MARTIN BOUNDARY OF RANDOM WALKS WITH UNBOUNDED JUMPS IN HYPERBOLIC GROUPS

BY SÉBASTIEN GOUËZEL

IRMAR, Université de Rennes 1

Given a probability measure on a finitely generated group, its Martin boundary is a natural way to compactify the group using the Green function of the corresponding random walk. For finitely supported measures in hyperbolic groups, it is known since the work of Ancona and Gouëzel–Lalley that the Martin boundary coincides with the geometric boundary. The goal of this paper is to weaken the finite support assumption. We first show that, in any nonamenable group, there exist probability measures with exponential tails giving rise to pathological Martin boundaries. Then, for probability measures with superexponential tails in hyperbolic groups, we show that the Martin boundary coincides with the geometric boundary by extending Ancona's inequalities. We also deduce asymptotics of transition probabilities for symmetric measures with superexponential tails.

1. Introduction. Consider a probability measure μ on a finitely generated group Γ, whose support generates Γ as a semigroup (we say that μ is *admissible*). The Green function associated to μ is $G_\mu(x, y) = G(x, y) = \sum_{n=0}^\infty \mu^n(x^{-1}y)$. The Green function is defined so that the random walk with transition probabilities $p(a, b) = \mu(a^{-1}b)$ starting from x spends an average time $G(x, y)$ at y. We will always assume that this sum is finite (i.e., the random walk is transient). The function G contains a lot of information about the transition probabilities and the asymptotic properties of the random walk. Moreover, it is at the heart of the potential theory of μ, making it possible to describe all positive harmonic functions through the notion of *Martin boundary*.

The Martin boundary $\partial_\mu \Gamma$ is defined as follows: a sequence of points $y_n \in \Gamma$ going to infinity converges in $\Gamma \cup \partial_\mu \Gamma$ if and only if, for all z, the sequence $K_{y_n}(z) = G(z, y_n)/G(e, y_n)$ converges, where e denotes the identity of the group. One can associate to any $\xi \in \partial_\mu \Gamma$ the corresponding Martin kernel $K_\xi(z) = \lim K_{y_n}(z)$. This function is superharmonic (i.e., if P_μ denotes the Markov operator associated to μ, then $P_\mu K_\xi \le K_\xi$), and any positive superharmonic function on Γ can be decomposed as an integral of the kernels K_ξ with respect to some finite measure on $\Gamma \cup \partial_\mu \Gamma$ (the decomposition is unique if one requires that the measure is supported on Γ and on the minimal part of the Martin boundary, made of those ξ

whose kernel K_ξ is harmonic and minimal among positive harmonic functions). See, for instance, [5, 11, 12].

Describing concretely the Martin boundary in specific examples is difficult, especially in nonamenable situations. A landmark result in this direction is a theorem by Ancona [1] showing that, for finitely supported probability measures in (nonelementary) hyperbolic groups, the Martin boundary coincides with the geometric boundary of the group. His result is not restricted to probability measures: the Green function and the Martin boundary can be defined for any finite measure μ, and Ancona's result is true for any measure μ such that $r\mu$ has a finite Green function for some $r > 1$ [we will say that such a μ has the property Anc$_*$, since this property is called $(*)$ in Ancona's paper]. Ancona's proof is based on an inequality saying that, in hyperbolic groups, the Green function of a measure with finite support and property Anc$_*$ is essentially multiplicative along geodesics: there exists a constant C such that, for any x, y, z on a geodesic of the group (in this order), one has

(1.1) $$C^{-1}G(x, y)G(y, z) \leq G(x, z) \leq CG(x, y)G(y, z).$$

While the first inequality is true for any random walk in any group, the second one is highly nontrivial. It is used by Ancona to show that the Martin boundary coincides with the geometric boundary. It also plays an important role in the article [3] by Blachère, Haïssinsky and Mathieu: they prove that this inequality is necessary and sufficient so that a natural distance associated to the random walk, the *Green distance*, is hyperbolic (and they prove several properties of the harmonic measure at infinity under this condition). It is also instrumental in the articles [7, 8] by Gouëzel and Lalley, where the asymptotics of transition probabilities in hyperbolic groups are determined (note that the authors need to extend Ancona inequalities to some measures that do not satisfy Anc$_*$). All those results rely on the finiteness of the support of the measure μ.

Our goal in this article is to see to what extent the previous results can be extended to measures with infinite support. The tails of the measure, that is, the speed at which $\mu(B(e, n)^c)$ tends to 0 [where $B(e, n)^c$ denotes the complement of the ball centered at e of radius n, for some word distance in the group] will play an important role in the results. We will say that a measure has exponential tails if there exists $K > 1$ such that, for large enough n, $\mu(B(e, n)^c) \leq K^{-n}$. We will say that μ has superexponential tails if this condition is true for all $K > 1$. Equivalently, μ has exponential tails if, for some $\delta > 0$, the sum $\sum_{g \in \Gamma} e^{\delta|g|}\mu(g)$ is finite (where $|g|$ is the distance from e to g in a word metric), and μ has superexponential tails if this sum is finite for all $\delta > 0$.

Our first result shows that one cannot expect a reasonable description of the Martin boundary if one only demands an exponential decay of the tails:

THEOREM 1.1. *Consider a nonamenable finitely generated group Γ, and a sequence y_n going to infinity in Γ. There exists an admissible symmetric probabil-*

ity measure μ on Γ, *with exponential tails*, *such that* y_n *does not converge in the Martin boundary* $\partial_\mu \Gamma$.

This implies in particular that there exist uncountably many possible different Martin boundaries for measures with exponential tails, by a standard diagonal argument.

If the tails have a better behavior (i.e., if they are superexponential), we can extend Ancona's results:

THEOREM 1.2.   *In a nonelementary hyperbolic group* Γ, *consider an admissible measure satisfying* Anc$_*$, *with superexponential tails. Then it satisfies Ancona inequalities* (1.1). *In particular, its Martin boundary coincides with the geometric boundary of the group.*

It follows that all the results of [3] describing the geometry of the harmonic measure (and in particular its pointwise dimension), originally obtained for finitely supported measures, still hold for measures with superexponential tails.

As we explained before, the results of [7, 8] require Ancona inequalities for measures that do not satisfy Anc$_*$. We extend their results to measures with superexponential tails.

THEOREM 1.3.   *In a nonelementary hyperbolic group* Γ, *consider an admissible measure* μ *with superexponential tails and finite Green function. Assume that one of the following conditions is satisfied*:

1. *The measure* μ *is symmetric*.
2. *The group* Γ *is a free group on finitely many generators*.
3. *The group* Γ *is a cocompact lattice of* PSL(2, ℝ).

Then μ *satisfies Ancona inequalities* (1.1). *In particular, its Martin boundary coincides with the geometric boundary of the group.*

It is likely that the above conditions (μ symmetric or Γ planar) are not necessary for this theorem, but this is unknown even in the case of a finitely supported μ. The above conditions are precisely those that are used in [7, 8] to obtain (for finitely supported measures) Ancona inequalities and a description of the Martin boundary.

The motivation for the results of [7, 8] was to obtain asymptotics of transition probabilities for random walks. We deduce the corresponding statement in our setting.

THEOREM 1.4.   *In a nonelementary hyperbolic group* Γ, *consider a symmetric admissible probability measure* μ *with superexponential tails. Denote by* $R > 1$ *the inverse of the spectral radius of the corresponding random walk. For any*

$x, y \in \Gamma$, *there exists* $C(x, y) > 0$ *such that the transition probabilities* $p^n(x, y)$ *of the random walk at time n satisfy*

$$p^n(x, y) \sim C(x, y) R^{-n} n^{-3/2}$$

if the walk is aperiodic. If the walk is periodic, this asymptotics holds for even (*resp., odd*) *n if the distance from x to y is even* (*resp., odd*).

This result is new even for random walks on free groups. Note that, even in the finitely supported case, the proof requires the symmetry of the measure since the very end of the argument relies on spectral properties of the Markov operator.

The paper is organized as follows. In Section 2, we recall basic properties of the Green function. Section 3 is devoted to the construction of pathological Green functions for measures with exponential tails, proving in particular Theorem 1.1. The main idea of the construction is that, even with exponential tails, one can ensure that the most likely way to reach some point is by doing a direct jump. This makes it possible to prescribe very precisely the asymptotics of the Green function. Finally, Section 4 is devoted to the positive results in hyperbolic groups, for measures with superexponential tails. Ancona's arguments to get his inequality rely on a subtle induction that does not seem generalizable to the infinite support situation. We will rather use a lemma of [8] (see Lemma 4.4 below) showing that some upper bounds on relative Green functions imply Ancona inequalities. Such upper bounds are more manageable, and can be proved for infinitely supported measures as we will show.

2. The Green function.

Consider a finite admissible measure μ on a finitely generated group Γ. We will always assume that its Green function $G(x, y) = \sum \mu^n(x^{-1}y)$ is finite for some x, y (and, therefore, for all x, y by admissibility). Denote by P_μ the operator associated to μ, given by $P_\mu f(x) = \sum \mu(x^{-1}y) f(y)$—when μ is a probability measure, this is simply the Markov operator associated to the corresponding random walk. Even when μ is not a probability measure, we will use probabilistic notation such as $p^n(x, y) = \mu^n(x^{-1}y)$, and think of $G(x, y) = \sum (P_\mu^n \delta_y)(x)$ as an average time spent at y if one starts from x.

The Green function can also be formulated in terms of paths. Let $\tau = (x, x_1, \ldots, x_{n-1}, y)$ be a path of length n from x to y, we define its μ-weight (or simply weight) $\pi_\mu(\tau) = \pi(\tau)$ by

$$\pi(\tau) = \prod_{i=0}^{n-1} p(x_i, x_{i+1}),$$

where $x_0 = x$ and $x_n = y$ by convention, and we write $p(a, b) = \mu(a^{-1}b)$. We think of $\pi(\tau)$ as the "probability" to follow the path τ. By definition, $G(x, y) = \sum \pi(\gamma)$, where the sum is over all paths from x to y.

If Ω is a subset of Γ, one defines the restricted Green function $G(x, y; \Omega)$ as $\sum \pi(\gamma)$ where the sum is over all paths $\gamma = (x, x_1, \ldots, x_{n-1}, y)$ such that $x_i \in \Omega$ for $1 \leq i \leq n - 1$. If A is a subset of Γ and $x, y \notin A$, one has

$$G(x, y) = G(x, y; A^c) + \sum_{a \in A} G(x, a; A^c)G(a, y)$$

$$= G(x, y; A^c) + \sum_{a \in A} G(x, a)G(a, y; A^c),$$

where A^c denotes the complement of A. Indeed, the first (resp., second) formula is proved by splitting a path from x to y according to its first (resp., last) visit to A if it exists, the remaining trajectories giving the contribution $G(x, y; A^c)$. If all trajectories from x to y have to go through A, this contribution vanishes. This is used crucially in the usual arguments for finitely supported measures, where one uses wide enough "barriers" A between x and y, that any trajectory from x to y has to visit. In the infinite support situation, the contribution $G(x, y; A^c)$ will always be present.

More generally, if Ω is a subset of Γ containing x and y, the above formula holds restricted to Ω, that is,

$$G(x, y; \Omega) = G(x, y; \Omega \cap A^c) + \sum_{a \in A \cap \Omega} G(x, a; \Omega \cap A^c)G(a, y; \Omega)$$

(2.1)

$$= G(x, y; \Omega \cap A^c) + \sum_{a \in A \cap \Omega} G(x, a; \Omega)G(a, y; \Omega \cap A^c).$$

Let d be a word distance on Γ coming from a finite symmetric generating set. If x and y are at distance d, there is a path from x to y with weight bounded from below by C^{-d}, and staying close to a geodesic segment from x to y. We deduce that, for any z,

(2.2) $$C^{-d(x,y)} \leq G(x, z)/G(y, z) \leq C^{d(x,y)},$$

and similar inequalities hold for the Green function restricted to any set containing a fixed size neighborhood of a geodesic segment from x to y. These inequalities are called Harnack inequalities.

