsilon^{-1/2}$), the decomposition $f = Xu + h$ also gives a global control on $\|If\|_{\ell^\infty}$ of the form

$$\|If\|_{\ell^\infty} \leq C\varepsilon^\tau. \tag{1-3}$$

Indeed, if one integrates $f = Xu + h$ along a closed orbit of any length, the contribution of Xu vanishes and one is left with a bound $\|h\|_{C^0} \leq C\varepsilon^\tau$. The bound (1-3) holds in particular if $If(\gamma) = 0$ for all γ with $\ell(\gamma) \leq \varepsilon^{-1/2}$. This statement illustrates quantitatively the fact that the quantities $If(\gamma)$ for different γ are far from being independent.

Remark 1.3. In Theorem 1.2, the constants β, C, τ depend on the Anosov flow under consideration, but in a locally uniform way: given an Anosov flow, one can find such parameters that work for any flow in a neighborhood of the initial flow. The local uniformity can be checked either directly from the proof, or using a (Hölder-continuous) orbit-conjugacy between the initial flow and the perturbed one.

Remark 1.4. It could be interesting to extend the positive and the finite approximate Livšic theorems to other regularities like H^s spaces for $s > (n + 1)/2$ but we were unable to do so.

1B. *X-ray transform for the geodesic flow.* If (M, g) is a smooth closed Riemannian manifold, we set $\mathcal{M} := SM$, where SM is the unit tangent bundle, and denote by X the geodesic vector field on SM. We will always assume that the geodesic flow is Anosov on SM and we say that (M, g) is an *Anosov Riemannian manifold*. It is a well-known fact that a negatively curved manifold has Anosov geodesic flow. We will denote by $\mathcal{C}$ the set of free homotopy classes on M; they are in one-to-one correspondence with the set of conjugacy classes of $\pi_1(M, \cdot)$. If (M, g) is Anosov, we know by [Klingenberg 1974] that given a free homotopy class $c \in \mathcal{C}$, there exists a unique closed geodesic $\gamma \in \mathcal{G}$ belonging to the free homotopy class c. In other words, $\mathcal{G}$ and $\mathcal{C}$ are in one-to-one correspondence. As a consequence, we will rather see the x-ray transform as a map $I^g : C^0(SM) \to \ell^\infty(\mathcal{C})$ and we will drop the index g if the context is clear.

If $f \in C^\infty(M, \otimes_S^m T^*M)$ is a symmetric tensor, then by Section 2, we can see f as a function $\pi_m^* f \in C^\infty(SM)$, where $\pi_m^* f(x, v) := f_x(v, \dots, v)$. The x-ray transform I_m of f is simply defined by $I_m f := I \circ \pi_m^* f$. In other words, it consists in integrating the tensor f along closed geodesics by plugging m-times the speed vector in f. This map I_m may appear in different contexts. In particular, I_2 is well known to be the differential of the marked length spectrum and it was studied in [Guillarmou and Lefeuvre 2019] to prove its rigidity, thus partially answering the conjecture of [Burns and Katok 1985].

The natural operator of derivation of symmetric tensors is $D := \sigma \circ \nabla$, where ∇ is the Levi-Civita connection and σ is the operator of symmetrization of tensors (see Section 2). Any smooth tensor $f \in C^\infty(M, \otimes_S^m T^*M)$ can be uniquely decomposed as $f = Dp + h$, where $p \in C^\infty(M, \otimes_S^{m-1} T^*M)$ and $h \in C^\infty(M, \otimes_S^m T^*M)$ is a *solenoidal tensor* i.e., a tensor such that $D^*h = 0$, where D^* is the formal adjoint of D. We say that Dp is the *potential part* of the tensor f. We will see that $I_m(Dp) = 0$. In other words, the potential tensors are always in the kernel of the x-ray transform. We will say that I_m is *solenoidal injective*, or in short s-injective, if injective when restricted to

$$C_{\text{sol}}^\infty(M, \otimes_S^m T^*M) := C^\infty(M, \otimes_S^m T^*M) \cap \ker(D^*).$$

Note that we will often add an index sol to a functional space on tensors to denote the fact that we are considering the intersection with $\ker D^*$.

It is conjectured that I_m is s-injective for all Anosov Riemannian manifolds, in any dimension and without any assumption on the curvature. Under the additional assumption that the sectional curvatures are nonpositive, the *Pestov energy identity* allows us to show injectivity (see [Guillemin and Kazhdan 1980a; Croke and Sharafutdinov 1998] for the original proofs). Without any assumption on the curvature, this is still true for surfaces by [Paternain, Salo, and Uhlmann 2014; Guillarmou 2017]. In higher dimensions, it holds for $m = 0, 1$ (see [Dairbekov and Sharafutdinov 2003]) but remains an open question for higher-order tensors without any assumption on the curvature. However, it is already known that $C_{\text{sol}}^\infty(M, \otimes_S^m T^*M) \cap \ker(I_m)$ is finite-dimensional.

We will also prove a stability estimate on I_m.

Theorem 1.5. *Assume I_m is s-injective. Then for all $0 < \beta < \alpha < 1$ there exists $\theta_1 := \theta(\alpha, \beta) > 0$ and $C := C(\alpha, \beta) > 0$ such that if $f \in C_{\text{sol}}^\alpha(M, \otimes_S^m T^*M)$ is a solenoidal symmetric m-tensor such that $\|f\|_{C^\alpha} \leq 1$, then $\|f\|_{C^\beta} \leq C \|I_m f\|_{\ell^\infty}^{\theta_1}$.*

Actually, if I_m is not known to be injective, one still has the previous estimate by taking f solenoidal and orthogonal to the kernel of I_m. Combining this estimate with Theorem 1.2 (and more specifically (1-3)), we immediately obtain the following.

Theorem 1.6. *Assume I_m is s-injective. Then for all $0 < \beta < \alpha < 1$ there exist $\theta_2 := \theta(\alpha, \beta) > 0$ and $C := C(\alpha, \beta) > 0$ such that for any $L > 0$ large enough if $f \in C^\alpha_{\mathrm{sol}}(M, \otimes^m_S T^*M)$ is a solenoidal symmetric m-tensor such that $\|f\|_{C^\alpha} \leq 1$, and $I_m f(\gamma) = 0$ for all closed geodesics $\gamma \in \mathcal{C}$ such that $\ell(\gamma) \leq L$, then $\|f\|_{C^\beta} \leq C L^{-\theta_2}$.*

Even in the case where $f \in C^\alpha(M)$ is a function on M, this result seemed to be previously unknown.

2. On symmetric tensors

We describe elementary properties of symmetric tensors on Riemannian manifolds. This is a background section for which we also refer to [Guillemin and Kazhdan 1980b; Dairbekov and Sharafutdinov 2010].

2A. *Definitions and first properties.*

2A1. *Symmetric tensors in Euclidean space.* Let E be a Euclidean $(n+1)$-dimensional vector space endowed with a metric g and let $(e_1, \ldots, e_{n+1})$ be an orthonormal basis. We say that a tensor $f \in \otimes^m E^*$ is symmetric if $f(v_1, \ldots, v_m) = f(v_{\tau(1)}, \ldots, v_{\tau(m)})$ for all $v_1, \ldots, v_m \in E$ and $\tau \in \mathfrak{S}_m$, the group of permutations of order m. We denote by $\otimes^m_S T^*E$ the vector space of symmetric m-tensors on E. There is a natural projection $\sigma : \otimes^m E^* \to \otimes^m_S E^*$ given by

$$\sigma(v_1^* \otimes \cdots \otimes v_m^*) = \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} v_{\tau(1)}^* \otimes \cdots \otimes v_{\tau(m)}^*$$

for all $v_1^*, \ldots, v_m^* \in E^*$. The metric g induces a scalar product $\langle \cdot, \cdot \rangle$ on $\otimes^m E^*$ by declaring the basis $(e_{i_1}^* \otimes \cdots \otimes e_{i_m}^*)_{1 \leq i_1, \ldots, i_m \leq n+1}$ to be orthonormal which yields

$$\langle u_1^* \otimes \cdots \otimes u_m^*, v_1^* \otimes \cdots \otimes v_m^* \rangle = \prod_{i=1}^m g^{-1}(u_i^*, v_i^*),$$

where g^{-1} is the dual metric, that is, the natural metric on E^* which makes the musical isomorphism $\sharp : E \to E^*$ an isometry. Since σ is self-adjoint with respect to this metric, it is an orthogonal projection. Let $(g_{ij})_{1 \leq i,j \leq n+1}$ denote the metric g in the coordinates $(x_1, \ldots, x_{n+1})$. Then the metric can be expressed as

$$\langle f, h \rangle = \sum_{i_1, \ldots, i_m = 1}^{n+1} f_{i_1 \cdots i_m} h^{i_1 \cdots i_m},$$

where $h^{i_1 \cdots i_m} = \sum_{j_1, \ldots, j_m = 1}^{n+1} g^{i_1 j_1} \cdots g^{i_m j_m} h_{j_1 \cdots j_m}$. We define the trace $\mathrm{Tr}_g : \otimes^m_S E^* \to \otimes^{m-2}_S E^*$ of a symmetric tensor by

$$\mathrm{Tr}_g(f) = \sum_{i=1}^{n+1} f(e_i, e_i, \cdot, \ldots, \cdot).$$

In coordinates, $\mathrm{Tr}_g(f)(v_2, \ldots, v_m) = \mathrm{Tr}(g^{-1} f(\cdot, \cdot, v_2, \ldots, v_m))$. Its adjoint with respect to the scalar products is the map $I : \otimes_S^{m-2} E^* \to \otimes_S^m E^*$ given by $I(u) = \sigma(g \otimes u)$.

Symmetric tensors can also be seen as homogeneous polynomials on the unit sphere of the Euclidean space. We denote by $\mathbb{S}_E$ the n-dimensional unit sphere on (E, g) and by dS the Riemannian measure on the sphere induced by the metric $g|_{\mathbb{S}_E}$. We define $\pi_m : (x, v) \mapsto (x, \otimes^m v)$ for $v \in E$; it induces a canonical morphism $\pi_m^* : \otimes_S^m E^* \to C^\infty(\mathbb{S}_E)$ given by $\pi_m^* f(v) = f(v, \ldots, v)$. Its formal adjoint is $\langle \pi_m^* f, h \rangle_{L^2(\mathbb{S}_E, dS)} = \langle f, \pi_{m*} h \rangle_{\otimes^m T^* M}$, where $f \in \otimes_S^m T^* M$ and $h \in C^\infty(\mathbb{S}_E)$. In coordinates,

$$(\pi_{m*}h)_{i_1 \cdots i_m} := \pi_{m*}h(\partial_{i_1}, \ldots, \partial_{i_m}) = \sum_{j_1, \ldots, j_m = 1}^{n+1} g_{i_1 j_1} \cdots g_{i_m j_m} \int_{\mathbb{S}_E} h(v) v_{j_1} \cdots v_{j_m} \, dS. \qquad (2\text{-}1)$$

Also note that (2-1) can be rewritten intrinsically as,

$$\text{for all } u_1, \ldots, u_m \in E, \quad \pi_{m*}h(u_1, \ldots, u_m) = \int_{\mathbb{S}_E} h(v) g(v, u_1) \cdots g(v, u_m) \, dv. \qquad (2\text{-}2)$$

The map $\pi_{m*}\pi_m^*$ is an isomorphism we will study in the next section. Also note that $\pi_m^*(\sigma f) = \pi_m^* f$ (since all the antisymmetric parts of the tensor f vanish by plugging m times the same vector v).

