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Statistical properties of a skew product with a curve of neutral points

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Abstract. We study a skew product with a curve of neutral points. We show that there exists a unique absolutely continuous invariant probability measure, and that the Birkhoff averages of a sufficiently smooth observable converge to a normal law or a stable law, depending on the average of the observable along the neutral curve.

1. *Introduction*

Let $T : M \to M$ be a map on a compact manifold. While uniformly hyperbolic or uniformly expanding dynamics are well understood, problems arise when there are neutral fixed points (where the differential of T has an eigenvalue equal to 1). The one-dimensional case has been thoroughly studied, particularly when T has only one neutral fixed point (see [**LSV99**] and references therein). The normal form at the fixed point dictates the asymptotics of the dynamics and, in particular, the speed of mixing and the convergence of Birkhoff sums to limit laws [**Gou04, Zwe03**].

In this article, we study the same type of phenomenon, but in higher dimension. In contrast to [**Hu01, PY01**] (where the case of isolated fixed points is considered), our models admit a whole invariant neutral curve. We show that the one-dimensional results remain essentially true.

More precisely, for $\alpha > 0$, define a map T_α on [0, 1] by

$$T_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } 0 \leqslant x \leqslant \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} < x \leqslant 1. \end{cases}$$

It has a neutral fixed point at 0, behaving like $x(1 + x^\alpha)$. This map admits an absolutely continuous invariant measure, which is of finite mass if and only if $\alpha < 1$. To mix different such behaviours, we consider a skew product, similar to the Alves–Viana map [**Via97**] but where the unimodal maps are replaced by T_α. Let $\alpha : S^1 \to (0, \infty)$ be a map with minimum $\alpha_{\min}$ and maximum $\alpha_{\max}$. Assume that:

(1) α is C^2;
(2) $0 < \alpha_{\min} < \alpha_{\max} < 1$;
(3) α takes the value $\alpha_{\min}$ at a unique point $\omega_0 \in S^1$, with $\alpha''(\omega_0) > 0$;
(4) $\alpha_{\max} < \frac{3}{2}\alpha_{\min}$ (which implies $\alpha_{\max} < \alpha_{\min} + \frac{1}{2}$).

These conditions are, for example, satisfied by $\alpha(\omega) = \alpha_{\min} + \varepsilon(1 + \sin(2\pi\omega))$ where $\alpha_{\min} \in (0, 1)$ and ε is small enough.

We define a map T on $S^1 \times [0, 1]$ by

$$T(\omega, x) = (F(\omega), T_{\alpha(\omega)}(x)) \tag{1}$$

where $F(\omega) = 4\omega$.

The qualitative behaviour of T can be described as follows. In a compact set disjoint from $S^1 \times \{0\}$, say $S^1 \times [\frac{1}{10}, 1]$, T is uniformly expanding. Hence, the interesting points are the points $\mathbf{x} = (\omega, \varepsilon)$ with small ε. Such a point takes a long time to reach $S^1 \times [\frac{1}{10}, 1]$ since each map $T_{\alpha(\omega')}$ has a neutral fixed point at 0. The iterates of $\mathbf{x}$ will feel the strongest expansion essentially when they are of the form (ω', ε') with ω' close to ω_0 (where the neutral point is the least neutral). Therefore, the precise behaviour of the map T will depend on a strong way on the behaviour of $\alpha(\omega')$ for ω' close to ω_0, and on the value of $\alpha_{\min}$. This explains the conditions $\alpha_{\min} < 1$ and $\alpha''(\omega_0) > 0$, which are really important for our analysis. On the other hand, the other conditions $\alpha_{\max} < 1$ and $\alpha_{\max} < \frac{3}{2}\alpha_{\min}$ are merely technical. They could probably be removed at the expense of greater technicalities in the proofs.

In the following, we will generalize the one-dimensional results on the maps T_α to this skew product T. First of all, in §2, we prove that there exists a unique absolutely continuous invariant probability measure m, whose density h is, in fact, Lipschitz on every compact subset of $S^1 \times (0, 1]$ (Theorem 2.10). In §3, we prove limit theorems for abstract Markov maps (using a method essentially due to [**MT04**] and recalled in Appendix A, and estimates of [**AD01b**] and [**Gou04**]). Finally, in §§4 and 5, we study the limit laws of Birkhoff sums for the skew product T, and we obtain the convergence to a normal law or a stable law, depending on the value of $\alpha_{\min}$. We obtain the following theorem (see Theorem 5.1 for more details).

THEOREM 1.1. *Set*

$$A = \frac{1}{4(\alpha_{\min}^{3/2}\sqrt{\pi/2\alpha''(x_0)})^{1/\alpha_{\min}}} \int_{S^1 \times \{1/2\}} h \, d\,\mathrm{Leb}, \tag{2}$$

where h is the density of the absolutely continuous invariant probability measure.

Let f be a Lipschitz function on $S^1 \times [0, 1]$, with $\int f \, dm = 0$. Write $c = \int_{S^1 \times \{0\}} f \, d\,\mathrm{Leb}$ and $S_n f = \sum_{k=0}^{n-1} f \circ T^k$. Then:

- *if $\alpha_{\min} < \frac{1}{2}$, there exists $\sigma^2 \geqslant 0$ such that $(1/\sqrt{n})S_n f \to \mathcal{N}(0, \sigma^2)$;*
- *if $\frac{1}{2} \leqslant \alpha_{\min} < 1$ and $c = 0$, then there exists $\sigma^2 \geqslant 0$ such that $(1/\sqrt{n})S_n f \to \mathcal{N}(0, \sigma^2)$;*
- *if $\alpha_{\min} = \frac{1}{2}$ and $c \neq 0$, then $S_n f/\sqrt{(c^2 A/4) n (\ln n)^2} \to \mathcal{N}(0, 1)$;*
- *if $\frac{1}{2} < \alpha_{\min} < 1$ and $c \neq 0$, then $S_n f/n^{\alpha_{\min}}\sqrt{\alpha_{\min} \ln n} \to Z$, where the random variable Z has an explicit stable distribution.*

An interesting feature of this example is that its study involves the sophisticated mixing properties of F, particularly a multiple decorrelation property, proved in Appendix B using [**Pèn02**].

Remark. Theorems of [**Gou04**] could be used instead of the method of [**MT04**] to get the limit laws. However, the proof of [**Gou04**] is much more complicated than the elementary method of [**MT04**], and less versatile. Among others, an advantage of this new method is that it can easily be extended to stable laws of index 1, in contrast to [**Gou04**].

In fact, the previous results remain true for a much larger family of maps. Although we will only give the proofs for the previous maps for the sake of simplicity, we indicate now the more general results that can be proved with the same arguments.

We first define the generalizations of the maps T_α. For $\alpha \in (0,1)$, consider a map $\bar{T}_\alpha : [0,1] \to [0,1]$ such that $\bar{T}_\alpha$ is an increasing diffeomorphism between $[0, x_\alpha)$ and $[0,1)$ (for some $0 < x_\alpha < 1$) and between $[x_\alpha, 1]$ and $[0,1]$. Assume that $\alpha \mapsto x_\alpha$ is C^1, that the map $(x,\alpha) \mapsto \bar{T}'_\alpha(x)$ is C^1 on the sets $\{0 < x < x_\alpha\}$ and $\{x_\alpha \leqslant x \leqslant 1\}$, and that $\bar{T}'_\alpha(x) > 1$ for all $x \neq 0$. We also need to prescribe the behaviour of $\bar{T}_\alpha$ close to 0. Let $\varepsilon_0 > 0$. Assume that $\bar{T}_\alpha(x) = x + c_\alpha x^{1+\alpha}(1 + f_\alpha(x))$ for $x \in [0, \varepsilon_0]$, where $c_\alpha > 0$ depends continuously on α, $f_\alpha(0) = 0$ and $(x,\alpha) \mapsto f_\alpha(x)$ is continuous on $[0,\varepsilon_0] \times (0,1)$. Finally, assume that $\bar{T}_\alpha$ is C^3 on $(0,\varepsilon_0]$ with non-positive Schwarzian derivative and that the partial derivatives of the function $(x,\alpha) \mapsto \bar{T}'_\alpha(x)$ are bounded by $C_\varepsilon x^{\alpha-1}$ on $(0,\varepsilon_0] \times (\varepsilon, 1-\varepsilon)$ for all $\varepsilon > 0$.

Let $\alpha : S^1 \to (0,1)$ be a C^1 map. Let $\bar{F} : S^1 \to S^1$ be a C^2 uniformly expanding map, such that $\bar{F}'(\omega) > \bar{T}'_{\alpha(\omega)}(x)$ for all $\omega \in S^1$ and $x \in [0,1]$. This ensures that the map $\bar{T}$ defined on $S^1 \times [0,1]$ by $\bar{T}(\omega, x) = (\bar{F}\omega, \bar{T}_{\alpha(\omega)}x)$ is partially hyperbolic. The arguments of §2 apply to $\bar{T}$, and show that $\bar{T}$ admits an absolutely continuous invariant probability measure $\bar{m}$, which is ergodic and whose density is Lipschitz on every compact subset of $S^1 \times (0,1]$.

To obtain limit theorems, we need additional assumptions. Let $\alpha_{\min}$ be the minimal value taken by the function α, and $\alpha_{\max}$ its maximal value. Assume that $\alpha_{\max} < \frac{3}{2}\alpha_{\min}$, and that $\mathrm{Leb}\{\omega \in S^1 \mid |\alpha(\omega) - \alpha_{\min}| < \varepsilon\} \sim C\varepsilon^\gamma$ for some $C > 0$ and $\gamma \geqslant 0$. This is, for example, the case when α is C^2 and takes the value $\alpha_{\min}$ at a unique point ω_0 with $\alpha''(\omega_0) > 0$ (and, in this case, $\gamma = \frac{1}{2}$). This holds more generally if $\alpha'(\omega_0), \dots, \alpha^{(p-1)}(\omega_0) = 0$ and $\alpha^{(p)}(\omega_0) > 0$ for some $p \in \mathbb{N}$ (and, in this case, $\gamma = 1/p$). The following analogue of Theorem 1.1 then holds. Let f be a Lipschitz function on $S^1 \times [0,1]$. Denote by μ the probability measure on S^1 which is absolutely continuous and $\bar{F}$-invariant. Let $c = \int_{S^1 \times \{0\}} f \, d\mu$. If $\alpha_{\min} < \frac{1}{2}$, or $\frac{1}{2} \leqslant \alpha_{\min} < 1$ and $c = 0$, then $S_n f / \sqrt{n}$ converges in distribution to a normal law $\mathcal{N}(0, \sigma^2)$ for some $\sigma^2 \geqslant 0$. On the other hand, if $\alpha_{\min} = \frac{1}{2}$ and $c \neq 0$, then $S_n f / n^{\alpha_{\min}} (\ln n)^{\gamma + 1/2}$ converges in distribution to a normal law, and if $\alpha_{\min} > \frac{1}{2}$ and $c \neq 0$, then $S_n f / n^{\alpha_{\min}} (\ln n)^\gamma$ converges in distribution to a stable law, which can be explicitly given in terms of μ and of the density of $\bar{m}$.

Remark. An important assumption of our arguments is the fact that the maps T_α are Markov. This is heavily used in our computations of return times. With the present techniques, it is unlikely that this assumption could be removed.

In this article, $a(n) \sim b(n)$ means that $a(n)/b(n) \to 1$ when $n \to \infty$. The integral with respect to a probability measure will sometimes be denoted by $E(\cdot)$. Finally, $\lfloor x \rfloor$ will denote the integer part of x. From this point on, we will only deal with the skew product T, and not its generalization $\bar{T}$.

2. *Invariant measure*

An important property of the map T, that will be used thoroughly in what follows, is that it is Markov: there exists a partition of the space such that every element of this partition is mapped by T on a union of elements of this partition. In fact, we will consider T_Y (the induced map on $Y = S^1 \times (\frac{1}{2}, 1]$), which is also Markov and expanding, contrary to T. We will apply to T_Y classical results on expanding Markov maps (also called *Gibbs–Markov* maps), which we recall below.

2.1. *Markov maps and invariant measures.*

Let $(Y, \mathcal{B}, m_Y)$ be a standard probability space, endowed with a bounded metric d. A non-singular map T_Y defined on Y is said to be a *Markov map* if there exists a finite or countable partition α of Y such that for all $a \in \alpha$, $m_Y(a) > 0$, $T_Y(a)$ is a union (mod 0) of sets of α, and $T_Y : a \to T_Y(a)$ is invertible. In this case, α is a *Markov partition* for T_Y.

A Markov map T_Y (with a Markov partition α) is a *Gibbs–Markov map* [**Aar97**] if:

(1) T_Y has the big image property: $\inf_{a\in\alpha} m_Y(T_Y(a)) > 0$;

(2) there exists $\lambda > 1$ such that for all $a \in \alpha$, for all $x, y \in a$, $d(T_Y x, T_Y y) \geqslant \lambda d(x, y)$;

(3) let g be the inverse of the Jacobian of T_Y, i.e. on a set $a \in \alpha$, $g(x) = (dm_Y/d(m_Y \circ (T_Y)_{|a}))(x)$, then there exists $C > 0$ such that for all $a \in \alpha$, for almost all $x, y \in a$,

$$\left|1 - \frac{g(x)}{g(y)}\right| \leqslant C d(T_Y x, T_Y y). \tag{3}$$

This definition is slightly more general than the definition of [**Aar97**]: the distance $d = d_\tau$ considered there is given by $d_\tau(x, y) = \tau^{s(x,y)}$ where $\tau < 1$ and $s(x, y)$ is the separation time of x and y, i.e.

$$s(x, y) = \inf\{n \in \mathbb{N} \mid \nexists a \in \alpha, T^n x \in a, T^n y \in a\}. \tag{4}$$

The proof of [**Aar97**, Theorem 4.7.4] still works in our context, and gives the following.

THEOREM 2.1. *Let T_Y be a transitive Gibbs–Markov map (for all $a, b \in \alpha$, there exists $n \in \mathbb{N}$, $m_Y(T_Y^n a \cap b) > 0$) such that* $\mathrm{Card}(\alpha_*) < \infty$*, where α_* is the partition generated by the images $T_Y(a)$ for $a \in \alpha$. Then T_Y is ergodic, and there exists a unique absolutely continuous (with respect to m_Y) invariant probability measure, denoted by μ_Y.*

Moreover, $\mu_Y = h m_Y$ where the density h is bounded and bounded away from 0*, and Lipschitz on every set of α_*.*

2.2. *Preliminary estimates.*

To apply Theorem 2.1, we will construct a Markov partition, and control the distortion of the inverse branches of T_Y.

