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LIMIT THEOREMS FOR COUPLED INTERVAL MAPS

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We prove a local limit theorem for Lipschitz continuous observables on a weakly coupled lattice of piecewise expanding mixing interval maps. The core of the paper is a proof that the spectral radii of the Fourier-transfer operators for such a system are strictly less than 1. This extends the approach of [9] where the ordinary transfer operator was studied.

Keywords: Coupled map lattice; piecewise expanding map; spectral gap; local limit theorem.

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1. Results

This paper deals with the issue of probabilistic limit theorems in dynamical systems, i.e. limit theorems for the Birkhoff sums $S_n f = \sum_{k=0}^{n-1} f \circ T^k$, where T is a probability preserving transformation of a space X and $f : X \to \mathbb{R}$ is an appropriate measurable function. There are currently many techniques available to prove the central limit theorem $S_n f/\sqrt{n} \to \mathcal{N}(0, \sigma^2)$, let us mention for example elementary techniques, martingales, spectral arguments. On the other hand, if one is interested in the local limit theorem $\mu\{S_n f \in [a, b]\} \sim \frac{|b-a|}{\sigma\sqrt{2\pi n}}$, the scope of possible techniques is much more narrow: all known proofs rely on spectral analysis of transfer operators. Therefore, the class of systems for which a local limit theorem is proved is much smaller.

We are interested in limit theorems for coupled map lattices. The only previous result in this context is [1], where central limit theorem, moderate deviations principle and a partial large deviations principle were established under strong analyticity assumptions on the local map and the coupling. In this paper, we establish central and local limit theorems for coupled mixing interval maps under much weaker assumptions. More precisely, we study the same class of systems as in [9]. We emphasize on local limit theorem, since it is the most demanding result. But our method, relying on spectral analysis of transfer operators, gives other limit theorems, see Remark 1.4 below.

Let us recall the setup from [9]. Given a compact interval $I \subset \mathbb{R}$ we will consider the phase space $\Omega := I^{\mathbb{Z}^d}$. In the following we always assume without loss of generality that $I = [0,1]$.

The single site dynamics is given by a map $\tau : I \to I$. We assume τ to be a continuous, piecewise $\mathcal{C}^2$ map from I to I with singularities at $\zeta_1, \dots, \zeta_{N-1} \in (0,1)$ in the sense that τ is monotone and $\mathcal{C}^2$ on each component of $I \backslash \{\zeta_0 = 0, \zeta_1, \dots, \zeta_{N-1}, \zeta_N = 1\}$. We assume further that τ', τ'' are bounded, that $\inf |\tau'| > 2$ and that τ is mixing, which means that the Perron-Frobenius operator of τ acting on L^1_{dx} has no other unimodular eigenvalue than the simple eigenvalue 1. Next, we define the unperturbed dynamics $T_0 : \Omega \to \Omega$ by $[T_0(x)]_{\mathbf{p}} := \tau(x_{\mathbf{p}})$.

To define the perturbed dynamics we introduce couplings $\Phi_\varepsilon : \Omega \to \Omega$ of the form $\Phi_\varepsilon(\mathbf{x}) := \mathbf{x} + A_\varepsilon(\mathbf{x})$. We say that such a coupling has range r and strength ε if for all $\mathbf{k}, \mathbf{p}, \mathbf{q} \in \mathbb{Z}^d$

$$|(A_\varepsilon)_{\mathbf{p}}|_\infty \leq 2\varepsilon, \quad |(DA_\varepsilon)_{\mathbf{qp}}|_\infty \leq 2\varepsilon, \quad |\partial_{\mathbf{k}}(DA_\varepsilon)_{\mathbf{qp}}|_\infty \leq 2\varepsilon, \tag{1.1}$$

and $\partial_{\mathbf{p}} \Phi_{\varepsilon,\mathbf{q}} = 0$ whenever $|\mathbf{p} - \mathbf{q}| > r$. The diffusive nearest neighbor coupling used in [10], and in much of the numerical literature, is defined by

$$[\Phi_\varepsilon(\mathbf{x})]_{\mathbf{p}} = x_{\mathbf{p}} + \frac{\varepsilon}{2d} \sum_{|\mathbf{p}-\mathbf{q}|=1} (x_{\mathbf{q}} - x_{\mathbf{p}}), \quad \mathbf{p} \in \mathbb{Z}^d, \tag{1.2}$$

and it is a trivial example of such a coupling with range $r = 1$ and strength ε. The dynamics $T_\varepsilon : \Omega \to \Omega$ that we wish to investigate is then defined as

$$T_\varepsilon := \Phi_\varepsilon \circ T_0. \tag{1.3}$$

Let m denote Lebesgue measure on the interval I. The following result is proved in [9]:

Theorem 1.1. *For each $r \in \mathbb{N}$, there exists $\varepsilon_0(r) > 0$ such that, for any coupling Φ_ε of range r and strength $0 \leq \varepsilon \leq \varepsilon_0(r)$, there exists a unique measure μ_ε such that, for $m^{\otimes \mathbb{Z}^d}$-almost every point $\mathbf{x}$,*

$$\frac{1}{n} \sum_{k=0}^{n-1} \delta_{T_\varepsilon^k \mathbf{x}} \to \mu_\varepsilon. \tag{1.4}$$

This measure μ_ε in fact has many additional properties: it is the unique invariant measure in the class $\mathcal{B}$ of measures of bounded variation (that we will define later), it is exponentially mixing both in time and space, and the convergence (1.4) holds for μ-almost every point whenever μ is a measure of bounded variation.

In this paper, we prove the following theorem.

Theorem 1.2. *For each $r \in \mathbb{N}$, there exists $0 < \varepsilon_1(r) \leq \varepsilon_0(r)$ satisfying the following property. Let Φ_ε be a coupling of range r and strength $0 \leq \varepsilon \leq \varepsilon_1(r)$, and let μ_ε denote the corresponding invariant measure given by Theorem* 1.1*. Let $f : \Omega \to \mathbb{R}$ be a Lipschitz function depending on a finite number of coordinates, with $\int f \, d\mu_\varepsilon = 0$.*

Central limit theorem. There exists $\sigma^2 \geq 0$ such that $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f \circ T_\varepsilon^k$ converges in distribution to $\mathcal{N}(0, \sigma^2)$, with respect to the measure μ_ε. Moreover, $\sigma^2 = 0$ if and only if there exists a measurable function $u : \Omega \to \mathbb{R}$ such that $f = u - u \circ T_\varepsilon$ μ_ε-almost everywhere.

Local limit theorem. Assume in addition that, whenever $u : \Omega \to \mathbb{R}$ is measurable and $\lambda \in \mathbb{R}^*$, the function $f - u + u \circ T_\varepsilon \mod \lambda$ is not μ_ε-almost everywhere constant — we say that f is aperiodic*. In particular, the variance σ^2 in the central limit theorem is nonzero. Then, for any compact interval $J \subset \mathbb{R}$,*

$$\sigma \sqrt{2\pi n} \cdot \mu_\varepsilon \{x : S_n f(x) \in J\} \to |J|. \tag{1.5}$$

Here, $|J|$ denotes the length of the interval J.

It is probably possible to weaken the assumptions, by replacing the finite range interaction by a short range interaction, and by allowing the function f to depend on all coordinates but with an exponentially small influence of far away coordinates (by mimicking the techniques of [9, Sec. 5]). On the other hand, it is unclear whether it is possible to remove the continuity assumption on τ (notice that, for finite range interactions, this condition is not required in [9]).

On the technical level, Theorem 1.2 is a consequence of a spectral description of perturbed transfer operators acting on a suitable Banach space, that we now describe. Denote by $\mathcal{M}$ the set of complex Borel measures on Ω where Ω is equipped with the product topology.

Let $\mathcal{C}$ be a set of objects "acting on functions depending on finitely many coordinates", defined as follows. An element of $\mathcal{C}$ is a family (μ_Λ), where Λ goes through the finite subsets of $\mathbb{Z}^d$, such that μ_Λ is a complex measure on I^Λ, and such that if $\Lambda' \subset \Lambda$ then the projection of μ_Λ on $I^{\Lambda'}$ is $\mu_{\Lambda'}$. Formally, $\mathcal{C}$ is the projective limit of the spaces of complex measures on I^Λ, Λ finite subsets of $\mathbb{Z}^d$. This is a complex vector space, and we will not use any topology on it. Note that there is a canonical inclusion of $\mathcal{M}$ in $\mathcal{C}$. If u is a bounded measurable function depending on a finite number of coordinates, and $\mu \in \mathcal{C}$, then it is possible to define canonically $u\mu \in \mathcal{C}$.

If $\mu \in \mathcal{C}$ and φ is a bounded measurable function depending on a finite number of coordinates, it is possible to define $\mu(\varphi)$ as $\mu_\Lambda(\varphi)$ whenever Λ is large enough. If φ depends on finitely many coordinates, then $\varphi \circ T_\varepsilon$ also depends on finitely many coordinates. This implies that, for any $\mu \in \mathcal{C}$, there exists a unique $\nu \in \mathcal{C}$ such that, for any φ,

$$\nu(\varphi) = \mu(\varphi \circ T_\varepsilon). \tag{1.6}$$

We write $\nu = P_\varepsilon \mu$. Thus, P_ε is a linear operator from $\mathcal{C}$ to $\mathcal{C}$. It is the so-called *transfer operator* of the map T_ε. The image under P_ε of a measure is still a measure.

Theorem 1.3. *There exists a subspace $\mathcal{D}$ of $\mathcal{C}$ endowed with a complete norm $\|\cdot\|$ with the following properties. First, $\mathcal{D}$ contains the set of measures with bounded variation, and for any $\mu \in \mathcal{D}$, $|\mu(1)| \leq \|\mu\|$. Moreover, for any finite subset Λ of $\mathbb{Z}^d$, there exists a constant $C(\Lambda)$ such that, for any $u : \Omega \to \mathbb{C}$ depending only on coordinates in Λ and Lipschitz, for any $\mu \in \mathcal{D}$, $u\mu$ also belongs to $\mathcal{D}$ and*

$$\|u\mu\| \leq C(\Lambda)(\mathrm{Lip}(u) + |u|_\infty) \|\mu\|, \tag{1.7}$$

where $\mathrm{Lip}(u)$ *denotes the best Lipschitz constant of u.* (*The norm* $\| \, . \, \|$ *is defined in Eq.* (2.20).)

