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Ann. I. H. Poincaré – PR 41 (2005) 997–1024
ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.elsevier.com/locate/anihpb

Berry–Esseen theorem and local limit theorem for non uniformly expanding maps

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Received 28 June 2004; accepted 24 September 2004

Available online 29 January 2005

Abstract

In Young towers with sufficiently small tails, the Birkhoff sums of Hölder continuous functions satisfy a central limit theorem with speed $O(1/\sqrt{n})$, and a local limit theorem. This implies the same results for many non uniformly expanding dynamical systems, namely those for which a tower with sufficiently fast returns can be constructed.

Résumé

Dans les tours de Young ayant des queues suffisamment petites, les sommes de Birkhoff des fonctions hölderiennes satisfont le théorème central limite avec vitesse $O(1/\sqrt{n})$ et le théorème de la limite locale. Par conséquent, de nombreux systèmes dynamiques non uniformément dilatants satisfont les mêmes conclusions : il suffit de pouvoir construire une tour avec des retours à la base suffisamment rapides.

MSC: 37A30; 37A50; 37C30; 37E05; 47A56; 60F15

Keywords: Non uniformly expanding maps; Berry–Esseen theorem; Local limit theorem

1. Results

1.1. Introduction

Let $T : X \to X$ be a probability preserving transformation and $f : X \to \mathbb{R}$. The functions $f \circ T^k$, for $k \in \mathbb{N}$, are identically distributed random variables, and it is an important problem in ergodic theory to see whether they satisfy the same kind of limit theorems as independent random variables.

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0246-0203/$ – see front matter
doi:10.1016/j.anihpb.2004.09.002

Many results are known when T is uniformly expanding or uniformly hyperbolic (without or with singularities, in the Markov or non Markov case), and f is Hölder continuous. In this case, it is indeed often possible to construct a space of functions containing f on which the transfer operator associated to T has a spectral gap. Therefore, the spectral perturbation method, introduced by Nagaev in the case of Markov chains, makes it possible to mimic the probabilistic proofs on independent variables. In this way, it is possible to get distributional convergence (to normal laws or stable laws), and more subtle results such as the speed of convergence (also called the Berry-Esseen theorem) or the local limit theorem (see for example [30,17,8,3]). These results, in turn, have important consequences concerning the asymptotic behavior of the system [2,33].

On the other hand, when the system is not uniformly expanding or uniformly hyperbolic, it is not possible to use directly the aforementioned spectral method. Consequently, other methods have been devised to handle the distributional convergence of Birkhoff sums. Among many techniques, the most flexible one is probably the martingale argument of Gordin (see for example [23,26,10,11,36]). Some results have also been obtained on the speed in the central limit theorem, by direct estimates (see [22,27]). However, there is currently no result concerning the local limit theorem, which is not surprising since the proof of this theorem requires a heavy Fourier machinery, even in the probabilistic case, and is not easily accessible to elementary methods.

The aim of this article is to prove the local limit theorem and the Berry–Esseen theorem for Hölder functions in the setting of *Young towers* [35], where the decay of correlations is not exponential and the transfer operator has no spectral gap. The Young towers are abstract spaces which can be used to model many non uniformly expanding maps, for example the Pomeau–Manneville maps in dimension 1 studied by Liverani, Saussol and Vaienti [24], the Viana map (for which a tower is built in [5]), or the unimodal maps for which the critical point does not return too quickly close to itself [9]. Thus, all these maps also satisfy the local limit theorem, and the central limit theorem with speed $O(1/\sqrt{n})$. These results also apply in non uniformly hyperbolic settings, with the techniques of [34].

The proof is spectral: it uses perturbations of transfer operators, as in [17], but applied to first return transfer operators associated to an induced map, as defined by Sarig in [32]. The method is related to [14], with a more systematic use of Banach algebra techniques.

1.2. Results in Young towers

A *Young tower* [35] is a probability space (X, m) with a partition $(B_{i,j})_{i\in I, j<\varphi_i}$ of X by positive measure subsets, where I is finite or countable and $\varphi_i \in \mathbb{N}^*$, together with a nonsingular map $T : X \to X$ satisfying the following properties.

1. $\forall i \in I, \forall 0 \leqslant j < \varphi_i - 1$, T is a measure preserving isomorphism between $B_{i,j}$ and $B_{i,j+1}$.
2. For every $i \in I$, T is an isomorphism between B_{i,φ_i-1} and $B := \bigcup_{k\in I} B_{k,0}$.
3. Let φ be the function equal to φ_i on $B_{i,0}$, whence T^φ is a function from B to itself. Let $s(x, y)$ be the separation time of the points x and $y \in B$ under T^φ, i.e. $s(x, y) = \inf\{n \mid \exists i \neq j,\ (T^\varphi)^n(x) \in B_{i,0},\ (T^\varphi)^n(y) \in B_{j,0}\}$.
   As T^{φ_i} is an isomorphism between $B_{i,0}$ and B, it is possible to consider the inverse g_m of its jacobian with respect to the measure m. We assume that there exist constants $\beta < 1$ and $C > 0$ such that $\forall x, y \in B_{i,0}$, $|\log g_m(x) - \log g_m(y)| \leqslant C\beta^{s(x,y)}$.
4. The map T preserves the measure m.
5. The partition $\bigvee_0^\infty T^{-n}((B_{i,j}))$ separates the points.

The notion of Young tower has been introduced by Young in [34,35] as a model for non uniformly expanding dynamical systems. The non uniformity is measured by the *size of tails* $m\{x \in B \mid \varphi(x) > n\}$: if this quantity is very small, then most points enjoy some expansion before time n, when they first return to the basis. This expansion, in turn, is sufficient to study statistical properties of the system, including decay of correlations. Young has proved that, if $m[\varphi > n] = O(1/n^\beta)$ for some $\beta > 1$, then the correlations of sufficiently regular functions (see the definition of $C_\tau(X)$ below) decay like $O(1/n^{\beta-1})$. In particular, if $\beta > 2$, these correlations are summable, and a martingale method can be used to prove that a central limit theorem holds.

We extend the separation time s to the whole tower, by setting $s(x, y) = 0$ if x and y are not in the same set $B_{i,j}$, and $s(x, y) = s(x', y') + 1$ otherwise, where x' and y' are the next iterates of x and y in B. For $0 < \tau < 1$, set

$$C_\tau(X) = \left\{ f : X \to \mathbb{R} \mid \exists C > 0,\ \forall x, y \in X,\ \left| f(x) - f(y) \right| \leqslant C\tau^{s(x,y)} \right\}.$$

This space has a norm $\|f\|_\tau = \inf\{C \mid \forall x, y \in X, |f(x) - f(y)| \leqslant C\tau^{s(x,y)}\} + \|f\|_\infty$.

The following theorem is well known and can for example be proved using martingale techniques (see [35, Theorem 4]).

Theorem 1.1. *Let $\tau < 1$. Assume that $m[\varphi > n] = O(1/n^\beta)$ with $\beta > 2$. Let $f \in C_\tau(X)$ have a vanishing integral. Then there exists $\sigma^2 \geqslant 0$ such that*

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f \circ T^k \to \mathcal{N}(0, \sigma^2).$$

Moreover, $\sigma^2 = 0$ if and only if f is a coboundary, i.e. there exists a measurable function g such that $f = g - g \circ T$ almost everywhere.

The main results of this article are Theorems 1.2 and 1.3. To formulate the first one, we will need the following definition:

Definition. A map $f : X \to \mathbb{R}$ is *periodic* if there exist $\rho \in \mathbb{R}$, $g : X \to \mathbb{R}$ measurable, $\lambda > 0$ and $q : X \to \mathbb{Z}$, such that $f = \rho + g - g \circ T + \lambda q$ almost everywhere. Otherwise, it is *aperiodic*.

Theorem 1.2 (local limit theorem). *Let $\tau < 1$. Assume that $m[\varphi > n] = O(1/n^\beta)$ with $\beta > 2$. Let $f \in C_\tau(X)$ have a vanishing integral, and let σ^2 be given by Theorem 1.1.*

Assume that f is aperiodic. This implies in particular $\sigma^2 > 0$. Then, for any bounded interval $J \subset \mathbb{R}$, for any real sequence k_n with $k_n/\sqrt{n} \to \kappa \in \mathbb{R}$, for any $u \in C_\tau(X)$, for any $v : X \to \mathbb{R}$ measurable,

$$\sqrt{n}\, m\left\{ x \in X \mid S_n f(x) \in J + k_n + u(x) + v(T^n x) \right\} \to |J| \frac{\mathrm{e}^{-\kappa^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}}.$$

The function on the right is the density of $\mathcal{N}(0, \sigma^2)$: this theorem (for $u = v = 0$ and $k_n = \kappa\sqrt{n}$) means that

$$m\left\{ \frac{1}{\sqrt{n}} S_n f \in \kappa + \frac{J}{\sqrt{n}} \right\} \sim P\left(\mathcal{N}(0, \sigma^2) \in \kappa + \frac{J}{\sqrt{n}} \right).$$

Hence, it shows that $S_n f/\sqrt{n}$ behaves like $\mathcal{N}(0, \sigma^2)$ at the local level (contrary to Theorem 1.1 which deals with the global level). It is important that f is aperiodic. Otherwise, f could be integer valued, and the theorem could not hold, e.g. for $k_n = 0$, $u = v = 0$ and $J = [1/3, 2/3]$.

For $f : X \to \mathbb{R}$, define a function f_B on B by

$$f_B(x) = \sum_{k=0}^{\varphi(x)-1} f(T^k x). \tag{1}$$

In the probabilistic case, the Berry–Esseen theorem, giving the speed of convergence in the central limit theorem, holds under an L^3 moment condition [12]. In the dynamical setting, we will need the same kind of hypothesis, but on the function f_B. Note that, since $|f_B| \leqslant \|f\|_\infty \varphi$ and $\beta > 2$, we always have $f_B \in L^2(B)$.

Theorem 1.3 (speed in the central limit theorem). *Let $\tau < 1$. Assume that $m[\varphi > n] = O(1/n^\beta)$ with $\beta > 2$. Let $f \in C_\tau(X)$ have a vanishing integral, and σ^2 be given by Theorem* 1.1.

Assume that $\sigma^2 > 0$, and that there exists $0 < \delta \leqslant 1$ such that $\int |f_B|^2 1_{|f_B|>z}\,\mathrm{d}m = O(z^{-\delta})$ when $z \to \infty$. If $\delta = 1$, assume also that $\int f_B^3 1_{|f_B|\leqslant z}\,\mathrm{d}m = O(1)$. Then there exists $C > 0$ such that $\forall n \in \mathbb{N}^*$, $\forall a \in \mathbb{R}$,

$$\left| m\left\{ x \;\Big|\; \frac{1}{\sqrt{n}} S_n f(x) \leqslant a \right\} - P\big(\mathcal{N}(0, \sigma^2) \leqslant a\big) \right| \leqslant \frac{C}{n^{\delta/2}}.$$

When $f_B \in L^p$ for some $2 < p \leqslant 3$, then the conditions of the theorem are satisfied for $\delta = p - 2$. In particular, when $f_B \in L^3$, we obtain a convergence with speed $O(1/\sqrt{n})$, which is the usual Berry–Esseen theorem. Note also that, for any $f \in C_\tau(X)$, the conditions of the theorem are satisfied for $\delta = \beta - 2$ if $2 < \beta < 3$, and for $\delta = 1$ if $\beta > 3$. The formulation we have given is more precise than the usual Berry–Esseen theorem, in view of the applications, where an L^p condition would not be optimal (see for example Theorem 1.5). In fact, the conditions of the theorem on f_B correspond to necessary and sufficient conditions to get a central limit theorem with speed $O(n^{-\delta/2})$ in the probabilistic (independent identically distributed) setting, as shown in [19, Theorem 3.4.1].

Remark. Using the same methods, it is possible to prove the same results in a more general setting, namely maps for which a first return map is *Gibbs–Markov* in the sense of [1]. For the sake of simplicity, we will only consider Young towers.

1.3. Applications

1.3.1. General setting

Let (X, d) be a locally compact separable metric space, endowed with a Borel probability measure μ, and $T : X \to X$ a nonsingular map for which μ is ergodic. Assume that there exist a bounded subset B of X with $\mu(B) > 0$, a finite or countable partition (mod 0) $(B_i)_{i\in I}$ of B, with $\mu(B_i) > 0$, and integers $\varphi_i > 0$ such that:

1. $\forall i \in I$, T^{φ_i} is an isomorphism between B_i and B.
2. $\exists\lambda > 1$ such that, $\forall i \in I$, $\forall x, y \in B_i$, $d(T^{\varphi_i}x, T^{\varphi_i}y) \geqslant \lambda d(x, y)$.
3. $\exists C > 0$ such that, $\forall i \in I$, $\forall x, y \in B_i$, $\forall k < \varphi_i$, $d(T^k x, T^k y) \leqslant C d(T^{\varphi_i}x, T^{\varphi_i}y)$.
4. $\exists\theta > 0$ and $D > 0$ such that, $\forall i \in I$, the jacobian g_μ defined on B_i by $g_\mu(x) = \mathrm{d}\mu/\mathrm{d}(\mu \circ T^{\varphi_i}_{|B_i})$ satisfies: for all $x, y \in B_i$, $|\log g_\mu(x) - \log g_\mu(y)| \leqslant D d(T^{\varphi_i}x, T^{\varphi_i}y)^\theta$.

Denote by φ the function on B equal to φ_i on each B_i. If $\mu\{x \mid \varphi(x) > n\}$ is summable, we can define a space $X' = \{(y, j) \mid y \in B,\ j < \varphi(x)\}$, and a map $T' : X' \to X'$ by $T'(y, j) = (y, j + 1)$ if $j < \varphi(x) - 1$ and $T'(y, j) = (T^{\varphi(y)}(y), 0)$ otherwise. Define also $\pi : X' \to X$ by $\pi(y, j) = T^j(y)$. Then $\pi \circ T' = T \circ \pi$.

Set $\mu' = \sum_{n=0}^\infty T'^n_*(\mu|B \cap \{\varphi > n\})$: it is a measure of finite mass on X', not necessarily T'-invariant. Young has proved in [35, Theorem 1] that there exists a unique invariant probability measure m' on X' which is absolutely continuous with respect to μ'. It is ergodic, and (X', T', μ') is a Young tower in the sense of Section 1.2. The measure $m = \pi_*(m')$ is T-invariant, absolutely continuous and ergodic.

If $f : X \to \mathbb{R}$ is Hölder continuous, then $f' := f \circ \pi : X' \to \mathbb{R}$ belongs to $C_\tau(X')$ for τ close enough to 1. Moreover, the Birkhoff sums $\sum_{k=0}^{n-1} f \circ T^k$ and $\sum_{k=0}^{n-1} f' \circ T'^k$ have the same distribution with respect respectively to m and m'. Hence, Theorems 1.1, 1.2 and 1.3 on the function f' in the Young tower (X', T', m') imply the same results on the function f in (X, T, m).

To apply these theorems, we have to check their assumptions. The condition $m[\varphi > n] = O(1/n^\beta)$ with $\beta > 2$ corresponds simply to the requirement

$$\sum_{\varphi_i > n} \mu(B_i) = O\left(\frac{1}{n^\beta}\right) \quad \text{for some } \beta > 2.$$

To apply Theorem 1.2, we additionally have to check that the function f' is aperiodic for T', which can be complicated when the extension X' is not explicitly described. On the other hand, the aperiodicity of f may be easier to check, using for example the information at the periodic points. In this case, the following abstract theorem ensures that f' is automatically aperiodic, whence we can apply Theorem 1.2.