We will write $T_\omega^n = T_{\alpha(F^{n-1}\omega)} \circ \cdots \circ T_{\alpha(\omega)}$, whence $T^n(\omega, x) = (F^n\omega, T_\omega^n(x))$. Write also $d((\omega_1, x_1), (\omega_2, x_2)) = |\omega_1 - \omega_2| + |x_1 - x_2|$. A point of $S^1 \times [0, 1]$ will be denoted by $\mathbf{x} = (\omega, x)$. Finally, set $d_{\text{vert}}((\omega_1, x_1), (\omega_2, x_2)) = |x_2 - x_1|$.

Define $X_0(\omega) = 1$, $X_1(\omega) = \frac{1}{2}$, and for $n \geqslant 2$, $X_n(\omega)$ is the preimage in $[0, \frac{1}{2}]$ of $X_{n-1}(F\omega)$ by $T_{\alpha(\omega)}$. These X_n will be useful in the construction of a Markov partition for T, in §2.3.

PROPOSITION 2.2. *There exists $C > 0$ such that for all $n \in \mathbb{N}^*$, for all $\omega \in S^1$,*

$$\frac{1}{Cn^{1/\alpha_{\min}}} \leqslant X_n(\omega) \leqslant \frac{C}{n^{1/\alpha_{\max}}}. \tag{5}$$

Proof. Write $Z_1 = \frac{1}{2}$ and $V(Z_{n+1}) = Z_n$ where $V(x) = x(1 + 2^{\alpha_{\max}} x^{\alpha_{\min}})$. We easily check inductively that $Z_n \leqslant X_n(\omega)$ for every ω, since $V(x) \geqslant T_{\alpha(\omega)}(x)$ for every ω. It is thus sufficient to estimate Z_n to get the minoration. As $V(x) \geqslant x$, the sequence Z_n is decreasing, and non-negative. Hence, it tends to a fixed point of V, necessarily 0.

We have

$$\begin{aligned}\frac{1}{Z_n^{\alpha_{\min}}} &= \frac{1}{Z_{n+1}^{\alpha_{\min}}}(1 + 2^{\alpha_{\max}} Z_{n+1}^{\alpha_{\min}})^{-\alpha_{\min}} = \frac{1}{Z_{n+1}^{\alpha_{\min}}}(1 - \alpha_{\min} 2^{\alpha_{\max}} Z_{n+1}^{\alpha_{\min}} + o(Z_{n+1}^{\alpha_{\min}})) \\ &= \frac{1}{Z_{n+1}^{\alpha_{\min}}} - \alpha_{\min} 2^{\alpha_{\max}} + o(1).\end{aligned}$$

A summation gives $1/Z_m^{\alpha_{\min}} \sim m\, \alpha_{\min} 2^{\alpha_{\max}}$, whence $Z_m \sim C/m^{1/\alpha_{\min}}$, which concludes the minoration.

The majoration is similar, using a sequence Z_n' with $Z_n' \geqslant X_n(\omega)$. □

We fix once and for all a large enough constant D. The following definition is analogous to a definition of Viana [**Via97**].

Definition 2.3. Let $\psi : K \to [0, 1]$, where K is a subinterval of S^1. We say that the graph of ψ is an *admissible curve* if ψ is C^1 with $|\psi'| \leqslant D$.

PROPOSITION 2.4. *Let ψ be an admissible curve, defined on K with $|K| < \frac{1}{4}$, and included in $K \times [0, \frac{1}{2}]$ or $K \times (\frac{1}{2}, 1]$. Then the image of ψ by T is still an admissible curve.*

Proof. Let (u, v) be a tangent vector at (ω, x) with $|v| \leqslant D|u|$, we have to check that its image (u', v') by $DT(\omega, x)$ still satisfies $|v'| \leqslant D|u'|$.

Assume first that $x \leqslant \frac{1}{2}$, whence $u' = 4u$ and $v' = (1 + (2x)^{\alpha(\omega)}(\alpha(\omega) + 1))v + x \ln(2x)\alpha'(\omega)(2x)^{\alpha(\omega)}u$. As $\alpha(\omega) \leqslant \alpha_{\max} \leqslant 1$, we get $|v'| \leqslant 3|v| + C|u|$ for a constant C (which depends only on $\|\alpha'\|_\infty$). Thus,

$$\frac{|v'|}{|u'|} \leqslant \frac{3}{4}\frac{|v|}{|u|} + \frac{C}{4}. \tag{6}$$

This will give $|v'|/|u'| \leqslant D$ if $\frac{3}{4}D + C/4 \leqslant D$, which is true if D is large enough.

Assume then that $x > \frac{1}{2}$. Then $u' = 4u$ and $v' = 2v$, and there is nothing to prove. □

COROLLARY 2.5. *Let (ω_1, x_1) and (ω_2, x_2) be two points in $S^1 \times [0, \frac{1}{2}]$ with $|x_1 - x_2| \leqslant D|\omega_1 - \omega_2|$ and $|\omega_1 - \omega_2| \leqslant \frac{1}{8}$. Then their images satisfy $|x_1' - x_2'| \leqslant D|\omega_1' - \omega_2'|$.*

Proof. Use a segment between the two points: it is an admissible curve. Hence, its image is still admissible. □

2.3. *The Markov partition.* Set $Y = S^1 \times (\frac{1}{2}, 1]$. For $\mathbf{x} \in Y$, set $\varphi_Y(\mathbf{x}) = \inf\{n > 0 \mid T^n(\mathbf{x}) \in Y\}$: this is the first return time to Y, everywhere finite. The map $T_Y(\mathbf{x}) := T^{\varphi_Y(\mathbf{x})}(\mathbf{x})$ is the map induced by T on Y. We will show that T_Y is a Gibbs–Markov map, by constructing an appropriate Markov partition.

If I is an interval of S^1, we will abusively write $I \times [X_{n+1}, X_n]$ for $\{(\omega, x) \mid \omega \in I, x \in [X_{n+1}(\omega), X_n(\omega)]\}$.

Set $I_n(\omega) = [X_{n+1}(\omega), X_n(\omega)]$ (or $\{\omega\} \times [X_{n+1}(\omega), X_n(\omega)]$, depending on the context). By definition of X_n, T maps $\{\omega\} \times I_n(\omega)$ bijectively on $\{F\omega\} \times I_{n-1}(F\omega)$. Thus, the interval $I_n(\omega)$ returns to $[\frac{1}{2}, 1]$ in exactly n steps.

Let $Y_n(\omega)$ be the preimage in $[\frac{1}{2}, 1]$ of $X_{n-1}(F\omega)$ under $T_{\alpha(\omega)}$. Thus, the interval $J_n(\omega) = [Y_{n+1}(\omega), Y_n(\omega)]$ returns to $[\frac{1}{2}, 1]$ in n steps.

We fix once and for all $0 < \varepsilon_0 < \frac{1}{8}$, *small enough so that* $D\varepsilon_0$ *is less than the length of every interval* $I_1(\omega)$. (This condition will be useful in distortion estimates.)

Let q *be large enough so that* $1/4^q < \varepsilon_0$, *and consider* $A_{s,n} = [s/4^{q+n}, (s+1)/4^{q+n}] \times J_n$, *for* $n \in \mathbb{N}^*$ *and* $0 \leqslant s \leqslant 4^{q+n} - 1$: this set is mapped by T^n on $[s/4^q, (s+1)/4^q] \times [\frac{1}{2}, 1]$. Let $K_0, \dots, K_{4^q-1}$ be the sets $[i/4^q, (i+1)/4^q] \times [\frac{1}{2}, 1]$. Then the map T_Y is an isomorphism between each $A_{s,n}$ and some K_i. Consequently, the map T_Y is Markov for the partition $\{A_{s,n}\}$, and it has the big image property.

To apply Theorem 2.1, we need expansion (for (2) in the definition of Gibbs–Markov maps) and distortion control (for (3)). The expansion is given by the next proposition, and the distortion is estimated in §2.4.

On the intervals $[X_3(\omega), X_1(\omega)]$, the derivative of $T_{\alpha(\omega)}$ is greater than 1, whence greater than a constant $2 > \lambda > 1$, independent of ω.

For (ω_1, x_1) and $(\omega_2, x_2) \in S^1 \times [0, 1]$, set

$$d'((\omega_1, x_1), (\omega_2, x_2)) = a|x_1 - x_2| + |\omega_1 - \omega_2| \tag{7}$$

where $a = (1 - \lambda/4)/D$.

PROPOSITION 2.6. *On each* $A_{s,n}$, *the map* T^n *is expanding by at least* λ *for the distance* d'.

Proof. For $n = 1$ (the points return directly to $S^1 \times [\frac{1}{2}, 1]$), everything is linear and the result is clear. Assume that $n \geqslant 2$. Take (ω_1, x_1) and $(\omega_2, x_2) \in A_{s,n}$, with, for example, $x_2 \geqslant x_1$.

Since $(\omega_1, x_1) \in A_{s,n}$, this point returns to $S^1 \times [\frac{1}{2}, 1]$ after exactly n iterations. Since $x_1 \leqslant x_2$ and (ω_2, x_2) returns to $S^1 \times [\frac{1}{2}, 1]$ after exactly n iterations, the point (ω_2, x_1) takes at least n iterations to come back to $S^1 \times [\frac{1}{2}]$. Therefore, we can apply Corollary 2.5 $n - 1$ times to (ω_1, x_1) and (ω_2, x_1). We get that in vertical distance,

$$d_{\text{vert}}(T^n(\omega_1, x_1), T^n(\omega_2, x_1)) \leqslant D|F^n\omega_1 - F^n\omega_2|. \tag{8}$$

In particular, $T^n_{\omega_2}(x_1) \geqslant T^n_{\omega_1}(x_1) - D\varepsilon_0 \geqslant \frac{1}{2} - D\varepsilon_0$. Thus, by the definition of ε_0,

$$T^n(\omega_2, x_1) \in I_i(F^n\omega_2) \quad \text{for } i = 0 \text{ or } 1. \tag{9}$$

Taking the preimage under T, this implies that $T^{n-1}(\omega_2, x_1) \in [X_3(F^{n-1}\omega_2), X_1(F^{n-1}\omega_2)]$. Moreover, $T^{n-1}(\omega_2, x_2) \in [X_2(F^{n-1}\omega_2), X_1(F^{n-1}\omega_2)] \subset [X_3(F^{n-1}\omega_2), X_1(F^{n-1}\omega_2)]$. Since each map T_α is expanding, we also have $d_{\text{vert}}(T^{n-1}(\omega_2, x_1), T^{n-1}(\omega_2, x_2)) \geqslant |x_1 - x_2|$. We apply once more T, which expands at least by λ on $[X_3(F^{n-1}\omega_2), X_1(F^{n-1}\omega_2)]$ by definition of λ, and get

$$d_{\text{vert}}(T^n(\omega_2, x_1), T^n(\omega_2, x_2)) \geqslant \lambda |x_1 - x_2|. \tag{10}$$

Finally,

$$\begin{aligned} d'(T^n(\omega_1, x_1), T^n(\omega_2, x_2)) &\\ &= a d_{\text{vert}}(T^n(\omega_1, x_1), T^n(\omega_2, x_2)) + |F^n\omega_1 - F^n\omega_2| \\ &\geqslant a d_{\text{vert}}(T^n(\omega_2, x_1), T^n(\omega_2, x_2)) - a d_{\text{vert}}(T^n(\omega_1, x_1), T^n(\omega_2, x_1)) \\ &\quad + |F^n\omega_1 - F^n\omega_2| \\ &\geqslant a\lambda|x_1 - x_2| - aD|F^n\omega_1 - F^n\omega_2| + |F^n\omega_1 - F^n\omega_2|. \end{aligned}$$

The proposition will be proved if $(1 - aD)|F^n\omega_1 - F^n\omega_2| \geqslant \lambda|\omega_1 - \omega_2|$. Indeed, we have

$$(1-aD)|F^n\omega_1 - F^n\omega_2| = (1-aD)4^n|\omega_1 - \omega_2| \geqslant (1-aD)4|\omega_1 - \omega_2| = \lambda|\omega_1 - \omega_2|. \qquad \square$$

2.4. *Distortion bounds.*

LEMMA 2.7. *There exists a constant $E > 0$ such that for all $n > 0$, for all $\omega_1, \omega_2 \in S^1$ with $|\omega_1 - \omega_2| \leqslant \varepsilon_0/4^n$, for all $x_1 \in J_n(\omega_1)$ with $T^{n-1}_{\omega_2} x_1 \leqslant \frac{1}{2}$,*

$$|\ln(T^n_{\omega_1})'(x_1) - \ln(T^n_{\omega_2})'(x_1)| \leqslant E|F^n\omega_1 - F^n\omega_2|. \tag{11}$$

Proof. We use Corollary 2.5 n times and get for $0 \leqslant k \leqslant n$ that $|T^k_{\omega_1} x_1 - T^k_{\omega_2} x_1| \leqslant D|F^k\omega_1 - F^k\omega_2|$.

In particular, for $k = n$, $|T^n_{\omega_1} x_1| \geqslant \frac{1}{2}$, whence $|T^n_{\omega_2} x_1| \geqslant \frac{1}{2} - D\varepsilon_0$. Consequently, $T^n(\omega_2, x_1) \in I_i(F^n\omega_2)$ for some $i \in \{0, 1\}$, by definition of ε_0. Applying T^{-k}, we get $T^{n-k}(\omega_2, x_1) \in I_{i+k}(F^{n-k}\omega_2)$.