The first visit Green function is $F(x, y) = G(x, y; \{y\}^c)$. It only takes into account the first visits to y. When μ is a probability measure, $F(x, y)$ is the probability to reach y starting from x. One has $G(x, y) = F(x, y)G(y, y) = F(x, y)G(e, e)$. Moreover, $F(x, y)G(y, z) \leq G(x, z)$ (since the concatenation of a path from x to y with a path from y to z gives a path from x to z). Hence,

(2.3) $$G(x, y)G(y, z) \leq G(e, e)G(x, z).$$

This shows that the left inequality in (1.1) is always true.

3. Pathological constructions in nonamenable groups.   Let Γ be a finitely generated nonamenable group. In this section, we construct admissible symmetric probability measures with exponential tails that behave in a pathological way regarding their Green functions and Martin boundaries.

The basic idea is the following. We start from a symmetric probability measure ν supported by a finite generating set of Γ, and we add Dirac masses, with a very small mass but supported far away from the identity. If we adjust carefully the weights, the way to reach some far away points with highest probability is to jump directly onto them (possibly with some short jumps), since an accumulation of small jumps has a lower probability that one single big jump. In this way, we will prescribe the behavior of the Green function at different scales.

This type of behavior is reminiscent of Lévy processes on $\mathbb{R}$: when such a process is large, this is typically due to one single large jump, the sum of the other jumps being negligible. We are constructing a kind of Lévy process on Γ, but with exponential tails. The reason behind this counterintuitive phenomenon (in $\mathbb{R}$, Lévy processes need to have heavy tails) is that exponentially small tails can still dominate the diffusive behavior since the diffusion is also exponentially small in nonamenable groups.

The precise construction is as follows. Let $\rho < 1$ be the spectral radius of the random walk given by ν. It is also the norm of the associated Markov operator P_ν since ν is symmetric. Let us fix a decreasing sequence r_i (the exponential weights) with $e^{r_0}\rho < 1$ and $\lim r_i = r > 0$. Let us also fix a sequence n_i tending very quickly to infinity, and a symmetric measure μ_i supported on the ball $B(e, n_i)$. Let

$$\mu = \nu + \sum e^{-r_i n_i}\mu_i \quad \text{and} \quad \mu' = \mu/\mu(\Gamma).$$

The probability measure μ' is symmetric, and has exponential tails of order r. We will see that we can prescribe the behavior of its Green function. Since most interesting things happen with one jump, we may equivalently work with μ' or μ. It will be more convenient to formulate the estimates for μ.

The fact that r_i is strictly decreasing is a central point of the construction. Roughly speaking, if one uses only the measures μ_i with $i \leq I$, then a jump of size $n \leq n_I$ is made with probability at most $e^{-r_I n}$. This implies that a point at a large distance n of e will be reached with probability roughly $e^{-r_I n}$. Let us take $n = n_{I+1}$, and x a point in the support of μ_{I+1}. It can be reached by a direct jump, with probability of the order of $e^{-r_{I+1} n}$, which is much bigger than $e^{-r_I n}$ since $r_{I+1} < r_I$. Hence, direct jumps are more likely than a combination of small jumps, as desired.

The rigorous version of this argument is slightly more complicated: using the measures μ_i with $i \leq I$, one can in fact reach a point at distance n with a probability at most $C(s)e^{-sn}$ for any $s < r_I$. Hence, we need to introduce another sequence: we fix once and for all $s_{i+1} \in (r_{i+1}, r_i)$ (we also require that $s_{i+1} < 2r_{i+1}$ for

technical reasons). In the following, we will always assume that n_i grows quickly enough so that

(3.1) $$r_{i+1} n_{i+1} \geq s_i n_i \geq r_i n_i + i + 1$$

and

(3.2) $$\frac{1}{1 - \rho e^{r_0}} \sum e^{-(r_i - s_{i+1}) n_i} \leq \frac{1}{2}.$$

Since $r_i - s_{i+1} > 0$, this can easily be guaranteed. From this point on, the letter C will denote a constant that can vary from one line to the other, but does not depend on the choices we have made provided the conditions (3.1) and (3.2) are satisfied.

Let us estimate the Green function $G(e, x)$ associated to μ. This is the sum of the weights of paths from e to x. We will group together those paths corresponding to the same sequence of measures ν or μ_i. This is most conveniently done in terms of Markov operators as follows. We will write $P = P_\nu$ and $P_i = P_{\mu_i}$ for the operators associated, respectively, to ν and μ_i. They satisfy $P_\mu = P + \sum e^{-r_i n_i} P_i$. Developing P_μ^n and grouping together the successive occurrences of P, we get

$$G(e, x) = \sum_n \langle P_\mu^n \delta_x, \delta_e \rangle$$

$$= \sum_{\ell=0}^{\infty} \sum_{a_0, i_1, a_1, \ldots, i_\ell, a_\ell} \langle P^{a_0} e^{-r_{i_1} n_{i_1}} P_{i_1} P^{a_1} \cdots P^{a_{\ell-1}} e^{-r_{i_\ell} n_{i_\ell}} P_{i_\ell} P^{a_\ell} \delta_x, \delta_e \rangle.$$

Each term in the double sum corresponds to the weight of several trajectories. We will say that the associated sequence $t = (a_0, i_1, a_1, \ldots, a_\ell)$ is a *template* for this set of trajectories. The norm of P^a on $\ell^2(\Gamma)$ is bounded by ρ^a, and the norm of P_i is at most 1. Hence, the sum of the weights of trajectories in a template t is bounded by its weight $\pi(t)$ defined by

$$\pi(t) = \rho^{a_0 + \cdots + a_\ell} e^{-r_{i_1} n_{i_1}} \cdots e^{-r_{i_\ell} n_{i_\ell}}.$$

Summing over the templates, we obtain

(3.3) $$G(e, x) \leq {\sum}' \pi(t),$$

where the notation $\sum'$ indicates that we can remove from the sum all those templates that give a vanishing contribution to $G(e, x)$, that is, those for which no trajectory can go from e to x.

It is not clear that the Green function of μ is well defined, since μ is not a probability measure. We can use (3.3) to show its finiteness, uniformly in x. We have

(3.4) $$\sum_t \pi(t) \leq \sum_\ell \left(\sum_{a=0}^{\infty} \rho^a \right)^{\ell+1} \left(\sum_i e^{-r_i n_i} \right)^\ell.$$

The sum over ℓ is a geometric series. It is finite if its general term is < 1, that is, $\frac{1}{1-\rho}\sum e^{-r_i n_i} < 1$. This is a consequence of (stronger) condition (3.2). As $G(e, x) \le \sum \pi(t)$, this shows that $G(e, x)$ is well-defined and uniformly bounded.

We need more notation regarding templates. Given a template $t = (a_0, i_1, a_1, \ldots, a_\ell)$, define its length $|t| = \sum a_k + \sum n_{i_k}$: any trajectory in the template ends at a point at distance at most $|t|$ of the origin. Let also $\max t = \sup i_k$ give the size of the biggest jump in t. We will write $t_1 \cdot t_2$ for the concatenation of two templates t_1 and t_2. It satisfies $\pi(t_1 \cdot t_2) = \pi(t_1)\pi(t_2)$.

The crucial estimates for template weights are the following.

LEMMA 3.1.   *For every integers i and n,*

$$(3.5) \qquad \sum_{\max t \ge i} \pi(t) \le C e^{-r_i n_i}$$

and

$$(3.6) \qquad \sum_{\max t < i, |t| \ge n} \pi(t) \le C e^{-s_i n}.$$

As a consequence, for every $i \in \mathbb{N}$ and for every $z \in \Gamma$,

$$(3.7) \qquad G(e, z) \le C e^{-r_i n_i} + C e^{-s_i |z|}.$$

Inequality (3.5) controls what happens when there is at least one big jump, while (3.6) controls the combination of several small jumps. The last inequality (3.7) is a consequence of the other two. Note that, if $|z|$ is comparable to n_i, then the second term in (3.7) is negligible compared to the first one since $s_i n_i - r_i n_i \to +\infty$ by (3.1). This shows rigorously that the most efficient way to visit z is to do one big jump rather than many small jumps, as we already explained informally.

PROOF OF LEMMA 3.1.   Let us first show (3.5). A template t with $\max t \ge i$ can be decomposed as $t = t_1 \cdot (j) \cdot t_2$ where t_1 and t_2 are shorter templates and j corresponds to a jump of size $n_j \ge n_i$. Therefore,

$$\sum_{\max t \ge i} \pi(t) \le \left(\sum_{t_1} \pi(t_1) \right) \left(\sum_{j=i}^{\infty} e^{-r_j n_j} \right) \left(\sum_{t_2} \pi(t_2) \right).$$

The first sum and the last sum are finite by (3.4). The middle one is bounded by $C e^{-r_i n_i}$ thanks to (3.1). This proves (3.5).

Let us now show (3.6). Writing $t = (a_0, i_1, \ldots, a_\ell)$, the corresponding sum is

$$\sum_{\max t < i, |t| \ge n} e^{-s_i(a_0 + \cdots + a_\ell + n_{i_1} + \cdots + n_{i_\ell})} \left(\rho e^{s_i} \right)^{a_0 + \cdots + a_\ell} e^{-(r_{i_1} - s_i)n_{i_1}} \cdots e^{-(r_{i_\ell} - s_i)n_{i_\ell}}.$$

The first factor is $e^{-s_i|t|} \leq e^{-s_i n}$. This yields a bound

$$e^{-s_i n} \sum_{\max t < i} \left(\rho e^{s_i}\right)^{a_0 + \cdots + a_\ell} e^{-(r_{i_1} - s_i)n_{i_1}} \cdots e^{-(r_{i_\ell} - s_i)n_{i_\ell}}$$

$$= e^{-s_i n} \sum_\ell \left(\sum_{a=0}^{\infty} (\rho e^{s_i})^a\right)^{\ell+1} \left(\sum_{j=0}^{i-1} e^{-(r_j - s_i)n_j}\right)^\ell.$$

This is again a geometric series. Let us bound e^{s_i} with e^{r_0} in the first factor, and $e^{-(r_j - s_i)n_j}$ with $e^{-(r_j - s_{j+1})n_j}$ in the second factor. We get that the general term of this geometric series is bounded by

$$\frac{1}{1 - \rho e^{r_0}} \sum_{j \geq 0} e^{-(r_j - s_{j+1})n_j}.$$

Condition (3.2) guarantees that this is $\leq 1/2$. Hence, the geometric series is uniformly bounded, yielding a bound $Ce^{-s_i n}$. This proves (3.6).

Let us finally prove (3.7) using (3.3). To go from e to z, the templates with $\max t \geq i$ give an overall contribution at most $Ce^{-r_i n_i}$, by (3.5). On the other hand, if $\max t < i$, then it is possible to reach z using a trajectory in the template only if $|t| \geq |z|$. By (3.6), those terms contribute at most $Ce^{-s_i|z|}$. □

This lemma implies that, in general, there is no Ancona inequality (1.1) in non-amenable groups, for measures with exponential tails.

PROPOSITION 3.2. *Let Γ be a finitely generated nonamenable group. There exists on Γ an admissible symmetric probability measure μ' with exponential tails whose Green function $G' = G_{\mu'}$ does not satisfy Ancona inequalities: there is no constant C such that $G'(x, z) \leq CG'(x, y)G'(y, z)$ for any $x, y, z \in \Gamma$ on a geodesic in this order.*

PROOF. We use the previous construction, with $\mu_i = (\delta_{z_i} + \delta_{z_i^{-1}})/2$ where z_i is a point at distance n_i of e. We will assume that n_i is even, and we will denote by y_i the midpoint of a geodesic segment from e to z_i. We will show that

$$(3.8) \qquad G'(e, z_i) \geq Ce^{-r_i n_i}$$

and

$$(3.9) \qquad G'(e, z) \leq Ce^{-s_i n_i/2}$$

for any z with $d(e, z) = n_i/2$. Hence, $G'(e, y_i)G'(y_i, z_i) \leq C^2 e^{-s_i n_i} = o(G'(e, z_i))$, contradicting any Ancona inequality.

Inequality (3.8) is obvious since the Green function is bounded from below by the contribution of single jumps: $G'(e, z_i) \geq \mu'(z_i) = \mu(\Gamma)^{-1}e^{-r_i n_i}/2$.

As $G' \le G$, inequality (3.9) follows from (3.7) since $|z| = n_i/2$. [The first term in (3.7) is dominated by the second term since we have requested that $s_i < 2r_i$.]