For any $r \in \mathbb{N}$, there exists $\varepsilon_1(r) > 0$ such that, if Φ_ε is a coupling with range r and strength $0 \leq \varepsilon \leq \varepsilon_1(r)$, *then the following holds*:

- *If $\mu \in \mathcal{D}$, then $P_\varepsilon \mu \in \mathcal{D}$ and $\|P_\varepsilon \mu\| \leq C \|\mu\|$ for some constant C. In fact, the operator P_ε has a simple eigenvalue at* 1 *and the rest of its spectrum is contained in a disk of radius* < 1.
- *Let f be a Lipschitz function depending on a finite number of coordinates. Then the map $t \mapsto P_{t,\varepsilon} = P_\varepsilon(e^{itf} \cdot)$ is an analytic map from $\mathbb{R}$ to $\mathcal{L}(\mathcal{D})$, the set of continuous linear operators on $\mathcal{D}$. If f is aperiodic, then the spectral radius of $P_{t,\varepsilon}$ is < 1 for any $t \neq 0$.*

The derivation of Theorem 1.2 from Theorem 1.3 is classical. Note, however, that some objects from $\mathcal{D}$ occurring in the proof are not known to be measures, so that one cannot directly cite [11], for example. So we will sketch the details of the proof in Appendix A, because this seems clearer than applying an abstract result like e.g. [7, Theorem VII.1.8] or [4, Corollary III.11].

Remark 1.4. Theorem 1.3 implies even more precise results: the limit theorems of Theorem 1.2 hold not only for μ_ε, but also for any probability measure μ which belongs to $\mathcal{D}$ (and in particular for any probability measure of bounded variation). In addition, further refinements of the central limit theorem hold. For example, the speed of convergence in the central limit theorem is $O(1/\sqrt{n})$, a renewal theorem holds, as well as a large deviation inequality (see again [4] for further details). One can also derive in the same way as in [1] the moderate deviations principle.

The rest of the paper will be devoted to the proof of Theorem 1.3. The main problem will be to get a Lasota–Yorke inequality with compactness, since the space of measures of bounded variation is not compact in the space of finite measures. We will therefore use artificial extensions as in [9], but we will lose control in the "central box" due to the factor e^{itf}. This loss will be compensated by the fact that, in large but finite boxes, the measures of bounded variation form a compact subset of the set of finite measures. Technically, we will have to take larger and larger boxes as t increases, but this causes no harm.

2. Functional Analytic Constructions

2.1. *Abstract tools*

We will need the following lemma.

Lemma 2.1. *Let T be a transformation preserving a probability measure μ. Let $n > 0$. Then a function f is aperiodic for T if and only if $S_n f$ is aperiodic for T^n.*

Proof. If f is periodic, then there exist $c, d > 0$ and u measurable such that $f = u - u \circ T + d \bmod c$. Therefore, $S_n f = u - u \circ T^n + nd \bmod c$, hence $S_n f$ is periodic.

Conversely, assume that $S_n f$ is periodic for T^n, i.e. $S_n f = u - u \circ T^n + d \bmod c$. Then $S_n(f - u + u \circ T) = d \bmod c$. For any function v, $S_n v$ is cohomologous to nv (since $v \circ T^k$ is cohomologous to v). Therefore, there exists a function w such that

$$n(f - u + u \circ T) = S_n(f - u + u \circ T) + w - w \circ T = d + w - w \circ T \bmod c. \quad (2.1)$$

Therefore, f is cohomologous to a constant modulo c/n, and f is periodic. □

We will also need the following formula on the essential spectral radius.

Lemma 2.2. *Let Q be a continuous linear operator on a complex Banach space $(B, \|\cdot\|)$. Assume that there exists a semi-norm $\|\cdot\|_w$ on B such that any sequence x_n in B with $\|x_n\| \leq 1$ contains a Cauchy subsequence for $\|\cdot\|_w$. Assume moreover that there exist $\sigma > 0$ and $C > 0$ such that, for any $x \in B$,*

$$\|Qx\| \leq \sigma \|x\| + C \|x\|_w . \quad (2.2)$$

Then the essential spectral radius of Q is at most σ.

This is a version of a theorem by Hennion [3], where one does not need to be able to iterate the operator for the weak norm (in the forthcoming application, the operator Q will indeed not be continuous for the weak norm).

Proof. Let $M > 0$ be such that $\|Qx\| \leq M \|x\|$. Notice also that there exists by assumption a constant $C > 0$ such that $\|x\|_w \leq C \|x\|$ for all $x \in B$. It allows to define a new seminorm on B by $\|x\|'_w = \sum_{n \geq 0} (2M)^{-n} \|Q^n x\|_w$. It satisfies the same compactness assumptions as $\|\cdot\|_w$. Moreover, Q is continuous for this seminorm. We can therefore iterate the equation $\|Qx\| \leq \sigma \|x\| + C \|x\|'_w$, and get an estimate

$$\|Q^n x\| \leq \sigma^n \|x\| + C_n \|x\|'_w . \quad (2.3)$$

The aforementioned theorem of Hennion [3, Corollaire 1] gives the conclusion. □

2.2. Measures of bounded variation

The concept of measures of bounded variation will play a central role. For $\mu \in \mathcal{M}$ and $\mathbf{p} \in \mathbb{Z}^d$, we define

$$\mathrm{Var}_{\mathbf{p}}\mu := \sup_{|\varphi|_{C^0(\Omega)} \le 1} \mu(\partial_{\mathbf{p}}\varphi). \tag{2.4}$$

Here, the sup is restricted to functions which are C^1 in $x_{\mathbf{p}}$, depending only on a finite number of coordinates. Let also

$$\mathrm{Var}\mu = \sup_{\mathbf{p} \in \mathbb{Z}^d} \mathrm{Var}_{\mathbf{p}}\ \mu. \tag{2.5}$$

The set $\mathcal{B} := \{\mu \in \mathcal{M} :\ \mathrm{Var}\ \mu < \infty\}$ consists of measures whose finite dimensional marginals are absolutely continuous with respect to Lebesgue and the density is a function of bounded variation. In fact, "Var" is a norm and, with this norm, $\mathcal{B}$ is a Banach space.

We define also in the same way, for any subset Λ of $\mathbb{Z}^d$ and any measure μ_Λ on I^Λ,

$$\mathrm{Var}_\Lambda \mu_\Lambda = \sup_{\mathbf{p} \in \Lambda} \mathrm{Var}_{\mathbf{p}}\ \mu_\Lambda. \tag{2.6}$$

We also need the usual total variation norm on complex measures:

$$|\mu| := \sup_{|\varphi|_{C^0(\Omega)} \le 1} \mu(\varphi). \tag{2.7}$$

Just like in [8, Sec. 3.3] one checks easily that

$$|\mu| \le \frac{1}{2}\mathrm{Var}_{\mathbf{p}}\ \mu, \quad \mathbf{p} \in \mathbb{Z}^d. \tag{2.8}$$

For $\mu \in \mathcal{M}$, let $A(\mu)$ denote its absolute value, it is a positive measure.

Lemma 2.3. *If* $\mu \in \mathcal{B}$, *then* $A(\mu) \in \mathcal{B}$ *and* $\mathrm{Var}\ A(\mu) \le \mathrm{Var}(\mu)$.

Proof. When μ is a measure with bounded variation on an interval, then the formula $\mathrm{Var}A(\mu) \le \mathrm{Var}(\mu)$ is a direct consequence of the formula

$$\mathrm{Var}(\mu) = \inf_{f\, dm = d\mu} \sup_{x_1 < \cdots < x_k} \sum |f(x_{i+1}) - f(x_i)|. \tag{2.9}$$

Indeed, if $d\mu = f dm$ then $dA(\mu) = |f| dm$, and the formula $||f|(x_{i+1}) - |f|(x_i)| \le |f(x_{i+1}) - f(x_i)|$ implies the conclusion.

In dimension n, the variation of a measure can be written as the integral of one-dimensional variations (see e.g. (43) in [8]). Hence, the result is implied by the one-dimensional result.

Consider now a measure $\mu \in \mathcal{B}$. If $A(\mu) = 0$, there is nothing to do. Otherwise, we can assume without loss of generality that $A(\mu)$ is a probability measure. There exists a measurable function φ, of absolute value almost everywhere equal to one, such that $\mu = \varphi A(\mu)$. Let ψ be a C^1 test function depending on a finite number of coordinates and bounded by 1, and let $\mathbf{q} \in \mathbb{Z}^d$. For any finite box Λ (containing all the coordinates on which ψ depends), the finite dimensional result implies

$$A(\pi_\Lambda \mu)(\partial_{\mathbf{q}} \psi) \leq \mathrm{Var}(A(\pi_\Lambda \mu)) \leq \mathrm{Var}(\pi_\Lambda \mu) \leq \mathrm{Var}(\mu), \tag{2.10}$$

where $\pi_\Lambda \mu$ is the projection of μ on I^Λ. Let φ_Λ denote the conditional expectation (for the measure $A(\mu)$) of the function φ with respect to the σ-algebra of sets depending only on coordinates in Λ. Then $\pi_\Lambda(\mu) = \pi_\Lambda(\varphi A\mu) = \varphi_\Lambda \pi_\Lambda(A\mu)$. Therefore, $A(\pi_\Lambda \mu) = |\varphi_\Lambda| \pi_\Lambda(A\mu)$. Hence, (2.10) reads

$$\int |\varphi_\Lambda| \partial_{\mathbf{q}} \psi \, d(A\mu) \leq \mathrm{Var}(\mu). \tag{2.11}$$

When the box Λ increases, the sequence of functions φ_Λ converges in $L^1(A\mu)$ to φ, by the martingale convergence theorem. Therefore, $|\varphi_\Lambda|$ converges to $|\varphi| = 1$. Taking the limit in (2.11), we get

$$\int \partial_{\mathbf{q}} \psi \, d(A\mu) \leq \mathrm{Var}(\mu). \tag{2.12}$$

