



RANDOM WALK SPEED IS A PROPER FUNCTION ON TEICHMÜLLER SPACE

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ABSTRACT. Consider a closed surface M with negative Euler characteristic, and an admissible probability measure on the fundamental group of M with a finite first moment. Corresponding to each point in the Teichmüller space of M , there is an associated random walk on the hyperbolic plane. We show that the speed of this random walk is a proper function on the Teichmüller space of M , and we relate the growth of the speed to the Teichmüller distance to a basepoint. One key argument is an adaptation of Gouëzel's pivoting techniques to actions of a fixed group on a sequence of hyperbolic metric spaces.

1. INTRODUCTION AND STATEMENTS OF RESULTS

Let M be a compact oriented surface with negative Euler characteristic with a basepoint p , and let $\Gamma = \pi_1(M, p)$. Let μ be a probability measure on Γ that is *admissible*, i.e., the semigroup generated by the support of μ is equal to Γ . Further assume that μ has a finite first moment. Consider a random walk $Z_n = g_1 \cdots g_n$, where g_i are i.i.d. elements of Γ with distribution μ . Fixing a complete hyperbolic metric ρ on M , define

$$\ell(\rho) := \lim_{n \rightarrow \infty} \frac{|Z_n|_\rho}{n},$$

where $|Z_n|_\rho$ denotes the ρ -length of the unique hyperbolic geodesic representing the free homotopy class of the element Z_n . The limit above exists almost surely, and is well defined. The quantity $\ell(\rho)$ is called the *speed (or drift)* of the random walk on (M, ρ) .

Let $\mathcal{T}(M)$ be the *Teichmüller space* of marked complete hyperbolic metrics on M . Our first result is concerned with the quantitative behavior of $\ell(\rho)$ as one varies the hyperbolic metric ρ on M , in other words as $[\rho]$ varies over points in the Teichmüller space $\mathcal{T}(M)$. When one moves in Teichmüller space, some curves get longer but others get shorter, so the behavior is not obvious. However,

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one expects that most curves get longer, so one should expect $\ell(\rho)$ to tend to infinity as ρ diverges to infinity in Teichmüller space. There is a difficulty, though, that most curves become more and more parallel to each other (up to orientation) when ρ converges to a point at infinity, as they align asymptotically with the measured foliation at infinity. This means that consecutive steps of the random walk are likely to be both large, but in opposite directions, thereby cancelling each other effectively and not contributing to the speed. Our main theorem shows that the former effect dominates the latter: The speed indeed tends to infinity at infinity. However, this discussion hints at the fact that this is not straightforward, and indeed our proof is rather indirect.

THEOREM A (Speed is a proper function on Teichmüller space). *Assume that the probability measure μ on Γ is admissible and has a finite first moment. Then the function $\ell : \mathcal{T}(M) \rightarrow [0, +\infty)$ is proper.*

Recall that a function is *proper* if the preimage of any compact set is compact. In our particular case, since ℓ is non-negative, properness is equivalent to the statement that for each $C > 0$ there exists a compact set $K \subset \mathcal{T}(M)$ such that $\ell(\rho) > C$ for all $[\rho] \notin K$.

We can actually be more precise, by comparing the speed of a random walk to the Teichmüller distance in the Teichmüller space. Let $(\mathcal{T}(M), \text{dist}_{\text{Teich}})$ denote the Teichmüller space of M endowed with the Teichmüller metric (see Section 2 for the definitions). Then, we have

THEOREM B. *For each $[\rho_0] \in \mathcal{T}(M)$ there exists a constant $c > 0$ such that*

$$\ell(\rho) \geq c \text{dist}_{\text{Teich}}([\rho], [\rho_0])$$

for all $[\rho] \in \mathcal{T}(M)$.

Our strategy to prove the properness of the drift function in Theorem A and the quantitative bound in Theorem B is through a compactification argument: we get representations at boundary points, and argue that the drift there is nonzero. Then, by continuity (up to a natural rescaling), we deduce that the drift *inside* the Teichmüller space is positive. Moreover, using extremal length bounds we prove that the natural rescaling factor grows at least linearly along all Teichmüller rays.

It is also possible to derive an upper bound of $\ell(\rho) \leq c \exp(2 \text{dist}_{\text{Teich}}([\rho], [\rho_0]))$ by a direct application of Wolpert's lemma. This direct upper bound can be further improved using the work of Choi–Rafi [8] and is discussed in the follow up article [2] by the first named author.

Note that the speed can also be considered as a Lyapunov exponent. More precisely, if $[\rho] \in \mathcal{T}(M)$ is a point in Teichmüller space, we can consider $[\rho]$ as a conjugacy class of a discrete, faithful representation $\rho : \pi_1(M) \rightarrow \text{PSL}(2, \mathbb{R})$. Indeed, if we fix a matrix norm $\|\cdot\|$ on $\text{PSL}(2, \mathbb{R})$, we have

$$\ell(\rho) = \lim_{n \rightarrow +\infty} \frac{1}{n} \int \log \|\rho(g_1) \cdots \rho(g_n)\| \, d\mu(g_1) \cdots d\mu(g_n),$$

which is the Lyapunov exponent of the random walk on $\text{PSL}(2, \mathbb{R})$.

A related argument was used in [12] to study continuity of Lyapunov exponents for certain meromorphic families of $\pi_1(M)$ -representations in $SL(2, \mathbb{C})$, with the same idea of looking at the scaled limiting action on an \mathbb{R} -tree. One simple situation where both approaches can be used is when the hyperbolic structure degenerates by only pinching a simple closed geodesic: this degeneracy can also be described by a meromorphic family of representations using plumbing co-ordinates around the resulting noded surface. In this situation, the non-Archimedean exponent defined in [12] should be zero and our bound in Theorem A considered in plumbing co-ordinates, should give information on the error term in [12, Theorem A]. Thus in the context of surface group representations into $PSL(2, \mathbb{R})$, the setting of our continuity result below is more general.

The main feature of this argument is that the representations at boundary points do not live on the same space as the original representations: the group acts on an \mathbb{R} -tree instead of the hyperbolic disk. These representations are constructed in [4] and [25] (following previous work [10] and [24]). In particular, the topological type of Gromov boundaries changes in the limit. This means that the usual continuity argument for the drift, relying on the convergence of stationary measures on the boundary (see [18]), does not work. However, we are able to obtain the continuity of the drift in this context thanks to the pivotal times argument of [15]. The notions in the next Theorem are defined in Section 2.2.