<div align="right">□</div>

We now turn to the proof of Theorem 1.1. Starting from a sequence y_n going to infinity, we wish to construct the measures μ and μ' (using the above construction) so that $G' = G_{\mu'}$ is such that, for some point z, the sequence $G'(z, y_n)/G'(e, y_n)$ does not converge. We will write $G' = G_{\mu'}$ and $G = G_\mu$.

We need to fix an additional sequence $s_i' \in (r_i, s_i)$, for instance the middle of this interval, to get some additional freedom. Taking a subsequence of y_n, we can assume that

$$(3.10) \qquad (s_i'/r_i - 1)|y_i| \to \infty, \qquad (1 - s_i'/s_i)|y_i| \to \infty.$$

Let $n_i = (s_i'/r_i)|y_i|$. One has $y_i \in B(e, n_i)$ by construction. The condition (3.10) ensures that, for any C, for large enough i, a point y with $|y| \le |y_i| + C$ belongs to $B(e, n_i)$. Taking a further subsequence of y_i if necessary, we can also assume that growth conditions (3.1) and (3.2) are satisfied by n_i.

To get the divergence of $G'(z, y_i)/G'(e, y_i)$ for some point z, we will choose the measures μ_i so that the limits of this sequence are different along even and odd values of i (with a limit of the order of 1 along odd i, and a small limit along a subsequence of even i). For i even, we let $\mu_i = (\delta_{y_i} + \delta_{y_i^{-1}})/2$. The choice of μ_i for odd i is postponed, let us first see the consequences of our choice for even i. The statements we will give now are valid for any choice of μ_i for odd i, with the only restriction that it has to be a probability measure, supported in $B(e, n_i)$.

Let us describe the asymptotics of $G(e, zy_i)$ for any fixed z.

LEMMA 3.3. *There exists a function* $\Phi : \Gamma \to (0, +\infty)$, *tending to* 0 *at infinity, such that for every* z *there exist infinitely many even indices* i *for which*

$$G(e, zy_i) \le \Phi(z)e^{-r_i n_i}.$$

Let us stress that the function Φ does not depend on the choice of μ_i for odd i.

PROOF OF LEMMA 3.3.   The idea is that, to go from e to zy_i, the random walk will most likely make one big jump of size n_i (corresponding to the measure μ_i), with weight $e^{-r_i n_i}/2$, and several small jumps. If z is large enough, a large number of small jumps will be needed, giving a small contribution $\Phi(z)$. The other cases (no big jump, or too many big jumps) will have a very small contribution. In this proof, i will implicitly be restricted to even values.

For the rigorous computation, we start from the bound (3.3) and cut the sum into several pieces. We should specify in which piece a template $t = (a_0, i_1, \ldots, a_\ell)$ goes.

- We put in J_1 the templates with $\max t > i$.

- We put in J_2 the templates where at least two jumps i_k are equal to i.
- We put in J_3 the templates with $\max t < i$ for which a trajectory can go from e to zy_i.
- Finally, we put in J_4 the remaining templates, that is, those with a single jump of size n_i and other shorter jumps, for which a trajectory can go from e to zy_i.

Denote by Σ_p the sum corresponding to templates in J_p. We will show that, for $p \leq 3$, one has $\Sigma_p = o(e^{-r_i n_i})$ when i tends to infinity, and that for infinitely many indices i one has $\Sigma_4 \leq \Psi(z)e^{-r_i n_i}$ for some function Ψ tending to 0 at infinity. The result follows with $\Phi = 2\Psi$.

Inequality (3.5) implies that $\Sigma_1 \leq Ce^{-r_{i+1} n_{i+1}}$. As $r_{i+1} n_{i+1} > r_i n_i + i + 1$ by (3.1), this is negligible compared to $e^{-r_i n_i}$, as desired.

A template $t \in \Sigma_2$ can be decomposed as $t = t_1 \cdot (i) \cdot t_2 \cdot (i) \cdot t_3$, for some templates t_1, t_2 and t_3. Since the sum of the weights of all templates is bounded, we obtain

$$\Sigma_2 \leq Ce^{-r_i n_i} Ce^{-r_i n_i} C.$$

This is again negligible with respect to $e^{-r_i n_i}$.

A template t in J_3 satisfies $|t| \geq |zy_i|$ and $\max t < i$. Hence, (3.6) gives the bound $\Sigma_3 \leq Ce^{-s_i |zy_i|}$. We have

$$s_i |zy_i| - r_i n_i \geq s_i (|y_i| - |z|) - r_i n_i = s_i (|y_i| - |z|) - s_i' |y_i|$$
$$= s_i \left(\left(1 - \frac{s_i'}{s_i} \right) |y_i| - |z| \right).$$

As $(1 - s_i'/s_i)|y_i| \to \infty$ by (3.10), this tends to infinity. Hence,

$$(3.11) \qquad\qquad e^{-s_i |zy_i|} = o(e^{-r_i n_i}).$$

This shows that Σ_3 is negligible with respect to $e^{-r_i n_i}$.

It remains to estimate Σ_4. A template $t \in J_4$ can be decomposed uniquely as $t = t_1 \cdot (i) \cdot t_2$, for some templates t_1 and t_2 with maximum $< i$. If this template contributes to $G(e, zy_i)$, then zy_i can be written as $uy_i^{\pm 1}v$ with $|u| \leq |t_1|$ and $|v| \leq |t_2|$. Denote by $\varphi_i(z)$ the minimum of the quantities $|u| + |v|$ over all decompositions $zy_i = uy_i^{\pm 1}v$, we get $|t_1| + |t_2| \geq \varphi_i(z)$. In particular, $|t_1| \geq \varphi_i(z)/2$ or $|t_2| \geq \varphi_i(z)/2$. It follows that

$$\Sigma_4 \leq 2 \left(\sum_{\max t_1 < i, |t_1| \geq \varphi_i(z)/2} \pi(t_1) \right) e^{-r_i n_i} \left(\sum_{t_2} \pi(t_2) \right).$$

The first sum is bounded by $Ce^{-s_i \varphi_i(z)/2} \leq Ce^{-r \varphi_i(z)/2}$ by (3.6), and the last sum is uniformly bounded. Hence,

$$\Sigma_4 \leq Ce^{-r\varphi_i(z)/2} e^{-r_i n_i}.$$

To conclude, we have to show that $\varphi_i(z)$ is large for infinitely many values of i, if z is far away from e. Let $A > 0$, let us denote by B_i the set of z that can be written as $u y_i^{\pm 1} v y_i^{-1}$ for some u and v with $|u| + |v| \le A$. The set B_i is finite, with cardinality at most $f(A) = 2(\operatorname{Card} B(e, A))^2$. If $z \notin B_i$, it satisfies $\varphi_i(z) > A$ by definition. The points with $\limsup \varphi_i(z) \le A$ belong to $\bigcup_n \bigcap_{i>n} B_i$. This is an increasing union of sets of cardinality at most $f(A)$, hence it has cardinality at most $f(A)$. This shows that, apart from finitely many exceptions, $\limsup \varphi_i(z) > A$; hence, $\Sigma_4 \le C e^{-rA/2} e^{-r_i n_i}$ for infinitely many i's. $\square$

Let us fix a point z away from the origin, so that $\Phi(z)$ is suitably small (how small will be seen later in the proof). We now define the measures μ_i for odd i. If i is large enough, $z y_i \in B(e, n_i)$ thanks to (3.10). For those i's, let

$$\mu_i = \tfrac{1}{4}(\delta_{y_i} + \delta_{z y_i} + \delta_{y_i^{-1}} + \delta_{(z y_i)^{-1}}).$$

The choice of μ_i for smaller i is not relevant (take, e.g., $\mu_i = \delta_e$).

If i is large and odd, Lemma 3.1 gives $G(e, z y_i) \le C e^{-r_i n_i} + C e^{-s_i |z y_i|}$. By (3.11), the second term is negligible with respect to the first one. Hence, $G(e, z y_i) \le C e^{-r_i n_i}$. In the same way $G(e, y_i) \le C e^{-r_i n_i}$.

The Green function $G' = G_{\mu'}$ is bounded by $G = G_\mu$. For i large and odd, we obtain $G'(e, z y_i) \le C e^{-r_i n_i}$ and $G'(e, y_i) \le C e^{-r_i n_i}$. As it is possible to jump directly from e to $z y_i$ or y_i with weight $\mu(\Gamma)^{-1} e^{-r_i n_i}/4$, corresponding lower bounds hold. In particular, there exists a constant C_0 such that, for i large and odd,

$$\frac{G'(e, z y_i)}{G'(e, y_i)} \in [C_0^{-1}, C_0].$$

For infinitely many (even) values of i, we have $G'(e, z y_i) \le \Phi(z) e^{-r_i n_i}$ by Lemma 3.3. Moreover, $G'(e, y_i) \ge C^{-1} e^{-r_i n_i}$ [since one can jump directly from e to y_i with weight $\mu(\Gamma)^{-1} e^{-r_i n_i}/2$]. Hence, for those values of i, there exists a constant C_1 such that

$$\frac{G'(e, z y_i)}{G'(e, y_i)} \le C_1 \Phi(z).$$

We can finally specify the choice of z: as Φ tends to 0 at infinity, we may choose z such that $C_1 \Phi(z) < C_0^{-1}$. The previous estimates imply that

$$\liminf_i \frac{G'(e, z y_i)}{G'(e, y_i)} \le C_1 \Phi(z) < C_0^{-1} \le \limsup \frac{G'(e, z y_i)}{G'(e, y_i)}.$$

In particular, the sequence $G'(e, z y_i)/G'(e, y_i)$ does not converge when i tends to infinity. Equivalently, $G'(z^{-1}, y_i)/G'(e, y_i)$ does not converge. This completes the proof of Theorem 1.1.

4. Positive results in hyperbolic groups.

4.1. *Preliminaries.*   A hyperbolic group is a finitely generated group in which geodesic triangles are δ-thin for some δ, that is, each side of the triangle is included in the δ-neighborhood of the union of the other sides. This notion is independent of the choice of the generating system (albeit the constant δ does change with the generating system). See, for instance, [6]. This essentially means that finite configurations of points in the group resemble finite configurations of points in a tree—this intuition is made precise by the following classical theorem.

THEOREM 4.1.   *For any $n \in \mathbb{N}$ and $\delta > 0$, there exists a constant $C = C(n, \delta)$ with the following property. Consider a subset A of a δ-hyperbolic group, of cardinality at most n. There exists a map Φ from A to a metric tree such that, for any $x, y \in A$,*

$$d(x, y) - C \le d\big(\Phi(x), \Phi(y)\big) \le d(x, y).$$

Another intuition is that δ-hyperbolic spaces resemble the usual hyperbolic space $\mathbb{H}^m$. Again, this is made precise by the following theorem [4]. We will write $d_{\mathbb{H}}$ for the hyperbolic distance in $\mathbb{H}^m$, and $|x|_{\mathbb{H}} = d_{\mathbb{H}}(x, O)$ where O is a fixed reference point in $\mathbb{H}^m$.

THEOREM 4.2.   *Consider a hyperbolic group Γ. If m is large enough, there exist a mapping $\Psi : \Gamma \to \mathbb{H}^m$ and $\lambda > 0$, $C > 0$ such that, for all $x, y \in \Gamma$,*

$$\big|\lambda d_{\mathbb{H}}\big(\Psi(x), \Psi(y)\big) - d(x, y)\big| \le C.$$

Ancona's original strategy [1] to prove Ancona inequalities (1.1) for finitely supported measures, based on a subtle induction, is apparently difficult to extend to measures with infinite support. We will rather rely on the strategy of [8], and in particular on the following lemma (see the proofs of Theorems 4.1 and 4.3 in [8]). We recall that the relative Green function $G(x, y; \Omega)$ has been defined in Section 2.

DEFINITION 4.3.   Let μ be an admissible measure with finite Green function on a hyperbolic group. It satisfies *pre-Ancona inequalities* if, for all $K > 0$, there exists n_0 such that, for all $n \ge n_0$, for all points x, y, z on a geodesic segment (in this order) with $d(x, y) \in [n, 100n]$ and $d(y, z) \in [n, 100n]$, one has $G(x, z; B(y, n)^c) \le K^{-n}$.