□

An element μ of $\mathcal{B}$ gives rise to an element $(\mu_\Lambda)_\Lambda$ of $\mathcal{C}$ canonically by taking the induced measure on every finite subset of $\mathbb{Z}^d$. It satisfies

$$\sup_\Lambda \mathrm{Var}_\Lambda(\mu_\Lambda) < \infty. \tag{2.13}$$

Lemma 2.4. *Conversely, consider an element* $(\mu_\Lambda)_\Lambda$ *of* $\mathcal{C}$ *satisfying* (2.13)*. Then it comes from an element of* $\mathcal{B}$*.*

Proof. Let Λ_n be an increasing sequence of boxes. Define a measure μ_n on $I^{\mathbb{Z}^d}$ by $\mu_n = \mu_{\Lambda_n} \otimes m^{\otimes \mathbb{Z}^d \setminus \Lambda_n}$. The sequence μ_n has uniformly bounded variation. Let μ be one of its weak limits. Its marginal on each box Λ coincides with μ_Λ by construction. □

For $u : \Omega \to \mathbb{R}$ and $\mathbf{p} \in \mathbb{Z}^d$, let

$$\mathrm{Lip}_{\mathbf{p}}(u) = \sup_{\mathbf{x} \in I^{\mathbb{Z}^d \setminus \{\mathbf{p}\}}} \sup_{x_{\mathbf{p}} \neq x'_{\mathbf{p}} \in I} \frac{u(x_{\mathbf{p}}, \mathbf{x}) - u(x'_{\mathbf{p}}, \mathbf{x})}{|x_{\mathbf{p}} - x'_{\mathbf{p}}|}. \tag{2.14}$$

Lemma 2.5. *For any* $u : \Omega \to \mathbb{R}$ *depending on a finite number of coordinates, any* $\mu \in \mathcal{B}$ *and any* $\mathbf{p} \in \mathbb{Z}^d$,

$$\mathrm{Var}_{\mathbf{p}}(u\mu) \leq \sup |u| \mathrm{Var}_{\mathbf{p}}(\mu) + \mathrm{Lip}_{\mathbf{p}}(u)|\mu|. \tag{2.15}$$

Proof. In one dimension, this is a consequence of [8, Lemma 2.2(b)] and the fact that a Lipschitz function is differentiable almost everywhere and is equal to the integral of its derivative. This extends to finite boxes by (43) in [8]. Taking the supremum over finite boxes yields the conclusion of the lemma. □

2.3. *A family of extensions*

For $\mathbf{p} \in \mathbb{Z}^d$, denote by $\mathcal{B}_{\mathbf{p}}$ the set of measures μ in $\mathcal{B}$ such that, whenever a test function φ does not depend on the coordinate $\mathbf{p}$, then $\mu(\varphi) = 0$.

We can now define a family of extensions. We adapt the construction of [9, Sec. 3], the main difference being that we keep a central part of the measure on a finite subset of $\mathbb{Z}^d$.

Let Λ be a finite subset of $\mathbb{Z}^d$, we define a space $\mathcal{E}(\Lambda)$ as follows. An element of $\mathcal{E}(\Lambda)$ is a family $\mu = (\mu_c, (\mu_{\mathbf{p}})_{\mathbf{p}\in\mathbb{Z}^d\setminus\Lambda})$ such that μ_c is a measure of the form $\nu \otimes m^{\mathbb{Z}^d\setminus\Lambda}$ where ν is a measure on I^Λ, and $\mu_{\mathbf{p}} \in \mathcal{B}_{\mathbf{p}}$. Here, m denotes Lebesgue measure on I. We assume moreover

$$\|\mu\| := \max\left(\operatorname{Var}(\mu_c), \sup_{\mathbf{p}\in\mathbb{Z}^d\setminus\Lambda} \operatorname{Var}(\mu_{\mathbf{p}})\right) < \infty. \tag{2.16}$$

On $\mathcal{E}(\Lambda)$, we also define a "weak norm" by

$$\|\mu\|_w = |\mu_c|. \tag{2.17}$$

The unit ball of $(\mathcal{E}(\Lambda), \|\cdot\|)$ is relatively compact for the seminorm $\|\cdot\|_w$.

There is a canonical projection from $\mathcal{E}(\Lambda)$ to $\mathcal{C}$, given by the sum of the measures μ_c and $(\mu_{\mathbf{p}})_{p\in\mathbb{Z}^d\setminus\Lambda}$. We will denote it by $\Pi_{\mathcal{E}(\Lambda)}$ or simply by Π. We describe now a (non-canonical) redistribution process introduced in [9]. Let B be a subset of $\mathbb{Z}^d$, of cardinality $J \in [0, \infty]$. Let $\sigma : [0, J) \to B$ be an enumeration of the points in B. For $j \leq J$, let $B_j = \sigma[0, j)$. In particular, $B_0 = \emptyset$ and $B_J = B$. If $\mu \in \mathcal{B}$, define measures $\mu_{\mathbf{p}}$, for $\mathbf{p} \in B$, by

$$\mu_{\mathbf{p}} = \pi_{\mathbb{Z}^d\setminus B_j}\mu \otimes m^{\otimes B_j} - \pi_{\mathbb{Z}^d\setminus B_{j+1}}\mu \otimes m^{\otimes B_{j+1}}, \quad \text{where } j = \sigma^{-1}(\mathbf{p}). \tag{2.18}$$

By construction, $\mu = \pi_{\mathbb{Z}^d\setminus B}\mu \otimes m^{\otimes B} + \sum_{\mathbf{p}\in B} \mu_{\mathbf{p}}$, and $\mu_{\mathbf{p}} \in \mathcal{B}_{\mathbf{p}}$ satisfies $\operatorname{Var}(\mu_{\mathbf{p}}) \leq 2\operatorname{Var} \mu$. We say that $\pi_{\mathbb{Z}^d\setminus B}\mu \otimes m^{\otimes B}$ is the part of μ remaining at the end of the redistribution process.

Using this process for $B = \mathbb{Z}^d\setminus\Lambda$, we obtain a map H_Λ which associates to any $\mu \in \mathcal{B}$ an element $H_\Lambda(\mu) \in \mathcal{E}(\Lambda)$. It satisfies $\|H_\Lambda(\mu)\| \leq 2\operatorname{Var}(\mu)$, and $\Pi \circ H_\Lambda = \mathrm{Id}$.

Finally, let f be a Lipschitz function depending on a finite number of coordinates, and let $t \in \mathbb{R}$. Assume that the function tf depends only on coordinates in Λ. For each $n \in \mathbb{N}$, we define on $\mathcal{E}(\Lambda)$ an operator $Q_{t,\varepsilon,n,\Lambda}$, which is a (non-canonical) lift of $P^n_{t,\varepsilon}$ on $\mathcal{C}$. Starting from $\mu = (\mu_c, (\mu_{\mathbf{p}})) \in \mathcal{E}(\Lambda)$, apply first $P^n_{t,\varepsilon}$ to each measure μ_c and $\mu_{\mathbf{p}}$. Then, redistribute the mass as follows:

- For $\operatorname{dist}(\mathbf{p}, \Lambda) > nr$, distribute $P^n_{t,\varepsilon}\mu_{\mathbf{p}}$ to $B = \{\mathbf{q} : |\mathbf{q} - \mathbf{p}| \leq nr\}$. The points of B are all outside of Λ. Moreover, since $\mu_{\mathbf{p}} \in \mathcal{B}_{\mathbf{p}}$ and tf depends only on coordinates in Λ, we have $\pi_{\mathbb{Z}^d\setminus B}(P^n_{t,\varepsilon}\mu_{\mathbf{p}}) = 0$, i.e. there is no mass remaining at the end of this redistribution process.
- For the other measures, use H_Λ.

We get as in [9, Lemma 3.1]

$$\|Q_{t,\varepsilon,n,\Lambda}\mu\| \leq 2B(\Lambda,n,r)\sup\Big(\mathrm{Var}(P^n_{t,\varepsilon}\mu_c), \sup_{\mathbf{p}\in\mathbb{Z}^d\setminus\Lambda}\mathrm{Var}(P^n_{t,\varepsilon}\mu_{\mathbf{p}})\Big) \tag{2.19}$$

with $B(\Lambda,n,r) = \#\{\mathbf{q}\in\mathbb{Z}^d : \mathrm{dist}(\mathbf{q},\Lambda)\leq nr\} + \#\{\mathbf{q}\in\mathbb{Z}^d : |\mathbf{q}|\leq nr\}$, since every new measure receives a contribution from a number of sites bounded by $B(\Lambda,n,r)$. Note that we have written n as an index and not an exponent, in $Q_{t,\varepsilon,n,\Lambda}$, since these operators are not the powers of a single operator due to the (non-canonical) redistribution process.

Note that the extension $\mathcal{E}(\emptyset)$ is at the heart of the proof of [9].

2.4. *Construction of a canonical extension*

Let Λ be a finite subset of $\mathbb{Z}^d$. Let $\mathcal{E}(\Lambda)_0 \subset \mathcal{E}(\Lambda)$ be the kernel of $\Pi_{\mathcal{E}(\Lambda)}$, i.e. the elements of $\mathcal{E}(\Lambda)$ which induce the zero measure on the basis. This is a closed subspace of $\mathcal{E}(\Lambda)$, we can therefore consider the quotient space $\mathcal{D}(\Lambda) := \mathcal{E}(\Lambda)/\mathcal{E}(\Lambda)_0$ with its canonical norm. The map $\Pi_{\mathcal{E}(\Lambda)}$ induces a map $\Pi_{\mathcal{D}(\Lambda)} : \mathcal{D}(\Lambda)\to\mathcal{C}$, which is injective. In this way, we can therefore consider $\mathcal{D}(\Lambda)$ as a subspace of $\mathcal{C}$.