For $x \leqslant \frac{1}{2}$ and $\omega \in S^1$, write $G(\omega, x) = \ln T'_{\alpha(\omega)}(x) = \ln(1 + (\alpha(\omega) + 1)(2x)^{\alpha(\omega)})$. Then

$$\frac{\partial G}{\partial x}(\omega, x) = \frac{(\alpha(\omega) + 1)\alpha(\omega) 2^{\alpha(\omega)} x^{\alpha(\omega) - 1}}{1 + (\alpha(\omega) + 1)(2x)^{\alpha(\omega)}} \leqslant C x^{\alpha_{\min} - 1}$$

and

$$\left|\frac{\partial G}{\partial \omega}(\omega, x)\right| = \left|\frac{\alpha'(\omega)(2x)^{\alpha(\omega)} + (\alpha(\omega) + 1)\alpha'(\omega)\ln(2x)(2x)^{\alpha(\omega)}}{1 + (\alpha(\omega) + 1)(2x)^{\alpha(\omega)}}\right| \leqslant C.$$

Note that $T^k(\omega_1, x_1) \in I_{n-k}(F^k\omega_1)$ and $T^k(\omega_2, x_1) \in I_{n-k+i}(F^k\omega_2)$ with $i \leqslant 1$. Hence, Proposition 2.2 shows that the second coordinates of $T^k(\omega_1, x_1)$ and $T^k(\omega_2, x_1)$ are at least $1/C(n - k + 1)^{1/\alpha_{\min}}$. On the set of points (ω, x) with $x \geqslant 1/C(n - k + 1)^{1/\alpha_{\min}}$, the estimates on the partial derivatives of G show that this function is $C(n-k+1)^{1/\alpha_{\min} - 1}$-Lipschitz. Therefore,

$$\begin{aligned} |G(T^k(\omega_1, x_1)) - G(T^k(\omega_2, x_1))| &\leqslant C(n - k + 1)^{1/\alpha_{\min} - 1} d(T^k(\omega_1, x_1), T^k(\omega_2, x_1)) \\ &\leqslant C(n - k + 1)^{1/\alpha_{\min} - 1}(1 + D)|F^k\omega_1 - F^k\omega_2| \\ &\leqslant C(n - k + 1)^{1/\alpha_{\min} - 1}(1 + D)4^k|\omega_1 - \omega_2|. \end{aligned}$$

Finally,

$$\begin{aligned}|\ln(T^n_{\omega_1})'(x_1) - \ln(T^n_{\omega_2})'(x_1)| &\leqslant \sum_{k=0}^{n-1} |G(T^k(\omega_1, x_1)) - G(T^k(\omega_2, x_1))| \\ &\leqslant C4^n|\omega_1 - \omega_2| \sum_{k=0}^{n-1} (n-k+1)^{1/\alpha_{\min}-1} 4^{k-n} \\ &\leqslant C|F^n\omega_1 - F^n\omega_2| \sum_{l=1}^{\infty} (l+1)^{1/\alpha_{\min}-1} 4^{-l}.\end{aligned}$$

The last sum is finite, which concludes the proof. □

For $n \geqslant 2$, write $J_n^+(\omega) = [Y_{n+2}(\omega), Y_n(\omega)]$. Thus, if $n \geqslant 1$, $J_{n+1}^+(\omega)$ is the preimage of $I_n^+(F\omega)$, defined by $I_n^+(F\omega) = [X_{n+2}(F\omega), X_n(F\omega)]$. These intervals will appear naturally in distortion controls, since we have seen in the proof of Lemma 2.7 that, if we move away horizontally from a point in $J_n(\omega_1)$, we find a point in $J_{n+i}(\omega_2)$ for $i \in \{0, 1\}$, i.e. in $J_n^+(\omega_2)$.

LEMMA 2.8. *There exists a constant C such that for all $n > 0$, for all $\omega \in S^1$, for all $x, y \in J_n^+(\omega)$,*

$$|\ln(T^n_\omega)'(x) - \ln(T^n_\omega)'(y)| \leqslant C|T^n_\omega(x) - T^n_\omega(y)|.$$

Proof. Recall that the Schwarzian derivative of an increasing diffeomorphism g of class C^3 is

$$Sg(x) = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2.$$

The composition of two functions with non-positive Schwarzian derivative still has a non-positive Schwarzian derivative.

For $\tau > 0$, the Koebe principle [**dMvS93**, Theorem IV.1.2] states that, if $Sg \leqslant 0$ and $J \subset J'$ are two intervals such that $g(J')$ contains a τ-scaled neighbourhood of $g(J)$ (i.e. the intervals on the left and on the right of $g(J)$ in $g(J')$ have length at least $\tau|g(J)|$), then there exists a constant $K(\tau)$ such that

$$|\ln g'(x) - \ln g'(y)| \leqslant K(\tau)\frac{|x-y|}{|J|}, \quad \forall x, y \in J. \tag{12}$$

This implies that the distortion of g is bounded on J. Hence it is possible to replace the bound on the right-hand side with $K'(\tau)(|g(x) - g(y)|/|g(J)|)$.

In our case, if $0 < \alpha < 1$, the left branch of T_α has non-positive Schwarzian derivative, since $T_\alpha''' < 0$ and $T_\alpha' > 0$. In particular, let g be the composition of the (analytic extensions to $(0, +\infty)$ of the) left branches of $T_{\alpha(F^{n-1}\omega)}, \dots, T_{\alpha(F\omega)}$, and of the right branch of $T_{\alpha(\omega)}$. Then, on J_n^+, we have $T^n_\omega = g$, and $g : (\frac{1}{2}, +\infty) \to (0, +\infty)$ has non-positive Schwarzian derivative.

We want to see that $|\ln(T^n_\omega)'(x) - \ln(T^n_\omega)'(y)| \leqslant C|T^n_\omega(x) - T^n_\omega(y)|$. For this, we apply the Koebe principle to $J = J_n^+$ and $J' = [\frac{1}{2} + \delta, 2]$ for δ very small. Then $g(J) = [X_2, 1]$ while $g(J')$ contains $[\delta', 2]$, where $\delta' > 0$ is arbitrarily small if δ is small enough. As the X_2 are uniformly bounded away from 0, there exists $\tau > 0$ (independent of ω and n) such that $g(J')$ contains a τ-scaled neighbourhood of $g(J)$. The Koebe principle then gives the desired result. □

PROPOSITION 2.9. *There exists a constant C such that, for every $A_{s,n}$, for every (ω_1, x_1) and $(\omega_2, x_2) \in A_{s,n}$,*

$$\left| \frac{\det DT^n(\omega_1, x_1)}{\det DT^n(\omega_2, x_2)} - 1 \right| \leqslant Cd(T^n(\omega_1, x_1), T^n(\omega_2, x_2)). \tag{13}$$

Proof. The matrix $DT^n(\omega, x)$ is upper triangular, with 4^n in the upper left corner. Thus, we have to show that

$$|\ln(T^n_{\omega_1})'(x_1) - \ln(T^n_{\omega_2})'(x_2)| \leqslant Cd(T^n(\omega_1, x_1), T^n(\omega_2, x_2)). \tag{14}$$

Assume, for example, that $x_2 \geqslant x_1$, which implies that $T^k_{\omega_2}(x_1) \leqslant \frac{1}{2}$ for $k = 0, \dots, n-1$. Lemma 2.7 can be applied to x_1, ω_1 and ω_2. Moreover, (9) implies that $x_1 \in J_n^+(\omega_2)$.

Write

$$\begin{aligned} |\ln(T^n_{\omega_2})'(x_2) - \ln(T^n_{\omega_1})'(x_1)| &\leqslant |\ln(T^n_{\omega_2})'(x_2) - \ln(T^n_{\omega_2})'(x_1)| \\ &\quad + |\ln(T^n_{\omega_2})'(x_1) - \ln(T^n_{\omega_1})'(x_1)| \\ &\leqslant Cd(T^n(\omega_2, x_2), T^n(\omega_2, x_1)) + E|F^n\omega_2 - F^n\omega_1| \end{aligned}$$

by Lemmas 2.8 and 2.7. For the first term,

$$\begin{aligned} d(T^n(\omega_2, x_2), T^n(\omega_2, x_1)) &\leqslant d(T^n(\omega_2, x_2), T^n(\omega_1, x_1)) + d(T^n(\omega_1, x_1), T^n(\omega_2, x_1)) \\ &\leqslant d(T^n(\omega_2, x_2), T^n(\omega_1, x_1)) + (D+1)|F^n\omega_1 - F^n\omega_2| \end{aligned}$$

using admissible curves.

As $|F^n\omega_1 - F^n\omega_2| \leqslant d(T^n(\omega_1, x_1), T^n(\omega_2, x_2))$, we get the conclusion. □

2.5. *Construction of the invariant measure.* The previous estimates and Theorem 2.1 easily give that T_Y admits an invariant measure, with Lipschitz density. Inducing gives an invariant measure for T, whose density is Lipschitz on each set $S^1 \times (X_{n+1}, X_n)$. However, this does not exclude discontinuities on $S^1 \times X_n$, which is not surprising since T itself has a discontinuity on $S^1 \times \{\frac{1}{2}\}$, and T^n is discontinuous on $S^1 \times X_n$.

However, in the one-dimensional case, Liverani *et al.* [**LSV99**] have proved that the density is really continuous everywhere, since they constructed it as an element of a cone of continuous functions. This fact remains true here, as shown in the following.

THEOREM 2.10. *The map T admits a unique absolutely continuous invariant probability measure dm. Moreover, this measure is ergodic. Finally, the density $h = dm/d\,\mathrm{Leb}$ is Lipschitz on every compact subset of $S^1 \times (0, 1]$.*

Proof. Consider the map T_Y induced by T on $Y = S^1 \times (\frac{1}{2}, 1]$. It is Markov for the partition $\alpha = \{A_{s,n}\}$, and transitive for this partition since $T_Y^2(a) = Y$ for all $a \in \alpha$. Moreover, it is expanding for d' on each set of the partition (Proposition 2.6) and its distortion is Lipschitz (Proposition 2.9, and d equivalent to d').

Theorem 2.1 shows that T_Y admits a unique absolutely continuous invariant probability measure $dm_Y = h\, d\mathrm{Leb}$, which is ergodic. Moreover, the density h is Lipschitz (for the distance d', whence for the usual one) on each element of the partition α_* generated by the sets $T_Y(a)$, i.e. on the sets K_i.

To construct an invariant measure for the initial map T, we use the classical induction process [**Aar97**, §1.1.5]: let φ_Y be the return time to Y under T, then $\mu = \sum_{n=0}^{\infty} T_*^n(m_Y \mid \varphi_Y > n)$ is invariant. To check that the new measure has finite mass, we have to see that $\sum m_Y(\varphi_Y > n) < \infty$. As dm_Y and $d\,\mathrm{Leb}$ are equivalent, we check it for $d\,\mathrm{Leb}$. We have

$$\mathrm{Leb}(\varphi_Y > n) = \mathrm{Leb}\left(S^1 \times \left[\frac{1}{2}, Y_{n+1}\right]\right) = \frac{1}{2}\mathrm{Leb}(S^1 \times [0, X_n]) \leqslant \frac{1}{2}\frac{C}{n^{1/\alpha_{\max}}},$$

using Proposition 2.2. As $\alpha_{\max} < 1$, this is summable.

We know that h is Lipschitz on the sets $[s/4^q, (s+1)/4^q] \times [\frac{1}{2}, 1]$, we have to prove the continuity on $\{s/4^q\} \times [\frac{1}{2}, 1]$, which is not hard: these numbers $s/4^q$ are artificial, since they depend on the arbitrary choice of a Markov partition on S^1. We can do the same construction using sets other than the $A_{s,n}$. For example, set $A'_{s,n} = [\frac{1}{3} + s/4^{q+n}, \frac{1}{3} + (s+1)/4^{q+n}] \times J_n$, and $K'_i = [\frac{1}{3} + i/4^q, \frac{1}{3} + (i+1)/4^q]$. Since $\frac{1}{3}$ is a fixed point of F, the map T_Y is Markov for the partition $\{A'_{s,n}\}$, and each of these sets is mapped on a set K'_i. Thus, the same arguments as above apply, and prove that h is Lipschitz on each set K'_i. Since the boundaries of the sets K_i and K'_i are different, this shows that h is, in fact, Lipschitz on $S^1 \times [\frac{1}{2}, 1]$.

We show now that h is Lipschitz on $S^1 \times [X_2, 1]$. Note that it is slightly incorrect to say that h is Lipschitz, since h is defined only almost everywhere. Nevertheless, if we prove that $|h(\mathbf{x}) - h(\mathbf{y})| \leqslant Cd(\mathbf{x}, \mathbf{y})$ for almost all $\mathbf{x}$ and $\mathbf{y}$, then there will exist a unique version of h that really is Lipschitz. Thus, all of the equalities we will write until the end of this proof will be true only almost everywhere.

Let $A^+_{s,n} = [s/4^{q+n}, (s+1)/4^{q+n}] \times J^+_n$: T^n is a diffeomorphism between $A^+_{s,n}$ and $K^+_i = [i/4^q, (i+1)/4^q] \times [X_2, 1]$. We fix some $K^+ = K^+_i = I \times [X_2, 1]$, and we show that h is Lipschitz on K^+. Let us denote by $A^+_{s_1,n_1}, A^+_{s_2,n_2}, \dots$ the sets $A^+_{s,n}$ whose image under T^n is K^+, and by $U_j : K^+ \to A^+_{s_j,n_j}$ the inverse of the restriction of T^{n_j} to $A^+_{s_j,n_j}$. Let T_Y be the map induced by T on $Y = S^1 \times [\frac{1}{2}, 1]$. Then $h\ d\mathrm{Leb}_{|Y}$ is invariant under T_Y. This implies that, for each $\mathbf{x} \in I \times [\frac{1}{2}, 1]$,

$$h(\mathbf{x}) = \sum JU_j(\mathbf{x}) h(U_j\mathbf{x}) \tag{15}$$

where JU_j is the Jacobian of U_j.

Let $Z = S^1 \times [X_2, 1]$, and T_Z be the map induced by T on Z. Since $h\ d\mathrm{Leb}_{|Z}$ is also invariant under T_Z, we have the same kind of equation as above. For $\mathbf{x} \in I \times [X_2, \frac{1}{2}]$, all its preimages under T_Z are in $S^1 \times [\frac{1}{2}, 1]$, and the invariance gives that

$$h(\mathbf{x}) = \sum JU_j(\mathbf{x}) h(U_j\mathbf{x}). \tag{16}$$

We have shown that, for every $\mathbf{x} \in S^1 \times [X_2, 1]$,

$$h(\mathbf{x}) = \sum JU_j(\mathbf{x}) h(U_j\mathbf{x}). \tag{17}$$

This means that h is invariant under some kind of transfer operator, even though it is not a genuine transfer operator since the images of the maps U_j are not disjoint, and since they do not cover the space. In particular, the images of the U_j are included in $S^1 \times [\frac{1}{2}, 1]$, and we already know that h is Lipschitz on this set.

The bounds of the previous sections still apply to the distortion of the U_j, and their expansion. In particular, $|1 - JU_j(\mathbf{y})/JU_j(\mathbf{x})| \leqslant Cd(\mathbf{x}, \mathbf{y})$ for a constant C independent of j, and $|h(U_j\mathbf{x}) - h(U_j\mathbf{y})| \leqslant Cd(U_j\mathbf{x}, U_j\mathbf{y}) \leqslant C'd(\mathbf{x}, \mathbf{y})$ (since h is Lipschitz on the image of U_j). Thus,

$$\begin{aligned}|h(\mathbf{x}) - h(\mathbf{y})| &\leqslant \sum |JU_j(\mathbf{x})h(U_j\mathbf{x}) - JU_j(\mathbf{y})h(U_j\mathbf{y})| \\ &\leqslant \sum |JU_j(\mathbf{x})| \left|1 - \frac{JU_j(\mathbf{y})}{JU_j(\mathbf{x})}\right| |h(U_j\mathbf{x})| + \sum |JU_j(\mathbf{y})||h(U_j\mathbf{x}) - h(U_j\mathbf{y})| \\ &\leqslant Cd(\mathbf{x}, \mathbf{y}) \sum |JU_j(\mathbf{x})| + C'd(\mathbf{x}, \mathbf{y}) \sum |JU_j(\mathbf{y})|.\end{aligned}$$

It remains to prove that $\sum |JU_j(\mathbf{x})|$ is bounded. The bound on distortion gives $JU_j(\mathbf{x}) \asymp \mathrm{Leb}(\mathrm{Im}\, U_j)$, whence $\sum JU_j(\mathbf{x}) \leqslant C \sum \mathrm{Leb}(\mathrm{Im}\, U_j)$, which is finite since every point of $S^1 \times [\frac{1}{2}, 1]$ is in the image of at most two maps U_j.