LEMMA 4.4.   *Let μ be an admissible measure on a hyperbolic group. Assume that μ satisfies pre-Ancona inequalities. Then it satisfies Ancona inequalities (1.1).*

This lemma justifies the name "pre-Ancona inequalities." It is proved in [8] as follows. Assume that x, y, z are given along a geodesic, and one wants to prove that $G(x, z) \leq CG(x, y)G(y, z)$. One constructs a string of beads along a geodesic segment $[x, z]$, the size of a bead being proportional to its distance to y. Then, using pre-Ancona inequalities, one shows inductively that the weight of trajectories avoiding any bead is comparatively small. It follows that most weight comes from trajectories passing in a bead within distance $O(1)$ of y, as desired.

To prove Ancona inequalities, our strategy will always be to show that pre-Ancona inequalities are satisfied.

4.2. *Ancona inequalities for measures satisfying* Anc_*.   In this paragraph, we prove Theorem 1.2. Consider an admissible measure μ on a hyperbolic group, with superexponential tails and satisfying Anc_*, we will show that it satisfies pre-Ancona inequalities. We have to show that, for any points x, y, z on a geodesic in this order with $n \leq d(x, y), d(y, z) \leq 100n$, the Green function $G(x, z; B(y, n)^c)$ decays superexponentially fast in terms of n.

We express things in terms of operators. Let $P = P_\mu$ be the operator associated to μ. We decompose P as $A_n + B_n$ where A_n corresponds to jumps of size at most $n/2$, and B_n to the bigger jumps. On ℓ^2, they satisfy $\|A_n\| \leq \|P\| \leq \mu(\Gamma)$ (which is finite since μ has well-defined tails), and $\|B_n\|$ decays superexponentially fast in terms of n by assumption.

Let us fix a constant C_0. The Green function $G(x, z)$ is the sum of the weights $\pi(\tau)$ of all paths τ from x to z. The contribution of paths with length at most $C_0 n$ is

$$\sum_{k=0}^{C_0 n} P^k \delta_z(x) = \sum_{k=0}^{C_0 n} (A_n + B_n)^k \delta_z(x) \leq \sum_{k=0}^{C_0 n} \|(A_n + B_n)^k\|$$

$$\leq \sum_{k=0}^{C_0 n} \sum_{\ell=0}^{k} \binom{k}{\ell} \|A_n\|^\ell \|B_n\|^{k-\ell}.$$

By Anc_*, there exists $r > 1$ such that the measure $r\mu$ has a finite Green function. The contribution to $G(x, z)$ of paths longer than $C_0 n$ is

$$\sum_{k>C_0 n} p^k(x, z) \leq r^{-C_0 n} \sum_{k>C_0 n} r^k p^k(x, z) \leq r^{-C_0 n} G_{r\mu}(x, z).$$

The quantity $G_{r\mu}(x, z)$ grows at most exponentially in terms of n, thanks to Harnack inequality (2.2) and since $d(x, z) \leq 200n$. Hence, we obtain from some constant D_0 independent of C_0

$$(4.1) \qquad G(x, z) \leq \sum_{k=0}^{C_0 n} \sum_{\ell=0}^{k} \binom{k}{\ell} \|A_n\|^\ell \|B_n\|^{k-\ell} + r^{-C_0 n} D_0^n.$$

Let us now estimate $G(x, z; B(y, n)^c)$. Consider a trajectory from x to z outside of $B(y, n)$ with jumps bounded by $n/2$. Putting geodesics between the successive points of the trajectory, one obtains a path from x to z avoiding $B(y, n/2)$. This path is exponentially long (since this is the case in hyperbolic space, to which the group can be compared thanks to Theorem 4.2). Hence, the number of jumps is at least $Ce^{\alpha n}/n \geq Ce^{\beta n}$. It follows that, among trajectories of length at most $C_0 n$, it is necessary to have a jump larger than $n/2$ if n is large enough. This shows that, in (4.1), the terms with $k = \ell$ (i.e., coming from A_n^k) do not contribute to $G(x, z; B(y, n)^c)$. This equation gives

$$G(x, z; B(y, n)^c) \leq \sum_{k=0}^{C_0 n} \sum_{\ell=0}^{k-1} \binom{k}{\ell} \|A_n\|^\ell \|B_n\|^{k-\ell} + r^{-C_0 n} D_0^n.$$

As $k - \ell \geq 1$, we can bound $\|B_n\|^{k-\ell}$ with $\|B_n\|$, yielding

$$G(x, z; B(y, n)^c) \leq \|B_n\| \sum_{k=0}^{C_0 n} \sum_{\ell=0}^{k-1} \binom{k}{\ell} \|A_n\|^\ell + r^{-C_0 n} D_0^n$$

$$\leq \|B_n\| \sum_{k=0}^{C_0 n} (\|A_n\| + 1)^k + r^{-C_0 n} D_0^n$$

$$\leq \|B_n\| \sum_{k=0}^{C_0 n} (\|P\| + 1)^k + r^{-C_0 n} D_0^n$$

$$\leq \|B_n\| D_1^{C_0 n} + r^{-C_0 n} D_0^n,$$

for some constant D_1 independent of C_0.

Let us complete the proof. Fix $K > 1$, we want to show that $G(x, z; B(y, n)^c) \leq 2K^{-n}$ if n is large enough. First, we choose C_0 with $r^{-C_0} D_0 < K^{-1}$, so that the second term in the previous equation is bounded by K^{-n}. Then, as $\|B_n\|$ decays superexponentially, we have $\|B_n\| D_1^{C_0 n} \leq K^{-n}$ if n is large enough.

REMARK 4.5. If the measure μ has finite support, the proof simplifies drastically since there is no trajectory from x to z with length at most $C_0 n$ avoiding $B(y, n)$. Hence, one gets a very simple proof of Ancona's original results [1] (most of the complexity is in fact hidden in Lemma 4.4).

4.3. *Ancona inequalities in the free group.* In this paragraph, we prove the second item of Theorem 1.3: in a free group, an admissible measure μ with superexponential tails and finite Green function satisfies Ancona inequalities. Since Ancona inequalities for finitely supported measures are trivial in the free group, the only difficulty comes from long jumps. The trick we will devise to handle those long jumps (replacing a trajectory involving a long jump by a longer trajectory with short jumps) will be used several times in the rest of the paper.

By Lemma 4.4, it suffices to show that μ satisfies pre-Ancona inequalities. Consider three points x, y, z on a geodesic in this order with $n \leq d(x, y), d(y, z) \leq 100n$, we want to show that $G(x, z; B(y, n)^c)$ is superexponentially small. We may assume without loss of generality that $y = e$. We will first give the proof assuming for simplicity that μ gives positive mass to every generator of the group.

Denote by $Z_0, \ldots, Z_N$ the finitely many connected components of $\Gamma - B(e, n/2)$, with $x \in Z_0$ and $z \in Z_N$. Let also $A_i = Z_i \cap (\Gamma - B(e, n))$.

Consider a trajectory $\tau = (x_0 = x, x_1, \ldots, x_{k-1}, x_k = z)$ of the random walk from x to z, avoiding $B(e, n)$. It cannot stay forever in A_0, let us say that the first jump outside of A_0 is from x_i to x_{i+1}. We associate to τ a modified trajectory $m(\tau)$ (again from x to z) as follows. Let a and b be different elements in the support of μ. Let τ_i be a geodesic from x_i to e, with length $n_i = |x_i|$, and let τ_{i+1} be a geodesic from e to x_{i+1}, with length $n_{i+1} = |x_{i+1}|$. We let

$$
\begin{aligned}
(4.2) \qquad m(\tau) = \big(x_0, \ldots, x_{i-1}, (\tau_i), a, a^{-1}, \ldots, a, a^{-1}, \\
b, b^{-1}, \ldots, b, b^{-1}, (\tau_{i+1}), x_{i+2}, \ldots, x_k = z\big),
\end{aligned}
$$

where we put n_i copies of a, a^{-1} and n_{i+1} copies of b, b^{-1}. The interest of this insertion is that the map $\tau \to m(\tau)$ is one-to-one: if one knows $m(\tau)$, then the number of a, a^{-1} following the first return to e gives n_i. In the same way, one can determine n_{i+1}. Removing the pieces of length $n_i - 1$ before the first return to e, and $n_{i+1} - 1$ after the last return to e, one recovers the initial trajectory τ.

To get $m(\tau)$, we removed a big jump of τ, and we added $3(n_i + n_{i+1})$ jumps of length 1 (with weight uniformly bounded from below, by a constant C_0^{-1}). We obtain

$$
\pi\big(m(\tau)\big) \geq \pi(\tau) C_0^{-3(n_i + n_{i+1})} / \pi(x_i, x_{i+1}).
$$

For any constant K, there exists C_K such that $\pi(e, u) = \mu(u) \leq C_K K^{-|u|}$ since μ has superexponential tails. Hence, we get

$$
\pi(\tau) \leq \pi\big(m(\tau)\big) C_0^{3(n_i + n_{i+1})} C_K K^{-d(x_i, x_{i+1})}.
$$

Since x_i and x_{i+1} belong to different connected components of $\Gamma - B(e, n/2)$, we have $d(x_i, x_{i+1}) \geq |x_i| + |x_{i+1}| - n$. As $|x_i| \geq n$ and $|x_{i+1}| \geq n$, this gives $d(x_i, x_{i+1}) \geq (|x_i| + |x_{i+1}|)/2 = (n_i + n_{i+1})/2$. We get

$$
\pi(\tau) \leq \pi\big(m(\tau)\big) C_0^{3(n_i + n_{i+1})} C_K K^{-(n_i + n_{i+1})/2}.
$$

If K is large enough so that $C_0^3 K^{-1/4} \leq 1$, we obtain

$$
\pi(\tau) \leq \pi\big(m(\tau)\big) C_K K^{-(n_i + n_{i+1})/4} \leq \pi\big(m(\tau)\big) C_K K^{-n/2}.
$$

The map $\tau \mapsto m(\tau)$ is one-to-one. Summing over all trajectories from x to z outside of $B(e, n)$, we obtain

$$
G\big(x, z; B(e, n)^c\big) \leq C_K K^{-n/2} G(x, z).
$$

Since $d(x, z) \leq 200n$, we have $G(x, z) \leq C^n$ by Harnack inequalities (2.2). As K can be arbitrarily large, this shows that $G(x, z; B(e, n)^c)$ is smaller than any exponential, as desired. This completes the proof of pre-Ancona inequalities when μ gives positive mass to all generators.

In the general case, one has to tweak the definition of the modified trajectory $m(\tau)$ to ensure that $m(\tau)$ has positive weight, while retaining the injectivity of the map $\tau \mapsto m(\tau)$. One can, for instance, proceed as follows. To each generator s, let us associate a path σ_s from e to s with $\pi(\sigma_s) > 0$—such a path exists since μ is admissible. Then, in the definition of $m(\tau)$, one replaces the geodesic $\tau_i = s_1 \cdots s_{n_i}$ with the concatenation $\tilde{\tau}_i$ of the paths $\sigma_{s_1} \cdots \sigma_{s_{n_i}}$. In the same way, one replaces τ_{i+1} with the corresponding path $\tilde{\tau}_{i+1}$. Note that $\pi(\tilde{\tau}_i) \geq C_1^{-n_i}$ and $\pi(\tilde{\tau}_{i+1}) \geq C_1^{-n_{i+1}}$ for some constant C_1, since the lengths of $\tilde{\tau}_i$ and $\tilde{\tau}_{i+1}$ are bounded, respectively, by Cn_i and Cn_{i+1}.

A problem that may appear with this construction is that the first return to e in $m(\tau)$ might happen before the end of $\tilde{\tau}_i$, so that the reconstitution of τ from $m(\tau)$ is problematic. To avoid this problem, one may add a loop γ from e to itself, with $\pi(\gamma) > 0$, that does not appear when one concatenates paths σ_s along a geodesic segment. In the end, one chooses for $m(\tau)$ the trajectory

$$(4.3) \quad \begin{aligned} \big(x_0, \ldots, x_{i-1}, (\tilde{\tau}_i), (\gamma), (\alpha), \ldots, (\alpha), (\beta), \ldots, (\beta), \\ (\gamma), (\tilde{\tau}_{i+1}), x_{i+2}, \ldots, x_k = z\big), \end{aligned}$$

where α and β are two fixed distinct loops from e to e with positive weight, and one puts $|\tilde{\tau}_i|$ terms α and $|\tilde{\tau}_{i+1}|$ terms β. By construction, $\tau \mapsto m(\tau)$ is one-to-one and $\pi(m(\tau)) \geq \pi(\tau)C_2^{n_i + n_{i+1}}/\pi(x_i, x_{i+1})$ for some constant C_2. The rest of the argument goes through.