Since $\Pi_{\mathcal{E}(\Lambda)}\circ Q_{t,\varepsilon,n,\Lambda} = P^n_{t,\varepsilon}\circ\Pi_{\mathcal{E}(\Lambda)}$, the operator $Q_{t,\varepsilon,n,\Lambda}$ leaves $\mathcal{E}(\Lambda)_0$ invariant, and induces therefore a map $\bar{Q}_{t,\varepsilon,n,\Lambda}$ on $\mathcal{D}(\Lambda)$. An interesting consequence of this construction is that $\bar{Q}_{t,\varepsilon,n,\Lambda} = \bar{Q}^n_{t,\varepsilon,1,\Lambda}$, i.e. we are really dealing with the powers of a single operator. This is due to the fact that the non-canonicity in the redistribution process is killed by the quotient, any redistribution would induce the same map on $\mathcal{D}(\Lambda)$.

Proposition 2.6. *If Λ,Λ' are two finite subsets of $\mathbb{Z}^d$, then the subsets $\Pi_{\mathcal{D}(\Lambda)}(\mathcal{D}(\Lambda))$ and $\Pi_{\mathcal{D}(\Lambda')}(\mathcal{D}(\Lambda'))$ of $\mathcal{C}$ are equal, and the induced norms are equivalent.*

Proof. It is sufficient to prove this for $\Lambda'\subset\Lambda$. Consider $\Lambda'\subset\Lambda$, and construct a continuous linear map from $\mathcal{E}(\Lambda)$ to $\mathcal{E}(\Lambda')$ by redistributing the mass of μ_c in any convenient way. This induces a map from $\mathcal{D}(\Lambda)$ to $\mathcal{D}(\Lambda')$. Conversely, starting from an element of $\mathcal{E}(\Lambda')$, we can consider $\mu_c+\sum_{\mathbf{p}\in\Lambda\setminus\Lambda'}\mu_{\mathbf{p}}$ and redistribute it in any way, to get an element of $\mathcal{E}(\Lambda)$. Going to the quotient gives a canonical map from $\mathcal{D}(\Lambda')$ to $\mathcal{D}(\Lambda)$, which is inverse to the previous one. Hence, we have constructed a canonical isomorphism between $\mathcal{D}(\Lambda)$ and $\mathcal{D}(\Lambda')$, which commutes with the projections $\Pi_{\mathcal{E}(\Lambda)}$ and $\Pi_{\mathcal{E}(\Lambda')}$. We get the proposition by projecting everything in $\mathcal{C}$. □

Let $\mathcal{D}\subset\mathcal{C}$ be obtained by projecting any $\mathcal{D}(\Lambda)$. It is independent of the choice of Λ. We consider on it the norm given by the projection of the norm on $\mathcal{D}(\emptyset)$ — any other choice would give an equivalent norm. This is the space described in Theorem 1.3.

In another way, the norm of an element $\mu \in \mathcal{C}$ is the infimum of the quantity

$$\max\Big(|\mu(1)|, \sup_{\mathbf{p}\in\mathbb{Z}^d} \mathrm{Var}(\mu_{\mathbf{p}})\Big) \tag{2.20}$$

over all decompositions $\mu = \mu(1)m^{\otimes\mathbb{Z}^d} + \sum_{p\in\mathbb{Z}^d} \mu_{\mathbf{p}}$ where $\mu_{\mathbf{p}} \in \mathcal{B}_{\mathbf{p}}$. The elements of $\mathcal{D}$ are exactly those elements of $\mathcal{C}$ for which such a decomposition exists with finite (2.20).

3. Proof of the Main Theorem

From this point on, we fix a range r. The next lemma gives a contraction estimate for the action of P_ε^n on the measures $\mu \in \mathcal{B}_{\mathbf{p}}$. This is essentially contained in [9], one simply has to check that the only variations involved in the computation are those of points close to $\mathbf{p}$.

Lemma 3.1. *There exist $\alpha, \rho \in (0,1)$, $\varepsilon_1 = \varepsilon_1(r) > 0$ and $C = C(r) > 0$ such that, for any coupling Φ_ε of range r and strength $0 \le \varepsilon \le \varepsilon_1$, for all $\mathbf{p} \in \mathbb{Z}^d$, for all $n \in \mathbb{N}$, and for all $\mu \in \mathcal{B}_{\mathbf{p}}$,*

$$|P_\varepsilon^n \mu| \le C\alpha^{2n} \sup_{\mathbf{j}\in\mathbb{Z}^d} \rho^{|\mathbf{j}-\mathbf{p}|}\mathrm{Var}_{\mathbf{j}}(\mu). \tag{3.1}$$

The precise choice of the constants and the details of the proof are provided in Sec. 4.

Lemma 3.2. *There exist $\alpha \in (0,1)$ and $\varepsilon_1 = \varepsilon_1(r) > 0$ such that, for any coupling Φ_ε of range r and strength $0 \le \varepsilon \le \varepsilon_1$, and for any Lipschitz function f depending only on a finite number of coordinates, there exists $C > 0$ such that, for all $\mu \in \mathcal{B}$, for all $n \in \mathbb{N}$, and for all $t \in \mathbb{R}$,*

$$\mathrm{Var}(P_{t,\varepsilon}^n \mu) \le \alpha^{2n}\mathrm{Var}(\mu) + C(1+|t|)|\mu|. \tag{3.2}$$

Proof. By Lemma 2.5, we have

$$\mathrm{Var}(e^{itf}\mu) \le \mathrm{Var}(\mu) + C|t||\mu|. \tag{3.3}$$

Moreover, [8, Proposition 4.1 for $\theta = 1$] implies that

$$\mathrm{Var}(P_\varepsilon\mu) \le \alpha^2\mathrm{Var}(\mu) + C|\mu|. \tag{3.4}$$

Using these two equations and a geometric series, we get the conclusion.[a] □

We fix from now on the value of $\varepsilon_1(r)$ as the minimum of those given in the two previous lemmas, it will satisfy the conclusion of Theorem 1.3. We denote also by α the maximum of the values given in the previous lemmas. Fix now a coupling Φ_ε

[a] Referring to [8, Proposition 4.1] in this proof we make use of the assumption that $\inf|\tau'| > 2$. However, $\mathrm{Var}(P_{t,\varepsilon}^\ell\mu)$ could be estimated, for each fixed ℓ, just as for $\ell = 1$ in (3.3) and (3.4), and with $\alpha \in (0,1)$ chosen such that $\alpha^{-2\ell} < \inf|(\tau^\ell)'|$ one would recover (3.2).

of range r and strength $0 \le \varepsilon \le \varepsilon_1(r)$, as well as a Lipschitz function f depending only on a finite number of coordinates, say coordinates in a box $[-A, A]^d$. All the constants that we will construct from this point on are allowed to depend on f as well as r, τ.

Lemma 3.3. *There exist $C_0 > 1$, $\ell > 0$ such that, for all $\mathbf{p} \in \mathbb{Z}^d$, for all $n \in \mathbb{N}$ with $|\mathbf{p}| > \ell n$, for all $\mu \in \mathcal{B}_{\mathbf{p}}$,*

$$\mathrm{Var}(P^n_{t,\varepsilon}\mu) \le C_0(1 + |t|)^2 \alpha^n \mathrm{Var}(\mu). \tag{3.5}$$

Proof. Write $n = a + b$ where $a = n/2$ or $(n-1)/2$ depending on whether n is even or odd. By (3.2),

$$\mathrm{Var}(P^n_{t,\varepsilon}\mu) \le \alpha^{2a}\mathrm{Var}(P^b_{t,\varepsilon}\mu) + C(1 + |t|)|P^b_{t,\varepsilon}\mu|. \tag{3.6}$$

For the first term, (3.2) again gives $\mathrm{Var}(P^b_{t,\varepsilon}\mu) \le C(1 + |t|)\mathrm{Var}(\mu)$. For the second one, by (3.1) and Lemma 2.5,

$$\begin{aligned} |P^b_{t,\varepsilon}\mu| = |P^b_{\varepsilon}(e^{itS_b f}\mu)| &\le C\alpha^{2b} \sup_{\mathbf{j}\in\mathbb{Z}^d} \rho^{|\mathbf{j}-\mathbf{p}|}\mathrm{Var}_{\mathbf{j}}(e^{itS_b f}\mu) \\ &\le C\alpha^{2b} \sup_{\mathbf{j}\in\mathbb{Z}^d} \rho^{|\mathbf{j}-\mathbf{p}|}(\mathrm{Var}\ \mu + \mathrm{Lip}_{\mathbf{j}}(e^{itS_b f})|\mu|). \end{aligned}$$

Define a distance on Ω by $d(\mathbf{x}, \mathbf{y}) = \sup_{\mathbf{q}\in\mathbb{Z}^d} |x_{\mathbf{q}} - y_{\mathbf{q}}|$. It does not define the product topology but, nevertheless, there exists a constant C such that $|f(\mathbf{x}) - f(\mathbf{y})| \le Cd(\mathbf{x}, \mathbf{y})$ (since f is Lipschitz and depends on a finite number of coordinates). Moreover, we have $d(T_0\mathbf{x}, T_0\mathbf{y}) \le Cd(\mathbf{x}, \mathbf{y})$, since τ is Lipschitz, as well as $d(\Phi_\varepsilon\mathbf{x}, \Phi_\varepsilon\mathbf{y}) \le Cd(\mathbf{x}, \mathbf{y})$. Hence,

$$\mathrm{Lip}_{\mathbf{j}}(e^{itS_b f}) \le |t| \sum_{k=0}^{b-1} \mathrm{Lip}_{\mathbf{j}}(f \circ T^k_\varepsilon) \le |t| \sum_{k=0}^{b-1} C^k \le |t|C^b. \tag{3.7}$$

Moreover, if $|\mathbf{j}| > rb + A$, the function $e^{itS_b f}$ does not depend on the coordinate $\mathbf{j}$, hence $\mathrm{Lip}_{\mathbf{j}}(e^{itS_b f}) = 0$. Finally,

$$|P^b_{t,\varepsilon}\mu| \le C\alpha^{2b}\mathrm{Var}\ \mu + C\alpha^{2b} \sup_{|\mathbf{j}|\le rb+A} |t|\rho^{|\mathbf{j}-\mathbf{p}|}C^b|\mu|. \tag{3.8}$$

If $|\mathbf{p}| > \ell n$ for some large enough ℓ, we have $\rho^{|\mathbf{p}|-rb-A}C^b \le 1$, and we get $|P^b_{t,\varepsilon}\mu| \le C(1 + |t|)\alpha^{2b}\mathrm{Var}\ \mu$. Together with (3.6), this proves the lemma. □

Fix $t \in \mathbb{R}$. Let Λ_N be the box $[-\ell N, \ell N]^d$, with ℓ from the previous lemma. Let N be large enough so that $C_0(1 + |t|)^2\alpha^N \le \frac{1}{4B(\Lambda_N,N,r)}$. We will now work in the extension $\mathcal{E} = \mathcal{E}(\Lambda_N)$, and study the operator $Q = Q_{t,\varepsilon,N,\Lambda_N}$.