Theorem 1.4. *Let $T': X' \to X'$ be a probability preserving map on a probability space (X', m'). Let (X, m) be a standard probability space, $T: X \to X$ an ergodic probability preserving map, and $\pi: X' \to X$ a map with countable fibers, such that $m = \pi_*(m')$ and $T \circ \pi = \pi \circ T'$. Let $f: X \to \mathbb{R}$. Then*

- *The function f is a coboundary for T if and only the function $f \circ \pi$ is a coboundary for T'.*
- *The function f is aperiodic for T if and only if the function $f \circ \pi$ is aperiodic for T'.*

1.3.2. Examples

Recently, many maps have been shown to fit in the previous setting. For example, [5, Theorem 3] shows that the Alves–Viana map, given by

$$T: \begin{cases} S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}, \\ (\omega, x) \mapsto \big(16\omega, a - x^2 + \varepsilon \sin(2\pi\omega)\big) \end{cases}$$

satisfies these assumptions (for any $\beta > 2$) when 0 is preperiodic for the map $x \mapsto a - x^2$, and ε is small enough. In fact, any map close enough to T in the C^3-topology also satisfies them.

In the one-dimensional case, [9] shows that many unimodal maps of the interval also satisfy these hypotheses: it is sufficient that the returns of the critical point close to itself occur at a slow enough rate.

Finally, we will discuss with more details the case of the Pomeau–Manneville maps, studied among many others by Liverani, Saussol and Vaienti [24]. They form an interesting class of applications, since the influence of the fixed point 0 becomes more and more important when α increases. The explicit formula (2) is not important, what matters is only the local behavior around the fixed point. Hence, all the following results can be extended to a much larger class of examples but, for the sake of simplicity, we will only consider the following maps.

Let $\alpha \in (0, 1/2)$, and consider $T: [0,1] \to [0,1]$ given by

$$T(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } 0 \leqslant x \leqslant 1/2, \\ 2x - 1 & \text{if } 1/2 < x \leqslant 1. \end{cases} \tag{2}$$

This map has a parabolic fixed point at 0, and is expanding elsewhere. It has a unique absolutely continuous invariant probability measure m, whose density is Lipschitz on any interval of the form $(\varepsilon, 1]$ [24, Lemma 2.3].

Theorem 1.5. *Let $0 < \alpha < 1/2$, and let $f: [0,1] \to \mathbb{R}$ be a Hölder function with vanishing integral, which cannot be written as $g - g \circ T$. Then f satisfies a central limit theorem with variance $\sigma^2 > 0$.*

- *If $\alpha < 1/3$, or $f(0) = 0$ and there exists $\gamma > \alpha - 1/3$ such that $|f(x)| \leqslant K x^\gamma$, then there exists $C > 0$ such that $\forall n \in \mathbb{N}^*$, $\forall a \in \mathbb{R}$,*

$$\left| m\left\{ x \;\Big|\; \frac{1}{\sqrt{n}} S_n f(x) \leqslant a \right\} - P\big(\mathcal{N}(0, \sigma^2) \leqslant a\big) \right| \leqslant \frac{C}{\sqrt{n}}.$$

- *If $1/3 < \alpha < 1/2$, $f(0) = 0$ and there exists $\gamma > 0$ such that $|f(x)| \leqslant K x^\gamma$ and $\delta := \frac{1}{\alpha - \gamma} - 2 \in (0, 1)$, then there exists $C > 0$ such that $\forall n \in \mathbb{N}^*$, $\forall a \in \mathbb{R}$,*

$$\left| m\left\{ x \;\Big|\; \frac{1}{\sqrt{n}} S_n f(x) \leqslant a \right\} - P\big(\mathcal{N}(0, \sigma^2) \leqslant a\big) \right| \leqslant \frac{C}{n^{\delta/2}}.$$

- *If* $1/3 < \alpha < 1/2$ *and* $f(0) \neq 0$, *then there exists* $C > 0$ *such that* $\forall n \in \mathbb{N}^*$, $\forall a \in \mathbb{R}$,

$$\left| m\left\{x \;\middle|\; \frac{1}{\sqrt{n}} S_n f(x) \leqslant a\right\} - P\big(\mathcal{N}(0,\sigma^2) \leqslant a\big)\right| \leqslant \frac{C}{n^{(1/2\alpha)-1}}.$$

Moreover, if f *is aperiodic, it satisfies the local limit theorem.*

Proof. Let $x_0 = 1$, and x_{n+1} be the preimage of x_n in $[0, 1/2]$. Let y_{n+1} be the preimage of x_n in $(1/2, 1]$: the intervals $B_n = (y_{n+1}, y_n]$ form a partition of $B = (1/2, 1]$ and, if $\varphi_n = n$, all the hypotheses of Section 1.3 are satisfied. Moreover, $m(B_n) \sim \frac{C}{n^{1/\alpha+1}}$ and $x_n \sim \frac{D}{n^{1/\alpha}}$ for constants $C, D > 0$ [24]. In particular, $m(\varphi > n) = O(1/n^\beta)$ for $\beta = 1/\alpha > 2$.

Let f be Hölder on $[0, 1]$. If $f(0) \neq 0$, then $f_B = nf(0) + o(n)$ on B_n. Otherwise, let $\gamma > 0$ be such that $|f(x)| \leqslant Kx^\gamma$. Reducing γ if necessary, we can assume that $\gamma < \alpha$. Then it is easy to check that $|f_B| \leqslant Cn^{1-\gamma/\alpha}$ on B_n.

Using these estimates, we can check the integrability assumptions of Theorem 1.3 for $\delta = 1$ in the first case, $\frac{1}{\alpha-\gamma} - 2$ in the second case, and $\frac{1}{\alpha} - 2$ in the third case. Hence, Theorem 1.3 implies the desired estimates on the speed in the central limit theorem.

Finally, the local limit theorem is a direct consequence of Theorem 1.2. □

The aperiodicity assumption is a priori not easy to check, since the periodicity equality $f = g - g \circ T + \rho + \lambda q$ is assumed to hold only almost everywhere. However, under suitable regularity assumptions on f, it is possible to prove that this equality holds everywhere (see e.g. [3] for locally constant f, [15] for Hölder f). For example, if T is given by (2), then $f = \log|T'| - \int \log|T'|$ is aperiodic.

In Section 2, we will prove Theorem 1.4, and show that it is sufficient to prove Theorems 1.2 and 1.3 in mixing Young towers (i.e., such that the return times φ_i satisfy $\gcd(\varphi_i) = 1$). The rest of paper is devoted to the proof of these theorems. In Section 3, we prove an abstract spectral result on perturbations of series of operators. In Section 4, we apply this result to first return transfer operators, to get the key result Theorem 4.6. We then use this estimate in the last two sections to prove respectively the local limit Theorem 1.2 and the Berry–Esseen Theorem 1.3.

2. Preliminary reductions

2.1. Proof of Theorem 1.4

Proof of the coboundary result. If f is a coboundary, i.e. $f = g - g \circ T$, then $f' := f \circ \pi$ can be written as $f' = g' - g' \circ T'$, where $g' = g \circ \pi$. However, the converse is not immediate: if $f' = g' - g' \circ T'$, the function g' is a priori not constant on the fibers $\pi^{-1}(x)$, which prevents us from writing $g' = g \circ \pi$.

We use the following characterization of coboundaries: *Let* T *be an endomorphism of a probability space* (X, m). *Then a measurable function* f *on* X *can be written as* $g - g \circ T$ *if and only if*

$$\forall \varepsilon > 0, \exists C > 0,\ \forall n \geqslant 1, \quad m\big\{x \in X \mid \big|S_n f(x)\big| \geqslant C\big\} \leqslant \varepsilon. \tag{3}$$

This characterization, due to Schmidt, is proved for example in [4].

If f' is a coboundary, then (3) is satisfied by f' in X', whence it is also satisfied by f in X (since this condition only involves distributions). Thus, f can be written as $g - g \circ T$. □

Proof of the aperiodicity result. If f is periodic on X, i.e. $f = \rho + g - g \circ T + \lambda q$ where q is integer-valued, then $f \circ \pi = \rho + (g \circ \pi) - (g \circ \pi) \circ T' + \lambda(q \circ \pi)$, i.e. $f \circ \pi$ is periodic. On the other hand, if $f \circ \pi = \rho' + g' - g' \circ T' + \lambda q'$, it is not necessarily possible to write directly $g' = g \circ \pi$. The proof of the periodicity of f will use ideas of [4]. We can assume for example that $\lambda = 2\pi$. Replacing m' by one of its ergodic components, we can also assume that m' is ergodic.

Since the projection π has countable fibers, there exists a measurable subset A of X' such that π is an isomorphism between A and X, and $m'(A) > 0$. Define a function $\tilde{g}$ on X by $\tilde{g}(x) = g'(x')$, where x' is the unique preimage of x in A. Replacing f by $f - \tilde{g} + \tilde{g} \circ T - \rho'$, and g' by $g' - \tilde{g} \circ \pi$, we can assume without loss of generality that $g' = 0$ on A and $\rho' = 0$.

For $x \in X$, let $W_n(x)$ be the measure on S^1 given by

$$W_n(x) = \frac{1}{n} \sum_{k=1}^{n} \delta\left(e^{iS_k f(x)}\right)$$

where $\delta(y)$ is the Dirac mass at y. For $u \in C^0(S^1)$, it is possible by compactness to find a subsequence n_k such that

$$\int_{S^1} u \, dW_{n_k}(x) \to L(u)(x) \quad \text{weak} * \text{ in } L^\infty(X).$$

It is possible to obtain this convergence for a dense countable set of functions in $C^0(S^1)$, by a diagonal argument. By passing to a further subsequence, it is also possible to guarantee that $\frac{1}{n} \sum_{k=1}^{n} \int_{S^1} u \, dW_{n_k}(x) \to L(u)(x)$ on a set $Y \subset X$ with $m(Y) = 1$, by Komlos' Theorem [21]. By density, we get the same convergence for any $u \in C^0(S^1)$.

For $x \in Y$, the map $u \in C^0(S^1) \mapsto L(u)(x) \in \mathbb{R}$ is a nonnegative continuous linear functional sending 1 to 1, thus given by a probability measure P_x. Moreover, these measures satisfy $P_{Tx}(S) = P_x(e^{if(x)} S)$ for any Borel subset S of S^1, since $W_n(Tx)(S) = W_n(x)(e^{if(x)} S) \pm \frac{2}{n}$.

For some $\varepsilon > 0$, we will prove that

$$m\left\{x \mid P_x\left(\{1\}\right) \geqslant \varepsilon\right\} \geqslant \varepsilon. \tag{4}$$

If $x' \in A \cap T'^{-k}(A)$, then $e^{iS_k f \circ \pi(x')} = e^{i(g'(x') - g' \circ T'^k(x'))} = 1$, i.e. $S_k f \circ \pi(x') \in 2\pi\mathbb{Z}$. Hence,

$$\begin{aligned}\int_X W_n(x)\left(\{1\}\right) dm(x) &= \frac{1}{n} \sum_{k=1}^{n} \int_X 1\left(S_k f(x) \in 2\pi\mathbb{Z}\right) dm(x) \geqslant \frac{1}{n} \sum_{k=1}^{n} \int_{X'} 1(A \cap T'^{-k} A) \, dm'(x') \\ &= \int_{X'} 1_A \cdot \left(\frac{1}{n} \sum_{k=1}^{n} 1_A \circ T'^k\right) dm'(x') \to m'(A)^2 > 0\end{aligned}$$

by Birkhoff Theorem. Thus, for large enough n, $\int_X W_n(x)(\{1\}) \geqslant 3\varepsilon > 0$, whence $\int (\frac{1}{n} \sum_{k=1}^{n} W_{n_k}(x))(\{1\}) \geqslant 2\varepsilon$ for large enough n. Since $(\frac{1}{n} \sum_{k=1}^{n} W_{n_k}(x))(\{1\}) \leqslant 1$, we get

$$m\left\{x \,\middle|\, \left(\frac{1}{n} \sum_{k=1}^{n} W_{n_k}(x)\right)\left(\{1\}\right) \geqslant \varepsilon\right\} \geqslant \varepsilon.$$

Thus, the set $C = \{x \mid \limsup(\frac{1}{n} \sum_{k=1}^{n} W_{n_k}(x))(\{1\}) \geqslant \varepsilon\}$ satisfies $m(C) \geqslant \varepsilon$. Finally, $P_x(\{1\}) \geqslant \varepsilon$ on C, and this proves (4).

Define a measure μ on $X \times S^1$, by $\mu(U \times V) = \int_U P_x(V) \, dm(x)$. Then μ is invariant under the action of $T_f : (x, y) \mapsto (T(x), e^{-if(x)} y)$, and [4] proves that, for almost every ergodic component P of μ, there exists a compact subgroup H of S^1 and a map $\omega : X \to S^1$ such that, denoting by m_H the Haar measure of H,

$$\forall U \times V \subset X \times S^1, \quad P(U \times V) = \int_U m_H\left(\omega(x) V\right) dm(x).$$

Moreover, in this case, it is possible to write $e^{if(x)} = \frac{\omega(Tx)}{\omega(x)} \psi(x)$, where ψ takes its values in H.

If, for all component P of μ, we had $H = S^1$, then $P = m \otimes \mathrm{Leb}$ for all P, whence $\mu = m \otimes \mathrm{Leb}$. This is a contradiction, since $\mu(C \times \{1\}) > 0$, while $(m \otimes \mathrm{Leb})(C \times \{1\}) = 0$. Thus, for some choice of P, $H = \mathbb{Z}/k\mathbb{Z}$, whence $\psi(x)^k = 1$, and $\mathrm{e}^{\mathrm{i}kf(x)} = \omega(Tx)^k/\omega(x)^k$. Thus, f is periodic on X. □

Remark. The proof only shows that the period of f on X divides the period of $f \circ \pi$ on X', not that they are equal. In fact, it is not hard to construct examples of Young towers where the two periods are different. Moreover, the result is not true without the assumption that the fibers are countable.

2.2. Reduction to the mixing case

In the proofs in Young towers, it is often useful to assume that the tower is mixing, i.e. $\gcd(\varphi_i) = 1$. This restriction may seem technical, but it is important (for example, without it, there is no decay of correlations any more). For limit theorems, however, it is irrelevant: *Theorems* 1.1, 1.2 *and* 1.3 *for mixing towers imply the same results for general towers.*

Proof. We assume that Theorems 1.1, 1.2 and 1.3 are true in any mixing Young tower. Let X be a non-mixing Young tower, with $N = \gcd(\varphi_i) > 1$, and let $f \in C_\tau(X)$ be of vanishing integral. For $k = 0, \dots, N-1$, set $Z_k = \bigcup B_{i,j}$ for $j \equiv k \bmod N$. Then, for every k, (Z_k, T^N) is a mixing Young tower, to which we can apply Theorems 1.1, 1.2 and 1.3.

On Z_k, we consider the function f_k given by $\sum_{i=0}^{N-1} f(T^i x)$, i.e. $(S_N f)_{|Z_k}$. Then $\int_{Z_k} f_k = \int_X f = 0$. Theorem 1.1 applied to f_k on (Z_k, T^N) gives a constant σ_k such that, on Z_k,

$$\frac{1}{\sqrt{sN}} S_{sN} f \to \mathcal{N}(0, \sigma_k^2).$$

Writing an integer n as $sN + r$ with $r < N$, we get that $\frac{1}{\sqrt{n}} S_n f \to \mathcal{N}(0, \sigma_k^2)$ on Z_k. Finally, if $x \in Z_0$, $S_n f(x)$ and $S_n f(T^k x)$ differ by at most $2k\|f\|_\infty$. Thus, $\frac{1}{\sqrt{n}} S_n f - \frac{1}{\sqrt{n}} S_n f \circ T^k$ tends to 0 in probability on Z_0, which shows that $\sigma_k = \sigma_0$. Writing σ for this common number, we get that $\frac{1}{\sqrt{n}} S_n f \to \mathcal{N}(0, \sigma^2)$ on X. Moreover, if $\sigma = 0$, then f_0 is a coboundary for T^N, i.e. $f_0 = g - g \circ T^N$ where $g : Z_0 \to \mathbb{R}$ is measurable. We extend g to the whole tower: if $(x, j) \in X$, with $j = sN + r$ and $r < N$, set $g(x, j) = g(x, sN) - \sum_{i=0}^{r-1} f(x, sN + i)$. It is then easy to check that $f = g - g \circ T$. This proves Theorem 1.1.