THEOREM C (Speed is lower semi-continuous for a converging sequence of actions on uniformly hyperbolic spaces). *Let Γ be a countable group, and μ_k be a sequence of probability measures on Γ converging pointwise to a probability measure μ_∞ . Let $(X_k, o_k)_{k \in \mathbb{N} \cup \{\infty\}}$ be a uniformly hyperbolic sequence of pointed metric spaces, and let $\rho_k : \Gamma \rightarrow \text{Isom}(X_k)$, $k = 1, 2, \dots$ be a sequence of isometric actions of Γ converging to an action ρ_∞ . Assume that μ_∞ is non-elementary for ρ_∞ .*

Consider for each k a random walk $Z_n^{(k)} = g_1^{(k)} \cdots g_n^{(k)}$, where $g_1^{(k)}, \dots, g_n^{(k)}, \dots$ is an i.i.d. sequence with common distribution μ_k . Then, one has

$$\liminf \ell(\mu_k) \geq \ell(\mu_\infty),$$

where for each $k \in \mathbb{N} \cup \{\infty\}$ we define $\ell(\mu_k) \in [0, \infty]$ as the almost sure limit

$$\ell(\mu_k) = \lim_{n \rightarrow +\infty} \frac{1}{n} \text{dist}(o_k, \rho_k(Z_n^{(k)})o_k).$$

Returning to the setting of Theorem A, it is well known that $\ell(\rho) > 0$ for all ρ and the random walk driven by μ converges to the boundary almost surely. In other words, the limit

$$X_\infty = \lim_{n \rightarrow \infty} \rho(Z_n)o,$$

exists almost surely and X_∞ is in the visual boundary $\partial\mathbb{H}$ of \mathbb{H} , (cf. [17]). This limit defines a *hitting measure* on $\partial\mathbb{H}$ as follows: for any Borel set $U \subset \partial\mathbb{H}$,

$$\nu_\rho(U) := \mathbb{P}\left(\lim_{n \rightarrow \infty} \rho(Z_n)o \in U\right).$$

The measure ν_ρ is the unique μ -stationary measure on the visual boundary for the ρ -action. Properties of the stationary measures associated to random walks with finite support are quite subtle, as illustrated by the following. Here $\dim(\nu_\rho) \in [0, 1]$ denotes the Hausdorff dimension of the stationary measure ν_ρ associated to the representation ρ .

CONJECTURE (Singularity Conjecture). *If μ is admissible and has finite support, then there exists $\kappa < 1$ such that*

$$\dim(\nu_\rho) \leq \kappa$$

for all $[\rho] \in \mathcal{T}(M)$.

The conjecture above, stated in [11, Conjecture 1.21] and more generally in [19], remains open in spite of some recent progress made in [20]. We remark that for all ρ there exists μ with infinite support on Γ such that $\dim(\nu_\rho) = 1$, in fact it can be arranged that ν_ρ is absolutely continuous. This follows from the Furstenberg-Lyons-Sullivan discretization of Brownian motion [22], and also from more general results of Connell and Muchnik [9].

The speed of the random walk is closely related to the Hausdorff dimension of the stationary measure. Work in [28] shows that:

$$\dim(\nu_\rho) = \frac{h}{\ell(\rho)},$$

where h denotes the entropy of the random walk.

Therefore Theorem A immediately translates into a statement about the behavior of the Hausdorff dimension of the stationary measure:

COROLLARY 1.1 (Dimension drop of stationary measures). *For each $\varepsilon > 0$ there exists a compact $K \subset \mathcal{T}(M)$ such that $\dim(\nu_\rho) < \varepsilon$ for all $[\rho] \notin K$.*

Proof. This follows immediately from Theorem A and the above formula for $\dim(\nu_\rho)$. \square

2. PRELIMINARIES

2.1. Teichmüller Geometry. We now recall the definitions of the Teichmüller space and Teichmüller distance. General references on the subject are [16] and [27].

Let $M = M_{g,n}$ be a compact surface of genus g with n boundary components. A complex structure on M is an atlas of charts $z_\alpha : U_\alpha \rightarrow \mathbb{C}$, where at the intersection $z_\alpha(U_\alpha \cap U_\beta)$ the transition maps $z_\beta \circ z_\alpha^{-1}$ are biholomorphisms.

The Teichmüller space $\mathcal{T}(M)$ of M is defined as the equivalence classes of complex structures on M , where two complex structures X and Y are *equivalent* if there is a biholomorphism $f : X \rightarrow Y$ which is isotopic to the identity. Equivalently, $\mathcal{T}(M)$ can be thought of as the space of equivalence classes of hyperbolic structures on M , where two hyperbolic structures X and Y are equivalent if there is an isometry $f : X \rightarrow Y$ that is isotopic to identity. Yet another definition is given by considering the representations of the fundamental group

$\Gamma = \pi_1(M)$. Let $G = \text{Isom}^+(\mathbb{H})$ be the orientation preserving isometries of the upper half-plane. The Teichmüller space of M is also the space of discrete faithful representations from Γ into G considered up to conjugation by elements of G . We denote the equivalence class of a representation $\rho : \Gamma \rightarrow G$ with $[\rho]$. We endow $\mathcal{T}(M)$ with the subspace topology inherited from the space of representations from Γ to G .

In order to define a metric on the Teichmüller space, we switch back to the complex analytic definition and consider a map $f : X \rightarrow Y$ that sends a complex structure X on M to a complex structure Y . Let f_z and $f_{\bar{z}}$ denote the partial derivatives with respect to z and \bar{z} , where z is a complex co-ordinate with respect to X . Let us define

$$K_f(p) := \frac{|f_z(p)| + |f_{\bar{z}}(p)|}{|f_z(p)| - |f_{\bar{z}}(p)|}$$

the *quasi-conformal dilatation* of f at $p \in X$. This quantity is well defined as a change of coordinate is conformal.