We denote by j_ξ the multiplication by ξ, that is, $j_\xi : f \mapsto \xi \otimes f$, and by i_ξ the contraction, that is, $i_\xi : f \mapsto u(\xi^\sharp, \cdot, \ldots, \cdot)$. The adjoint of i_ξ on symmetric tensors with respect to the L^2-scalar product is σj_ξ, that is

$$\text{for all } f \in \otimes_S^{m-1} E^*, \ h \in \otimes_S^m E^*, \quad \langle \sigma j_\xi f, h \rangle = \langle f, i_\xi h \rangle.$$

The space $\otimes_m^S E^*$ can thus be decomposed as the direct sum

$$\otimes_S^m E^* = \mathrm{ran}(\sigma j_\xi|_{\otimes_S^{m-1} E^*}) \oplus^\perp \ker(i_\xi|_{\otimes_S^m E^*}).$$

We denote by $\pi_{\ker i_\xi}$ the projection onto the right space, parallel to the left space. We will need the following.

Lemma 2.1. *For all $f, h \in \otimes_S^m E^*$,*

$$C_{n,m} \int_{\langle \xi, v \rangle = 0} \pi_m^* f(v) \pi_m^* h(v) \, dS_\xi(v) = \langle \pi_{\ker i_\xi} \pi_{m*} \pi_m^* \pi_{\ker i_\xi} f, h \rangle,$$

where

$$C_{n,m} = \int_0^\pi \sin^{n-1+2m}(\varphi) \, d\varphi = \sqrt{\pi} \frac{\Gamma((n+2m)/2)}{\Gamma((n+1+2m)/2)}$$

and dS_ξ is the canonical measure induced on the $(n-1)$-dimensional sphere $\mathbb{S}_{E,\xi} := \mathbb{S}_E \cap \{\langle \xi, v \rangle = 0\}$.

Proof. We can write $h = \sigma j_\xi h_1 + h_2$, where $h_1 \in \otimes_S^{m-1} E^*$, $h_2 \in \ker(i_\xi|_{\otimes_S^m T^* M})$. Note that $\pi_m^*(\sigma j_\xi h_1)(v) = \pi_m^*(j_\xi h_1)(v) = \langle \xi, v \rangle \pi_{m-1}^* h_1(v)$ and this vanishes on $\{\langle \xi, v \rangle = 0\}$ (and the same holds for f). In other words, $\pi_m^* h = \pi_m^* \pi_{\ker i_\xi}$ on $\{\langle \xi, v \rangle = 0\}$. We are thus left to check that, for $f, h \in \ker i_\xi$,

$$C_{n,m} \int_{\langle \xi, v \rangle = 0} \pi_m^* f(v) \pi_m^* h(v) \, dS_\xi(v) = \int_{\mathbb{S}_E} \pi_m^* f(v) \pi_m^* h(v) \, dS(v).$$

We will use the coordinates $v' = (v, \varphi) \in \mathbb{S}_{E,\xi} \times [0, \pi]$ on $\mathbb{S}_E$, which allow us to take the decomposition $v' = \sin(\varphi)v + \cos(\varphi)\xi^\sharp/|\xi|$. Then the measure on $\mathbb{S}_E$ disintegrates as $dS = \sin^{n-1}(\varphi)\, d\varphi\, dS_\xi(v)$. Also note that $\pi_m^* f(v + \cos(\varphi)\xi^\sharp/|\xi|) = \pi_m^* f(v)$. Then, if $C_{n,m} := \int_0^\pi \sin^{n-1+2m}(\varphi)\, d\varphi$, we obtain

$$
\int_{\langle \xi, v\rangle = 0} \pi_m^* f(v) \pi_m^* h(v)\, dS_\xi(v)
$$

$$
= C_{n,m}^{-1} \int_0^\pi \sin^{n-1+2m}(\varphi)\, d\varphi \int_{\langle \xi, v\rangle = 0} \pi_m^* f(v) \pi_m^* h(v)\, dS_\xi(v)
$$

$$
= C_{n,m}^{-1} \int_0^\pi \int_{\langle \xi, v\rangle = 0} \pi_m^* f(\sin(\varphi)v + \cos(\varphi)\xi^\sharp/|\xi|) \pi_m^* h(\sin(\varphi)v + \cos(\varphi)\xi^\sharp/|\xi|) \sin^{n-1}(\varphi)\, d\varphi\, dS_\xi(v)
$$

$$
= C_{n,m}^{-1} \int_{\mathbb{S}_E} \pi_m^* f(v') \pi_m^* h(v')\, dS(v'). \qquad\qquad \square
$$

2A2. *Spherical harmonics.* Let $\Delta|_{\mathbb{S}_E} := \operatorname{div}_{\mathbb{S}_E} \nabla_{\mathbb{S}_E}$ be the Laplacian on the unit sphere $\mathbb{S}_E$ induced by the metric $g|_{\mathbb{S}_E}$ and Δ be the usual Laplacian on E induced by g. Let

$$
L^2(\mathbb{S}_E) = \bigoplus_{m=0}^{+\infty} \Omega_m
$$

be the spectral break-up in spherical harmonics, where $\Omega_m := \ker(\Delta|_{\mathbb{S}_E} + m(m+n-1))$ are the eigenspaces of the Laplacian. We denote by E_m the vector space of trace-free symmetric m-tensors, where the trace is, as before, taken over the first two coordinates.

Lemma 2.2. *The map $\pi_m^* : E_m \to \Omega_m$ is an isomorphism and $\pi_{m*}\pi_m^*|_{E_m} = \lambda_{m,n}\mathbb{1}_{E_m}$ for some constant $\lambda_{m,n} \neq 0$.*

This also shows that, up to rescaling by the constant $\lambda_{m,n}$, $\pi_m^* : E_m \to \Omega_m$ is an isometry. One could be more accurate and actually show that the maps

$$
\pi_m^* : \otimes_S^m E^* \to \bigoplus_{k=0}^{[m/2]} \Omega_{m-2k}, \quad \pi_{m*} : \bigoplus_{k=0}^{[m/2]} \Omega_{m-2k} \to \otimes_S^m E^* \tag{2-3}
$$

are isomorphisms, where $[m/2]$ stands for the integer part of $m/2$. This follows from the (unique) decomposition of a symmetric tensor into a trace-free part and a remainder (which lies in the image of the adjoint of Tr). More precisely, by iterating this process, one can decompose u as $u = \sum_{k=0}^{[m/2]} I^k(u_k)$, where $I : \otimes_S^\bullet E^* \to \otimes_S^{\bullet+2} E^*$ is the adjoint of Tr with respect to the scalar products and $u_k \in \otimes_S^{m-2k} E^*$, $\operatorname{Tr}(u_k) = 0$ and $\pi_m^* I^k(u_k) \in \Omega_{m-2k}$. Then (2-3) is an immediate consequence of the previous lemma. The map $\pi_{m*}\pi_m^*$ acts by scalar multiplication on each component $I^k(u_k)$ (but with a different constant though, so $\pi_{m*}\pi_m^*$ is not a multiple of the identity). Since we will only need the fact that $\pi_{m*}\pi_m^*$ is an isomorphism, we do not provide further details.

2A3. *Symmetric tensors on a Riemannian manifold.*

Decomposition in solenoidal and potential tensors. We now consider the Riemannian manifold (M, g) and denote by $d\mu$ the Liouville measure on the unit tangent bundle SM. All the previous definitions naturally extend to the vector bundle $TM \to M$. For $f, h \in C^\infty(M, \otimes_S^m T^*M)$, we define the L^2-scalar

product

$$\langle f, h \rangle = \int_M \langle f_x, h_x \rangle_x \, d \operatorname{vol}(x),$$

where $\langle \cdot, \cdot \rangle_x$ is the scalar product on $T_x M$ introduced in the previous paragraph and d vol is the Riemannian measure induced by g. The map $\pi_m^* : C^\infty(M, \otimes_S^m T^*M) \to C^\infty(SM)$ is the canonical morphism given by $\pi_m^* f(x, v) = f_x(v, \ldots, v)$, whose formal adjoint with respect to the two L^2-inner products (on $L^2(SM, d\mu)$ and $L^2(\otimes_S^m T^*M, d$ vol$)$) is π_{m*}; i.e., $\langle \pi_m^* f, h \rangle_{L^2(SM, d\mu)} = \langle f, \pi_{m*} h \rangle_{L^2(\otimes_S^m T^*M, d \text{ vol})}$.

If ∇ denotes the Levi-Civita connection, we set $D := \sigma \circ \nabla : C^\infty(M, \otimes_S^m T^*M) \to C^\infty(M, \otimes_S^{m+1} T^*M)$, which is the symmetrized covariant derivative. Its formal adjoint with respect to the L^2-scalar product is $D^* = -\operatorname{Tr}(\nabla)$, where the trace is taken with respect to the two first indices, like in Section 2A1. One has the following relation between the geodesic vector field X on SM and the operator D:

Lemma 2.3. $$X\pi_m^* = \pi_{m+1}^* D.$$

The operator D can be seen as a differential operator of order 1. Its principal symbol is given by $\sigma(D)(x, \xi) f \mapsto \sigma(\xi \otimes f) = \sigma j_\xi f$ (see [Sharafutdinov 1994, Theorem 3.3.2]).

Lemma 2.4. *D is elliptic. It is injective on tensors of odd order, and its kernel is reduced to $\mathbb{R}\sigma(g^{\otimes m/2})$ on even tensors.*

When m is even, we will denote by $K_m = c_m \sigma(g^{\otimes m/2})$, with $c_m > 0$, a unitary vector in the kernel of D.

Proof. We fix $(x, \xi) \in T^*M$. For a tensor $u \in \otimes_s^m T_x^*M$, using the fact that the antisymmetric part of $\xi \otimes u$ vanishes in the integral, we have

$$\langle \sigma(D)u, \sigma(D)u \rangle = \int_{\mathbb{S}_x^n} \langle \xi, v \rangle^2 \pi_m^* u^2(v) \, dS_x(v) = |\xi|^2 \int_{\mathbb{S}_x^n} \langle \xi/|\xi|, v \rangle^2 \pi_m^* u^2(v) \, dS_x(v) > 0,$$

unless $u \equiv 0$. Since $\otimes_s^m T_x^*M$ is finite-dimensional, the map

$$(u, \xi/|\xi|) \mapsto \langle \sigma(D)(x, \xi/|\xi|)u, \sigma(D)(x, \xi/|\xi|)u \rangle,$$

defined on the compact set $\{u \in \otimes_s^m T_x^*M : |u|^2 = 1\} \times \mathbb{S}^n$ is bounded and attains its lower bound $C^2 > 0$ (which is independent of x). Thus $\|\sigma(x, \xi)\| \geq C|\xi|$, so the operator is uniformly elliptic and can be inverted (on the left) modulo a compact remainder: there exist pseudodifferential operators Q, R of respective orders $-1, -\infty$ such that $QD = 1 + R$.