Lemma 3.4. *There exists a constant $C > 0$ such that, for all $\mu \in \mathcal{E}$,*

$$\|Q\mu\| \le \frac{1}{2}\|\mu\| + C\|\mu\|_w. \tag{3.9}$$

Proof. By (2.19), we have

$$\|Q\mu\| \leq 2B(\Lambda_N, N, r) \sup\Big(\mathrm{Var}(P^N_{t,\varepsilon}\mu_c), \sup_{\mathbf{p}\in\mathbb{Z}^d\setminus\Lambda_N} \mathrm{Var}(P^N_{t,\varepsilon}\mu_{\mathbf{p}})\Big). \tag{3.10}$$

Moreover, (3.5) shows that $\mathrm{Var}(P^N_{t,\varepsilon}\mu_{\mathbf{p}}) \leq C_0(1+|t|)^2\alpha^N \|\mu\|$, while (3.2) gives $\mathrm{Var}(P^N_{t,\varepsilon}\mu_c) \leq \alpha^{2N}\|\mu\| + C(1+|t|)\|\mu\|_w$. We get

$$\|Q\mu\| \leq 2B(\Lambda_N, N, r)\max(C_0(1+|t|)^2\alpha^N \|\mu\|,\ \alpha^{2N}\|\mu\| + C(1+|t|)\|\mu\|_w), \tag{3.11}$$

which yields the desired conclusion by the choice of N. □

This is a Lasota–Yorke inequality for the operator Q. The main advantage of this construction is that, since the unit ball of $(\mathcal{E}(\Lambda), \|\cdot\|)$ is relatively compact for the seminorm $\|\cdot\|_w$, we get from Lemma 2.2 that the essential spectral radius of Q is at most $1/2$. To show that the spectral radius of Q is less than 1, it is therefore sufficient to check that there is no eigenvalue of modulus ≥ 1.

Lemma 3.5. *Let $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$. Let $\mu \in \mathcal{E}$ satisfy $Q\mu = \lambda\mu$. Then $\mu = 0$, or $|\lambda| = 1$ and tf is periodic.*

Proof. Let $\nu = \Pi(\mu) \in \mathcal{C}$. We will first check that it belongs to $\mathcal{B}$. By Lemma 2.4, it is sufficient to check that the variations of the measures ν_Λ are uniformly bounded.

Let φ be a smooth function depending on a finite number of coordinates, bounded by 1, and fix $\mathbf{q}$. Fix $K \geq A$ such that $|\mathbf{q}| \leq K$ and that φ depends only on the coordinates $\mathbf{p}$ with $|\mathbf{p}| \leq K$. For $n \in \mathbb{N}$ which is a multiple of N we have, since μ is an eigenfunction of Q,

$$|\nu(\partial_{\mathbf{q}}\varphi)| \leq |P^n_{t,\varepsilon}\mu_c(\partial_{\mathbf{q}}\varphi)| + \sum_{|\mathbf{p}|\leq K+rn} |P^n_{t,\varepsilon}\mu_{\mathbf{p}}(\partial_{\mathbf{q}}\varphi)|.$$

Indeed, if $|\mathbf{p}| > K + rn$, then

$$P^n_{t,\varepsilon}\mu_{\mathbf{p}}(\partial_{\mathbf{q}}\varphi) = \mu_{\mathbf{p}}(e^{itS_nf}(\partial_{\mathbf{q}}\varphi)\circ T^n_\varepsilon) = 0 \tag{3.12}$$

since $e^{itS_nf}(\partial_{\mathbf{q}}\varphi)\circ T^n_\varepsilon$ does not depend on $x_{\mathbf{p}}$. We get

$$|\nu(\partial_{\mathbf{q}}\varphi)| \leq \mathrm{Var}(P^n_{t,\varepsilon}\mu_c) + \sum_{|\mathbf{p}|\leq K+rn} \mathrm{Var}(P^n_{t,\varepsilon}\mu_{\mathbf{p}}). \tag{3.13}$$

Let $\ell' = \max(\ell, r) + 1$ with ℓ as in Lemma 3.3, and let $k(\mathbf{p})$ be the integer part of $|\mathbf{p}|/\ell'$. If $n \geq K$, then $k(\mathbf{p}) \leq n$ whenever $|\mathbf{p}| \leq K + rn$. Then, by (3.2),

$$\mathrm{Var}(P^n_{t,\varepsilon}\mu_{\mathbf{p}}) \leq C(1+|t|)\mathrm{Var}(P^{k(\mathbf{p})}_{t,\varepsilon}\mu_{\mathbf{p}}). \tag{3.14}$$

We can then use (3.5) since $|\mathbf{p}| > \ell k(\mathbf{p})$. We get

$$\mathrm{Var}(P^n_{t,\varepsilon}\mu_{\mathbf{p}}) \leq C(1+|t|)^3\alpha^{k(\mathbf{p})}\mathrm{Var}(\mu_{\mathbf{p}}). \tag{3.15}$$

Finally,

$$|\nu(\partial_{\mathbf{q}}\varphi)| \leq C(1+|t|)\mathrm{Var}(\mu_c) + C(1+|t|)^3 \sum_{|p|\leq K+rn} \alpha^{k(\mathbf{p})}\mathrm{Var}(\mu_{\mathbf{p}}). \tag{3.16}$$

This last sum is bounded uniformly in K and n. This proves that the variations of the measures ν_Λ are uniformly bounded, i.e. $\nu \in \mathcal{B}$.

If $\nu = 0$, the marginal of ν on Λ_N vanishes, i.e. $\mu_c = 0$. Therefore, $\|\mu\|_w = 0$. The Lasota–Yorke inequality (3.9) then gives $\mu = 0$.

Assume now that $\nu \neq 0$. We will prove that $|\lambda| = 1$ and that tf is periodic. The measure ν satisfies $P_{t,\varepsilon}^N \nu = \lambda\nu$. The absolute value $A(\nu)$ of the measure ν belongs to $\mathcal{B}$, by Lemma 2.3. It satisfies

$$A(\nu) = |\lambda|^{-1} A(P_{t,\varepsilon}^N \nu) \leq P_\varepsilon^N A(\nu), \tag{3.17}$$

where the last inequality is obtained by the following direct computation:

$$\begin{aligned} A(P_{t,\varepsilon}^N\nu)(\varphi) &= \sup_{|g|\leq 1} |(P_{t,\varepsilon}^N\nu)(g\cdot\varphi)| = \sup_{|g|\leq 1} \left|\nu\left(e^{itS_Nf}\cdot g\circ T_\varepsilon^N \cdot \varphi\circ T_\varepsilon^N\right)\right| \\ &\leq \sup_{|g|\leq 1} |\nu(g\cdot\varphi\circ T_\varepsilon^N)| = P_\varepsilon^N A(\nu)(\varphi). \end{aligned} \tag{3.18}$$

Since $A(\nu)$ and $P_\varepsilon^N A(\nu)$ have the same mass, this yields $A(\nu) = P_\varepsilon^N A(\nu)$ and $|\lambda| = 1$. Since $A(\nu)$ belongs to $\mathcal{B}$, it has to be a scalar multiple of the SRB measure μ_ε, see Theorem 1.1. In particular, ν is absolutely continuous with respect to μ_ε, and the Radon–Nikodym derivative $g = \frac{d\nu}{d\mu_\varepsilon}$ is a function of almost everywhere constant modulus. Since we assume ν to be nonzero, we have $|g| \neq 0$ almost everywhere. Then

$$P_\varepsilon^N\left(e^{itS_Nf}\frac{g}{g\circ T_\varepsilon^N}\mu_\varepsilon\right) = \frac{1}{g}P_\varepsilon^N(e^{itS_Nf}g\mu_\varepsilon) = \frac{1}{g}P_{t,\varepsilon}^N(\nu) = \lambda\frac{1}{g}\nu = \lambda\mu_\varepsilon. \tag{3.19}$$

In particular,

$$1 = \left|\int e^{itS_Nf}\frac{g}{g\circ T_\varepsilon^N}\,d\mu_\varepsilon\right| \leq \int\left|e^{itS_Nf}\frac{g}{g\circ T_\varepsilon^N}\right|\,d\mu_\varepsilon = 1. \tag{3.20}$$

Therefore, we have equality in the inequality, and $e^{itS_Nf}\frac{g}{g\circ T_\varepsilon^N}$ is almost everywhere equal to a constant of modulus 1. This shows that tS_Nf is periodic for T_ε^N. By Lemma 2.1, tf is periodic for T_ε. □

Proof of Theorem 1.3. The extension $\mathcal{D}$ is as described in Sec. 2.4. The formula (2.20) for the norm clearly gives $|\mu(1)| \leq \|\mu\|$.