For the local limit theorem, let us assume that f is aperiodic, and take u, v and k_n as in the assumptions of Theorem 1.2. We show that the functions f_k are also aperiodic. Otherwise, for example, $f_0 = \rho + g - g \circ T^N + \lambda q$, where g and q are defined on Z_0. We extend g and q to the whole tower: for $(x, j) \in X$ with $j = sN + r$ and $0 < r < N$, set $g(x, j) = g(x, sN) - \sum_0^{r-1} (f(x, sN + i) - \frac{\rho}{N}) + \lambda q(x, sN)$, and $q(x, j) = 0$. Then $f = \frac{\rho}{N} + g - g \circ T + \lambda q$, which is a contradiction. Thus, all the functions f_k are aperiodic. We can apply to them Theorem 1.2 in the mixing tower (Z_k, T^N), and get that

$$\sqrt{sN}\, m\big\{x \in Z_k \mid S_{sN} f(x) \in J + k_{sN} + u(x) + v(T^{sN} x)\big\} \to m(Z_k)|J| \frac{\mathrm{e}^{-\kappa^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}}.$$

Summing over k, we get the conclusion of Theorem 1.2 for f, and for the times of the form sN. For times of the form $sN + r$ with $0 < r < N$, we use the same result for $f \circ T^r$, $u - S_r f$, $v \circ T^r$ and the sequence k_{sN+r}, and get that

$$\sqrt{sN + r}\, m\big\{x \in X \mid S_{sN} f \circ T^r(x) \in J + k_{sN+r} + u(x) - S_r f(x) + v(T^{sN+r} x)\big\} \to |J| \frac{\mathrm{e}^{-\kappa^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}}.$$

As $S_{sN} f \circ T^r(x) + S_r f(x) = S_{sN+r} f(x)$, this concludes the proof of Theorem 1.2.

Finally, the central limit theorem with speed is deduced from the same result on each Z_k, for times of the form sN. We extend the result to arbitrary times: writing n as $sN+r$ with $r<N$, we have $|S_{sN+r}f - S_{sN}f| \leqslant r\|f\|_\infty$. This introduces an error, of order $O(1/\sqrt{n}) \leqslant O(1/n^{\delta/2})$. □

Remark. For this proof, it was important to have a strong version of the local limit theorem, involving functions u and v.

Theorem 1.1 is proved in [35, Theorem 4]. The rest of the paper is devoted to the proof of Theorems 1.2 and 1.3 in mixing Young towers. *From this point on, X will be a mixing Young tower, i.e.* $\gcd(\varphi_i)=1$.

3. An abstract result

If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two sequences indexed by $\mathbb{N}$, we denote by $(a_n)\star(b_n)$ the sequence $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. If $(a_n)_{n\in\mathbb{Z}}$ and $(b_n)_{n\in\mathbb{Z}}$ are two summable sequences indexed by $\mathbb{Z}$, we also define their convolution $c_n=(a_n)\star(b_n)$ by $c_n=\sum_{k=-\infty}^{\infty} a_k b_{n-k}$. Hence, if $\sum a_n z^n$ and $\sum b_n z^n$ are series with summable coefficients, the coefficient of z^n in $(\sum a_k z^k)(\sum b_k z^k)$ is given by $(a_n)\star(b_n)$ (more precisely, it is given by the nth term $((a_k)\star(b_k))_n$ of this sequence, but we will often abuse notations and write simply $(a_n)\star(b_n)$). Finally, we write $\mathbb{D}=\{z\in\mathbb{C} \mid |z|<1\}$ and $\overline{\mathbb{D}}=\{z\in\mathbb{C} \mid |z|\leqslant 1\}$.

The goal of this section is to prove the following theorem:

Theorem 3.1. *Let $\beta>2$. Let R_n, for $n\in\mathbb{N}^*$, be operators on a Banach space E, with $\sum_{k=n+1}^{\infty}\|R_k\| = O(1/n^\beta)$. Set $R(z)=\sum R_n z^n$, and assume that 1 is a simple isolated eigenvalue of $R(1)$, while $I-R(z)$ is invertible for $z\in\overline{\mathbb{D}}-\{1\}$. Let P be the spectral projection associated to $R(1)$ and the eigenvalue 1, and assume that $PR'(1)P=\mu P$ with $\mu>0$.*

Let $R_n(t)$ be operators on E (for t in some interval $[-\alpha,\alpha]$ with $\alpha>0$) such that $\sum_{k=n+1}^{\infty}\|R_k(t)-R_k\| \leqslant C|t|/n^{\beta-1}$. Set $R(z,t)=\sum z^n R_n(t)$. Let $\lambda(z,t)$ be the eigenvalue close to 1 of $R(z,t)$, for (z,t) close to $(1,0)$. We assume that $\lambda(1,t)=1-M(t)$ with $M(t)\sim ct^2$ for some constant c with $\operatorname{Re}(c)>0$.

Then, for small enough t, $I-R(z,t)$ is invertible for all $z\in\mathbb{D}$. Let us denote its inverse by $\sum T_n(t)z^n$. Then there exist $\alpha'>0$, $d>0$ and $C>0$ such that, for every $t\in[-\alpha',\alpha']$, for every $n\in\mathbb{N}^*$,

$$\left\| T_n(t) - \frac{1}{\mu}\left(1-\frac{1}{\mu}M(t)\right)^n P \right\| \leqslant \frac{C}{n^{\beta-1}} + C|t|\left(\frac{1}{n^{\beta-1}}\right)\star(1-dt^2)^n. \tag{5}$$

In the application of Theorem 3.1 to the proof of Theorems 1.2 and 1.3, the operators R_n will describe the returns to the basis B, and will be easily understood, as well as their perturbations $R_n(t)$. On the other hand, T_n will describe all the iterates at time n, and $T_n(t)$ will be closely related to the characteristic function $E(e^{itS_n f})$. Thus, (5) will enable us to describe precisely $E(e^{itS_n f})$, and this information will be sufficient to get Theorems 1.2 and 1.3.

3.1. Banach algebras and Wiener Lemma

In this paragraph, we define some Banach algebras which will be useful in the following estimates. We postpone the proofs of the properties of these algebras to the Appendix.

A Banach algebra $\mathcal{A}$ is a complex Banach space with an associative multiplication $\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ such that $\|AB\|\leqslant\|A\|\|B\|$, and a neutral element. The set of invertible elements is then an open subset of $\mathcal{A}$, on which the inversion is continuous.

Let $\mathcal{C}$ be a Banach algebra. If $\gamma>1$, we write $\mathcal{O}_\gamma(\mathcal{C})$ for the set of formal series $\sum_{n=-\infty}^{\infty} A_n z^n$, where $A_n\in\mathcal{C}$ and $\|A_n\|=O(1/|n|^\gamma)$ when $n\to\pm\infty$, endowed with the standard product of power series, corresponding to the convolution of the sequences (A_n) and (B_n). It admits the naive norm $\sup_{n\in\mathbb{Z}}(|n|+1)^\gamma\|A_n\|$, for which it is not a Banach algebra. However, there exists a norm, equivalent to the previous one, which makes $\mathcal{O}_\gamma(\mathcal{C})$ into a Banach algebra (Proposition A.1). Moreover, this algebra satisfies a Wiener Lemma: if $A(z)=\sum A_n z^n\in\mathcal{O}_\gamma(\mathcal{C})$ is such that $A(z)$ is invertible for every $z\in S^1$, then A is invertible in $\mathcal{O}_\gamma(\mathcal{C})$ (Theorem A.3).

We will also use the Banach algebra $\mathcal{O}_\gamma^+(\mathcal{C})$, given by the set of series $\sum A_n z^n\in\mathcal{O}_\gamma(\mathcal{C})$ such that $A_n=0$ for $n<0$. It is a closed subalgebra of $\mathcal{O}_\gamma(\mathcal{C})$, and it satisfies also a Wiener Lemma (Theorem A.4).

Notation. If $f:[-\alpha,\alpha]\times\mathbb{Z}\to\mathbb{R}_+$ for some $\alpha>0$, and $\mathcal{C}$ is a Banach algebra, we denote by $O_{\mathcal{C}}(f(t,n))$ the set of series $\sum_{n=-\infty}^{\infty}c_n(t)z^n$ where $c_n:[-\alpha,\alpha]\to\mathcal{C}$ is such that there exists $\alpha'>0$ and $C>0$ such that

$$\forall t\in[-\alpha',\alpha'],\ \forall n\in\mathbb{Z},\quad \|c_n(t)\|\leqslant Cf(t,n).$$

We will often omit the subscript in $O_{\mathcal{C}}$. As usual, we will often write $\sum c_n(t)z^n=O(f(t,n))$ instead of the more correct formulation $\sum c_n(t)z^n\in O(f(t,n))$. We will also write $O(g(n))$ for the set of series $\sum A_n z^n$ with $\|A_n\|\leqslant Cg(n)$ for some constant C. This is a particular case of the previous notation, where the functions $f(t,n)$ are independent of t. Until the end of Section 3, the notation O will always have this signification.

Remark. The notations $\mathcal{O}$ and O should not be confused: there are similarities (which is why we have used the same letter), but the calligraphic notation $\mathcal{O}$ indicates additionally a Banach algebra. In this case, we can for example use the continuity of inversion.

With these notations, we can reformulate Theorem A.3 as follows: if $\sum A_n z^n=O(1/(|n|+1)^\gamma)$ for $\gamma>1$, and $\sum A_n z^n$ is invertible for every $z\in S^1$, then $(\sum A_n z^n)^{-1}=O(1/(|n|+1)^\gamma)$. The fact that $\mathcal{O}_\gamma(\mathcal{C})$ is a Banach algebra also implies that, for $\gamma>1$,

$$O\left(\frac{1}{(|n|+1)^\gamma}\right)\star O\left(\frac{1}{(|n|+1)^\gamma}\right)\subset O\left(\frac{1}{(|n|+1)^\gamma}\right),\tag{6}$$

i.e., if two series $\sum A_n z^n$ and $\sum B_n z^n$ (with $A_n,B_n\in\mathcal{C}$) satisfy

$$\sup_{n\in\mathbb{Z}}\big(|n|+1\big)^\gamma\|A_n\|<\infty\quad\text{and}\quad\sup_{n\in\mathbb{Z}}\big(|n|+1\big)^\gamma\|B_n\|<\infty,$$

then the series $\sum C_n z^n:=(\sum A_n z^n)(\sum B_n z^n)$ also satisfies $\sup_{n\in\mathbb{Z}}(|n|+1)^\gamma\|C_n\|<\infty$.

3.2. Preliminary technical estimates

For notational convenience, we will often write t instead of $|t|$ in what follows. Equivalently, the reader may consider that the proofs are written for $t\geqslant 0$. We will also write $1/|n|^\gamma$ instead of $1/(|n|+1)^\gamma$, discarding the problem at $n=0$.

Lemma 3.2. *When $\gamma>1$ and $d>0$,*

$$O\left(\frac{1}{|n|^\gamma}\right)\star O\big(1_{n\geqslant 0}t^2(1-dt^2)^n\big)\subset O\left(\frac{1}{|n|^\gamma}+1_{n\geqslant 0}t^2\left(1-\frac{d}{2}t^2\right)^n\right).$$

Proof. For $n<0$, the coefficient in the convolution is less than $\sum_{k=0}^{\infty}t^2(1-dt^2)^k\frac{1}{|n|^\gamma}\leqslant\frac{C}{|n|^\gamma}$. For $n\geqslant 0$, it is less than

$$\sum_{k=-\infty}^{n/2}\frac{1}{|k|^\gamma}t^2(1-dt^2)^{n/2}+\sum_{k=n/2}^{n}\frac{1}{(n/2)^\gamma}t^2(1-dt^2)^{n-k}\leqslant Ct^2(1-dt^2)^{n/2}+\frac{C}{n^\gamma}.$$

Finally, as $\sqrt{1-dt^2}\leqslant(1-\frac{d}{2}t^2)$, we get the conclusion. □

Lemma 3.3. *Let $\gamma > 1$ and $d > 0$. Let $G_t(z) = O(t/|n|^\gamma + 1_{n\geqslant 0}t^3(1-dt^2)^n)$, and assume that $F(z) = O(1/|n|^\gamma)$ is invertible for every $z \in S^1$. Then $[F(z) + G_t(z)]^{-1} = F(z)^{-1} + O(\frac{t}{|n|^\gamma} + 1_{n\geqslant 0}t^3(1 - \frac{d}{64}t^2)^n)$.*

Proof. We first assume that $F(z) = 1$. Setting $H_t(z) = \sum_{n\in\mathbb{Z}} \frac{t}{|n|^\gamma} z^n + \sum_{n\in\mathbb{N}} t^3(1-dt^2)^n z^n$, the norm of the coefficients of $[1 + G_t(z)]^{-1}$ is less than the coefficients of $[1 - H_t(z)]^{-1}$. Thus, it is sufficient to consider $1/(1 - tK(z) - t^3/(1 - (1-dt^2)z))$ where $K(z) = \sum z^n/|n|^\gamma$. Note that

$$\frac{1}{1 - tK(z) - t^3/(1-(1-dt^2)z)} = \frac{1}{1 - tK(z) - t^3} \frac{1 - (1-dt^2)z}{1 - (1-dt^2)[1 + t^3/(1 - tK(z) - t^3)]z}. \tag{7}$$

For small enough t, $|(1-dt^2)[1 + t^3/(1 - tK(z) - t^3)]| < 1$, whence

$$\begin{aligned}
&\frac{1 - (1-dt^2)z}{1 - (1-dt^2)[1 + t^3/(1 - tK(z) - t^3)]z} \\
&\quad = \big(1 - (1-dt^2)z\big) \sum_{n=0}^\infty (1-dt^2)^n \left[1 + \frac{t^3}{1 - tK(z) - t^3}\right]^n z^n \\
&\quad = 1 + \frac{tz(1-dt^2)}{1 - tK(z) - t^3} t^2 \sum_{n=0}^\infty (1-dt^2)^n \left[1 + \frac{t^3}{1 - tK(z) - t^3}\right]^n z^n.
\end{aligned} \tag{8}$$

We first study the sum. Let $\mathcal{A} = \mathcal{O}_\gamma(\mathbb{C})$ be the Banach algebra of the series whose coefficients are $O(1/|n|^\gamma)$. As $K(z) \in \mathcal{A}$, we have $1 - tK(z) - t^3 = 1 + O_\mathcal{A}(t)$. Since the inversion is Lipschitz on a Banach algebra, we get $t^3/(1 - tK(z) - t^3) = t^3 + O_\mathcal{A}(t^4)$, whence $\|[1 + t^3/(1 - tK(z) - t^3)]^n\|_\mathcal{A} \leqslant C(1 + 2t^3)^n$. Let us estimate the coefficient of z^p in $t^2 \sum_{n=0}^\infty (1-dt^2)^n [1 + t^3/(1 - tK(z) - t^3)]^n z^n$. This is at most

$$\begin{aligned}
t^2 \sum_{n=0}^\infty (1-dt^2)^n \left(\left[1 + \frac{t^3}{1 - tK(z) - t^3}\right]^n\right)_{-n+p} &\leqslant \sum_{n=0}^\infty t^2 (1-dt^2)^n \frac{(1+2t^3)^n}{|-n+p|^\gamma} \\
&\leqslant \sum_{n=0}^\infty t^2 \left(1 - \frac{d}{2}t^2\right)^n \frac{1}{|-n+p|^\gamma}
\end{aligned}$$

for t small enough so that $(1-dt^2)(1+2t^3) \leqslant (1 - \frac{d}{2}t^2)$. We find the same expression as in the convolution between $O(1_{n\geqslant 0}t^2(1 - \frac{d}{2}t^2)^n)$ and $O(1/|n|^\gamma)$, that we have already estimated in Lemma 3.2. Thus, we get at most $O(1/|p|^\gamma + 1_{p\geqslant 0}t^2(1 - \frac{d}{4}t^2)^p)$.