We have proved that h is Lipschitz on $S^1 \times [X_2, 1]$, except maybe on $\{s/4^q\} \times [X_2, 1]$. As above, using another Markov partition, we exclude the possibility of discontinuities there. Thus, h is Lipschitz on $S^1 \times [X_2, 1]$.

To prove that h is Lipschitz on $S^1 \times [X_k, 1]$, we do exactly the same thing, except that we consider $[Y_{n+k}, Y_n]$ instead of $J_n^+ = [Y_{n+2}, Y_n]$. As above, writing $U_1, U_2, \dots$ for the inverse branches of T^n defined on a set $[s/4^{n+q}, (s+1)/4^{n+q}] \times [Y_{n+k}, Y_n]$ and whose image is $K' = [i/4^q, (i+1)/4^q] \times [X_k, 1] = I \times [X_k, 1]$, we show that $h(\mathbf{x}) = \sum JU_j(\mathbf{x})h(U_j\mathbf{x})$ for $\mathbf{x} \in K'$. In fact, for $\mathbf{x} \in I \times [X_l, X_{l-1}]$, we use the invariance of $h\ d\mathrm{Leb}$ under the map induced by T on $S^1 \times [X_l, 1]$ to prove this equality. We conclude finally as above, using the fact that h is Lipschitz on $S^1 \times [\frac{1}{2}, 1]$, which contains the images of the U_j.

This concludes the proof, since every compact subset of $S^1 \times (0, 1]$ is contained in $S^1 \times [X_k, 1]$ for large enough k. □

3. *Limit theorems for Markov maps*

We want to establish limit theorems for Birkhoff sums. In this direction, we give in this section an abstract result, valid for a map that induces a Gibbs–Markov map on a subset of the space (which is the case of our skew product). Related limit theorems have been proved in [**Gou04**], but we will show here a slightly different result, which requires more control on the return time φ but is more elementary, using Theorem A.1 proved in Appendix A and inspired by results of Melbourne and Török [**MT04**] for flows.

If $Z_0, \dots, Z_{n-1}, \dots$ are independent identically distributed random variables with zero mean, the sums $B_n{}^{-1} \sum_{k=0}^{n-1} Z_k$ (where B_n is a real sequence) converge to a non-trivial limit distribution in the following cases: if $Z_k \in L^2$, there is convergence to a normal law for $B_n = \sqrt{n}$. There is also convergence to a normal law, but with a different normalization, if $P(|Z_k| > x) = x^{-2}l(x)$ with $L(x) := 2\int_1^x (l(u)/u)\, du$ unbounded and slowly varying (i.e. $L : (0, \infty) \to (0, \infty)$ satisfies $\lim_{x\to\infty} L(ax)/L(x) = 1$ for all $a > 0$); this is, in particular, true when l itself is slowly varying. Finally, if $P(Z_k > x) = (c_1 + o(1))x^{-p}L(x)$ and $P(Z_k < -x) = (c_2 + o(1))x^{-p}L(x)$, where L is slowly varying and $p \in (0, 2)$, we have convergence (for a good choice of B_n) to a limit law called stable law. It is a remarkable fact that, in this probabilistic setting, these sufficient conditions for convergence are also necessary (see, e.g., [**Fel66**, Theorem XVII.5.1a]).

In the dynamical setting, we will prove the same kind of limit theorems, still with three possible cases: L^2, normal non-standard and stable. The normalizations will, moreover, be the same as in the probabilistic setting. However, we will only give sufficient conditions for convergence, the converse seems definitely out of reach.

THEOREM 3.1. *Let $T : X \to X$ be an ergodic transformation preserving a probability measure m. Assume that there exists a subset Y of X with $m(Y) > 0$ and a countable partition α of Y, such that the first return map $T_Y(x) = T^{\varphi(x)}(x)$ (where $\varphi(x) = \inf\{n > 0 \mid T^n(x) \in Y\}$) is Gibbs–Markov for the measure $m_{|Y}$ and the partition α. Assume, moreover, that φ is constant on each element of α.*

Let $f : X \to \mathbb{R}$ be an integrable map with $\int f = 0$, such that $f_Y(y) := \sum_{n=0}^{\varphi(y)-1} f(T^n y)$ satisfies

$$\sum_{a\in\alpha} m(a) Df_Y(a) < \infty \tag{18}$$

where

$$Df_Y(a) = \inf\{C > 0 \mid \forall x, y \in a, |f_Y(x) - f_Y(y)| \leqslant Cd(x, y)\}. \tag{19}$$

Then we have the following.

- *Assume that $f_Y \in L^2$. Assume, moreover, that φ satisfies one of the following hypotheses:*
  - *$\varphi \in L^2$;*
  - *$m(\varphi > x) = x^{-p} L(x)$ where L is slowly varying and $p \in (1, 2]$.*

  Then there exists $\sigma^2 \geqslant 0$ such that $(1/\sqrt{n}) S_n f \to \mathcal{N}(0, \sigma^2)$.
- *Assume that $m(|f_Y| > x) = x^{-2} l(x)$, with $L(x) := 2\int_1^x (l(u)/u)\,du$ unbounded and slowly varying. Assume, moreover, that $m(\varphi > x) = (c + o(1))x^{-2} l(x)$ with $c > 0$. Let $B_n \to \infty$ satisfy $nL(B_n) = B_n^2$. Then ${B_n}^{-1} S_n f \to \mathcal{N}(0, 1)$.*
- *Assume that $m(f_Y > x) = (c_1 + o(1))x^{-p} L(x)$ and $m(f_Y < -x) = (c_2 + o(1))x^{-p} L(x)$ where L is a slowly varying function, $p \in (1, 2)$, and $c_1, c_2 \geqslant 0$ with $c_1 + c_2 > 0$. Assume also that $m(\varphi > x) = (c_3 + o(1))x^{-p} L(x)$ with $c_3 > 0$. Let $B_n \to \infty$ satisfy $nL(B_n) = B_n^p$. Then ${B_n}^{-1} S_n f \to Z$ where the random variable Z has a characteristic function given by*

  $$E(e^{itZ}) = e^{-c|t|^p (1 - i\beta \operatorname{sgn}(t) \tan(p\pi/2))} \tag{20}$$

  with $c = (c_1 + c_2)\Gamma(1 - p)\cos(p\pi/2)$ and $\beta = (c_1 - c_2)/(c_1 + c_2)$.

In the second case of the theorem, when l itself is slowly varying, then L is automatically slowly varying.

Proof. The idea is to use Theorem A.1: we have to check all of its hypotheses. We will use the notation of this theorem and, in particular, write $E_Y(u) = \int_Y u\,dm/m(Y)$.

We first treat the third case (stable law), using the results of [**AD01b**] (and the generalizations of [**Gou04**]). Let $s(x, y)$ be the separation time of x and y defined in (4), $\tau = 1/\lambda$ and $d_\tau = \tau^s$ the corresponding metric. Since every iteration of T_Y expands by at least λ, we get $d(x, y) \leqslant C d_\tau(x, y)$. In particular, we can assume without loss of generality that $d = d_\tau$, which is the setting of [**AD01b**] and [**Gou04**].

Let P be the transfer operator associated to T_Y (it is defined by $\int u \cdot v \circ T_Y = \int P(u) \cdot v$), and let $P_t(u) = P(e^{itf_Y} u)$. Let $\mathcal{L}$ be the space of bounded Lipschitz functions (i.e., such that there exists C such that $|g(x) - g(y)| \leqslant C d(x, y)$ for all $a \in \alpha$, for all $x, y \in a$). Since $\sum m(a) D f_Y(a) < \infty$, [**Gou04**, Theorem 3.8] (which is a strengthening of Theorem 5.1 in [**AD01b**]) ensures that, for small enough t, P_t acting on $\mathcal{L}$ has an eigenvalue $\lambda(t) = e^{-(c/m(Y))|t|^p(1 - i\beta \operatorname{sgn}(t) \tan(p\pi/2)) L(|t|^{-1})(1+o(1))}$ close to 1, and the remaining part of the spectrum of P_t is uniformly bounded away from 1.

We will use this information to estimate $E_Y(\varphi e^{i(t/B_n) S^Y_{\lfloor nm(Y) \rfloor} f_Y})$. Since φ is Lipschitz and integrable, $P\varphi \in \mathcal{L}$ by [**AD01b**, Proposition 1.4]. Let $k(n) = \lfloor nm(Y) \rfloor - 1$. Then

$$E_Y(\varphi e^{i(t/B_n) S^Y_{k(n)} f_Y \circ T_Y}) = E_Y(P\varphi \cdot e^{i(t/B_n) S^Y_{k(n)} f_Y}) = E_Y(P^{k(n)}_{t/B_n} P\varphi)$$
$$= E_Y(\varphi) \lambda(t/B_n)^{k(n)} + o(1).$$

The slow variation of L implies that, for all $t \neq 0$,

$$k(n) \frac{1}{m(Y)} \left| \frac{t}{B_n} \right|^p L(B_n/|t|) \sim |t|^p \frac{n}{B_n^p} L(B_n) \to |t|^p. \tag{21}$$

Hence, we get

$$\lambda\left(\frac{t}{B_n}\right)^{k(n)} \to e^{-c|t|^p(1 - i\beta \operatorname{sgn}(t) \tan(p\pi/2))}. \tag{22}$$

This shows that $E_Y(\varphi e^{i(t/B_n) S^Y_{k(n)} f_Y \circ T_Y}) \to E_Y(Z) E(e^{itZ})$, where the random variable Z is as in the statement of the theorem. Hence,

$$E_Y(\varphi e^{i(t/B_n)(S^Y_{\lfloor nm(Y) \rfloor} f_Y - f_Y)}) \to E_Y(Z) E(e^{itZ}). \tag{23}$$

Moreover, the difference between this term and $E_Y(\varphi e^{i(t/B_n) S^Y_{\lfloor nm(Y) \rfloor} f_Y})$ is bounded by $E_Y(\varphi |e^{-i(t/B_n) f_Y} - 1|)$, which tends to 0 by dominated convergence. Thus,

$$E_Y(\varphi e^{i(t/B_n) S^Y_{\lfloor nm(Y) \rfloor} f_Y}) \to E_Y(\varphi) E(e^{itZ}). \tag{24}$$

This is (52). Moreover, since L is slowly varying, the equation $nL(B_n) = B_n^p$ implies that $\inf_{r \geqslant n}(B_r/B_n) > 0$ (using for example the Potter bounds [**BGT87**, Theorem 1.5.6]).

Hypothesis 2 of Theorem A.1 is satisfied for $b = 1$, according to Birkhoff's theorem applied to $\varphi - E_Y(\varphi)$ (and because T_Y is ergodic, which is a consequence of the ergodicity of T). Finally, the hypothesis on the distribution of φ ensures, by [**AD01b**, Theorem 6.1], that $(S^Y_{\lfloor nm(Y) \rfloor} \varphi - nm(Y) E_Y(\varphi))/B_n$ converges in distribution. Thus, (53′) is satisfied. We can use Theorem A.1, and get that $S_n f / B_n \to Z$. This concludes the proof of the third case of Theorem 3.1.

The proof of the second case of Theorem 3.1 is exactly the same, using Theorem 3.1 in [**AD01a**] instead of Theorem 5.1 in [**AD01b**] to show the convergence in distribution of $S^Y_{\lfloor nm(Y) \rfloor} f_Y / B_n$ and $(S^Y_{\lfloor nm(Y) \rfloor} \varphi - nm(Y) E_Y(\varphi))/B_n$.

In the first case ($f_Y \in L^2$), the proof is again identical when $\varphi \in L^2$, with $B_n = \sqrt{n}$: indeed, in [**GH88**] it was proved that the Birkhoff sums of f_Y and φ satisfy a classical central limit theorem. However, when $m(\varphi > x) = x^{-p}L(x)$, we have to check in a different way the hypotheses 2 and 3 of Theorem A.1. Theorem 6.1 of [**AD01b**] ensures that, if B'_n is given by

$$nL(B'_n) = (B'_n)^p, \tag{25}$$

then $(S_n^Y\varphi - nE_Y(\varphi))/B'_n$ converges in distribution. Assume for the moment that, in the natural extension of T_Y, for all $b > \frac{1}{2}$, for almost all $x \in Y$,

$$\frac{1}{|N|^b}\sum_{k=0}^{N-1} f_Y(T_Y^k x) \to 0 \tag{26}$$

when $N \to \pm\infty$. Then hypothesis 2 of Theorem A.1 is satisfied for any $b > \frac{1}{2}$. Let $\kappa > 0$ be very small. As L is slowly varying, $L(B'_n) = O((B'_n)^\kappa)$, whence Equation (25) gives $B'_n = O(n^{1/(p-\kappa)})$. Thus, if $b < p/2$, we have $B'_n = O(B_n^{1/b})$, which implies (53′).