As to the injectivity of D, if $Df = 0$ for some tensor $f \in \mathcal{D}'(M, \otimes_S^m T^*M)$, then f is smooth and $\pi_{m+1}^* Df = X\pi_m^* f = 0$. By ergodicity of the geodesic flow, $\pi_m^* f = c \in \Omega_0$ is constant. If m is odd, then $\pi_m^* f(x, v) = -\pi_m^* f(x, -v)$ so $f \equiv 0$. If m is even, then, by Section 2A2, $f = I^{m/2}(u_{m/2})$, where $u_{m/2} \in \otimes_S^0 E^* \simeq \mathbb{R}$ so $f = c' \sigma(g^{\otimes m/2})$. $\qquad\square$

By classical elliptic theory, the ellipticity of D implies that

$$H^s(M, \otimes_S^m T^*M) = D(H^{s+1}(M, \otimes_S^{m-1} T^*M)) \oplus \ker D^*|_{H^s(M, \otimes_S^m T^*M)}, \tag{2-4}$$

and the decomposition still holds in the smooth category and in the $C^{k,\alpha}$-topology for $k \in \mathbb{N}$, $\alpha \in (0, 1)$. This is the content of the following theorem:

Theorem 2.5 (tensor decomposition). *Let $s \in \mathbb{R}$ and $f \in H^s(M, \otimes_S^m T^*M)$. Then, there exists a unique pair of symmetric tensors $(p, h) \in H^{s+1}(M, \otimes_S^{m-1} T^*M) \times H^s(M, \otimes_S^m T^*M)$ such that $f = Dp + h$ and $D^*h = 0$. Moreover, if $m = 2l + 1$ is odd, $\langle p, K_{2l} \rangle = 0$.*

Tensorial distributions. The spaces $H^s(M, \otimes_S^m T^*M)$ that have been mentioned so far are the L^2-based Sobolev spaces of order $s \in \mathbb{R}$. They can be defined in coordinates (each coordinate of the tensor has to be in $H_{\text{loc}}^s(\mathbb{R})$) or more intrinsically by setting $H^s(M, \otimes_S^m T^*M) := (\mathbb{1} + D^*D)^{-s/2} L^2(M, \otimes_S^m T^*M)$. These two definitions are equivalent by [Shubin 2001, Proposition 7.3], following the properties of the operator $\mathbb{1} + D^*D$ (it is elliptic, invertible, positive). In the same fashion, the spaces $L^p(M, \otimes_S^m T^*M)$ for $p \geq 1$ can be defined in coordinates. Note that the maps

$$\pi_m^* : H^s(M, \otimes_S^m T^*M) \to H^s(SM), \quad \pi_{m*} : H^s(SM) \to H^s(M, \otimes_S^m T^*M)$$

are bounded for all $s \in \mathbb{R}$ (and they are bounded on L^p-spaces for $p \geq 1$). The operator π_{m*} acts by duality on distributions, namely

$$\pi_{m*} : C^{-\infty}(SM) \to C^{-\infty}(M, \otimes_S^m T^*M), \quad \langle \pi_{m*} f_1, f_2 \rangle := \langle f_1, \pi_m^* f_2 \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the distributional pairing.

The projection on solenoidal tensors. When m is even, we denote by $\Pi_{K_m} := \langle K_m, \cdot \rangle K_m$ the orthogonal projection on $\mathbb{R}K_m$. We define $\Delta_m := D^*D + \varepsilon(m)\Pi_{K_m}$, where $\varepsilon(m) = 1$ for m even, $\varepsilon(m) = 0$ for m odd. The operator Δ_m is an elliptic differential operator of order 2 which is invertible; as a consequence, its inverse is also pseudodifferential of order -2 (see [Shubin 2001, Theorem 8.2]). We can thus define the operator

$$\pi_{\ker D^*} := \mathbb{1} - D\Delta_m^{-1}D^*. \tag{2-5}$$

One can check that this is exactly the L^2-orthogonal projection on solenoidal tensors, it is a pseudodifferential operator of order 0 (as a composition of pseudodifferential operators).

Since $\sigma(D)(x, \xi) = \sigma j_\xi$, we know by Section 2A1 that given $(x, \xi) \in T^*M$, the space $\otimes_S^m T_x^*M$ breaks up as the direct sum

$$\otimes_S^m T_x^*M = \text{ran}(\sigma(D)(x, \xi)|_{\otimes_S^{m-1} T_x^*M}) \oplus \ker(\sigma(D^*)(x, \xi)|_{\otimes_S^m T_x^*M})$$

$$= \text{ran}(\sigma j_\xi|_{\otimes_S^{m-1} T_x^*M}) \oplus \ker(i_\xi|_{\otimes_S^m T_x^*M}).$$

We recall that $\pi_{\ker i_\xi}$ is the projection on $\ker(i_\xi|_{\otimes_S^m T_x^*M})$ parallel to $\text{ran}(\sigma j_\xi|_{\otimes_S^{m-1} T_x^*M})$.

Lemma 2.6. *The principal symbol of $\pi_{\ker D^*}$ is $\sigma_{\pi_{\ker D^*}} = \pi_{\ker i_\xi}$.*

Proof. First, observe that

$$D\Delta_m^{-1}D^*D\Delta_m^{-1}D^* = D\Delta_m^{-1}(\Delta_m - \varepsilon(m)\Pi_{K_m})\Delta_m^{-1}D^*$$

$$= D\Delta_m^{-1}D^* - \varepsilon(m)D\Delta_m^{-1}\Pi_{K_m}\Delta_m^{-1}D^*.$$

The second operator is smoothing so at the principal symbol level

$$\sigma_{(D\Delta_m^{-1}D^*)^2} = \sigma_{D\Delta_m^{-1}D^*}^2 = \sigma_{D\Delta_m^{-1}D^*},$$

which implies that $\sigma_{D\Delta_m^{-1}D^*}$ is a projection. Moreover, $\sigma_{D\Delta_m^{-1}D^*} = \sigma_D\sigma_{\Delta_m^{-1}}\sigma_{D^*} = \sigma j_\xi \sigma_{\Delta_m^{-1}} i_\xi$, so it is the projection onto ran σj_ξ with kernel ker i_ξ. Since $\pi_{\ker D^*} = \mathbb{1} - D\Delta_m^{-1}D^*$, the result is immediate. $\qquad\square$

3. On Livšic-type theorems

We will denote by $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ the Riemannian distance on $\mathcal{M}$ inherited from the Riemannian metric g. The α-Hölder norm of f is defined by

$$\|f\|_{C^\alpha} := \sup_{x\in\mathcal{M}} |f(x)| + \sup_{x,y\in\mathcal{M},\, x\neq y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} = \|f\|_\infty + \|f\|_\alpha.$$

In a series of inequalities, we will sometimes write $A \lesssim B$ to denote the fact that there exists a universal constant $C > 0$ such that $A \leq C \cdot B$. Note that a constant $C > 0$ may still appear from time to time and, as usual, it may change from one line to another.

3A. *Properties of Anosov flows.* We refer to the exhaustive [Katok and Hasselblatt 1995] and the book [Hasselblatt and Fisher 2019] for an introduction to hyperbolic dynamics.

3A1. *Stable and unstable manifolds.* The *global stable* and *unstable manifolds* $W^s(x)$, $W^u(x)$ are defined by

$$W^s(x) = \{x' \in \mathcal{M} : d(\varphi_t(x), \varphi_t(x')) \to_{t\to+\infty} 0\},$$
$$W^u(x) = \{x' \in \mathcal{M} : d(\varphi_t(x), \varphi_t(x')) \to_{t\to-\infty} 0\}.$$

For $\varepsilon > 0$ small enough, we define the *local stable* and *unstable manifolds* $W_\varepsilon^s(x) \subset W^s(x)$ and $W_\varepsilon^u(x) \subset W^u(x)$ by

$$W_\varepsilon^s(x) = \{x' \in W^s(x) : \text{for all } t \geq 0,\ d(\varphi_t(x), \varphi_t(x')) \leq \varepsilon\},$$
$$W_\varepsilon^u(x) = \{x' \in W^u(z) : \text{for all } t \geq 0,\ d(\varphi_{-t}(x), \varphi_{-t}(x')) \leq \varepsilon\}.$$

For all $\varepsilon > 0$ small enough, there exists $t_0 > 0$ such that

$$\text{for all } x \in \mathcal{M}, \text{ for all } t \geq t_0, \quad \varphi_t(W_\varepsilon^s(x)) \subset W_\varepsilon^s(\varphi_t(x)), \quad \varphi_{-t}(W_\varepsilon^u(x)) \subset W_\varepsilon^u(\varphi_{-t}(x)), \tag{3-1}$$

and

$$T_x W_\varepsilon^s(x) = E_s(x), \quad T_x W_\varepsilon^u(x) = E_u(x).$$

3A2. *Classical properties.* The main tool we will use to construct suitable periodic orbits is the following classical shadowing property of Anosov flows. Part of the proof can be found in [Katok and Hasselblatt 1995, Corollary 18.1.8] and [Hasselblatt and Fisher 2019, Theorem 5.3.2]. The last bound is a consequence of hyperbolicity and can be found in [Hasselblatt and Fisher 2019, Proposition 6.2.4]. For the sake of simplicity, we will write $\gamma = [xy]$ if γ is an orbit segment with endpoints x and y.

Theorem 3.1. *There exist $\varepsilon_0 > 0$, $\theta > 0$ and $C > 0$ with the following property. Consider $\varepsilon < \varepsilon_0$ and a finite or infinite sequence of orbit segments $\gamma_i = [x_i \, y_i]$ of length T_i greater than 1 such that, for any n, $d(y_n, x_{n+1}) \leq \varepsilon$. Then there exists a genuine orbit γ and times τ_i such that γ restricted to $[\tau_i, \tau_i + T_i]$ shadows γ_i up to $C\varepsilon$. More precisely, for all $t \in [0, T_i]$, one has*

$$d(\gamma(\tau_i + t), \gamma_i(t)) \leq C\varepsilon e^{-\theta \min(t, T_i - t)}. \tag{3-2}$$

Moreover, $|\tau_{i+1} - (\tau_i + T_i)| \leq C\varepsilon$. Finally, if the sequence of orbit segments γ_i is periodic, then the orbit γ is periodic.

Remark 3.2. In this theorem, we could also allow the first orbit segment γ_i to be infinite on the left, and the last orbit segment γ_j to be infinite on the right. In this case, (3-2) should be replaced by its obvious reformulation: assuming that γ_i is defined on $(-\infty, 0]$ and γ_j on $[0, +\infty)$, we would get for some $\tilde{\tau}_{i+1}$ within $C\varepsilon$ of τ_{i+1} and all $t \geq 0$

$$d(\gamma(\tilde{\tau}_{i+1} - t), \gamma_i(-t)) \leq C\varepsilon e^{-\theta t} \tag{3-3}$$

and

$$d(\gamma(\tau_j + t), \gamma_j(t)) \leq C\varepsilon e^{-\theta t}.$$

In particular, if γ_0 is an orbit segment $[x \, y]$ with $d(y, x) \leq \varepsilon_0$, then applying the above theorem to $\gamma_i := \gamma_0$ for all $i \in \mathbb{Z}$, one gets a periodic orbit that shadows γ_0; this is the *Anosov closing lemma*. We will also use thoroughly the version with two orbit segments that are repeated to get a periodic orbit.

3A3. *Cover by parallelepipeds.* We will now fix ε_0 small enough so that the previous propositions are guaranteed. For $\varepsilon \leq \varepsilon_0$, we define the set $W_\varepsilon(x) := \bigcup_{y \in W_\varepsilon^u(x)} W_\varepsilon^s(x)$. We can cover the manifold $\mathcal{M}$ by a finite union of flow boxes $\mathcal{U}_i := \bigcup_{t \in (-\delta, \delta)} \varphi_t(\Sigma_i)$, where $\Sigma_i := W_{\varepsilon_0}(x_i)$ and $x_i \in \mathcal{M}$.

We denote by $\pi_i : \mathcal{U}_i \to \Sigma_i$ the projection by the flow on the transverse section and we define $\mathrm{t}_i : \mathcal{U}_i \to \mathbb{R}$ such that $\pi_i(x) = \varphi_{\mathrm{t}_i(x)}(x)$ for $x \in \mathcal{U}_i$. We will need the following lemma:

Lemma 3.3. *The maps π_i, t_i are Hölder-continuous.*

Proof. This is actually a general fact related to the Hölder regularity of the foliation and the smoothness of the flow.