The *quasi-conformal dilatation* of f is defined as

$$K_f = \sup_{p \in X} \{K_f(p)\}.$$

The map f is called *quasiconformal* if $K_f < \infty$. One can see that $K_f \geq 1$, and $K_f = 1$ if and only if f is conformal. The *Teichmüller distance* $\text{dist}_{\text{Teich}}$ on $\mathcal{T}(M)$ is defined as follows:

$$\text{dist}_{\text{Teich}}(X, Y) = \frac{1}{2} \inf_{f \approx id} \{\log(K_f) | f : X \rightarrow Y\},$$

where f is any quasi-conformal map isotopic to the identity.

Let γ be a non-trivial essential simple closed curve on M , and (X, dz) be a complex structure on M . The *extremal length* of γ on X is defined as

$$\text{Ext}_X(\gamma) = \sup \frac{L_\sigma^2(\gamma)}{A(\sigma)},$$

where the supremum is over all conformal metrics $\sigma(z)|dz|$, and

$$L_\sigma(\gamma) = \inf_{\gamma \sim \gamma'} \int_{\gamma'} \sigma(z)|dz|, \quad A(\sigma) = \int_X \sigma^2(z)|dz|^2.$$

2.2. Random walks on hyperbolic spaces. In this subsection we provide the context for Theorem C. Background on random walks on groups is available in [21], [3], and [30]. A general reference on hyperbolic spaces is [14].

Let (X, dist) be a metric space. The Gromov product between points x, y in X with respect to a third point o is defined as

$$(x, y)_o = \frac{1}{2} (\text{dist}(o, x) + \text{dist}(o, y) - \text{dist}(x, y)).$$

The space X is called δ -*hyperbolic* (or *hyperbolic* for short) for some $\delta \geq 0$ if for all $w, x, y, z \in X$, it holds that

$$\min((x, y)_w, (y, z)_w) \leq (x, z)_w + \delta.$$

A sequence of pointed metric spaces (X_k, o_k) is called *uniformly hyperbolic* if there exists $\delta \geq 0$ such that X_k is δ -hyperbolic for all k .

Let X be a hyperbolic space, and let Γ be a finitely generated group which acts on X by isometries. Let μ be a probability measure on Γ which is *admissible*, that is, the semi group generated by the support of μ is equal to Γ . The measure μ determines a random walk on Γ by taking

$$Z_n = g_1 g_2 \cdots g_n,$$

where g_i are i.i.d elements of Γ with common distribution μ . Fixing a base point $x \in X$ defines a sequence of points $Z_n x = g_1 g_2 \cdots g_n x$ in X , and it is called a random walk in X driven by μ .

We say that μ has a *finite first moment* if $\sum_{g \in \Gamma} |g| \mu(g) < \infty$, where $|\cdot|$ is some word metric on Γ with respect to a finite generating set.

We now consider the case where $X = (\mathbb{H}, \text{dist})$ is the hyperbolic plane, with the distance function induced from the hyperbolic metric $ds^2 = (dx^2 + dy^2)/y^2$. Fix a base point $o \in \mathbb{H}$, and let $G = \text{Isom}^+(\mathbb{H})$ be the group of orientation preserving isometries of \mathbb{H} . Given a discrete faithful representation $\rho : \Gamma \rightarrow G$ we consider the *speed (or linear drift)* of the induced random walk $\rho(Z_n)o$ on \mathbb{H} , which is the number $\ell(\rho)$ (given by Kingman's subadditive ergodic theorem) such that almost surely

$$\ell(\rho) := \lim_{n \rightarrow +\infty} \frac{1}{n} \text{dist}(o, \rho(Z_n)o).$$

This definition is equivalent to the one given in the introduction (see e.g., [13]). We claim that ℓ is well defined as a function on $\mathcal{T}(M)$: suppose that $\tilde{\rho} = g^{-1}\rho g$ and notice that

$$\text{dist}(o, \tilde{\rho}(Z_n)o) = \text{dist}(o, g^{-1}\rho(Z_n)g o) = \text{dist}(g o, \rho(Z_n)g o).$$

The claim now follows from the observation that the right-hand side of the above equation differs from $\text{dist}(o, \rho(Z_n)o)$ in absolute value by at most $2 \text{dist}(o, g o)$.

We now describe the set-up for Theorem C from the introduction.

DEFINITION 2.1. Consider a group Γ , and a sequence of pointed metric spaces $(X_k, o_k)_{k \in \mathbb{N} \cup \{\infty\}}$, each of them endowed with an isometric action ρ_k of Γ . We say that this sequence of actions converges if, for each $g \in \Gamma$, the distance $\text{dist}(o_k, \rho_k(g)o_k)$ converges to $\text{dist}(o_\infty, \rho_\infty(g)o_\infty)$ as $k \rightarrow \infty$.

We say that a probability measure μ on Γ is *non-elementary* for an isometric action ρ on a hyperbolic space if there exist two elements in the semigroup generated by the support of μ which act through ρ as two independent loxodromic isometries, i.e., their sets of fixed points at infinity are disjoint.

Let Γ be a countable group, μ_k be a sequence of probability measures on Γ which converges pointwise to a probability measure μ_∞ . Let $(X_k, o_k)_{k \in \mathbb{N} \cup \{\infty\}}$ be a uniformly hyperbolic sequence of pointed metric spaces, and $\rho_k : \Gamma \rightarrow \text{Isom}(X_k)$, $k \in \mathbb{N}$ be a sequence of isometric actions of Γ that converges to an action ρ_∞ . Assume that μ_∞ is non-elementary for ρ_∞ .

Consider for each k a random walk $Z_n^{(k)} = g_1^{(k)} \cdots g_n^{(k)}$ where $g_i^{(k)}$ are i.i.d. elements with common distribution μ_k . For each $k \in \mathbb{N} \cup \{\infty\}$, we define $\ell(\mu_k) \in [0, +\infty]$ as the almost sure limit

$$\ell(\mu_k) = \lim_{n \rightarrow +\infty} \frac{1}{n} \text{dist}\left(o_k, \rho_k(Z_n^{(k)})o_k\right).$$

Theorem C states that for such a sequence of actions, $\liminf \ell(\mu_k) \geq \ell(\mu_\infty)$. We will deduce Theorem C from the more precise Proposition 4.6, that gives uniform exponential large deviation estimates along the sequence μ_k .

3. PROOFS OF THEOREMS A AND B

We will now explain how we can deduce Theorems A and B from the semi-continuity statement of Theorem C.