As $tz(1-dt^2)/(1 - tK(z) - t^3) = O(t/|n|^\gamma)$, another convolution yields that (8) is

$$1 + O\left(\frac{t}{|n|^\gamma} + 1_{n\geqslant 0}t^3\left(1 - \frac{d}{8}t^2\right)^n\right).$$

Multiplying by $1/(1 - tK(z) - t^3) = 1 + O(t/|n|^\gamma)$ gives that (7) $= 1 + O(t/|n|^\gamma + 1_{n\geqslant 0}t^3(1 - \frac{d}{16}t^2)^n)$. This concludes the proof in the case $F(z) = 1$.

We now handle the case of an arbitrary $F(z)$. Note that

$$\big[F(z) + G_t(z)\big]^{-1} = \big[1 + F(z)^{-1}G_t(z)\big]^{-1} F(z)^{-1}.$$

The Wiener Lemma A.3 implies that $F(z)^{-1} = O(1/|n|^\gamma)$, whence Lemma 3.2 gives $F(z)^{-1}G_t(z) = O(t/|n|^\gamma + 1_{n\geqslant 0}t^3(1 - \frac{d}{2}t^2)^n)$. Thus, the case $F(z) = 1$ yields

$$\big[1 + F(z)^{-1}G_t(z)\big]^{-1} = 1 + O\left(\frac{t}{|n|^\gamma} + 1_{n\geqslant 0}t^3\left(1 - \frac{d}{32}t^2\right)^n\right).$$

Another convolution with $F(z)^{-1}$ gives the result. □

3.3. Proof of Theorem 3.1

Let $M(t)$ be as in Theorem 3.1. We fix once and for all $d>0$ such that $|1-\frac{1}{\mu}M(t)|\leqslant 1-dt^2$ for small enough t, and we restrict the range of t so that this inequality is true. The invertibility of $R(z,t)$ for $z\in\mathbb{D}$ and small enough t is proved in [14, Proposition 2.7].

To estimate the eigenvalues using Banach algebra techniques, we will need that the eigenvalue close to 1 of $R(z)$ is defined on the whole circle S^1, which is not a priori the case. Consequently, we use the construction in the second step of the proof of Theorem 2.4 in [14]: we replace $R(z)$ by $\tilde{R}(z)=\sum_{-\infty}^{\infty}\tilde{R}_n z^n$, such that it has a unique eigenvalue $\tilde{\lambda}(z)$ close to 1 for $z\in S^1$, equal to 1 only for $z=1$, with $\sum_{|k|>n}\|\tilde{R}_k\|=O(1/n^\beta)$, and such that $R(z)=\tilde{R}(z)$ for z close to 1 on S^1. We also set $\tilde{R}(z,t)=(\tilde{R}(z)-R(z))+\sum R_n(t)z^n$. For small enough t, $\tilde{R}(z,t)$ has for all $z\in S^1$ a unique eigenvalue $\tilde{\lambda}(z,t)$ close to 1.

Lemma 3.4. *We have*

$$\frac{1-\tilde{\lambda}(z,t)}{1-(1-(1/\mu)M(t))z}=\frac{1-\tilde{\lambda}(z)}{1-z}+O\left(\frac{t}{|n|^{\beta-1}}+1_{n\geqslant 0}t^3\left(1-\frac{d}{2}t^2\right)^n\right).$$

Proof. We write $K(z,t)=(\tilde{\lambda}(1,t)-\tilde{\lambda}(z,t))/(1-z)-(1-\tilde{\lambda}(z))/(1-z)$. Recall that $\tilde{\lambda}(1,t)=\lambda(1,t)=1-M(t)$. Then, writing $B(z)=(1-\tilde{\lambda}(z))/(1-z)$, we have

$$\tilde{\lambda}(z,t)=1-M(t)+(z-1)\big(K(z,t)+B(z)\big).$$

Thus,

$$\begin{aligned}\frac{1-\tilde{\lambda}(z,t)}{1-(1-(1/\mu)M(t))z}-B(z)&=\frac{M(t)+(1-z)(K(z,t)+B(z))-[1-(1-(1/\mu)M(t))z]B(z)}{1-(1-(1/\mu)M(t))z}\\&=K(z,t)\frac{1-z}{1-(1-(1/\mu)M(t))z}+M(t)\frac{1-zB(z)/\mu}{1-(1-(1/\mu)M(t))z}\\&=\mathrm{I}+\mathrm{II}.\end{aligned}$$

For I,

$$\frac{1-z}{1-(1-(1/\mu)M(t))z}=1-\frac{1}{\mu}M(t)\sum_{n=1}^{\infty}\left(1-\frac{1}{\mu}M(t)\right)^{n-1}z^n \tag{9}$$

is in $1+O(1_{n\geqslant 0}t^2(1-dt^2)^n)$. We multiply it by $K(z,t)$. Set $\mathcal{A}=\mathcal{O}_{\beta-1}(\mathrm{Hom}(E))$ (the Banach algebra of functions whose coefficients are in $O(1/|n|^{\beta-1})$ for $n\in\mathbb{Z}$). We have $(\tilde{R}(z,t)-\tilde{R}(1,t))/(z-1)=(\tilde{R}(z)-\tilde{R}(1))/(z-1)+O_{\mathcal{A}}(t)$. The proof of [14, Lemma 2.6], but in the algebra $\mathcal{A}$ and with tildes everywhere, applies (using Theorem A.3 to ensure that the inverses remain in $\mathcal{A}$). It gives $(\tilde{\lambda}(z,t)-\tilde{\lambda}(1,t))/(z-1)=(\tilde{\lambda}(z)-\tilde{\lambda}(1))/(z-1)+O_{\mathcal{A}}(t)$, i.e. $K(z,t)=O(t/|n|^{\beta-1})$. Hence, Lemma 3.2 yields

$$I=O\left(\frac{t}{|n|^{\beta-1}}+1_{n\geqslant 0}t^3\left(1-\frac{d}{2}t^2\right)^n\right).$$

Since $(\tilde{R}(z)-\tilde{R}(1))/(z-1)=O(1/|n|^\beta)$, we prove that $B(z)=O(1/|n|^\beta)$ as in the third step of the proof of Theorem 2.4 in [14] (but in the Banach algebra $\mathcal{O}_\beta(\mathrm{Hom}(E))$). Since $1-zB(z)/\mu$ vanishes at 1 (Step 7 of the proof of Lemma 3.1 in [16]) and is in $O(1/|n|^\beta)$, it can be written as $(1-z)C(z)$ where $C(z)=O(1/|n|^{\beta-1})$. To obtain II, we multiply $C(z)$ by $M(t)(1-z)/(1-(1-(1/\mu)M(t))z)=O(t^2+1_{n\geqslant 0}t^4(1-dt^2)^n)$. Lemma 3.2 yields

$$\mathrm{II}=O\left(\frac{t^2}{|n|^{\beta-1}}+1_{n\geqslant 0}t^4\left(1-\frac{d}{2}t^2\right)^n\right). \qquad \square$$

Corollary 3.5. *We have*

$$\left(\frac{I-\tilde{R}(z,t)}{1-(1-(1/\mu)M(t))z}\right)^{-1}=\left(\frac{I-\tilde{R}(z)}{1-z}\right)^{-1}+O\left(\frac{t}{|n|^{\beta-1}}+1_{n\geqslant 0}t^3\left(1-\frac{d}{256}t^2\right)^n\right).$$

Proof. Let $\tilde{P}(z,t)$ be the spectral projection associated to the eigenvalue $\tilde{\lambda}(z,t)$ of $\tilde{R}(z,t)$, and $\tilde{Q}(z,t)=I-\tilde{P}(z,t)$. Set $\mathcal{A}=\mathcal{O}_{\beta-1}(\mathrm{Hom}(E))$. Then $\tilde{R}(z,t)=\tilde{R}(z)+O_{\mathcal{A}}(t)$ by assumption. As $\mathcal{A}$ satisfies the Wiener Lemma A.3, the integral expression of the projection $\tilde{P}(z,t)$ shows that $\tilde{P}(z,t)=\tilde{P}(z)+O_{\mathcal{A}}(t)$. Moreover, $I-\tilde{R}(z,t)\tilde{Q}(z,t)=I-\tilde{R}(z)\tilde{Q}(z)+O_{\mathcal{A}}(t)$, whence $(I-\tilde{R}(z,t)\tilde{Q}(z,t))^{-1}=(I-\tilde{R}(z)\tilde{Q}(z))^{-1}+O_{\mathcal{A}}(t)$, since $I-\tilde{R}(z)\tilde{Q}(z)$ is invertible in $\mathcal{A}$ by Theorem A.3, and the inversion is Lipschitz.

As

$$I-\tilde{R}(z,t)=\left(1-\tilde{\lambda}(z,t)\right)\tilde{P}(z,t)+\left(I-\tilde{R}(z,t)\tilde{Q}(z,t)\right)\tilde{Q}(z,t),$$

Lemmas 3.3 and 3.4 yield that

$$\begin{aligned}
&\left(\frac{I-\tilde{R}(z,t)}{1-(1-(1/\mu)M(t))z}\right)^{-1}\\
&\quad=\frac{1-(1-(1/\mu)M(t))z}{1-\tilde{\lambda}(z,t)}\tilde{P}(z,t)+\left(1-\left(1-\frac{1}{\mu}M(t)\right)z\right)\left(I-\tilde{R}(z,t)\tilde{Q}(z,t)\right)^{-1}\tilde{Q}(z,t)\\
&\quad=\left[\frac{1-z}{1-\tilde{\lambda}(z)}+O\left(\frac{t}{|n|^{\beta-1}}+1_{n\geqslant 0}t^3\left(1-\frac{d}{128}t^2\right)^n\right)\right]\left[\tilde{P}(z)+O\left(\frac{t}{|n|^{\beta-1}}\right)\right]\\
&\qquad+\left[(1-z)+O(1_{n=0}t^2)\right]\left[\left(I-\tilde{R}(z)\tilde{Q}(z)\right)^{-1}+O\left(\frac{t}{|n|^{\beta-1}}\right)\right]\left[\tilde{Q}(z)+O\left(\frac{t}{|n|^{\beta-1}}\right)\right]\\
&\quad=\frac{1-z}{1-\tilde{\lambda}(z)}\tilde{P}(z)+(1-z)\left(I-\tilde{R}(z)\tilde{Q}(z)\right)^{-1}\tilde{Q}(z)+O\left(\frac{t}{|n|^{\beta-1}}+1_{n\geqslant 0}t^3\left(1-\frac{d}{256}t^2\right)^n\right)\\
&\quad=\left(\frac{I-\tilde{R}(z)}{1-z}\right)^{-1}+O\left(\frac{t}{|n|^{\beta-1}}+1_{n\geqslant 0}t^3\left(1-\frac{d}{256}t^2\right)^n\right). \qquad \square
\end{aligned}$$

Corollary 3.6. *We have*

$$\left(\frac{I-R(z,t)}{1-(1-(1/\mu)M(t))z}\right)^{-1}=\left(\frac{I-R(z)}{1-z}\right)^{-1}+O\left(\frac{t}{|n|^{\beta-1}}+1_{n\geqslant 0}t^3\left(1-\frac{d}{512}t^2\right)^n\right).$$

Proof. Let χ_1,χ_2 be a C^∞ partition of unity of S^1 such that $R(z)=\tilde{R}(z)$ on the support of χ_1. Since χ_1 is C^∞, $\chi_1(z)=O(1/|n|^{\beta-1})$. Writing $A(z,t)=((I-R(z,t))/(1-(1-(1/\mu)M(t))z))^{-1}$, Corollary 3.5 ensures that

$$\chi_1(z)A(z,t)=\chi_1(z)A(z,0)+O\left(\frac{t}{|n|^{\beta-1}}+1_{n\geqslant 0}t^3\left(1-\frac{d}{512}t^2\right)^n\right).$$

Concerning χ_2, we can modify R outside of its support so that $I-R(z)$ is everywhere invertible on S^1. Let $\mathcal{A}=\mathcal{O}_{\beta-1}(\mathrm{Hom}(E))$. Since $I-R(z,t)=I-R(z)+O_{\mathcal{A}}(t)$, Theorem A.3 yields $(I-R(z,t))^{-1}=(I-R(z))^{-1}+O_{\mathcal{A}}(t)$. Hence,

$$\chi_2(z)A(z,t)=\chi_2(z)A(z,0)+O\left(\frac{t}{|n|^{\beta-1}}\right). \qquad \square$$

In fact, the functions in the previous corollary are defined on the whole disk $\overline{\mathbb{D}}$, i.e. their coefficients for $n<0$ vanish. However, during the proof, it was important to work in a less restrictive context, for example to introduce partitions of unity.

Proof of Theorem 3.1. Set

$$F_t(z) = \left(\frac{I - R(z,t)}{1 - (1 - (1/\mu)M(t))z}\right)^{-1} - \left(\frac{I - R(z)}{1 - z}\right)^{-1}$$

and

$$\left(\frac{I - R(z)}{1 - z}\right)^{-1} = \frac{1}{\mu}P + (1 - z)A(z)$$

where $A(z) = O(1/n^{\beta-1})$, by [32, Theorem 1] or [16, Theorem 1.1].

Then

$$\begin{aligned}\sum T_n(t)z^n &= \big(I - R(z,t)\big)^{-1} \\ &= \frac{1}{1 - (1 - (1/\mu)M(t))z}\frac{1}{\mu}P + \frac{1 - z}{1 - (1 - (1/\mu)M(t))z}A(z) + \frac{1}{1 - (1 - (1/\mu)M(t))z}F_t(z) \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

The coefficient of z^n in I is $\frac{1}{\mu}(1 - \frac{1}{\mu}M(t))^n P$. So, we have to bound the coefficients of II and III to conclude.

By (9) and Lemma 3.2, the coefficients of II belong to

$$\big[1 + O\big(t^2(1 - dt^2)^n\big)\big] \star O\left(\frac{1}{n^{\beta-1}}\right) \subset O\left(\frac{1}{n^{\beta-1}} + t^2\left(1 - \frac{d}{2}t^2\right)^n\right).$$

For III, we get by Corollary 3.6 that the coefficient of z^n is bounded by

$$O\big((1 - dt^2)^n\big) \star O\left(\frac{t}{n^{\beta-1}} + t^3\left(1 - \frac{d}{512}t^2\right)^n\right).$$

The convolution between $(1 - dt^2)^n$ and $t^3(1 - \frac{d}{512}t^2)^n$ is bounded by the convolution between $(1 - \frac{d}{512}t^2)^n$ and $t^3(1 - \frac{d}{512}t^2)^n$, which gives $nt^3(1 - \frac{d}{512}t^2)^n$. This is less than $Ct(1 - \frac{d}{1024}t^2)^n$, since

$$\frac{n}{2}(1 - ct^2)^n \leqslant \sum_{k=0}^{n/2}(1 - ct^2)^k(1 - ct^2)^{n/2} \leqslant (1 - ct^2)^{n/2}\frac{1}{ct^2}.$$

As $t(1 - \frac{d}{1024}t^2)^n \leqslant t(1 - \frac{d}{1024}t^2)^n \star (\frac{1}{n^{\beta-1}})$, we get a bound of the form stated in the theorem. □

4. The key estimate

In this section, the assumptions are as in Theorem 1.1, i.e. X is a Young tower whose return time φ satisfies $m[\varphi > n] = O(1/n^\beta)$ with $\beta > 2$. We also assume that $\gcd(\varphi_i) = 1$. Take also $f \in C_\tau(X)$ with $\int f\,\mathrm{d}m = 0$.

