AN INTERVAL MAP WITH A SPECTRAL GAP ON LIPSCHITZ FUNCTIONS, BUT NOT ON BOUNDED VARIATION FUNCTIONS

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(Communicated by Carlangelo Liverani)

ABSTRACT. We construct a uniformly expanding map of the interval, preserving Lebesgue measure, such that the corresponding transfer operator admits a spectral gap on the space of Lipschitz functions, but does not act continuously on the space of bounded variation functions.

To study the statistical properties of uniformly expanding maps, a very successful strategy is to find a Banach space on which the transfer operator of the map admits a spectral gap (see e.g. [1] and references therein). In dimension $d \geq 2$, several Banach spaces have been considered, suited to different situations. However, in one dimension, it is commonly acknowledged that the space $\mathcal{BV}$ of bounded variation functions is the best space to work with (see nevertheless [4] for examples of suitable function spaces when the derivative is only Hölder continuous).

The purpose of this note is to exhibit a very specific example showing that, quite surprisingly, the behavior on the space Lip of Lipschitz functions can be better than the behavior on $\mathcal{BV}$: we will construct a uniformly expanding map T of the interval $I = [0, 1)$, preserving Lebesgue measure, such that the transfer operator $\mathcal{L}$ associated to T (defined by duality by $\int f \cdot g \circ T \, \mathrm{dLeb} = \int \mathcal{L} f \cdot g \, \mathrm{dLeb}$) admits a spectral gap on Lip, while it does not act continuously on $\mathcal{BV}$.

The main motivation for this short note is the articles [2] and [3], in which the authors prove results on empirical processes assuming that the transfer operator admits a spectral gap on Lip. Since similar results were already known when the transfer operator admits a spectral gap on $\mathcal{BV}$, our example shows that these articles really have new applications.

Our example is a Markov map of the interval, with (infinitely many) full branches. The difficulty in constructing such an example is that the usual mechanisms that imply continuity on Lip also imply continuity on $\mathcal{BV}$: if $\mathcal{L}_v$ denotes the part of the transfer operator corresponding to an inverse branch v of T (so that $\mathcal{L} = \sum_v \mathcal{L}_v$), one has $\|\mathcal{L}_v\|_{\mathcal{BV} \to \mathcal{BV}} \leq C \|\mathcal{L}_v\|_{\mathrm{Lip} \to \mathrm{Lip}}$, for some universal constant C. To take this fact into account, our example will therefore involve pairs of branches v and w such that $\mathcal{L}_v$ and $\mathcal{L}_w$ behave badly on Lip, while the sum $\mathcal{L}_v + \mathcal{L}_w$ behaves nicely, thanks to nontrivial compensations.

If f is a function defined on $[0,1)$, we will write $\|f\|_{\mathrm{Lip}} = \sup|f| + \mathrm{Lip}(f)$, where $\mathrm{Lip}(f) = \sup|f(x) - f(y)|/|x - y|$.

Let us fix a sequence $(a_n)_{n \in \mathbb{N}^*}$ of positive real numbers, with $\sum a_n < 1/4$, as well as an integer $N > 0$. Our construction will depend on them, and we will show that a good choice of these parameters gives the required properties.

Let $I_n = [4\sum_{k=1}^{n-1} a_k, 4\sum_{k=1}^{n} a_k)$. We decompose this interval (of length $4a_n$) into two subintervals of length $2a_n$ that we denote respectively by $I_n^{(1)}$ and $I_n^{(2)}$. We have

$$\int_0^1 a_n(1 + 2\cos^2(2\pi n^4 x))\, \mathrm{d}x = \int_0^1 a_n(1 + 2\sin^2(2\pi n^4 x))\, \mathrm{d}x = 2a_n.$$

We can therefore define two maps v_n and w_n on $[0,1)$, with respective images $I_n^{(1)}$ and $I_n^{(2)}$, such that $v_n'(x) = a_n(1 + 2\cos^2(2\pi n^4 x))$ and $w_n'(x) = a_n(1 + 2\sin^2(2\pi n^4 x))$. We define T on I_n by imposing that v_n and w_n are two inverse branches of T. It remains to define T on $J := [4\sum_{n=1}^{\infty} a_n, 1)$. We cut this interval into N subintervals of equal length, and let T send each of these intervals affinely onto $[0,1)$.