3.1. Proof of Theorem A. Fix a finite symmetric generating set $F \subset \Gamma$ containing the identity. By [4, Proposition 2.1] there exists for each ρ a basepoint $o_\rho \in \mathbb{H}$ such that

$$\max_{\gamma \in F} \text{dist}(o_\rho, \rho(\gamma)o_\rho) = \min_{x \in \mathbb{H}} \max_{\gamma \in F} \text{dist}(x, \rho(\gamma)x).$$

We define the rescaling factor $R_\rho = R_{\rho, F}$ as the common value of both sides of the equation above. From the definition, it follows that R_ρ is continuous and proper on $\mathcal{T}(M)$. Consider the rescaled distance $\text{dist}_\rho = R_\rho^{-1} \text{dist}$ on \mathbb{H} .

Recall that an \mathbb{R} -tree is a non-empty metric space which is 0-hyperbolic and such that every pair of points is joined by a unique geodesic. The following is the main result of [4] and [25] (following previous work [10] and [24]).

LEMMA 3.1. *Each sequence ρ_n such that $[\rho_n]$ leaves every compact subset in $\mathcal{T}(M)$, has a subsequence ρ_{n_k} with $n_k \rightarrow +\infty$ such that ρ_{n_k} when viewed as an action on \mathbb{H} endowed with the distance $\text{dist}_{\rho_{n_k}}$ and the basepoint $o_{\rho_{n_k}}$, converges to an action ρ_T on an \mathbb{R} -tree (T, dist_T, o_T) .*

Furthermore, the group $\rho_T(\Gamma)$ acts minimally on T (i.e., there is no proper closed invariant subtree), and for any arc I in T the set of $\gamma \in \Gamma$ such that ρ_T stabilizes I is a virtually abelian subgroup of Γ .

We now verify that the action ρ_T is non-elementary.

LEMMA 3.2 (Non-elementary action on the \mathbb{R} -tree). *Let ρ_T be a representation of Γ into the isometry group of an \mathbb{R} -tree (T, dist_T) with the property that stabilizers of arcs are virtually abelian.*

Then there exist $\gamma_1, \gamma_2 \in \Gamma$ such that $\rho_T(\gamma_1)$ and $\rho_T(\gamma_2)$ are loxodromic isometries of T along geodesics whose intersection is either empty or a compact arc.

Proof. The action of $\rho_T(\Gamma)$ is irreducible in the sense that there is no global fixed point on the boundary at infinity (see [26, Proposition 2.6]). The existence of the two required loxodromic elements now follows from [7, Proposition 3.7]. \square

In view of the above lemmas, Theorem A follows immediately from Theorem C and in fact we obtain a lower bound in terms of the rescaling factor R_ρ .

THEOREM 3.3. *There exists a constant $c > 0$ such that $\ell(\rho) \geq cR_\rho$ for all $[\rho] \in \mathcal{F}(M)$.*

Proof. Suppose by contradiction that we may find a sequence of representations ρ_n for which $\ell(\rho_n)/R_{\rho_n}$ tends to zero. Extracting a subsequence, we may assume that ρ_{n_k} , viewed as an action on \mathbb{H} endowed with the distance $\text{dist}_{\rho_{n_k}}$, converges to a non-elementary action, either on a tree if ρ_n escapes to infinity by Lemma 3.1, or on \mathbb{H} itself otherwise.

Notice that $\ell(\rho_n)/R_{\rho_n}$ is the speed of $\rho(Z_n)o$ with respect to the distance dist_ρ . We may apply Theorem C to deduce that the liminf of this speed is bounded below by the speed in the limiting action. As the speed of a non-elementary action is always positive, we deduce that this liminf is positive, which is a contradiction. \square

3.2. Proof of Theorem B. In view of Theorem 3.3, to prove Theorem B it suffices to find a set of curves F such that the rescaling factor $R_\rho = R_{\rho,F}$ can be bound from below by a multiple of $\text{dist}_{\text{Teich}}([\rho_0], [\rho])$. We fix $F \subset \Gamma$ to be a subset that is finite, symmetric and filling.

We denote by $\text{length}_\rho(\gamma)$ the hyperbolic length of the geodesic representative of γ , that is

$$\text{length}_\rho(\gamma) = \min_{x \in \mathbb{H}} \text{dist}(x, \rho(\gamma)x).$$

We observe that

$$(3.1) \quad R_\rho = \min_{x \in \mathbb{H}} \max_{\gamma \in F} \text{dist}(x, \rho(\gamma)x) \geq \max_{\gamma \in F} \text{length}_\rho(\gamma).$$

So to obtain Theorem B, we need a lower bound for the right-hand side above.

Let $\text{Ext}_\rho(\gamma)$ denote the extremal length of the curve γ under the conformal structure provided by ρ . As proven by Maskit [23, Corollary 3], we have

$$(3.2) \quad \frac{1}{2} \text{length}_\rho(\gamma) e^{\text{length}_\rho(\gamma)/2} \geq \text{Ext}_\rho(\gamma),$$

so it suffices to obtain a lower bound on extremal length. This will be obtained from a result of Walsh [29, Lemma 3]. While we do not need the specific details in Walsh, we include some of them for coherence.

For a unit area quadratic differential q based at some basepoint $[\rho_0]$, denote $R(q; t)$ the point in Teichmüller space obtained after following a Teichmüller ray for time $t > 0$ in the direction provided by q . Let $V(q)$ (respectively $H(q)$) be the vertical (respectively horizontal) foliation of q . The union of vertical saddle connections of $V(q)$ is called its critical graph. The complement of the critical graph decomposes into finitely many components (the number bounded above in terms of the Euler characteristic) each of which is either a cylinder C or a minimal component V with every leaf dense. The transverse measure restricted to a minimal component V is a linear combination $\sum_j m_{V,j}$ of distinct ergodic measures $m_{V,j}$. Each pair $V_j = (V, m_{V,j})$ is said to be an indecomposable component of $V(q)$.

Walsh proved the following inequality

$$(3.3) \quad e^{-2t} \text{Ext}_{R(q;t)}(\gamma) \geq E_q^2(\gamma),$$

where

$$E_q^2(\gamma) = \sum_{V_j} \frac{i(V_j, \gamma)^2}{i(V_j, H(q))}$$

in which $i(*, *)$ denotes the geometric intersection number.

We will use the fact that F is filling to derive a uniform (over q) lower bound on $\max_{\gamma \in F} E_q(\gamma)$.