Hence, to conclude the proof, we just have to check (26). In [**Gou04**, Lemma 3.4] it was proved that $Pf_Y \in \mathcal{L}$, and has a vanishing integral. If T_Y is mixing, then 1 is the only eigenvalue of P of modulus 1, and P has a spectral gap, whence $P^n f_Y \to 0$ exponentially fast. In particular, $\int f_Y \circ T_Y^n \cdot f_Y = \int (P^n f_Y)\cdot f_Y = O((1-\delta)^n)$ for some $0 < \delta < 1$. Thus, as $f_Y \in L^2$, [**Kac96**, Theorem 16] gives that, for every $b > \frac{1}{2}$, $(1/N^b)\sum_{k=0}^{N-1} f_Y(T_Y^k x) \to 0$ almost everywhere when $N \to \infty$. In the natural extension, $\int f_Y \circ T_Y^{-n} \cdot f_Y = \int f_Y \cdot f_Y \circ T_Y^n$ decays also exponentially fast, whence the same argument gives that $(1/|N|^b)\sum_{k=0}^{N-1} f_Y(T_Y^k x) \to 0$ when $N \to -\infty$. Hence, (26) is satisfied if T_Y is mixing. In the general case, there exists a decomposition $Y = Y_1 \cup \cdots \cup Y_d$ such that T_Y maps Y_i to Y_{i+1} for $1 \leqslant i \leqslant d-1$, and Y_d to Y_1, and such that T_Y^d is mixing on each Y_i (with $m(Y_i) = 1/d$ for $1 \leqslant i \leqslant d$) (see [**Aar97**]). In particular, set $g = d\sum_{i=1}^d (\int_{Y_i} f_Y)1_{Y_i}$ and $h = f_Y - g$: this function satisfies $P^{dn}h = O((1-\delta)^{dn})$, using the same argument as above on each Y_i, since $\int_{Y_i} h = 0$ for all i. Hence, $P^n h = O((1-\delta)^n)$. In particular, $(1/|N|^b)\sum_{k=0}^{N-1} h(T_Y^k x) \to 0$ when $N \to \pm\infty$, as above. Moreover, $\sum_{i=1}^d (\int_{Y_i} f_Y) = \int f_Y = 0$, whence $\sum_{k=0}^{N-1} g(T_Y^k x)$ is uniformly bounded. This implies that $(1/|N|^b)\sum_{k=0}^{N-1} f_Y(T_Y^k x) \to 0$ almost everywhere when $N \to \pm\infty$. □

4. *Asymptotic behaviour of X_n*

We return to the study of the skew product (1). To prove limit theorems using Theorem 3.1, we will need to estimate $m(\varphi_Y > n)$, which is directly related to the speed of convergence of X_n to 0. This section will be devoted to the proof of the following theorem.

THEOREM 4.1. *When $n \to +\infty$,*

$$\left(\frac{n}{\sqrt{\ln n}}\right)^{1/\alpha_{\min}} X_n \to \frac{1}{(2^{\alpha_{\min}}\,{\alpha_{\min}}^{3/2}\sqrt{\pi/2\alpha''(x_0)})^{1/\alpha_{\min}}} \tag{27}$$

almost everywhere and in L^1.

The proof of this theorem is a quite involved computation, which relies on the following lemma.

LEMMA 4.2. *We have*

$$E(e^{-(\alpha-\alpha_{\min})w}) \sim \sqrt{\frac{\pi}{2\alpha''(x_0)}}\frac{1}{\sqrt{w}} \quad \textit{when } w \to \infty. \tag{28}$$

Proof. Write $\beta = \alpha - \alpha_{\min}$, and $f(b) = \mathrm{Leb}\{\omega \mid \beta(\omega) \in [0, b)\}$. In a neighbourhood of ω_0 (the unique point where α takes its minimal value $\alpha_{\min}$), α behaves like the parabola $\alpha_{\min} + (\alpha''(\omega_0)/2)(\omega - \omega_0)^2$, whence $f(b) \sim \sqrt{2/\alpha''(x_0)}\sqrt{b}$ when $b \to 0$.

Writing P_β for the distribution of β, an integration by parts gives

$$\begin{aligned} E(e^{-(\alpha-\alpha_{\min})w}) &= \int_0^\infty e^{-bw}\, dP_\beta(b) = w\int_0^\infty e^{-bw} f(b)\, db = \int_0^\infty e^{-u} f(u/w)\, du \\ &= \frac{1}{\sqrt{w}}\int_0^\infty e^{-u}(\sqrt{w} f(u/w))\, du. \end{aligned}$$

However $e^{-u}(\sqrt{w} f(u/w)) \to e^{-u}\sqrt{2/\alpha''(x_0)}\sqrt{u}$ when $w \to \infty$. There exists a constant E such that $f(u) \leqslant E\sqrt{u}$ (this is clear in a neighbourhood of 0, and elsewhere since f is bounded), whence $e^{-u}(\sqrt{w} f(u/w)) \leqslant Ee^{-u}\sqrt{u}$ integrable. By dominated convergence,

$$\int_0^\infty e^{-u}(\sqrt{w} f(u/w))\, du \to \sqrt{\frac{2}{\alpha''(x_0)}}\int_0^\infty e^{-u}\sqrt{u}\, du = \sqrt{\frac{2}{\alpha''(x_0)}}\frac{\sqrt{\pi}}{2}. \qquad \square$$

Proof of Theorem 4.1. As in the proof of Proposition 2.2, we write

$$\frac{1}{X_n(F\omega)^{\alpha_{\min}}} = \frac{1}{X_{n+1}(\omega)^{\alpha_{\min}}} - \alpha_{\min} 2^{\alpha_{\min}}(2X_{n+1}(\omega))^{\alpha(\omega)-\alpha_{\min}} + O(X_{n+1}(\omega)^{2\alpha(\omega)-\alpha_{\min}}). \tag{29}$$

Proposition 2.2 gives

$$X_{n+1}(\omega)^{2\alpha(\omega)-\alpha_{\min}} \leqslant X_{n+1}(\omega)^{\alpha_{\min}} \leqslant \frac{C}{(n+1)^{\alpha_{\min}/\alpha_{\max}}} \leqslant \frac{C}{\sqrt{n+1}} \tag{30}$$

as $\alpha_{\min}/\alpha_{\max} \geqslant \frac{1}{2}$ by hypothesis. Thus,

$$\frac{1}{X_{n+1}(\omega)^{\alpha_{\min}}} - \frac{1}{X_n(F\omega)^{\alpha_{\min}}} = 2^{\alpha_{\min}}\alpha_{\min}(2X_{n+1}(\omega))^{\alpha(\omega)-\alpha_{\min}} + O(1/\sqrt{n}).$$

Summing from 1 to n, we get a constant P (independent of ω) such that

$$\frac{1}{X_n(\omega)^{\alpha_{\min}}} \geqslant 2^{\alpha_{\min}}\alpha_{\min}\left[\sum_{k=1}^{n}(2X_k(F^{n-k}\omega))^{\alpha(F^{n-k}\omega)-\alpha_{\min}} - P\sqrt{n}\right] \tag{31}$$

$$\frac{1}{X_n(\omega)^{\alpha_{\min}}} \leqslant 2^{\alpha_{\min}}\alpha_{\min}\left[\sum_{k=1}^{n}(2X_k(F^{n-k}\omega))^{\alpha(F^{n-k}\omega)-\alpha_{\min}} + P\sqrt{n}\right]. \tag{32}$$

Equation (31) and Proposition 2.2 imply that

$$\frac{\sqrt{\ln n}}{n}\frac{1}{2^{\alpha_{\min}}\alpha_{\min}X_n(\omega)^{\alpha_{\min}}} \geqslant \frac{\sqrt{\ln n}}{n}\sum_{k=1}^{n}\left(\frac{2C^{-1}}{k^{1/\alpha_{\min}}}\right)^{\alpha(F^{n-k}\omega)-\alpha_{\min}} - P\sqrt{\frac{\ln n}{n}} =: A_n(\omega). \tag{33}$$

We first study the convergence of A_n. The functions α and $\alpha \circ F^{n-k}$ have the same distribution since F preserves Lebesgue measure. Thus, by Lemma 4.2,

$$E\left(\left(\frac{2C^{-1}}{k^{1/\alpha_{\min}}}\right)^{\alpha\circ F^{n-k}-\alpha_{\min}}\right) \sim \sqrt{\frac{\pi}{2\alpha''(x_0)}}\frac{1}{\sqrt{\ln(k^{1/\alpha_{\min}})-\ln(2C^{-1})}} \sim \sqrt{\frac{\pi\,\alpha_{\min}}{2\alpha''(x_0)}}\frac{1}{\sqrt{\ln k}}.$$

Summing, we get that

$$E(A_n) \to C_1 := \sqrt{\frac{\pi\,\alpha_{\min}}{2\alpha''(x_0)}}, \tag{34}$$

since $\sum_{k=2}^{n}(1/\sqrt{\ln k}) \sim n/\sqrt{\ln n}$.

We will need L^p estimates, for $p \geqslant 1$. To get them, we use a result of Pène [**Pèn02**], recalled in Appendix B. Let us denote by $\|g\|$ the Lipschitz norm of a function $g : S^1 \to \mathbb{R}$, i.e. $\|g\| = \sup_{x\in S^1}|g(x)| + \sup_{x\neq y}|g(x)-g(y)|/|x-y|$.

We define $f_k(\omega) = (2C^{-1}/k^{1/\alpha_{\min}})^{\alpha(\omega)-\alpha_{\min}}$, and $g_k = f_k - E(f_k)$. Thus, $A_n = (\sqrt{\ln n}/n)\sum_{k=1}^{n} f_k \circ F^{n-k} - P\sqrt{\ln n/n}$. As $g_k' = \ln(2C^{-1}/k^{1/\alpha_{\min}})\alpha' f_k$, there exists a constant L such that, for $k \leqslant n$, $\|g_k\| \leqslant L\ln n$. As a consequence, Theorem B.1 applied to $g_k/(L\ln n)$ gives

$$\|A_n - E(A_n)\|_p = \frac{\sqrt{\ln n}}{n}L\ln n\left\|\sum_{k=1}^{n} g_k\circ F^{n-k}/(L\ln n)\right\|_p \leqslant \frac{\sqrt{\ln n}}{n}L\ln n K_p\sqrt{n},$$

i.e.

$$\|A_n - E(A_n)\|_p \leqslant L_p\sqrt{\frac{\ln^3 n}{n}}. \tag{35}$$

This implies, in particular, that A_n converges almost everywhere to C_1. Namely, if $\delta > 0$,

$$\mathrm{Leb}\{|A_n - E(A_n)| > \delta\} \leqslant \int \frac{|A_n - E(A_n)|^4}{\delta^4} \leqslant \frac{L_4^4}{\delta^4}\left(\frac{\ln^3 n}{n}\right)^{4/2} \tag{36}$$

which is summable, and $E(A_n) \to C_1$.

We have

$$\begin{aligned} A_n(\omega) &\geqslant \frac{\sqrt{\ln n}}{n}\left[\sum_{k=1}^{n}\left(\frac{2C^{-1}}{k^{1/\alpha_{\min}}}\right)^{\alpha_{\max}-\alpha_{\min}} - P\sqrt{n}\right] \geqslant \frac{\sqrt{\ln n}}{n}[Kn^{2-\alpha_{\max}/\alpha_{\min}} - P\sqrt{n}] \\ &\geqslant K'\frac{\sqrt{\ln n}}{n}n^{2-\alpha_{\max}/\alpha_{\min}} \end{aligned}$$

since $\alpha_{\max}/\alpha_{\min} < \frac{3}{2}$. Thus,

$$\left\|\frac{1}{A_n}\right\|_\infty \leqslant K''\frac{n^{\alpha_{\max}/\alpha_{\min}-1}}{\sqrt{\ln n}}. \tag{37}$$

Note that $E(A_n)$ tends to $C_1 \neq 0$, whence $1/E(A_n)$ is bounded. Thus,

$$\begin{aligned} \left\|\frac{1}{A_n} - \frac{1}{E(A_n)}\right\|_p &\leqslant \left\|\frac{1}{A_n}\right\|_\infty \frac{1}{E(A_n)}\|A_n - E(A_n)\|_p \leqslant K'''\frac{n^{\alpha_{\max}/\alpha_{\min}-1}}{\sqrt{\ln n}}L_p\sqrt{\frac{\ln^3 n}{n}} \\ &= M_p\frac{\ln n}{n^\kappa} \end{aligned}$$

where $\kappa = \frac{3}{2} - \alpha_{\max} / \alpha_{\min} > 0$. In particular, $1/A_n$ tends to $1/C_1$ in every L^p. Equation (33) shows that

$$\left(\frac{n}{\sqrt{\ln n}}\right)^{1/\alpha_{\min}} X_n \leqslant \frac{1}{(2^{\alpha_{\min}}\alpha_{\min} A_n)^{1/\alpha_{\min}}}. \tag{38}$$

The right-hand side tends to

$$C_2 := \frac{1}{(2^{\alpha_{\min}}\alpha_{\min}{}^{3/2}\sqrt{\pi/2\alpha''(x_0)})^{1/\alpha_{\min}}} \tag{39}$$

in every L^p and, in particular, in L^1. Thus,

$$\overline{\lim}\, E\left(\left(\frac{n}{\sqrt{\ln n}}\right)^{1/\alpha_{\min}} X_n\right) \leqslant C_2. \tag{40}$$

Moreover, A_n converges almost everywhere to C_1, whence (38) yields that, almost everywhere,

$$\overline{\lim}\left(\frac{n}{\sqrt{\ln n}}\right)^{1/\alpha_{\min}} X_n(\omega) \leqslant C_2. \tag{41}$$

Set $Q = \sup_n(1/E(A_n)) + 1$, we estimate $\mathrm{Leb}\{1/A_n \geqslant Q\}$. If $p \geqslant 1$,

$$\mathrm{Leb}\left\{\frac{1}{A_n} \geqslant Q\right\} \leqslant \mathrm{Leb}\left\{\left|\frac{1}{A_n} - \frac{1}{E(A_n)}\right| \geqslant 1\right\} \leqslant E\left(\left|\frac{1}{A_n} - \frac{1}{E(A_n)}\right|^p\right) \leqslant \left(M_p \frac{\ln n}{n^\kappa}\right)^p.$$

In particular, choosing p large enough gives

$$\mathrm{Leb}\left\{\frac{1}{A_n} \geqslant Q\right\} \leqslant \frac{M}{n^5}. \tag{42}$$

Setting $Q' = Q/2^{\alpha_{\min}}\alpha_{\min}$, (38) thus yields that

$$\mathrm{Leb}\left\{X_n \geqslant \left(\frac{Q'\sqrt{\ln n}}{n}\right)^{1/\alpha_{\min}}\right\} \leqslant \frac{M}{n^5}. \tag{43}$$

Consequently,

$$U_n := \left\{\omega \,\middle|\, \exists \sqrt{n} \leqslant k \leqslant n \text{ with } X_k(F^{n-k}\omega) \geqslant \left(\frac{Q'\sqrt{\ln k}}{k}\right)^{1/\alpha_{\min}}\right\}$$

has a measure at most $\sum_{\sqrt{n}}^{n}(M/k^5) \leqslant M'/n^2$ (since Leb is invariant under F^{n-k}). Finally, Borel–Cantelli ensures that there is a full measure subset of S^1 on which $\omega \notin U_n$ for large enough n.

Set

$$A'_n(\omega) = \frac{\sqrt{\ln n}}{n}\left[\sum_{k=1}^{n}\left(\frac{2(Q'\sqrt{\ln k})^{1/\alpha_{\min}}}{k^{1/\alpha_{\min}}}\right)^{\alpha(F^{n-k}\omega)-\alpha_{\min}} + (P+1)\sqrt{n}\right]. \tag{44}$$

As for A_n, we show that $A'_n \to C_1$ in every L^p and almost everywhere.