For the sake of simplicity, we drop the index i in this proof. Let us first prove the Hölder-continuity for x close to Σ and x' close to x. We fix $p \in \Sigma$ and choose smooth local coordinates $\psi : B(p, \eta) \to \mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^{n_s} \times \mathbb{R}^{n_u}$ around p (and centered at 0) so that $d\psi_p(X) = \partial_{x_0}$. This choice guarantees that in a neighborhood of 0, the flow is transverse to the hyperplane $\{0\} \times \mathbb{R}^{n_s + n_u}$. We still denote by Σ_η its image $\psi(\Sigma_\eta) \subset \mathbb{R}^{n+1}$, which is a submanifold of Hölder regularity (the index η indicates that we consider the same objects intersected with the ball $B(x, \eta)$). Moreover, there exists a Hölder-continuous homeomorphism $\Phi : S \to \Sigma_\eta$, where $S \subset \{0\} \times \mathbb{R}^{n_s + n_u}$ (since Σ_η is a submanifold of M with Hölder regularity). We consider $\hat{\varphi} : (-\delta, \delta) \times S \to \varphi_{(-\delta, \delta)}(S) =: V \supset \Sigma_\eta$ defined by $\hat{\varphi}(t, z) = \varphi_t(0, z)$, which is a smooth diffeomorphism. Note that, for $(0, z) \in S$, t satisfies $(\mathrm{t}(z), z) = \hat{\varphi}^{-1}(\Phi(z))$. So it is Hölder-continuous on S. Then $z \mapsto \pi(0, z) = \varphi_{\mathrm{t}(z)}(0, z)$ is Hölder-continuous on S too.

We denote by $\pi_S : V \to S$ the projection and by $t_S : V \to S$ the time such that $\pi_S(x) = \varphi_{t_S(x)}(x)$. These two maps are smooth by the implicit function theorem since the flow is transverse to S. Moreover, we have $\pi(x) = \pi|_S(\pi_S(x))$ so π is Hölder-continuous. And $t(x) = t_S(x) + t|_S(\pi_S(x))$, so t is Hölder-continuous too. Note that by compactness of Σ, this procedure can be done with only a finite number of charts, thus ensuring the uniformity of the constants. Thus, π_i, t_i are Hölder-continuous in a neighborhood of Σ. Now, in order to obtain the continuity on the whole cube $\mathcal{U}$, one can repeat the same argument i.e., write the projection as the composition of a first projection on a smooth small section S defined in a neighborhood of Σ with the actual projection on Σ. This provides the desired result. $\qquad\square$

3B. *Proof of the approximate Livšic theorem.* We now deal with the proof of Theorem 1.2.

3B1. *A key lemma.* The following lemma states that we can find a sufficiently dense and yet separated orbit in the manifold $\mathcal{M}$. The separation can only hold transversally to the flow direction, and is defined as follows. Recall that $W_\varepsilon(x) := \bigcup_{y \in W_\varepsilon^u(x)} W_\varepsilon^s(x)$. Then we say that a set S is ε-*transversally separated* if, for any $x \in S$, we have $S \cap W_\varepsilon(x) = \{x\}$.

Lemma 3.4. *Consider a transitive Anosov flow on a compact manifold. There exist β_s, $\beta_d > 0$ such that the following holds. Let $\varepsilon > 0$ be small enough. There exists a periodic orbit $\mathcal{O}(x_0) := (\varphi_t x_0)_{0 \le t \le T}$ with $T \le \varepsilon^{-1/2}$ such that this orbit is ε^{β_s}-transversally separated and $(\varphi_t x_0)_{0 \le t \le T-1}$ is ε^{β_d}-dense. If $\kappa > 0$ is some fixed constant, then one can also require that there exists a piece of $\mathcal{O}(x_0)$ of length $\le C(\kappa)$ which is κ-dense in the manifold.*

Proof. We could give a combinatorial construction in terms of Markov partitions and carefully chosen sequences of symbols in the symbolic dynamics representation of the flow. However, controlling rigorously the boundary effects on separation is delicate. Instead, we give a geometric construction solely based on the shadowing theorem. It is easy to obtain an ε^{β_d}-dense orbit by concatenating orbit segments thanks to the shadowing theorem. However, separation is harder to enforce. In this proof, we introduce several constants, but none of them will depend on ε.

Let us fix two periodic points p_1 and p_2 with different orbits $\mathcal{O}(p_1)$ and $\mathcal{O}(p_2)$ of respective lengths ℓ_1 and ℓ_2. By the shadowing theorem and transitivity, there exists an orbit γ_- which is asymptotic to $\mathcal{O}(p_1)$ in negative time and to $\mathcal{O}(p_2)$ in positive time. Also, there exists an orbit γ_+ which is asymptotic to $\mathcal{O}(p_2)$ in negative time and to $\mathcal{O}(p_1)$ in positive time. On γ_-, fix a point z_0, and $\rho_0 > 0$ small enough so that $\gamma_- \cup \gamma_+$ meets $W_{3\rho_0}(z_0)$ only at z_0, and $\mathcal{O}(p_1)$ and $\mathcal{O}(p_2)$ are at distance $> 3\rho_0$ of z_0. Denote by C_0 the constant C in the shadowing theorem, Theorem 3.1. Reducing ρ_0 if necessary, we can assume $\rho_0 < \varepsilon_0$, where ε_0 is given by Theorem 3.1. Let us also fix a large constant C_1, on which our construction will depend.

We truncate γ_- in positive time, stopping it at a fixed time where it is within distance $\rho_0/(2C_0)$ of p_2 to get an orbit γ'_-. Let t_- be the largest time in $(-\infty, -2C_1|\log \varepsilon|]$, where $\gamma'_-(t)$ is within distance ε of p_1. As the orbit γ'_- converges exponentially quickly in negative time to $\mathcal{O}(p_1)$ by hyperbolicity, one has $d(\gamma'_-(t), \mathcal{O}(p_1)) \le \varepsilon$ for $t \le -2C_1|\log \varepsilon|$ if C_1 is large enough. Hence, one needs to wait at most ℓ_1 before being ε-close to p_1. This shows that the time t_- satisfies $t_- = -2C_1|\log \varepsilon| + O(1)$.

In the same way, we truncate γ_+ in negative time at a fixed time for which it is within distance $\rho_0/(2C_0)$ of p_2, obtaining an orbit γ'_+. We denote by t_+ the smallest time in $[2C_1|\log \varepsilon|, +\infty)$ with $d(\gamma'_+(t), p_1) \le \varepsilon$. It satisfies $t_+ = 2C_1|\log \varepsilon| + O(1)$.

As the flow is transitive, it has a dense orbit. Therefore, for any x, y, there exists an orbit $\gamma_{x,y}$ starting from a point within distance $\rho_0/(2C_0)$ of x, ending at a point within distance $\rho_0/(2C_0)$ of y, and with length $\in [1, T_0]$, where T_0 is fixed and independent of x and y.

To any x, we associate an orbit as follows. Start with γ'_-, then follow $\gamma_{p_2, \varphi_{-C_1|\log \varepsilon|}x}$, then follow the orbit of x between times $-C_1|\log \varepsilon|$ and $C_1|\log \varepsilon|$, then follow $\gamma_{\varphi_{C_1|\log \varepsilon|}x, p_2}$, then follow γ'_+. In this sequence, the distance between an endpoint of a piece and the starting point of the next one is always less than ρ_0/C_0. Hence, Theorem 3.1 applies and yields an infinite orbit γ'_x that follows the above pieces of orbits up to ρ_0. If C_1 is large enough, (3-2) implies that x is within distance at most ε of γ'_x. The inequality (3-3) shows that $\gamma'_-(t_-)$ and the corresponding point x_- on γ'_x are within distance $e^{-\theta t_-}$. If C_1 is large enough, this is bounded by ε since $t_- = -2C_1|\log \varepsilon| + O(1)$. Therefore, $d(x_-, p_1) \le 2\varepsilon$. In the same way, the point x_+ on γ'_x matching $\gamma'_+(t_+)$ is within distance ε of $\gamma'_+(t_+)$ and therefore within distance 2ε of p_1. Let us truncate γ'_x between x_- and x_+ to get an orbit segment γ_x of length $6C_1|\log \varepsilon| + O(1)$, starting and ending within 2ε of p_1, and passing within ε of x.

Let $\beta_d = 1/(3 \dim(\mathcal{M}))$. We define a sequence of points of $\mathcal{M}$ as follows. Let x_1 be an arbitrary point for which the $C(\kappa)$-beginning of its orbit is $\kappa/2$-dense, to guarantee in the end that the last condition of the lemma is satisfied. If γ_{x_1} is not $\varepsilon^{\beta_d}/2$-dense, we choose another point x_2 which is not in the $\varepsilon^{\beta_d}/2$-neighborhood of γ_{x_1}. Then $\gamma_{x_1} \cup \gamma_{x_2}$ contain both x_1 and x_2 in their ε-neighborhood, and therefore in their $\varepsilon^{\beta_d}/2$-neighborhood. If $\gamma_{x_1} \cup \gamma_{x_2}$ is still not $\varepsilon^{\beta_d}/2$-dense, then we add a third piece of orbit γ_{x_3}, and so on. By compactness, this process stops after finitely many steps, giving a finite sequence $x_1, \ldots, x_N$.

As all γ_{x_i} start and end with p_1 up to 2ε, we can glue the sequence

$$\ldots, \gamma_{x_N}, \gamma_{x_1}, \gamma_{x_2}, \ldots, \gamma_{x_N}, \gamma_{x_1}, \ldots$$

thanks to Theorem 3.1. We get a periodic orbit γ which shadows them up to $2C_0\varepsilon$. We claim this orbit satisfies the requirements of the lemma. We should check its length, its density, and its separation.

Let us start with the length. The points x_i are separated by at least $\varepsilon^{\beta_d}/3$. The balls of radius $\varepsilon^{\beta_d}/6$ are disjoint, and each has a volume $\ge c\varepsilon^{\beta_d \cdot \dim(\mathcal{M})} = c\varepsilon^{1/3}$. We get that the number N of points x_i is bounded by $C\varepsilon^{-1/3}$. As each piece γ_{x_i} has length at most $C|\log \varepsilon|$, it follows that the total length of γ is bounded by $C|\log \varepsilon|\varepsilon^{-1/3} \le \varepsilon^{-1/2}$.

Let us check the density. By construction, the union of the γ_{x_i} is $\varepsilon^{\beta_d}/2$-dense. As γ approximates each γ_{x_i} within $2C_0\varepsilon$, it follows that γ is $2C_0\varepsilon + \varepsilon^{\beta_d}/2$ dense, and therefore ε^{β_d}-dense. In the statement of the lemma, we require the slightly stronger statement that if one removes a length-1 piece at the end of the orbit, it remains ε^{β_d}-dense. Such a length-1 piece in γ_{x_N} consists of points that are within 2ε of $\mathcal{O}(p_1)$. They are approximated within ε^{β_d} by the start and end of all the other γ_{x_i}.