Main Theorem. *If $a_n = 1/(100n^3)$ and $N = 4$, then the map T preserves Lebesgue measure and the associated transfer operator has a spectral gap on the space of Lipschitz functions, with a simple eigenvalue at 1 and no other eigenvalue of modulus 1. On the other hand, the transfer operator does not act continuously on the space of bounded variation functions.*

For the proof, we will work with general values of a_n and N, and specialize them only at the end of the argument.

Lemma 1. *The map T preserves Lebesgue measure.*

Proof. At a point x, the sum of the derivatives of the inverse branches of T is equal to

$$\sum_{n=1}^{\infty} (v_n'(x) + w_n'(x)) + N \cdot |J|/N$$

$$= \sum_{n=1}^{\infty} a_n(2 + 2\cos^2(2\pi n^4 x) + 2\sin^2(2\pi n^4 x)) + |J|$$

$$= \sum_{n=1}^{\infty} 4a_n + |J| = 1. \qquad \square$$

Let $\mathcal{L}$ be the transfer operator of T associated to Lebesgue measure, given by $\mathcal{L}f(x) = \sum_{T(y)=x} f(y)/T'(y)$. Let also

$$\mathcal{L}_n f(x) = \sum_{T(y)=x,\, y \in I_n} f(y)/T'(y) = v_n'(x)f(v_n x) + w_n'(x)f(w_n x),$$

and $\mathcal{M}f(x) = \sum_{T(y)=x,\, y \in J} f(y)/T'(y)$, so that $\mathcal{L} = \sum_n \mathcal{L}_n + \mathcal{M}$.

Lemma 2. *Assume that $\limsup n^4 a_n = \infty$. Then, for any $k \geq 1$, the operator $\mathcal{L}^k$ does not act continuously on the space $\mathcal{BV}$ of bounded variation functions.*

Proof. Let us fix an inverse branch v of T with $v(I) \subset J$ (v is affine and its slope is $|J|/N$). Fix $k > 0$. Let χ_n be the characteristic function of the interval $v^{k-1} \circ v_n(I)$. Its variation is bounded by 2. Moreover, $\mathcal{L}^k \chi_n(x) = a_n(1 +$

$2\cos^2(2\pi n^4 x))(|J|/N)^{k-1}$, hence the variation of $\mathcal{L}^k \chi_n$ is at least $C(k)a_n n^4$, for some $C(k) > 0$. This concludes the proof. $\square$

The next lemma is the crucial lemma: it ensures that $\mathcal{L}_n$ behaves well on Lip, while we have seen (in the proof of Lemma 2) that it is not the case on $\mathcal{BV}$. We should emphasize that each branch of $\mathcal{L}_n$ behaves badly on Lip: it is the addition of the two branches that gives a better behavior, thanks to the compensations of the bumps of v'_n and w'_n.

Lemma 3. *We have* $\|\mathcal{L}_n f\|_{\mathrm{Lip}} \leq a_n(32\pi n^4 a_n + 8)\|f\|_{\mathrm{Lip}}.$

Proof. Let us decompose $\mathcal{L}_n/a_n$ as

$$\mathcal{L}_n f(x)/a_n = (1 + 2\cos^2(2\pi n^4 x))f(v_n x) + (1 + 2\sin^2(2\pi n^4 x))f(w_n x)$$
$$= (f(v_n x) + 3f(w_n x)) + 2\cos^2(2\pi n^4 x)(f(v_n x) - f(w_n x))$$
$$=: \mathcal{L}_n^{(1)} f(x) + 2\,\mathcal{L}_n^{(2)} f(x).$$

We have $\left\|\mathcal{L}_n^{(1)} f\right\|_{C^0} \leq 4\|f\|_{C^0}$. Moreover, since v_n and w_n are contracting by a factor at least $3a_n$, we have $\mathrm{Lip}(\mathcal{L}_n^{(1)} f) \leq 12a_n \mathrm{Lip}(f) \leq 3\mathrm{Lip}(f)$. Hence, $\left\|\mathcal{L}_n^{(1)} f\right\|_{\mathrm{Lip}} \leq 4\|f\|_{\mathrm{Lip}}$.