Let Λ be a fixed box, we want to check Eq. (1.7), i.e.

$$\|u\mu\| \leq C(\Lambda)(\mathrm{Lip}(u) + |u|_\infty)\,\|\mu\| \tag{3.21}$$

whenever u is Lipschitz continuous and depends only on coordinates in Λ. Let us first work in the extension $\mathcal{E}(\Lambda)$. If $\mu = (\mu_c, (\mu_{\mathbf{p}})_{p\in\mathbb{Z}^d\setminus\Lambda}) \in \mathcal{E}(\Lambda)$, then all the measures $u\mu_{\mathbf{p}}$ still belong to $\mathcal{B}_{\mathbf{p}}$, and they satisfy $\mathrm{Var}(u\mu_{\mathbf{p}}) \leq (\mathrm{Lip}\,u + |u|_\infty)\mathrm{Var}\mu_{\mathbf{p}}$ by Lemma 2.5. Moreover, $u\mu_c$ is still of the form $\nu\otimes m^{\mathbb{Z}^d\setminus\Lambda}$, and its variation is at most $(\operatorname{Lip} u + |u|_\infty)\mathrm{Var}\mu_c$, again by Lemma 2.5. Hence, the multiplication by u is well defined on $\mathcal{E}(\Lambda)$ and its norm is at most $(\operatorname{Lip} u + |u|_\infty)$. This multiplication leaves $\mathcal{E}(\Lambda)_0$ invariant, hence induces an operator on the quotient space $\mathcal{D}(\Lambda)$ with the same bound on its norm (see Sec. 2.4). Since $\mathcal{D}(\Lambda)$ is isomorphic to $\mathcal{D}$ by Proposition 2.6, this proves (3.21).

In order to get analyticity of $t \mapsto P_{t,\varepsilon}$, it is enough to prove that the map $M_t(\mu) = e^{itf}\mu$ depends analytically on t. For this, we only have to check that the series expansion

$$\sum_{n\geq 0} \frac{(itf)^n}{n!}\mu \tag{3.22}$$

is well defined for any $\mu \in \mathcal{D}$. But this is a direct consequence of (3.21), since

$$\begin{aligned}\left\|\frac{(itf)^n}{n!}\mu\right\| &\leq C(\operatorname{Lip}(f^n) + |f^n|_\infty)\,\|\mu\|\,\frac{|t|^n}{n!}\\ &\leq C(n\operatorname{Lip}(f) + |f|_\infty)|f|_\infty^{n-1}\,\|\mu\|\,\frac{|t|^n}{n!}.\end{aligned} \tag{3.23}$$

This gives analyticity of M_t, and its series expansion.

In [9], it is proved that, in the extension $\mathcal{E}(\emptyset)$, the operator $Q_{0,\varepsilon,N,\emptyset}$ (which is a lift of $P_{0,\varepsilon}^N$ on $\mathcal{C}$) has a simple eigenvalue at 1 for sufficiently large N, the rest of its spectrum being contained in a disk of radius < 1: this is indeed an easy consequence of Lemma 3.1. After a quotient by $\mathcal{E}(\emptyset)_0$ (which is left invariant by $Q_{0,\varepsilon,N,\emptyset}$), this implies that P_ε^N acts continuously on $\mathcal{D}$, has a simple eigenvalue at 1 and the rest of its spectrum is contained in a disk of radius < 1. The same is then true for the operator P_ε itself.

Consider now $t \neq 0$ and assume that f is aperiodic. For a suitable N and a suitable box Λ_N, Lemma 3.5 shows that the spectral radius of $Q_{t,\varepsilon,N,\Lambda_N}$ is < 1 on $\mathcal{E}(\Lambda_N)$. On the quotient $\mathcal{D} \cong \mathcal{E}(\Lambda_N)/\mathcal{E}(\Lambda_N)_0$, this implies that the spectral radius of $P_{t,\varepsilon}^N$ is < 1. Therefore, $P_{t,\varepsilon}$ also has a spectral radius < 1. □

4. Proof of Lemma 3.1

We introduce a family of additional "local" norms: for $\rho \in (0,1)$ (to be chosen later), for any $\mathbf{p} \in \mathbb{Z}^d$, $\Lambda \subset \mathbb{Z}^d$ and $\mu \in \mathcal{B}$ let

$$\mathrm{Var}^{\mathbf{p}}(\mu) = \sup_{\mathbf{j}\in\mathbb{Z}^d} \rho^{|\mathbf{j}-\mathbf{p}|}\mathrm{Var}_{\mathbf{j}}(\mu), \tag{4.1}$$

$$\mathrm{Var}^{\Lambda}(\mu) = \sup_{\mathbf{p}\in\Lambda} \mathrm{Var}^{\mathbf{p}}(\mu). \tag{4.2}$$

Observe that $\mathrm{Var}(\mu) = \sup_{\mathbf{p}} \mathrm{Var}^{\mathbf{p}}(\mu) = \mathrm{Var}^{\mathbb{Z}^d}(\mu)$. In this section we denote $\Lambda(\mathbf{p}, n) = \{\mathbf{q} : |\mathbf{q} - \mathbf{p}| \leq rn\}$, so the range r will often be suppressed in the notation.

For the proof of Lemma 3.1 we need two further lemmas that will be proved later. Let $\lambda_1 = \frac{1}{2}\inf|\tau'| > 1$ and denote by $\alpha_0 \in (0,1)$ the mixing rate of τ.

Lemma 4.1. (Localized Lasota–Yorke type estimate) *For any* $\lambda \in (1, \lambda_1)$, *for any range* r *and any* $\rho \in (0,1)$, *there are* $\varepsilon_2 > 0$ *and* $C > 0$ *such that, for any coupling* Φ_ε *of range* r *and strength* $0 \le \varepsilon \le \varepsilon_2$, *for all* $m \in \mathbb{N}$, *for all* $\mathbf{p} \in \mathbb{Z}^d$, *and for all* $\nu \in \mathcal{B}$,

$$\mathrm{Var}^{\mathbf{p}}(P_\varepsilon^m \nu) \le C\big(\lambda^{-m}\mathrm{Var}^{\mathbf{p}}(\nu) + |\nu|\big) \le 2C\mathrm{Var}^{\mathbf{p}}(\nu). \tag{4.3}$$

Lemma 4.2. *For any range* r *and any* $\rho \in (0,1)$, *there are* $\varepsilon_3 > 0$ *and* $C > 0$ *such that, for any coupling* Φ_ε *of range* r *and strength* $0 \le \varepsilon \le \varepsilon_3$, *for all* $m \in \mathbb{N}$, *for all* $\mathbf{p} \in \mathbb{Z}^d$, *and for all* $\nu \in \mathcal{B}_{\mathbf{p}}$,

$$|P_\varepsilon^m \nu| \le C(\alpha_0^m \mathrm{Var}_{\mathbf{p}}(\nu) + m\varepsilon \mathrm{Var}^{\mathbf{p}}(\nu))\,. \tag{4.4}$$

Proof of Lemma 3.1. We can precise the choice of the constants appearing in the lemma: let $\lambda \in (1, \lambda_1)$ be fixed, then choose $\alpha, \alpha_1, \rho \in (0,1)$ such that

$$\sqrt{\max\{\lambda^{-1}, \alpha_0\}} < \alpha_1 < \rho^{2r}\alpha^2. \tag{4.5}$$

The maximal coupling strength ε_1 will have to be taken smaller than ε_2 and ε_3 from the previous lemmas, and even smaller in the calculation below.

Before getting into the proof of Lemma 3.1, let us establish a preliminary inequality in the extension $\mathcal{E}(\emptyset)$, using Lemmas 4.1 and 4.2. Let $Q := Q_{t,\varepsilon,2m,\emptyset}$ be the lift of $P_{t,\varepsilon}^{2m}$ described in Sec. 2.2. It redistributes mass from a site $\mathbf{q}$ to sites in $\Lambda(\mathbf{q}, 2m)$ only. We claim that there exist $m \in \mathbb{N}$ and $\varepsilon_1 > 0$ such that, whenever the coupling strength is at most ε_1, for each $\Gamma \subseteq \mathbb{Z}^d$ and each $\tilde{\nu} \in \mathcal{E}(\emptyset)$ with $\tilde{\nu}_c = 0$,

$$\sup_{\mathbf{j}\in\mathbf{q}+\Gamma} \mathrm{Var}^{\mathbf{q}+\Gamma}\big((Q\tilde{\nu})_{\mathbf{j}}\big) \le \alpha_1^{2m} \sup_{\mathbf{j}\in\Lambda(\mathbf{q},2m)+\Gamma} \mathrm{Var}^{\Lambda(\mathbf{q},2m)+\Gamma}(\tilde{\nu}_{\mathbf{j}}). \tag{4.6}$$

In view of the redistribution mechanism described in (2.19), we have

$$\begin{aligned}\sup_{\mathbf{j}\in\mathbf{q}+\Gamma} \mathrm{Var}^{\mathbf{q}+\Gamma}((Q\tilde{\nu})_{\mathbf{j}}) &\le Cm^d \sup_{\mathbf{j}\in\Lambda(\mathbf{q},2m)+\Gamma} \mathrm{Var}^{\mathbf{q}+\Gamma}(P_\varepsilon^{2m}\tilde{\nu}_{\mathbf{j}})\\ &\le Cm^d \sup_{\mathbf{i},\mathbf{j}\in\Lambda(\mathbf{q},2m)+\Gamma} \big(\lambda^{-m}\mathrm{Var}^{\mathbf{i}}(P_\varepsilon^m\tilde{\nu}_{\mathbf{j}}) + |P_\varepsilon^m\tilde{\nu}_{\mathbf{j}}|\big)\\ &\le Cm^d \sup_{\mathbf{i},\mathbf{j}\in\Lambda(\mathbf{q},2m)+\Gamma} (\lambda^{-m}\mathrm{Var}^{\mathbf{i}}(\tilde{\nu}_{\mathbf{j}}) + \alpha_0^m\mathrm{Var}_{\mathbf{j}}(\tilde{\nu}_{\mathbf{j}}) + m\varepsilon\,\mathrm{Var}^{\mathbf{j}}(\tilde{\nu}_{\mathbf{j}})),\end{aligned} \tag{4.7}$$

where we used the Lasota–Yorke type inequality (4.3) and the estimate (4.4). Hence,

$$\begin{aligned}\sup_{\mathbf{j}\in\mathbf{q}+\Gamma} \mathrm{Var}^{\mathbf{q}+\Gamma}((Q\tilde{\nu})_{\mathbf{j}}) &\le Cm^d \sup_{\mathbf{i},\mathbf{j}\in\Lambda(\mathbf{q},2m)+\Gamma} (\lambda^{-m}\mathrm{Var}^{\mathbf{i}}(\tilde{\nu}_{\mathbf{j}}) + (\alpha_0^m + m\varepsilon)\mathrm{Var}^{\mathbf{j}}(\tilde{\nu}_{\mathbf{j}}))\\ &\le Cm^d\,(\lambda^{-m} + \alpha_0^m + m\varepsilon) \sup_{\mathbf{j}\in\Lambda(\mathbf{q},2m)+\Gamma} \mathrm{Var}^{\Lambda(\mathbf{q},2m)+\Gamma}(\tilde{\nu}_{\mathbf{j}}).\end{aligned} \tag{4.8}$$

Choosing m sufficiently large and then ε_1 sufficiently small, (4.6) follows.