LEMMA 3.4. *Given a basepoint $[\rho_0] \in \mathcal{T}(M)$ there is some $c > 0$ such that*

$$\inf_{q \in T^1([\rho_0])} \max_{\gamma \in F} E_q(\gamma) > c,$$

where the infimum is taken over all unit area quadratic differentials at $[\rho_0]$.

Proof. For any $q \in T^1([\rho_0])$, we have $i(V_j, H(q)) \leq i(V(q), H(q)) = \text{Area}(q) = 1$. This implies $E_q^2(\gamma) \geq \sum_{V_j} i(V_j, \gamma)^2$.

Assume we have a sequence q_n of unit area quadratic differentials at $[\rho_0]$ such that $\max_{\gamma \in F} E_{q_n}(\gamma)$ converges to 0 for all j . Since the space of unit area quadratic differentials at a basepoint is compact we can pass to a subsequence that converges to some q . Furthermore, since geometric intersection number is continuous we have $E_q(\gamma) = 0$ for all $\gamma \in F$. In particular, this implies $i(V(q), \gamma) = 0$ for all $\gamma \in F$. This is impossible because F is a filling set. \square

We use the above lemma to get the following global lower bound on the maximal lengths over F .

LEMMA 3.5. *Given a basepoint $[\rho_0] \in \mathcal{T}(M)$ there are some $c_1, c_2 > 0$ such that, for all $[\rho] \in \mathcal{T}(M)$,*

$$\max_{\gamma \in F} \text{Ext}_\rho(\gamma) \geq c_1 e^{2 \text{dist}_{\text{Teich}}([\rho_0], [\rho])}$$

and hence

$$\max_{\gamma \in F} \text{length}_\rho(\gamma) \geq c_2 \text{dist}_{\text{Teich}}([\rho_0], [\rho]),$$

where $\text{dist}_{\text{Teich}}$ denotes the Teichmüller distance.

Proof. Let q be such that $[\rho] = R(q; \text{dist}_{\text{Teich}}([\rho_0], [\rho]))$. By Equation (3.3) we have

$$\max_{\gamma \in F} \text{Ext}_\rho(\gamma) \geq e^{2 \text{dist}([\rho_0], [\rho])} \max_{\gamma \in F} E_q^2(\gamma),$$

and so by Lemma 3.4 we get the first inequality. By Equation (3.2) we have

$$\max_{\gamma \in F} \frac{1}{2} \text{length}_\rho(\gamma) + \log \max_{\gamma \in F} \frac{1}{2} \text{length}_\rho(\gamma) \geq 2 \text{dist}([\rho_0], [\rho]) + \log(c_1),$$

so the second inequality in the lemma is asymptotically satisfied for any c_2 slightly smaller than 2. Furthermore, given any bounded domain we can choose c_2 small enough so the inequality is satisfied. \square

Together with (3.1), Lemma 3.5 gives a lower bound $R_\rho \geq c_2 \text{dist}_{\text{Teich}}([\rho_0], [\rho])$. With Theorem 3.3, this concludes the proof of Theorem B.

4. PROOF OF THEOREM C

In this section, we prove Theorem C. To have lighter notation, we will keep the action implicit and write go_k instead of $\rho_k(g)o_k$. Since there is only one possible action for each basepoint o_k , this should not create confusion.

4.1. Schottky sets. Following [15], a finite set of isometries S acting on a Gromov hyperbolic space X , is said to be (η, C, D) -Schottky if the following three conditions are satisfied:

1. For all $x, y \in X$ the proportion of $a \in S$ such that $(x, ay)_o \leq C$ is at least $1 - \eta$.
2. For all $x, y \in X$ the proportion of $a \in S$ such that $(x, a^{-1}y)_o \leq C$ is at least $1 - \eta$.
3. For all $a \in S$ one has $\text{dist}(o, ao) \geq D$.

The following is a sufficient condition for a set S to be Schottky with certain parameters, which depends on checking conditions involving only a finite number of points. Since the notion of convergence we use in Definition 2.1 only gives control over finitely many points at a time, this criterion will enable us to construct finite sets which are Schottky sets uniformly along a converging family of representations.

LEMMA 4.1 (Schottky set criterion). *Let (X, dist) be a δ -hyperbolic metric space with a basepoint $o \in X$. Suppose S is a finite symmetric set of isometries of X such that $c_1 + 2\delta < c_2/2$ where $c_1 = \max_{g \neq h, g, h \in S} (go, ho)_o$, and $c_2 = \min_{g \in S} \text{dist}(o, go)$.*

Then S is an (η, C, D) -Schottky set with $\eta = \frac{2}{\#S}$ and $C = c_1 + 3\delta$ and $D = c_2$.

Proof. Let $\varepsilon \in (2\delta, c_2/2 - c_1)$ and for each $g \in S$ set

$$V(g) = \{x \in X : (x, go)_o \geq c_1 + \varepsilon\}.$$

Claim 1: If $g \neq h$ then $V(g) \cap V(h) = \emptyset$.

Indeed if $x \in V(g) \cap V(h)$ then one would have

$$c_1 + \varepsilon \leq \min\{(x, go)_o, (x, ho)_o\} \leq (go, ho)_o + \delta \leq c_1 + \delta,$$

contradicting the fact that $\delta < \varepsilon$.

Claim 2: If $x \notin V(g^{-1})$ then $gx \in V(g)$.

To see this observe that from the first condition one has

$$\frac{\text{dist}(o, x) + \text{dist}(o, g^{-1}o) - \text{dist}(x, g^{-1}o)}{2} < c_1 + \varepsilon,$$

while if $gx \notin V(g)$ we would have

$$\frac{\text{dist}(o, gx) + \text{dist}(o, go) - \text{dist}(gx, go)}{2} < c_1 + \varepsilon.$$

Taking the sum this would imply

$$c_2 \leq \text{dist}(o, go) < 2c_1 + 2\varepsilon,$$

contradicting the fact that $\varepsilon < \frac{1}{2}c_2 - c_1$.

Claim 3: If $x \in V(g)$ and $y \in V(h)$ for $g \neq h$ then $(x, y)_o \leq c_1 + 2\delta$.