4.4. *Ancona inequalities for symmetric measures.*

In this paragraph, we prove the first item of Theorem 1.3: in a hyperbolic group, a symmetric admissible measure μ with superexponential tails and finite Green function satisfies Ancona inequalities. By Lemma 4.4, it suffices to show that it satisfies pre-Ancona inequalities. Consider three points x, y, z on a geodesic in this order with $n \leq d(x, y), d(y, z) \leq 100n$; we want to show that $G(x, z; B(y, n)^c)$ is superexponentially small. We may assume without loss of generality that $y = e$.

The proof follows the strategy in [7], Theorem 2.3: we will construct several barriers so that most trajectories from x to z will visit them. The construction is made in $\mathbb{H}^m$, using an approximate embedding Ψ of Γ inside $\mathbb{H} = \mathbb{H}^m$ given by Theorem 4.2. We will think of $\mathbb{H}^m$ using the model of the unit ball in $\mathbb{R}^m$, hence its boundary is identified with the unit sphere S^{m-1}. We denote by O the center of the unit ball in $\mathbb{R}^m$. Changing the generators of the group if necessary, we may assume that μ gives positive mass to all of them. We will need to choose at some point in the proof some very small ε, and we will denote by C a generic constant that does not depend on ε.

We will use the following easy lemma of hyperbolic geometry.

LEMMA 4.6.   *There exist $\alpha > 0$ and $C > 0$ with the following property*: *for any points a and b in a ball $B_{\mathbb{H}}(u, |u|_{\mathbb{H}}/9)$ of $\mathbb{H}^m$, the angle between $[Oa]$ and $[Ob]$ is at most $Ce^{-\alpha |u|_{\mathbb{H}}}$.*

The hyperbolic geodesic from $\Psi(x)$ to $\Psi(z)$ can be extended biinfinitely. Composing Ψ with a hyperbolic isometry, we can assume that the center O of the unit ball in $\mathbb{R}^m$ belongs to this geodesic, and that $\Psi(e)$ is at a bounded distance of O. Let ξ denote the limit in negative time of this geodesic.

To an angle $\theta \in (0, \pi)$, we associate the union $Y(\theta)$ of all semiinfinite geodesics $[O\zeta)$ (with $\zeta \in S^{m-1}$) making an angle θ with $[O\xi)$ (its boundary at infinity is the set of points of S^{m-1} at distance θ of ξ). This is the boundary of a cone based at O. Let $Z(\theta)$ be the union of all hyperbolic balls $B_{\mathbb{H}}(u, |u|_{\mathbb{H}}/10)$ for $u \in Y(\theta)$. This is a thickening of $Y(\theta)$, thicker and thicker close to infinity. It cuts $\mathbb{H}^m$ into two connected components.

LEMMA 4.7.   *If u and v are two points in the two components of $\mathbb{H}^m - Z(\theta)$, one has*

$$d_{\mathbb{H}}(u, v) \geq \big(|u|_{\mathbb{H}} + |v|_{\mathbb{H}}\big)/11.$$

PROOF.   The hyperbolic geodesic from u to v intersects $Y(\theta)$ at a single point w. It satisfies $d_{\mathbb{H}}(u, v) = d_{\mathbb{H}}(u, w) + d_{\mathbb{H}}(w, v)$. By assumption, $u \notin B_{\mathbb{H}}(w, |w|_{\mathbb{H}}/10)$, hence $d_{\mathbb{H}}(u, w) \geq |w|_{\mathbb{H}}/10$. Trivially, $d_{\mathbb{H}}(u, w) \geq |u|_{\mathbb{H}} - |w|_{\mathbb{H}}$. For any $t \in [0, 1]$, we obtain

$$d_{\mathbb{H}}(u, w) \geq t|w|_{\mathbb{H}}/10 + (1 - t)\big(|u|_{\mathbb{H}} - |w|_{\mathbb{H}}\big).$$

Let $t = 10/11$, so that the terms involving $|w|_{\mathbb{H}}$ cancel each other. We are left with $d_{\mathbb{H}}(u, w) \geq |u|_{\mathbb{H}}/11$. Since an analogous estimate is true for v, this completes the proof. $\square$

Let $A(\theta) = B(e, n)^c \cap \Psi^{-1}(Z(\theta)) \subset \Gamma$ be the set of points of Γ outside of $B(e, n)$ whose image under Ψ belongs to $Z(\theta)$. The previous lemma shows that, if a trajectory in Γ jumps past $A(\theta)$, it has to make a big jump.

Let $N = \lfloor e^{\varepsilon n} \rfloor$. In $X = [0, \pi]$, let $X_i = [(2i - 1)/N, 2i/N]$ for $1 \leq i \leq N$. For any $\theta_i \in X_i$ and $\theta_{i+1} \in X_{i+1}$, the visual angle from O between two points in $Y(\theta_i)$ and $Y(\theta_{i+1})$ is at least $e^{-\varepsilon n}$. It follows from Lemma 4.6 that, if ε is small enough and if n is large enough, the angle between two points in $Z(\theta_i)$ and $Z(\theta_{i+1})$ is at least $e^{-\varepsilon n}/2$. This shows in particular that $A(\theta_i)$ and $A(\theta_{i+1})$ are disjoint.

LEMMA 4.8.   *If ε is small enough, there exist angles $\theta_i \in X_i$ such that, for all $0 \leq i \leq N$,*

$$(4.4) \qquad\qquad \sum_{u \in A_i, v \in A_{i+1}} G(u, v)^2 \leq 1/4,$$

where G is the Green function associated to μ and we denoted $A_0 = \{x\}$, $A_{N+1} = \{z\}$ and $A_i = A(\theta_i)$ for $1 \le i \le N$.

This lemma shows that one can choose barriers so that the weight of trajectories going from one barrier to the next is small. This will guarantee that trajectories visiting all barriers have a superexponentially small weight. It will remain to handle trajectories jumping past barriers—we will use Lemma 4.7 to show that the jumps have to be large, implying that these trajectories contribute again with a very small weight thanks to the argument of Section 4.3.

PROOF OF LEMMA 4.8. The proof is similar to that of Lemma 2.6 in [7]; the difference is that we are considering thicker barriers. For $a \in \Gamma$, let $X_i(a)$ be the set of angles $\theta \in X_i$ such that $a \in A(\theta)$. If one shows that

$$(4.5) \qquad \mathrm{Leb}(X_i(a)) \le Ce^{-\alpha|a|}$$

for some α independent of ε, the remaining part of the argument of [7] will apply verbatim. We sketch very quickly the rest of the argument in [7] for the convenience of the reader.

Using hyperbolicity, one checks that a supermultiplicative function H with $\sum_{x \in \Gamma} H(e, x) < \infty$ has bounded sum on any sphere $\mathbb{S}^k$, that is, $\sum_{x \in \mathbb{S}^k} H(e, x) \le C$ uniformly in k, where C does not depend on H. This estimate applies to $H_r(e, x) = G_{r\mu}(e, x)G_{r\mu}(x, e)$ for any $r < 1$. Letting r tend to 1 and using the symmetry of μ, we obtain $\sum_{x \in \mathbb{S}^k} G(e, x)^2 \le C$. Hence, the function $G(e, x)$ is not in $\ell^2(\Gamma)$, but close. In particular, if A is a subset such that $\mathrm{Card}(A \cap \mathbb{S}^k)$ is exponentially smaller than $\mathbb{S}^k$, one expects that typically $\sum_{x \in A} G(e, x)^2$ will be finite (and small if A is thin enough). Of course, this might not be true for all such subsets A, but it will be true for most subsets A in a suitable sense. The lemma is proved by showing that, if one chooses θ_i randomly in X_i, then the estimate (4.4) holds with positive probability. This follows from the combination of inequality (4.5) with the estimate $\sum_{x \in \mathbb{S}^k} G(e, x)^2 \le C$.

It remains to prove (4.5). Since distances in the group and in hyperbolic space are equivalent, it is sufficient to show the corresponding estimate in $\mathbb{H}$, that is, for all $u \in \mathbb{H}$,

$$\mathrm{Leb}\{\theta : u \in Z(\theta)\} \le Ce^{-\alpha|u|_\mathbb{H}}.$$

For $u \in Z(\theta)$, there exists $v \in Y(\theta)$ such that $d_\mathbb{H}(u, v) \le |v|_\mathbb{H}/10$. Since $|v|_\mathbb{H}/10 \le (d_\mathbb{H}(u, v) + |u|_\mathbb{H})/10$, we get $d_\mathbb{H}(u, v) \le |u|_\mathbb{H}/9$, that is, $v \in B_\mathbb{H}(u, |u|_\mathbb{H}/9)$. Lemma 4.6 shows that the trace at infinity of this ball gives rise to an exponentially small angle. This completes the proof. $\square$

Let us prove the pre-Ancona inequalities. The Green function $G(x, z; B(e, n)^c)$ is the sum of the weights $\pi(\tau)$ of the trajectories τ from x to z avoiding $B(e, n)$.

We will say that such a trajectory is walking if it visits in this order the barriers $A_1, \ldots, A_N$ constructed in Lemma 4.8, and jumping otherwise.

Decomposing walking trajectories according to their first visits to the barriers, we get that their contribution to $G(x, z; B(e, n)^c)$ is bounded by

$$\sum_{a_1 \in A_1, \ldots, a_N \in A_N} G(x, a_1) G(a_1, a_2) \cdots G(a_{N-1}, a_N) G(a_N, z).$$

Using the estimate (4.4) on barriers and Cauchy–Schwarz inequality, one shows that this is bounded by $2^{-N} \leq 2^{-e^{\varepsilon n} + 1}$ (see the beginning of the proof of Lemma 2.6 in [7]). Hence, the contribution of walking trajectories is smaller than any exponential, as desired.

Consider now a jumping trajectory $\tau = (x_0 = x, x_1, \ldots, x_{k-1}, x_k = z)$, and assume that the first jump past a barrier happens at index i, from x_i to x_{i+1}. One associates to τ a modified trajectory $m(\tau)$ as in Section 4.3 [see equation (4.2) there—as we assume that μ gives positive weight to the generators, there is no need to use the more complicated definition (4.3)]. Lemma 4.7 shows that there exists a constant C such that $d(x_i, x_{i+1}) \geq C^{-1}(|x_i| + |x_{i+1}|)$. This is sufficient for all the computations of Section 4.3. It follows that the contribution of jumping trajectories is smaller than any exponential, as desired.

4.5. *Ancona inequalities in Fuchsian groups.*

In this paragraph, we prove the third item of Theorem 1.3: an admissible measure μ with superexponential tails and finite Green function on a cocompact lattice Γ of $\mathrm{PSL}(2, \mathbb{R})$ satisfies Ancona inequalities. Since the argument follows rather closely the previous subsection, we will only sketch the argument. Note that Γ is quasi-isometric with $\mathbb{H}^2$, giving an identification of the boundary $\partial \Gamma$ with the circle S^1. The planarity of $\mathbb{H}^2$ will be essential.

Again, we want to prove pre-Ancona inequalities between points x, y and z with $n \leq d(x, y), d(y, z) \leq 100n$, and we may assume that $y = e$. As in the previous subsection, we will construct several barriers between x and z, and treat separately trajectories that visit all the barriers (walking trajectories) and trajectories that jump past a barrier (jumping trajectories).