Finally, let us check the more delicate separation, which has motivated the finer details of the construction as we will see now. Let β_s be suitably large. We want to show that any two points x, y of γ within distance ε^{β_s} are on the same local flow line. Since the expansion of the flow is at most exponential, for any

$t \le 20C_1|\log \varepsilon|$, we have $d(\varphi_t x, \varphi_t y) \le \varepsilon$ if β_s is large enough. In the piece of γ of length $10C_1|\log \varepsilon|$ starting at x, there is an interval $[t_1, t_2]$ of length $4C_1|\log \varepsilon| + O(1)$ during which $\varphi_t x$ is within distance at most $\rho_0/2$ of $\mathcal{O}(p_1)$, corresponding to the junction between the orbits γ_{x_i} and $\gamma_{x_{i+1}}$, where i is such that x belongs to the shadow of $\gamma_{x_{i-1}}$. For $t \in [t_1, t_2]$, one also has $d(\varphi_t y, \mathcal{O}(p_1)) \le \rho_0$ as the orbits follow each other up to ε. Note that in each γ_j the consecutive time spent close to $\mathcal{O}(p_1)$ is bounded by $2C_1|\log \varepsilon|$ as we have forced a passage close to p_2 (and therefore far away from $\mathcal{O}(p_1)$) after this time in the construction. It follows that also for y the time interval $[t_1, t_2]$ has to correspond to a junction between two orbits γ_{x_j} and $\gamma_{x_{j+1}}$. Consider the smallest times t and t' after the junctions for which $\varphi_t(x)$ and $\varphi_{t'}(y)$ are $2\rho_0$-close to z_0. Since the orbit γ'_- meets $W_{3\rho_0}(z_0)$ at the single point z_0, these times have to correspond to each other; i.e., the orbits are synchronized up to an error $O(\varepsilon)$. To conclude, it remains to show that $i = j$. Suppose by contradiction $i < j$ for instance. The orbit of x follows γ_{x_i} up to $2C_0\varepsilon$, the orbit of y follows γ_{x_j} up to $2C_0\varepsilon$, and the orbits of x and y are within ε of each other. We deduce that γ_{x_i} and γ_{x_j} follow each other up to $(4C_0 + 1)\varepsilon$. Since x_j is within ε of γ_{x_j}, it follows that x_j is within $(4C_0 + 2)\varepsilon$ of γ_{x_i}. This is a contradiction with the construction, as we could have added the point x_j only if it was not in the ε^{β_d}-neighborhood of γ_{x_i}, and $\varepsilon^{\beta_d} > (4C_0 + 2)\varepsilon$ if ε is small enough. $\qquad\square$

3B2. *Construction of the approximate coboundary.* Let us now prove Theorem 1.2. The result is obvious if ε is bounded away from 0 by taking $u = 0$ and $h = f$. Hence, we can assume that ε is small enough to apply Lemma 3.4, with $\kappa = \varepsilon_0$. On the orbit $\mathcal{O}(x_0)$ given by this lemma, we define a function $\tilde{u}$ by $\tilde{u}(\varphi_t x_0) = \int_0^t f(\varphi_s x_0)\, ds$. Note that it may not be continuous at x_0. As a consequence, we will rather denote by $\mathcal{O}(x_0)$ the set $(\varphi_t x_0)_{0 \le t \le T-1}$ (which satisfies the required properties of density and transversality) in order to avoid problems of discontinuity.

Lemma 3.5. *There exist $\beta_1, C > 0$ independent of ε such that $\|\tilde{u}\|_{C^{\beta_1}(\mathcal{O}(x_0))} \le C$.*

Proof. We first study the Hölder regularity of $\tilde{u}$; namely we want to control $|\tilde{u}(x) - \tilde{u}(y)|$ by $Cd(x, y)^{\beta_1}$ for some well-chosen exponent β_1, when $d(x, y) \le \varepsilon_0$ (where ε_0 is the scale under which the shadowing theorem, Theorem 3.1, holds). If x and y are on the same local flow line, then the result is obvious since f is bounded by 1, so we are left to prove that $\tilde{u}$ is transversally C^{β_1}. Consider $x = \varphi_{t_0} x_0 \in \mathcal{O}(x_0)$ and $y = \varphi_{t_0+t} \in W_{\varepsilon_0}(x)$. By transversal separation of $\mathcal{O}(x_0)$, these points satisfy $d(x, y) \ge \varepsilon^{\beta_s}$. We can close the segment $[xy]$; i.e., we can find a periodic point p such that $d(p, x) \le Cd(x, y)$ with period $t_p = t + \tau$, where $|\tau| \le Cd(x, y)$, which shadows the segment. Then

$$|\tilde{u}(y) - \tilde{u}(x)| \le \underbrace{\left| \int_0^t f(\varphi_s x)\, ds - \int_0^{t_p} f(\varphi_s p)\, ds \right|}_{= \text{(I)}} + \underbrace{\left| \int_0^{t_p} f(\varphi_s p)\, ds \right|}_{= \text{(II)}}.$$

The first term (I) is bounded by $Cd(x, y)^{\beta_1'}$ for some $\beta_1' > 0$ depending on the dynamics, whereas the second term (II) is bounded — by assumption — by εt_p. But $\varepsilon t_p \lesssim \varepsilon t \lesssim \varepsilon T \lesssim \varepsilon^{1/2} \lesssim d(x, y)^{1/2\beta_s}$. We thus obtain the desired result with $\beta_1 := \min(\beta_1', \frac{1}{2}\beta_s)$.

We now prove that $\tilde{u}$ is bounded for the C^0-norm. We know that there exists a segment of the orbit $\mathcal{O}(x_0)$ — call it S — of length $\le C$ which is ε_0-dense in $\mathcal{M}$. In particular, for any $x \in \mathcal{O}(x_0)$, there exists

$x_S \in S$ with $d(x, x_S) \leq \varepsilon_0$, and therefore $|\tilde{u}(x) - \tilde{u}(x_S)| \leq Cd(x, x_S)^{\beta_1} \leq C\varepsilon_0^{\beta_1}$ thanks to the Hölder control of the previous paragraph. Using the same argument with x_0, we get, as $\tilde{u}(x_0) = 0$,

$$|\tilde{u}(x)| = |\tilde{u}(x) - \tilde{u}(x_0)| \leq |\tilde{u}(x) - \tilde{u}(x_S)| + |\tilde{u}(x_S) - \tilde{u}((x_0)_S)| + |\tilde{u}(x_0) - \tilde{u}((x_0)_S)|.$$

The first and last term are bounded by $C\varepsilon_0^{\beta_1}$, and the middle one is bounded by C as S has a bounded length and $\|f\|_{C^0} \leq 1$.   □

For each i, we extend the function $\tilde{u}$ (defined on $\mathcal{O}(x_0)$) to a Hölder function u_i on Σ_i by the formula $u_i(x) = \sup \tilde{u}(y) - \|\tilde{u}\|_{C^{\beta_1}(\mathcal{O}(x_0))} d(x, y)^{\beta_1}$, where the supremum is taken over all $y \in \mathcal{O}(x_0)$. With this formula, it is classical that the extension is Hölder-continuous, with $\|u_i\|_{C^{\beta_1}(\Sigma_i)} \leq \|\tilde{u}\|_{C^{\beta_1}(\mathcal{O}(x_0))}$. We then push the function u_i by the flow in order to define it on $\mathcal{U}_i$ by setting, for $x \in \Sigma_i$, $\varphi_t x \in \mathcal{U}_i$, $u_i(\varphi_t x) = u_i(x) + \int_0^t f(\varphi_s x)\,ds$. Note that by Lemma 3.3, the extension is still Hölder with the same regularity. We now set $u := \sum_i u_i \theta_i$ and $h := f - Xu = -\sum_i u_i X\theta_i$. The functions $X\theta_i$ are uniformly bounded in C^∞, independently of ε so the functions $u_i X\theta_i$ are in C^{β_1} with a Hölder norm independent of $\varepsilon > 0$, and thus $\|h\|_{C^{\beta_1}} \leq C$.

Lemma 3.6. $\|h\|_{C^{\beta_1/2}} \leq \varepsilon^{\beta_3/2}.$

Proof. We claim that h vanishes on $\mathcal{O}(x_0)$: indeed, on $\mathcal{U}_i \cap \mathcal{O}(x_0)$ one has $u_i \equiv \tilde{u}$ and thus $h = -\tilde{u} \sum_i X\theta_i = -\tilde{u}X \sum_i \theta_i = -\tilde{u}X\mathbf{1} = 0$. Since $\mathcal{O}(x_0)$ is ε^{β_d}-dense and $\|h\|_{C^{\beta_1}} \leq C$, we get that $\|h\|_{C^0} \leq C\varepsilon^{\beta_1\beta_d} = C\varepsilon^{\beta_3}$, where $\beta_3 = \beta_1\beta_d$. By interpolation, we eventually obtain that $\|h\|_{C^{\beta_1/2}} \leq \varepsilon^{\beta_3/2}$.   □

Proof of Theorem 1.2. The previous lemma provides the desired estimate on the remainder h. This completes the proof of Theorem 1.2.   □

4. Generalized geodesic x-ray transform

From now on, we will use the dual decomposition of the cotangent space $T^*\mathcal{M} = E_0^* \oplus E_u^* \oplus E_s^*$, where $E_0^*(E_u \oplus E_s) = 0$, $E_s^*(E_s \oplus \mathbb{R}X) = 0$, $E_u^*(E_u \oplus \mathbb{R}X) = 0$. If $A^{-\top}$ denotes the inverse transpose of a linear operator A, then the dual estimates to (1-2) are

$$\begin{aligned}
|d\varphi_t^{-\top}(x) \cdot \xi|_{\varphi_t(x)} &\leq Ce^{-\lambda t}|\xi|_x \quad \text{for all } t > 0,\ \xi \in E_s^*(x), \\
|d\varphi_t(x) \cdot \xi|_{\varphi_t(x)} &\leq Ce^{-\lambda|t|}|\xi|_x \quad \text{for all } t < 0,\ \xi \in E_u^*(x),
\end{aligned} \tag{4-1}$$

where $|\cdot|_x$ is now g^{-1}, the dual metric to g (which makes the musical isomorphism $\flat : T\mathcal{M} \to T^*\mathcal{M}$ an isometry). For the sake of simplicity, we now assume that X generates a contact Anosov flow; the results of this section will be applied to the case of an Anosov geodesic flow. It would actually be sufficient to assume that the flow is Anosov, preserves a smooth measure and is mixing for this measure. Note that a contact Anosov flow is exponentially mixing by [Liverani 2004]. We will denote by μ the *normalized* volume form induced by the contact 1-form. In the case of a geodesic flow, μ is nothing but the Liouville volume form. By $L^2(\mathcal{M})$, we will always refer to the space $L^2(\mathcal{M}, d\mu)$. The orthogonal projection on the constant function is denoted by $\mathbf{1} \otimes \mathbf{1}$.

4A. *Resolvent of the flow at* 0. By [Faure and Sjöstrand 2011], we know that the resolvents $R_\pm(\lambda) :=$ $(X \pm \lambda)^{-1} : \mathcal{H}_\pm^s \to \mathcal{H}_\pm^s$ (initially defined for $\Re(\lambda) > 0$) admit a meromorphic extension to the half-space $\{\Re(\lambda) > -cs\}$ — where $\mathcal{H}_\pm^s$ are *anisotropic Sobolev spaces* — and thus $R_\pm(\lambda) : C^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$ admit a meromorphic extension to the whole complex plane. For $\Re(\lambda) > 0$, $R_\pm(\lambda) : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ are bounded and the expression $R_+(\lambda)$ is given by

$$R_+(\lambda) = (X + \lambda)^{-1} = \int_0^{+\infty} e^{-\lambda t} e^{-tX} \, dt, \tag{4-2}$$

where $e^{-tX} f(x) = f(\varphi_{-t}(x))$ for $f \in C^\infty(\mathcal{M})$, $x \in \mathcal{M}$.