The goal of this section is to estimate precisely $\int_X \mathrm{e}^{\mathrm{i}tS_n f} \cdot u \cdot v \circ T^n\,\mathrm{d}m$ when u, v are functions on X. We will use the same kind of perturbative ideas as in [30] and [17], but applied to transfer operators associated to induced maps on the basis B of the tower (see [32]). Separating the different return times, it will be possible to use the abstract Theorem 3.1, to finally get the key estimate Theorem 4.6.

4.1. *First return transfer operators*

Let $\hat{T}$ be the transfer operator associated to T, defined by $\int u \cdot v \circ T\,\mathrm{d}m = \int \hat{T}u \cdot v\,\mathrm{d}m$. When u is integrable on X, it is given by

$$\hat{T}^n u(x) = \sum_{T^n y = x} g_m^{(n)}(y)u(y),$$

where $g_m^{(n)}$ is the inverse of the jacobian of T^n at y.

As the basis B of the tower plays a particular role, we will decompose the trajectories of the preimages of x under T^n, keeping track of the moments their iterates fall again in B. More formally, we introduce the following operators:

$$R_n u(x) = \sum_{\substack{T^n y = x \\ y \in B,\, Ty, \ldots, T^{n-1} y \notin B,\, T^n y \in B}} g_m^{(n)}(y) u(y),$$

$$T_n u(x) = \sum_{\substack{T^n y = x \\ y \in B,\, T^n y \in B}} g_m^{(n)}(y) u(y),$$

$$A_n u(x) = \sum_{\substack{T^n y = x \\ y \in B,\, Ty, \ldots, T^n y \notin B}} g_m^{(n)}(y) u(y),$$

$$B_n u(x) = \sum_{\substack{T^n y = x \\ y, \ldots, T^{n-1} y \notin B,\, T^n y \in B}} g_m^{(n)}(y) u(y),$$

$$C_n u(x) = \sum_{\substack{T^n y = x \\ y, \ldots, T^n y \notin B}} g_m^{(n)}(y) u(y).$$

The interpretations of these operators are as follows: R_n takes into account the first returns to B, while T_n takes all returns into account. Hence,

$$T_n = \sum_{k_1 + \cdots + k_l = n} R_{k_1} \cdots R_{k_l}. \tag{10}$$

The operators B_n and A_n see respectively the beginning and the end of the trajectories, outside of B. Thus, if x is fixed and y satisfies $T^n y = x$ and $\{y, Ty, \ldots, T^n y\} \cap B \neq \emptyset$, we can consider the first b iterates of y, until it enters in B (this corresponds to B_b), then some successive returns to B, during a time k (this corresponds to T_k), and finally a iterations outside of B (corresponding to A_a). Thus,

$$\hat{T}^n = C_n + \sum_{a+k+b=n} A_a T_k B_b. \tag{11}$$

The operator C_n takes into account the points y such that $\{y, Ty, \ldots, T^n y\} \cap B = \emptyset$.

We perturb these operators, setting (for $X = A, B, C, R, T$, and $t \in \mathbb{R}$)

$$X_n(t)(\cdot) = X_n(\mathrm{e}^{\mathrm{i} t S_n f} \cdot).$$

Eq. (10) remains true with t everywhere:

$$T_n(t) = \sum_{k_1 + \cdots + k_l = n} R_{k_1}(t) \cdots R_{k_l}(t). \tag{12}$$

Let $\hat{T}(t)$ be the perturbation of $\hat{T}$ given by $\hat{T}(t)(\cdot) = \hat{T}(\mathrm{e}^{\mathrm{i}tf} \cdot)$. Then the following analogue of (11) holds:

$$\hat{T}(t)^n = C_n(t) + \sum_{a+k+b=n} A_a(t) T_k(t) B_b(t). \tag{13}$$

Let f_B be given by (1). As $f_B = S_n f$ on $\{y \in B \mid \varphi(y) = n\}$, we get $R_n(t)(u) = R_n(\mathrm{e}^{\mathrm{i}t f_B} u)$.

For $z \in \overline{\mathbb{D}}$, write $R(z) = \sum R_n z^n$, and

$$R(z,t) = \sum_{n=0}^{\infty} R_n(t) z^n. \tag{14}$$

Let T_B be the first return map induced by T on B, i.e. $T_B(x) = T^{\varphi(x)}(x)$. Then (14) corresponds to considering all the preimages of a point in B under T_B, whence $R(1)$ is the transfer operator $\hat{T}_B$ associated to T_B, and

$$R(1,t)(u) = \hat{T}_B(\mathrm{e}^{\mathrm{i}t f_B} u). \tag{15}$$

When u is a function on a subset Z of X, we denote by $D_\tau u(Z)$ the best Hölder constant of u on Z, i.e.

$$D_\tau u(Z) = \inf\{C > 0 \mid \forall x, y \in Z,\ |u(x) - u(y)| \leqslant C\tau^{s(x,y)}\} \tag{16}$$

where $s(x, y)$ is the separation time of x and y.

The operators R_n and $R_n(t)$ act on $C_{\tau'}(B)$ for any $\tau \leqslant \tau' < 1$. Take η such that

$$0 < \eta < \min(1/2, \beta - 2) \tag{17}$$

and set $\nu = \tau^\eta$. For technical reasons, we will let the operators act on $C_\nu(B)$. We regroup in the following lemma all the estimates we will need later. Their proofs are rather straightforward, but sometimes lengthy. Hence, we will not give the details of the proofs, and rather give references to articles where similar estimates are proved.

Lemma 4.1. *The operators R_n and $R_n(t)$ acting on $C_\nu(B)$ satisfy the following estimates*:

(1) $\sum_{k=n+1}^{\infty} \|R_k\| = O(1/n^\beta)$.
(2) *The operator $R(z,t)$ satisfies a Doeblin–Fortet inequality*

$$\|R(z,t)^n u\| \leqslant C(1 + |t|)(\nu|z|)^n \|u\| + C|z|^n \|u\|_{L^1}. \tag{18}$$

In particular, the spectral radius of $R(z,t)$ is $\leqslant |z|$, and its essential spectral radius is $\leqslant \nu|z|$. Thus, $I - R(z,t)$ is not invertible if and only if 1 is an eigenvalue of $R(z,t)$, and this can happen only for $|z| = 1$.
(3) *$R(1)$ has a simple isolated eigenvalue at 1 (the eigenspace is the space of constant functions), and $I - R(z)$ is invertible for $z \neq 1$.*
(4) *There exists $C > 0$ such that, for any $t \in \mathbb{R}$, for every $n \in \mathbb{N}^*$,*

$$\sum_{k=n+1}^{\infty} \|R_k(t) - R_k(0)\| \leqslant C \frac{|t|}{n^{\beta-1}}.$$

(5) *For every $t \in \mathbb{R}$, there exists $C = C(t)$ such that, for any $t' \in \mathbb{R}$, for every $n \in \mathbb{N}^*$, $\|R_n(t') - R_n(t)\| \leqslant C|t' - t|^{1/2}/n^{\beta-1}$.*

Proof. The first assertion is a consequence of [14, Lemma 4.2]: it gives $\|R_n\| = O(m[\varphi = n])$.

The inequality (18) is similar to [3, Proposition 2.1]. In this article, the hypothesis is that $D_\nu f_B(B) < \infty$, but the proofs work in fact as soon as $\sum m(B_{i,0}) D_\nu f_B(B_{i,0}) < \infty$. In our case, $D_\nu f_B(B_{i,0}) \leqslant C\varphi_i$, whence $\sum m(B_{i,0}) D_\nu f_B(B_{i,0}) \leqslant C \sum m(B_{i,0})\varphi(B_{i,0}) = C$ by Kac's Formula. Since the injection $C_\nu(B) \to L^1(B)$ is compact, the statement on the essential spectral radius of $R(z,t)$ is then a consequence of Hennion's Theorem [18].

The third assertion is [14, Lemma 4.3].

The two remaining assertions are proved by direct estimates, similar to the estimates in [3, Theorem 2.4]. The last one holds in $C_{\sqrt{\tau}}(X)$ but not in $C_\tau(X)$. This is the reason of the requirement $\nu \geqslant \sqrt{\tau}$. □

We will also need the following estimates on $A_a(t)$, $B_b(t)$ and $C_n(t)$, acting on $C_\nu(X)$.

Lemma 4.2. *Let $u, v \in L^\infty(X)$. There exists a constant C such that, for any $t \in \mathbb{R}$, for any $a \in \mathbb{N}$,*

$$\left| \int A_a(t)(u)v - \int A_a(u)v \right| \leqslant C \frac{|t|}{a^{\beta-1}} \|u\|_\infty \|v\|_\infty. \tag{19}$$

Moreover, $\int A_a(u)v = O(1/a^\beta)$, and

$$\sum_{a=0}^{\infty} \int A_a(1_B)v = \int v. \tag{20}$$

Proof. Define a function v' on B by $v'(x) = v(T^a x)$ if $\varphi(x) > a$, and 0 otherwise. Changing variables, we get $\int A_a(t)(u)v = \int_{\{\varphi > a\}} \mathrm{e}^{\mathrm{i}tS_a f} u v'$. Since $|\mathrm{e}^{\mathrm{i}tS_a f} - 1| \leqslant |t| a \|f\|_\infty$ and $m(\varphi > a) \leqslant C/a^\beta$, this implies (19).

Moreover, $|\int A_a(u)v| \leqslant m(\varphi > a)\|u\|_\infty \|v\|_\infty \leqslant (C/a^\beta)\|u\|_\infty \|v\|_\infty$. Finally, $\int A_a(1_B)v = \int_{T^a\{\varphi > a\}} v$. Since the sets $T^a\{\varphi > a\}$ form a partition of X, (20) readily follows. □

Lemma 4.3. *For $t \in \mathbb{R}$, $B_b(t) = B_b + O(|t|/b^{\beta-1})$, where $\|B_b\| = O(1/b^\beta)$ and $\forall u \in C_\nu(X)$,*

$$\sum_{b=0}^{\infty} \int_B B_b u \, \mathrm{d}m = \int_X u \, \mathrm{d}m.$$

Proof. Let Λ_b be the set of points that enter into B after exactly b iterations, so that $B_b(t)(u)$ takes the values of u on Λ_b into account. As in [3, Theorem 2.4], we check that $\|B_b(t) - B_b\| \leqslant C|t| m(\Lambda_b) D_\nu S_b f(\Lambda_b)$. As $D_\nu S_b f \leqslant Cb$ and $m(\Lambda_b) = m[\varphi > b] = O(1/b^\beta)$, we get indeed that $\|B_b(t) - B_b\| \leqslant C|t|/b^{\beta-1}$.

Moreover, we check as in [14, Lemma 4.2.] that $\|B_b\| = O(m(\Lambda_b)) = O(1/b^\beta)$. Finally, as $\int_B B_b u \, \mathrm{d}m = \int_{\Lambda_b} u \, \mathrm{d}m$ we get $\sum_b \int_B B_b u \, \mathrm{d}m = \int_X u \, \mathrm{d}m$. □

Lemma 4.4. *Let $u \in L^\infty(X)$. There exists a constant $C > 0$ such that, for any $t \in \mathbb{R}$, for any $n \in \mathbb{N}$,*

$$\left\| C_n(t)(u) \right\|_{L^1} \leqslant \frac{C}{n^{\beta-1}} \|u\|_\infty.$$

Proof. Set $X_{n+1} = X \setminus \bigcup_{i=0}^n T^{-i} B$ and $Z_{n+1} = T^n(X_{n+1})$. The function $C_n(u)$ vanishes outside of Z_{n+1}. Let $x \in Z_{n+1}$ and let y be its preimage under T^n. Then $|C_n(t)(u)(x)| = |u(y)| \leqslant \|u\|_\infty$. Hence, $\|C_n(t)(u)\|_{L^1} \leqslant m(Z_{n+1})\|u\|_\infty$. Finally, $m(Z_{n+1}) = m(X_{n+1}) = \sum_{p=n+1}^{\infty} m(\varphi > p) = O(1/n^{\beta-1})$. □

4.2. Result for the induced map

Eq. (15) and (2) in Lemma 4.1 imply that the perturbed transfer operator $R(1,t) = \hat{T}_B(t) = \hat{T}_B(\mathrm{e}^{\mathrm{i}tf_B} \cdot)$ has a spectral gap. Thus, the classical methods of [17] apply to it, and yield an asymptotic expansion of the maximal eigenvalue of $\hat{T}_B(t)$ (this implies a central limit theorem for T_B and the function f_B, but we are not interested in it here). To estimate the speed in the central limit theorem, we will need a rather precise asymptotic expansion of this eigenvalue, given by the following proposition.

Proposition 4.5. *Assume that T and f satisfy the assumptions of Theorem 1.1 in a mixing Young tower. Let σ^2 be the variance in this theorem. Then, for small enough t, $R(1,t)$ has a unique eigenvalue $\lambda(1,t)$ close to 1. It can be written as $\lambda(1,t) = 1 - \frac{\sigma^2}{2m(B)} L(t)$ for a function L such that $L(t) \sim t^2$ when $t \to 0$.*

Write $E_B(u) = \frac{1}{m(B)} \int_B u$, and define a function a on B by $a = (I - \hat{T}_B)^{-1}(\hat{T}_B f_B)$. Then

$$\lambda(1,t) = E_B(\mathrm{e}^{\mathrm{i}tf_B}) - t^2 E_B(a f_B) + O(t^3). \tag{21}$$

Proof. The fact that $\lambda(1,t) = 1 - \frac{\sigma^2}{2m(B)}(t^2 + o(t^2))$ is a consequence of [14, Theorem 3.7]. It remains to prove (21). As everything takes place in B, we can multiply m by a constant, and assume that $m(B) = 1$. Set $R_t = R(1,t)$, and let ξ_t be the eigenfunction of R_t corresponding to its eigenvalue λ_t close to 1. We normalize it so that $\int \xi_t = 1$. We will also write $R = R_0 = \hat{T}_B$.

Lemma 3.4 of [14] states that there exists a constant C such that, if $g : B \to \mathbb{R}$ is integrable,

$$\|Rg\|_\nu \leqslant C\left(\|g\|_{L^1} + \sum_I m(B_{i,0}) D_\nu g(B_{i,0})\right) \tag{22}$$

where $D_\nu g(B_{i,0})$ is the best ν-Hölder constant of g on $B_{i,0}$, defined in (16). In particular, $Rf_B \in C_\nu(B)$.