Let us now turn to $\mathcal{L}_n^{(2)}$. It satisfies $\left\|\mathcal{L}_n^{(2)} f\right\|_{C^0} \leq 2\|f\|_{C^0}$. Moreover, for any x, y

$$\mathcal{L}_n^{(2)} f(x) - \mathcal{L}_n^{(2)} f(y) = (\cos^2(2\pi n^4 x) - \cos^2(2\pi n^4 y))(f(v_n x) - f(w_n x))$$
$$+ \cos^2(2\pi n^4 y)((f(v_n x) - f(v_n y)) - (f(w_n x) - f(w_n y))).$$

Since the derivative of $\cos^2(2\pi n^4 x)$ is bounded by $4\pi n^4$, the first term is at most $4\pi n^4|x-y||f(v_n x)-f(w_n x)|$. The distance between $v_n x$ and $w_n x$ being at most the length of I_n, i.e., $4a_n$, we therefore get a bound $16\pi n^4 a_n|x-y|\mathrm{Lip}(f)$. For the second term, $\cos^2(2\pi n^4 y)$ is bounded by 1, and $|f(v_n x) - f(v_n y)| \leq 3a_n \mathrm{Lip}(f)|x-y| \leq \mathrm{Lip}(f)|x-y|$. Similarly, $|f(w_n x) - f(w_n y)| \leq \mathrm{Lip}(f)|x-y|$. This second term is therefore bounded by $2|x-y|\mathrm{Lip}(f)$. We obtain

$$\mathrm{Lip}(\mathcal{L}_n^{(2)} f) \leq (16\pi n^4 a_n + 2)\mathrm{Lip}(f).$$

This proves the lemma. $\square$

Let us fix once and for all $a_n = c/n^3$, with c small enough so that

$$\sum_{n=1}^{\infty} \|\mathcal{L}_n f\|_{\mathrm{Lip}} \leq \frac{1}{2}\|f\|_{\mathrm{Lip}}. \tag{1}$$

Using Lemma 3, one can check that $c = 1/100$ is sufficient.

Lemma 4. *If* $N \geq 4$, *the operator* $\mathcal{L}$ *satisfies*

$$\|\mathcal{L} f\|_{\mathrm{Lip}} \leq \frac{3}{4}\|f\|_{\mathrm{Lip}} + \|f\|_{C^0}.$$

Proof. By construction, $\mathcal{M}$ satisfies the inequalities $\|\mathcal{M} f\|_{C^0} \leq |J|\|f\|_{C^0}$ and $\mathrm{Lip}(\mathcal{M} f) \leq |J|\mathrm{Lip}(f) \cdot \frac{|J|}{N}$ (since every inverse branch of T on J contracts by a factor $|J|/N$). With (1), we obtain

$$\|\mathcal{L} f\|_{\mathrm{Lip}} \leq \left(\frac{1}{2} + \frac{|J|^2}{N}\right)\|f\|_{\mathrm{Lip}} + |J|\|f\|_{C^0}.$$

This gives the desired conclusion if $N \geq 4$, since $|J| \leq 1$. $\square$

Proof of the main theorem. By Lemma 2, the choice of a_n ensures that the transfer operator $\mathcal{L}$ does not act continuously on $\mathcal{BV}$. On the other hand, since the inclusion of Lip in C^0 is compact, Lemma 4 and Hennion's theorem show that the essential spectral radius of $\mathcal{L}$ acting on Lip is $\leq 3/4 < 1$.

It remains to study the eigenvalues of modulus 1 of $\mathcal{L}$. Let f be a nonzero eigenfunction for such an eigenvalue λ. The function f is Lipschitz on $[0, 1)$, hence it can be continuously extended to $[0, 1]$. Let $x \in [0, 1]$ be such that $|f(x)|$ is maximal. Assume first $x < 1$, then

$$|f(x)| = |\mathcal{L} f(x)| = \Big| \sum_{T(y)=x} f(y)/T'(y) \Big|$$
$$\leq \sum_{T(y)=x} |f(y)|/T'(y) \leq \sum_{T(y)=x} \sup |f|/T'(y) = |f(x)|.$$

There is equality everywhere in these inequalities. Hence, $|f(y)| = |f(x)|$ for any preimage y of x, and the complex numbers $f(y)$ all have the same argument. This shows that f is constant on the set $T^{-1}(x)$. Applying the same argument to $\mathcal{L}^n$, we see that f is constant on $T^{-n}(x)$. This set being more and more dense as n tends to infinity, this shows that f is constant, concluding the proof.

The case $x = 1$ is analogous, using left limits everywhere. $\square$

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