In a neighborhood of 0, we can thus write the Laurent expansions

$$R_+(\lambda) = R_0^+ + \frac{\mathbf{1} \otimes \mathbf{1}}{\lambda} + \mathcal{O}(\lambda), \quad R_-(\lambda) = R_0^- - \frac{\mathbf{1} \otimes \mathbf{1}}{\lambda} + \mathcal{O}(\lambda), \tag{4-3}$$

where $R_0^+ : \mathcal{H}_+^s \to \mathcal{H}_+^s$, $R_0^- : \mathcal{H}_-^s \to \mathcal{H}_-^s$ are bounded. Since $H^s \subset \mathcal{H}_\pm^s \subset H^{-s}$, we obtain that $R_0^\pm :$ $H^s \to H^{-s}$ are bounded and thus $(R_0^+)^* : H^s \to H^{-s}$ is bounded too. Moreover, it is easy to check that formally $(R_0^+)^* = -R_0^-$ (i.e., the operators coincide on $C^\infty(\mathcal{M})$), in the sense that for all $f_1, f_2 \in C^\infty(\mathcal{M})$, $\langle R_0^- f_1, f_2 \rangle_{L^2(\mathcal{M})} = \langle f_1, -R_0^+ f_2 \rangle_{L^2(\mathcal{M})}$. Since $C^\infty(\mathcal{M})$ is dense in $H^s(\mathcal{M})$, we obtain that $(R_0^+)^* = -R_0^-$ on $H^s(\mathcal{M})$, in the sense that for all $f_1, f_2 \in H^s(\mathcal{M})$, $\langle R_0^- f_1, f_2 \rangle_{L^2(\mathcal{M})} = \langle f_1, -R_0^+ f_2 \rangle_{L^2(\mathcal{M})}$.

Also note that, as operators $C^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$, one has

$$X R_0^+ = R_0^+ X = \mathbb{1} - \mathbf{1} \otimes \mathbf{1}, \quad X R_0^- = R_0^- X = \mathbb{1} - \mathbf{1} \otimes \mathbf{1}. \tag{4-4}$$

For the sake of simplicity, we will write $R_0 := R_0^+$. We introduce the operator

$$\Pi := R_0 + R_0^*, \tag{4-5}$$

the sum of the two holomorphic parts of the resolvent. An easy computation, using (4-3), proves that $\Pi(\mathbf{1}) = 0$ and the image $\Pi(C^\infty(\mathcal{M}))$ is orthogonal to the constants. We recall:

Theorem 4.1. [Guillarmou 2017, Theorem 1.1] *For all $s > 0$, the operator $\Pi : H^s(\mathcal{M}) \to H^{-s}(\mathcal{M})$ is bounded, self-adjoint and satisfies*:

(1) *For all $f \in H^s(\mathcal{M})$, we have $X\Pi f = 0$.*

(2) *For all $f \in H^s(\mathcal{M})$ such that $Xf \in H^s(\mathcal{M})$, we have $\Pi X f = 0$.*

If $f \in H^s(\mathcal{M})$ with $\langle f, \mathbf{1} \rangle_{L^2} = 0$, then $f \in \ker \Pi$ if and only if there exists a solution $u \in H^s(\mathcal{M})$ to the cohomological equation $Xu = f$, and u is unique modulo constants.

There exist two other characterizations of the operator Π that are more tractable and which we detail in the next proposition. We set $\Pi_\lambda := \mathbf{1}_{(-\infty, \lambda]}(-iX)$.

Proposition 4.2. *For $f_1, f_2 \in C^\infty(\mathcal{M})$ such that $\langle f, \mathbf{1} \rangle_{L^2} = 0$:*

(1) $\langle \Pi f_1, f_2 \rangle = 2\pi \partial_\lambda|_{\lambda=0} \langle \Pi_\lambda f_1, f_2 \rangle$.

(2) $\langle \Pi f_1, f_2 \rangle = \int_{-\infty}^{+\infty} \langle f_1 \circ \varphi_t, f_2 \rangle \, dt$.

Proof. (1) For $f_1, f_2 \in C^\infty(\mathcal{M})$ such that $\int_\mathcal{M} f_i \, d\mu = 0$, we have using Stone's formula, for $\delta > 0$,

$$\langle \Pi_{\lambda+\delta} f_1, f_2 \rangle - \langle \Pi_{\lambda-\delta} f_1, f_2 \rangle = \langle \mathbf{1}_{[\lambda-\delta,\lambda+\delta]} f_1, f_2 \rangle$$
$$= \frac{1}{2\pi} \int_{\lambda-\delta}^{\lambda+\delta} \langle (R_+(-i\lambda) - R_-(i\lambda)) f_1, f_2 \rangle \, d\lambda.$$

Dividing by 2δ and passing to the limit $\delta \to 0^+$, we obtain $\partial_\lambda|_{\lambda=0} \langle \Pi_\lambda f_1, f_2 \rangle = \frac{1}{2\pi} \langle (R_0^+ - R_0^-) f_1, f_2 \rangle = \frac{1}{2\pi} \langle \Pi f_1, f_2 \rangle$.

(2) Thanks to the exponential decay of correlations (see [Liverani 2004]), one can apply Lebesgue's dominated convergence theorem in the limit $\lambda \to 0^+$ in the expression

$$\langle \Pi f_1, f_2 \rangle = \lim_{\lambda \to 0^+} \int_{-\infty}^{+\infty} e^{-\lambda|t|} \langle f_1 \circ \varphi_{-t}, f_2 \rangle \, dt,$$

and the result is then immediate. Note that a polynomial decay would have been sufficient.   □

The quantity $\langle \Pi f, f \rangle$ is sometimes referred to in the literature as the *variance* of the flow. In particular, it enjoys the following positivity property:

Lemma 4.3. *The operator* $\Pi : H^s(\mathcal{M}) \to H^{-s}(\mathcal{M})$ *is positive in the sense of quadratic forms, namely* $\langle \Pi f, f \rangle_{L^2} \geq 0$ *for all real-valued* $f \in H^s(\mathcal{M})$.

There are different ways of proving this lemma, related to the different characterizations of the operator Π. We only detail one of them which is in the dynamical spirit of this article. Another way could be to use the first item of Proposition 4.2 and the fact that the spectral measure Π_λ is nondecreasing.

Proof. By density, it is sufficient to prove the lemma for a real-valued $f \in C^\infty(\mathcal{M})$. We will actually show that for $\lambda > 0$

$$\left\langle \left(R_+(\lambda) - \frac{\mathbf{1} \otimes \mathbf{1}}{\lambda} \right) f, f \right\rangle = \langle R_+(\lambda) f, f \rangle - \frac{1}{\lambda} \left(\int_\mathcal{M} f \, d\mu \right)^2 \geq 0.$$

The same arguments being valid for $R_-(\lambda)$, we will deduce the result by taking the limit $\lambda \to 0^+$. By Parry's formula [1988, Section 3], we know that

$$\langle R_+(\lambda) f, f \rangle = \lim_{T \to \infty} \frac{1}{N(T)} \sum_{\ell(\gamma) \leq T} e^{\int_\gamma J^u} \frac{1}{\ell(\gamma)} \int_0^{\ell(\gamma)} R_+(\lambda) f(\varphi_t z) f(\varphi_t z) \, dt, \qquad (4\text{-}6)$$

where γ is a periodic orbit, $z \in \gamma$, $\ell(\gamma)$ is the length of γ and $N(T) = \sum_{\ell(\gamma) \leq T} e^{\int_\gamma J^u}$ is a normalizing coefficient, where

$$J^u : x \mapsto \partial_t \det d\varphi_t(x)|_{E_u(x)}|_{t=0}$$

is the unstable Jacobian (or the *geometric potential*). Let us fix a closed orbit γ and a base point $z \in \gamma$. We set $\tilde{f}(t) := f(\varphi_t z)$, which we see as a smooth function, ℓ-periodic on $\mathbb{R}$ (with $\ell := \ell(\gamma)$). Since $R_+(\lambda)$ commutes with X, $R_+(\lambda)$ acts as a Fourier multiplier on functions defined on γ. As a consequence, if we

take the decomposition $\tilde{f}(t) = \sum_{n \in \mathbb{Z}} c_n e^{2i\pi nt/\ell}$, we have

$$
\begin{aligned}
R_+(\lambda)\tilde{f}(t) &= \int_0^{+\infty} e^{-\lambda s} \tilde{f}(t+s) \, ds \\
&= \sum_{n \in \mathbb{Z}} c_n e^{2i\pi nt/\ell} \int_0^{+\infty} e^{-(\lambda - 2i\pi n/\ell)s} \, ds \\
&= \sum_{n \in \mathbb{Z}} \frac{c_n(\lambda + 2i\pi n/\ell)}{\lambda^2 + 4\pi^2 n^2/\ell^2} e^{2i\pi nt/\ell}.
\end{aligned}
$$

Then

$$
\langle R_+(\lambda)\tilde{f}, \tilde{f} \rangle_{L^2} = \frac{1}{\ell} \int_0^\ell R_+(\lambda)\tilde{f}(t)\tilde{f}(t) \, dt = \sum_{n \in \mathbb{Z}} \frac{|c_n|^2(\lambda + 2i\pi n/\ell)}{\lambda^2 + 4\pi^2 n^2/\ell^2} = \lambda \sum_{n \in \mathbb{Z}} \frac{|c_n|^2}{\lambda^2 + 4\pi^2 n^2/\ell^2},
$$

by oddness of the imaginary part of the sum. In particular

$$
\frac{1}{\ell} \int_0^\ell R_+(\lambda)\tilde{f}(t)\tilde{f}(t) \, dt \geq \frac{|c_0|^2}{\lambda} = \frac{1}{\lambda} \left(\frac{1}{\ell} \int_0^\ell \tilde{f}(t) \, dt \right)^2. \tag{4-7}
$$

Inserting (4-7) into (4-6), and then applying Jensen's convexity inequality we have

$$
\begin{aligned}
\langle R_+(\lambda)f, f \rangle &\geq \lambda^{-1} \lim_{T \to \infty} \frac{1}{N(T)} \sum_{\ell(\gamma) \leq T} e^{\int_\gamma J^u} \left(\frac{1}{\ell(\gamma)} \int_0^{\ell(\gamma)} f(\varphi_t z) \, dt \right)^2 \\
&\geq \lambda^{-1} \lim_{T \to \infty} \left(\frac{1}{N(T)} \sum_{\ell(\gamma) \leq T} e^{\int_\gamma J^u} \frac{1}{\ell(\gamma)} \int_0^{\ell(\gamma)} f(\varphi_t z) \, dt \right)^2 = \frac{1}{\lambda} \left(\int_{SM} f \, d\mu \right)^2,
\end{aligned}
$$

where we used again Parry's formula in the last equality. $\qquad \square$

4B. The normal operator. We now consider a smooth closed manifold (M, g) with Anosov geodesic flow and define $\mathcal{M} := SM$, the unit tangent bundle (with respect to the metric g). We introduce

$$
\Pi_m := \pi_{m*}(\Pi + \mathbf{1} \otimes \mathbf{1})\pi_m^*. \tag{4-8}
$$

Recall from Section 2A3 that given $(x, \xi) \in T^*M$, the space $\otimes_S^m T_x^*M$ decomposes as the direct sum

$$
\begin{aligned}
\otimes_S^m T_x^*M &= \mathrm{ran}(\sigma_D(x, \xi)|_{\otimes_S^{m-1} T_x^*M}) \oplus \ker(\sigma_{D^*}(x, \xi)|_{\otimes_S^m T_x^*M}) \\
&= \mathrm{ran}(\sigma j_\xi|_{\otimes_S^{m-1} T_x^*M}) \oplus \ker(i_\xi|_{\otimes_S^m T_x^*M}).
\end{aligned}
$$

The projection on the right space parallel to the left space is denoted by $\pi_{\ker i_\xi}$ and $\mathrm{Op}(\pi_{\ker i_\xi}) = \pi_{\ker D^*} + S$ by Lemma 2.6, where $S \in \Psi^{-1}$ and Op is any quantization on M (see [Shubin 2001, Section 6.4] for instance). Here, Ψ^m denotes the set of pseudodifferential operators of order $m \in \mathbb{R}$ and we will denote by S^m the class of usual symbols of order m. Given $P \in \Psi^m$, we will denote by σ_m its principal symbol. The following structure theorem is crucial in the sequel. It can be seen as a more intrinsic version of [Sharafutdinov, Skokan, and Uhlmann 2005, Theorem 2.1].