Let us show that, in $C_\nu(B)$,

$$R\left(\frac{\mathrm{e}^{\mathrm{i}tf_B} - 1}{t}\right) = \mathrm{i}R(f_B) + O(t). \tag{23}$$

The Taylor Formula gives

$$\frac{\mathrm{e}^{\mathrm{i}tf_B} - 1}{t} - \mathrm{i}f_B = -tf_B^2 \int_0^1 (1-u)\,\mathrm{e}^{\mathrm{i}tuf_B}\,\mathrm{d}u.$$

We use this formula to bound $D_\nu(\frac{\mathrm{e}^{\mathrm{i}tf_B}-1}{t} - \mathrm{i}f_B)(B_{i,0})$, where $B_{i,0}$ is an element of the partition of B. Set $n = \varphi(B_{i,0})$ the return time on $B_{i,0}$. As $D_\nu(h_1h_2) \leqslant D_\nu(h_1)\|h_2\|_\infty + \|h_1\|_\infty D_\nu(h_2)$,

$$\begin{aligned} &D_\nu\left(\frac{\mathrm{e}^{\mathrm{i}tf_B} - 1}{t} - \mathrm{i}f_B\right)(B_{i,0}) \\ &\quad\leqslant |t|\left\|(f_B)_{|B_{i,0}}\right\|_\infty D_\nu f_B(B_{i,0}) + |t|\left\|(f_B)_{|B_{i,0}}\right\|_\infty^2 \int_0^1 D_\nu(\mathrm{e}^{\mathrm{i}tuf_B})(B_{i,0})\,\mathrm{d}u. \end{aligned}$$

The first term is $\leqslant |t|n^2$. For the second term, take C such that $|\mathrm{e}^{\mathrm{i}s} - 1| \leqslant C|s|^\eta$ for any $s \in \mathbb{R}$ (where η was defined in (17)). Then, if $x, y \in B_{i,0}$,

$$\begin{aligned} |\mathrm{e}^{\mathrm{i}tuf_B(x)} - \mathrm{e}^{\mathrm{i}tuf_B(y)}| &= |\mathrm{e}^{\mathrm{i}tu(f_B(x) - f_B(y))} - 1| \leqslant C\left|tu\big(f_B(x) - f_B(y)\big)\right|^\eta \\ &\leqslant C\left|D_\tau f_B(B_{i,0})\tau^{s(x,y)}\right|^\eta \leqslant Cn^\eta \nu^{s(x,y)}, \end{aligned}$$

whence $D_\nu(\mathrm{e}^{\mathrm{i}tuf_B})(B_{i,0}) \leqslant Cn^\eta$. An integration yields

$$D_\nu\left(\frac{\mathrm{e}^{\mathrm{i}tf_B} - 1}{t} - \mathrm{i}f_B\right)(B_{i,0}) \leqslant |t|n^{2+\eta}.$$

Eq. (22) gives that

$$\left\|R\left(\frac{\mathrm{e}^{\mathrm{i}tf_B} - 1}{t} - \mathrm{i}f_B\right)\right\|_\nu \leqslant C\left(\left\|\frac{\mathrm{e}^{\mathrm{i}tf_B} - 1}{t} - \mathrm{i}f_B\right\|_{L^1} + \sum m[\varphi = n]|t|n^{2+\eta}\right).$$

As $|\mathrm{e}^{\mathrm{i}tf_B} - 1 - \mathrm{i}tf_B| \leqslant t^2 f_B^2$, with f_B^2 integrable, the first term is $O(t)$. For the second term, $\sum m[\varphi = n]n^{2+\eta} = \sum(m[\varphi > n-1] - m[\varphi > n])n^{2+\eta} \leqslant C\sum m[\varphi > n]n^{1+\eta}$. Since $m[\varphi > n] = O(1/n^\beta)$, this sum is finite by definition of η. This proves (23).

We return to the study of the eigenvalue λ_t. As $\lambda_t\xi_t = R_t\xi_t$, we get after integration that

$$\lambda_t = E(\mathrm{e}^{\mathrm{i}tf_B}) + \int (\mathrm{e}^{\mathrm{i}tf_B} - 1)(\xi_t - 1). \tag{24}$$

As $f_B \in L^2$ and $\int f_B = 0$, we have

$$E(e^{itf_B}) = 1 + it\int f_B - \frac{t^2}{2}\int f_B^2 + o(t^2) = 1 - \frac{t^2}{2}\int f_B^2 + o(t^2). \tag{25}$$

Moreover, by (4) in Lemma 4.1, $\xi_t - 1 = O(\|R_t - R\|) = O(t)$ in $C_\nu(B)$, hence in L^2, and $e^{itf_B} - 1 = itf_B + o(t) = O(t)$ in L^2. Consequently, $\int (e^{itf_B} - 1)(\xi_t - 1) = O(t^2)$, which implies that $\lambda_t = 1 + O(t^2)$.

Thus,

$$\frac{\xi_t - 1}{t} = \frac{\lambda_t \xi_t - \xi_0}{t} + O(t) = (R_t - R_0)\frac{\xi_t - \xi_0}{t} + R\frac{\xi_t - \xi_0}{t} + \frac{R_t - R_0}{t}\xi_0 + O(t).$$

As $R_t - R_0 = O(t)$ and $(\xi_t - \xi_0)/t$ is bounded, $(R_t - R_0)(\xi_t - \xi_0)/t = O(t)$. Moreover, $((R_t - R_0)/t)\xi_0 = R((e^{itf_B} - 1)/t)$. Hence,

$$(I - R)\frac{\xi_t - \xi_0}{t} = R\left(\frac{e^{itf_B} - 1}{t}\right) + O(t).$$

As $Rf_B \in C_\nu(B)$, and $I - R$ is invertible on the functions of $C_\nu(B)$ with vanishing integral ((3) in Lemma 4.1), it is possible to define $a = (I - R)^{-1}(Rf_B) \in C_\nu(B)$. Then, using (23),

$$(I - R)\left(\frac{\xi_t - \xi_0}{t} - ia\right) = R\left(\frac{e^{itf_B} - 1}{t} - if_B\right) + O(t) = O(t).$$

As the inverse of $I - R$ is continuous on the functions with zero integral, we get that, in $C_\nu(B)$ (hence in L^∞),

$$\xi_t = 1 + tia + O(t^2).$$

As $e^{itf_B} = 1 + itf_B + O(t^2)$ in L^1 since f_B^2 is integrable, we get

$$\int (e^{itf_B} - 1)(\xi_t - 1) = -t^2\int f_B a + O(t^3).$$

Eq. (24) yields the desired conclusion. □

4.3. The key estimate

Theorem 4.6. *Let X be a Young tower with $m[\varphi > n] = O(1/n^\beta)$ for $\beta > 2$, and $\gcd(\varphi_i) = 1$. Let $f \in C_\tau(X)$ be of zero integral. Let σ and $L(t)$ be given by Proposition 4.5, and assume $\sigma > 0$. Then there exist $\alpha > 0$, $C > 0$ and $d > 0$ such that, for any $u \in C_\tau(X)$ and $v \in L^\infty(X)$, for any $n \in \mathbb{N}^*$, for any $t \in [-\alpha, \alpha]$,*

$$\left|\int_X e^{itS_n f}\cdot u \cdot v \circ T^n \,dm - \left(1 - \frac{\sigma^2}{2}L(t)\right)^n \left(\int_X u\,dm\right)\left(\int_X v\,dm\right)\right|$$
$$\leqslant C\left[\frac{1}{n^{\beta-1}} + |t|\left(\frac{1}{n^{\beta-1}}\right) \star (1 - dt^2)^n\right]\|u\|\|v\|_\infty.$$

Proof. Set $R(z,t) = \sum R_n(t)z^n$, we want to apply Theorem 3.1 to $R(z,t)$. Proposition 4.5 gives the behavior of the eigenvalue of $R(1,t)$, while Lemma 4.1 shows the required estimates. Finally, the spectral projection P of $R(1,0)$ is the projection on the constant functions on B, given by $Pg = (\int_B g\,dm)/m(B)$. It satisfies $PR'(1)P = \frac{1}{m(B)}P$ [14, Lemma 4.4].

Consequently, we can apply Theorem 3.1 with $M(t) = \frac{\sigma^2}{2m(B)}L(t)$ and $\mu = \frac{1}{m(B)}$. As $\sum T_n(t)z^n = (I - R(z,t))^{-1}$ by (12), we get that there exists an error term $E(n,t)$ such that $T_n(t) = m(B)(1 - \frac{\sigma^2}{2}L(t))^n P + E(n,t)$, with

$$\|E(n,t)\| \leqslant C\left[\frac{1}{n^{\beta-1}} + |t|\left(\frac{1}{n^{\beta-1}}\right) \star (1 - dt^2)^n\right] =: e(n,t). \tag{26}$$

In the following, C and d will denote generic constants, that may vary finitely many times. In particular, we may write inequalities like $10e(n,t) \leqslant e(n,t)$. With this convention, Lemma 3.2 implies that

$$\left(\frac{1}{n^{\beta-1}}\right) \star e(n,t) = O\big(e(n,t)\big). \tag{27}$$

Let $u \in C_\tau(X)$ and $v \in L^\infty(X)$. To simplify the expressions, we will assume that these functions are of norm at most 1. Then (13) implies that

$$\begin{aligned}\int_X \mathrm{e}^{\mathrm{i}tS_nf} \cdot u \cdot v \circ T^n &= \int_X \hat{T}^n(\mathrm{e}^{\mathrm{i}tS_nf}u)v = \int_X \hat{T}(t)^n(u)v \\ &= \sum_{a+k+b=n} \int_X A_a(t)T_k(t)B_b(t)(u)v + \int_X C_n(t)(u)v \\ &= \sum_{a+k+b=n} \int_X A_a(t)m(B)\left(1-\frac{\sigma^2}{2}L(t)\right)^k PB_b(t)(u)v \\ &\quad + \sum_{a+k+b=n} \int_X A_a(t)E(k,t)B_b(t)(u)v + \int_X C_n(t)(u)v. \end{aligned} \tag{28}$$

By Lemma 4.4, $|\int_X C_n(t)(u)v| \leqslant C/n^{\beta-1} \leqslant e(n,t)$.

Let us bound the second sum of (28). If $h \in C_\nu(X)$, the function $A_a(t)h$ is supported in $T^a\{y \in B \mid \varphi(y) > a\}$, whose measure is $m[\varphi > a] = O(1/a^\beta)$. Thus, $|\int_X A_a(t)h| \leqslant (C/a^\beta)\|h\|_\infty \leqslant (C/a^{\beta-1})\|h\|$. Moreover, $\|E(k,t)B_b(t)(u)\| \leqslant e(k,t)\|B_b(t)\| \leqslant e(k,t)C/b^{\beta-1}$ by Lemma 4.3. Thus,

$$\begin{aligned}\left|\sum_{a+k+b=n} \int_X A_a(t)E(k,t)B_b(t)(u)v\right| &\leqslant \sum_{a+k+b=n} \frac{C}{a^{\beta-1}} e(k,t) \frac{C}{b^{\beta-1}} \\ &\leqslant C\left(\frac{1}{n^{\beta-1}}\right) \star e(n,t) \star \left(\frac{1}{n^{\beta-1}}\right) \leqslant e(n,t)\end{aligned} \tag{29}$$

by (27).

The first sum of (28) can be written as

$$I = \sum_{a+k+b=n} \left(\int_X A_a(t)(1_B)v\right)\left(1-\frac{\sigma^2}{2}L(t)\right)^k \left(\int_B B_b(t)(u)\right).$$

Using Lemmas 4.2 and 4.3 and convolving, we find a sequence w_n such that $w_n = O(1/n^\beta)$, $\sum w_n = (\int u)(\int v)$, and

$$I = \left(w_n + O\left(\frac{|t|}{n^{\beta-1}}\right)\right) \star \left(1-\frac{\sigma^2}{2}L(t)\right)^k.$$

As $L(t) \sim t^2$ when $t \to 0$, the term coming from $O(|t|/n^{\beta-1})$ is bounded by $|t|(1/n^{\beta-1}) \star (1-dt^2)^n \leqslant e(n,t)$. Moreover, for $x, y \in \mathbb{R}$,

$$|\mathrm{e}^x - \mathrm{e}^y| \leqslant |x-y|\,\mathrm{e}^{\max(x,y)}. \tag{30}$$

Thus,

$$\left|\sum_{k=0}^{n} w_{n-k}\left(1-\frac{\sigma^2}{2}L(t)\right)^k - \sum_{k=0}^{n} w_{n-k}\left(1-\frac{\sigma^2}{2}L(t)\right)^n\right|$$
$$\leqslant \sum_{k=0}^{n} \frac{C}{(n-k)^\beta}(n-k)\left|\ln\left(1-\frac{\sigma^2}{2}L(t)\right)\right|\left(1-\frac{\sigma^2}{2}L(t)\right)^k$$
$$\leqslant \sum_{k=0}^{n} Ct^2(1-dt^2)^k\frac{1}{(n-k)^{\beta-1}} \leqslant e(n,t).$$

Hence, up to $O(e(n,t))$, the integral $\int e^{itS_n f}\cdot u\cdot v\circ T^n$ is equal to $\sum_{k=0}^n w_{n-k}(1-\frac{\sigma^2}{2}L(t))^n$. Finally,

$$\left|\left(1-\frac{\sigma^2}{2}L(t)\right)^n \int_X u\int_X v - \sum_{k=0}^{n} w_{n-k}\left(1-\frac{\sigma^2}{2}L(t)\right)^n\right|$$
$$=\left(1-\frac{\sigma^2}{2}L(t)\right)^n\left|\sum_{k=n+1}^{\infty} w_k\right| \leqslant C\sum_{k=n+1}^{\infty}\frac{1}{k^\beta} \leqslant \frac{C}{n^{\beta-1}} \leqslant e(n,t).$$

This concludes the proof. □

When $u=v=1$, Theorem 4.6 states that, for small t, the characteristic function of $S_n f$ behaves essentially like $(1-\frac{\sigma^2}{2}L(t))^n$, which is very similar to the characteristic function of the sum of n independent identically distributed random variables. Hence, using this estimate, it will be possible to use the classical probabilistic proofs to get the local limit theorem or the Berry–Esseen theorem. However, some care is still required to check that the error term in Theorem 4.6 is sufficiently small so that these proofs still work.

5. Proof of the local limit theorem

5.1. Periodicity problems

This paragraph is related to the end of Section 3 of [3]. The differences come mainly from the inductive process and the fact that we are considering series of operators instead of a single operator.

Let X be a Young tower with $\gcd(\varphi_i)=1$, $\tau<1$ and $f\in C_\tau(X)$. For $t\in\mathbb{R}$, e^{itf} is said to be *cohomologous* to a constant $\lambda\in S^1$ if there exists $\omega: X\to S^1$ such that $e^{itf}=\lambda(\omega\circ T)/\omega$ almost everywhere.

Proposition 5.1. *Let $t\in\mathbb{R}$, and $z\in S^1$. The following assertions are equivalent*:

1. e^{itf} *is cohomologous to* z^{-1}.
2. 1 *is an eigenvalue of the operator* $R(z,t)$ *acting on* $C_\nu(B)$.

Proof. Suppose first that there exists a nonzero $\omega\in C_\nu(B)$ such that $R(z,t)\omega=\omega$. As $R(z,t)(v)=\hat{T}_B(z^\varphi e^{itf_B}v)$, the adjoint operator of $R(z,t)$ in L^2 is $W(v)=z^{-\varphi}e^{-itf_B}v\circ T_B$. Then $\|W\omega-\omega\|_{L^2}^2=\|W\omega\|_{L^2}^2-\|\omega\|_{L^2}^2$. As T_B preserves the measure, we have $\|\omega\|_{L^2}=\|W\omega\|_{L^2}$, whence $\|W\omega-\omega\|_{L^2}=0$. Consequently, $\omega=z^{-\varphi}e^{-itf_B}\omega\circ T_B$ almost everywhere. Taking the modulus, $|\omega|=|\omega|\circ T_B$, and the ergodicity of T_B gives that $|\omega|$ is almost everywhere constant. We can assume that $|\omega|=1$. We extend the function ω to X by setting

$$\omega(x,k)=\omega(x,0)e^{itf(x,0)}\cdots e^{itf(x,k-1)}z^k.$$

Then, for $k < \varphi(x) - 1$, we have by construction $\omega \circ T(x,k)/\omega(x,k) = z\,\mathrm{e}^{\mathrm{i}tf(x,k)}$. Moreover, for $k = \varphi(x) - 1$,

$$\omega \circ T(x,k)/\omega(x,k) = \omega \circ T_B(x,0)/\omega(x,k) = \mathrm{e}^{\mathrm{i}tf_B(x,0)} z^{k+1} \omega(x,0)/\omega(x,k) = \mathrm{e}^{\mathrm{i}tf(x,k)} z.$$

Thus, $\mathrm{e}^{\mathrm{i}tf} = z^{-1}\omega \circ T/\omega$ almost everywhere.