Let us now prove Lemma 3.1. As P_ε contracts the total variation norm, it suffices to prove the lemma for multiples $n = k2m$ of the fixed integer $2m$ which satisfies

(4.6). So let $\mu \in \mathcal{B}_{\mathbf{p}}$, define an element $\tilde{\mu}$ which has only zero components except for $\tilde{\mu}_{\mathbf{p}} = \mu$ and observe first that

$$\begin{aligned}
|P_\varepsilon^{k2m}\mu| &\leq \sum_{\mathbf{q}\in\mathbb{Z}^d} \left|(Q^k\tilde{\mu})_{\mathbf{q}}\right| \leq \frac{1}{2} \sum_{\mathbf{q}\in\Lambda(\mathbf{p},k2m)} \mathrm{Var}_{\mathbf{q}}((Q^k\tilde{\mu})_{\mathbf{q}}) \\
&\leq C\cdot(k2m)^d \cdot \sup_{\mathbf{q}\in\Lambda(\mathbf{p},k2m)} \mathrm{Var}_{\mathbf{q}}((Q^k\tilde{\mu})_{\mathbf{q}}), \qquad (4.9)
\end{aligned}$$

where we used the fact that each application of $Q := Q_{t,\varepsilon,2m,\emptyset}$ redistributes mass from a site $\mathbf{q}$ to sites in $\Lambda(\mathbf{q}, 2m)$ only.

Applying (4.6) repeatedly and observing that $\tilde{\mu}_{\mathbf{j}} = 0$ if $\mathbf{j} \neq \mathbf{p}$ and $\tilde{\mu}_{\mathbf{p}} = \mu$, we obtain

$$\begin{aligned}
\sup_{\mathbf{q}\in\Lambda(\mathbf{p},k2m)} \mathrm{Var}_{\mathbf{q}}\big((Q^k\tilde{\mu})_{\mathbf{q}}\big) &\leq \sup_{\mathbf{q}\in\Lambda(\mathbf{p},k2m)} \alpha_1^{k2m} \sup_{\mathbf{j}\in\Lambda(\mathbf{q},k2m)} \mathrm{Var}^{\Lambda(\mathbf{q},k2m)}(\tilde{\mu}_{\mathbf{j}}) \\
&= \alpha_1^{k2m} \sup_{\mathbf{q}\in\Lambda(\mathbf{p},k2m)} \mathrm{Var}^{\Lambda(\mathbf{q},k2m)}(\tilde{\mu}_{\mathbf{p}}) \\
&\leq \alpha_1^{k2m}\rho^{-2rk2m}\mathrm{Var}^{\mathbf{p}}(\mu). \qquad (4.10)
\end{aligned}$$

Together with (4.9) this yields $|P_\varepsilon^n\mu| \leq C\,n^d(\alpha_1\rho^{-2r})^n\mathrm{Var}^{\mathbf{p}}(\mu)$, which finishes the proof of Lemma 3.1 in view of the choice of the constants in (4.5). □

Proof of Lemma 4.1. We will prove

$$\mathrm{Var}^{\mathbf{p}}(P_\varepsilon\nu) \leq \lambda^{-1}\mathrm{Var}^{\mathbf{p}}(\nu) + C|\nu|. \qquad (4.11)$$

From this, (4.3) follows by induction.[b]

Observe first that

$$\mathrm{Var}^{\mathbf{p}}(P_0\nu) \leq \lambda_1^{-1}\mathrm{Var}^{\mathbf{p}}(\nu) + C|\nu|, \qquad (4.12)$$

where $\lambda_1 = \frac{1}{2}\inf|\tau'|$. This is a simple consequence of the Lasota–Yorke inequality for the single site map, compare e.g. the proof of Lemma 3.2 in [8]. We will show that[c]

$$\mathrm{Var}^{\mathbf{p}}((\Phi_\varepsilon)_*\nu) \leq (1 + C\varepsilon)\mathrm{Var}^{\mathbf{p}}(\nu). \qquad (4.13)$$

We first notice that under a mild bound on the coupling strength ε, the coupling assumption (1.1) ensures that the infinite matrix $D\Phi_\varepsilon(\mathbf{x})$ is invertible. Moreover,

[b] It is only here where we use the assumption $\inf|\tau'| > 2$. For $1 < \inf|\tau'| \leq 2$ this reduction to the case $m = 1$ is not possible, see also the remarks in [9, Footnote 12].

[c] We write $F_*\nu$ for the push-forward of the measure ν under the map F, i.e. the measure given by $(F_*\nu)(A) = \nu(F^{-1}A)$. Note that this object is sometimes denoted by $F^*\nu$ in [8, 9].

taking C large enough to get $1_{|\mathbf{i}-\mathbf{j}|\leq r} \leq \frac{C}{2}\rho^{4|\mathbf{i}-\mathbf{j}|}$ for all $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$, the second and third parts of this assumption can be rewritten as

$$|(DA_\varepsilon)_{\mathbf{ij}}|_\infty \leq C\varepsilon\rho^{4|\mathbf{i}-\mathbf{j}|}, \quad |\partial_{\mathbf{k}}(DA_\varepsilon)_{\mathbf{ij}}|_\infty \leq C\varepsilon\rho^{4|\mathbf{i}-\mathbf{j}|}. \tag{4.14}$$

A direct computation using these estimates (see for example p. 300 in [5]) gives that $B(\mathbf{x}) := (D\Phi_\varepsilon(\mathbf{x}))^{-1}$ satisfies

$$|b_{\mathbf{ii}}|_\infty \leq 1 + C\varepsilon, \quad |b_{\mathbf{ij}}|_\infty \leq C\varepsilon\rho^{2|\mathbf{j}-\mathbf{i}|}, \quad |\partial_{\mathbf{i}} b_{\mathbf{ij}}|_\infty \leq C\varepsilon\rho^{2|\mathbf{j}-\mathbf{i}|}. \tag{4.15}$$

We can then follow the proof of Lemma 3.3 in [8] with some modifications. For all $\mathbf{j}, \mathbf{p} \in \mathbb{Z}^d$,

$$\begin{aligned}
&\rho^{|\mathbf{j}-\mathbf{p}|}((\Phi_\varepsilon)_*\nu)(\partial_{\mathbf{j}}\varphi) \\
&\quad = \rho^{|\mathbf{j}-\mathbf{p}|}\sum_{\mathbf{i}\in\mathbb{Z}^d} \nu\big(\partial_{\mathbf{i}}(\varphi\circ\Phi_\varepsilon)\, b_{\mathbf{ij}}\big) \\
&\quad = \rho^{|\mathbf{j}-\mathbf{p}|}\sum_{\mathbf{i}\in\mathbb{Z}^d} \nu\big(\partial_{\mathbf{i}}(\varphi\circ\Phi_\varepsilon\cdot b_{\mathbf{ij}})\big) - \rho^{|\mathbf{j}-\mathbf{p}|}\sum_{\mathbf{i}\in\mathbb{Z}^d} \nu\big(\varphi\circ\Phi_\varepsilon\cdot\partial_{\mathbf{i}} b_{\mathbf{ij}}\big) \\
&\quad \leq \rho^{|\mathbf{j}-\mathbf{p}|}\sum_{\mathbf{i}\in\mathbb{Z}^d} \mathrm{Var}_{\mathbf{i}}(\nu)\cdot|b_{\mathbf{ij}}|_\infty + |\nu|\rho^{|\mathbf{j}-\mathbf{p}|}\cdot\sum_{\mathbf{i}\in\mathbb{Z}^d}|\partial_{\mathbf{i}} b_{\mathbf{ij}}|_\infty \\
&\quad \leq \rho^{|\mathbf{j}-\mathbf{p}|}\mathrm{Var}_{\mathbf{j}}(\nu) + C\,\varepsilon\Bigg(\mathrm{Var}^{\mathbf{p}}(\nu)\sum_{\mathbf{i}\in\mathbb{Z}^d}\rho^{|\mathbf{j}-\mathbf{p}|-|\mathbf{i}-\mathbf{p}|+2|\mathbf{j}-\mathbf{i}|} + |\nu|\sum_{\mathbf{i}\in\mathbb{Z}^d}\rho^{|\mathbf{j}-\mathbf{p}|+2|\mathbf{j}-\mathbf{i}|}\Bigg) \\
&\quad \leq (1 + C\varepsilon)\mathrm{Var}^{\mathbf{p}}(\nu).
\end{aligned}$$

This yields (4.13) and finishes the proof of Lemma 4.1. □

Proof of Lemma 4.2. The proof follows closely the corresponding one in [9]. For each $\mathbf{p} \in \mathbb{Z}^d$ define a coupling map $\Phi_{\varepsilon,\mathbf{p}} : \Omega \to \Omega$ where site $\mathbf{p}$ is decoupled from all other sites,

$$(\Phi_{\varepsilon,\mathbf{p}}(\mathbf{x}))_{\mathbf{q}} = \begin{cases} x_{\mathbf{p}} & \text{if } \mathbf{q} = \mathbf{p}, \\ \big(\Phi_\varepsilon(\mathbf{x}_{\mathbb{Z}^d\setminus\{\mathbf{p}\}}, a)\big)_{\mathbf{q}} & \text{if } \mathbf{q} \neq \mathbf{p}, \end{cases} \tag{4.16}$$

where a is an arbitrary point in I. Denote by $P_{\varepsilon,\mathbf{p}}$ the Perron–Frobenius operator of $\Phi_{\varepsilon,\mathbf{p}} \circ T_0$. We will show that, for each $\nu \in \mathcal{B}$,

$$|(\Phi_\varepsilon)_*\nu - (\Phi_{\varepsilon,\mathbf{p}})_*\nu| \leq C\varepsilon\mathrm{Var}^{\mathbf{p}}(\nu). \tag{4.17}$$

Then, making use of the fact that $|P_\varepsilon| = |P_{\varepsilon,\mathbf{p}}| = 1$ and of estimate (4.3), a simple telescoping argument yields

$$|P_\varepsilon^m\nu - P_{\varepsilon,\mathbf{p}}^m\nu| \leq C\, m\varepsilon\mathrm{Var}^{\mathbf{p}}(\nu), \tag{4.18}$$

and (4.4) follows once we have shown that $|P_{\varepsilon,\mathbf{p}}^m\nu| \leq C\alpha_0^m\mathrm{Var}_{\mathbf{p}}(\nu)$ for any $\nu \in \mathcal{B}_{\mathbf{p}}$. But this is proved precisely as in [9, pp. 40–41], where α_0 is the mixing rate for the single site map.