Let ω be such that $\omega \notin U_n$ for large enough n, and $A'_n(\omega) \to C_1$ (these properties are true almost everywhere). Then, for large enough n, equation (32) and the fact that $X_k(F^{n-k}\omega) \leqslant (Q'\sqrt{\ln k}/k)^{1/\alpha_{\min}}$ for $\sqrt{n} \leqslant k \leqslant n$, yield that

$$\frac{1}{2^{\alpha_{\min}}\alpha_{\min} X_n(\omega)^{\alpha_{\min}}} \leqslant \left[\sum_{k=1}^{\sqrt{n}} 1 + \sum_{k=\sqrt{n}}^{n}\left(\frac{2(Q'\sqrt{\ln k})^{1/\alpha_{\min}}}{k^{1/\alpha_{\min}}}\right)^{\alpha(F^{n-k}\omega)-\alpha_{\min}} + P\sqrt{n}\right]$$
$$\leqslant \frac{n}{\sqrt{\ln n}} A'_n(\omega) \sim \frac{n}{\sqrt{\ln n}} C_1.$$

Therefore,

$$\underline{\lim}\left(\frac{n}{\sqrt{\ln n}}\right)^{1/\alpha_{\min}} X_n(\omega) \geqslant C_2. \tag{45}$$

Equations (41) and (45) prove that $(n/\sqrt{\ln n})^{1/\alpha_{\min}} X_n$ tends almost everywhere to C_2. We get the convergence in L^1 from the inequality (40) and the following elementary lemma. □

LEMMA 4.3. *Let f_n be non-negative functions on a probability space, with $f_n \to f$ almost everywhere, and $\overline{\lim}\, E(f_n) \leqslant E(f) < \infty$. Then $f_n \to f$ in L^1.*

Proof. Write $g_n = f_n + f - |f - f_n| \geqslant 0$. Fatou's lemma gives $E(\underline{\lim}\, g_n) \leqslant \underline{\lim}\, E(g_n)$. Therefore,

$$2E(f) \leqslant \overline{\lim}\, E(f_n) + E(f) - \overline{\lim}\, E(|f - f_n|).$$

Consequently, the hypotheses imply that $\overline{\lim}\, E(|f - f_n|) \leqslant 0$. □

5. *Limit theorems*

Set

$$A = \frac{1}{4({\alpha_{\min}}^{3/2}\sqrt{\pi/2\alpha''(x_0)})^{1/\alpha_{\min}}} \int_{S^1\times\{1/2\}} h\, d\mathrm{Leb}, \tag{46}$$

where h is the density of m with respect to Leb.

In this section, we prove the following theorem.

THEOREM 5.1. *Let f be a Hölder function on $S^1 \times [0, 1]$, with $\int f\, dm = 0$. Write $c = \int_{S^1\times\{0\}} f\, d\mathrm{Leb}$. Then:*

- *if $\alpha_{\min} < \frac{1}{2}$, there exists $\sigma^2 \geqslant 0$ such that $(1/\sqrt{n})S_n f \to \mathcal{N}(0, \sigma^2)$;*
- *if $\frac{1}{2} \leqslant \alpha_{\min} < 1$ and $c = 0$, assume also that there exists $\gamma > (\alpha_{\max}/\alpha_{\min})(\alpha_{\min} - \frac{1}{2})$ such that $|f(\omega, x) - f(\omega, 0)| \leqslant Cx^\gamma$; then there exists $\sigma^2 \geqslant 0$ such that $(1/\sqrt{n})S_n f \to \mathcal{N}(0, \sigma^2)$;*
- *if $\alpha_{\min} = \frac{1}{2}$ and $c \neq 0$, then $S_n f/\sqrt{(c^2 A/4)n(\ln n)^2} \to \mathcal{N}(0, 1)$;*
- *if $\frac{1}{2} < \alpha_{\min} < 1$ and $c \neq 0$, then $S_n f/n^{\alpha_{\min}}\sqrt{\alpha_{\min}\ln n} \to Z$, where the random variable Z has a characteristic function given by*

$$E(e^{itZ}) = e^{-A|c|^{1/\alpha_{\min}}\Gamma(1-1/\alpha_{\min})\cos(\pi/2\alpha_{\min})|t|^{1/\alpha_{\min}}(1-i\,\mathrm{sgn}(ct)\tan(\pi/2\alpha_{\min}))}. \tag{47}$$

The random variable Z in the last case has a so-called stable distribution of exponent $1/\alpha_{\min}$ and parameters $A|c|^{1/\alpha_{\min}}\Gamma(1 - 1/\alpha_{\min})\cos(\pi/2\alpha_{\min})$ and $\mathrm{sgn}(c)$ (see, e.g., [**Fel66**, Ch. XVII] for general background on stable laws).

To prove this theorem, we will use Theorem 3.1. For this, we need a control of $m(\varphi_Y > n)$ which comes from the asymptotic behaviour of X_n proved in Theorem 4.1. It will also be necessary to estimate $m(f_Y > x)$, through the study of the integrability of f_Y (Lemmas 5.3 and 5.4).

In the rest of this section, f will be a Hölder function on $S^1 \times [0, 1]$, fixed once and for all. Recall that $f_Y(y) = \sum_{k=0}^{\varphi_Y(y)-1} f(T^k y)$, where φ_Y is the first return time to $Y = S^1 \times (\frac{1}{2}, 1]$.

5.1. *Estimates on measures.*

LEMMA 5.2. *We have*

$$m(\varphi_Y > n) \sim \left(\frac{\sqrt{\ln n}}{n}\right)^{1/\alpha_{\min}} A \tag{48}$$

where A is given by (46).

Proof. We have

$$\begin{aligned} m(\varphi_Y > n) &\\ &= \int_{S^1} \int_{1/2}^{Y_{n+1}(\omega)} h(\omega, u)\, du\, d\omega = \int_{S^1} \int_0^{X_n(F\omega)/2} h\left(\omega, \frac{1}{2} + u\right) du\, d\omega \\ &= \int_{S^1} \frac{X_n(F\omega)}{2} h\left(\omega, \frac{1}{2}\right) d\omega + \int_{S^1} \int_0^{X_n(F\omega)/2} \left[h\left(\omega, \frac{1}{2} + u\right) - h\left(\omega, \frac{1}{2}\right)\right] du\, d\omega \\ &= I + II. \end{aligned}$$

As

$$\left(\frac{n}{\sqrt{\ln n}}\right)^{1/\alpha_{\min}} X_n(F\omega) \to \frac{1}{(2^{\alpha_{\min}} {\alpha_{\min}}^{3/2} \sqrt{\pi/2\alpha''(x_0)})^{1/\alpha_{\min}}}$$

in L^1 and almost everywhere (Theorem 4.1) and $h(\omega, \frac{1}{2})$ is bounded, we get that $I \sim (\sqrt{\ln n}/n)^{1/\alpha_{\min}} A$. Moreover, for large enough n, $|h(\omega, \frac{1}{2} + u) - h(\omega, \frac{1}{2})| \leqslant \varepsilon$, whence $II = o(\sqrt{\ln n}/n)^{1/\alpha_{\min}}$. □

LEMMA 5.3. *If* $\alpha_{\min} < \frac{1}{2}$, *then* $f_Y \in L^2(Y, dm)$.

Proof. We have

$$\begin{aligned} \int f_Y^2\, dm &\leqslant C \sum m(\varphi_Y = n) n^2 = C \sum (m(\varphi_Y > n-1) - m(\varphi_Y > n)) n^2 \\ &\leqslant C \sum m(\varphi_Y > n) n \end{aligned}$$

which is summable since $m(\varphi_Y > n) \sim A(\sqrt{\ln n}/n)^{1/\alpha_{\min}}$ with $1/\alpha_{\min} > 2$. □

LEMMA 5.4. *Assume that* $\int_{S^1 \times \{0\}} f = 0$. *Let* $\gamma \in (0, \alpha_{\max})$ *be such that* $|f(\omega, x) - f(\omega, 0)| \leqslant C x^\gamma$. *If* $1 < p < \min(2/\alpha_{\min}, 1/\alpha_{\min}(1 - \gamma/\alpha_{\max}))$, *then* $f_Y \in L^p(Y, dm)$.

Proof. As h is bounded on Y, it is sufficient to prove that $f_Y \in L^p(Y, d\mathrm{Leb})$.

Assume first that $f \equiv 0$ on $S^1 \times \{0\}$. Then, if $\mathbf{x} = (\omega, x)$ satisfies $\varphi_Y(\mathbf{x}) = n$, we have $f_Y(\mathbf{x}) = \sum_0^{n-1} f(T^k \mathbf{x})$. If $k \geqslant 1$, $T_\omega^k(x) \leqslant X_{n-k}(F^k\omega) \leqslant C/(n-k)^{1/\alpha_{\max}}$, whence $|f(T^k\mathbf{x})| \leqslant C/(n-k)^{\gamma/\alpha_{\max}}$, and a summation yields that $|f_Y(\mathbf{x})| \leqslant Cn^{1-\gamma/\alpha_{\max}}$. This bound tends to infinity when $n \to \infty$, but sufficiently slowly so that f_Y still belongs to L^p. More precisely,

$$\begin{aligned}\int |f_Y|^p &\leqslant C \sum m(\varphi_Y = n) n^{p(1-\gamma/\alpha_{\max})} \\ &\leqslant C \sum m(\varphi_Y > n) n^{p(1-\gamma/\alpha_{\max})-1}.\end{aligned}$$

As $m(\varphi_Y > n) \sim A(\sqrt{\ln n}/n)^{1/\alpha_{\min}}$, this last series is summable as soon as

$$-\frac{1}{\alpha_{\min}} + p\left(1 - \frac{\gamma}{\alpha_{\max}}\right) - 1 < -1, \tag{49}$$

which is the case by assumption on p.

Assume now that f has a vanishing integral on S^1. Let $g(\omega, x) = f(\omega, 0)$. The function $f - g$ vanishes on $S^1 \times \{0\}$, whence $f_Y - g_Y \in L^p$ according to the first part of this proof. Consequently, it is sufficient to prove that $g_Y \in L^p$. Write $\chi(\omega) = f(\omega, 0)$ and $S_n\chi(\omega) = \sum_{k=0}^{n-1} \chi(F^k\omega)$: then $g_Y(\omega, x) = S_{\varphi_Y(\omega,x)}\chi(\omega)$.

Let $M_n\chi(\omega) = \max_{k\leqslant n} |S_k\chi(\omega)|$. Let $\delta > 0$, and $l = (1+\delta)/\delta$, so that $1/l + 1/(1+\delta) = 1$. We have

$$\begin{aligned}\int_{\{\varphi_Y \geqslant 2\}} |g_Y|^p &= \sum_{n=2}^{\infty} \int_{S^1} \int_{1/2+X_n(F\omega)/2}^{1/2+X_{n-1}(F\omega)/2} |S_n\chi(\omega)|^p \, du \, d\omega \\ &\leqslant \sum_{k=1}^{\infty} \int_{S^1} \int_{1/2+X_{2^k}(F\omega)/2}^{1/2+X_{2^{k-1}}(F\omega)/2} |M_{2^k}\chi(\omega)|^p \, du \, d\omega \\ &\leqslant \sum_{k=1}^{\infty} \int_{S^1} X_{2^{k-1}}(F\omega)|M_{2^k}\chi(\omega)|^p \, d\omega \leqslant \sum_{k=1}^{\infty} \|X_{2^{k-1}} \circ F\|_{1+\delta} \|M_{2^k}\chi\|_{lp}^p,\end{aligned}$$

where the last inequality is Hölder inequality. If δ is small enough, $lp > 2$, whence Corollary B.4 yields that $\|M_{2^k}\chi\|_{lp} \leqslant Ck^{(lp-1)/lp}\sqrt{2^k}$. Moreover,

$$\|X_{2^{k-1}} \circ F\|_{1+\delta} = \|X_{2^{k-1}}\|_{1+\delta} \leqslant \left(\int X_{2^{k-1}}\right)^{1/(1+\delta)} \sim C\left(\frac{\sqrt{\ln(2^{k-1})}}{2^{k-1}}\right)^{1/(1+\delta)\,\alpha_{\min}}$$

by Theorem 4.1. Thus, $\int |g_Y|^p < \infty$ if $1/(1+\delta)\,\alpha_{\min} > p/2$, and it is possible to choose δ such that this inequality is true, since $1/\alpha_{\min} > p/2$ by hypothesis. □

5.2. *Proof of Theorem 5.1.* To apply Theorem 3.1, we first check the condition (18). Let θ be the Hölder exponent of f. We will work with the distance $d_{\lambda^{-\theta}}(x, y) = \lambda^{-\theta s(x,y)}$. For this distance, T_Y is a Gibbs–Markov map.

FACT. *If f is θ-Hölder on $S^1 \times [0, 1]$, then*

$$\sum m[A_{s,n}] Df_Y(A_{s,n}) < \infty. \tag{50}$$

Recall that $Df_Y(A_{s,n})$ (defined in Theorem 3.1) is the best Lipschitz constant of f_Y on $A_{s,n}$, here for the distance $d_{\lambda^{-\theta}}$.

Proof of the fact. Take (ω_1, x_1) and $(\omega_2, x_2) \in A_{s,n}$ with, for example, $x_2 \geqslant x_1$. Then (9) implies that $x_1 \in J_n^+(\omega_2)$, and Corollary 2.5 applied $n-1$ times proves that, for $0 \leqslant k \leqslant n$, $d(T^k(\omega_1, x_1), T^k(\omega_2, x_1)) \leqslant D|F^k\omega_1 - F^k\omega_2|$. Moreover, $d(T^k(\omega_2, x_1), T^k(\omega_2, x_2)) \leqslant d(T^n(\omega_2, x_1), T^n(\omega_2, x_2))$ (since each map $T_{\alpha(\omega)}$ is expanding).