The basic ingredient for the barriers is constructed in [7], Appendix A: it is shown there that, for any finite family of disjoint subintervals $I^{(1)}, \ldots, I^{(N)}$ of S^1, one can find for $1 \leq i \leq N$ paths $X_n^{(i)}$ in the Cayley graph of Γ starting from e such that:

- One has $d(X_k^{(i)}, X_{k+1}^{(i)}) \leq 1$.
- The path $X_k^{(i)}$ converges to a point in $I^{(i)}$ when $k \to \infty$.
- There exist $\alpha > 0$ and $C > 0$ such that

$$(4.6) \quad G(e, X_k^{(i)}) \leq C e^{-\alpha k} \quad \text{and} \quad G(X_k^{(i)}, X_\ell^{(j)}) \leq C e^{-\alpha(k+\ell)} \qquad \text{for all } i \neq j.$$

- For some $s > 0$, one has $d(e, X_k^{(i)}) \sim sk$.

The constant C in the third item depends on N, while the other constants do not. The paths $X_k^{(i)}$ are constructed as typical trajectories of *another* (symmetric) random walk. The inequalities for G only rely on the supermultiplicativity (2.3) of the Green function of μ (and a version of Kingman's ergodic theorem)—in particular, the finiteness of the support of μ is not required.

Given such trajectories, one can replace each point $X_k^{(i)}$ by a ball $B(X_k^{(i)}, C)$ of some fixed radius C. This yields barriers that random walks with finite range cannot avoid, as in [7]. The inequalities in (4.6) guarantee that such barriers satisfy an inequality similar to (4.4). However, such a thickening does not imply that a jump past the barrier has to be long. Let us define a thicker barrier by $Z_i = \bigcup_k B(X_k^{(i)}, ck)$, where $c \leq 1$ is a suitably small constant, and let $A_i = Z_i \cap (\Gamma - B(e, n))$.

As in Lemma 4.7, one shows that jumps above such barriers have to be long. It follows that jumping trajectories will give a contribution to $G(x, z; B(e, n)^c)$ that is smaller than any exponential, as in Section 4.3.

To control the contribution of walking trajectories, it only remains to prove that an inequality similar to (4.6) holds: if n is large enough,

$$(4.7) \qquad \sum_{u \in A_i, v \in A_j} G(u, v)^2 \leq 1/4.$$

To prove this estimate, consider two points u and v in A_i and A_j. They belong to balls $B(X_k^{(i)}, ck)$ and $B(X_\ell^{(j)}, c\ell)$. Note first that

$$n \leq |u| \leq |X_k^{(i)}| + ck \leq (1 + c)k.$$

In particular, $k \geq n/2$. In the same way, $\ell \geq n/2$. Thanks to Harnack inequalities (2.2), we have

$$G(u, v) \leq C_0^{d(u, X_k^{(i)})} C_0^{d(X_\ell^{(j)}, v)} G(X_k^{(i)}, X_\ell^{(j)}) \leq C_0^{ck+c\ell} C e^{-\alpha(k+\ell)}.$$

If c is small enough, this is bounded by $C e^{-\alpha(k+\ell)/2}$. Hence, we get

$$\sum_{u \in A_i, v \in A_j} G(u, v)^2 \leq C \sum_{k,\ell \geq n/2} \operatorname{Card} B(X_k^{(i)}, ck) \operatorname{Card} B(X_\ell^{(j)}, c\ell) C e^{-\alpha(k+\ell)}.$$

If c is small enough, $\operatorname{Card} B(X_k^{(i)}, ck) = \operatorname{Card} B(e, ck)$ grows at most like $e^{\alpha k/2}$. The estimate (4.7) follows for large n.

4.6. *Strong Ancona inequalities.*

The proof of Theorem 1.4 on the asymptotics of transition probabilities involves a reinforcement of Ancona inequalities, called strong Ancona inequalities and defined as follows.

DEFINITION 4.9. An admissible measure μ with finite Green measure on a hyperbolic group satisfies strong Ancona inequalities if it satisfies Ancona inequalities and, additionally, there exist constants $C > 0$ and $\rho > 0$ such that, for all points x, x', y, y' whose configuration is approximated by a tree as follows:

$$
\begin{array}{ll}
y & \\
\quad y' & \xrightarrow{\hspace{5cm}} \quad x \\
& \xleftarrow{\hspace{5cm}} \quad x' \\
& \geq n
\end{array}
$$
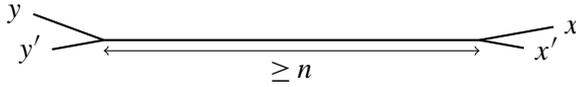
one has

(4.8) $$\left| \frac{G(x,y)/G(x',y)}{G(x,y')/G(x',y')} - 1 \right| \leq Ce^{-\rho n}.$$

Usual Ancona inequalities ensure that $(G(x,y)/G(x',y))/(G(x,y')/G(x',y'))$ [the quantity on the left-hand side of (4.8)] is bounded from above and from below. Strong Ancona inequalities strengthen this by saying that it is exponentially close to 1, in terms of the distance between $\{x, x'\}$ and $\{y, y'\}$.

In this paragraph, we will prove the following theorem.

THEOREM 4.10. *In a hyperbolic group* Γ, *consider an admissible measure* μ *with finite Green function and superexponential tails. Assume that* μ *satisfies pre-Ancona inequalities. Then it satisfies strong Ancona inequalities.*

Quantitative inequalities such as strong Ancona inequalities are instrumental to get asymptotics of transition probabilities. Indeed, the following holds. Consider an admissible symmetric probability measure μ on a hyperbolic group, let R denote the inverse of the spectral radius of the corresponding random walk, and assume that the measures $r\mu$ (for $1 \leq r \leq R$) satisfy strong Ancona inequalities, uniformly in r (i.e., with the same C and the same ρ). If the random walk generated by μ is aperiodic, it follows that $p^n(x, y) \sim C(x, y) R^{-n} n^{-3/2}$ for all $x, y \in \Gamma$. If μ is periodic, this is true for even n (resp., odd n) if the distance from x to y is even (resp., odd). This statement follows from [8], Theorem 9.1 and [7], Theorem 3.1.

PROOF OF THEOREM 1.4. Consider an admissible symmetric probability measure μ with superexponential tails in a hyperbolic group Γ. Let R denote the inverse of its spectral radius.

It follows from the discussion in the previous paragraph that, to prove Theorem 1.4, it suffices to prove strong Ancona inequalities for the measures $r\mu$, uniformly in $1 \leq r \leq R$. Pre-Ancona inequalities have been proved in Section 4.4 for each of those measures, hence they also satisfy strong Ancona inequalities by Theorem 4.10. The only remaining problem is the uniformity of those inequalities for $1 \leq r \leq R$. One checks in the proof of Theorem 4.10 that the constants C and ρ one obtains only depend on the constants in the pre-Ancona inequalities and in the Harnack inequalities. The pre-Ancona inequalities for $R\mu$ imply the same

inequalities for $r\mu$ for any r, since $r\mu \leq R\mu$. Hence, the pre-Ancona inequalities are uniform. Moreover, it is clear that the Harnack inequality are also uniform in r. $\square$

The rest of this subsection is devoted to the proof of Theorem 4.10. The argument dates back to Anderson and Schoen [2]. For finitely supported measures, the methods of [2] were adapted to the free group by Ledrappier [10], and then to any hyperbolic group by Izumi, Neshveyev and Okayasu [9]. The idea is to define a sequence of shrinking domains on which two given positive harmonic functions (with a common normalization) have to be closer and closer, by an inductive argument: one shows that two positive harmonic functions defined on one of those domains have a common significant part on a smaller domain. One can then subtract this common part to both functions in the smaller domain, and repeat the argument. In particular, one always works with positive harmonic functions, but defined on smaller and smaller domains.

While we will essentially follow the same strategy, the difficulty in the case of infinitely supported measures is that harmonicity becomes a global property, involving the whole group: it will not be possible to work with functions defined only on subdomains, we will need to keep track of the behavior of functions in the whole group. We will retain positivity in the smaller domains, but we will also need quantitative controls everywhere in the group.

The proof will involve not only global Ancona inequalities, but also Ancona inequalities for Green functions restricted to some classes of domains (as defined in Section 2).

DEFINITION 4.11. Let H_0 be a constant. Let $[x, z]$ be a geodesic in Γ, and let $y \in [x, z]$. We say that a subset Ω of Γ is H_0-hourglass-shaped around x, y, z if, for any $w \in [x, z]$, the ball $B(w, H_0 + d(w, y)/2)$ is included in Ω.

The proof of Ancona inequalities from pre-Ancona inequalities (that we described briefly after Lemma 4.4) still works in H_0-hourglass-shaped domains, since it shows that most trajectories flow along the hourglass. This implies the following lemma (this is Theorem 4.1 in [8]).

LEMMA 4.12. *Consider an admissible measure μ satisfying pre-Ancona inequalities in a hyperbolic group. Let H_0 be large enough. There exists $C > 0$ such that, for any domain Ω which is H_0-hourglass-shaped around three points x, y, z on a geodesic (in this order), the Green function relative to Ω satisfies Ancona inequalities, that is,*

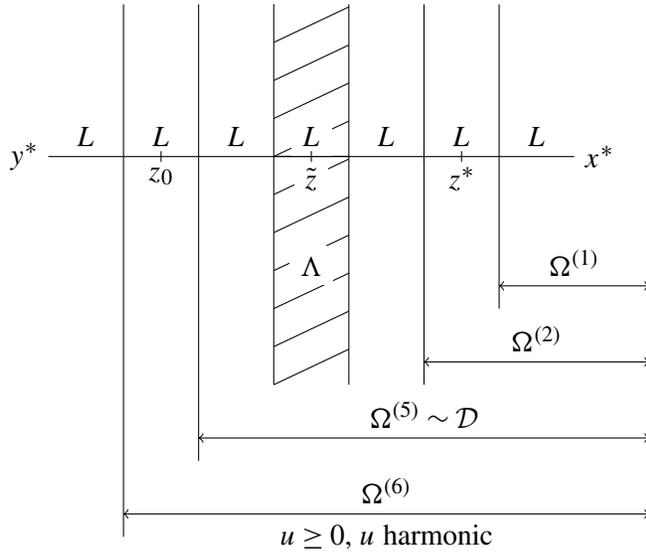$$G(x, z; \Omega) \leq C G(x, y; \Omega) G(y, z; \Omega).$$

FIG. 1. *The domains in Lemma 4.13.*

From this point on, we fix an admissible measure μ with superexponential tails, which satisfies pre-Ancona inequalities. We will prove that it satisfies strong Ancona inequalities. We fix the constant H_0 given by Lemma 4.12 for this measure.

The next lemma gives the basic inductive step for the proof of Theorem 4.10. For $u, v, z \in \Gamma$, we write $(u, v)_z$ for their Gromov product, given by $(u, v)_z = (d(u, z) + d(v, z) - d(u, v))/2$. This is essentially the length of the part that is common to two geodesics $[z, u]$ and $[z, v]$.

LEMMA 4.13. *There exists $C_1 > 1$ such that, for any $D > 0$, the following holds if L is a large enough even integer. Consider a geodesic segment γ between two points x^* and y^*, of length $7L$. Let $\Omega^{(j)} = \{z : (y_*, z)_{x_*} \le jL\}$ for $1 \le j \le 6$ (this is essentially the set of points whose projection on γ is at distance at most jL of x^*) and let z^* be the point at distance $3L/2$ of x^* on γ. Let $\mathcal{H}$ be the set of functions $u : \Gamma \to \mathbb{R}$ satisfying the following properties:*

1. *the function u is positive on $\Omega^{(6)}$;*
2. *for all $z \in \Gamma$, one has $|u(z)| \le D^{d(z, z^*)} u(z^*)$;*
3. *the function u is harmonic on $\Omega^{(6)}$, that is, $u(z) = \sum_{w \in \Gamma} p(z, w) u(w)$ for all $z \in \Omega^{(6)}$ (note that the previous property ensures that this sum is well defined, since μ has superexponential tails);*
4. *the function $|u(z)|$ is bounded by a finite linear combination of functions $G(z, t_i)$.*

Then there exists a domain $\mathcal{D}$, included in $\Omega^{(6)}$ and including $\Omega^{(5)}$ such that, for all $z \in \Omega^{(1)}$, for all $u \in \mathcal{H}$,

$$C_1^{-1} \leq \frac{u(z)}{G(z, z^*; \mathcal{D})u(z^*)} \leq C_1.$$

Note that the Green function $G(z, z^*; \mathcal{D})$ satisfies a Harnack inequality on $\Omega^{(1)}$, of the form $G(z, z^*; \mathcal{D}) \leq C_0^{d(z,z')} G(z', z^*; \mathcal{D})$ where the constant C_0 only depends on μ. Therefore, the conclusion of the lemma implies that, for all $z, z' \in \Omega^{(1)}$, one has

$$u(z) \leq C_1^2 C_0^{d(z,z')} u(z').$$

This inequality should be compared to the second assumption on u, involving an arbitrarily large constant D. Hence, the lemma asserts that a weak growth control implies in fact a much stronger growth control (but on a smaller domain). This remark will be crucial to check inductively the assumptions of the lemma.