Theorem 4.4. Π_m *is a pseudodifferential operator of order* -1 *with principal symbol*

$$\sigma_m := \sigma_{\Pi_m} : (x, \xi) \mapsto \frac{2\pi}{C_{n,m}} |\xi|^{-1} \pi_{\ker i_\xi} \pi_{m*} \pi_m^* \pi_{\ker i_\xi},$$

with $C_{n,m} = \int_0^\pi \sin^{n-1+2m}(\varphi) \, d\varphi$.

Proof. The fact that Π_m is pseudodifferential was proved in [Guillarmou 2017]. All that is left to compute is the principal symbol of Π_m. According to the proof in [Guillarmou 2017, Theorem 3.1], we can only consider the integral in time between $(-\varepsilon, \varepsilon)$. Namely, given $\chi \in C_c^\infty(\mathbb{R})$ a smooth cutoff function around 0 whose support is contained in $(-\varepsilon, \varepsilon)$, one has

$$\Pi_m = \pi_{m*} \int_{-\varepsilon}^\varepsilon \chi(t) e^{-tX} \, dt \pi_m^* - \pi_{m*} R_0^+ \int_0^{+\infty} \chi'(t) e^{-tX} \, dt \pi_m^*$$
$$- \pi_{m*} R_0^- \int_{-\infty}^0 \chi'(t) e^{-tX} \, dt \pi_m^* + \left(1 - \int_{-\infty}^{+\infty} \chi(t) \, dt\right) \pi_{m*} \mathbf{1} \otimes \mathbf{1} \pi_m^*.$$

On the right-hand side, the last term is obviously smoothing. Following the same computations as in [Guillarmou 2017, Theorem 3.1], one can prove that the second and the third terms are also smoothing (this stems from an argument on the wavefront set of the kernel of these operators, using the fact that there are no conjugate points in the manifold). Thus, the pseudodifferential behavior of the operator Π_m is encapsulated by the first term whose kernel has a support living in a neighborhood of the diagonal in $M \times M$. In the following, $\varepsilon > 0$ is chosen small enough (less than the injectivity radius at the point x).

Let us consider a smooth section $f_1 \in C^\infty(M, \otimes_S^m T^*M)$ defined in a neighborhood of $x \in M$ and $f_2 \in \otimes_S^m T_x^*M$. Then

$$\langle \sigma_m(x_0, \xi) f_1, f_2 \rangle_{x_0} = \lim_{h \to 0} h^{-1} e^{-iS(x_0)/h} \langle \Pi_m(e^{iS(x)/h} f_1), f_2 \rangle_{x_0}$$
$$= \lim_{h \to 0} h^{-1} e^{-iS(x_0)/h} \langle \Pi \pi_m^*(e^{iS(x)/h} f_1), \pi_m^* f_2 \rangle_{L^2(S_{x_0}M)},$$

where $\xi = dS(x) \neq 0$. Here, it is assumed that $\mathrm{Hess}_x S$ is nondegenerate. We obtain

$$\langle \sigma_m(x, \xi) f_1, f_2 \rangle_{x_0}$$
$$= \lim_{h \to 0} h^{-1} \int_{\mathbb{S}^n} \int_{-\varepsilon}^{+\varepsilon} e^{i/h(S(\gamma(t))-S(x))} \pi_m^* f_1(\gamma(t), \dot{\gamma}(t)) \pi_m^* f_2(x_0, v) \chi(t) \, dt \, dv$$
$$= \lim_{h \to 0} h^{-1} \int_{\mathbb{S}^{n-1}} \left(\int_0^\pi \int_{-\varepsilon}^{+\varepsilon} e^{i/h(S(\gamma(t))-S(x))} \pi_m^* f_1(\gamma(t), \dot{\gamma}(t)) \pi_m^* f_2(x_0, v) \sin^{n-1}(\varphi) \chi(t) \, dt \, d\varphi \right) du,$$

where χ is a cutoff function with support in $(-\varepsilon, \varepsilon)$, γ is the geodesic such that $\gamma(0) = x$, $\dot{\gamma}(0) = v$, and we have decomposed $v = \cos(\varphi)n + \sin(\varphi)u$ with $n = \xi^\sharp/|\xi| = dS(x)^\sharp/|dS(x)|$, $u \in \mathbb{S}^{n-1}$. We apply the stationary phase lemma [Zworski 2012, Theorem 3.13] uniformly in the u-variable, where $u \in \mathbb{S}^{n-1}$. For fixed u, the phase is $\Phi : (t, \varphi) \mapsto S(\gamma(t)) - S(x)$ so $\partial_t \Phi(t, \varphi) = dS(\dot{\gamma}(t))$. More generally if $\widetilde{\Phi} : (t, v) \mapsto S(\gamma(t)) - S(x)$ denotes the map defined for any $v \in \mathbb{S}^n$, then

$$\partial_v \widetilde{\Phi}(t, v) \cdot V = d\pi(d\varphi_t(x, v) \cdot V) \quad \text{for all } V \in \mathbb{V},$$

where $\mathbb{V} = \ker d\pi_0$, with $\pi_0 : SM \to M$ the natural projection. Since (M, g) has no conjugate points, $d\pi(d\varphi_t(x, v)) \cdot V \neq 0$ as long as $t \neq 0$ and $V \in \mathbb{V} \setminus \{0\}$. And $dS(\dot{\gamma}(0)) = dS(\cos(\varphi)n + \sin(\varphi)u) = \cos(\varphi)|dS(x)| = 0$ if and only if $\varphi = \frac{\pi}{2}$. So the only critical point of Φ is $\left(t = 0, \varphi = \frac{\pi}{2}\right)$. Let us also remark that

$$\mathrm{Hess}_{(0,\pi/2)} \Phi = \begin{pmatrix} \mathrm{Hess}_x S(u, u) & -|dS(x)| \\ -|dS(x)| & 0 \end{pmatrix}$$

is nondegenerate with determinant $-|\xi|^2$, so the stationary phase lemma can be applied and we get

$$\int_0^\pi \int_{-\varepsilon}^{+\varepsilon} e^{i/h(S(\gamma(t)) - S(x_0))} \pi_m^* f_1(\gamma(t), \dot{\gamma}(t)) \pi_m^* f_2(x_0, v) \sin^{n-1}(\varphi) \, dt \, d\varphi$$
$$\sim_{h \to 0} 2\pi h |\xi|^{-1} \pi_m^* f_1(x_0, u) \pi_m^* f_2(x_0, u).$$

Eventually, we obtain

$$\langle \sigma_m(x, \xi) f_1, f_2 \rangle_{x_0} = \frac{2\pi}{|\xi|} \int_{\{\langle \xi, v \rangle = 0\}} \pi_m^* f_1(v) \pi_m^* f_2(v) \, dS_\xi(v),$$

where dS_ξ is the canonical measure induced on the $(n-1)$-dimensional sphere $\mathbb{S}_x M \cap \{\langle \xi, v \rangle = 0\}$. The sought result then follows from Lemma 2.1. $\qquad\square$

4C. Ellipticity, injectivity on solenoidal tensors.

Lemma 4.5. *The operator Π_m is elliptic on solenoidal tensors; that is, there exist pseudodifferential operators Q and R of respective orders 1 and $-\infty$ such that*

$$Q \Pi_m = \pi_{\ker D^*} + R.$$

Proof. We define

$$\tilde{q}(x, \xi) = \begin{cases} 0 & \text{on } \mathrm{ran}(\sigma j_\xi), \\ \frac{1}{2\pi} C_{n,m} |\xi| (\pi_{\ker i_\xi} \pi_{m*} \pi_m^* \pi_{\ker i_\xi})^{-1} & \text{on } \ker(i_\xi), \end{cases}$$

and $q(x, \xi) = (1 - \chi(x, \xi))\tilde{q}(x, \xi)$ for some cutoff function $\chi \in C_c^\infty(T^*M)$ around the zero section. By construction, $\mathrm{Op}(q)\Pi_m = \pi_{\ker D^*} - R'$ with $R' \in \Psi^{-1}$. Let $r' = \sigma_{R'}$ and define $a \sim \sum_{k=0}^\infty r'^k$. Then $\mathrm{Op}(a)$ is a microlocal inverse for $\mathbb{1} - R'$; that is, $\mathrm{Op}(a)(\mathbb{1} - R') \in \Psi^{-\infty}$. Since $R'D = 0$, we obtain that $R' = R' \pi_{\ker D^*}$ and thus

$$\underbrace{\mathrm{Op}(a) \, \mathrm{Op}(q)}_{= Q} \Pi_m = \mathrm{Op}(a)(\mathbb{1} - R') \pi_{\ker D^*} = \pi_{\ker D^*} + R,$$

where R is a smoothing operator. $\qquad\square$

From now on, we assume that the x-ray transform is injective on solenoidal tensors.

Lemma 4.6. *If I_m is solenoidal injective, then Π_m is injective on $H_{\mathrm{sol}}^s(M, \otimes_S^m T^*M)$ for all $s \in \mathbb{R}$.*

Proof. We fix $s \in \mathbb{R}$. We assume that $\Pi_m f = 0$ for some $f \in H_{\mathrm{sol}}^s(M, \otimes_S^m T^*M)$. By ellipticity of the operator, we get that $f \in C_{\mathrm{sol}}^\infty(M, \otimes_S^m T^*M)$, and

$$\langle \Pi_m f, f \rangle_{L^2} = \langle \Pi \pi_m^* f, \pi_m^* f \rangle_{L^2} + \left(\int_{SM} \pi_m^* f \, d\mu \right)^2$$

$$= \langle (1 + \Delta_m)^{-s} \Pi \pi_m^* f, \pi_m^* f \rangle_{H^s} + \left(\int_{SM} \pi_m^* f \, d\mu \right)^2 = 0.$$

Here, the Laplacian Δ_m is the one introduced in Section 2A3. The scalar product on H^s is $\langle f, h \rangle_{H^s} :=$ $\langle (1+\Delta_m)^{s/2} f, (1+\Delta_m)^{s/2} h \rangle_{L^2}$. By Lemma 4.3, since $\langle \Pi \pi_m^* f, \pi_m^* f \rangle \geq 0$, we obtain that $\int_{SM} \pi_m^* f \, d\mu = 0$. Moreover, $(1+\Delta_m)^{-s} \Pi$ is bounded and positive (hence self-adjoint) on H^s so there exists a square root $R : H^s \to H^s$, that is, a bounded positive operator satisfying $(1+\Delta_m)^{-s} \Pi = R^* R$, where R^* is the adjoint on H^s. Then

$$\langle (1+\Delta_m)^{-s} \Pi \pi_m^* f, \pi_m^* f \rangle_{H^s} = 0 = \| R \pi_m^* f \|_{H^s}^2.$$

This yields $(1+\Delta_m)^{-s} \Pi \pi_m^* f = 0$, so $\Pi \pi_m^* f = 0$. By Theorem 4.1, there exists $u \in C^\infty(SM)$ such that $\pi_m^* f = Xu$, so $f \in \ker I_m \cap \ker D^*$. By s-injectivity of the x-ray transform, we get $f \equiv 0$. $\qquad\square$

A direct consequence of Lemmas 4.6 and 4.5 is:

Theorem 4.7. *If I_m is solenoidal injective, then there exists a pseudodifferential operator Q' of order 1 such that $Q' \Pi_m = \pi_{\ker D^*}$.*