Thus, for $0 \leqslant k \leqslant n$,

$$\begin{aligned} d(T^k(\omega_1, x_1), T^k(\omega_2, x_2)) &\leqslant d(T^k(\omega_1, x_1), T^k(\omega_2, x_1)) + d(T^k(\omega_2, x_1), T^k(\omega_2, x_2)) \\ &\leqslant D|F^k\omega_1 - F^k\omega_2| + d(T^n(\omega_2, x_1), T^n(\omega_2, x_2)) \\ &\leqslant D|F^n\omega_1 - F^n\omega_2| + d(T^n(\omega_1, x_1), T^n(\omega_2, x_1)) \\ &\quad + d(T^n(\omega_1, x_1), T^n(\omega_2, x_2)) \\ &\leqslant D|F^n\omega_1 - F^n\omega_2| + D|F^n\omega_1 - F^n\omega_2| \\ &\quad + d(T^n(\omega_1, x_1), T^n(\omega_2, x_2)) \\ &\leqslant (1+2D) d(T^n(\omega_1, x_1), T^n(\omega_2, x_2)). \end{aligned}$$

We deduce that

$$\begin{aligned} |f_Y(\omega_1, x_1) - f_Y(\omega_2, x_2)| &\leqslant \sum_{k=0}^{n-1} |f(T^k(\omega_1, x_1)) - f(T^k(\omega_2, x_2))| \\ &\leqslant \sum_{k=0}^{n-1} Cd(T^k(\omega_1, x_1), T^k(\omega_2, x_2))^\theta \\ &\leqslant C'nd(T^n(\omega_1, x_1), T^n(\omega_2, x_2))^\theta. \end{aligned}$$

As T_Y is expanding for the distance d' (defined in (7), and equivalent to d), we have

$$\begin{aligned} d(T^n(\omega_1, x_1), T^n(\omega_2, x_2)) &\leqslant Cd_{\lambda^{-1}}(T^n(\omega_1, x_1), T^n(\omega_2, x_2)) \\ &= C\lambda d_{\lambda^{-1}}((\omega_1, x_1), (\omega_2, x_2)), \end{aligned}$$

whence $d(T^n(\omega_1, x_1), T^n(\omega_2, x_2))^\theta \leqslant Cd_{\lambda^{-\theta}}((\omega_1, x_1), (\omega_2, x_2))$.

Thus, $Df_Y(A_{s,n}) \leqslant Cn$, and

$$\sum m(A_{s,n}) Df_Y(A_{s,n}) \leqslant C \sum m(\varphi_Y = n)n = C < +\infty, \tag{51}$$

by Kac's formula. □

Proof of Theorem 5.1. In the case $\alpha_{\min} < \frac{1}{2}$, Lemma 5.3 gives that $f_Y \in L^2$. Moreover, $\varphi \in L^2$ (since $\varphi = g_Y$ for $g \equiv 1$, whence Lemma 5.3 also applies). We have already checked the condition (18), so we can apply (the first case of) Theorem 3.1. This yields the central limit theorem for f.

Assume now that $\frac{1}{2} \leqslant \alpha_{\min} < 1$ and that $c = 0$. Under the assumptions of the theorem, we can apply Lemma 5.4 with $p = 2$, and get that $f_Y \in L^2$. Moreover, Lemma 5.2 shows that $m[\varphi_Y > x] \sim (\sqrt{\ln x}/x)^{1/\alpha_{\min}} A$. We have checked all of the hypotheses of the first case of Theorem 3.1. Applying this theorem, we conclude the proof of the second case.

The third and fourth cases are analogous. Let us prove, for example, the fourth case, i.e. $\frac{1}{2} < \alpha_{\min} < 1$ and $c \neq 0$. Assume, for example, that $c > 0$. We estimate $m(f_Y > x)$.

FACT. *We have*

$$m(f_Y > x) \sim \left(\frac{c\sqrt{\ln x}}{x}\right)^{1/\alpha_{\min}} A \quad \text{and} \quad m(f_Y < -x) = o\left(\frac{\sqrt{\ln x}}{x}\right)^{1/\alpha_{\min}}.$$

Proof. We prove the estimate on $m(f_Y > x)$, the other being similar.

Let $g \equiv c$ on $S^1 \times [0,1]$. Then $g_Y = nc$ on $[\varphi_Y = n]$, which implies that $m(g_Y > nc) = m(\varphi_Y > n) \sim (\sqrt{\ln n}/n)^{1/\alpha_{\min}} A$ by Lemma 5.2.

In the general case, consider $j = f - g$, and let us prove that $m(|j_Y| > x) = o(\sqrt{\ln x}/x)^{1/\alpha_{\min}}$. As $f_Y = g_Y + j_Y$, it will give

$$m(g_Y > x(1+\varepsilon)) - m(|j_Y| > x\varepsilon) \leqslant m(f_Y > x) \leqslant m(g_Y > x(1-\varepsilon)) + m(|j_Y| > x\varepsilon),$$

which gives the conclusion.

Let $\gamma > 0$ with $\gamma < \min(\theta, \alpha_{\max})$ (where θ is the Hölder coefficient of f). Lemma 5.4 gives that $j_Y \in L^p$ if $p < \min(2/\alpha_{\min}, 1/\alpha_{\min}(1-\gamma/\alpha_{\max}))$. We can, in particular, choose $p > 1/\alpha_{\min}$. Then $m(|j_Y| > x) \leqslant \int (|j_Y|/x)^p = O(x^{-p})$, which concludes the proof of the fact. □

The same fact holds for φ_Y, with the same proof. Therefore, the assumptions of the third case of Theorem 3.1 are satisfied. This implies the desired result. □

A. *Appendix. Induced maps and limit theorems*

The aim of this section is to prove very general results stating that, if a function satisfies a limit theorem for an induced map, it also satisfies one for the initial map. Similar theorems have been proved in [**Gou04**], by spectral methods, under strong technical assumptions. We will describe here a more elementary method, essentially due to Melbourne and Török for flows [**MT04**]. Zweimüller has also used the same kind of arguments to study limit laws in dimension 1, see [**Zwe03**]. This new method does not imply all the results of [**Gou04**], but it can be used in settings where [**Gou04**] can not be applied.

If Y is a subset of a probability space (X, m), $T : X \to X$, and T_Y is the induced map on Y, we will write $S_n^Y g = \sum_{k=0}^{n-1} g \circ T_Y^k$: this is the Birkhoff sum of g, for the transformation T_Y. We will also write $E_Y(g) = \int_Y g/m[Y]$. Finally, for $t \in \mathbb{R}$, $\lfloor t \rfloor$ denotes the integer part of t.

THEOREM A.1. *Let $T : X \to X$ be an ergodic endomorphism of a probability space (X, m), and $f : X \to \mathbb{R}$ an integrable function with vanishing integral. Let $Y \subset X$ have positive measure. For $y \in Y$, write $\varphi(y) = \inf\{n > 0 \mid T^n(y) \in Y\}$ and $f_Y(y) = \sum_{k=0}^{\varphi(y)-1} f(T^k y)$.*

We assume the following properties.

(1) *There exists a sequence $B_n \to +\infty$, with $\inf_{r \geqslant n}(B_r/B_n) > 0$, such that (f_Y, φ) satisfies a mixing limit theorem for the normalization B_n: there exists a random variable Z such that, for every $t \in \mathbb{R}$,*

$$E_Y(\varphi e^{itS^Y_{\lfloor nm(Y)\rfloor} f_Y/B_n}) \to E_Y(\varphi) E(e^{itZ}). \tag{52}$$

(2) *There exists $b > 0$ such that, in the natural extension of T_Y, $(1/N^b)\sum_0^{N-1} f_Y(T_Y^k y)$ tends almost everywhere to 0 when $N \to \pm\infty$.*

(3) *The sequence $(S_n^Y \varphi - nE_Y(\varphi))/B_n^{1/b}$ is tight, in the following sense: for every $\varepsilon > 0$, there exists $A > 0$ and N_0 such that, for every $n \geqslant N_0$,*

$$m\left\{ y \in Y \,\middle|\, \left| \frac{S_n^Y \varphi - nE_Y(\varphi)}{B_n^{1/b}} \right| \geqslant A \right\} \leqslant \varepsilon. \tag{53}$$

Then the function f also satisfies a limit theorem:

$$E(e^{itS_n f/B_n}) \to E(e^{itZ}), \tag{54}$$

i.e. $S_n f/B_n$ tends in distribution to Z.

The hypotheses of the theorem are tailor-made so that the following proof works, but they are, in fact, often satisfied in natural cases. Let us comment on these three hypotheses.

(1) The convergence (52) is very often satisfied when f_Y satisfies a limit theorem. Namely, the martingale proofs or spectral proofs of limit theorems automatically give this kind of convergence†.

(2) The natural extension is useful so that we can let N tend to $-\infty$, and consider T_Y^{-1} in the proof. Generally, Birkhoff's theorem yields that this assumption is satisfied for $b = 1$. This is often sufficient. However, sometimes, it is important to have better estimates. It is then possible to use [**Kac96**, Theorem 16], for example: this theorem ensures that, if the correlations of $f_Y \in L^2$ decay at least like $O(1/n)$, then the hypothesis is satisfied for any $b > \frac{1}{2}$ (for $N \to -\infty$, use the fact that $\int f_Y \cdot f_Y \circ T_Y^n = \int f_Y \circ T_Y^{-n} \cdot f_Y$, and apply the result to T_Y^{-1}).

(3) The third assumption is weaker than

$$\exists B'_n = O(B_n^{1/b}) \quad \text{such that } \frac{S_n^Y \varphi - nE_Y(\varphi)}{B'_n} \text{ converges in distribution.} \tag{53$'$}$$

Moreover, φ is often simpler than f_Y. Since f_Y satisfies a limit theorem (this is more or less the first hypothesis), this is also often the case of φ, which implies (53′). Thus, (53′), and hence (53), are satisfied quite generally.

Proof of Theorem A.1. We can assume that $m(Y) < 1$.

Without loss of generality, we can replace T by its natural extension and assume that T is invertible. We will identify X with $\{(y, i) \mid y \in Y, i \in \{0, \dots, \varphi(y) - 1\}\}$. In this notation, for $i < \varphi(y) - 1$, $T(y, i) = (y, i + 1)$, while $T(y, \varphi(y) - 1) = (T_Y(y), 0)$. Note that $E_Y(\varphi) = 1/m(Y)$ by Kac's formula. Let π be the projection from X to Y, given by $\pi(y, i) = y$.

In this proof, we will write $S_t f(x)$, even when t is not an integer, for $S_{\lfloor t \rfloor} f(x)$. In the same way, T^t should be understood as $T^{\lfloor t \rfloor}$. We also extend B_n to $\mathbb{R}_+$, setting $B_t := B_{\lfloor t \rfloor}$.

As T is ergodic, T_Y is also ergodic [**Aar97**, Proposition 1.5.2]. Birkhoff's theorem gives that

$$S_n^Y \varphi = \frac{n}{m(Y)} + o(n) \tag{55}$$

almost everywhere on Y. For $y \in Y$ and $N \in \mathbb{N}$, let $n(y, N)$ be the greatest integer n such that $S_n^Y \varphi(y) \leqslant N$. If y is such that $S_n^Y \varphi(y) = n/m(Y) + o(n)$ (which is true almost everywhere), then $n(y, N)$ is finite for every N, and $n(y, N)/m(Y) \sim N$, i.e.

$$\frac{n(y, N)}{Nm(Y)} \to 1. \tag{56}$$

† In fact, Zweimüller pointed out that the distributional convergence of $S_{\lfloor nm(Y) \rfloor}^Y f_Y / B_n$ to Z always implies (52): this is a consequence of the proof of [**Aar97**, Proposition 3.6.1].

Since $\int_X e^{it(S_N^Y f_Y)\circ\pi} = \int_Y \varphi e^{itS_N^Y f_Y}$, (52) yields that

$$\frac{(S^Y_{Nm(Y)} f_Y)\circ\pi}{B_N} \to Z \tag{57}$$

in distribution on X. The idea of the proof will be to see that $(S^Y_{Nm(Y)} f_Y)\circ\pi$ and $S_N f$ are close (this is not surprising, since one iteration of T_Y corresponds roughly to $1/m(Y)$ iterations of T). This will give that $S_N f/B_N$ tends to Z.

We write

$$S_N f(y,i) = (S_N f(y,i) - S_N f(y,0)) + (S_N f(y,0) - S^Y_{n(y,N)} f_Y(y)) \\ + (S^Y_{n(y,N)} f_Y(y) - S^Y_{Nm(Y)} f_Y(y)) + S^Y_{Nm(Y)} f_Y(y).$$

The last term, equal to $(S^Y_{Nm(Y)} f_Y)\circ\pi$, satisfies a limit theorem by (57). To conclude the proof, we will see that the three other terms, divided by B_N, tend to 0 in probability.

The second and third terms depend only on y. Thus, the following lemma will be useful to prove that they tend to 0 on X.

LEMMA A.2. *Let f_n be a sequence of functions on Y, tending to* 0 *in probability on Y. Then $f_n\circ\pi$ tends to* 0 *in probability on X.*

Proof. Take $\varepsilon > 0$. As $f_n \to 0$ in probability, the measure of $E_n := \{y \in Y \mid |f_n(y)| \geqslant \varepsilon\}$ tends to 0. As $\varphi \in L^1$, dominated convergence yields that $\int_{E_n} \varphi \to 0$, i.e. the measure of $\pi^{-1}(E_n)$ tends to 0. However, $\pi^{-1}(E_n)$ is exactly the set where $|f_n\circ\pi| \geqslant \varepsilon$. □

FACT. $B_N^{-1}(S_N f(y,i) - S_N f(y,0))$ *tends to* 0 *in probability on X.*

Proof. Set $V_N(y) = \sum_{i=0}^{\varphi(y)-1} |f\circ T^N(y,i)|$ on Y. Then $\|V_N\|_{L^1(Y)} = \|f\circ T^N\|_{L^1(X)} = \|f\|_{L^1(X)}$ since T preserves the measure. Thus, V_N/B_N tends to 0 in $L^1(Y)$, and in probability. Lemma A.2 yields that $B_N^{-1} V_N\circ\pi$ tends to 0 in probability on X.

As $S_N f(y,i) - S_N f(y,0) = \sum_N^{N+i-1} f(T^k(y,0)) - \sum_0^{i-1} f(T^k(y,0))$, we get $|S_N f(y,i) - S_N f(y,0)| \leqslant V_N(y) + V_0(y)$. Thus, $B_N^{-1}(S_N f(y,i) - S_N f(y,0))$ is bounded by a function going to 0 in probability. □

FACT. $B_N^{-1}(S_N f(y,0) - S^Y_{n(y,N)} f_Y(y))$ *tends to* 0 *in probability on X.*

Proof. By Lemma A.2, it is sufficient to prove it on Y. Set $H(y,i) = \left|\sum_{j=0}^{i-1} f(y,j)\right|$. Then

$$|S_N f(y,0) - S^Y_{n(y,N)} f_Y(y)| = H\circ T^N(y,0). \tag{58}$$

Since T preserves the measure m, for any $a > 0$,

$$m\left\{y \in Y \,\middle|\, \frac{1}{B_N} H\circ T^N(y,0) \geqslant a\right\} \leqslant m\left\{x \in X \,\middle|\, \frac{1}{B_N} H\circ T^N(x) \geqslant a\right\} \\ = m\left\{x \in X \,\middle|\, \frac{1}{B_N} H(x) \geqslant a\right\}.$$

Since H is measurable and $B_N \to \infty$, this measure tends to 0 when $N \to \infty$. □

FACT. $B_N^{-1}(S^Y_{n(y,N)} f_Y - S^Y_{Nm(Y)} f_Y)$ *tends to* 0 *in probability on X when $N \to \infty$.*

Proof. By Lemma A.2, it is sufficient to prove it on Y.