PROOF OF LEMMA 4.13.   Let $D > 0$ be fixed, we will show the conclusion of the lemma if L is large enough. We will write $o_L(1)$ for a term that may depend on D and L, and tends to 0 when L tends to infinity (with fixed D). We will also write C for generic constants that do not depend on D. In particular, the constants in various Harnack inequalities will be denoted by C_0.

STEP 1.   *There exists a domain $\mathcal{D}$, containing $\Omega^{(5)}$ and contained in a fixed size neighborhood of $\Omega^{(5)}$, such that for all $z, z' \in \mathcal{D}$ there exists a path in $\mathcal{D}$ from z to z' with weight at least $C_0^{-d(z,z')}$.*

PROOF.   The set $\Omega^{(5)}$ is convex, up to a constant K_0: any geodesic between two points in $\Omega^{(5)}$ is contained in its neighborhood $B(\Omega^{(5)}, K_0)$. Let K_1 be such that any generator can be written as the product of at most K_1 elements in the support of μ. Between any points $z, z' \in \Omega^{(5)}$, there exists a path staying in $B(\Omega^{(5)}, K_0 + K_1)$ of length at most $K_1 d(z, z')$ whose transitions are all in a finite subset of the support of μ. The weight of this path is therefore at least $\bar{C}_0^{-d(z,z')}$, for some $\bar{C}_0 > 0$.

For all $z \in B(\Omega^{(5)}, K_0 + K_1)$, choose a point ζ_z in $\Omega^{(5)}$ with $d(z, \zeta_z) \leq K_0 + K_1$, and choose two paths τ_z and τ_z', respectively, from z to ζ_z and from ζ_z to z, with uniformly bounded length, and weight uniformly bounded from below. Let finally $\mathcal{D}$ be the union of all the (points visited by the) paths τ_z and τ_z'.

This set satisfies the required properties. Indeed, fix w and $w' \in \mathcal{D}$, we construct a path from w to w' with weight at least $C_0^{-d(w,w')}$ as follows. First, let z be such that $w \in \tau_z \cup \tau_z'$, and z' be such that $w' \in \tau_{z'} \cup \tau_{z'}'$. We can go from w to ζ_z in $\tau_z \cup \tau_z' \subset \mathcal{D}$ with weight bounded from below, then from ζ_z to $\zeta_{z'}$ in $B(\Omega^{(5)}, K_0 +$

$K_1) \subset \mathcal{D}$ with weight at least $\bar{C}_0^{-d(\zeta_z,\zeta_{z'})} \geq C^{-1}\bar{C}_0^{-d(w,w')}$, and then from $\zeta_{z'}$ to w' in $\tau_{z'} \cup \tau'_{z'} \subset \mathcal{D}$ with weight bounded from below. The concatenation of these three paths stays in $\mathcal{D}$ and has weight at least $C_0^{-d(w,w')}$ for some C_0, as desired.   □

We deduce in particular of the properties of $\mathcal{D}$ that, for all $z, z' \in \mathcal{D}$,

$$(4.9) \qquad G(z, z^*; \mathcal{D}) \geq C_0^{-d(z',z^*)} G(z, z'; \mathcal{D})$$

since a path from z to z' can be extended in $\mathcal{D}$ by a path from z' to z^* with weight at least $C_0^{-d(z',z^*)}$.

Let u be a function in $\mathcal{H}$.

STEP 2.   *For all $z \in \Omega^{(2)}$,*

$$(4.10) \qquad u(z) = \sum_{w \in \Omega^{(6)} - \mathcal{D}} G(z, w; \mathcal{D})u(w) + o_L(1)G(z, z^*; \mathcal{D})u(z^*).$$

One interest of this formula is that the values of u appearing on the right-hand side are all positive since $w \in \Omega^{(6)}$.

PROOF OF STEP 2.   We start from z and follow the random walk given by μ until time n, stopping it when one exits $\mathcal{D}$. Since u is harmonic on $\mathcal{D}$, the average value of u at time n coincides with $u(z)$, that is,

$$(4.11) \qquad u(z) = \sum_{w \notin \mathcal{D}} G_{\leq n}(z, w; \mathcal{D})u(w) + \sum_{w \in \mathcal{D}} p^n(z, w; \mathcal{D})u(w),$$

where $G_{\leq n}(z, w; \mathcal{D})$ is the sum of the weights of all paths from z to w of length at most n that stay in $\mathcal{D}$ except maybe at the last step, and $p^n(z, w; \mathcal{D})$ is the same quantity but for paths of length exactly n. Note that $G_{\leq n}(z, w; \mathcal{D})$ converges to $G(z, w; \mathcal{D})$ when n tends to infinity.

By assumption, the function $|u|$ is bounded by a linear combination of functions $G(z, t_i)$. For each of those functions, $\sum_{w \in \Gamma} p^n(z, w)G(w, t_i)$ tends to 0 when n tends to infinity (since this is the sum of the weights of paths from z to t_i of length at least n). It follows that the last sum in (4.11) converges to 0 with n. If u were positive, one would readily deduce that $u(z) = \sum_{w \notin \mathcal{D}} G(z, w; \mathcal{D})u(w)$ by passing to the limit. However, since u can be negative on the complement of $\Omega^{(6)}$, we should be more careful. To justify the limit and equation (4.10), it suffices to show that

$$\sum_{w \notin \Omega^{(6)}} G(z, w; \mathcal{D})|u(w)| \leq o_L(1)G(z, z^*; \mathcal{D})u(z^*).$$

Denoting by z' the last point in $\mathcal{D}$ of a trajectory from z to w, this sum can be written as

$$\sum_{w \notin \Omega^{(6)}} \sum_{z' \in \mathcal{D}} G(z, z'; \mathcal{D})p(z', w)|u(w)|.$$

Bounding $|u(w)|$ by $u(z^*)D^{d(w,z^*)}$ and using inequality (4.9), we get that this is at most

$$\sum_{w \notin \Omega^{(6)}} \sum_{z' \in \mathcal{D}} G(z, z^*; \mathcal{D}) C_0^{d(z',z^*)} p(z', w) D^{d(w,z^*)} u(z^*).$$

The required factor $G(z, z^*; \mathcal{D})u(z^*)$ can be factorized out, one should show that the remaining term is $o_L(1)$. The measure μ has superexponential tails. Hence, for any K, one has $p(z', w) \le K^{-d(z',w)}$ if L is large enough (since the jump from z' to w has size at least $L/2$). Hence, it suffices to show that

$$\sum_{w \notin \Omega^{(6)}} \sum_{z' \in \mathcal{D}} C_0^{d(z',z^*)} D^{d(w,z^*)} K^{-d(z',w)} = o_L(1).$$

Let z_0 be the point on γ at distance $3L/2$ of y^*. By hyperbolicity, any geodesic segment from w to z' passes within bounded distance of z_0, and its length is at least $L/2$. Hence,

$$d(z', z^*) \le d(z', z_0) + d(z_0, z^*) \le d(z', w) + 7L \le d(z', w) + 14d(z', w)$$
$$= 15d(z', w).$$

Moreover, $d(w, z^*) \le d(w, z') + d(z', z^*) \le 16d(z', w)$. Writing $n = d(z', w)$, we deduce that the above sum is bounded by

$$\sum_{n=L/2}^{\infty} \mathrm{Card}\{(z' \in \mathcal{D}, w \notin \Omega^{(6)}) : d(z', w) = n\}(C_0^{15} D^{16} K^{-1})^n.$$

If z' and w are at distance n, they both belong to the ball $B(z_0, n + C)$. Hence, $\mathrm{Card}\{(z', w) : d(z', w) = n\}$ grows at most exponentially fast, let us say that it is bounded by C_2^n. If K was chosen so that $C_2 C_0^{15} D^{16} K^{-1} < 1$, the above series is converging, and can be made arbitrarily small by increasing L, as desired. $\quad\square$

STEP 3. *Define a domain $\Lambda = \Omega^{(4)} - \Omega^{(3)}$. For all $z \in \Omega^{(2)}$,*

$$u(z) = \sum_{w \in \Omega^{(6)} - \mathcal{D}} \sum_{w' \in \Lambda} G(z, w'; \mathcal{D}) G(w', w; \mathcal{D} - \Lambda) u(w)$$

(4.12)
$$+ o_L(1) G(z, z^*; \mathcal{D}) u(z^*) + o_L(1) u(z).$$

PROOF. We start from expression (4.10). By (2.1), every term $G(z, w; \mathcal{D})$ can be decomposed as

$$G(z, w; \mathcal{D}) = \sum_{w' \in \Lambda} G(z, w'; \mathcal{D}) G(w', w; \mathcal{D} - \Lambda) + G(z, w; \mathcal{D} - \Lambda),$$

by considering the last visit of a trajectory to Λ if it exists. We have to show that the contribution of the terms $G(z, w; \mathcal{D} - \Lambda)$ is negligible. Let us consider a trajectory

τ from z to w that does not visit Λ, it has to jump past Λ. Say that the first jump happens from a point w_i to a point w_{i+1}.

If $w_{i+1} = w$, that is, the trajectory has jumped directly out of $\mathcal{D}$, then we can use the same argument as in Step 2 since we are considering a trajectory ending with a very big jump. The same argument shows that the overall contribution of those trajectories to (4.10) is bounded by $o_L(1)G(z, z^*; \mathcal{D})u(z^*)$.

Assume now that $w_{i+1} \neq w$, and in particular $w_{i+1} \in \mathcal{D}$. Let $\tilde{z}$ be the middle point of Λ, located on γ at distance $7L/2$ of x^*. As in Section 4.3, we define a modified trajectory $m(\tau)$ by removing the big jump, and replacing it with two almost geodesic trajectories in $\mathcal{D}$ from w_i to $\tilde{z}$ and from $\tilde{z}$ to w_{i+1}. The construction of $\mathcal{D}$ in Step 1 ensures that one can find such trajectories, with positive weight. One also adds loops around $\tilde{z}$, counting the lengths of the trajectories from w_i to $\tilde{z}$ and from $\tilde{z}$ to w_{i+1}, to make sure that the map $\tau \mapsto m(\tau)$ is one-to-one. As in Section 4.3, one verifies that the weight of $m(\tau)$ is larger than the weight of τ [the ratio $\pi(m(\tau))/\pi(\tau)$ even tends to infinity when L tends to infinity]. Summing over all those trajectories, we get that their weight is bounded by $o_L(1)G(z, w; \mathcal{D})$.