Proof. The operator Π_m is elliptic of order -1 on $\ker D^*$, and thus is Fredholm as an operator $H^s_{\mathrm{sol}}(M, \otimes_S^m T^* M) \to H^{s+1}_{\mathrm{sol}}(M, \otimes_S^m T^* M)$ for all $s \in \mathbb{R}$. It is self-adjoint on $H^{-1/2}_{\mathrm{sol}}(M, \otimes_S^m T^* M)$, and thus Fredholm of index 0 (the index being independent of the Sobolev space considered, see [Shubin 2001, Theorem 8.1]) and injective, and thus invertible on $H^s_{\mathrm{sol}}(M, \otimes_S^m T^* M)$. We multiply the equality $Q \Pi_m = \pi_{\ker D^*} + R$ on the right by $Q' := \pi_{\ker D^*} \Pi_m^{-1} \pi_{\ker D^*}$:

$$Q \Pi_m Q' = Q \underbrace{\Pi_m \pi_{\ker D^*}}_{= \Pi_m} \Pi_m^{-1} \pi_{\ker D^*} = Q \pi_{\ker D^*} = Q' + R Q'.$$

As a consequence, $Q' = Q \pi_{\ker D^*} +$ smoothing so it is a pseudodifferential operator of order 1, and $Q' \Pi_m = \pi_{\ker D^*}$. $\qquad\square$

This yields the following stability estimate:

Lemma 4.8. *If I_m is solenoidal injective, then for all $s \in \mathbb{R}$ there exists a constant $C := C(s) > 0$ such that,*

$$\text{for all } f \in H^s_{\mathrm{sol}}(M, \otimes_S^m T^* M), \quad \| f \|_{H^s} \leq C \| \Pi_m f \|_{H^{s+1}}.$$

4D. *Stability estimates for the x-ray transform.* Before going on with the proof of Theorem 1.5, let us recall the definition *Hölder–Zygmund spaces*. Let $\psi \in C_c^\infty(\mathbb{R})$ be a smooth cutoff function with support in $[-2, 2]$ and such that $\psi \equiv 1$ on $[-1, 1]$. For $j \in \mathbb{N}$, we introduce the functions $\varphi_j \in C_c^\infty(T^* M)$ defined by $\varphi_0(x, \xi) := \psi(|\xi|)$, $\varphi_j(x, \xi) := \psi(2^{-j} |\xi|) - \psi(2^{-j+1} |\xi|)$, for $j \geq 1$ with $(x, \xi) \in T^* M$, $|\cdot|$ being the norm induced by g on the cotangent bundle. Since φ_j is a symbol in $S^{-\infty}$, one observes that the operators $\mathrm{Op}(\varphi_j)$ are smoothing.

For $s \in \mathbb{R}$, we define $C_*^s(M)$, the *Hölder–Zygmund space of order s* as the completion of $C^\infty(M)$ with respect to the norm

$$\| u \|_{C_*^s} := \sup_{j \in \mathbb{N}} 2^{js} \| \mathrm{Op}(\varphi_j) u \|_{L^\infty},$$

and we recall (see [Taylor 1991, Appendix A, A.1.8] for instance) that a pseudodifferential operator $P \in \Psi^m(M)$ of order $m \in \mathbb{R}$ is bounded as an operator $C_*^{s+m}(M) \to C_*^s(M)$ for all $s \in \mathbb{R}$. Note that

the previous definition of Hölder–Zygmund spaces can be easily generalized to sections of a vector bundle. When $s \in (0, 1)$, it is a well-known fact that the space $C_*^s(M)$ coincides with $C^s(M)$, the space of Hölder-continuous functions, with equivalent norms $\|u\|_{C_*^s} \asymp \|u\|_{C^s}$. The Hölder–Zygmund spaces correspond to the Besov spaces $B_{q,r}^s(M)$, with $q = r = +\infty$, while the Sobolev spaces $H^s(M)$ correspond to the choice $q = r = 2$. Here

$$\|u\|_{B_{q,r}^s} := \left(\sum_{j=0}^{+\infty} \|2^{sj} \, \mathrm{Op}(\varphi_j)u\|_{L^q}^r \right)^{1/r}$$

In particular, Lemma 4.8 can be upgraded to:

Lemma 4.9. *If I_m is solenoidal injective, then for all $s \in \mathbb{R}$ there exists a constant $C := C(s) > 0$ such that*

$$\text{for all } f \in C_{*,\mathrm{sol}}^s(M, \otimes_S^m T^*M), \quad \|f\|_{C_*^s} \leq C \|\Pi_m f\|_{C_*^{s+1}}.$$

Eventually, we will need this last result:

Lemma 4.10. *For all $s > 0$, the operator $\Pi : C_*^s(SM) \to C_*^{-s-(n+1)/2}(SM)$ is bounded.*

Proof. Fix $\varepsilon > 0$ small enough. Then

$$C_*^s \hookrightarrow H^{s-\varepsilon} \xrightarrow{\Pi} H^{-s+\varepsilon} \hookrightarrow C_*^{-s-(n+1)/2+\varepsilon} \hookrightarrow C_*^{-s-(n+1)/2}$$

by Sobolev embeddings. □

We can now deduce from the previous work the stability estimate of Theorem 1.5.

Proof of Theorem 1.5. We assume that $f \in C_{\mathrm{sol}}^\alpha(M, \otimes_S^m T^*M)$ is such that $\|f\|_{C^\alpha} \leq 1$. By Theorem 1.2, we can write $\pi_m^* f = Xu + h$, with $u, Xu, h \in C^{\alpha'}$, where $0 < \alpha' < \alpha$ and $\|h\|_{C^{\alpha'}} \lesssim \|I_m f\|_{\ell^\infty}^\tau$.

We have

$$\|f\|_{C_*^{-1-\alpha'-(n+1)/2}} \lesssim \|\Pi_2 f\|_{C_*^{-\alpha'-(n+1)/2}} \qquad \text{(by Lemma 4.9)}$$

$$\lesssim \|\Pi \pi_2^* f\|_{C_*^{-\alpha'-(n+1)/2}} \lesssim \|\Pi(Xu+h)\|_{C_*^{-\alpha'-(n+1)/2}} \lesssim \|\Pi h\|_{C_*^{-\alpha'-(n+1)/2}}$$

$$\lesssim \|h\|_{C^{\alpha'}} \qquad \text{(by Lemma 4.10)}$$

$$\lesssim \|I_2 f\|_{\ell^\infty}^\tau \qquad \text{(by Theorem 1.2).}$$

Using $\|f\|_{C^\alpha} \leq 1$ and interpolating C^β between $C^{-1-\alpha'-(n+1)/2}$ and C^α, one obtains the desired result. □

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References

[Burns and Katok 1985] K. Burns and A. Katok, "Manifolds with nonpositive curvature", *Ergodic Theory Dynam. Systems* **5**:2 (1985), 307–317. MR Zbl

[Croke and Sharafutdinov 1998] C. B. Croke and V. A. Sharafutdinov, "Spectral rigidity of a compact negatively curved manifold", *Topology* **37**:6 (1998), 1265–1273. MR Zbl

[Dairbekov and Sharafutdinov 2003] N. S. Dairbekov and V. A. Sharafutdinov, "Some problems of integral geometry on Anosov manifolds", *Ergodic Theory Dynam. Systems* **23**:1 (2003), 59–74. MR Zbl

[Dairbekov and Sharafutdinov 2010] N. S. Dairbekov and V. A. Sharafutdinov, "Conformal Killing symmetric tensor fields on Riemannian manifolds", *Mat. Tr.* **13**:1 (2010), 85–145. MR Zbl

[Faure and Sjöstrand 2011] F. Faure and J. Sjöstrand, "Upper bound on the density of Ruelle resonances for Anosov flows", *Comm. Math. Phys.* **308**:2 (2011), 325–364. MR Zbl

[Guillarmou 2017] C. Guillarmou, "Invariant distributions and x-ray transform for Anosov flows", *J. Differential Geom.* **105**:2 (2017), 177–208. MR Zbl

[Guillarmou and Lefeuvre 2019] C. Guillarmou and T. Lefeuvre, "The marked length spectrum of Anosov manifolds", *Ann. of Math.* (2) **190**:1 (2019), 321–344. MR Zbl

[Guillemin and Kazhdan 1980a] V. Guillemin and D. Kazhdan, "Some inverse spectral results for negatively curved 2-manifolds", *Topology* **19**:3 (1980), 301–312. MR Zbl

[Guillemin and Kazhdan 1980b] V. Guillemin and D. Kazhdan, "Some inverse spectral results for negatively curved *n*-manifolds", pp. 153–180 in *Geometry of the Laplace operator* (Honolulu, HI, 1979), Proc. Sympos. Pure Math. **36**, Amer. Math. Soc., Providence, RI, 1980. MR Zbl

[Hasselblatt and Fisher 2019] B. Hasselblatt and T. Fisher, *Hyperbolic flows*, European Mathematical Society, Zürich, 2019. Zbl

[Katok 1990] S. Katok, "Approximate solutions of cohomological equations associated with some Anosov flows", *Ergodic Theory Dynam. Systems* **10**:2 (1990), 367–379. MR Zbl

[Katok and Hasselblatt 1995] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications **54**, Cambridge University Press, Cambridge, 1995. MR Zbl

[Klingenberg 1974] W. Klingenberg, "Riemannian manifolds with geodesic flow of Anosov type", *Ann. of Math.* (2) **99** (1974), 1–13. MR Zbl

[Liverani 2004] C. Liverani, "On contact Anosov flows", *Ann. of Math.* (2) **159**:3 (2004), 1275–1312. MR Zbl

[Livšic 1972] A. N. Livšic, "Cohomology of dynamical systems", *Izv. Akad. Nauk SSSR Ser. Mat.* **36** (1972), 1296–1320. In Russian; translated in *Math. USSR-Izv.* **6**:6 (1972), 1278–1301. MR Zbl

[de la Llave et al. 1986] R. de la Llave, J. M. Marco, and R. Moriyón, "Canonical perturbation theory of Anosov systems and regularity results for the Livšic cohomology equation", *Ann. of Math.* (2) **123**:3 (1986), 537–611. MR Zbl

[Lopes and Thieullen 2005] A. O. Lopes and P. Thieullen, "Sub-actions for Anosov flows", *Ergodic Theory Dynam. Systems* **25**:2 (2005), 605–628. MR Zbl

[Parry 1988] W. Parry, "Equilibrium states and weighted uniform distribution of closed orbits", pp. 617–625 in *Dynamical systems* (College Park, MD, 1986–87), edited by J. C. Alexander, Lecture Notes in Math. **1342**, Springer, 1988. MR Zbl

[Paternain, Salo, and Uhlmann 2014] G. P. Paternain, M. Salo, and G. Uhlmann, "Spectral rigidity and invariant distributions on Anosov surfaces", *J. Differential Geom.* **98**:1 (2014), 147–181. MR Zbl

[Sharafutdinov 1994] V. A. Sharafutdinov, *Integral geometry of tensor fields*, VSP, Utrecht, 1994. MR Zbl

[Sharafutdinov, Skokan, and Uhlmann 2005] V. Sharafutdinov, M. Skokan, and G. Uhlmann, "Regularity of ghosts in tensor tomography", *J. Geom. Anal.* **15**:3 (2005), 499–542. MR

[Shubin 2001] M. A. Shubin, *Pseudodifferential operators and spectral theory*, 2nd ed., Springer, 2001. MR Zbl

[Taylor 1991] M. E. Taylor, *Pseudodifferential operators and nonlinear PDE*, Progress in Mathematics **100**, Birkhäuser, Boston, MA, 1991. MR Zbl

[Zworski 2012] M. Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics **138**, Amer. Math. Soc., Providence, RI, 2012. MR Zbl

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