For $n < 0$, write $S_n^Y f_Y = \sum_1^{|n|} f_Y \circ T_Y^{-j}$. Then, setting $\nu(y, N) = n(y, N) - Nm(Y)$,

$$S_{n(y,N)}^Y f_Y(y) - S_{Nm(Y)}^Y f_Y(y) = S_{\nu(y,N)}^Y f_Y(T_Y^{Nm(Y)}(y)). \tag{59}$$

Let $A > 0$ and $N \in \mathbb{N}$, we will estimate the measure of $\{y \mid \nu(y, N) \geqslant AB_N^{1/b}\}$. Take $\alpha > 0$ such that $m(Y) + \alpha < 1$. Assume first that $AB_N^{1/b} > \alpha N$. Then,

$$\{y \mid \nu(y, N) \geqslant AB_N^{1/b}\} \subset \{y \mid n(y, N) \geqslant (m(Y) + \alpha)N\}. \tag{60}$$

By (55), the measure of this set tends to 0 when $N \to \infty$. Assume next that $AB_N^{1/b} \leqslant \alpha N$. Since $E_Y(\varphi) = 1/m(Y)$, we get

$$\{y \mid \nu(y, N) \geqslant AB_N^{1/b}\} = \{n(y, N) \geqslant AB_N^{1/b} + Nm(Y)\} = \{S^Y_{AB_N^{1/b}+Nm(Y)}\varphi \leqslant N\}$$

$$= \left\{ \frac{S^Y_{AB_N^{1/b}+Nm(Y)}\varphi - (AB_N^{1/b} + Nm(Y))E_Y(\varphi)}{(B_{pAB_N^{1/b}+Nm(Y)})^{1/b}} \leqslant -\frac{A}{m(Y)} \left(\frac{B_N}{B_{AB_N^{1/b}+Nm(Y)}} \right)^{1/b} \right\}.$$

Moreover, $AB_N^{1/b} + Nm(Y) \leqslant (m(Y) + \alpha)N \leqslant N$. By assumption, there exists $c > 0$ such that, for all $n \leqslant r$, $B_r/B_n \geqslant c$. In particular, $B_N/B_{AB_N^{1/b}+Nm(Y)} \geqslant c$. Hence,

$$\{y \mid \nu(y, N) \geqslant AB_N^{1/b}\} \subset \left\{ \frac{S^Y_{AB_N^{1/b}+Nm(Y)}\varphi - (AB_N^{1/b} + Nm(Y))E_Y(\varphi)}{(B_{AB_N^{1/b}+Nm(Y)})^{1/b}} \leqslant -\frac{Ac^{1/b}}{m(Y)} \right\}.$$

Consequently, if A is large enough, assumption 3 yields that $m\{y \mid \nu(y, N) \geqslant AB_N^{1/b}\} \leqslant \varepsilon$ for large enough N. We handle in the same way the set of points where $\nu(y, N) \leqslant -AB_N^{1/b}$. We have thus proved

$$\forall \varepsilon > 0, \exists A > 0, \exists N_0 > 0, \forall N \geqslant N_0, \quad m\{y \mid |\nu(y, N)| \geqslant AB_N^{1/b}\} \leqslant \varepsilon. \tag{61}$$

Set $W_N(y) = B_N^{-1} S_{\nu(y,N)} f_Y(T_Y^{Nm(Y)}(y))$, we will show that it tends to 0 in distribution, which will conclude the proof, by (59). Take $a > 0$, we will show that $m(|W_N| > a) \to 0$ when $N \to \infty$.

Let $\varepsilon > 0$. Assumption 2 ensures that there exists $\widetilde{Y}$ with $m(\widetilde{Y}) \geqslant m(Y) - \varepsilon$ and N_1 such that $(1/|N|^b)|S_N^Y f_Y| \leqslant \varepsilon$ on $\widetilde{Y}$, for every $|N| \geqslant N_1$. Define $Y'_N = \{y \in Y \mid |\nu(y, N)| < N_1\}$ and $Y''_N = \{y \in Y \mid |\nu(y, N)| \geqslant N_1\}$. We estimate first the contribution of Y'_N.

Set $\psi(y) = \sum_{-N_1}^{N_1-1} |f_Y \circ T_Y^j(y)|$. If $y \in Y'_N$, then $|W_N(y)| \leqslant \psi(T_Y^{Nm(Y)}(y))/B_N$. Therefore,

$$\begin{aligned} m\{y \in Y'_N \mid |W_N(y)| \geqslant a\} &\leqslant m\{y \in Y'_N \mid |\psi(T_Y^{Nm(Y)} y)| \geqslant aB_N\} \\ &= m\{y \in Y'_N \mid |\psi(y)| \geqslant aB_N\}. \end{aligned}$$

Since ψ is measurable, this quantity tends to 0 when $N \to \infty$. In particular, if N is large enough, it is at most ε.

We then estimate the contribution of Y''_N. Set $\widetilde{Y}''_N = Y''_N \cap T_Y^{-Nm(Y)}(\widetilde{Y})$, it satisfies $m(\widetilde{Y}''_N) \geqslant m(Y''_N) - \varepsilon$. Thus,

$$m(|W_N| \geqslant a) \leqslant m\{y \in \widetilde{Y}''_N \mid |W_N(y)| \geqslant a\} + 2\varepsilon. \tag{62}$$

On $\widetilde{Y}''_N$, $|\nu(y, N)| \geqslant N_1$, whence $(1/|\nu(y, N)|^b)|S^Y_{\nu(y,N)} f_Y(T_Y^{Nm(Y)} y)| \leqslant \varepsilon$. Thus,

$$|W_N(y)| \leqslant \varepsilon \frac{|\nu(y, N)|^b}{B_N} = \varepsilon\left(\frac{|\nu(y, N)|}{B_N^{1/b}}\right)^b.$$

Consequently,

$$m(|W_N| \geqslant a) \leqslant m\left(\frac{|\nu(y, N)|}{B_N^{1/b}} \geqslant \left(\frac{a}{\varepsilon}\right)^{1/b}\right) + 2\varepsilon. \tag{63}$$

This equation together with (61) implies that $m(|W_N| \geqslant a) \to 0$ when $N \to \infty$. □

The three facts we have just proved imply that $S_N f(y, i)/B_N - S^Y_{Nm(Y)} f_Y(y)/B_N \to 0$ in distribution on X. As $S^Y_{Nm(Y)} f_Y(y)/B_N \to Z$ in distribution on X, by (57), this concludes the proof. □

B. *Appendix. Multiple decorrelations and L^p-boundedness*

The following theorem has been useful in this paper.

THEOREM B.1. *Let $F : \omega \to 4\omega$ on the circle S^1. Then, for every $p \in [1, \infty)$, there exists a constant K_p such that, for every $n \in \mathbb{N}$, for every $f_0, \dots, f_{n-1} : S^1 \to \mathbb{R}$ bounded by* 1*, of zero average and* 1*-Lipschitz,*

$$\left\| \sum_{k=0}^{n-1} f_k \circ F^k \right\|_p \leqslant K_p \sqrt{n}. \tag{64}$$

This result has essentially been proved by Pène in [**Pèn02**], in a much broader context. Her proof depends on a property of multiple decorrelations, which is implied by the spectral gap of the transfer operator.

LEMMA B.2. *Let $\|f\|$ be the Lipschitz norm of the function f on the circle S^1. Then, for every $m, m' \in \mathbb{N}$, there exist $C > 0$ and $\delta < 1$ such that, for every $N \in \mathbb{N}$, for every increasing sequences $(k_1, \dots, k_m)$ and $(l_1, \dots, l_{m'})$, for every Lipschitz functions $G_1, \dots, G_m, H_1, \dots, H_{m'}$,*

$$\left| \mathrm{Cov}\left(\prod_{i=1}^m G_i \circ F^{k_i}, \prod_{j=1}^{m'} H_j \circ F^{N+l_j} \right) \right| \leqslant C \left(\prod_{i=1}^m \|G_i\| \right) \left(\prod_{j=1}^{m'} \|H_j\| \right) \delta^{N-k_m}. \tag{65}$$

Here $\mathrm{Cov}(u, v) = \int uv - \int u \int v$.

Proof. Let $\widehat{F}$ be the transfer operator associated to F, and acting on Lipschitz functions. It is known that it admits a spectral gap and that its iterates are bounded, i.e. there exist constants $M > 0$ and $\delta < 1$ such that $\|\widehat{F}^n f\| \leqslant M\|f\|$, and $\|\widehat{F}^n f\| \leqslant M\delta^n \|f\|$ if $\int f = 0$.

We can assume that $N \geqslant k_m$ (otherwise $\delta^{N-k_m} \geqslant 1$, and the inequality (65) becomes trivial). Then, writing $\varphi = \prod_{i=1}^m G_i \circ F^{k_i}$ and $\psi = \prod_{j=1}^{m'} H_j \circ F^{l_j}$, we get

$$\begin{aligned} |\mathrm{Cov}(\varphi, \psi \circ F^N)| &= \left| \int \left(\varphi - \int \varphi\right) \psi \circ F^N \right| = \left| \int \widehat{F}^N \left(\varphi - \int \varphi\right) \psi \right| \\ &\leqslant \left\| \widehat{F}^N \left(\varphi - \int \varphi\right) \right\| \|\psi\|_\infty. \end{aligned}$$

However,

$$\begin{aligned}\widehat{F}^N(\varphi) &= \widehat{F}^N\left(\prod G_i \circ F^{k_i}\right)\\ &= \widehat{F}^{N-k_m}(G_m \widehat{F}^{k_m-k_{m-1}}(G_{m-1}\widehat{F}^{k_{m-1}-k_{m-2}}(\ldots \widehat{F}^{k_2-k_1}(G_1))\ldots)\\ &=: \widehat{F}^{N-k_m}(\chi).\end{aligned}$$

As the iterates of $\widehat{F}$ are bounded on Lipschitz functions, we get a bound on the Lipschitz norm of χ: $\|\chi\| \leqslant M^{m-1}\prod \|G_i\|$. Moreover, $\int \chi = \int \varphi$, whence

$$\begin{aligned}\left\|\widehat{F}^N\left(\varphi - \int \varphi\right)\right\| = \left\|\widehat{F}^{N-k_m}\left(\chi - \int \chi\right)\right\| &\leqslant M\delta^{N-k_m}\left\|\chi - \int \chi\right\|\\ &\leqslant M\delta^{N-k_m}M^{m-1}\prod \|G_i\|.\end{aligned}$$

□

When p is an even integer, Theorem B.1 is then a consequence of [**Pèn02**, Lemma 2.3.4]. The Hölder inequality gives the general case.

Remark. The same result holds for Hölder functions instead of Lipschitz functions, with the same proof.

We will also need the following result.

THEOREM B.3. *Let T be a measure preserving transformation on a space X. Let $f : X \to \mathbb{R}$ and $p > 2$ be such that*

$$\exists C > 0, \forall n \in \mathbb{N}^*, \quad \|S_n f\|_p \leqslant C\sqrt{n}. \tag{66}$$

Write $M_n f(x) = \sup_{1\leqslant k\leqslant n} |S_k f(x)|$. Then there exists a constant K such that

$$\forall n \geqslant 2, \quad \|M_n f\|_p \leqslant K(\ln n)^{(p-1)/p}\sqrt{n}. \tag{67}$$

Proof. Let $n \in \mathbb{N}^*$. Let $k < 2^n$, and write its binary decomposition $k = \sum_{j=0}^{n-1}\varepsilon_j 2^j$, with $\varepsilon_j \in \{0,1\}$. Set $q_j = \sum_{l=j}^{n-1}\varepsilon_l 2^l$ (in particular, $q_0 = k$ and $q_n = 0$). Then $S_k f = \sum_{j=0}^{n-1}(S_{q_j} f - S_{q_{j+1}} f)$. Consequently, the convexity inequality $(a_0+\cdots+a_{n-1})^p \leqslant n^{p-1}(a_0^p + \cdots + a_{n-1}^p)$ gives

$$|S_k f|^p \leqslant n^{p-1}\sum_{j=0}^{n-1}|S_{q_j} f - S_{q_{j+1}} f|^p. \tag{68}$$

Note that q_{j+1} is of the form $\lambda 2^{j+1}$ with $0 \leqslant \lambda \leqslant 2^{n-j-1}-1$, and q_j is equal to q_{j+1} or $q_{j+1}+2^j$. Thus,

$$|S_k f|^p \leqslant n^{p-1}\sum_{j=0}^{n-1}\left(\sum_{\lambda=0}^{2^{n-j-1}-1}|S_{\lambda 2^{j+1}+2^j} f - S_{\lambda 2^{j+1}} f|^p\right). \tag{69}$$

The right-hand term is independent of k, and gives a bound on $|M_{2^n-1} f|^p$. Moreover,

$$\int |S_{\lambda 2^{j+1}+2^j} f - S_{\lambda 2^{j+1}} f|^p = \int |S_{2^j} f|^p \leqslant C^p \sqrt{2^j}^p. \tag{70}$$

Therefore, we get

$$\int |M_{2^n-1}f|^p \leqslant n^{p-1}\sum_{j=0}^{n-1} 2^{n-j-1}C^p 2^{pj/2} \leqslant Kn^{p-1}2^n 2^{(p/2-1)n} = Kn^{p-1}\sqrt{2^n}^p.$$

For times of the form 2^n-1, this is a bound of the form $\|M_t\|_p \leqslant K(\ln t)^{(p-1)/p}\sqrt{t}$. To get the same estimate for an arbitrary time t, it is sufficient to choose n with $2^{n-1} \leqslant t < 2^n$, and to note that $M_t \leqslant M_{2^n-1}$. □

COROLLARY B.4. *Let $F : \omega \to 4\omega$ on the circle S^1, let $\chi : S^1 \to \mathbb{R}$ be a Hölder function with 0 average, and let $p > 2$. Write $M_n\chi(x) = \sup_{1\leqslant k\leqslant n} |S_k\chi(x)|$. Then there exists a constant K such that*

$$\|M_n\chi\|_p \leqslant K(\ln n)^{(p-1)/p}\sqrt{n}, \quad \forall n \geqslant 2. \tag{71}$$

Proof. Theorem B.1 (or rather the remark following it, for the Hölder case) shows that $\|S_n\chi\|_p \leqslant C\sqrt{n}$. Consequently, Theorem B.3 gives the conclusion. □

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