By hyperbolicity one has

$$\min((x, go)_o, (x, ho)_o) \leq (go, ho)_o + \delta \leq c_1 + \delta.$$

Since $(x, go)_o \geq c_1 + \varepsilon > c_1 + \delta$ this implies that $(x, ho)_o \leq c_1 + \delta$. From this we obtain

$$\min((y, ho)_o, (x, y)_o) \leq (x, ho)_o + \delta \leq c_1 + 2\delta,$$

but since $(y, ho)_o \geq c_1 + \varepsilon > c_1 + 2\delta$ this implies $(x, y)_o \leq c_1 + 2\delta$ as claimed.

Claim 4: S is (η, C, D) -Schottky for the constants in the statement.

Let us check the first property in the definition of Schottky sets, as the second one follows by symmetry of S and the third one comes from the definition of c_2 . Given $x, y \in X$ let $a_1, a_2 \in S$ be distinct and such that $x \notin V(a_i^{-1})$ for $i = 1, 2$. By Claim 1 the a_i are chosen among at least $\#S - 1$ elements of S . By Claim 2 one has $a_i x \in V(a_i)$ for $i = 1, 2$. By hyperbolicity and Claim 3 one has

$$\min((a_1 x, y)_o, (a_2 x, y)_o) \leq (a_1 x, a_2 x)_o + \delta \leq c_1 + 3\delta = C.$$

This implies that either $(a_1 x, y)_o \leq C$ or $(a_2 x, y)_o \leq C$. Hence the subset of S consisting of elements with $(ax, y)_o > C$ can have at most two elements. \square

4.2. Proof of Theorem C. In this paragraph, we prove Theorem C. Let us fix a sequence of pointed δ -hyperbolic spaces (X_k, o_k) endowed with actions of a group Γ , and assume that ρ_k converges to ρ_∞ in the sense of Definition 2.1. Let also (μ_k) be probability measures on Γ such that μ_k converges pointwise to μ_∞ and the action of μ_∞ through ρ_∞ is non-elementary on X_∞ .

The following lemma is a classical application of a ping-pong argument.

LEMMA 4.2. *Let $\eta > 0$. Then there exists $C > 0$ such that, for any $D > 0$, there exist N and a finite symmetric set S in Γ in the support of μ_∞^N such that $\#S \geq 2/\eta$ and*

$$(4.1) \quad \max_{g \neq h, g, h \in S} (go_\infty, ho_\infty)_{o_\infty} < C - 3\delta, \quad \min_{g \in S} \text{dist}(o_\infty, go_\infty) > D.$$

Proof. This follows readily from the proof techniques of [6, Proposition A.2] or [15, Proposition 3.12]. \square

Let $\eta > 0$. For suitable C and D , we can consider a set S as in Lemma 4.2. By definition of converging actions, for large n the inequalities in (4.1) also hold for ρ_k . By Lemma 4.1, it follows that $\rho_k(S)$ is an (η, C, D) -Schottky set, uniformly for all large enough k . We can then use this Schottky set as in [15], to obtain quantitative estimates that are uniform in k . As a first example, let us get a uniform version of [15, Lemma 4.14].

LEMMA 4.3. *Let $\varepsilon > 0$. There exists E such that, for all large k , for any $g \in \Gamma$,*

$$\mathbb{P}(\forall n, \text{dist}(o_k, gZ_n^{(k)} o_k) \geq \text{dist}(o_k, go_k) - E) \geq 1 - \varepsilon.$$

The intuition behind this lemma is that, given a Schottky set, then jumps from this Schottky set will most of the time go towards infinity, yielding linear progress from any starting point (and in particular small probability to go back towards the origin). This is proved in [15] using the notion of pivotal times. There is a pedagogical difficulty here: it would not make sense to repeat exactly

all the pivotal times computations of [15], but we can not expect our readers to be very familiar with this article. As a middle ground, we have decided to extract a black box from [15], in the form of the following lemma:

LEMMA 4.4. *Let $\delta > 0$, $C > 0$, $N > 0$, $\alpha > 0$, $\varepsilon > 0$. Then there exists $n_0 = n_0(\delta, C, N, \alpha, \varepsilon)$ with the following property. Consider a probability measure μ on the group G of isometries of a δ -hyperbolic space X , and a set $S \subseteq G$ which is $(1/100, C, 20C + 100\delta + 1)$ -Schottky. Assume that μ^N gives mass at least α to each element of S . Then, for any isometry $g \in G$, for any $o \in X$, there exists a set U of probability at least $1 - \varepsilon$ in $(\Omega, \mathbb{P}) = (G^{\mathbb{N}}, \mu^{\otimes \mathbb{N}})$ such that, for each $\omega \in U$, there exists $j \leq n_0$ with*

$$\forall n \geq n_0, \text{dist}(o, gZ_n o) \geq \text{dist}(o, gZ_j o) - 2C - 6\delta,$$

where Z_n is the position of the random walk at time n .

This technical lemma is unfortunately not stated explicitly in [15], but it is proved there, as the first step of the proof of [15, Lemma 4.14]. The reader may either go and check in [15] that the n_0 given there indeed only depends on $\delta, C, N, \alpha, \varepsilon$, or trust us and accept this lemma as a black box.

Proof of Lemma 4.3. Let $\eta = 1/100$. Let C be given by Lemma 4.2 for this value of η . Take $D = 20C + 100\delta + 1$. By Lemma 4.2 and the discussion that follows it, we obtain a symmetric set $S \subseteq \Gamma$ in the support of μ_∞^N for some N , such that $\rho_k(S)$ is (η, C, D) Schottky for all large k . For large enough k , all the measures μ_k^N give a weight bounded away from zero to all elements of S , say bounded from below by some $\alpha > 0$, as μ_k converges pointwise to μ_∞ .

Let $n_0 = n_0(\delta, C, N, \alpha, \varepsilon/2)$ be given by Lemma 4.4. Then this lemma applies uniformly to all measures μ_k for large k : for all $g \in \Gamma$, there exists a set $U = U(k, g)$ of probability at least $1 - \varepsilon/2$, and some $j = j(k, g, \omega) \leq n_0$ such that, on U ,

$$(4.2) \quad \forall n \geq n_0, \text{dist}(o_k, gZ_n^{(k)} o_k) \geq \text{dist}(o_k, gZ_j^{(k)} o_k) - 2C - 6\delta.$$

It remains to control the n_0 first steps. Let F be a finite subset of Γ such that, with probability $> 1 - \varepsilon/2$, for all $i \leq n_0$, then $Z_i^{(\infty)}$ belongs to F . This property also holds for large enough k , by pointwise convergence of μ_k to μ_∞ . There are finitely many points $(g o_\infty)_{g \in F}$. By convergence of the actions, all the distances $\text{dist}(o_k, g o_k)_{g \in F}$ are uniformly bounded for large k , by a constant C' .