It remains to prove (4.17). Here we can follow closely the proof of Lemma 3.2a) in [9]. Indeed, let $F_t := t\Phi_{\varepsilon,\mathbf{p}} + (1-t)\Phi_\varepsilon$ and $\Delta_{\mathbf{q}} := (\Phi_{\varepsilon,\mathbf{p}} - \Phi_\varepsilon)_{\mathbf{q}}$. Just as in [9] one shows that, for each test function φ,

$$\big((\Phi_{\varepsilon,\mathbf{p}})_*\nu - (\Phi_\varepsilon)_*\nu\big)(\varphi) = \int_0^1 \sum_{\mathbf{q}\in\mathbb{Z}^d} (F_t)_*\big(\Delta_{\mathbf{q}}\cdot\nu\big)(\partial_{\mathbf{q}}\varphi)dt. \tag{4.19}$$

As, in our case, $\Delta_{\mathbf{q}} = 0$ if $|\mathbf{q}-\mathbf{p}| > r$, we conclude

$$\begin{aligned} |(\Phi_{\varepsilon,\mathbf{p}})_*\nu - (\Phi_\varepsilon)_*\nu| &\le \sum_{|\mathbf{q}-\mathbf{p}|\le r} \sup_{0\le t\le 1} \mathrm{Var}_{\mathbf{q}}\big((F_t)_*(\Delta_{\mathbf{q}}\cdot\nu)\big) \\ &\le C \sum_{|\mathbf{q}-\mathbf{p}|\le r} \mathrm{Var}^{\mathbf{q}}(\Delta_{\mathbf{q}}\cdot\nu), \end{aligned} \tag{4.20}$$

where we used (4.13) (which applies as well to $(F_t)_*$) for the second inequality. Hence, by Lemma 2.5,

$$\begin{aligned} |(\Phi_{\varepsilon,\mathbf{p}})_*\nu - (\Phi_\varepsilon)_*\nu| &\le C \sum_{|\mathbf{q}-\mathbf{p}|\le r} \Big(|\Delta_{\mathbf{q}}|_\infty \mathrm{Var}^{\mathbf{q}}(\nu) + \sup_{\mathbf{j}\in\mathbb{Z}^d} \rho^{|\mathbf{j}-\mathbf{q}|}\mathrm{Lip}_j(\Delta_{\mathbf{q}})|\nu|\Big) \\ &\le C\varepsilon \mathrm{Var}^{\mathbf{q}}(\nu) \end{aligned}$$

in view of assumption (1.1). This is (4.17) and finishes the proof of the lemma. □

Appendix A. Proof of Theorem 1.2 Assuming Theorem 1.3

The operator P_ε has a simple eigenvalue at 1, and the corresponding eigenfunction is the invariant measure μ_ε obtained in [9]. By classical analytic perturbation theory, the operator $P_{t,\varepsilon}$ has for small t a unique eigenvalue $\lambda(t)$ close to 1, which is still simple. Let Π_t denote the corresponding spectral projection, and $\mu_{t,\varepsilon} = \Pi_t(\mu_\varepsilon)$. There exist $\delta < 1$ and $C > 0$ such that, for all small enough t, for all $n \in \mathbb{N}$,

$$\left|\int e^{itS_n f} d\mu_\varepsilon - \lambda(t)^n \mu_{t,\varepsilon}(1)\right| \le C\delta^n. \tag{A.1}$$

Hence, a precise description of the eigenvalue $\lambda(t)$ will imply a central limit theorem for the Birkhoff sums $S_n f$.

Let $\nu_t = \mu_{t,\varepsilon}/\mu_{t,\varepsilon}(1)$. Differentiating the equality $P_{t,\varepsilon}\nu_t = \lambda(t)\nu_t$ and using $\left.\frac{dP_{t,\varepsilon}}{dt}\right|_{t=0} = P_\varepsilon(if\cdot)$, we get

$$P_\varepsilon(if\mu_\varepsilon) + P_\varepsilon(\nu_0') = \lambda'(0)\mu_\varepsilon + \nu_0'. \tag{A.2}$$

Integrating the function 1 with respect to this equality, we obtain

$$i\int f d\mu_\varepsilon + \int d\nu_0' = \lambda'(0) + \int d\nu_0'. \tag{A.3}$$

Since $\int f\, d\mu_\varepsilon = 0$, we therefore have $\lambda'(0) = 0$.

Differentiating twice $P_{t,\varepsilon}\nu_t = \lambda(t)\nu_t$ yields

$$P_\varepsilon(-f^2\mu_\varepsilon) + 2P_\varepsilon(if\nu_0') + P_\varepsilon(\nu_0'') = \nu_0'' + \lambda''(0)\mu_\varepsilon. \tag{A.4}$$

Integrating the function 1 yields

$$\lambda''(0) = -\int f^2 d\mu_\varepsilon + 2i\int f\, d\nu_0'. \tag{A.5}$$

From (A.2), we have $\nu_0' = P_\varepsilon\nu_0' + P_\varepsilon(if\mu_\varepsilon)$. Iterating this equation gives

$$\nu_0' = P_\varepsilon^n\nu_0' + i\sum_{k=1}^{n} P_\varepsilon^k(f\mu_\varepsilon). \tag{A.6}$$

Since $\nu_t(1) = 1$, we have $\nu_0'(1) = 0$. The space $\{\mu \in \mathcal{D} : \mu(1) = 0\}$ is closed and P_ε leaves this space invariant, therefore its spectral radius on this space is < 1. This implies that $P_\varepsilon^n\nu_0'$ converges exponentially fast to 0. In the same way, $(f\mu_\varepsilon)(1) = 0$, hence $P_\varepsilon^k(f\mu_\varepsilon)$ converges exponentially fast to 0 in $\mathcal{D}$. Letting n tend to infinity in (A.6), we get $\nu_0' = i\sum_{k=1}^\infty P_\varepsilon^k(f\mu_\varepsilon)$. In particular, $\nu_0'(f) = i\sum_{k=1}^\infty \int f\cdot f\circ T_\varepsilon^k\, d\mu_\varepsilon$, and this series converges exponentially fast. From (A.5), we obtain

$$\lambda''(0) = -\int f^2\, d\mu_\varepsilon - 2\sum_{k=1}^\infty \int f\cdot f\circ T_\varepsilon^k\, d\mu_\varepsilon. \tag{A.7}$$

Moreover,

$$\begin{aligned}
\int \left(\sum_{k=0}^{n-1} f\circ T_\varepsilon^k\right)^2 d\mu_\varepsilon &= n\int f^2\, d\mu_\varepsilon + 2\sum_{k=1}^{n}(n-k)\int f\cdot f\circ T_\varepsilon^k\, d\mu_\varepsilon \\
&= -n\lambda''(0) - 2\sum_{k=1}^\infty k\int f\cdot f\circ T_\varepsilon^k\, d\mu_\varepsilon + O(\delta^n) \\
&= -n\lambda''(0) + O(1).
\end{aligned}$$

Since this integral is non-negative, this shows that $\lambda''(0) \le 0$. Hence, we can write $\lambda''(0) = -\sigma^2$ for some $\sigma \ge 0$. Furthermore, if $\lambda''(0) = 0$, then S_nf is bounded in L^2, which implies that f can be written as $u - u\circ T_\varepsilon$ in L^2 (see e.g. [6]). This proves the nondegeneracy criterion in Theorem 1.2.

Since $\lambda(t) = -\sigma^2t^2/2 + o(t^2)$, $\lambda(t/\sqrt{n})^n \to e^{-\sigma^2t^2/2}$. Together with (A.1), this shows that $S_nf/\sqrt{n}$ converges in distribution to $\mathcal{N}(0,\sigma^2)$ and proves the central limit theorem.

The method to derive the local limit theorem from the description of $\lambda(t)$ for small t and the control of the spectral radius of $P_{t,\varepsilon}$ for all $t \neq 0$ is also classical. We know indeed (see for example [2]) that it is sufficient to prove that

$$\sigma\sqrt{2\pi n} \int_\Omega h \circ S_n f \, d\mu_\varepsilon \to \int_{\mathbb{R}} h \, dm \tag{A.8}$$

for any L^1 function h whose Fourier transform is continuous and null outside a compact interval. One has in this case

$$\sigma\sqrt{2\pi n} \int_\Omega h \circ S_n f \, d\mu_\varepsilon = \sigma\sqrt{\frac{n}{2\pi}} \int_{-K}^{K} \int_\Omega e^{itS_n f} \, d\mu_\varepsilon \, \hat{h}(t) \, dt. \tag{A.9}$$

The limit term is obtained from application of (A.1) in a neighborhood of $t = 0$, while the rest of the integral goes uniformly to 0 since the spectral radius of $P_{t,\varepsilon}$ is < 1 for any $t \neq 0$. The reader is referred to [4] for further details.

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