Conversely, suppose that a measurable function ω satisfies $\mathrm{e}^{\mathrm{i}tf} = z^{-1}\omega \circ T/\omega$. The previous calculations give $\mathrm{e}^{\mathrm{i}tf_B} = z^{-\varphi}\omega \circ T_B/\omega$. The operator $R(z,t) = \hat{T}_B(z^{\varphi}\,\mathrm{e}^{\mathrm{i}tf_B}\cdot)$ acts on L^1, and satisfies

$$R(z,t)(\omega) = \hat{T}_B(z^{\varphi}\,\mathrm{e}^{\mathrm{i}tf_B}\omega) = \hat{T}_B(\omega \circ T_B) = \omega.$$

But, in Lemma 4.1, we have seen that $R(z,t)$ satisfies a Doeblin–Fortet inequality between the spaces $C_\nu(B)$ and $L^1(B)$. [20] ensures that the eigenfunctions of $R(z,t)$ in $L^1(B)$ for the eigenvalue 1 are in fact in $C_\nu(B)$, i.e. $\omega \in C_\nu(B)$. □

Corollary 5.2. *The set $\mathfrak{A} := \{t \in \mathbb{R} \mid \mathrm{e}^{\mathrm{i}tf}$ is cohomologous to a constant$\}$ is a closed subgroup of $\mathbb{R}$. Moreover, for every $t \in \mathfrak{A}$, there exists a unique $z(t) \in S^1$ such that $\mathrm{e}^{\mathrm{i}tf}$ is cohomologous to $z(t)$. Finally, the map $t \mapsto z(t)$ is a continuous morphism from $\mathfrak{A}$ to S^1.*

Proof. The set $\mathfrak{A}$ is clearly a subgroup of $\mathbb{R}$. If $\mathrm{e}^{\mathrm{i}tf}$ is cohomologous simultaneously to z and z', then $z' = (\mathrm{e}^{\mathrm{i}h\circ T}/\mathrm{e}^{\mathrm{i}h})z$ for some function $h : X \to \mathbb{R}$. As T is mixing [35, Theorem 1(iii)], the only constant s satisfying $g \circ T = sg$ for some nonzero function g is 1. This implies that $z = z'$. The map $t \mapsto z(t)$ is thus well defined, and it is clearly a group morphism.

It remains to check that $\mathfrak{A}$ is closed and that $z(t)$ is continuous. Let t_n be a sequence of $\mathfrak{A}$ converging to $T \in \mathbb{R}$. Let Z be a cluster point of the sequence $z(t_n)$. By (5) in Lemma 4.1, $(z,t) \mapsto R(z,t)$ is a continuous map with values in $\mathrm{Hom}(C_\nu(B))$. If $I - R(Z^{-1}, T)$ were invertible, then $I - R(z_n^{-1}, t_n)$ would also be invertible for large enough n, which is a contradiction by the previous proposition. Thus, $I - R(Z^{-1}, T)$ is not invertible, whence 1 is an eigenvalue of $R(Z^{-1}, T)$ by quasi-compactness. This implies that $T \in \mathfrak{A}$ and $Z = z(T)$, once again by the previous proposition. □

Consequently, there are three cases to be considered for the local limit theorem: $\mathfrak{A}$ is either $\mathbb{R}$, or $\{0\}$, or a discrete subgroup of $\mathbb{R}$. If it is $\mathbb{R}$, [25] ensures that f can be written as $g - g \circ T$, hence $\sigma = 0$ in the central limit theorem, and there is nothing to prove. If $\mathfrak{A} = \{0\}$, it is not possible to write f as $\rho + g - g \circ T + \lambda q$, where $\rho \in \mathbb{R}$, $g : X \to \mathbb{R}$ is measurable, $\lambda > 0$ and $q : X \to \mathbb{Z}$, i.e. f is aperiodic. This case is handled by Theorem 1.2. Finally, $\mathfrak{A} = 2\pi\mathbb{Z}$ means that $f = \rho + g - g \circ T + q$ where q takes integer values, and that there is no such expression where q takes its values in $n\mathbb{Z}$ with $n \geqslant 2$. This is dealt with in Theorem 5.3.

5.2. The aperiodic case

Proof of Theorem 1.2. Let f be an aperiodic function on X, as in the hypotheses of Theorem 1.2. Then, for every $(z,t) \in (\overline{\mathbb{D}} \times \mathbb{R}) - \{(1,0)\}$, $I - R(z,t)$ is invertible: for $|z| < 1$, the spectral radius of $R(z,t)$ is at most $|z| < 1$ ((2) in Lemma 4.1), and for $|z| = 1$ this comes from Proposition 5.1 and Corollary 5.2, since $\mathfrak{A} = \{0\}$.

Let $\alpha > 0$ be given by Theorem 4.6: we control the behavior of the integrals when $|t| \leqslant \alpha$. Take $K > 0$. We will show the following fact: *there exists $C > 0$ such that, for every $|t| \in [\alpha, K]$, for every $n \in \mathbb{N}^*$, for all functions $u \in C_\tau(X)$ and $v \in L^\infty(X)$,*

$$\left| \int_X \mathrm{e}^{\mathrm{i}tS_nf} \cdot u \cdot v \circ T^n \right| \leqslant \frac{C}{n^{\beta-1}} \|u\| \|v\|_\infty. \tag{31}$$

Let us write $\mathcal{A} = \mathcal{O}^+_{\beta-1}(\mathrm{Hom}(C_\nu(B)))$ (the Banach algebra of series $\sum_0^\infty A_n z^n$ where $A_n \in \mathrm{Hom}(C_\nu(B))$ and $\|A_n\| = O(1/n^{\beta-1})$, with the norm $\|\sum A_n z^n\| = \sup_{n\in\mathbb{N}}(n+1)^{\beta-1}\|A_n\|$). The map $t \mapsto R(z,t)$ is continuous from $[-K,-\alpha] \cup [\alpha,K]$ to $\mathcal{A}$ ((5) in Lemma 4.1). Moreover, $I - R(z,t)$ is invertible on $\overline{\mathbb{D}}$ for t in these intervals. Theorem A.4 shows that $(I-R(z,t))^{-1}\in\mathcal{A}$, and the continuity of the inversion even yields that $t\mapsto(I-R(z,t))^{-1}$ is continuous. By compactness, there exists C such that $\|(I-R(z,t))^{-1}\|_{\mathcal{A}}\leqslant C$ for $|t|\in[\alpha,K]$. As $(I-R(z,t))^{-1}=\sum T_n(t)z^n$, this implies that $\|T_n(t)\|\leqslant C/n^{\beta-1}$, uniformly in n and t.

We have

$$\int_X \mathrm{e}^{\mathrm{i}tS_nf}\cdot u\cdot v\circ T^n=\int_X C_n(t)(u)v+\sum_{a+k+b=n}\int_X A_a(t)T_k(t)B_b(t)(u)v.$$

By Lemma 4.4, $|\int_X C_n(t)(u)v|\leqslant(C/n^{\beta-1})\|u\|_\infty\|v\|_\infty$. By Lemmas 4.2 and 4.3,

$$\begin{aligned}\left|\int A_a(t)T_k(t)B_b(t)(u)v\right|&\leqslant\frac{C}{a^{\beta-1}}\|T_k(t)B_b(t)(u)\|_\infty\|v\|_\infty\leqslant\frac{C}{a^{\beta-1}}\|T_k(t)\|\|B_b(t)\|\|u\|\|v\|_\infty\\&\leqslant\frac{C}{a^{\beta-1}}\frac{C}{k^{\beta-1}}\frac{C}{b^{\beta-1}}\|u\|\|v\|_\infty.\end{aligned}$$

Thus,

$$\left|\int_X \mathrm{e}^{\mathrm{i}tS_nf}\cdot u\cdot v\circ T^n\right|\leqslant C\left(\frac{1}{n^{\beta-1}}\right)\star\left(\frac{1}{n^{\beta-1}}\right)\star\left(\frac{1}{n^{\beta-1}}\right)\|u\|\|v\|_\infty\leqslant\frac{C}{n^{\beta-1}}\|u\|\|v\|_\infty,$$

and (31) is proved.

We prove now the local limit theorem, using the method of Breiman [7]. Take u, v and k_n as in the assumptions of Theorem 1.2. Let $\psi\in L^1(\mathbb{R})$ be such that its Fourier transform $\hat\psi$ is supported in $[-K,K]$. Then $\psi(x)=\frac{1}{2\pi}\int_{-K}^{K}\hat\psi(t)\,\mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}t$, whence

$$\begin{aligned}\sqrt{n}\,E\big(\psi(S_nf-k_n-u-v\circ T^n)\big)&=\frac{\sqrt{n}}{2\pi}\int_{-K}^{K}\hat\psi(t)E\big(\mathrm{e}^{\mathrm{i}t(S_nf-k_n-u-v\circ T^n)}\big)\,\mathrm{d}t\\&=\frac{\sqrt{n}}{2\pi}\int_{-\alpha}^{\alpha}\hat\psi(t)\,\mathrm{e}^{-\mathrm{i}tk_n}E\big(\mathrm{e}^{\mathrm{i}tS_nf}\,\mathrm{e}^{-\mathrm{i}tu}\,\mathrm{e}^{-\mathrm{i}tv\circ T^n}\big)\,\mathrm{d}t\\&\quad+\frac{\sqrt{n}}{2\pi}\int_{\alpha\leqslant|t|\leqslant K}\hat\psi(t)\,\mathrm{e}^{-\mathrm{i}tk_n}E\big(\mathrm{e}^{\mathrm{i}tS_nf}\,\mathrm{e}^{-\mathrm{i}tu}\,\mathrm{e}^{-\mathrm{i}tv\circ T^n}\big)\,\mathrm{d}t.\end{aligned}\tag{32}$$

For $\alpha\leqslant|t|\leqslant K$, the norms $\|\mathrm{e}^{-\mathrm{i}tu}\|$ and $\|\mathrm{e}^{-\mathrm{i}tv}\|_\infty$ remain bounded. Hence, (31) implies that $|E(\mathrm{e}^{\mathrm{i}tS_nf}\,\mathrm{e}^{-\mathrm{i}tu}\times\mathrm{e}^{-\mathrm{i}tv\circ T^n})|\leqslant C/n^{\beta-1}$. Therefore, the second integral tends to 0. For the first integral, we approximate $E(\mathrm{e}^{\mathrm{i}tS_nf}\times\mathrm{e}^{-\mathrm{i}tu}\,\mathrm{e}^{-\mathrm{i}tv\circ T^n})$ by $(1-\frac{\sigma^2}{2}L(t))^n\int\mathrm{e}^{-\mathrm{i}tu}\int\mathrm{e}^{-\mathrm{i}tv}$. By Theorem 4.6, the error term is bounded by

$$C\sqrt{n}\int_{-\alpha}^{\alpha}\left(\frac{1}{n^{\beta-1}}+|t|\left(\frac{1}{n^{\beta-1}}\right)\star(1-dt^2)^n\right)\mathrm{d}t.$$

Let us show that this integral tends to 0. This is clear for the first term. For the second term, we cut the integral in two pieces. For $|t|\leqslant 1/\sqrt{n}$, the convolution is bounded (since $\frac{1}{n^{\beta-1}}$ is summable and $(1-dt^2)^n\leqslant 1$), whence the integral is $\leqslant C\sqrt{n}\int_{|t|\leqslant 1/\sqrt{n}}|t|\,\mathrm{d}t\to 0$. For $|t|\geqslant 1/\sqrt{n}$, Lemma 3.2 gives that the convolution is bounded by $1/(|t|n^{\beta-1})+|t|(1-dt^2)^n$, whence the integral is less than

$$C\sqrt{n}\int_{1/\sqrt{n}\leqslant|t|\leqslant\alpha}\frac{1}{|t|n^{\beta-1}}+|t|(1-dt^2)^n\,\mathrm{d}t\leqslant C\sqrt{n}\,\frac{\ln n}{n^{\beta-1}}+C\sqrt{n}\left[\frac{(1-dt^2)^{n+1}}{-2d(n+1)}\right]_{1/\sqrt{n}}^{\alpha}\to 0.$$

Finally, we have proved that

$$\sqrt{n}\, E\big(\psi(S_n f - k_n - u - v\circ T^n)\big) = \frac{\sqrt{n}}{2\pi}\int_{-\alpha}^{\alpha} \hat{\psi}(t)\,\mathrm{e}^{-\mathrm{i}tk_n}\left(1-\frac{\sigma^2}{2}L(t)\right)^n E(\mathrm{e}^{-\mathrm{i}tu})E(\mathrm{e}^{-\mathrm{i}tv})\,\mathrm{d}t + o(1).$$

But

$$\begin{aligned}
&\frac{\sqrt{n}}{2\pi}\int_{-\alpha}^{\alpha} \hat{\psi}(t)\,\mathrm{e}^{-\mathrm{i}tk_n}\left(1-\frac{\sigma^2}{2}L(t)\right)^n E(\mathrm{e}^{-\mathrm{i}tu})E(\mathrm{e}^{-\mathrm{i}tv})\,\mathrm{d}t\\
&\quad= \frac{1}{2\pi}\int_{-\alpha\sqrt{n}}^{\alpha\sqrt{n}} \hat{\psi}\left(\frac{t}{\sqrt{n}}\right)\mathrm{e}^{-\mathrm{i}tk_n/\sqrt{n}}\left(1-\frac{\sigma^2}{2}L\left(\frac{t}{\sqrt{n}}\right)\right)^n E(\mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{n}}u})E(\mathrm{e}^{-\mathrm{i}\frac{t}{\sqrt{n}}v})\,\mathrm{d}t\\
&\quad\to \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\psi}(0)\,\mathrm{e}^{-\mathrm{i}t\kappa}\,\mathrm{e}^{-\frac{\sigma^2}{2}t^2}\,\mathrm{d}t,
\end{aligned}$$

by dominated convergence. We have used the fact that $L(t)\sim t^2$ close to 0, and in particular, if α is small enough, $(1-\frac{\sigma^2}{2}L(t/\sqrt{n}))^n \leqslant (1-\sigma^2t^2/(4n))^n \leqslant \mathrm{e}^{-\sigma^2t^2/4}$, which gives the domination.

Set $\chi(\kappa) = \mathrm{e}^{-\kappa^2/(2\sigma^2)}/(\sigma\sqrt{2\pi})$. We have proved that, for any ψ in L^1 with $\hat{\psi}$ compactly supported,

$$\sqrt{n}\, E\big(\psi(S_n f - k_n - u - v\circ T^n)\big) \to \chi(\kappa)\int_{\mathbb{R}} \psi(x)\,\mathrm{d}x. \tag{33}$$

Eq. (33) can then be extended to a larger class of functions by density arguments (see [7]), and this larger class contains in particular the characteristic functions of bounded intervals. This concludes the proof. □

5.3. The periodic case

The following theorem gives the local limit theorem when the group $\mathfrak{A}$ of Paragraph 5.1 is a discrete subgroup of $\mathbb{R}$, for example $2\pi\mathbb{Z}$.

Theorem 5.3 (local limit theorem, periodic case). *Let X be a Young tower with* $\gcd(\varphi_i)=1$. *Assume that $m[\varphi>n]=O(1/n^\beta)$ with $\beta>2$. Let $\tau<1$. Take $f\in C_\tau(X)$ of zero integral, and σ^2 given by Theorem* 1.1.

Assume that $f=\rho+q$ where q takes integer values and $\rho\in\mathbb{R}$, but that f cannot be written as $f=\rho'+g-g\circ T+\lambda q'$, where $\lambda\in\mathbb{N}-\{1\}$ and $q':X\to\mathbb{Z}$ (this implies in particular $\sigma>0$). Then, for every sequence k_n with $k_n-n\rho\in\mathbb{Z}$ such that $k_n/\sqrt{n}\to\kappa\in\mathbb{R}$,

$$\sqrt{n}\, m\big\{x\in X \mid S_n f(x) = k_n\big\} \to \frac{\mathrm{e}^{-\kappa^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}}.$$

Proof. This is essentially the same proof as that of Theorem 1.2, but we use the Fourier transform on $\mathbb{Z}$ (i.e. Fourier series) instead of the Fourier transform on $\mathbb{R}$.