We obtain a set $V = V(k)$ of probability at least $1 - \varepsilon/2$ on which

$$(4.3) \quad \forall i \leq n_0, \text{dist}(o_k, Z_i^{(k)} o_k) \leq C'.$$

The set $U(k, g) \cap V(k)$ has probability at least $1 - \varepsilon$. We claim that, on this set, we have for all n the inequality

$$\text{dist}(o_k, gZ_n^{(k)} o_k) \geq \text{dist}(o_k, g o_k) - 2C - 6\delta - C',$$

proving the lemma with $E = 2C + 6\delta + C'$. Let us check this claim. First, if $n \leq n_0$, then $gZ_n^{(k)} o_k$ is within distance C' of $g o_k$ by (4.3), and the result is obvious.

Then, for $n \geq n_0$, the result follows from the inequality (4.2) together with the fact that $gZ_j^{(k)}o_k$ is within distance C' of go_k as $j \leq n_0$. \square

Let us now proceed to the lower estimate of the drift. As above, we extract a black box result from [15], specifying which properties of the measures are used.

LEMMA 4.5. *Let $\delta > 0$, $\eta > 0$, $\alpha \in (0, 1)$, $C > 0$, $N > 0$, $A > 0$, with $\eta A \geq C$. Consider also a nonnegative real random variable Q , and $r \geq 0$ with*

$$(4.4) \quad r < (1 - 40\eta) \frac{\mathbb{E}(Q)}{NA} - 2\eta.$$

There exist n_0 and $\kappa > 0$ only depending on these quantities, with the following property.

Consider a probability measure μ on the group G of isometries of a δ -hyperbolic space X , and a set $S \subseteq G$ which is $(\eta, C, 20C + 100\delta + 1)$ -Schottky. Assume that $\mu^{2N} \geq \alpha\mu_S^2$, where μ_S is the uniform probability measure on S . Assume moreover that the random walk $Z^{(v)}$ driven by the probability measure $\nu = (\mu^{2N} - \alpha\mu_S^2)/(1 - \alpha)$ satisfies, for any $g \in G$, the estimate

$$\mathbb{P}(\forall n \geq 0, \text{dist}(o, gZ_n^{(v)}o) \geq \text{dist}(o, go) - \eta NA) \geq 1 - \eta.$$

Finally, assume that the length of the jumps of μ^{NA} are stochastically bounded below by Q : for all k , we have

$$\sum_{g: \text{dist}(o, go) \geq k} \mu^{NA}(g) \geq \mathbb{P}(Q \geq k).$$

Then, for all $n \geq n_0$,

$$\mathbb{P}(\text{dist}(o, Z_n o) \leq rn) \leq e^{-\kappa n},$$

where Z_n is the position of the random walk at time n .

Again, this lemma is not stated exactly in this form in [15], but it is proved there at the end of Section 5.3. The strategy is to decompose the walk along successive time intervals of length roughly NA (for which the size of the jumps is bounded below by Q) interspersed with Schottky jumps that put the former in general position. This ensures that the progress towards infinity is bounded below by a sum of independent random variables distributed like Q , up to controlled error terms. The precise condition on r that shows up at the end of [15, Section 5.3] is

$$r + \eta < \left((1 - \eta) \frac{\mathbb{E}(Q)}{NA} - \eta \right) (1 - 22\eta)(1 - 17\eta),$$

which follows from (4.4).

Let us deduce from this result uniform large deviations estimates along a converging sequence of actions. We return to the standing assumptions of this paragraph, with a converging sequence of actions of Γ on hyperbolic spaces X_k , and a pointwise converging sequence of probability measures μ_k on Γ such that the action of μ_∞ on X_∞ is non-elementary.

PROPOSITION 4.6. *Let $\ell(\mu_\infty)$ be the drift of the random walk on X_∞ . Let $a < \ell(\mu_\infty)$. Then there exists $\kappa > 0$ such that, for all large enough k , for all $n \in \mathbb{N}$,*

$$\mathbb{P}\left(\text{dist}(o_k, Z_n^{(k)} o_k) \leq an\right) \leq e^{-\kappa n}.$$

Proof. Let $b \in (a, \ell(\mu_\infty))$. Let $\eta > 0$ be small enough (how small will be prescribed at the end of the proof), with $b + \eta < \ell(\mu_\infty)$. Let C be given by Lemma 4.2 for this value of η . Take $D = 20C + 100\delta + 1$. By Lemma 4.2 and the discussion that follows it, we obtain a symmetric set $S \subseteq \Gamma$ in the support of μ_∞^N for some N , such that $\rho_k(S)$ is (η, C, D) Schottky for all large k . For large enough k , all the measures μ_k^N give a weight bounded away from zero to all elements of S , as μ_k converges pointwise to μ_∞ . In particular, for some $\alpha > 0$, one has $\mu_k^{2N} \geq 2\alpha\mu_S^2$, where μ_S is the uniform measure on S .

The probability measures $\nu_k := (\mu_k^{2N} - \alpha\mu_S^2)/(1 - \alpha)$ converge pointwise to $\nu_\infty = (\mu_\infty^{2N} - \alpha\mu_S^2)/(1 - \alpha)$. Moreover, ν_∞ acts in a non-elementary way on X_∞ through ρ_∞ , as it satisfies $\nu_\infty \geq \alpha\mu_S^2/(1 - \alpha)$ and therefore gives nonzero weight to independent loxodromic elements since S is Schottky. Therefore, Lemma 4.3 applies to the sequence of measures ν_k for $\varepsilon = \eta$, yielding some constant E . Let A be large enough that $\eta A \geq C$ and $\eta NA \geq E$.