It follows that the term we have to estimate, coming from (4.10), is bounded by

$$o_L(1) \sum_{w \in \Omega^{(6)} - \mathcal{D}} G(z, w; \mathcal{D})u(w).$$

Formula (4.10) shows that the sum is bounded by $u(z) + o_L(1)G(z, z^*; \mathcal{D})u(z^*)$. This completes the proof. $\square$

In expression (4.12), we can bound each factor $G(z, w'; \mathcal{D})$ using Ancona inequalities in the hourglass-shaped domain $\mathcal{D}$ if $z \in \Omega^{(1)}$. Indeed, a geodesic from $z \in \Omega^{(1)}$ to $w' \in \Lambda$ passes within bounded distance of z^* by hyperbolicity, and $\mathcal{D}$ is H_0-hourglass-shaped around z, z^*, w' if L is large enough. It follows from Lemma 4.12 that $G(z, w'; \mathcal{D}) = C_3^{\pm 1} G(z, z^*; \mathcal{D})G(z^*, w'; \mathcal{D})$ for some constant C_3 (this notation means that the ratio of those quantities belongs to $[C_3^{-1}, C_3]$). As all the relevant values $u(w)$ are positive, we obtain

$$u(z) = C_3^{\pm 1} G(z, z^*; \mathcal{D}) \sum_{w \in \Omega^{(6)} - \mathcal{D}} \sum_{w' \in \Lambda} G(z^*, w'; \mathcal{D})G(w', w; \mathcal{D} - \Lambda)u(w)$$
$$+ o_L(1)G(z, z^*; \mathcal{D})u(z^*) + o_L(1)u(z).$$

Applying again (4.12), but to the point $z^* \in \Omega^{(2)}$, we get that the double sum on the right-hand side of the first line is equal to $u(z^*) + o_L(1)u(z^*)$. This yields

$$u(z) = C_3^{\pm 1} G(z, z^*; \mathcal{D})u(z^*) + o_L(1)G(z, z^*; \mathcal{D})u(z^*) + o_L(1)u(z).$$

Let L be large enough so that the $o_L(1)$ terms are bounded by $\min(C_3^{-1}/2, 1/2)$. We obtain that the ratio between $u(z)$ and $G(z, z^*; \mathcal{D})u(z^*)$ is bounded from above and from below. This completes the proof of Lemma 4.13. $\square$

PROOF OF THEOREM 4.10.   Let us fix a large enough constant D (several conditions will appear in the proof below), and let L be given for this value of D by Lemma 4.13.

Starting with 4 points x, x', y, y' as in the statement of strong Ancona inequalities, we want to show that (4.8) holds. Let $\tilde{x}$ and $\tilde{y}$ denote the branching points of the tree between $\{x, x'\}$ and $\{y, y'\}$. We can without loss of generality assume that $d(\tilde{x}, \tilde{y})$ is of the form $7nL$ for some large integer n. We have to show that the functions $u_0(z) = G(z, y)/G(\tilde{x}, y)$ and $v_0(z) = G(z, y')/G(\tilde{x}, y')$ are exponentially close (in terms of n) in a domain containing x and x'.

Let γ be a geodesic of length $7nL$ from $\tilde{y}$ to $\tilde{x}$; we chop it into n pieces γ_i of length $7L$ (the piece γ_1 is closest to $\tilde{y}$). We will successively apply Lemma 4.13 along those pieces. We will denote by y_i^* and x_i^* the endpoints of γ_i, by z_i^* the point at distance $3L/2$ of x_i^* on γ_i, and by $\Omega_i^{(j)}$ the corresponding domains defined in Lemma 4.13 for $1 \leq j \leq 6$.

Harnack inequalities show that u_0 satisfies $|u_0(z)/u_0(z')| \leq C_0^{d(z,z')}$ for some constant C_0. In particular, if $D \geq C_0$, the function u_0 satisfies all the assumptions of Lemma 4.13 along the geodesic γ_1. We obtain a domain $\mathcal{D}_1$ (that does not depend on u_0) such that

$$(4.13) \qquad C_1^{-1} \leq \frac{u_0(z)}{G(z, z_1^*; \mathcal{D}_1)u_0(z_1^*)} \leq C_1,$$

for all $z \in \Omega_1^{(1)}$. Using (4.13) at the point $\tilde{x}$ and dividing, we get on $\Omega_1^{(1)}$

$$C_1^{-2} \leq \frac{u_0(z)}{G(z, z_1^*; \mathcal{D}_1)u_0(\tilde{x})/G(\tilde{x}, z_1^*; \mathcal{D}_1)} \leq C_1^2.$$

Let

$$\varphi_1(z) = \frac{1}{2C_1^2} \frac{G(z, z_1^*; \mathcal{D}_1)}{G(\tilde{x}, z_1^*; \mathcal{D}_1)} u_0(\tilde{x}).$$

We note that φ_1 depends on u_0 only through its value at $\tilde{x}$. By construction, we have on $\Omega_1^{(1)}$

$$(4.14) \qquad \varphi_1 \leq u_0/2 \leq C_1^4 \varphi_1.$$

In particular, the function $u_1 = u_0 - \varphi_1$ is positive on $\Omega_1^{(1)}$. It is also harmonic there. We will show that u_1 satisfies the assumptions of Lemma 4.13 with respect to the geodesic segment γ_2. Since assumption (4) is trivial, we only have to prove the growth control (2).

Let $z \in \Gamma$, we have to show that $|u_1(z)| \leq D^{d(z,z_2^*)}u_1(z_2^*)$. We start with the case $z \in \Omega_1^{(1)} - \{z_2^*\}$ (the case $z = z_2^*$ is trivial). By construction, $u_1(z) \geq 0$. Using (twice) (4.13), and thanks to Harnack inequality, we get

$$|u_1(z)| \leq u_0(z) \leq C_1 G(z, z_1^*; \mathcal{D}_1)u_0(z_1^*) \leq C_1 C_0^{d(z,z_2^*)} G(z_2^*, z_1^*; \mathcal{D}_1)u_0(z_1^*)$$

$$\leq C_1^2 C_0^{d(z,z_2^*)} u_0(z_2^*) \leq 2C_1^2 C_0^{d(z,z_2^*)} u_1(z_2^*).$$

If D is large enough so that $2C_1^2 C_0 \leq D$, we obtain $|u_1(z)| \leq D^{d(z,z_2^*)} u_1(z_2^*)$ for $z \in \Omega_1^{(1)} - \{z_2^*\}$, as desired. Assume now that $z \notin \Omega_1^{(1)}$. Thanks to Harnack inequalities,

$$G(z, z_1^*; \mathcal{D}_1) \leq G(z, z_1^*) \leq C_0^{d(z,z_1^*)} G(z_1^*, z_1^*) \leq C_2 C_0^{d(z,z_1^*)} G(z_1^*, z_1^*; \mathcal{D}_1)$$

for some $C_2 > 0$. Hence, $\varphi_1(z) \leq C_2 C_0^{d(z,z_1^*)} \varphi_1(z_1^*)$. As $\varphi_1(z_1^*) \leq u_0(z_1^*)$ by (4.14), we obtain

$$|u_1(z)| \leq |u_0(z)| + \varphi_1(z) \leq D^{d(z,z_1^*)} u_0(z_1^*) + C_2 C_0^{d(z,z_1^*)} u_0(z_1^*).$$

If D is large enough, this is bounded by $2D^{d(z,z_1^*)} u_0(z_1^*)$. Inequality (4.13) at $z = z_2^*$, combined with Harnack inequality, yields $u_0(z_1^*) \leq C_1 C_0^{d(z_1^*,z_2^*)} u_0(z_2^*)$. Since $u_0 \leq 2u_1$ on $\Omega_1^{(1)}$, we obtain

$$|u_1(z)| \leq 4C_1 D^{d(z,z_1^*)} C_0^{d(z_1^*,z_2^*)} u_1(z_2^*).$$

As $z \notin \Omega_1^{(1)}$, we have $d(z, z_2^*) \geq d(z, z_1^*) + L$, whereas $d(z_1^*, z_2^*) = 7L$. Hence,

$$|u_1(z)| \leq 4C_1 (C_0^7 D^{-1})^L D^{d(z,z_2^*)} u_1(z_2^*).$$

If D is large enough so that $4C_1 C_0^7 D^{-1} \leq 1$, we finally get $|u_1(z)| \leq D^{d(z,z_2^*)} u_1(z_2^*)$. This is the requested inequality.

We have shown that the function u_1 satisfies the assumptions of Lemma 4.13 along the geodesic segment γ_2. Hence, we may apply the same argument: we obtain a function φ_2 with $\varphi_2 \leq u_1/2 \leq C_1^4 \varphi_2$ on $\Omega_1^{(2)}$, only depending on u_1 through the value of $u_1(\tilde{x})$ [and, therefore, only depending on $u_0(\tilde{x})$]. Let $u_2 = u_1 - \varphi_2$, it again satisfies the assumptions of the lemma along γ_3, and we can continue the construction inductively.

In the end, we construct n functions $\varphi_1, \ldots, \varphi_n$ such that $u_0 = u_n + \varphi_1 + \cdots + \varphi_n$, only depending on $u_0(\tilde{x})$. As $u_k = u_{k-1} - \varphi_k \leq (1 - C_1^{-4}/2) u_{k-1}$, we have in particular $u_n \leq (1-\varepsilon)^n u_0$ on $\Omega_n^{(1)}$, for $\varepsilon = C_1^{-4}/2 > 0$. The same construction can be done starting from the function $v_0(z) = G(z, y')/G(\tilde{x}, y')$. Since $v_0(\tilde{x}) = u_0(\tilde{x}) = 1$, the functions φ_i that we get are the same. Hence, on $\Omega_n^{(1)}$,

$$|u_0(z) - v_0(z)| = |u_n(z) - v_n(z)| \leq (1-\varepsilon)^n (u_0(z) + v_0(z)).$$

Therefore,

$$|u_0(z)/v_0(z) - 1| \leq (1-\varepsilon)^n (u_0(z)/v_0(z) + 1).$$

This implies that $u_0(z)/v_0(z)$ is bounded by $(1 + (1-\varepsilon)^n)/(1 - (1-\varepsilon)^n) \leq 2/\varepsilon$, yielding

$$|u_0(z)/v_0(z) - 1| \leq C(1-\varepsilon)^n.$$

In other words,

$$\left| \frac{G(z, y)/G(\tilde{x}, y)}{G(z, y')/G(\tilde{x}, y')} - 1 \right| \le C(1 - \varepsilon)^n.$$

Using this inequality at $z = x$ and $z = x'$ (those points belong to $\Omega_n^{(1)}$), we get the conclusion of the theorem. $\square$

REFERENCES

[1] ANCONA, A. (1988). Positive harmonic functions and hyperbolicity. In *Potential Theory— Surveys and Problems* (*Prague*, 1987). *Lecture Notes in Math.* **1344** 1–23. Springer, Berlin. MR0973878

[2] ANDERSON, M. T. and SCHOEN, R. (1985). Positive harmonic functions on complete manifolds of negative curvature. *Ann. of Math.* (2) **121** 429–461. MR0794369

[3] BLACHÈRE, S., HAÏSSINSKY, P. and MATHIEU, P. (2011). Harmonic measures versus quasiconformal measures for hyperbolic groups. *Ann. Sci. Éc. Norm. Supér.* (4) **44** 683–721. MR2919980

[4] BONK, M. and SCHRAMM, O. (2000). Embeddings of Gromov hyperbolic spaces. *Geom. Funct. Anal.* **10** 266–306. MR1771428

[5] DYNKIN, E. B. (1969). The boundary theory of Markov processes (discrete case). *Uspehi Mat. Nauk* **24** 3–42. MR0245096

[6] GHYS, É. and DE LA HARPE, P., eds. (1990). *Sur les Groupes Hyperboliques D'après Mikhael Gromov. Progress in Mathematics* **83**. Birkhäuser, Boston, MA. MR1086648

[7] GOUËZEL, S. (2014). Local limit theorem for symmetric random walks in Gromov-hyperbolic groups. *J. Amer. Math. Soc.* **27** 893–928. MR3194496

[8] GOUËZEL, S. and LALLEY, S. P. (2013). Random walks on co-compact Fuchsian groups. *Ann. Sci. Éc. Norm. Supér.* (4) **46** 129–173. MR3087391

[9] IZUMI, M., NESHVEYEV, S. and OKAYASU, R. (2008). The ratio set of the harmonic measure of a random walk on a hyperbolic group. *Israel J. Math.* **163** 285–316. MR2391133

[10] LEDRAPPIER, F. (2001). Some asymptotic properties of random walks on free groups. In *Topics in Probability and Lie Groups*: *Boundary Theory. CRM Proc. Lecture Notes* **28** 117– 152. Amer. Math. Soc., Providence, RI. MR1832436

[11] SAWYER, S. A. (1997). Martin boundaries and random walks. In *Harmonic Functions on Trees and Buildings* (*New York*, 1995). *Contemp. Math.* **206** 17–44. Amer. Math. Soc., Providence, RI. MR1463727

[12] WOESS, W. (2000). *Random Walks on Infinite Graphs and Groups. Cambridge Tracts in Mathematics* **138**. Cambridge Univ. Press, Cambridge. MR1743100

IRMAR
CNRS UMR 6625
UNIVERSITÉ DE RENNES 1
35042 RENNES
FRANCE
E-MAIL: sebastien.gouezel@univ-rennes1.fr