If k and l are two integer numbers,

$$1_{k=l} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t(l-k)}\,\mathrm{d}t.$$

Applying this equation to $k_n - n\rho$ and $S_n f(x) - n\rho$ and integrating gives

$$m\{x \in X \mid S_n f(x) = k_n\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}tk_n} E(\mathrm{e}^{\mathrm{i}tS_n f})\,\mathrm{d}t.$$

This is an analogue of (32). From this point on, the proof of Theorem 1.2 applies. The only problem is to check that, on $[-\pi, -\alpha] \cup [\alpha, \pi]$, $I - R(z,t)$ is invertible for $z \in \overline{\mathbb{D}}$. This comes from the assumptions on f, which ensures that $I - R(z,t)$ is invertible as soon as $t \notin 2\pi\mathbb{Z}$, by Proposition 5.1 and Corollary 5.2, since $\mathfrak{A} = 2\pi\mathbb{Z}$. □

6. Proof of the central limit theorem with speed

Proof of Theorem 1.3. The Berry–Esseen Theorem [12] implies that the result will be proved if we show that, for some $c > 0$,

$$\int_{-c\sqrt{n}}^{c\sqrt{n}} \frac{1}{|t|} \left| E(\mathrm{e}^{\mathrm{i}(t/\sqrt{n})S_n f}) - \mathrm{e}^{-(\sigma^2/2)t^2} \right| \mathrm{d}t = O\left(\frac{1}{n^{\delta/2}}\right).$$

We first estimate the integral between $-1/n$ and $1/n$:

$$\begin{aligned}\int_{-1/n}^{1/n} \frac{1}{|t|} \left| E(\mathrm{e}^{\mathrm{i}(t/\sqrt{n})S_n f}) - \mathrm{e}^{-(\sigma^2/2)t^2} \right| \mathrm{d}t &\leqslant \int_{-1/n}^{1/n} \frac{1}{|t|} \left| E(\mathrm{e}^{\mathrm{i}(t/\sqrt{n})S_n f}) - 1 \right| \mathrm{d}t + \int_{-1/n}^{1/n} \frac{1}{|t|} |e^{-(\sigma^2/2)t^2} - 1|\,\mathrm{d}t \\ &\leqslant \int_{-1/n}^{1/n} \frac{1}{\sqrt{n}} E(|S_n f|)\,\mathrm{d}t + \int_{-1/n}^{1/n} \frac{\sigma^2}{2} |t|\,\mathrm{d}t.\end{aligned}$$

But $\int |S_n f| \leqslant n \int |f|$, whence we get $O(1/\sqrt{n})$ for this term.

Let $L(t)$ be given by Proposition 4.5. Then, for small enough c,

$$\begin{aligned}&\int_{1/n \leqslant |t| \leqslant c\sqrt{n}} \frac{1}{|t|} \left| E(\mathrm{e}^{\mathrm{i}(t/\sqrt{n})S_n f}) - \mathrm{e}^{-(\sigma^2/2)t^2} \right| \mathrm{d}t \\ &\quad\leqslant \int_{1/n \leqslant |t| \leqslant c\sqrt{n}} \frac{1}{|t|} \left| \left(1 - \frac{\sigma^2}{2} L\left(\frac{t}{\sqrt{n}}\right)\right)^n - \mathrm{e}^{-(\sigma^2/2)t^2} \right| \mathrm{d}t \\ &\qquad + \int_{1/n \leqslant |t| \leqslant c\sqrt{n}} \frac{1}{|t|} \left| E(\mathrm{e}^{\mathrm{i}(t/\sqrt{n})S_n f}) - \left(1 - \frac{\sigma^2}{2} L\left(\frac{t}{\sqrt{n}}\right)\right)^n \right| \mathrm{d}t.\end{aligned}$$

Let us show that the second term satisfies

$$\int_{1/n \leqslant |t| \leqslant c\sqrt{n}} \frac{1}{|t|} \left| E(\mathrm{e}^{\mathrm{i}(t/\sqrt{n})S_n f}) - \left(1 - \frac{\sigma^2}{2} L\left(\frac{t}{\sqrt{n}}\right)\right)^n \right| \mathrm{d}t = O\left(\frac{1}{\sqrt{n}}\right).$$

Set $e(n,t) = C[1/n^{\beta-1} + |t|(1/n^{\beta-1}) \star (1 - dt^2)^n]$. By Theorem 4.6,

$$\left| E(\mathrm{e}^{\mathrm{i}(t/\sqrt{n})S_n f}) - \left(1 - \frac{\sigma^2}{2} L\left(\frac{t}{\sqrt{n}}\right)\right)^n \right| \leqslant e\left(n, \frac{t}{\sqrt{n}}\right).$$

Thus, it is enough to prove that

$$\int_{1/n\leqslant|t|\leqslant c\sqrt{n}} \frac{e(n,t/\sqrt{n})}{|t|}\,\mathrm{d}t = O\left(\frac{1}{\sqrt{n}}\right).$$

For the term $1/n^{\beta-1}$ in $e(n,t)$, the integral is $C(\ln n)/n^{\beta-1}$, which is $O(1/\sqrt{n})$.

For the other term $c(n,t)=|t|(1/n^{\beta-1})\star(1-dt^2)^n$, we cut the integral in two pieces. For $|t|\leqslant 1$, the convolution is bounded (since $1/n^{\beta-1}$ is summable and $(1-dt^2/n)^n\leqslant 1$). It remains $\int_{1/n}^1 \frac{1}{|t|}\frac{|t|}{\sqrt{n}}\,\mathrm{d}t\leqslant\frac{1}{\sqrt{n}}$. For $|t|\geqslant 1$, we use Lemma 3.2, which gives that $c(n,t)\leqslant 1/(|t|n^{\beta-1})+|t|(1-\frac{d}{2}t^2)^n$. Thus,

$$\int_1^{\sqrt{n}} \frac{1}{|t|}\left|c\left(n,\frac{t}{\sqrt{n}}\right)\right|\mathrm{d}t \leqslant \int_1^{\sqrt{n}} \frac{\sqrt{n}}{t^2 n^{\beta-1}}\,\mathrm{d}t + \frac{1}{\sqrt{n}}\int_1^{\sqrt{n}} \mathrm{e}^{-dt^2/2}\,\mathrm{d}t = O\left(\frac{1}{\sqrt{n}}\right).$$

Finally, we have proved that

$$\int_{-c\sqrt{n}}^{c\sqrt{n}} \frac{1}{|t|}\left|E\left(\mathrm{e}^{\mathrm{i}(t/\sqrt{n})S_n f}\right)-\mathrm{e}^{-(\sigma^2/2)t^2}\right|\mathrm{d}t \leqslant \int_{|t|\leqslant c\sqrt{n}} \frac{1}{|t|}\left|\left(1-\frac{\sigma^2}{2}L\left(\frac{t}{\sqrt{n}}\right)\right)^n-\mathrm{e}^{-(\sigma^2/2)t^2}\right|\mathrm{d}t + O\left(\frac{1}{\sqrt{n}}\right). \tag{34}$$

We have only to deal with the powers of a function. Hence, it will be possible to use the same methods as in probability theory. More precisely, the study of the speed in the central limit theorem in [19, Theorem 3.4.1] uses the two following facts:

1. If a random variable Z satisfies $E(|Z|^2 1_{|Z|>z})=O(z^{-\delta})$ with $0<\delta\leqslant 1$ and, in the $\delta=1$ case, $E(Z^3 1_{|Z|\leqslant z})=O(1)$, then there exists a constant $\lambda^2\geqslant 0$ such that $E(\mathrm{e}^{\mathrm{i}tZ})=1-\frac{\lambda^2}{2}t^2(1+\gamma(t))$, with $\int_{-x}^x t^2|\gamma(t)|\,\mathrm{d}t=O(x^{3+\delta})$ when $x\to 0$.
2. If a function $\tilde\gamma(t)$ satisfies $\int_{-x}^x t^2|\tilde\gamma(t)|\,\mathrm{d}t=O(x^{3+\delta})$ with $0<\delta\leqslant 1$, and $\sigma^2\geqslant 0$, then $\Lambda(t):=t^2(1+\tilde\gamma(t))$ satisfies $\int_{|t|\leqslant c\sqrt{n}}\frac{1}{|t|}|(1-\frac{\sigma^2}{2}\Lambda(\frac{t}{\sqrt{n}}))^n-\mathrm{e}^{-(\sigma^2/2)t^2}|\,\mathrm{d}t=O(n^{-\delta/2})$.

By Proposition 4.5, the eigenvalue $\lambda(t)$ of $\hat T_B(t)$ is equal to $E_B(\mathrm{e}^{\mathrm{i}tf_B})+\alpha t^2+O(t^3)$ for some constant α. The fact 1 applied to the random variable $f_B: B\to\mathbb{R}$ implies that $E_B(\mathrm{e}^{\mathrm{i}tf_B})=1-\frac{\lambda^2}{2}t^2(1+\gamma(t))$ for some function $\gamma(t)$ satisfying $\int_{-x}^x t^2|\gamma(t)|\,\mathrm{d}t=O(x^{3+\delta})$. As $\lambda(t)=1-\frac{\sigma^2}{2m(B)}L(t)$, this implies that $L(t)=t^2(1+m(B)\frac{\lambda^2}{\sigma^2}\gamma(t)+O(t))$. Hence, we can write $L(t)=t^2(1+\tilde\gamma(t))$ with $\int_{-x}^x t^2|\tilde\gamma(t)|\,\mathrm{d}t=O(x^{3+\delta})$. Therefore, the fact 2 implies that $\int_{|t|\leqslant c\sqrt{n}}\frac{1}{|t|}|(1-\frac{\sigma^2}{2}L(\frac{t}{\sqrt{n}}))^n-\mathrm{e}^{-(\sigma^2/2)t^2}|\,\mathrm{d}t=O(n^{-\delta/2})$.

By (34), we obtain $\int_{-c\sqrt{n}}^{c\sqrt{n}}\frac{1}{|t|}|E(\mathrm{e}^{\mathrm{i}(t/\sqrt{n})S_n f})-\mathrm{e}^{-(\sigma^2/2)t^2}|\,\mathrm{d}t=O(n^{-\delta/2})$, which concludes the proof. □

Appendix. The Wiener Lemma

In this appendix, we prove that the algebra $\mathcal{O}_\gamma(\mathcal{C})$ introduced in Paragraph 3.1 is indeed a Banach algebra, and that it satisfies a Wiener Lemma (Theorem A.3).

Let $\mathcal{C}$ be a Banach algebra, and take $\gamma > 1$. Write $w_n = (n+1)^{-\gamma}$ for $n \geqslant 0$. There exists a constant c such that $(w_n) \star (w_n) \leqslant c w_n$. We define a norm on $\mathcal{O}_\gamma(\mathcal{C})$ by

$$\left\| \sum_{n\in\mathbb{Z}} A_n z^n \right\| = \left(\sum_{n\in\mathbb{Z}} \|A_n\| + c \sup_{n\geqslant 0} \frac{\|A_n\|}{w_n} \right) + \left(\sum_{n\in\mathbb{Z}} \|A_n\| + c \sup_{n\leqslant 0} \frac{\|A_n\|}{w_{|n|}} \right). \tag{A.1}$$

Proposition A.1. *Let $\mathcal{C}$ be a Banach algebra, and $\gamma > 1$. With the norm* (A.1), *$\mathcal{O}_\gamma(\mathcal{C})$ is a Banach algebra.*

Proof. The completeness is clear. It is sufficient to prove the submultiplicativity of the norm for one half of this norm, for example the first one. Let us write $\|\sum A_n z^n\|_1 = \sum \|A_n\|$, and $P_w(\sum A_n z^n) = \sup_{n\geqslant 0} \|A_n\|/w_n$. Then, if $A = \sum A_n z^n$ and $B = \sum B_n z^n$, we have $\|AB\|_1 \leqslant \|A\|_1 \|B\|_1$. Moreover, for $n \geqslant 0$,

$$\begin{aligned}
\frac{\|(AB)_n\|}{w_n} &\leqslant \frac{\sum_k \|A_k B_{n-k}\|}{w_n} \\
&\leqslant \frac{1}{w_n} \sum_{k=0}^{n} \|A_k B_{n-k}\| + \left(\sum_{k=-\infty}^{-1} \|A_k\| P_w(B) \frac{w_{n-k}}{w_n} \right) + \left(\sum_{k=-\infty}^{-1} \|B_k\| P_w(A) \frac{w_{n-k}}{w_n} \right) \\
&\leqslant P_w(A) P_w(B) \frac{(w \star w)_n}{w_n} + \|A\|_1 P_w(B) + \|B\|_1 P_w(A).
\end{aligned}$$

Thus, $P_w(AB) \leqslant c P_w(A) P_w(B) + \|A\|_1 P_w(B) + \|B\|_1 P_w(A)$. This gives the conclusion. □

We will now identify the characters of the commutative algebra $\mathcal{O}_\gamma(\mathbb{C})$, i.e. the algebra morphisms from $\mathcal{O}_\gamma(\mathbb{C})$ to $\mathbb{C}$. For $\lambda \in S^1$, we can define a character χ_λ on $\mathcal{O}_\gamma(\mathbb{C})$ by $\chi_\lambda(a) = \sum a_n \lambda^n$.

Proposition A.2. *The characters of $\mathcal{O}_\gamma(\mathbb{C})$ are exactly the χ_λ, for $\lambda \in S^1$.*

Proof. This result is given by Rogozin in [28], but there is a (density) problem in his argument, for $\gamma \notin \mathbb{N}$. A correction is given in [29], and a more direct argument can be found in [13, Theorem 1.2.12]. □

The following theorem has been thoroughly used in Section 3. It is a Wiener Lemma in the algebra $\mathcal{O}_\gamma(\mathbb{C})$.

Theorem A.3. *Let $\mathcal{C}$ be a Banach algebra, $\gamma > 1$, and $A(z) = \sum_{n\in\mathbb{Z}} A_n z^n \in \mathcal{O}_\gamma(\mathcal{C})$. Assume that, for every $z \in S^1$, $A(z)$ is an invertible element of $\mathcal{C}$. Then A is invertible in the Banach algebra $\mathcal{O}_\gamma(\mathcal{C})$.*

Proof. Gelfand's Theorem [31, Theorem 11.5 (c)] ensures that, if an element a of a commutative Banach algebra satisfies $\chi(a) \neq 0$ for every character χ, then a is invertible. With Proposition A.2, this gives Theorem A.3 for $\mathcal{O}_\gamma(\mathbb{C})$.

To handle the case of a general noncommutative Banach algebra, we use Theorem 3 of [6]. □

The same kind of Wiener Lemma holds in the algebra $\mathcal{O}_\gamma^+(\mathcal{C})$ (also defined in Section 3.1):

Theorem A.4. *Let $\mathcal{C}$ be a Banach algebra, $\gamma > 1$, and $A(z) = \sum_{n\in\mathbb{N}} A_n z^n \in \mathcal{O}_\gamma^+(\mathcal{C})$. Assume that, for every $z \in \overline{\mathbb{D}}$, $A(z)$ is an invertible element of $\mathcal{C}$. Then A is invertible in the Banach algebra $\mathcal{O}_\gamma^+(\mathcal{C})$.*

Proof. This is the same proof as in Theorem A.3 (but here, the characters on $\mathcal{O}_\gamma^+(\mathcal{C})$ are given by $\chi_\lambda(a) = \sum_{n=0}^{\infty} \lambda^n a_n$ for $\lambda \in \overline{\mathbb{D}}$). □

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