By subadditivity, we have $NA\ell(\mu_\infty) \leq \sum_g \mu_\infty^{NA}(g) \text{dist}(o_\infty, g o_\infty)$. In particular, as $b + \eta < \ell(\mu_\infty)$, we may find a finite subset $F \subseteq G$ such that

$$NA(b + \eta) < \sum_{g \in F} \mu_\infty^{NA}(g) \text{dist}(o_\infty, g o_\infty).$$

Let $\varepsilon > 0$. Let Q be the real distribution with an atom of mass $\mu_\infty^{NA}(g) - \varepsilon$ at $\text{dist}(o_\infty, g o_\infty) - \varepsilon$ for each $g \in F$, and the missing mass put at 0. For small enough ε , this random variable has expectation $> NA(b + \eta)$. Moreover, by convergence of μ_k to μ_∞ and ρ_k to ρ_∞ , the distribution of the size of the jumps of μ_k^{NA} through ρ_k is bounded below by Q , for all large enough k .

We apply Lemma 4.5 to these quantities $\delta, \eta, \alpha, C, N, A, Q$, with

$$r = (1 - 40\eta)(b + \eta) - 2\eta,$$

which is indeed $< (1 - 40\eta)\mathbb{E}(Q)/NA - 2\eta$. This lemma provides us with n_0 and $\kappa > 0$. By construction, for all large enough k , the measures μ_k all satisfy the assumptions of the lemma. It follows that, uniformly in k large, we have for all $n \geq n_0$

$$\mathbb{P}\left(\text{dist}(o_k, Z_n^{(k)} o_k) \leq rn\right) \leq e^{-\kappa n}.$$

When $\eta \rightarrow 0$, then $r = r(\eta)$ tends to $b > a$. Therefore, we may choose η with $r > a$. We get for all large k and all $n \geq n_0$ the estimate

$$(4.5) \quad \mathbb{P}\left(\text{dist}(o_k, Z_n^{(k)} o_k) \leq an\right) \leq e^{-\kappa n}.$$

It remains to handle each $n \in [1, n_0)$. For each such n , there exists g with $\mu_\infty^n(g) > 0$ and $\text{dist}(o_\infty, g o_\infty) > an$, as otherwise the drift $\ell(\mu_\infty)$ would be $\leq a$. These two inequalities still hold for large k . It follows that $\mathbb{P}(\text{dist}(o_k, Z_n^{(k)} o_k) \leq$

an) is bounded away from 1, uniformly for large k . Decreasing κ if necessary, we may therefore enforce (4.5) separately for each $n \in [1, n_0)$. \square

Proof of Theorem C. Let $a < \ell(\mu_\infty)$. Proposition 4.6 implies that, for all large k , one has almost surely eventually $\text{dist}(o_k, Z_n^{(k)} o_k) > an$. As $\ell(\mu_k)$ is the almost sure limit of $\text{dist}(o_k, Z_n^{(k)} o_k)/n$, we get $\ell(\mu_k) \geq a$ for large k . \square

5. SINGULARITY CONJECTURE AND OPEN QUESTIONS

In this last section we return to the singularity conjecture and dimension drop of stationary measures. Recall from the introduction that by the results of [28], the stationary measure ν_ρ is exact dimensional and its dimension is given by

$$(5.1) \quad \dim(\nu_\rho) = \frac{h}{\ell(\rho)},$$

where $h = h(\mu)$ is the asymptotic (or Avez) entropy defined by

$$h = \lim_{n \rightarrow +\infty} \frac{1}{n} H(Z_n),$$

and $H(Z) = -\sum_{g \in \text{supp}(Z)} \mathbb{P}(Z = g) \log(\mathbb{P}(Z = g))$ denotes the Shannon entropy of the random variable Z . Note that h does not depend on the representation ρ .

Recall the singularity conjecture from the introduction:

CONJECTURE 5.1. *If μ is admissible and has finite support then there exists $\delta < 1$ such that*

$$\dim(\nu_\rho) \leq \delta$$

for all $[\rho] \in \mathcal{F}(M)$.

Since the visual boundary is one-dimensional, equation (5.1) implies that $h \leq \ell(\rho)$ for all ρ . The singularity conjecture then amounts to this inequality being strict on all of $\mathcal{F}(M)$.

Let us record some basic properties of ℓ .

PROPOSITION 5.2. *The function $\ell : \mathcal{F}(M) \rightarrow (0, +\infty)$ is continuous.*

Proof. This follows immediately from Theorem C. An alternative argument is via the Furstenberg formula [19, Theorem 18] for speed and convergence of the stationary measures. \square

One basic result from $\ell(p)$ being continuous and proper is as follows

COROLLARY 5.3. *The functions $\ell : \mathcal{F}(M) \rightarrow [h, +\infty)$ and $\dim(\nu) : \mathcal{F}(M) \rightarrow (0, 1]$ attain their minimum and maximum respectively.*

It is natural to ask then the following question

QUESTION 5.4. *Does $\dim(\nu_\rho)$ attain its maximum at a unique point in $\mathcal{F}(M)$? Equivalently is $\ell : \mathcal{F}(M) \rightarrow (0, +\infty)$ minimized at a unique point?*

When the maximal dimension is 1 and μ is symmetric (i.e., $\mu(g) = \mu(g^{-1})$ for all g) the answer to the previous question is affirmative. That is, we have the following.

PROPOSITION 5.5. *If μ is admissible symmetric and has a finite first moment, then there exists at most one point $[\rho] \in \mathcal{T}(M)$ such that $\dim(\nu_\rho) = 1$.*

Proof. If $\dim(\nu_\rho) = 1$ then, under the assumption that μ is symmetric, in fact ν_ρ is absolutely continuous with respect to the visual measure (Lebesgue measure) on the boundary [5, Theorem 1.5]. Suppose $\dim(\nu_{\rho_1}) = \dim(\nu_{\rho_2}) = 1$.

There exists a quasi-conformal homeomorphism $\varphi : \mathbb{H} \rightarrow \mathbb{H}$ such that $\varphi(o) = o$ and $\varphi \circ \rho_1(\gamma) = \rho_2(\gamma) \circ \varphi$ for all $\gamma \in \Gamma$. The quasi-conformal map φ extends continuously to the visual boundary in a unique way. Denoting this extension by φ as well we have $\varphi_* \nu_{\rho_1} = \nu_{\rho_2}$.

This implies that the restriction of φ to the visual boundary is absolutely continuous. However this can only happen if $[\rho_1] = [\rho_2]$ (see [1]). \square

Another natural question to ask is the following one.

QUESTION 5.6. *Is the function $\ell : \mathcal{T}(M) \rightarrow (0, +\infty)$ (strictly) convex?*

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